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Non-linear dynamical systems

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NON-LINEAR DYNAMICAL SYSTEMS

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APPROVED FOR THE DEPARTMENT OF
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ABSTRACT

NON-LINEAR DYNAMICAL SYSTEMS

by Marina Marinas

The purpose of this thesis is to present the main definitions and results of discrete dynamical systems. In the first chapter we define the main notions of dynamical systems in R^1 and prove the fundamental theorem due to Sarkovskii. This theorem defines an hierarchy of periodic points for any continuous map. We also define the notion of chaotic dynamical systems and give an example of such a system. The second chapter extends some of the results of the first chapter into the complex plane. In particular, we define the notion of the Julia set, describe explosions on Julia sets and give examples of dynamical systems in the complex plane.

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Contents

Preface

There are three aspects of dynamical systems theory.

Historically, the subject is an outgrowth of the qualitative theory of differential equations and this latter has applications in the study of physical systems. More specifically, there are connections with theoretical dynamics and nonlinear mechanics.

Secondly, dynamical systems theory may well be considered as applied topology, and indeed general analytic and algebraic topology have all played fundamental roles in the development of the subject.

Finally, it is proposed that dynamical systems theory is an independent mathematical discipline with its own subject-matter, proof methods, and fundamental results.

It was Poincaré who first formulated and solved problems of dynamics as problems in topology.

Markov performed the logical next step of defining an essentially topological concept, containing as special case differential equations. This was called a dynamical system.

There are different types of dynamical systems such as differential equations and iterated functions. The former give examples of continuous dynamical systems, while the latter give examples of discrete dynamical systems.

Dynamical systems have applications in all branches of science (difference equations, differential equations, mathematical economics, classical mechanics, physics, biology, to name a few).

There are non-trivial connections between dynamical systems on the complex plane and fractals. In particular, the Julia sets of many dynamical systems in **C** are fractal sets.

The purpose of this thesis is to introduce the main definitions and results of discrete dynamical systems.

In the first chapter we define the main notions of dynamical systems in **R** and prove the fundamental theorem due to Sarkovskii. This theorem defines a hierarchy of periodic points of any continuous map. We also define the notion of chaotic dynamical systems and give an example of such a system.

The second chapter extends some of the results of the first chapter into the complex plane. In particular, we define the notion of the Julia set, describe explosions on Julia sets and give examples of dynamical systems in the complex domain.

CHAPTER 1.

DISCRETE DYNAMICAL SYSTEMS AND CHAOS.

In this chapter we shall discuss the main notions of discrete dynamical systems and shall prove the fundamental result known as Sarkovskii's Theorem.

We shall also give the definition of chaotic behaviour (or, simply chaos) and we shall present some examples of such behaviour.

To make the presentation self-contained, we will also include some basic definitions. They can be found in most of the articles on the subject and we do not give specific references in such instances.

1. Basic Definitions and Results for Dynamical Systems.

Throughout this chapter f denotes a continuous mapping from a topological space X to itself, where X is usually $\mathbf{R}^1$, the real numbers, or S^1, the unit circle in the plane.

Definition 1.1 $f^n(x) = (f \circ f \circ \ldots\ldots \circ f)(x)$, $n \geq 2$, $n \in \mathbf{Z}$.
Here "o" denotes the usual composition of functions.

Definition 1.2 A point $x_0 \in \mathbf{R}^1$ is called a ***fixed point*** for the mapping f if $f(x_0) = x_0$. The set of all fixed points for f is called the ***fixed set*** and will be denoted by F_f.

Example 1.1 Let $f(x) = \mu x(1 - x)$, $\mu \neq 0$. Let us find all the fixed points of this mapping. To do that, we have to solve the equation

$$\mu x(1 - x) = x$$

or

$$x(\mu - \mu x - 1) = 0.$$

The solutions of this equation are $x = 0$ and $x = (\mu-1)/\mu$. Therefore if $\mu = 1$, $f(x)$ has one fixed point $x = 0$, and if $\mu \neq 1$, $f(x)$ has two fixed points: $x_1 = 0$ and $x_2 = (\mu-1)/\mu$.

Remark. In some cases, the existence of fixed points can be proved without explicit calculations. As an example we have the following simple result:

Theorem 1.1 Let f be a continuous function from [0,1] to [0,1]. Then f has a fixed point in [0,1].

Proof: The intuitive idea behind the proof is obvious:

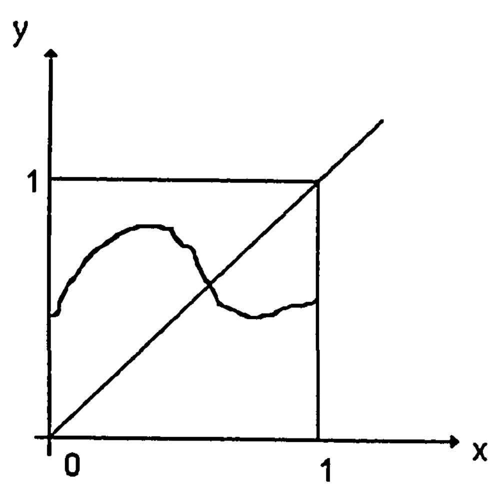

Fig.1

The diagonal must intersect the graph of the function.

Here is the formal proof:

Let $g(x) = f(x) - x$; $g(x)$ is a continuous function.

$g(0) = f(0)$

$g(1) = f(1) - 1$

If $f(1) = 1$ then 1 is a fixed point of f.

If $f(0) = 0$ then 0 is a fixed point of f.

Otherwise $g(0) > 0$ and $g(1) < 0$. So, by the Intermediate Value Theorem, there exists a point $c \in (0,1)$ such that $g(c) = 0$, i.e. $f(c) = c$.

#

Definition 1.3 A point x_0 is called a ***periodic point*** of period k for f if there exists a positive integer k such that $f^k(x_0) = x_0$.

The smallest such number k (if it exists) is called the ***prime period*** of f.

The set of all periodic points of f is called the ***periodic set*** and we will denote it by P_f.

A period will always mean the prime period unless otherwise stated.

Of course if $k = 1$, x_0 is a fixed point.

In general, it is very difficult to determine all periodic points of a given mapping. However, in some simple cases, explicit calculations can be carried through.

Example 1.2

a) Let S^1 be the unit circle in the plane.

$S^1 = \{ \alpha \in R^2 \mid \alpha = e^{i\theta},\ 0 \leq \theta < 2\pi \}$.

A point on S^1 can be parametrized by its angle θ.

Let $f_\mu(\alpha) = e^{i(\theta + 2\mu\pi)}$ where $\mu = 1/n$, $n \in \mathbf{N}$. (Note that f_μ is multiplication by a fixed primitive n-th root of unity.)

Then $f_\mu^n(\alpha) = e^{i(\theta + 2\pi)} = e^{i\theta} \cdot e^{i2\pi} = \alpha$ so that $f_\mu^n(\alpha) = \alpha$ for any α.

Hence all points on S^1 are periodic of period n for f.

b) Let $f(x) = -x^3$. Then $f^n(x) = (-1)^n x^{3^n}$ and $f^n(x) = x$ if $(-1)^n x^{3^n} = x$.

$x = 0$ is a solution of this equation but $f(0) = 0$ so that 0 is a fixed point for f(x).

The equation $(-1)^n x^{3^n - 1} = 1$ has solutions $x = -1$ and $x = 1$ if n is even and has no solution if n odd. Hence $x = -1$ and $x = 1$ are periodic points of period 2 and there are no other periodic points of f(x).

c) Let $f(x) = \mu x(1 - x)$. Then

$f^2(x) = \mu^2 x(1 - x)[1 - \mu x(1 - x)]$

$f^2(x) = x$ if $\mu^2 x(1 - x)[1 - \mu x(1 - x)] = x$

$x = 0$ is a solution of this equation but again $x = 0$ is fixed point of f.

