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Laurent polynomial representations of sl(n)

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ABSTRACT

LAURENT POLYNOMIAL REPRESENTATIONS OF $\mathfrak{sl}(n)$

by Hélène M. Payne

In this thesis,we study infinite dimensional representations of the special linear Lie algebras $\mathfrak{sl}(n)$. These representations arise naturally from the Weyl construction $e_{ij} \mapsto x_i \frac{\partial}{\partial x_j}$, where e_{ij} are the elementary matrices. (This definition of elementary matrix differs from the one in linear algebra.) Our main results provide explicit decomposition of indecomposable representations related to the Laurent polynomials $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$. In particular, we verify that the space of degree zero of homogeneous polynomials of three variables (resp. two variables) has length 7 (resp. 3) as an $\mathfrak{sl}(3)$-representation (resp. $\mathfrak{sl}(2)$-representation).

Dedication

To my children

Nikolas, Mattias, Celine, and Emma.

ACKNOWLEDGEMENTS

Going back to graduate school to study mathematics had long been a dream of mine. Now, after having had the opportunity to delve deeper into the world of mathematics, I must admit it has been every bit as enjoyable as I thought it would. Along the way, I studied various areas of pure mathematics, the courses taken being guided both by what interested me and by what fit into my schedule, as I juggled family life with four kids and pursued my mathematics studies. One of my favorite courses was abstract algebra, which was taught by Professor Grantcharov. Abstract algebra became the topic of my second qualifying exam and as I wished to gain a deeper understanding of the subject, I was fortunate to be offered the opportunity to work on Lie algebras with Professor Grantcharov as my thesis advisor. Studying Lie algebras and their representations proved much more difficult a subject than basic abstract algebra. I wanted a real challenge and got what I asked for.

This thesis would never have been written if it weren't for Professor Grantcharov. He is an incredibly patient teacher whether teaching in a classroom setting or on a one-to-one basis acting as a thesis advisor. The journey was at times both slow and frustrating. Professor Grantcharov always remained positive and supportive, patiently explaining the same concept two, three times, whatever it took to get over the hump or obstacle. He also was extremely accommodating and flexible about meeting times, and never gave up on me when I stalled for a couple of months trying to finish a long proof. He also taught me everything I now know on writing mathematically. He deserves a lot of credit for helping make this thesis become reality.

I also want to acknowledge Professor Blockus and Professor Peterson for all the effort they put in while being on my thesis committee and before that, on my second qualifying exam.

Professor Blockus is a fantastic teacher, organized and hard working. Taking her beginning topology class was one of the high points during my studies at San Jose State. If I could have found a way to continue with topology, it might have been my thesis topic instead of abstract algebra. I want to thank her for all her constructive input in writing this thesis, especially since she took the time to do this while serving as vice chair of the mathematics department.

Professor Peterson left a lasting impression on me. Although I never took a class from him, I know he is an outstanding teacher. He turned my qualifying exam and thesis defense into learning experiences, as he has a gift for explaining very difficult mathematical concepts in a clear and simple way. He helped me understand where my work on Lie algebras fits into the bigger picture. I want to thank him for his numerous contributions to this thesis.

Other Professors at San Jose State I would like to recognize are Dr. So for his incredible enthusiasm and energy in the classroom and Dr. Kellum for still loving mathematics.

Last but not least, I would like to thank my husband Alex and my children Nikolas, Mattias, Celine, and Emma for helping me make this happen. They always supported me, and having Alex at the thesis defense made all the difference.

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Chapter 1

Introduction

A main conceptual point in the development of theoretical physics in the 20^{th} century is its reunification with pure mathematics. This is apparent in both main physical theories in the first half of the century, namely general relativity and quantum mechanics, as well as in the main creations of the second half of the 20^{th} century such as quantum field theory and string theory. The notion of a group, more precisely that of a continuous group, has played a prominent role in these developments. This kind of group is usually named Lie group after the Norwegian mathematician Sophus Lie. A representation of a group is an interpretation of a more complicated group as a subgroup of the group of matrices. Mathematicians have studied representations since the early 20^{th} century. The first deep results related to the finite dimensional representations were obtained by Wilhelm Killing, Elie Cartan, and Hermann Weyl [Kil88, Car83, Wey46]. A striking discovery was made in the 50's: physicists, Murray Gell-Man, and others, realized that abstract theory of finite and infinite dimensional representations perfectly fits the framework of elementary particle theory.

The study of general (not necessarily finite dimensional) weight representations of Lie algebras emerged in the early 1980's as a part of a fundamental effort in the structure theory of representations of Lie algebras and Lie groups. This effort has been motivated significantly by theoretical physics. Weight representations have been studied in works of Georgia Benkart, Dan Britten, Suren Fernando, Vyacheslav Futorny, and Frank Lemire over the last 20 years [BBL97, BL87, Fer90, Fut87]. A major breakthrough in the representation theory of reductive Lie algebras was made by Olivier Mathieu [Mat00] in 2000 who classified all irreducible weight representations with finite weight multiplicities.

Following Mathieu's classification, it is natural to aim at the other classification problem: describing all weight (not necessarily irreducible) representations. Recently, Dimitar Grantcharov and Vera Serganova [GS06] discovered that the category of all weight representations is too "large," i.e., its indecomposable objects cannot be parameterized. In order to achieve parametrization one has to consider the subcategory of all weight representations that have bounded weight multiplicities. Such representations exist for two series of simple Lie algebras: the symplectic Lie algebras $\mathfrak{sp}(2n)$, $n \geq 2$, and the special linear Lie algebras $\mathfrak{sl}(n)$, $n \geq 2$. A complete classification of the indecomposable objects is obtained in the case of $\mathfrak{sp}(2n)$ in [GS06]. The case of $\mathfrak{sl}(n)$ turned out to be much more complicated and is still open.

A particular and important class of bounded weight representations is the class of Laurent polynomial representations. This class arises naturally from the Weyl representation of the general linear Lie algebra in terms of differential operators. In this thesis we describe explicitly the decomposition of the Laurent polynomial representations in the lower rank cases $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$. In particular we prove that these representations are indecomposable and have lengths 3 and 7 for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$, respectively. We believe that the low rank computations in this thesis will help to solve the open classification problem for the bounded weight representations of $\mathfrak{sl}(n)$.

The organization of the thesis is as follows. In Chapter 2 we fix the notations and conventions used in the thesis. All necessary definitions and results related to the Lie algebras and their representations is collected in Chapter 3. In Chapter 4 we introduce the generalized Weyl representations of $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$. In the following chapters we focus on the homogeneous Laurent polynomials. In Chapter 5 we show that the vector space $\widetilde{P_m}$ of homogeneous Laurent polynomials of degree m is a representation of $\mathfrak{sl}(2)$ of length 3. Our main result is stated and proved in Chapter 6. Namely, we explicitly find the decomposition of the homogeneous Laurent polynomials $\widetilde{P_0}$ of degree zero as an $\mathfrak{sl}(3)$-representation and verify that it has length 7, i.e., it has 7 irreducible subquotients.

Chapter 2

Notations and Conventions

All variables and symbols in this thesis will adhere to the conventions listed in this section unless specified otherwise. The set of integers is denoted $\mathbb{Z}$, and by $\mathbb{Z}_+$ we denote the non-negative integers. From Section 3.4.1 on, we fix the ground field F to be $\mathbb{C}$, the set of complex numbers. By $\mathrm{Hom}_F(V,W)$ we refer to the set of all F-homomorphisms from V to W or more specifically:

$$\mathrm{Hom}_F(V,W) := \{f : V \to W | f(ax+by) = af(x)+bf(y) \text{ for every } a,b \in F;\ x,y \in V\}.$$

We set $\mathrm{Hom}(V,W) := \mathrm{Hom}_{\mathbb{C}}(V,W)$. For elements of Laurent polynomial vector spaces expressed as linear combinations of Laurent polynomials, such as

$$g = \sum_{k_1+k_2+\ldots+k_r=m} a_{k_1,k_2,\ldots,k_r} x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r},$$

all coefficients $a_{k_1,k_2,\ldots,k_r}$ are assumed to be nonzero unless stated otherwise. We also assume that the reader is familiar with all basic definitions and results from linear algebra involving bases, dimension, homomorphism, the three Isomorphism Theorems, etc.

Chapter 3

Lie Algebras

3.1 Background

In this section we give an overview about Lie algebras. For more information, we refer the reader to [Hum72].

Definition 3.1.1. A *Lie algebra* is a vector space L over a field F, together with a binary operation $[.,.] : L \times L \to L$, called the Lie bracket or commutator, which satisfies the following axioms:

$$[ax + by, z] = a[x, z] + b[y, z] \tag{3.1}$$

$$[z, ax + by] = a[z, x] + b[z, y] \qquad \text{(bilinearity)}$$

$$[x, x] = 0 \tag{3.2}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \textit{(Jacobi identity)} \tag{3.3}$$

for all $x, y, z \in L$.

If identity (3.2) is applied to $[x + y, x + y]$ we obtain

$$\begin{aligned}[x + y, x + y] &= [x, x + y] + [y, x + y] \\ &= [x, x] + [x, y] + [y, x] + [y, y] \\ &= [x, y] + [y, x] \\ &= 0,\end{aligned}$$

and it follows that the Lie bracket is skew-symmetric, i.e.,

$$[x, y] = -[y, x]. \tag{3.4}$$

Conversely, if (3.4) is applied to $[x, x]$, we obtain identity (3.2):

$$[x, x] = -[x, x],$$

$$2[x, x] = 0, \text{ if char } F \neq 2, \text{ then}$$

$$[x, x] = 0.$$

Therefore identities (3.2) and (3.4) are identical.

3.2 Subalgebras of Lie Algebras

A subspace K of a Lie algebra L is called a *Lie subalgebra* if it is a Lie algebra itself. Equivalently, K is a Lie subalgebra of L if $[x, y] \in K$ whenever $x, y \in K$.

3.3 The General Linear Algebra, $\mathfrak{gl}(V)$

Let V be a vector space over F. We denote by End V, the space of all linear transformations $f : V \to V$. Notice that End V is a ring relative to the usual product operation. With the Lie bracket defined as $[x, y] = xy - yx$, it may easily be verified that End V is a Lie algebra over F, called the *general linear algebra* $\mathfrak{gl}(V)$. In the special case when V has dimension n over F, we have that End V, and thus $\mathfrak{gl}(V)$, has dimension n^2. If a basis is fixed for V, $\mathfrak{gl}(V)$ is isomorphic to the Lie algebra $\mathfrak{gl}(n)$ consisting of all $n \times n$ matrices over F. The Lie algebra $\mathfrak{gl}(n)$ is strongly related to the Lie group $GL(n)$, consisting of all $n \times n$ matrices whose determinant is nonzero.

We fix the *the standard basis* of $\mathfrak{gl}(n)$ consisting of the matrices e_{ij} (having 1 in the (i, j)-position and 0 elsewhere). It is easy to check $e_{ij}e_{kl} = \delta_{jk}e_{il}$, where

$\delta_{jk} = 1$ if $j = k$, and 0 otherwise. It follows that:

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}. \tag{3.5}$$

3.4 The Classical Lie Algebras

The *classical Lie algebras* are A_n, B_n, C_n, and D_n. Being subalgebras of $\mathfrak{gl}(V)$, they are named "classical" in reference to their relation to certain classical linear Lie groups. In addition to the classical Lie algebras there are five exceptional ones which will not be discussed in this thesis.

$A_n = \mathfrak{sl}(n+1)$: Let dim $V = n + 1$. $\mathfrak{sl}(n+1)$, the *special linear algebra* is the set of all $(n+1) \times (n+1)$ matrices having trace zero. It is closed under the bracket operation since:

$$\begin{aligned}
\mathrm{Tr}(xy) &= \sum_{i=1}^{n+1} (xy)_{ii} \\
&= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} x_{ij} y_{ji} \\
&= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} y_{ji} x_{ij} \\
&= \sum_{j=1}^{n+1} \sum_{i=1}^{n+1} y_{ji} x_{ij} \\
&= \sum_{j=1}^{n+1} (yx)_{jj} \\
&= \mathrm{Tr}(yx),
\end{aligned}$$

and

$$\begin{aligned}\mathrm{Tr}(x+y) &= \sum_{i=1}^{n+1}(x+y)_{ii} \\ &= \sum_{i=1}^{n+1}(x_{ii}+y_{ii}) \\ &= \sum_{i=1}^{n+1}x_{ii}+\sum_{i=1}^{n+1}y_{ii} \\ &= \mathrm{Tr}(x)+\mathrm{Tr}(y).\end{aligned}$$

The basis elements of A_n are $e_{ij}, i \neq j$, $(n(n+1)$ of them), and $h_i = e_{ii} - e_{i+1,i+1}, 1 \leq i \leq n$ (n of them), for a total of $(n+1)^2 - 1$ basis elements.