The other solutions are the roots of the equation $\mu^2(1-x) - \mu^3 x(1-x)^2 = 1$.

So the periodic points are solutions of a cubic equation and, therefore, depending on μ, f(x) has 1 or 3 periodic points of period 2.

It is clear that in order to determine periodic points of period $n \geq 3$, we will have to solve equations of order 6 or higher, which is impossible in general.

d) Let $g(\theta) = 4\theta$. We will show that the periodic points of g are dense in S^1.

We have $g^n(\theta) = 4^n\theta$ and, therefore, the periodic points of g are the solutions of the equation $4^n\theta = \theta + 2k\pi$ and $\theta = (2k\pi)/(4^n-1)$. To show that these points are dense in S^1 is equivalent to showing that the points $k/(4^n-1)$ are dense in [0,1].

For each n, the fractions $k/(4^n-1)$ divide the interval [0,1] into equal parts of length $1/(4^n - 1)$.
Any point $a \in [0,1]$ is in one of these intervals and, therefore, the distance between a and one of the numbers $k/(4^n - 1)$ cannot exceed the length of the interval, $1/(4^n - 1)$, but $\lim 1/(4^n - 1) = 0$ as n goes to infinity.
Therefore the set of points $\{k/(4^n - 1)\}$, n =1,2,...; k = 0,1,...,n-1, is dense in [0,1].

Definition 1.4 A ***discrete dynamical system*** is the set $\{f^n\}$, n = 0, 1, 2, 3,......., where f is a continuous map from the topological space X to itself.

To summarize, the explicit approach to dynamical systems even for simple functions becomes too unwieldy in general and qualitative considerations become more useful.

Definition 1.5 The ***orbit*** of a point x_0 for the function f(x) is the set of points $\{x_0, f(x_0), f^2(x_0), ...\}$
We can describe the orbit using the ***dynamical portrait.***

Example 1.3

a) $f(x) = -1/2\,x$.

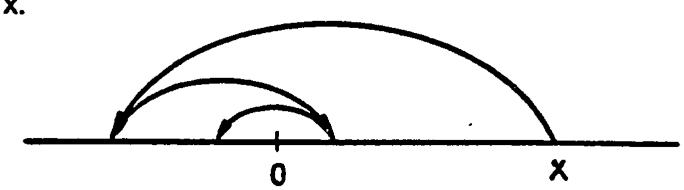

Fig. 2

b) $f(x) = 3x$.

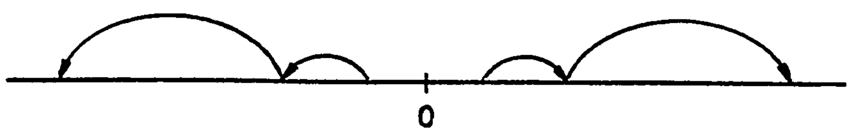

Fig. 3

c) $f(x) = x^2$.

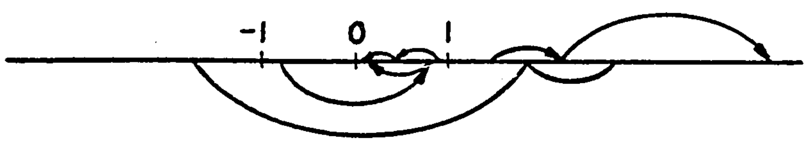

Fig. 4

From the diagrams we can see that these functions do not have periodic points, except for fixed points.

Another approach to analyzing dynamical systems is to use ***graphical analysis.***

Example 1.4

a) $f(x) = -x, \; -\infty < x < \infty$.

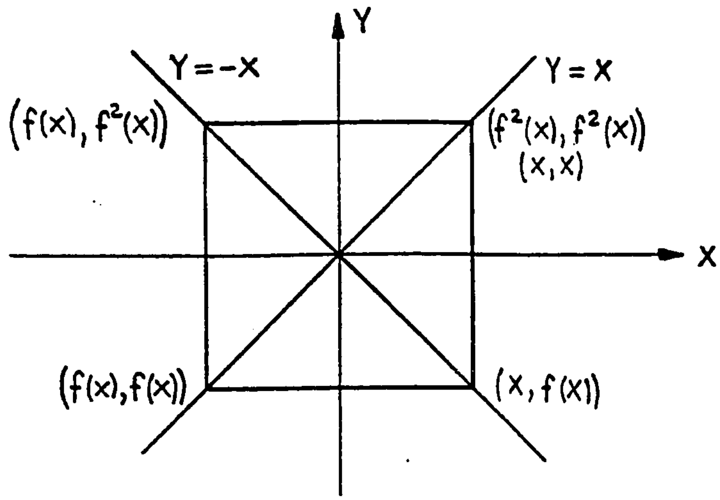

Fig. 5

b) $f(x) = x - x^2$.

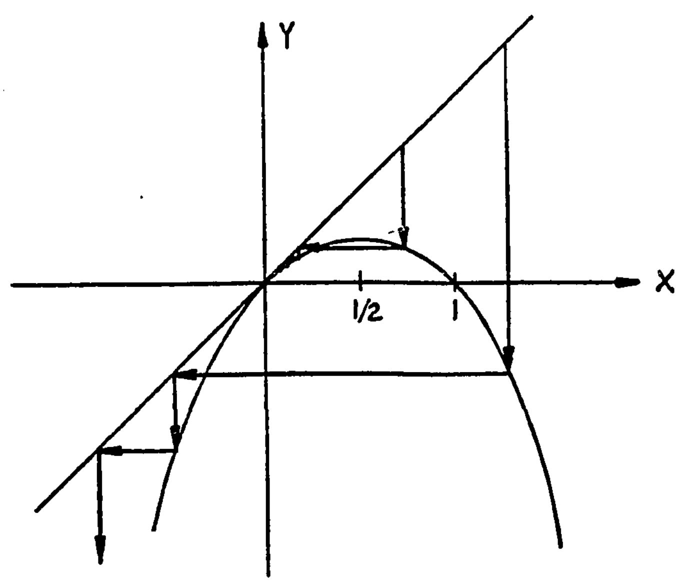

Fig.6

c) $f(x) = -x^3$.

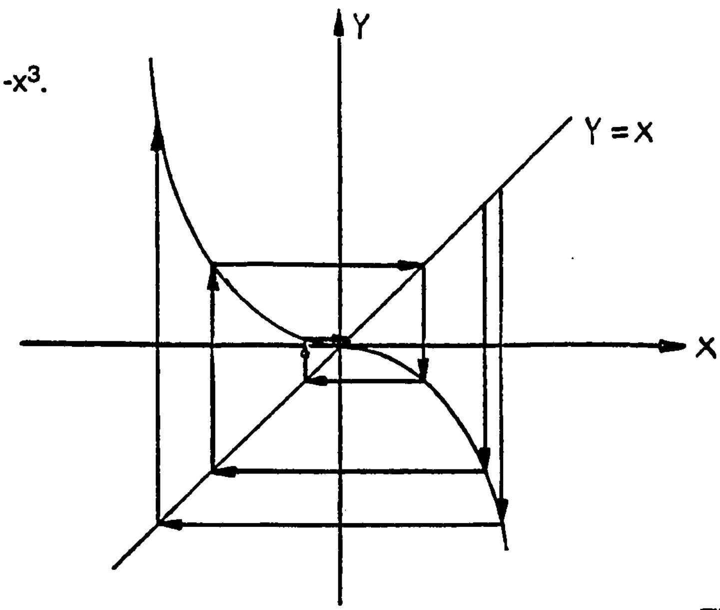

Fig. 7

From this graphical analysis we see that any point is periodic of period two for $f(x) = -x$, there are no periodic points for $f(x) = x - x^2$, and there are two periodic points of period two: — $x = -1$ and $x = 1$ — for $f(x) = -x^3$ but no other periodic points.

There are also some negative results about the existence of periodic points. For example:

Theorem 1.2 A self-homeomorphism of $\mathbf{R}^1$ can have no periodic points with period greater than 2.

Note. A homeomorphism is a continuous, one-to-one, onto mapping from $\mathbf{R}^1$ to $\mathbf{R}^1$ with a continuous inverse.

Proof: Let f be a homeomorphism. Then it is easy to show that f is monotone.

Assume that f is increasing and that $x < f(x)$. Then

$$f(x) < f^2(x),\ f^2(x) < f^3(x),\ f^3(x) < f^4(x), \ldots..$$

Therefore

$$x < f(x) < f^2(x) < f^3(x) < f^4(x) < \ldots\ldots\ldots$$

i.e. x is not a periodic point.

Assume f is increasing and that $x > f(x)$. Then

$$f(x) > f^2(x),\ f^2(x) > f^3(x),\ f^3(x) > f^4(x) \ldots\ldots.$$

Therefore

$$x > f(x) > f^2(x) > f^3(x) > f^4(x) > \ldots\ldots\ldots$$

i.e. x is not a periodic point.

Assume f is decreasing and that $x < f(x)$. Then

$$f(x) > f^2(x),\ f^2(x) < f^3(x),\ f^3(x) > f^4(x) > \ldots\ldots\ldots.$$

There are three cases:

1^0. $f^2(x) = x$. Then x has period 2.

2^0. $f^2(x) < x$. Then there are two possibilities for $f^3(x)$:

a) $f^3(x) \leq f(x)$ b) $f^3(x) > f(x)$.

Since f^{-1} is also decreasing, a) implies $f^2(x) \geq x$ (contradiction).

Then $f^3(x) > f(x)$ and continuing we have:

$$f^4(x) < f^2(x) < x < f(x) < f^3(x) < f^5(x) < \ldots\ldots\ldots$$

i.e. x is not a periodic point.

3^0. $f^2(x) > x$. Then there are two possibilities for $f^3(x)$:

a) $f^3(x) \geq f(x)$ b) $f^3(x) < f(x)$.

Since f^{-1} is also decreasing, a) implies $f^2(x) \leq x$ (contradiction).

Then $f^3(x) < f(x)$ and continuing we have:

$$x < f^2(x) < f^4(x) < f^5(x) < f^3(x) < f(x) < \ldots\ldots\ldots$$

i.e. x is not a periodic point.

The case when f is decreasing and $x > f(x)$ can be treated similarly.

Thus f cannot have periodic points with period greater than 2.

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Example 1.5 Homeomorphism of $\mathbf{R}^1$ with periodic points of period 2:

$$f(x) = -x;$$

Then $f^2(x) = x$ for any $x \in \mathbf{R}$ so that any point is periodic with period 2.

See also example 1.4 (c) above.

Definition 1.6 A point x_0 is called ***eventually periodic*** for f if there exists $m > 0$ such that $f^m(x_0)$ is a periodic point for f and none of the points x_0, $f(x_0)$, $f^2(x_0)$,.........$f^{m-1}(x_0)$ is periodic.

Example 1.6 Let $f(x) = x^2$. Then $f(-1) = 1$ and $f(1) = 1$, therefore $f^2(-1) = 1$ and $f^k(-1) = 1$ for $k \geq 1$ i.e. $f(-1)$ is fixed point of $f(x)$. Hence $x = -1$ is eventually fixed for f.

Definition 1.7 Assume f is differentiable and let x_0 be a periodic point of period n for f(x).

If $|(f^n)'(x_0)| \neq 1$, x_0 is called ***an hyperbolic periodic point.***

If $|(f^n)'(x_0)| < 1$, x_0 is called ***an attracting periodic point.***

If $|(f^n)'(x_0)| > 1$, x_0 is called ***a repelling periodic point.***

Example 1.7 Let $f(x) = 2(x - x^2)$. The fixed points of f(x) are the roots of the equation $2(x - x^2) = x$ i.e. $x = 0$ and $x = 1/2$.
Also $f'(x) = 2 - 4x$. Then $f'(0) = 2$ and $f'(1/2) = 0$ so that $| f'(0) | > 1$ and $| f'(1/2) | < 1$, i.e. $x = 0$ is an hyperbolic repelling fixed point for f and $x = 1/2$ is an hyperbolic attracting fixed point for f.

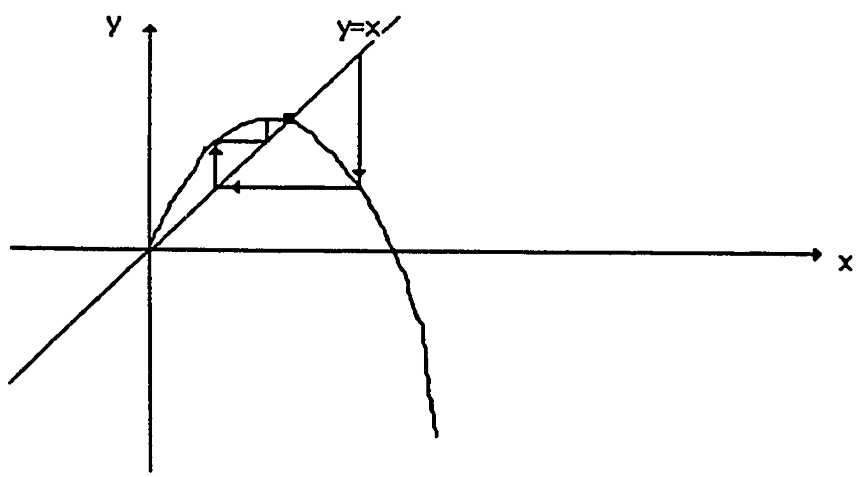


Fig. 8

Intuitively, if x_0 is an attracting periodic point of f, then there is a neighborhood of x_0 such that any point in it approaches x_0 upon iterations of f. On the other hand, if x_0 is a repelling periodic point of f, then there is a neighborhood of x_0 such that any point in it goes away from x_0 upon iterations of f.

2. Sarkovskii' s Theorem.

In this section, we will prove two theorems due to Sarkovskii.

Sarkovskii I. If $f : \mathbf{R}^1 \rightarrow \mathbf{R}^1$ is continuous and has a periodic point of period three, then f has periodic points of all other periods.

Proof:

Step 1. Let a be a periodic point of period 3 and let $b = f(a)$ and $c = f(b)$. Then $f(c) = a$. Let us assume that $a < b < c$. The other cases are similar.

Let $\mathbf{I}_0 = [a,b]$ and $\mathbf{I}_1 = [b,c]$.

Since f is a continuous function, for any $u \in [f(a), f(b)]$ there exists $\propto \in [a,b]$ such that $u = f(\propto)$. Hence $\mathbf{I}_1 \subset f(\mathbf{I}_0)$ (since $\mathbf{I}_1 = [f(a), f(b)]$).

Since $f(b)$, $f(c) \in f(\mathbf{I}_1)$ and $f(b) = c$, $f(c) = a$ it follows that $[a,c] \subset f(\mathbf{I}_1)$. But $\mathbf{I}_0$, $\mathbf{I}_1 \subset [a,c]$. Then $\mathbf{I}_0$, $\mathbf{I}_1 \subset f(\mathbf{I}_1)$.

Step 2.

$f(b) - b = c - b > 0$.

$f(c) - c = a - c < 0$.

Let $g(x) = f(x) - x$. Then $g(b) > 0$ and $g(c) < 0$. Hence, by the Intermediate Value Theorem, there exists a point $x \in [b,c]$ such that $g(x) = 0$ or $f(x) = x$. So f has a fixed point between b and c.