$C_n = \mathfrak{sp}(2n)$: Let dim $V = 2n$. A nondegenerate skewsymmetric form f, is defined by the matrix $s = \begin{bmatrix} \mathbf{0} & I_n \\ -I_n & \mathbf{0} \end{bmatrix}$, for which $f(v,w) = v^t s w$ with $v, w \in V$, $\mathbf{0}$ is the $n \times n$ zero matrix, and I_n the $n \times n$ identity matrix. Then $\mathfrak{sp}(2n)$, the *symplectic algebra*, consists of all endomorphisms x of V, satisfying $f(x(v),w) = -f(v,x(w))$ for all $v, w \in V$. It can be verified that this condition is equivalent to $sx = -x^t s$, which puts x on the form $x = \begin{bmatrix} m & p \\ q & -m^t \end{bmatrix}$ for which $p^t = p$ and $q^t = q$, and m, p, and q are $n \times n$ matrices. One may also verify that $\mathfrak{sp}(2n)$ is closed under the Lie bracket by checking that $s[x,y] = -[x,y]^t s$ whenever $sx = -x^t s$ and $sy = -y^t s$. We note that $\mathrm{Tr}(x) = 0$, making $\mathfrak{sp}(2n)$ a subalgebra of $\mathfrak{sl}(2n)$. Its basis consists of the diagonal elements $e_{ii} - e_{i+n,i+n}, 1 \leq i \leq n$ (n of them) and in addition, the basis elements $e_{ij} - e_{j+n,i+n}, 1 \leq i \neq j \leq n$ ($n^2 - n$ of them) take care of $m, -m^t$. For p we use the basis elements $e_{i,i+n}, 1 \leq i \leq n$ (n of them) for its diagonal elements, and $e_{i,n+j} + e_{j,n+i}, 1 \leq i < j \leq n$ ($\frac{1}{2}n(n-1)$ of them) for the nondiagonal elements. Similarly for the positions in q we use the basis elements $e_{i+n,i}, 1 \leq i \leq n$ (n of them) for its diagonal elements, and

$e_{i+n,j} + e_{j+n,i}, 1 \leq i < j \leq n$ ($\frac{1}{2}n(n-1)$ of them) for the nondiagonal elements. Hence, there are a total of $2n^2 + n$ basis elements in $\mathfrak{sp}(2n)$, or dim $\mathfrak{sp}(2n) = 2n^2 + n$. (This may be compared with the $4n^2 - 1$ basis elements of $\mathfrak{sl}(2n)$.)

$B_n = \mathfrak{o}(2n+1)$: Let dim $V = 2n+1$. A nondegenerate symmetric bilinear form f is defined by the matrix $s = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_n \\ \mathbf{0} & I_n & \mathbf{0} \end{bmatrix}$ for which $f(v,w) = v^t s w$ with $v, w \in V$ and $\mathbf{0}$ is the $n \times 1$, $1 \times n$ or $n \times n$ zero matrix, depending on its position in the matrix. Then $\mathfrak{o}(2n+1)$, the *orthogonal algebra of odd rank*, consists of all endomorphisms x of V, satisfying $f(x(v), w) = -f(v, x(w))$ for all $v, w \in V$, which is the same condition as for C_n. Again, it can be verified that this condition is equivalent to $sx = -x^t s$, which puts x on the form $x = \begin{bmatrix} 0 & b_1 & b_2 \\ -b_2^t & m & p \\ -b_1^t & q & -m^t \end{bmatrix}$ for which $p^t = -p$ and $q^t = -q$, and m, p, and q are $n \times n$ matrices, and b_1 and b_2 are $1 \times n$ matrices. Analogous to C_n, one may also verify that $\mathfrak{o}(2n+1)$ is closed under the Lie bracket by checking that $s[x,y] = -[x,y]^t s$ whenever $sx = -x^t s$ and $sy = -y^t s$. We also note that $\mathrm{Tr}(x) = 0$, making $\mathfrak{o}(2n)$ a subalgebra of $\mathfrak{sl}(2n)$. Its basis consists of the diagonal elements $e_{ii} - e_{i+n,i+n}, 2 \leq i \leq n+1$ (n of them). In addition there are $2n$ basis elements, $e_{1,n+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{n+i+1,1}$ ($1 \leq i \leq n$), involving row one and column one. The basis elements corresponding to the nondiagonal elements of m and $-m^t$ are $e_{i+1,j+1} - e_{j+n+1,i+n+1}, 1 \leq i \neq j \leq n$ ($n^2 - n$ of them). For p we use the basis elements $e_{i+1,j+n+1} - e_{j+1,i+n+1}, 1 \leq i < j \leq n$ ($\frac{1}{2}n(n-1)$ of them) and similarly for the positions in q we use the basis elements $e_{i+n+1,j+1} - e_{j+n+1,i+1}, 1 \leq i < j \leq n$ ($\frac{1}{2}n(n-1)$ of them). Hence, dim $\mathfrak{o}(2n+1) =$ dim $\mathfrak{sp}(2n)$.

$D_n = \mathfrak{o}(2n)$: Let dim $V = 2n$. A nondegenerate symmetric bilinear form f is defined by the matrix $s = \begin{bmatrix} \mathbf{0} & I_n \\ I_n & \mathbf{0} \end{bmatrix}$ for which $f(v, w) = v^t s w$ with $v, w \in V$. Then $\mathfrak{o}(2n)$, the *orthogonal algebra of even rank*, consists of all endomorphisms x of V, satisfying $f(x(v), w) = -f(v, x(w))$ for all $v, w \in V$, which is the same condition as for C_n, and B_n. Again, it can be verified that this condition is equivalent to $sx = -x^t s$, which for D_n puts x on the form $x = \begin{bmatrix} m & p \\ q & -m^t \end{bmatrix}$ for which $p^t = -p$ and $q^t = -q$, and m, p, and q are $n \times n$ matrices. Analogous to C_n and B_n, one may also verify that $\mathfrak{o}(2n)$ is closed under the Lie bracket by checking that $s[x, y] = -[x, y]^t s$ whenever $sx = -x^t s$ and $sy = -y^t s$. We also note that $\text{Tr}(x) = 0$, making $\mathfrak{o}(2n + 1)$ a subalgebra of $\mathfrak{sl}(2n + 1)$. Its basis consists of the diagonal elements $e_{ii} - e_{i+n,i+n}, 1 \leq i \leq n$ (n of them). In addition the basis elements corresponding to m and $-m^t$ are $e_{i,j} - e_{j+n,i+n}, 1 \leq i \neq j \leq n$ ($n^2 - n$ of them). For p we use the basis elements $e_{i,j+n} - e_{j,i+n}, 1 \leq i < j \leq n$ ($\frac{1}{2}n(n-1)$ of them) and similarly for the positions in q we use the basis elements $e_{i+n,j} - e_{j+n,i}, 1 \leq i < j \leq n$ ($\frac{1}{2}n(n-1)$ of them). Hence, there are a total of $2n^2 - n$ basis elements in $\mathfrak{o}(2n)$, or dim $\mathfrak{o}(2n)$=$2n^2 - n$.

In this thesis, we will only focus on the special linear algebras $A_1 = \mathfrak{sl}(2)$ and $A_2 = \mathfrak{sl}(3)$.

3.4.1 Relations of $\mathfrak{sl}(2)$

From here on, we fix the ground field F to be $\mathbb{C}$ and we fix the following basis elements of $\mathfrak{sl}(2)$; $e_{12} = e$, $e_{21} = f$, and $h_1 = h$, i.e.,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If we apply (3.5) to the basis elements e, f, and h, we obtain the Lie bracket multiplication table for $\mathfrak{sl}(2)$:

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

3.4.2 Relations of $\mathfrak{sl}(3)$

We fix the following basis elements of $\mathfrak{sl}(3)$:

$$e_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad e_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Its Lie bracket multiplication table is listed below:

$$[e_{12}, e_{13}] = 0, \quad [e_{12}, e_{23}] = e_{13}, \quad [e_{12}, e_{21}] = h_1, \quad [e_{12}, e_{31}] = -e_{32},$$

$$[e_{12}, e_{32}] = 0, \quad [e_{13}, e_{23}] = 0, \quad [e_{13}, e_{21}] = -e_{23}, \quad [e_{13}, e_{31}] = h_1 + h_2,$$

$$[e_{13}, e_{32}] = e_{12}, \quad [e_{23}, e_{21}] = 0, \quad [e_{23}, e_{31}] = e_{21}, \quad [e_{23}, e_{32}] = h_2,$$

$$[e_{21}, e_{31}] = 0, \quad [e_{21}, e_{32}] = -e_{31}, \quad [e_{31}, e_{32}] = 0, \quad [h_1, e_{12}] = 2e_{12},$$

$$[h_1, e_{13}] = e_{13}, \quad [h_1, e_{23}] = -e_{23}, \quad [h_1, e_{21}] = -2e_{21}, \quad [h_1, e_{31}] = -e_{31},$$

$$[h_1, e_{32}] = e_{32}, \quad [h_1, h_2] = 0, \quad [h_2, e_{12}] = -e_{12}, \quad [h_2, e_{13}] = e_{13},$$

$$[h_2, e_{23}] = 2e_{23}, \quad [h_2, e_{21}] = e_{21}, \quad [h_2, e_{31}] = -e_{31}, \quad [h_2, e_{32}] = -2e_{32}.$$

3.5 Roots of the Lie Algebras $\mathfrak{gl}(n)$ and $\mathfrak{sl}(n)$

Set

$$\mathfrak{h}_{\mathfrak{gl}(n)} := \left\{ \left. \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix} \right| b_1, b_2, \dots, b_n \in \mathbb{C} \right\}.$$

Then $\mathfrak{h}_{\mathfrak{gl}(n)}$ is a *Cartan subalgebra*, of $\mathfrak{gl}(n)$. For the general definition of a Cartan subalgebra, we refer the reader to [Hum72]. Notice that $\mathfrak{h}_{\mathfrak{gl}(n)}$ is an abelian Lie subalgebra of $\mathfrak{gl}(n)$, i.e., if $h_1, h_2 \in \mathfrak{h}_{\mathfrak{gl}(n)}$, $[h_1, h_2] = 0$. In particular, $[h, e_{ii}] = 0$, for all $h \in \mathfrak{h}_{\mathfrak{gl}(n)}$, and $1 \leq i \leq n$. If we let

$$h = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix} \in \mathfrak{h}_{\mathfrak{gl}(n)},$$

for $i \neq j$, we can define homomorphisms: $\alpha_{ij} : \mathfrak{h}_{\mathfrak{gl}(n)} \to \mathbb{C}$, by $\alpha_{ij}(h) = b_i - b_j$. The Lie bracket, $[h, e_{ij}]$, with $i \neq j$, can now be expressed in terms of α_{ij}: $[h, e_{ij}] = (b_i - b_j)e_{ij} = \alpha_{ij}(h)e_{ij}$. Set $\varepsilon_i : \mathfrak{h}_{\mathfrak{gl}(n)} \to \mathbb{C}$ to be the homomorphism defined by $\varepsilon_i(h) = b_i$. Then $\alpha_{ij} := \varepsilon_i - \varepsilon_j$. The elements, $\varepsilon_i - \varepsilon_j$, $i \neq j$, of the vector space $\mathfrak{h}^*_{\mathfrak{gl}(n)} := \mathrm{Hom}(\mathfrak{h}_{\mathfrak{gl}(n)}, \mathbb{C})$, are called *roots* of $\mathfrak{gl}(n)$. The collection $\Delta_{\mathfrak{gl}(n)}$ of all roots of $\mathfrak{gl}(n)$ is called the *root system* of $\mathfrak{gl}(n)$. More explicitly,

$$\Delta_{\mathfrak{gl}(n)} = \{\varepsilon_i - \varepsilon_j | 1 \leq i \neq j \leq n\}.$$

It is thus convenient to set $e_{\varepsilon_i - \varepsilon_j} := e_{ij}$, since it simplifies taking the Lie bracket of (non-diagonal) basis elements. Then for $\alpha, \beta \in \Delta_{\mathfrak{gl}(n)}$, we can directly verify that

$$[e_\alpha, e_\beta] = \begin{cases} e_{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta_{\mathfrak{gl}(n)}, \\ 0, & \text{if } \alpha + \beta \notin \Delta_{\mathfrak{gl}(n)}, \text{ and } \alpha + \beta \neq 0, \\ e_{ii} - e_{jj}, & \text{if } \alpha = -\beta = \varepsilon_i - \varepsilon_j. \end{cases}$$

Each root, α, in turn, gives rise to an α-*root space*,

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{gl}(n) | [h, x] = \alpha(h)(x); h \in \mathfrak{h}_{\mathfrak{gl}(n)}\} = \mathbb{C}e_\alpha,$$

and in particular if $\alpha = \varepsilon_i - \varepsilon_j$ and $e_\alpha = e_{ij}$ we have

$$\mathfrak{g}^{\varepsilon_i - \varepsilon_j} = \{x \in \mathfrak{g}(n) | [h, x] = (\varepsilon_i - \varepsilon_j)(h)(x), \text{for every h } \in \mathfrak{h}_{\mathfrak{gl}(n)}\} = \mathbb{C}e_{ij},$$

for $i \neq j$.