$f^2(a) - a = f(f(a)) - a = f(b) - a = c - a > 0$.

$f^2(b) - b = f(f(b)) - b = f(c) - b = a - b < 0$.

Let $g(x) = f^2(x) - x$. Then $g(a) > 0$ and $g(b) < 0$. Hence, by the Intermediate Value Theorem, there exists $x \in [a,b]$ such that $g(x) = 0$ or $f^2(x) = x$. Hence f^2

has fixed points between a and b. One can show that at least one of these points has prime period two, i.e. it is a periodic point of period two for f.

Step 3. If **I** is a closed interval and $\mathbf{I} \subset f(\mathbf{I})$ then f has a fixed point in **I**.

Proof: Let $\mathbf{I} = [a,b]$. Since $\mathbf{I} \subset f(\mathbf{I})$, there exist points $x_1, x_2 \in \mathbf{I}$ such that $f(x_1) < a$ and $f(x_2) > b$.

Let $g(x) = f(x) - x$. Then

$$g(x_1) = f(x_1) - x_1 < f(x_1) - a < 0$$

and $g(x_2) = f(x_2) - x_2 > f(x_2) - b > 0$.

By the Intermediate Value Theorem these inequalities ímply that there exists a point c between x_1 and x_2 such that $g(c) = 0$ or $f(c) = c$; hence c is a fixed point for f.

Step 4.

We have to show that f also has periodic points of period $n > 3$.

Let define a nested sequence of intervals $A_0, A_1, A_2, \ldots\ldots$ in $\mathbf{I}_1$ as follows:

Let $A_0 = \mathbf{I}_1$. $A_0 \subset f(A_0)$, since $\mathbf{I}_1 \subset f(\mathbf{I}_1)$. Then there exists $A_1 \subset A_0$ such that $f(A_1) = A_0$.

Since $A_1 \subset A_0$ and $f(A_1) = A_0$, there exists $A_2 \subset A_1$ such that $f(A_2) = A_1$.

Then $f^2(A_2) = f(A_1) = A_0$.

Continuing in this way, there exists $A_{n-2} \subset A_{n-3}$ such that $f(A_{n-2}) = A_{n-3}$.

Then $f^{n-2}(A_{n-2}) = f^{n-3}(f(A_{n-2})) = f^{n-3}(A_{n-3}) = \ldots\ldots\ldots = f^2(A_2) = A_0$.

So far we have constructed the nested sequence:

$\mathbf{I}_0 = A_0 \supset A_1 \supset A_2 \supset \ldots\ldots\ldots \supset A_{n-3} \supset A_{n-2}$.

Since $f^{n-2}(A_{n-2}) = A_0$ and $f(A_0) \supset \mathbf{I}_0$, it follows that $f^{n-1}(A_{n-2}) = f(A_0) \supset \mathbf{I}_0$. Then there exists $A_{n-1} \subset A_{n-2}$ such that $f^{n-1}(A_{n-1}) = \mathbf{I}_0$ which implies $f^n(A_{n-1}) = f(\mathbf{I}_0)$. But $\mathbf{I}_1 \subset f(\mathbf{I}_0)$, hence $\mathbf{I}_1 \subset f^n(A_{n-1})$.

Since $A_{n-1} \subset A_{n-2} \subset \mathbf{I}_1$, it follows that $A_{n-1} \subset f^n(A_{n-1})$ or $f^n(A_{n-1})$ covers A_{n-1}. This implies, using the result of Step 3, that f^n has a fixed point in A_{n-1}, say p; i.e. $p \in A_{n-1}$ such that $f^n(p) = p$.

Step 5. We can prove that p has actually prime period n (not less).

$p \in A_{n-1}$; hence $p \in \mathbf{I}_1$ and also

$p \in A_1$; hence $f(p) \in f(A_1)$, or $f(p) \in \mathbf{I}_1$;

$p \in A_2$; hence $f^2(p) \in f^2(A_2)$, or $f^2(p) \in \mathbf{I}_1$ etc., until

$p \in A_{n-2}$; hence $f^{n-2}(p) \in f^{n-2}(A_2)$, or $f^{n-2}(p) \in \mathbf{I}_1$;

$p \in A_{n-1}$; hence $f^{n-1}(p) \in f^{n-1}(A_{n-1})$, or $f^{n-1}(p) \in \mathbf{I}_0$.

Therefore $f^{n-1}(p) \neq p$ since $p \in \mathbf{I}_1$.

Assume $f^{n-i}(p) = p$ for some i, $2 \le i \le n-1$. Then $f^{n-1}(p) = f^{i-1}(p)$, $1 \le i-1 \le n-2$, or $f^{n-1}(p) \in \mathbf{I}_1$, which is false. Hence $f^{n-i}(p) \neq p$ for any i, $1 \le i \le n-1$, so that p has prime period n.

If $f^{n-1}(p)$ lies on the boundary of $\mathbf{I}_0$, then $f^{n-1}(p) = a$ or $f^{n-1}(p) = b$. Since $f^n(p) = p$, these yield $f(a) = p$ or $f(b) = p$. But $f(a) = b$ and $f(b) = c$ so that $p = b$ or $p = c$. Hence p has period 3 or $n = 3$. This is a contradiction since on step 4 we assumed $n > 3$. Therefore $f^{n-1}(p)$ is not on the boundary of $\mathbf{I}_0$.

#

Consider the following ordering of the natural numbers:

$3 > 5 > 7 > 9 > 11 > \dots\dots > 2 \cdot 3 > 2 \cdot 5 > 2 \cdot 7 > \dots\dots > 2^2 \cdot 3 > 2^2 \cdot 5$
$> 2^2 \cdot 7 > \dots\dots > 2^3 \cdot 3 > 2^3 \cdot 5 > 2^3 \cdot 7 > \dots\dots > 2^3 > 2^2 > 2 > 1.$

That is , first list all odd numbers except 1, followed by 2 times odds, 2^2 times the odds, 2^3 times the odds etc. This exhausts all the natural numbers with the exception of the powers of 2 which are listed last, in decreasing order.
This is ***Sarkovskii's ordering of the natural numbers.***

Sarkovskii II. If $f : \mathbf{R}^1 \to \mathbf{R}^1$ is continuous and has a periodic point of prime period k, then f also has a periodic point of period h for any h such that $k > h$, where $>$ is Sarkovskii's ordering of the natural numbers.

Proof:

Remark. In the following, the notation $I_1 \to I_2$ is used if $I_2 \subset f(I_1)$.

Step 1. If a sequence of intervals is such that $I_1 \to I_2 \to \dots\dots \to I_n \to I_1$, then f has a periodic point of period n in I_1.

Proof:

Since $I_1 \to I_2 \to \dots\dots \to I_n \to I_1$ we have:

$I_2 \subset f(I_1)$, $I_3 \subset f(I_2)$,.........$I_n \subset f(I_{n-1})$, $I_1 \subset f(I_n)$.

Hence $I_1 \subset f(I_n) \subset f^2(I_{n-1}) \subset f^3(I_{n-2}) \subset \dots\dots \subset f^{n-2}(I_3) \subset f^{n-1}(I_2) \subset f^n(I_1)$.

But if $I_1 \subset f^n(I_1)$ it follows by Step 3 of the previous proof that f^n has a fixed point in I_1, i.e. f has a periodic point of period n in I_1.

Step 2. Assume first that f has a periodic point x of period n with n odd such that $n > 1$. Suppose that f has no periodic points of odd period less than n.

If $\{x, f(x), f^2(x), \ldots\ldots f^{n-1}(x)\}$ is the orbit of x, rearrange its elements in increasing order and get $x_1, x_2, \ldots\ldots x_n$. Then $x_n = f^i(x)$ for some i, $0 \le i \le n-1$.

Hence $f(x_n) = f^{i+1}(x) < x_n$ (If $f^{i+1}(x) = x_n$, then $f(x_n) = x_n$, i.e. x_n has period 1, which would contradict our assumption).

Choose the largest i for which $f(x_i) > x_i$. Since $f(x_{i+1}) < x_{i+1}$ it follows that $f(x_{i+1}) \le x_i$. Since $f(x_i) > x_i$ it follows that $f(x_i) \ge x_{i+1}$. Then

$$f(x_{i+1}) \le x_i < x_{i+1} \le f(x_i).$$

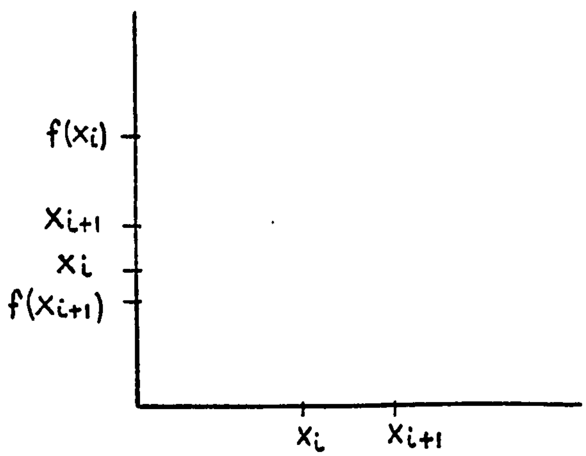


Fig.9

Let $\mathbf{I}_1 = [x_i, x_{i+1}]$. Then $\mathbf{I}_1 \subset f(\mathbf{I}_1)$ and therefore $\mathbf{I}_1 \to \mathbf{I}_1$.

If $f(x_{i+1}) = x_i$ and $f(x_i) = x_{i+1}$ then $f^2(x_i) = f(f(x_i)) = f(x_{i+1}) = x_i$.

But $x_i = f^k(x)$ for some k, $0 \le k \le n-1$. Then $f^2(f^k(x)) = f^k(x)$ or $f^k(f^2(x)) = f^k(x)$.

Hence $f^2(x) = x$. This is not possible since x has an odd period $n > 1$. Then either $f(x_{i+1}) < x_i$ or $f(x_i) > x_{i+1}$, so that $f(\mathbf{I}_1)$ contains at least one interval of the form $[x_j, x_{j+1}]$, call it $\mathbf{I}_2$. But $\mathbf{I}_2 \subset f(\mathbf{I}_1)$ implies $\mathbf{I}_1 \to \mathbf{I}_2$.

Continuing, we find $\mathbf{I}_3, \ldots\ldots, \mathbf{I}_k$ such that $\mathbf{I}_{j+1} \subset f(\mathbf{I}_j)$.

Since n is odd, one can show that there are more x_i's on one side of I_1 than on the other, so that some x_i's must change sides under the action of f and some must not. Consequently I_1 is in $f(I_k)$ or $f(I_k) \to I_1$ for some k. We thus have $I_1 \to I_2 \to .... \to I_k \to I_1$. Let k be the smallest number for which this happens, i.e. $I_1 \to I_2 \to ..... \to I_k \to I_1$ is the shortest path from I_1 to I_1 except, of course, $I_1 \to I_1$ after one iteration. We therefore obtain the diagram:

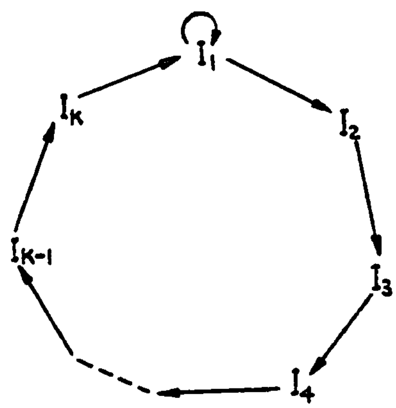
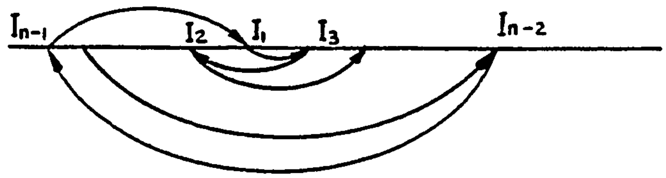


Fig. 10

If $k < n - 1$, then $k + 1 < n$ and the loops $I_1 \to I_2 \to .... \to I_k \to I_1$ or $I_1 \to I_2 \to .... \to I_k \to I_1 \to I_1$ give a periodic point of period k and k + 1. One of these numbers is odd. But we assumed that we cannot have a periodic point of odd period less than n. Hence $k = n - 1$.

Since k is the smallest integer that works, we cannot have $I_h \to I_j$ for any $j > h + 1$. It follows that the orbit of x must be ordered in R in one of two possible ways:



One possible ordering of $\mathbf{I}_j$. Fig. 11

The other is the mirror image

$I_{n-1} \rightarrow I_{n-4} \rightarrow I_{n-3} \rightarrow I_{n-2} \rightarrow I_{n-1}$.

Therefore we can extend the previous diagram to the following one:

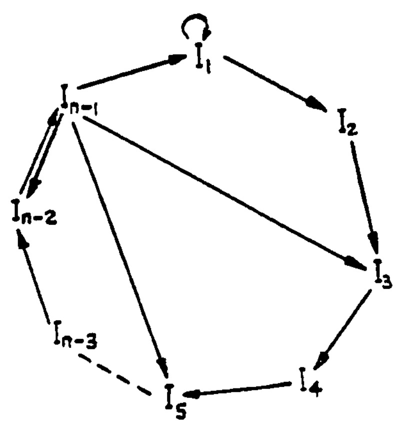


Fig. 12

Sarkovskii's Theorem for the special case of n odd is now immediate. Points with period larger than n are given by cycles of the form $\mathbf{I}_1 \rightarrow ..... \rightarrow \mathbf{I}_{n-1} \rightarrow \mathbf{I}_1 \rightarrow ...... \rightarrow \mathbf{I}_1$. Points with smaller even periods are given by cycles of the form $\mathbf{I}_{n-1} \rightarrow \mathbf{I}_{n-2} \rightarrow \mathbf{I}_{n-1}$ and so forth.

Similar considerations for all other cases complete the proof. For details see [1].

To show that Sarkovskii II is best possible, we will produce a map with a point with period 5 and no point with period 3.

We will define a function $f : [1,5] \to [1,5]$ such that $f(1) = 3$, $f(2) = 5$, $f(3) = 4$, $f(4) = 2$, $f(5) = 1$. Then

$f^5(1) = f^4(f(1)) = f^4(3) = f^3(f(3)) = f^3(4) = f^2(f(4)) = f^2(2) = f(f(2)) = f(5) = 1$,

so that 1 is a periodic point of period 5.

We take f to be linear between these integers, i.e. the graph of f is

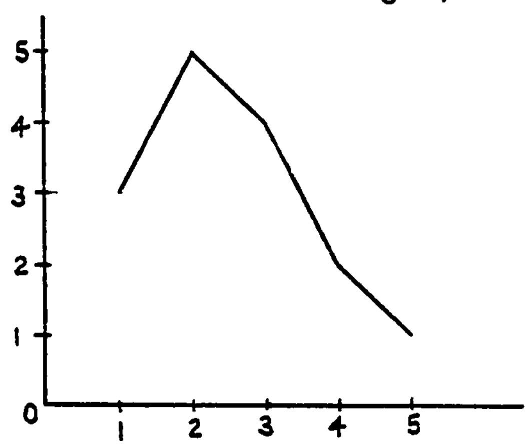


Fig.13

Then $f^3[1,2] = f^2(f[1,2]) = f^2[3,5] = f(f[3,5]) = f[1,4] = [2,5]$.