Theorem 3.5.1. *The general linear algebra* $\mathfrak{gl}(n)$ *can be written as a direct sum of its Cartan subalgebra and the root spaces* $\mathfrak{g}^{\varepsilon_i-\varepsilon_j}$*:*

$$\mathfrak{gl}(n) = \mathfrak{h}_{\mathfrak{gl}(n)} \oplus \left(\bigoplus_{\alpha\in\Delta_{\mathfrak{gl}(n)}} \mathfrak{g}^{\alpha} \right).$$

This decomposition is called the root space decomposition of $\mathfrak{gl}(n)$ *relative to* $\mathfrak{gl}(n)$*.*

Proof. Any element of x, of $\mathfrak{gl}(n)$ may be written as a linear combination of its basis elements:

$$x = \sum_{i\neq j} a_{ij}e_{ij} + \sum_{i=1}^{n} c_i e_{ii},$$

with $a_{ij}, c_i \in \mathbb{C}$. We see that $a_{ij}e_{ij} \in \mathbb{C}e_{ij} = \mathfrak{g}^{\varepsilon_i-\varepsilon_j}$ for each i, j with $i \neq j$. Furthermore, if we define the elements of h by: $b_i := c_i e_{ii}$ for $i = 1, ..., n$, it follows that $\sum_{i=1}^{n} c_i e_{ii} \in \mathfrak{h}_{\mathfrak{gl}(n)}$. Therefore, $\mathfrak{gl}(n)$ can be written as a direct sum of its Cartan subalgebra and the root spaces $\mathfrak{g}^{\varepsilon_i-\varepsilon_j}$. □

Root space decompositions exist for more general reductive Lie algebras, see for example §8 of [Hum72]. The roots $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots, \varepsilon_{n-1} - \varepsilon_n\}$ form a *basis of the root system* $\Delta_{\mathfrak{gl}(n)}$ of $\mathfrak{gl}(n)$. Namely, every root α in $\Delta_{\mathfrak{gl}(n)}$ can be expressed uniquely in the form $\alpha = \alpha_1(\varepsilon_1 - \varepsilon_2) + \cdots + \alpha_{n-1}(\varepsilon_{n-1} - \varepsilon_n)$, where all $\alpha_i \geq 0$ for all i, or $\alpha_i \leq 0$ for all i. For example, for $\mathfrak{gl}(5)$ we have:

$\varepsilon_2 - \varepsilon_5 = 0(\varepsilon_1 - \varepsilon_2) + 1(\varepsilon_2 - \varepsilon_3) + 1(\varepsilon_3 - \varepsilon_4) + 1(\varepsilon_4 - \varepsilon_5)$

In $\mathfrak{sl}(n)$ we fix the Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}(n)} = \mathfrak{h}_{\mathfrak{gl}(n)} \cap \mathfrak{sl}(n)$ of $\mathfrak{sl}(n)$. Moreover,

$$\mathfrak{h}_{\mathfrak{sl}(n)} := \left\{ \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix} \middle| \sum a_i = 0; a_1, a_2, \dots, a_n \in \mathbb{C} \right\}.$$

One can verify that $\mathfrak{h}_{\mathfrak{gl}(n)} \cong \mathfrak{h}_{\mathfrak{sl}(n)} \oplus \mathbb{C}I$, where

$$I = \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix},$$

the $n \times n$ identity matrix. The roots of $\mathfrak{sl}(n)$ are $\varepsilon_i - \varepsilon_j$ for $1 \leq i \neq j \leq n$. We thus verify that $\Delta_{\mathfrak{sl}(n)} = \Delta_{\mathfrak{gl}(n)}$.

3.5.1 The Roots of $\mathfrak{sl}(2)$

In $\mathfrak{sl}(2)$ we fix the Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}(2)} = \mathfrak{h}_{\mathfrak{gl}(2)} \cap \mathfrak{sl}(2)$. More explicitly, $\mathfrak{h}_{\mathfrak{sl}(2)}$ consists of all diagonal 2×2 matrices with zero trace:

$$\mathfrak{h}_{\mathfrak{sl}(2)} = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \middle| \, a_1 + a_2 = 0, a_1, a_2 \in \mathbb{C} \right\}.$$

Then, using arbitrary $h = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ in $\mathfrak{h}_{\mathfrak{sl}(2)}$, we can find the roots of $\mathfrak{sl}(2)$ by forming the Lie products $[h, e_{12}] = (a_1 - a_2)e_{12}$, and $[h, e_{21}] = (a_2 - a_1)e_{21}$. Hence, the roots of $\mathfrak{sl}(2)$ are $\pm(\varepsilon_1 - \varepsilon_2)$, or $\Delta_{\mathfrak{sl}(2)} = \{\pm(\varepsilon_1 - \varepsilon_2)\} = \Delta_{\mathfrak{gl}(2)}$. The basis of the root system is $\{\varepsilon_1 - \varepsilon_2\}$, and the dual to its Cartan subalgebra, $\mathfrak{h}^*_{\mathfrak{sl}(2)}$, is isomorphic to $\mathbb{C}(\varepsilon_1 - \varepsilon_2)$.

3.5.2 The Roots of $\mathfrak{sl}(3)$

We fix the following Cartan subalgebra, $\mathfrak{h}_{\mathfrak{sl}(3)}$ of $\mathfrak{sl}(3)$:

$$\mathfrak{h}_{\mathfrak{sl}(3)} = \left\{ \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \middle| \, \sum_{i=1}^{3} b_i = 0, b_1, b_2, b_3 \in \mathbb{C} \right\}.$$

The root system of $\mathfrak{sl}(3)$ is $\Delta_{\mathfrak{sl}(3)} = \{\pm(\varepsilon_i - \varepsilon_j) | 1 \leq i \neq j \leq 3\}$. It has a basis: $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$. We note that $\mathfrak{h}_{\mathfrak{sl}(3)} \subset \mathfrak{h}_{\mathfrak{gl}(3)}$. We also have

$$\begin{aligned} \mathfrak{h}^*_{\mathfrak{sl}(3)} &\cong \left\{ \sum_{i=1}^{2} c_i(\varepsilon_i - \varepsilon_{i+1}) | c_i \in \mathbb{C}, 1 \leq i \leq 2 \right\} \\ &= \left\{ \sum_{i=1}^{3} d_i \varepsilon_i \middle| \sum_{i=1}^{3} d_i = 0, \text{ with } d_i \in \mathbb{C} \text{ for all } 1 \leq i \leq 3, \right\}. \end{aligned}$$

3.6 Homomorphisms, Isomorphisms, and Automorphisms of Lie Algebras

3.6.1 Homomorphisms

A *Lie algebra homomorphism* $f : L \to L'$ is a $\mathbb{C}$-vector space homomorphism, i.e., $f(x + y) = f(x) + f(y)$ and $f(ax) = af(x)$, for all $x, y \in L$ and $a \in \mathbb{C}$, preserving the Lie bracket:

$$f([x, y]) = [f(x), f(y)].$$

3.6.2 Isomorphisms

A *Lie algebra isomorphism* $\phi : L \to L'$, is a vector space isomorphism which is also a Lie algebra homomorphism. (In particular, ϕ is one-to-one and onto),

3.6.3 Automorphisms

A *Lie algebra automorphism* of L is an isomorphism of L onto itself. We denote by $\mathrm{Aut}(L)$, the group of all automorphisms of L. Define $\mathrm{ad}\, x : L \to L$ by $\mathrm{ad}\, x(y) = [x, y]$ (see Section 3.8). Since $\mathrm{char}(\mathbb{C}) = 0$ and $\mathrm{ad}\, x$ is nilpotent, i.e.,

$(\text{ad}\, x)^k = 0$ for some $k > 0$, it can be shown that
$\exp(\text{ad}\, x) = 1 + \text{ad}\, x + (\text{ad}\, x)^2/2! + \ldots + (\text{ad}\, x)^n/n!$ is an automorphism. We refer the reader to §2.3 in [Hum72] for the proof. The automorphisms of L of the form $\exp(\text{ad}\, x)$ are called *inner*. The subgroup $\text{Inn}(L)$ generated by the inner automorphisms forms a normal subgroup of $\text{Aut}(L)$ (see §2.3 in [Hum72]).

3.7 Ideals

A subspace I of a Lie algebra L is called an *ideal* of L if $[x, y] \in L$ whenever $x \in L$, and $y \in I$.

3.7.1 The Derived Algebra

The *derived algebra* of a Lie algebra L, denoted $[L, L]$, consists of all linear combinations of commutators $[x, y]$, or more specifically,
$[L, L] = \text{Span}_{\mathbb{C}}\{[x, y] | x, y \in L\}$. We see that $[L, L]$ is an ideal of L, for if $x \in L$ and $y \in [L, L]$, surely $y \in L$ and by the definition of the derived algebra $[x, y] \in [L, L]$, making it an ideal of L.

3.7.2 Simple Lie Algebras

A Lie algebra is called *simple* if it has no ideals except itself and 0, and if $[L, L] \neq 0$. In particular, $[L, L] = L$.

3.8 Representation of Lie Algebras

A *representation* (*module*) of a Lie algebra L, is a Lie algebra homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$, i.e., a vector space homomorphism for which

$$\begin{aligned}\rho([x,y])(v) &= [\rho(x),\rho(y)](v) \\ &= \rho(x)\rho(y)(v) - \rho(y)\rho(x)(v),\end{aligned}$$

for all $x, y \in L$ and $v \in V$. An important representation is the *adjoint representation*, ad: $L \longrightarrow \mathfrak{gl}(L)$, for which $\operatorname{ad}(x)(y) = [x, y]$. It preserves the bracket since

$$\begin{aligned}[\operatorname{ad} x, \operatorname{ad} y](z) &= \operatorname{ad} x(\operatorname{ad} y(z)) - \operatorname{ad} y(\operatorname{ad} x(z)) \\ &= \operatorname{ad} x([y,z]) - \operatorname{ad} y([x,z]) \\ &= [x,[y,z]] - [y,[x,z]] \qquad \text{by (3.4)} \\ &= [x,[y,z]] + [[x,z],y] \\ &= [[x,y],z] \qquad \text{by the Jacobi identity} \\ &= \operatorname{ad}[x,y](z).\end{aligned}$$

In all cases when the map ρ is obvious, we will write for simplicity $x \cdot v$ instead of $\rho(x)v$, for $x \in L, v \in V$ and we will refer to V as "a representation of L." With the dot notation we have:

$$(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$$

$$x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$$

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v).$$

3.8.1 Subrepresentations

Let L be a Lie algebra over $\mathbb{C}$, and V be a representation of L. A *subrepresentation* (*submodule*) W of V, is a vector space which is an L-representation itself. In particular, $x \cdot v \in W$ whenever $x \in L$, and $v \in W$.

3.8.2 Restriction of a Representation

If K is a Lie subalgebra of a Lie algebra L, and V is an L-representation through the homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$, then the *restriction of* ρ *to* K, $\rho|_K : K \longrightarrow \mathfrak{gl}(V)$, is defined simply by $\rho|_K(x) = \rho(x)$ for $x \in K$.

3.8.3 Quotient Representations

Let W be a subrepresentation of V over a Lie algebra L. Then the quotient vector space V/W is a representation of L defined by $x \cdot (v + W) := x \cdot v + W$. This representation is called a *quotient representation.* The action is well-defined: If $v_1 + W = v_2 + W$, we must show that $x \cdot (v_1 + W) = x \cdot (v_2 + W)$. But this follows from the fact that $x \cdot (v_1 - v_2) \in W$.

3.8.4 Irreducible (Simple) Representations

A representation is called *irreducible* or *simple* if its only subrepresentations are itself and 0.

3.8.5 Completely Reducible (Semisimple) Representations

A representation V of a Lie algebra L is called *completely reducible* or *semisimple* if V is a direct sum of irreducible subrepresentations, or more specifically $V = \bigoplus_{i=1}^{n} V_i$ and V_i are irreducible for all i.

3.8.6 Indecomposable Representations

A representation V of a Lie algebra L is called *indecomposable*, if $V = V_1 \oplus V_2$ for some L-representations V_1 and V_2 implies that $V_1 = 0$ or $V_2 = 0$. In this thesis we will describe a special class of indecomposable infinite dimensional representations of $\mathfrak{sl}(n)$ for $n = 2$ and $n = 3$.

3.8.7 Weight Representations

A representation M of a Lie algebra L is a *weight representation* if $M = \bigoplus_{\mu \in \mathfrak{h}^*} M^\mu$, where

$$M^\mu := \{m \in M \mid hm = \mu(h)\, m, \text{ for every } h \in \mathfrak{h}\}$$

is the *μ-weight space of M* and $\mathfrak{h}$ is a Cartan subalgebra of L. A weight representation M is *bounded* if there is a constant C for which $\dim M^\mu < C$ for all $\mu \in \mathfrak{h}^*$, where $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ is the dual of $\mathfrak{h}$.

Chapter 4

Generalized Weyl Representations of $\mathfrak{gl}(n)$

Consider the following representation of $\mathfrak{gl}(n)$; $\mathcal{D} : \mathfrak{gl}(n) \to \mathfrak{gl}(V)$, where $V = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$, and the action on V is defined by

$\mathcal{D}(A) \cdot f = \sum a_{ij} x_i \frac{\partial}{\partial x_j}(f)$, where $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ for all $f \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$. It is called the *Weyl representation* of $\mathfrak{gl}(n)$, and was introduced by Hermann Weyl in the 1920's. For sake of completeness, we verify that $\mathcal{D}$ is a homomorphism. It is easy to show that $\mathcal{D}$ is a vector space homomorphism by checking that $\mathcal{D}(xA + yB) = x\mathcal{D}(A) + y\mathcal{D}(B)$ for $A, B \in \mathfrak{gl}(n)$ and $x, y \in \mathbb{C}$. We must verify the homomorphism axiom $\mathcal{D}[x, y] = [\mathcal{D}x, \mathcal{D}y]$ for basis elements $x = e_{ij}$ and $y = e_{kl}$:

$$\begin{aligned}
[\mathcal{D}(e_{ij}), \mathcal{D}(e_{kl})](f) &= [x_i \frac{\partial}{\partial x_j}, x_k \frac{\partial}{\partial x_l}](f) \\
&= x_i \frac{\partial}{\partial x_j}\left(x_k \frac{\partial f}{\partial x_l}\right) - x_k \frac{\partial}{\partial x_l}\left(x_i \frac{\partial f}{\partial x_j}\right) \\
&= \delta_{jk} x_i \frac{\partial f}{\partial x_l} - \delta_{il} x_k \frac{\partial f}{\partial x_j} + x_i x_k \frac{\partial^2 f}{\partial x_j \partial x_l} - x_k x_i \frac{\partial^2 f}{\partial x_l \partial x_j} \\
&= \mathcal{D}(\delta_{jk} e_{il} - \delta_{il} e_{kj})(f) \\
&= \mathcal{D}([e_{ij}, e_{kl}])(f)
\end{aligned}$$

4.1 Homogeneous Polynomials as Representations of $\mathfrak{gl}(n)$

Set $P_m := \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2}\dots x_n^{k_n} \,|\, \sum_{i=1}^n k_i = m; k_i \in \mathbb{Z}_+ \text{ for } i \in \{1,2,...,n\}\}$, and $\widetilde{P_m} := \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2}\dots x_n^{k_n} \,|\, \sum_{i=1}^n k_i = m; k_i \in \mathbb{Z} \text{ for } i \in \{1,2,...,n\}\}$.

Proposition 4.1.1. *For every non-negative integer m, P_m and $\widetilde{P_m}$ are bounded $\mathfrak{gl}(n)$-representations. Furthermore, $\widetilde{P_m}$ is infinite dimensional.*

Proof. The result is proved in [BL87] but for a purpose of completeness we sketch the proof. We first verify that $\widetilde{P_m}$ is a vector subspace of $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$ by checking that $af + bg \in \widetilde{P_m}$ whenever $f, g \in \widetilde{P_m}$ and $a, b \in \mathbb{C}$. Let $f = \sum_{k_1+\ldots+k_n=m} \alpha_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n}$ and $g = \sum_{k_1+\ldots+k_n=m} \beta_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n}$, where we allow some of the coefficients $\alpha_{k_1,\ldots,k_n}$ and $\beta_{k_1,\ldots,k_n}$ to be zero. Then

$$\begin{aligned}
af + bg &= a\left(\sum_{k_1+\ldots+k_n=m} \alpha_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n}\right) + b\left(\sum_{k_1+\ldots+k_n=m} \beta_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n}\right)\\
&= \sum_{k_1+\ldots+k_n=m} a\alpha_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n} + \sum_{k_1+\ldots+k_n=m} b\beta_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n}\\
&= \sum_{k_1+\ldots+k_n=m} \left(a\alpha_{k_1,\ldots,k_n} + b\beta_{k_1,\ldots,k_n}\right) x_1^{k_1}\ldots x_n^{k_n} \in \widetilde{P_m}.
\end{aligned}$$

Hence $\widetilde{P_m}$ is a vector space. Next, we verify that $\widetilde{P_m}$ is a Weyl subrepresentation of $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$ by showing that the basis elements of $\mathfrak{gl}(n)$ act invariantly on $\widetilde{P_m}$. Namely $e_{ij} \cdot f = x_i \dfrac{\partial f}{\partial x_j} \in \widetilde{P_m}$ for $f \in \widetilde{P_m}$. If $f = \sum_{k_1+\ldots+k_n=m} c_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n} \in \widetilde{P_m}$ then for $i \neq j$

$$\begin{aligned}
e_{ij} \cdot f &= x_i \frac{\partial f}{\partial x_j}\\
&= \left(x_i \frac{\partial}{\partial x_j}\right) \sum_{k_1+\ldots+k_n=m} c_{k_1,\ldots,k_n} x_1^{k_1}\ldots x_n^{k_n}\\
&= \sum_{k_1+\ldots+k_n=m} c_{k_1,\ldots,k_n} k_j x_1^{k_1}\ldots x_i^{k_i+1}\ldots x_j^{k_j-1}\ldots x_n^{k_n}
\end{aligned}$$

But since

$k_1 + \ldots + (k_i + 1) + \ldots + (k_j - 1) + \ldots + k_n = k_1 + \ldots + k_i + \ldots + k_j + \ldots + k_n = m$, we have that $x_i \dfrac{\partial f}{\partial x_j} \in \widetilde{P_m}$. One can easily check that

$e_{ii} \left(\sum_{k_1+\ldots+k_n=m} c_{k_1,\ldots,k_n} x_1^{k_1} \ldots x_n^{k_n} \right) = \sum_{k_1+\ldots+k_n=m} k_i c_{k_1,\ldots,k_n} x_1^{k_1} \ldots x_n^{k_n}$. Thus $e_{ii} \cdot f \in \widetilde{P_m}$ for $f \in \widetilde{P_m}$. The proof that P_m is a subrepresentation of $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, ..., x_n^{\pm 1}]$ is similar (see Section 3.8.2).

A basis of $\widetilde{P_m}$ is $\{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} | \sum k_i = m; k_i \in \mathbb{Z} \text{ for } i \in \{1, 2, ..., n\}\} = \{x_1^{k_1} x_2^{k_2} \ldots x_{n-1}^{k_{n-1}} x_n^{m-(k_1+k_2+\ldots+k_{n-1})} | k_i \in \mathbb{Z} \text{ for } i \in \{1, 2, ..., n-1\}\} \cong \mathbb{Z}^{n-1}$ and hence $\dim \widetilde{P_m} = \infty$.

It is not difficult to verify that the weight spaces of P_m and $\widetilde{P_m}$ are $P_m^\mu = \widetilde{P_m}^\mu = \mathbb{C} x_1^{k_1} \ldots x_n^{k_n}$ for $\mu = k_1 \varepsilon_1 + \ldots + k_n \varepsilon_n \in \mathfrak{h}^*_{\mathfrak{gl}(n)}$ and $k_1, ..., k_n \in \mathbb{Z}$. Now using that

$$\widetilde{P_m} = \bigoplus_{k_1+\ldots+k_n=m} \mathbb{C} x_1^{k_1} \ldots x_n^{k_n},$$

where the sum runs over all $k_i \in \mathbb{Z}$, we verify that $\widetilde{P_m}$ (and thus its subrepresentation P_m) is a weight $\mathfrak{gl}(n)$-representation. Finally, $\dim P_m^\mu = \dim \widetilde{P_m}^\mu$ is at most one, and therefore both P_m and $\widetilde{P_m}$ are bounded. □

We now consider the restriction of $\mathcal{D}$ on $\mathfrak{sl}(n)$, $\mathcal{D}|_{\mathfrak{sl}(n)} : \mathfrak{sl}(n) \to \mathfrak{gl}(V)$. Since the $\mathcal{D}$ is a $\mathfrak{gl}(n)$-representation, then the restriction of $\mathcal{D}$ to $\mathfrak{sl}(n)$, is an $\mathfrak{sl}(n)$ -representation. Hence, P_m and $\widetilde{P_m}$ are $\mathfrak{sl}(n)$-representations as well.

4.2 Homogeneous Laurent Polynomials as Representations of $\mathfrak{sl}(n)$

Lemma 4.2.1. *P_0 is isomorphic to the one-dimensional $\mathfrak{sl}(n)$-representation $\mathbb{C}$.*

Proof. By definition,

$P_0 = \text{Span}_{\mathbb{C}} \{x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} | k_1 + k_2 + \ldots + k_n = 0; k_1, k_2, ..., k_n \in \mathbb{Z}_+\}$. But since

$k_1 + k_2 + ... + k_n = 0$ and $k_1, k_2, ..., k_n \geq 0$, it follows that $k_1 = k_2 = ... = k_n = 0$, and $P_0 = \text{Span}_{\mathbb{C}}\{x_1^0 x_2^0 ... x_n^0\} \cong \mathbb{C}$. □

Lemma 4.2.2. *P_0 is an irreducible subrepresentation of $\widetilde{P_0}$.*

Proof. It is easy to check that $P_0 \subset \widetilde{P_0}$. P_0 is an $\mathfrak{sl}(n)$-representation (see 4.1.1 and earlier in this section). To show that P_0 is irreducible, suppose M is a subrepresentation of P_0, i.e., $M \subset P_0$ and suppose f is a nonzero element of M. Then $f \in P_0$ so $f = cx_1^0 x_2^0 ... x_n^0 = c$, where $c \in \mathbb{C}$. Let g be a nonzero element of P_0 i.e. $g = dx_1^0 x_2^0 ... x_n^0 = d$ for some $d \in \mathbb{C}$. We have $\frac{c}{d}(g) \in P_0$ but we also have $\frac{c}{d}(g) = \frac{c}{d}d = c \in M$. Hence $P_0 \subset M$ implying that $P_0 = M$ so P_0 is irreducible. □

In this thesis we will study the structure of the $\mathfrak{sl}(n)$-representation $\widetilde{P_m}$ for $n = 2$ and $n = 3$. In particular, we describe the quotient representation $\widetilde{P_m}/P_m$.

Chapter 5

Laurent Polynomial Weyl Representations of $\mathfrak{sl}(2)$

From now on we fix the constant m to be in $\mathbb{Z}_+$. In this chapter we also fix $n = 2$, i.e., consider $\mathfrak{sl}(2)$ only. In this case P_m and $\widetilde{P_m}$ are defined as:
$P_m = \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2} | k_1 + k_2 = m; k_1, k_2 \in \mathbb{Z}_+\}$ and
$\widetilde{P_m} = \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2} | k_1 + k_2 = m; k_1, k_2 \in \mathbb{Z}\}$.

5.1 Preparatory Results

Lemma 5.1.1. (i) *P_m is a vector space with basis $\{x_1^m, x_1^{m-1}x_2^1, ..., x_1^1x_2^{m-1}, x_2^m\}$. In particular, dim $P_m = m + 1$.*

(ii) *$\widetilde{P_m}$ is a vector space with basis $\{x_1^i x_2^{m-i} | i \in \mathbb{Z}\}$. In particular, dim $\widetilde{P_m} = \infty$.*

Proof. We begin with the proof for P_m followed by an analogous argument for $\widetilde{P_m}$.

(i) From the discussion in Section 4.1.1 we conclude that P_m and $\widetilde{P_m}$ are $\mathfrak{sl}(2)$-representations and in particular, vector spaces. The polynomials $x_1^i x_2^{m-i}, i = 0, ..., m$, are linearly independent by definition. It is easy to check that they span P_m. Indeed let f be an element in P_m. Then

$$f = \sum_{i_1+i_2=m; i_1,i_2 \geq 0} a_{i_1,i_2} x_1^{i_1} x_2^{i_2} = \sum_{i=0}^{m} \alpha_i x_1^i x_2^{m-i},$$

where some of a_{i_1,i_2} and α_i may be zero.Thus $\{x_1^m, x_1^{m-1}x_2^1, ..., x_1^1x_2^{m-1}, x_2^m\}$ is a basis of P_m. Since there are $(m+1)$ basis elements, we have dim $P_m = m+1$.

(ii) If f is an element of $\widetilde{P_m}$ then $f = \sum_{i_1+i_2=m} a_{i_1,i_2}x_1^{i_1}x_2^{i_2} = \sum_{i\in\mathbb{Z}} \alpha_i x_1^i x_2^{m-i}$, where again some of a_{i_1,i_2} and α_i may be zero, making the set of polynomials: $\{x_1^i x_2^{m-i} | i \in \mathbb{Z}\}$ a basis of $\widetilde{P_m}$, and dim $\widetilde{P_m} = \infty$ since dim $\widetilde{P_m} =$ dim $\mathbb{Z} = \infty$. □

Using Lemma 5.1.1, we can form the quotient vector space $\widetilde{P_m}/P_m = \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2} + P_m | k_1 + k_2 = m; k_1, k_2 \in \mathbb{Z}\}$. We will denote by P_m^+ and P_m^-, the following vector subspaces of $\widetilde{P_m}/P_m$:

$P_m^+ := \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2} + P_m | k_1 > m; k_1 + k_2 = m; k_1, k_2 \in \mathbb{Z}\}$ and

$P_m^- := \text{Span}_{\mathbb{C}}\{x_1^{k_1}x_2^{k_2} + P_m | k_2 > m; k_1 + k_2 = m; k_1, k_2 \in \mathbb{Z}\}$. We note that if $g \in P_m$, then the element $g + P_m$ of the quotient space $\widetilde{P_m}/P_m$ is the zero element since $g + P_m = P_m = 0$. Furthermore, $g + P_m = P_m = 0$ is an element of P_m^+ and P_m^- as well. We also fix the standard basis $\{e, f, h\}$ of $\mathfrak{sl}(2)$ introduced in Section 3.4.1.