$f^3[2,3] = f^2([2,3]) = f^2[4,5] = f(f[4,5]) = f[1,2] = [3,5]$.

$f^3[4,5] = f^2(f[4,5]) = f^2[1,2] = f(f[1,2]) = f[3,5] = [1,4]$.

Hence f^3 has no fixed points in any of these intervals.

Also we have $f^3[3,4] = f^2(f[3,4]) = f^2[2,4] = f(f[2,4]) = f[2,5] = [1,5]$ so that the graph of f^3 intersects the line $y = x$, i.e. f^3 has at least one fixed point in [3,4]. But since $f : [3,4] \to [2,4]$, $f : [2,4] \to [2,5]$, $f : [2,5] \to [1,5]$ are monotonically decreasing, f^3 is also monotonically decreasing on [3,4]. Hence the graph of f^3 intersects the line $y = x$ only once. Therefore the fixed point of f^3 is unique. On the other hand, by Theorem 1.1, f has a fixed point. Therefore the fixed point of f^3 must be the fixed point of f, not a periodic point of period 3. Hence there are no periodic points of period three for f.

3. Chaos.

Definition 3.1 The maps $f : A \to A$ and $g : B \to B$ are called ***topologically conjugate*** if there exists a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$. The homeomorphism h is called a ***topological conjugacy.***

Remark. Two maps which are topologically conjugate have the same dynamical behaviour. In particular, if x_0 is a periodic point of period k for f, i.e. $f^k(x_0) = x_0$ then $h(x_0)$ is a periodic point of period k for g. Indeed $g^k(h(x_0)) = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} \circ \ldots\ldots\ldots \circ h \circ f \circ h^{-1} (h(x_0)) = h \circ f^k \circ h^{-1} (h(x_0)) = h \circ f^k(x_0) = h(x_0)$.

Definition 3.2 $f: J \to J$ is called ***topologically transitive*** if for any pair of open sets U, V in J there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$ (i.e. starting anywhere in $\mathbf{R}^1$ we can get, upon iteration of f, as close as we want to any other point).

Definition 3.3 $f : J \to J$ has **sensitive dependence on initial conditions** if there exists $\delta > 0$ such that, for any $x \in J$ and for any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$ (i.e. there exists points arbitrarily close to x which eventually separate from x by at least δ under iterations of f).

Remark. S^1 becomes a metric space if we define $a = e^{i\Theta_a}$, $0 \leq \Theta_a < 2\pi$, $b = e^{i\Theta_b}$, $0 \leq \Theta_b < 2\pi$, and $dist(a,b) = |\Theta_a - \Theta_b|$.

Example 3.1 Let $g(\Theta) = 4\Theta$. Then $g^2(\Theta) = 4^2\Theta$.

The map g is topologically transitive since any arc in S^1 is expanded by g^k (for some k) to cover all of S^1. Also, since the distance between two points is four times bigger upon iteration of g, g has sensitive dependence on initial conditions.

Now we will give the main definition of this section.

Definition 3.4 If V is a set, $f : V \to V$ is called ***chaotic*** on V if:

1) f is topologically transitive;

2) f has sensitive dependence on initial conditions;

3) Periodic points of f are dense in V.

Theorem 3.1 If f and g are topologically conjugate and if f is chaotic then g is also chaotic.

Proof: Assume $f : A \to A$ and $g : B \to B$ and let $h : A \to B$ be a homeomorphism such that $h \circ f = g \circ h$.

a) Let $U, V \subset B$. Since h is continuous and onto, there exists open sets

$U^*, V^* \subset A$ such that $h(U^*) = U$, $h(V^*) = V$.

Since f is chaotic, it is topologically transitive, i.e. there exists $k > o$ such that $f^k(U^*) \cap V \neq \emptyset$.

Since $h \circ f = g \circ h$ we also have $g = h \circ f \circ h^{-1}$. Hence:

$g^k(U) = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} \circ \ldots\ldots\ldots \circ h \circ f \circ h^{-1}(U) = h \circ f^k \circ h^{-1}(U) = h \circ f^k(U^*)$.

$g^k(U) \cap V = h \circ f^k(U^*) \cap h(V^*) = h(f^k(U^*) \cap V^*) \neq \emptyset$ (the last equality holds since h is 1 - 1)

Therefore g is topologically transitive.

b) Let $x \in B$ and let N be any neighborhood of x.

$g^n(x) = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} \circ \ldots\ldots\ldots \circ h \circ f \circ h^{-1}(x) = h \circ f^n \circ h^{-1}(x)$.

Let $x^* = h^{-1}(x) \in A$. Then $g^n(x) = h \circ f^n(x^*)$.

Let also $N^* = h^{-1}(N)$. It is clear that $x^* \in N^*$.

Since f is chaotic, it has sensitive dependence on initial conditions. Therefore, there exists $\delta^* > 0$ such that for that x^*, and for its neighborhood N^*, there exists $y^* \in N$ and $n \geq 0$ such that $|f^n(x^*) - f^n(y^*)| > \delta^*$.

Let $y = h(y^*)$. Then

$g^n(y) = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} \circ \ldots\ldots\ldots \circ h \circ f \circ h^{-1}(y) = h \circ f^n \circ h^{-1}(y) = h \circ f^n(y)$.

Since h is a homeomorphism $|g^n(y) - g^n(x)| > \delta$ for some δ which depends on δ^*. Hence g has sensitive dependence on initial conditions.

c) Periodic points of f are dense in A , i.e., for any open set $U^* \subset A$, there exists

$x^* \in P_f$, $x^* \in U^*$. Then $f^k(x^*) = x^*$ for some k.

Let U be an open set in B. Since h is onto, there exists an open set $U^* \subset A$ such that $h(U^*) = U$.

There exists at least one periodic point, say x^* in U^* for f. Let $x = h(x^*)$. Then $x \in U$.

$g^k(x) = h \circ f \circ h^{-1} \circ h \circ f \circ h^{-1} \circ .... \circ h \circ f \circ h^{-1}(x) = h \circ f^k \circ h^{-1}(x) = h \circ f^k(x^*) = h(x^*) = x.$

Therefore x is a periodic point for g.

Then for any open set $U \subset B$, there exists $x \in U$ such that $x \in P_g$, i. e., periodic points of g are dense in B.

a), b), c) imply that g is chaotic on B.

#

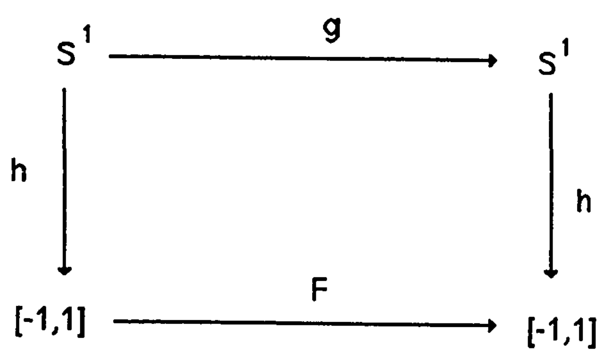
Example 3.2

$F(x) = 8x^4 - 8x^2 + 1$ is chaotic on the interval [-1 1].

Proof: Let $g(\Theta) = 4\Theta$ and $h(\Theta) = \cos\Theta$.

$g(\Theta)$ is chaotic (by Example 1.2 d) and by Example 3.1.)