Proposition 5.1.2. *P_m is an irreducible $\mathfrak{sl}(2)$-representation for every $m \in \mathbb{Z}_+$.*

Proof. We will begin by showing that P_m is an $\mathfrak{sl}(2)$-representation generated by the basis element $x_1^0 x_2^m = x_2^m$ through the repeated action of $e = x_1\frac{\partial}{\partial x_2}$ (see section 3.4.1). Suppose that $g \in P_m$. Then $g = \sum_{k_1+k_2=m} a_{k_1,k_2}x_1^{k_1}x_2^{k_2}$. We then have:

$e \cdot x_2^m = x_1\frac{\partial}{\partial x_2}x_2^m = mx_1x_2^{m-1}$. If we apply e on x_2^m twice, we obtain:

$e \cdot (e \cdot x_2^m) = e \cdot (mx_1x_2^{m-1}) = x_1\frac{\partial}{\partial x_2}(mx_1x_2^{m-1}) = m(m-1)x_1^2x_2^{m-2}$, and if e is applied k times on x_2^m, we obtain $e \cdot (e \cdot ...(e \cdot x_2^m) = m(m-1)...(m-k+1)x_1^k x_2^{m-k}$. We will denote by e^k the action of e on elements in P_m applied k times.

Remark 5.1.3. In fact, e^k are elements of the *universal enveloping algebra*, $\mathcal{U}(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$. For any Lie algebra L, its universal enveloping algebra is an associative algebra $\mathcal{U}$ containing L as a Lie subalgebra. Every L-representation is in fact a $\mathcal{U}$-representation. In this thesis we will intuitively make use of $\mathcal{U}(\mathfrak{sl}(2))$ in future. For more details, we refer the reader to Chapter V of [Hum72].

From the work above, we have that $e^k \cdot x_2^m = m(m-1)...(m-k+1)x_1^k x_2^{m-k}$, and thus the basis elements $x_1^k x_2^{m-k}$ are generated by action of $\dfrac{1}{m(m-1)...(m-k+1)}e^k = \dfrac{k!}{m!}e^k$ on x_2^m as: $\dfrac{k!}{m!}e^k \cdot x_2^m = x_1^k x_2^{m-k}$. Therefore,

$$\begin{aligned} g &= \sum_{k_1+k_2=m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \\ &= \sum_{k_1+k_2=m} a_{k_1,k_2} \left(\frac{k_1!}{m!} e^{k_1} \cdot x_2^m \right) \\ &= \left(\sum_{k_1+k_2=m} a_{k_1,k_2} \frac{k_1!}{m!} e^{k_1} \right) \cdot x_2^m \\ &= u \cdot x_2^m, \end{aligned} \tag{5.1}$$

where $u = \displaystyle\sum_{k_1+k_2=m} a_{k_1,k_2} \frac{k_1!}{m!} e^{k_1}$.

Now we show that x_2^m can be obtained from any element $\hat{g} \in P_m$ by consecutive application of elements of $\mathfrak{sl}(2)$. Let $\hat{g} = \displaystyle\sum_{k_1+k_2=m} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}$ with $b_{k_1,k_2} \in \mathbb{C}$, and let $\tilde{u} = \dfrac{1}{b_{\tilde{k},m-\tilde{k}}} \dfrac{1}{\tilde{k}!} f^{\tilde{k}}$, where $f = x_2 \dfrac{\partial}{\partial x_1}$ and $\tilde{k}$ =max$\{k_1 | b_{k_1,k_2} \neq 0\}$. Then the action of $\tilde{u}$ on the terms $b_{k_1,k_2} x_1^{k_1} x_2^{k_2}$ is:

$$\begin{aligned} \tilde{u} \cdot b_{k_1,k_2} x_1^{k_1} x_2^{k_2} &= \frac{1}{b_{\tilde{k},m-\tilde{k}}} \frac{1}{\tilde{k}!} \left(x_2 \frac{\partial}{\partial x_1} \right)^{\tilde{k}} \cdot (b_{k_1,k_2} x_1^{k_1} x_2^{k_2}) \\ &= \frac{1}{b_{\tilde{k},m-\tilde{k}}} \frac{1}{\tilde{k}!} x_2^{\tilde{k}} \frac{\partial^{\tilde{k}}}{\partial x_1^{\tilde{k}}} b_{k_1,k_2} x_1^{k_1} x_2^{k_2}. \end{aligned}$$

This is equal to zero whenever $k_1 < \tilde{k}$. For $k_1 = \tilde{k}$, we have:

$$\begin{aligned}
\tilde{u} \cdot b_{k_1,k_2} x_1^{k_1} x_2^{k_2} &= \frac{1}{b_{\tilde{k},m-\tilde{k}}} \frac{1}{\tilde{k}!} \left(x_2 \frac{\partial}{\partial x_1} \right)^{\tilde{k}} \cdot b_{\tilde{k},m-\tilde{k}} x_1^{\tilde{k}} x_2^{m-\tilde{k}} \\
&= \frac{1}{b_{\tilde{k},m-\tilde{k}}} \frac{1}{\tilde{k}!} x_2^{\tilde{k}} \frac{\partial^{\tilde{k}}}{\partial x_1^{\tilde{k}}} b_{\tilde{k},m-\tilde{k}} x_1^{\tilde{k}} x_2^{m-\tilde{k}} \\
&= \frac{1}{b_{\tilde{k},m-\tilde{k}}} \frac{1}{\tilde{k}!} b_{\tilde{k},m-\tilde{k}} \tilde{k}! x_2^{\tilde{k}} x_2^{m-\tilde{k}} \\
&= x_2^m .
\end{aligned}$$

Summarizing these results, we obtain:

$$\begin{aligned}
\tilde{u} \cdot \hat{g} &= \tilde{u} \cdot \left(\sum_{k_1+k_2=m} b_{k_1,k_2} x_1^{k_1} x_2^{k_2} \right) \\
&= x_2^m .
\end{aligned} \tag{5.2}$$

We have shown that $\tilde{u} \cdot \hat{g} = x_2^m$, for all nonzero $\hat{g} \in P_m$, for some $\tilde{u}$ in $\mathcal{U}(\mathfrak{sl}(2))$.

We will now prove that P_m is irreducible. Let M be a nontrivial subrepresentation of P_m, and let $w \in M$ be a nonzero element. We see that $w \in P_m$ since $M \subset P_m$. Therefore we have that $\tilde{u} \cdot w = x_2^m$ by Equation (5.2), so $x_2^m \in M$. Let g be an element in P_m. Then $g = u \cdot x_2^m$ by Equation 5.1. But $x_2^m \in M$, making g an element of M as well. Hence we have that $P_m \subset M$, and we must have that $M = P_m$ so P_m is irreducible. □

Proposition 5.1.4. $\widetilde{P_m}/P_m = P_m^+ \oplus P_m^-$

Proof. It is easy to check that $P_m^+ \cap P_m^- = \{0\}$. It remains to be shown that $\widetilde{P_m}/P_m = P_m^+ + P_m^-$. Suppose $\tilde{f} \in \widetilde{P_m}/P_m$. Then

$$\tilde{f} = \sum_{k_1+k_2=m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m = \sum_{k_1+k_2=m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m).$$

The sum may be split up into 3 parts, each one satisfying one of the following conditions: $k_1 > m$, $k_2 > m$, and $k_1, k_2 \leq m$:

$$\begin{aligned}
\tilde{f} &= \sum_{k_1+k_2=m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m) \\
&= \sum_{k_1+k_2=m, k_1>m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m) + \sum_{k_1+k_2=m, k_2>m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m) \\
&\quad + \sum_{k_1+k_2=m, k_1,k_2 \leq m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m) \\
&= (\sum_{k_1+k_2=m, k_1>m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2}) + P_m) + (\sum_{k_1+k_2=m, k_2>m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2}) + P_m) \\
&\quad + (\sum_{k_1+k_2=m, k_1,k_2 \leq m} (a_{k_1,k_2} x_1^{k_1} x_2^{k_2}) + P_m) \\
&= f_+ + f_- + 0,
\end{aligned}$$

where $f_+ \in P_m^+$ and $f_- \in P_m^-$. Thus, $\tilde{f} \in P_m^+ \oplus P_m^-$ and $\widetilde{P_m}/P_m = P_m^+ \oplus P_m^-$.

Our next goal is to prove that P_m^+ and P_m^- are irreducible subrepresentations of $\widetilde{P_m}/P_m$. □

Lemma 5.1.5. *$\widetilde{P_m}/P_m$, P_m^+ and P_m^- are $\mathfrak{sl}(2)$-representations.*

Proof. $\widetilde{P_m}/P_m$ is an $\mathfrak{sl}(2)$-representation because it is a quotient of two $\mathfrak{sl}(2)$-representations with $P_m \subset \widetilde{P_m}$ (see Section 3.8.3 and Proposition 4.1.1).

Next, we will show that P_m^+ is an $\mathfrak{sl}(2)$-representation. We begin by proving that P_m^+ is invariant under the action of non-diagonal elements of $\mathfrak{sl}(2)$: e_{ij} with $i \neq j$. Suppose f is an element of P_m^+. Then $f = \sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m$. We must show that $e_{ij} \cdot f \in P_m^+$ for $1 \leq i \neq j \leq 2$. We begin by showing that

$e_{12} \cdot f \in P_m^+$:

$$\begin{aligned} e_{12} \cdot f &= x_1 \frac{\partial}{\partial x_2} \cdot \left(\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m \right) \\ &= \sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} x_1 \frac{\partial}{\partial x_2} x_1^{k_1} x_2^{k_2} + P_m \\ &= \sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} k_2 x_1^{k_1+1} x_2^{k_2-1} + P_m \\ &= \sum_{i_1+i_2=m, i_1>m+1} a_{i_1-1,i_2+1} (i_2+1) x_1^{i_1} x_2^{i_2} + P_m \in P_m^+, \end{aligned}$$

by changing variables: $i_1 = k_1 + 1, i_2 = k_2 - 1$ and $i_1 + i_2 = m$ with $i_1 > m$. Next we show that $e_{21} \cdot f \in P_m^+$:

$$\begin{aligned} e_{21} \cdot f &= x_2 \frac{\partial}{\partial x_1} \cdot \left(\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m \right) \\ &= \sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} x_2 \frac{\partial}{\partial x_1} x_1^{k_1} x_2^{k_2} + P_m \\ &= \sum_{k_1+k_2=m, k_1>m+1} a_{k_1,k_2} x_2 \frac{\partial}{\partial x_1} x_1^{k_1} x_2^{k_2} + \\ &\quad \sum_{k_1+k_2=m, k_1=m+1} a_{k_1,k_2} x_2 \frac{\partial}{\partial x_1} x_1^{m+1} x_2^{-1} + P_m \\ &= \sum_{k_1+k_2=m, k_1>m+1} a_{k_1,k_2} k_1 x_1^{k_1-1} x_2^{k_2+1} + \\ &\quad \sum_{k_1+k_2=m, k_1=m+1} a_{k_1,k_2} (m+1) x_1^{(m+1)-1} x_2^{0} + P_m \\ &= \sum_{i_1+i_2=m, i_1>m} a_{i_1+1,i_2-1} (i_1+1) x_1^{i_1} x_2^{i_2} + 0 + P_m \in P_m^+. \end{aligned}$$

Note that the second term is zero since $x_1^{(m+1)-1} x_2^0 \in P_m$.

To account for the action of the $\mathfrak{sl}(2)$ diagonal element h, we easily check that $(e_{11} - e_{22}) \cdot f \in P_m^+$ if $f \in P_m^+$. □

Lemma 5.1.6. *Let f be a nonzero element of P_m^+, i.e.,* $f = \sum_{(k_1,k_2) \in I_f} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m$, *where* $I_f \subset \{(k_1,k_2) | k_1 + k_2 = m, k_1 > m\}$ *and* *choose* $j = \max\{k_1 | a_{k_1,k_2} \neq 0\}$. *Then*

$$\frac{(m+1)!}{j! a_{j,m-j}} e_{21}^{j-(m+1)} \cdot f = x_1^{m+1} x_2^{-1} + P_m$$

Proof.

$$\begin{aligned}
& \frac{(m+1)!}{j! a_{j,m-j}} e_{21}^{j-(m+1)} \cdot f \\
= \ & \frac{(m+1)!}{j! a_{j,m-j}} x_2^{j-(m+1)} \frac{\partial^{j-(m+1)} f}{\partial x_1^{j-(m+1)}} \\
= \ & \frac{(m+1)!}{j! a_{j,m-j}} x_2^{j-(m+1)} \frac{\partial^{j-(m+1)}}{\partial x_1^{j-(m+1)}} \left(\sum_{(k_1,k_2) \in I_f} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m \right) \\
= \ & \sum_{(k_1,k_2) \in I_f} a_{k_1,k_2} \frac{(m+1)!}{j! a_{j,m-j}} x_2^{j-(m+1)} \frac{\partial^{j-(m+1)}}{\partial x_1^{j-(m+1)}} x_1^{k_1} x_2^{k_2} + P_m \\
= \ & a_{j,m-j} \frac{(m+1)!}{j! a_{j,m-j}} x_2^{j-(m+1)} \frac{\partial^{j-(m+1)}}{\partial x_1^{j-(m+1)}} x_1^{j} x_2^{m-j} + \\
& \sum_{(k_1,k_2) \in I_f, j-(m+1) \leq k_1 < j} a_{k_1,k_2} \frac{(m+1)!}{j! a_{j,m-j}} x_2^{j-(m+1)} \frac{\partial^{j-(m+1)}}{\partial x_1^{j-(m+1)}} x_1^{k_1} x_2^{k_2} + \\
& \sum_{(k_1,k_2) \in I_f, m < k_1 < j-(m+1)} a_{k_1,k_2} \frac{(m+1)!}{j! a_{j,m-j}} x_2^{j-(m+1)} \frac{\partial^{j-(m+1)}}{\partial x_1^{j-(m+1)}} x_1^{k_1} x_2^{k_2} + P_m \\
= \ & a_{j,m-j} \frac{(m+1)!}{j! a_{j,m-j}} j(j-1) \ldots (m+2) x_1^{m+1} x_2^{-1} + \\
& \sum_{(k_1,k_2) \in I_f, j-(m+1) \leq k_1 < j} a_{k_1,k_2} \frac{(m+1)!}{j! a_{j,m-j}} \frac{k_1!}{(k_1 - j + m + 1)!} x_1^{k_1 - j + m + 1} x_2^{k_2 + j - m - 1} + \\
& 0 + P_m \\
= \ & \frac{(m+1)!}{j!} \frac{j!}{(m+1)!} x_1^{m+1} x_2^{-1} + 0 + 0 + P_m \\
= \ & x_1^{m+1} x_2^{-1} + P_m
\end{aligned}$$

□

Lemma 5.1.7. *Suppose* f *is an element of* P_m^+, *i.e.*,

$$f = \sum_{k_1 + k_2 = m, k_1 > m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} + P_m.$$

Then

$$f = \left(\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} \frac{(-1)^{k_1-(m+1)}}{(k_1-(m+1))!} e_{12}^{k_1-(m+1)} \right) \cdot \left(x_1^{m+1} x_2^{-1} + P_m \right)$$

Proof. We easily verify that:

$$\begin{aligned}
&\left(\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} \frac{(-1)^{k_1-(m+1)}}{(k_1-(m+1))!} e_{12}^{k_1-(m+1)} \right) \cdot \left(x_1^{m+1} x_2^{-1} + P_m \right) \\
= &\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} \frac{(-1)^{k_1-(m+1)}}{(k_1-(m+1))!} x_1^{k_1-(m+1)} \frac{\partial^{k_1-(m+1)}}{\partial x_2^{k_1-(m+1)}} x_1^{m+1} x_2^{-1} + P_m \\
= &\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} \frac{(-1)^{k_1-(m+1)}}{(k_1-(m+1))!} (-1)(-2)\cdots(-k_1+(m+1)) x_1^{k_1} x_2^{-1-k_1+m+1} \\
= &\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} \frac{(-1)^{k_1-(m+1)}}{(k_1-(m+1))!} (-1)^{k_1-(m+1)} (k_1-(m+1))! x_1^{k_1} x_2^{k_2} \\
= &\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \\
= &f
\end{aligned}$$

□

Lemma 5.1.8. *P_m^+ and P_m^- are irreducible subrepresentations of $\widetilde{P_m}/P_m$.*

Proof. Suppose $M \subset P_m^+$ is a nontrivial subrepresentation, and that $g \in M$ is a nonzero element. Then $g \in P_m^+$. Therefore, by Lemma 5.1.6: $\frac{(m+1)!}{j! a_{j,m-j}} e_{21}^{j-(m+1)} \cdot g = x_1^{m+1} x_2^{-1} + P_m$, for some $j > m+1$, and for some nonzero $a_{j,m-j} \in \mathbb{C}$. So $x_1^{m+1} x_2^{-1} + P_m \in M$. Now, suppose that $f \in P_m^+$. Then, by Lemma 5.1.7,

$$f = \left(\sum_{k_1+k_2=m, k_1>m} a_{k_1,k_2} \frac{(-1)^{k_1-(m+1)}}{(k_1-(m+1))!} e_{12}^{k_1-(m+1)} \right) \cdot \left(x_1^{m+1} x_2^{-1} + P_m \right).$$

Therefore, since $x_1^{m+1} x_2^{-1} + P_m \in M$, we have $f \in M$. We have shown that $P_m^+ \subset M$. Hence, $M = P_m^+$. A similar argument shows that P_m^- is irreducible. □

Lemma 5.1.9. *The vector space $\widetilde{P_m}$ is an infinite dimensional indecomposable $\mathfrak{sl}(2)$-representation.*

Proof. Section 4.1 and Proposition 4.1.1 show that $\widetilde{P_m}$ is an infinite dimensional $\mathfrak{sl}(2)$-representation. It remains to be shown that $\widetilde{P_m}$ is indecomposable. Suppose not, i.e., suppose $\widetilde{P_m} = M \oplus N$ for some $\mathfrak{sl}(2)$-representations M and N, and assume that $M, N \neq 0$. Elements in M and N are of the form $\sum_{k_1+k_2=m} a_{k_1,k_2} x_1^{k_1} x_2^{k_2}$. Suppose f is a nonzero element of N. Then

$$f = \sum_{(k_1,k_2)\in I_f} a_{k_1,k_2} x_1^{k_1} x_2^{k_2},$$

where $I_f \subset \{(k_1,k_2) | k_1 + k_2 = m\}$. We will show that applying the action of e_{12} and e_{21} on our element f, we obtain the element x_2^m which is an element of N since N is an $\mathfrak{sl}(2)$-representation.

Step 1.

Let $j = \max(\{k_2 | (k_1,k_2) \in I_f\}, 0)$. If $j > 0$, the action of e_{12}^j on f is as follows:

$$\begin{aligned} e_{12}^j \cdot f &= x_1^j \frac{\partial^j f}{\partial x_2^j} \\ &= x_1^j \frac{\partial^j}{\partial x_2^j} \left(\sum_{(k_1,k_2)\in I_f} a_{k_1,k_2} x_1^{k_1} x_2^{k_2} \right) \\ &= \sum_{(k_1,k_2)\in I_f} a_{k_1,k_2} x_1^j \frac{\partial^j}{\partial x_2^j} x_1^{k_1} x_2^{k_2}, \end{aligned}$$

and if $j = 0$, $e_{12}^j \cdot f = e_{12}^0 \cdot f = f$, i.e., e_{12}^j acts as the identity operator. We summarize the results of the differentiation $x_1^j \frac{\partial^j}{\partial x_2^j} x_1^{k_1} x_2^{k_2}$ when $j > 0$:

$$x_1^j \frac{\partial^j}{\partial x_2^j} x_1^{k_1} x_2^{k_2} = \begin{cases} 0 & \text{if } j > k_2 \geq 0, \\ j! x_1^m & \text{if } j = k_2 \geq 0, \\ \dfrac{(-k_2 + j - 1)!}{(-k_2 - 1)!} (-1)^j x_1^{k_1+j} x_2^{k_2-j} & \text{if } j > 0 > k_2. \end{cases}$$

After this step all x_2-exponents have become negative or zero, i.e.,

$$\begin{aligned} e_{12}^j \cdot f &= \sum_{j_1+j_2=0, j_2\le 0} b_{j_1,j_2} x_1^{j_1} x_2^{j_2} \\ &= g. \end{aligned}$$

The nonzero constants b_{j_1,j_2} are determined by: $b_{j_1,j_2} = a_{j_1,j_2}$ for $(j_1, j_2) \in I_f$ in the case $j = 0$. For $j > 0$ we have

$(j_1, j_2) = \{(k_1 + j, k_2 - j) | (k_1, k_2) \in I_f, k_2 = j \text{ or } k_2 < 0\}$ and

$b_{j_1,j_2} = c_{j_1-j,j_2+j} a_{j_1-j,j_2+j}$ where

$$c_{j_1-j,j_2+j} = \begin{cases} j! & \text{if } (j_1,j_2) = (m,0), \\ \dfrac{(-j_2-1)!}{(-j_2-j-1)!}(-1)^j & \text{otherwise.} \end{cases}$$

$g = \sum_{j_1+j_2=0, j_2\le 0} b_{j_1,j_2} x_1^{j_1} x_2^{j_2}$ is an element of N for which all x_2-exponents are negative or zero and all x_1-exponents are greater than or equal to m. We are guaranteed that at least one of the coefficients b_{j_1,j_2} is nonzero.

Step 2.

Let $k = \max(\{j_1 | b_{j_1,j_2} \neq 0\})$. The action of e_{21}^k on g is as follows:

$$\begin{aligned} e_{21}^k \cdot g &= x_2^k \frac{\partial^k g}{\partial x_1^k} \\ &= x_2^k \frac{\partial^k}{\partial x_1^k} \left(\sum_{j_1+j_2=m, j_2\le 0} b_{j_1,j_2} x_1^{j_1} x_2^{j_2} \right) \\ &= \sum_{j_1+j_2=m, j_2\le 0} b_{j_1,j_2} x_2^k \frac{\partial^k}{\partial x_1^k} x_1^{j_1} x_2^{j_2} \\ &= b_{k,m-k} x_2^k \frac{\partial^k}{\partial x_1^k} x_1^k x_2^{m-k} + \sum_{j_1+j_2=m, j_2\le 0, j_1<k} b_{j_1,j_2} x_2^k \frac{\partial^k}{\partial x_1^k} x_1^{j_1} x_2^{j_2} \\ &= b_{k,m-k} k! x_2^m + 0. \end{aligned}$$

Hence we have $b_{k,m-k} k! x_2^m \in N$, where $b_{k,m-k} \neq 0$ and $k > 0$. Therefore,

$x_2^m = \dfrac{1}{b_{k,m-k} k!} b_{k,m-k} k! x_2^m \in N.$

We have shown that x_2^m is an element of N. If we pick an arbitrary nonzero element of M and follow steps 1 and 2, we can show that x_2^m also is an element of M. But $\widetilde{P_m} = M \oplus N$ implying that $M \cap N = \{0\}$. This is a contradiction. We have shown that $\widetilde{P_m}$ is indecomposable. □

5.2 Decomposition of $\widetilde{P_m}$

Theorem 5.2.1. *The Laurent polynomial representation $\widetilde{P_m}$ is an infinite dimensional indecomposable representation containing P_m as an irreducible subrepresentation. The quotient representation $\widetilde{P_m}/P_m$ equals the direct sum, $P_m^+ \oplus P_m^-$ of two irreducible representations. In particular, $\widetilde{P_m}$ has length 3, i.e., it has 3 irreducible subquotients.*

Proof. $\widetilde{P_m}$ is an infinite dimensional indecomposable representation by Lemma 5.1.9. Proposition 4.1.1 shows that it contains P_m as a subspace, and P_m is an irreducible subrepresentation by Proposition 5.1.2. From Proposition 5.1.4 we have $\widetilde{P_m}/P_m = P_m^+ \oplus P_m^-$ and Lemma 5.1.8 shows that they are irreducible subrepresentations of $\widetilde{P_m}/P_m$. □

Chapter 6

Laurent Polynomial Weyl Representations of $\mathfrak{sl}(3)$.

Our main goal in this chapter is to find a decomposition of the representation $\widetilde{P_0}$. We will show that $\widetilde{P_0}$ is an indecomposable representation of length 7, i.e., it has 7 irreducible subquotients.

6.1 Preparatory Results

We will begin by considering the quotient space $\bar{P_0} = \widetilde{P_0}/P_0$. Elements in $\widetilde{P_0}/P_0$ are of the form: $\sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} x_{a_1}^{i_1} x_{a_2}^{i_2} x_{a_3}^{i_3} + P_0,$, with $J_f \subset \{(i_1,i_2,i_3)|i_1+i_2+i_3=0; i_1,i_2\neq 0\}$, and $(a_1,a_2,a_3)=(1,2,3),(2,3,1)$, or $(3,1,2)$. The zero element is: $0+P_0=P_0$.