$h(\Theta)$ is a homeomorphism and we have the following diagram:



Also $(h \circ g)(\Theta) = h(g(\Theta)) = h(4\Theta) = \cos 4\Theta$

$(F \circ h)(\Theta) = F(h(\Theta)) = F(\cos\Theta) = 8 \cos^4 \Theta - 8 \cos^2\Theta + 1 =$

$8 [(1+ \cos 2\Theta)/2]^2 - 8 [(1+ \cos 2\Theta)/2] +1 =$

$2 + 4 \cos 2\Theta + 2 \cos^2 2\Theta - 4 - 4 \cos 2\Theta + 1 = 2 \cos^2 2\Theta - 1 =$

$1+ \cos 4\Theta - 1 = \cos 4\Theta$,

so that $h \circ g = F \circ h$, i.e. F and g are topologically conjugate. Therefore by Theorem 3.1, F is chaotic.

CHAPTER 2.

EXAMPLES OF DISCRETE DYNAMICAL SYSTEMS IN THE COMPLEX PLANE.

1. Preliminary Results and Definitions.

Let **C** denote the usual complex plane.

Definition 1.1 A function $f(z)$ is ***analytic*** at z_0 if

$$\lim\ [\, f(z) - f(z_0)\,] \,/\, (z - z_0) \text{ exists as } z \to z_0 .$$

A function $f(z)$ is analytic on $U \subset \mathbf{C}$ if it is analytic at each point of U.

Definition 1.2 An open set $U \subset \mathbf{C}$ is ***simply connected*** if either $U = \mathbf{C}$ or there exists a one-to-one, onto, analytic map $f : U \to D$, where

$$D = \{ z \in \mathbf{C} \mid |z| < 1 \}.$$

Remark. All the definitions we gave in Chapter 1 for real-valued maps are valid for complex maps.

Definition 1.3 The ***Julia set*** of a complex analytic map F is the closure of the set of repelling periodic points of F and is denoted by $J(F)$.
(For the definition of a repelling periodic point, see Definition 1.7 in Chapter 1.)

Definition 1.4 The ***stable set*** of F, is the complement of the Julia set and is denoted by $S(F)$.

Definition 1.5 A set is ***perfect*** if every point in the set is an accumulation point or a limit point of other points in the set.

Definition 1.6 ***Basin of attraction*** of an attracting periodic point x is the set of all points which approach x under iterations of a given function f.

Properties of the Julia set:

1) J(F) is a closed set.

2) J(F) is completely invariant, i.e. it contains all forward images as well as preimages of J(F).

3) J(F) is a perfect set.

4) J(F) does not contain any attracting periodic point.

5) Theorem of Sullivan: If all singular values tend to ∞ under iterations of F, then J(F) = **C**. (Here singular values are critical values or asymptotic values.)

6) F is chaotic on J(F).

Remark. A complex analytic map separates the plane into two disjoint subsets: the stable set, where the dynamics are quite tame, and the Julia set, where the dynamics are chaotic.

2. Examples of Julia Sets.

Example 2.1 Let $Q_0(z) = z^2$.

If $|z| < 1$ then $Q_0^n(z) \to 0$; if $|z| > 1$ then $Q_0^n(z) \to \infty$, so the dynamics of Q_0 are quite tame off the unit circle.

On S^1 the map is chaotic since, if $z \in S^1$, then $z = e^{i\theta}$ and $Q_0(z) = e^{2i\theta}$, i.e., Q_0 transforms θ into 2θ. So the angular distance between two points is doubled upon iterations of Q_0, hence Q_0 has sensitive dependence on initial conditions. Any arc in S^1 is expanded by Q_0^k to cover all of S^1, hence Q_0 is topologically transitive. Also, periodic points of Q_0 are dense in S^1 as shown in example 2.1 in chapter 1.

Moreover $J(Q_0) = \{z \mid |z| = 1\} = S^1$ since: a) All the periodic points of Q_0 lie in S^1, b) All the periodic points are repelling since $|f'(z)| = 2|z| = 2 > 1$, and c) The periodic points of Q_0 are dense in S^1.

Properties of Julia set of $Q_0(z)$.

1) $J(Q_0)$ is completely invariant, i.e. it contains all forward images as well as preimages of Q_0.

2) If $z_0 \in J(Q_0)$ and U is any neighborhood of z_0, then for each $z \in C$, $z \neq 0$, there is an n such that $z \in Q_0^n(U)$. In other words, any neighborhood of any point in $J(Q_0)$ is smeared over the entire plane by iterations of Q_0.

3) Every point (exept 0) has a succession of preimages which converge to the Julia set. Thus, the Julia set is chaotic repelling for Q_0.

Remark. In general, J(F) is not a smooth curve as in case of Q_0. Usually, J(F) is much more complicated geometrically. The following example shows that the Julia set may be a Cantor set.

Exemple 2.2 Let $Q_c(z) = z^2 + c$.

If $|z| = r \geq |c| > 2$ then $Q_c^n(z) \to \infty$.

Proof: Let $Z = z^2 + c$. If $Z = X + iY$ and $z = x + iy$ then

$$X + iY = (x + iy)^2 + c_1 + ic_2.$$

Therefore

$$X = x^2 - y^2 + c_1$$
$$Y = 2xy + c_2$$

or

$$X - c_1 = x^2 - y^2$$
$$Y - c_2 = 2xy$$

so that we have

$$(X - c_1)^2 + (Y - c_2)^2 = (x^2 + y^2)^2$$

or

$$(X - c_1)^2 + (Y - c_2)^2 = r^4$$

if $x^2 + y^2 = r^2$ or $|z| = r$.

That means that Q_c maps the circle $|z| = r$ to the circle with center c and radius r^2. Since $r > 2$ we also have $r^2 > 2r$, hence the image circle lies in the exterior of the circle $|z| = r$. Therefore $|Q_c(z)| > |z|$ for all z with $|z| \geq |c|$ and repeating the procedure we have $Q_c^n(z) \to \infty$.

Remark. All the periodic points of Q_c lie inside the circle $|z| = |c|$.

We denote the set of points whose entire forward orbit lies inside the circle $|z| = |c|$ by L.

Theorem 2.1 If $|c|$ is sufficiently large, L is a Q_c-invariant Cantor set. All points in $C \setminus L$ tend to ∞ under iterations of Q_c.

Proof: Let g be the preimage of the circle $|z| = |c|$ under Q_c.

If $Z = z^2 + c$ then 0 is the only preimage of c, but all other points on $|z| = |c|$ have two preimages. Then g is a figure eight curve:

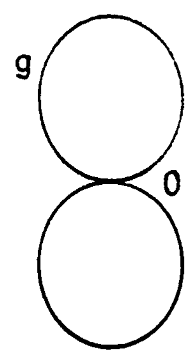


Fig. 14

and g is contained in the interior of the circle $|z| = |c|$. Indeed, assume that there exists a point z_0 of g outside the circle $|z| = |c|$. Then by the previous result, $|Q_c(z_0)| > |c|$. But $Q_c(z_0)$ should be on the circle $|z| = |c|$, since g is that circle's preimage.

The points between g and the circle $|z| = |c|$ are mapped into the exterior of $|z| = |c|$. Hence these points lie in the stable set.

Let $r < |c|$ such that g is contained in the interior of the disk $D = \{z \in \mathbf{C} \mid |z| \leq r\}$.

The preimage of D consists of two simply connected sets, one in each lobe of g.

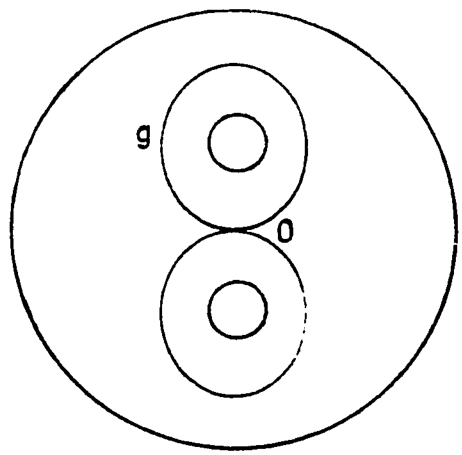


Fig. 15

It is easy to see that $Q_c^{-n}(D)$ consists of 2^n disks and that $Q_c^{-n}(D)$ is contained in $Q_c^{-n+1}(D)$. Also $L = \cap\ Q_c^{-n}(D)$ where the intersection is taken over all n.