Lemma 6.1.1. *The quotient space $\widetilde{P_0}/P_0$ is an $\mathfrak{sl}(3)$-representation.*

Proof. $\widetilde{P_0}$ and P_0 are $\mathfrak{sl}(3)$-representations with $P_0 \subset \widetilde{P_0}$ by Proposition 4.1.1 and Section 4.1. Hence it follows from Section 3.8.3 that the quotient vector space $\widetilde{P_0}/P_0$ is an $\mathfrak{sl}(3)$-representation. □

We consider the following subspaces of $\widetilde{P_0}/P_0$:

$$L_{a_1a_2}^{+} = \mathrm{Span}_{\mathbb{C}}\{x_{a_l}^{i_1} x_{a_2}^{i_2} x_{a_3}^{i_3} + P_0 | i_1, i_2 \geq 0\},$$

with $(a_1, a_2, a_3) = (1,2,3), (1,3,2)$, or $(2,3,1)$. Let us define P_0^+ by:

$$P_0^+ := \mathrm{Span}_{\mathbb{C}}\{x_1^{i_1}x_2^{i_2}x_3^{i_3} \in \widetilde{P_0} | i_j, i_k \geq 0; 1 \leq j \neq k \leq 3\}.$$

Lemma 6.1.2. (i) *P_0^+/P_0 is a subrepresentation of the quotient representation $\widetilde{P_0}/P_0$.*

(ii) *$L_{a_1a_2}^+$ are subrepresentations of P_0^+/P_0.*

(iii) *We have $P_0^+/P_0 = L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+$.*

Proof. We will begin by showing part (i). Suppose $f \in P_0^+/P_0$. Then

$$f = \sum_{a_i+a_j+a_k=0; a_p, a_q \geq 0, p\neq q} \alpha_{a_1,a_2,a_3} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0.$$

We will first verify the action of the non-diagonal elements of $\mathfrak{z}$ on elements of P_0^+/P_0. We must show that $e_{ij} \cdot f = x_i \dfrac{\partial f}{\partial x_j} \in P_0^+/P_0$, for $i \neq j$.

$$\begin{aligned}
e_{ij} \cdot f &= x_i \frac{\partial f}{\partial x_j} \\
&= x_i \frac{\partial}{\partial x_j}\left(\sum_{a_i+a_j+a_k=0; a_p, a_q \geq 0, p\neq q} \alpha_{a_1,a_2,a_3} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0\right) \\
&= \sum_{a_i+a_j+a_k=0; a_p, a_q \geq 0, p\neq q} x_i \frac{\partial}{\partial x_j}\left(\alpha_{a_1,a_2,a_3} x_i^{a_i} x_j^{a_j} x_k^{a_k}\right) + P_0 \\
&= \sum_{a_i+a_j+a_k=0; a_p, a_q \geq 0, p\neq q} \alpha_{a_1,a_2,a_3} x_i \frac{\partial}{\partial x_j} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0
\end{aligned}$$

It remains to be shown that $x_i \dfrac{\partial}{\partial x_j}\left(x_i^{a_i} x_j^{a_j} x_k^{a_k}\right) \in P_0^+$

Case 1: $a_i \geq 0, a_j \geq 0$.

Case 1.1: $a_i \geq 0, a_j = 0$.

$$x_i \frac{\partial}{\partial x_j}\left(x_i^{a_i} x_j^{a_j} x_k^{a_k}\right) = 0 \in P_0^+.$$

Case 1.2: $a_i \geq 0, a_j > 0$.

$$x_i \frac{\partial}{\partial x_j} \left(x_i^{a_i} x_j^{a_j} x_k^{a_k} \right) = a_j x_i^{a_i+1} x_j^{a_j-1} x_k^{a_k} \in P_0^+,$$

since $a_i + 1, a_j - 1 \geq 0$.

Case 2: $a_k \geq 0, a_j \geq 0$.

Case 2.1: $a_k \geq 0, a_j = 0$.

$$x_i \frac{\partial}{\partial x_j} \left(x_i^{a_i} x_j^{a_j} x_k^{a_k} \right) = 0 \in P_0^+.$$

Case 2.2: $a_k \geq 0, a_j > 0$.

$$x_i \frac{\partial}{\partial x_j} \left(x_i^{a_i} x_j^{a_j} x_k^{a_k} \right) = a_j x_i^{a_i+1} x_j^{a_j-1} x_k^{a_k} \in P_0^+,$$

since $a_k, a_j - 1 \geq 0$.

Case 3: $a_i \geq 0, a_k \geq 0$.

We must have $a_j \leq 0$ since $a_i + a_j + a_k = 0$.

Case 3.1: $a_j = 0$.

$$x_i \frac{\partial}{\partial x_j} \left(x_i^{a_i} x_j^{a_j} x_k^{a_k} \right) = 0 \in P_0^+.$$

Case 3.2: $a_j < 0$.

$$x_i \frac{\partial}{\partial x_j} \left(x_i^{a_i} x_j^{a_j} x_k^{a_k} \right) = a_j x_i^{a_i+1} x_j^{a_j-1} x_k^{a_k} \in P_0^+,$$

since $a_i + 1, a_k \geq 0$.

To account for the action of the $\mathfrak{sl}(3)$ diagonal elements h_i where $1 \leq i \leq 2$, we easily check that $(e_{ii} - e_{i+1,i+1}) \cdot f \in P_0^+/P_0$ if $f \in P_0^+/P_0$.

Now we will show part (ii), i.e., that $L_{a_1 a_2}^+$ are subrepresentations of P_0^+/P_0. Suppose $f \in L_{a_1 a_2}^+$. Then $f = \sum_{\sum k_i = 0; k_1, k_2 \geq 0} \alpha_{a_1, a_2, a_3} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0$. We must show that $e_{ij} \cdot f \in L^+_{a_1a_2}$, where $i \neq j$.

$$\begin{aligned}
e_{ij} \cdot f &= x_i \frac{\partial f}{\partial x_j} \\
&= x_i \frac{\partial}{\partial x_j} \left(\sum_{\sum k_i = 0; k_1, k_2 \geq 0} \alpha_{a_1,a_2,a_3} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 \right) \\
&= \sum_{\sum k_i = 0; k_1, k_2 \geq 0} x_i \frac{\partial}{\partial x_j} \left(\alpha_{a_1,a_2,a_3} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} \right) + P_0 \\
&= \sum_{\sum k_i = 0; k_1, k_2 \geq 0} \alpha_{a_1,a_2,a_3} x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 \\
&= \sum_{\sum k_i = 0; k_1, k_2 \geq 0} \alpha_{a_1,a_2,a_3} x_i \frac{\partial}{\partial x_j} \left(x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 \right)
\end{aligned}$$

It remains to be shown that $x_i \frac{\partial}{\partial x_j} \left(x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 \right) = x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 \in L^+_{a_1a_2}$. Since $f \in L^+_{a_1a_2}$, we have $k_1, k_2 \geq 0$.

Case 1: $j = a_1$, $i = a_2$ *or* $i = a_3$.

Case 1.1: $k_1 = 0$.

$$x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 = 0 + P_0 \in L^+_{a_1a_2}.$$

Case 1.2: $k_1 > 0$.

$$x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 = \begin{cases} k_1 x_{a_1}^{k_1-1} x_{a_2}^{k_2+1} x_{a_3}^{k_3} + P_0 \in L^+_{a_1a_2} & \text{if } i = a_2 \\ k_1 x_{a_1}^{k_1-1} x_{a_2}^{k_2} x_{a_3}^{k_3+1} + P_0 \in L^+_{a_1a_2} & \text{if } i = a_3, \end{cases}$$

since in each of these two cases, the a_1- and a_2-exponents are non-negative.

Case 2: $j = a_2$, $i = a_1$ *or* $i = a_3$.

Case 2.1: $k_2 = 0$.

$$x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 = 0 + P_0 \in L^+_{a_1a_2}.$$

Case 2.2: $k_2 > 0$.

$$x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 = \begin{cases} k_2 x_{a_1}^{k_1+1} x_{a_2}^{k_2-1} x_{a_3}^{k_3} + P_0 \in L^+_{a_1a_2} & \text{if } i = a_1 \\ k_2 x_{a_1}^{k_1} x_{a_2}^{k_2-1} x_{a_3}^{k_3+1} + P_0 \in L^+_{a_1a_2} & \text{if } i = a_3, \end{cases}$$

since in each of these two cases, the a_1- and a_2-exponents are non-negative.

Case 3: $j = a_3$, $i = a_1$ *or* $i = a_2$.

We must have $k_3 \leq 0$ since $k_1, k_2 \geq 0$ and $\sum k_i = 0$.

Case 3.1: $k_3 = 0$.

$$x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 = 0 + P_0 \in L^+_{a_1 a_2}.$$

Case 3.2: $k_3 < 0$.

$$x_i \frac{\partial}{\partial x_j} x_{a_1}^{k_1} x_{a_2}^{k_2} x_{a_3}^{k_3} + P_0 = \begin{cases} k_3 x_{a_1}^{k_1+1} x_{a_2}^{k_2} x_{a_3}^{k_3-1} + P_0 \in L^+_{a_1 a_2} & \text{if } i = a_1 \\ k_3 x_{a_1}^{k_1} x_{a_2}^{k_2+1} x_{a_3}^{k_3-1} + P_0 \in L^+_{a_1 a_2} & \text{if } i = a_2, \end{cases}$$

since in each of these two cases, the a_1- and a_2-exponents are non-negative.

We will now prove part (iii), i.e., $P_0^+/P_0 = L^+_{12} \oplus L^+_{13} \oplus L^+_{23}$. We verify that $L^+_{ij} \cap (L^+_{ik} + L^+_{jk}) = P_0$ for $(i, j, k) = (1, 2, 3), (2, 3, 1)$ or $(3, 1, 2)$. Indeed, let $f = \sum_{i_1+i_2+i_3=0} \alpha_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$ be such that $f + P_0 \in L^+_{12} \cap (L^+_{13} + L^+_{23})$ and $f \neq P_0$. Then we easily show that $\alpha_{i_1,i_2,i_3} \neq 0$ and $f \in L^+_{13} + L^+_{23}$ implies $i_3 \geq 0$. Also $f \in L^+_{12}$, $f \neq P_0$ and $\alpha_{i_1,i_2,i_3} \neq 0$ implies $i_3 < 0$. This is a contradiction. The proof is similar for the other identities.

It remains to be shown that $P_0^+/P_0 = L^+_{12} + L^+_{13} + L^+_{23}$. Elements in P_0^+/P_0 are of the form $\sum_{i_j, i_k \geq 0, j \neq k} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0$. Any element in P_0^+/P_0 also is in $\widetilde{P_0}/P_0$ since $P_0^+ \subset \widetilde{P_0}$. If $f \in P_0^+/P_0$ it can be written using four sums for which the terms have (i) $i_1, i_2 \geq 0, i_3 < 0$, (ii) $i_1, i_3 \geq 0, i_2 < 0$, (iii) $i_2, i_3 \geq 0, i_1 < 0$ and (iv)

$i_1 = i_2 = i_3 = 0$ as follows:

$$\begin{aligned} f &= \left(\sum_{i_1,i_2 \geq 0, i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + \sum_{i_1,i_3 \geq 0, i_2<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + \right. \\ &\quad \left. \sum_{i_2,i_3 \geq 0, i_1<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + \sum_{i_1=i_2=i_3=0} \beta_{0,0,0} x_1^0 x_2^0 x_3^0 \right) + P_0 \\ &= \left(\sum_{i_1,i_2 \geq 0, i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0 \right) + \left(\sum_{i_1,i_3 \geq 0, i_2<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0 \right) + \\ &\quad \left(\sum_{i_2,i_3 \geq 0, i_1<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0 \right) + (0 + P_0) \\ &= f_{12}^+ + f_{13}^+ + f_{23}^+ + 0 \\ &= f_{12}^+ + f_{13}^+ + f_{23}^+, \end{aligned}$$

where $f_{ij}^+ \in L_{ij}^+$ and $1 \leq i \neq j \leq 3$. Therefore $P_0^+/P_0 = L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+$. □

Next, we will consider the quotient representation $\widetilde{P_0}/P_0^+$. By the Third Isomorphism Theorem, we have $\widetilde{P_0}/P_0^+ \cong (\widetilde{P_0}/P_0)/(P_0^+/P_0)$. Elements in $\widetilde{P_0}/P_0^+$ are of the form $f_0 + P_0^+$, where $f_0 \in \widetilde{P_0}$. We set

$$L_i^+ := \mathrm{Span}_{\mathbb{C}}\{x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+ | a_j, a_k < 0\},$$

where $a_i + a_j + a_k = 0$ and $(i, j, k) = (1, 2, 3), (1, 3, 2)$, or $(2, 3, 1)$.

Proposition 6.1.3. *L_1^+, L_2^+, and L_3^+ are subrepresentations of $\widetilde{P_0}/P_0^+$. Moreover,*

$$\widetilde{P_0}/P_0^+ = L_1^+ \oplus L_2^+ \oplus L_3^+.$$

Proof. We will begin by showing that L_i^+ are representations. We will first verify the action of the non-diagonal elements of $\mathfrak{z}$ on elements of L_i^+. Suppose f is an element of L_i^+. Then $f = \sum_{a_j,a_k<0} \alpha_{a_i,a_j,a_k} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+$. We must show that

$e_{pq} \cdot f \in L_i^+$ for $1 \leq p \neq q \leq 3$.

$$\begin{aligned} e_{pq} \cdot f &= x_p \frac{\partial f}{\partial x_q} \\ &= x_p \frac{\partial}{\partial x_q} \left(\sum_{a_j, a_k < 0} \alpha_{a_i, a_j, a_k} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+ \right) \\ &= \sum_{a_j, a_k < 0} \alpha_{a_i, a_j, a_k} x_p \frac{\partial}{\partial x_q} \left(x_i^{a_i} x_j^{a_j} x_k^{a_k} \right) + P_0^+ \\ &= \sum_{a_j, a_k < 0} \alpha_{a_i, a_j, a_k} \left(x_p \frac{\partial}{\partial x_q} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+ \right). \end{aligned}$$

We now need to show that $x_p \frac{\partial}{\partial x_q} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+ \in L_i^+$.