Let $B = \{z \in \mathbf{C} \mid |z| \leq 1/2\}$. Then $Q_c(B) = \{z \in C \mid |z - c| \leq 1/4\}$.

Assume that $Q_c(B) \cap g = \emptyset$.

If $|\, Q_c'(z)\,| \leq 1$, then $2|z| \leq 1$ or $|z| \leq 1/2$, i.e., $z \in B$. Then any point z such that $|Q_c'(z)| \leq 1$ is mapped out of D. Hence $|Q_c'(z)| > 1$ for all $z \in L$. Therefore L is a Cantor set since it is closed, totally disconnected and perfect.

3. Explosion in Julia Sets.

An explosion is a sudden change from a nowhere dense Julia set to one which is the entire complex plane.

Example 3.1 The exponential family.

Let $E_m(z) = me^z$, $m > 0$.

<u>Case1</u>. If $m < 1/e$, $E_m(x) = me^x$, $x \in R$, has two fixed points, $p < 1$ and $q > 1$.

$E_m'(p) = me^p = p < 1$ and $E_m'(q) = me^q = q > 1$.

Hence p is attractive fixed point and q is repelling fixed point.

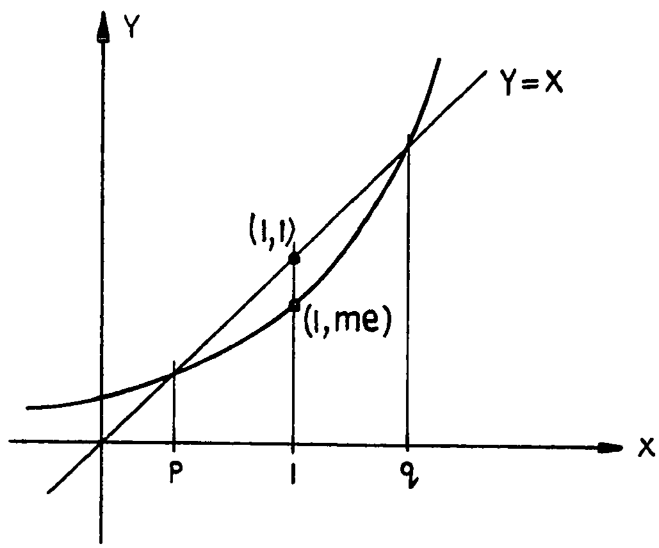


Fig. 16

Case2. If $m = 1/e$, $E_m(x) = me^x$, $x \in R$, has one fixed point $x = 1$.

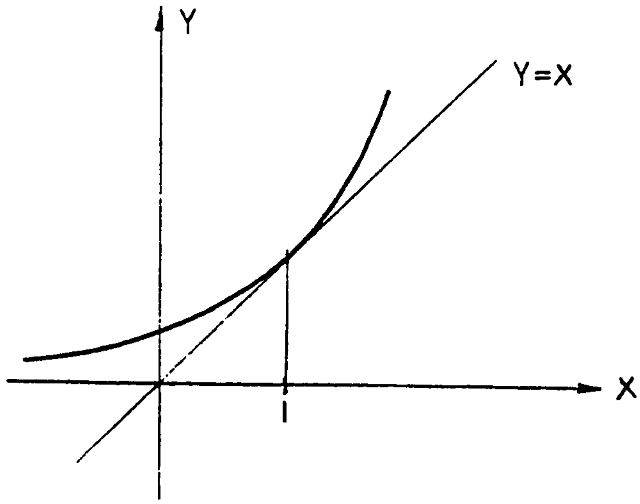


Fig. 17

Case 3. If $m > 1/e$, $E_m(x) = me^x$, $x \in R$, has no fixed points.

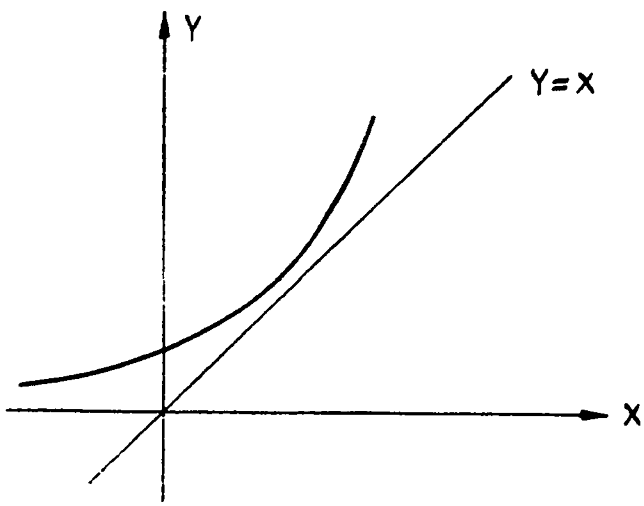


Fig. 18

Therefore, there is a ***bifurcation*** at $m = 1/e$.

If $m > 1/e$, then $\lim E_m^n(0) = \infty$ as $n \to \infty$. Zero is the only singular (asymptotic) point of $E_m(z)$, since $E_m(z)$ is an entire function, so there are no critical points; on the other hand, $\lim E_m(x) = 0$ as x approches $-\infty$. Hence, by the theorem of Sullivan, it follows that $J(E_m) = \mathbf{C}$ for $m > 1/e$.

If $m < 1/e$, the Julia set is not the entire plane since the attracting fixed point $p \in R$ must lie in $S(E_m)$.

From Fig 16., there is an $x_0 \in (p,q)$ such that $me^{x_0} < x_0$.

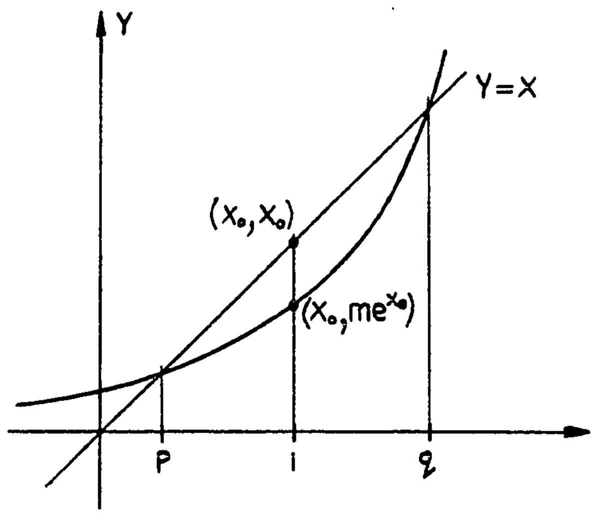


Fig. 19

Let $E_m(z) = X + iY$ and $z = x + iy$. Then $E_m(z) = me^z$ becomes $X + iY = me^{x+iy}$ or $X + iY = me^x(\cos y + i \sin y)$. Then $X = me^x \cos y$ and $Y = m e^x \sin y$.

If $x = x_0$, then $X^2 + Y^2 = m^2e^{2x_0}$, i.e. E maps vertical lines in **C** to circles with center at origin. In particular the line $x = x_0$ is mapped to the circle with center at origin and radius $r = |m|e^{x_0}$, $r < x_0$.

Hence, the half-plane $\mathrm{Re}\, z < x_0$ lies in the basin of attraction of p. Thus, this half-plane lies in $S(E_m)$.

Conclusions.

If $m > 1/e$, $J(E_m) = \mathbf{C}$.

If $m < 1/e$, $J(E_m)$ misses the entire half-plane $\mathrm{Re}\, z < x_0$.

This is an example of an explosion.

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