Case 1: $p = j$, $q = k$.

Case 1.1: $a_j = -1$.

$$\begin{aligned} x_j \frac{\partial}{\partial x_k} x_i^{a_i} x_j^{-1} x_k^{a_k} + P_0^+ &= a_k x_i^{a_i} x_j^0 x_k^{a_k - 1} + P_0^+ \\ &= P_0^+ \in L_i^+ \end{aligned}$$

Case 1.2: $a_j < -1$.

$$x_j \frac{\partial}{\partial x_k} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+ = a_k x_i^{a_i} x_j^{a_j + 1} x_k^{a_k - 1} + P_0^+ \in L_i^+,$$

since $a_j + 1$ and $a_k - 1$ are negative.

Case 2: $p = i$, $q = k$.

$$x_i \frac{\partial}{\partial x_k} x_i^{a_i} x_j^{-1} x_k^{a_k} + P_0^+ = a_k x_i^{a_i + 1} x_j^{a_j} x_k^{a_k - 1} + P_0^+ \in L_i^+,$$

since a_j and $a_k - 1$ are negative.

Case 3: $p = k$, $q = j$.

This case is analogous to Case 1.

Case 4: $p = i$, $q = j$.

This case is analogous to Case 2.

Case 5: $p = j,\ q = i$.

Note that $a_i \geq 2$ since $a_j, a_k \leq -1$.

Case 5.1: $a_j = -1$.

$$\begin{aligned} x_j \frac{\partial}{\partial x_i} x_i^{a_i} x_j^{-1} x_k^{a_k} + P_0^+ &= a_i x_i^{a_i - 1} x_j^0 x_k^{a_k} + P_0^+ \\ &= P_0^+ \in L_i^+ \end{aligned}$$

Case 5.2: $a_j < -1$.

$$x_j \frac{\partial}{\partial x_i} x_i^{a_i} x_j^{a_j} x_k^{a_k} + P_0^+ = a_i x_i^{a_i - 1} x_j^{a_j + 1} x_k^{a_k} + P_0^+ \in L_i^+,$$

since $a_j + 1$ and a_k are negative.

Case 6: $p = k,\ q = i$.

This case is analogous to Case 5.

To account for the action of the $\mathfrak{sl}(3)$ diagonal elements h_i, where $1 \leq i \leq 2$, we easily check that $(e_{ii} - e_{i+1,i+1}) \cdot f \in L_i^+$ if $f \in L_i^+$.

Next, we will show that $\widetilde{P_0}/P_0^+ = L_1^+ \oplus L_2^+ \oplus L_3^+$. We will first check that $L_i^+ \cap (L_j^+ + L_k^+) = P_0^+$ for $(i,j,k) = (1,2,3), (2,3,1)$ or $(3,1,2)$. The proof is analogous to the one for L_{ij}^+ (see Lemma 6.1.2). It remains to be shown that $\widetilde{P_0}/P_0^+ = L_1^+ + L_2^+ + L_3^+$. Suppose f is an element in $\widetilde{P_0}/P_0^+$. Then $f = \sum_{i_1+i_2+i_3=0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+$. We can split up f into four sums each satisfying one of the following conditions: (i) $i_1, i_2 < 0$, (ii) $i_1, i_3 < 0$, (iii) $i_2, i_3 < 0$, and (iv) $i_j, i_k \geq 0, j \neq k$. These cover all possible cases, since we require that $i_1 + i_2 + i_3 = 0$, preventing all three i_k's from being negative simultaneously. Furthermore, we note that $\beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \in P_0^+$, when i_1, i_2, i_3 satisfy the fourth condition, i.e., when

$i_j, i_k \geq 0$. We may then rewrite f as follows:

$$
\begin{aligned}
f &= \sum_{i_1+i_2+i_3=0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \\
&= \left(\sum_{i_1,i_2<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + \sum_{i_1,i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \right. \\
&\quad \left. + \sum_{i_2,i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + \sum_{i_j,i_k\geq 0; j\neq k} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \right) + P_0^+ \\
&= \left(\sum_{i_1,i_2<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) + \left(\sum_{i_1,i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) \\
&\quad + \left(\sum_{i_2,i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) + \left(\sum_{i_j,i_k\geq 0; j\neq k} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) \\
&= \left(\sum_{i_1,i_2<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) + \left(\sum_{i_1,i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) \\
&\quad + \left(\sum_{i_2,i_3<0} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) + P_0^+ \\
&= f_3^+ + f_2^+ + f_1^+,
\end{aligned}
$$

where $f_i^+ \in L_i^+$ for $1 \leq i \leq 3$. Hence, $\widetilde{P_0}/P_0^+ = L_1^+ \oplus L_2^+ \oplus L_3^+$ □

Lemma 6.1.4. *If* $i_1 + i_2 + i_3 = 0$, *then*

$$
e_{31}^j e_{12}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) = \begin{cases} \dfrac{(i_1+l)!}{(i_1+l-j)!} \dfrac{i_2!}{(i_2-l)!} x_1^{i_1+l-j} x_2^{i_2-l} x_3^{i_3+j}, & \text{if } l \leq i_2, j \leq i_1 + l \\ 0, & \text{otherwise} \end{cases}
$$

for every $j, l \in \mathbb{Z}_+$.

Proof. We will consider the different cases $l > i_2$; $l \leq i_2, j > i_1 + l$; and $l \leq i_2, j \leq i_1 + l$. *Case 1:* $l > i_2$.

$$
\begin{aligned}
e_{31}^j e_{12}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) &= e_{31}^j \cdot \left(x_1^l \frac{\partial^l}{\partial x_2^l} (x_1^{i_1} x_2^{i_2} x_3^{i_3}) \right) \\
&= 0
\end{aligned}
$$

Case 2: $l \leq i_2, j > i_1 + l$.

$$\begin{aligned} e_{31}^j e_{12}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) &= e_{31}^j \cdot \left(x_1^l \frac{\partial^l}{\partial x_2^l} (x_1^{i_1} x_2^{i_2} x_3^{i_3}) \right) \\ &= x_3^j \frac{\partial^j}{\partial x_1^j} \left(\frac{i_2!}{(i_2 - l)!} x_1^{i_1+l} x_2^{i_2-l} x_3^{i_3} \right) \\ &= 0 \end{aligned}$$

Case 3: $l \leq i_2, j \leq i_1 + l$.

$$\begin{aligned} e_{31}^j e_{12}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) &= e_{31}^j \cdot \left(x_1^l \frac{\partial^l}{\partial x_2^l} (x_1^{i_1} x_2^{i_2} x_3^{i_3}) \right) \\ &= x_3^j \frac{\partial^j}{\partial x_1^j} \left(\frac{i_2!}{(i_2 - l)!} x_1^{i_1+l} x_2^{i_2-l} x_3^{i_3} \right) \\ &= \frac{(i_1 + l)!}{(i_1 + l - j)!} \frac{i_2!}{(i_2 - l)!} x_1^{i_1+l-j} x_2^{i_2-l} x_3^{i^3+j}. \end{aligned}$$

□

Lemma 6.1.5. *For all* (i_1, i_2, i_3) *with* $i_1, i_2 \geq 0$ *and* $i_3 < 0$ *we have that*

$$e_{31}^{-i_3-1} e_{12}^{i_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) = (i_1 + i_2)! i_2! x_1 x_3^{-1}.$$

Proof. Let $l = i_2$ and $j = -i_3 - 1 = i_1 + i_2 - 1$. Then $l \leq i_2$ and $j = i_1 + i_2 - 1 = i_1 + l - 1 \leq i_1 + l$, so by Lemma 6.1.4:

$$\begin{aligned} & e_{31}^{-i_3-1} e_{12}^{i_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) \\ = \; & e_{31}^j e_{12}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) \\ = \; & \frac{(i_1 + l)!}{(i_1 + l - j)!} \frac{i_2!}{(i_2 - l)!} x_1^{i_1+l-j} x_2^{i_2-l} x_3^{i^3+j} \\ = \; & \frac{(i_1 + i_2)!}{(i_1 + i_2 - (i_1 + i_2 - 1))!} \frac{i_2!}{(i_2 - i_2)!} x_1^{i_1+i_2-(i_1+i_2-1)} x_2^{i_2-i_2} x_3^{i^3+-i_3-1} \\ = \; & \frac{(i_1 + i_2)!}{1!} \frac{i_2!}{0!} x_1 x_3^{-1} \\ = \; & (i_1 + i_2)! i_2! x_1 x_3^{-1}. \end{aligned}$$

□

Lemma 6.1.6. *Suppose* $f = x_1^{i_1}x_2^{i_2}x_3^{i_3}$, $i_1+i_2+i_3=0$ *and* $i_2, i_3 < 0$ *(in particular* $i_1 > 1$*), and let* $l, j \in \mathbb{Z}_+$. *Then*

$$e_{21}^j e_{31}^l \cdot f = \begin{cases} 0, & \text{if } j+l > i_1 \\ \dfrac{i_1!}{(i_1-(l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l}, & \text{otherwise.} \end{cases}$$

Proof. We will consider two different subcases when proving case one: $l > -(i_2+i_3) = i_1$ and $j+l > -(i_2+i_3) = i_1$. Then we will proceed to proving case two for which $j+l \leq -(i_2+i_3) = i_1$.

Case 1.1: $l > -(i_2+i_3) = i_1$. First note that $j \in Z_+$ and since $l > -(i_2+i_3) = i_1$, we have that $j+l > -(i_2+i_3) = i_1$, making it a subcase of Case 1. We obtain the following:

$$\begin{aligned} e_{21}^j e_{31}^l \cdot f &= e_{21}^j e_{31}^l \cdot (x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\ &= e_{21}^j \cdot x_3^l \frac{\partial^l}{\partial x_1^l}(x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\ &= 0, \end{aligned}$$

since $l > i_1$.

Case 1.2: $j+l > -(i_2+i_3) = i_1$ *and* $l \leq -(i_2+i_3) = i_1$.

$$\begin{aligned} e_{21}^j e_{31}^l \cdot f &= e_{21}^j e_{31}^l \cdot (x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\ &= e_{21}^j \cdot x_3^l \frac{\partial^l}{\partial x_1^l}(x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\ &= e_{21}^j \cdot x_3^l i_1(i_1-1)\cdot \ldots \cdot (i_1-l+1)(x_1^{i_1-l}x_2^{i_2}x_3^{i_3}) \\ &= i_1(i_1-1)\cdot \ldots \cdot (i_1-l+1) x_2^j \frac{\partial^j}{\partial x_1^j}(x_1^{i_1-l}x_2^{i_2}x_3^{i_3+l}) \\ &= 0, \end{aligned}$$

since $j > i_1 - l$. Hence Case 1 of Lemma 6.1.6 holds.

Case 2: $j+l \leq -(i_2+i_3) = i_1$.

$$\begin{aligned}
& e_{21}^{j}e_{31}^{l}\cdot f \\
= \;& e_{21}^{j}e_{31}^{l}\cdot(x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\
= \;& e_{21}^{j}\cdot x_3^{l}\frac{\partial^l}{\partial x_1^l}(x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\
= \;& e_{21}^{j}\cdot x_3^{l}i_1(i_1-1)\cdot\ldots\cdot(i_1-l+1)(x_1^{i_1-l}x_2^{i_2}x_3^{i_3}) \\
= \;& i_1(i_1-1)\cdot\ldots\cdot(i_1-l+1)x_2^{j}\frac{\partial^j}{\partial x_1^j}(x_1^{i_1-l}x_2^{i_2}x_3^{i_3+l}) \\
= \;& i_1(i_1-1)\cdot\ldots\cdot(i_1-l+1)(i_1-l)\cdot\ldots\cdot(i_1-(l+j)+1)x_1^{i_1-(l+j)}x_2^{i_2+j}x_3^{i_3+l} \\
= \;& \frac{i_1!}{(i_1-(l+j))!}x_1^{i_1-(l+j)}x_2^{i_2+j}x_3^{i_3+l},
\end{aligned}$$

proving Case 2 of Lemma 6.1.6.

□

Lemma 6.1.7. *Suppose* $f = x_1^{i_1}x_2^{i_2}x_3^{i_3}$*,* $i_1+i_2+i_3=0$*, and* $i_2, i_3 < 0$*. Then we have*

$$\begin{aligned}
e_{21}^{-i_2-1}e_{31}^{-i_3-1}\cdot f &= e_{21}^{-i_2-1}e_{31}^{-i_3-1}\cdot(x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\
&= \frac{i_1!}{2}x_1^{2}x_2^{-1}x_3^{-1}.
\end{aligned}$$

Proof. Let $j = -i_2-1$ and $l = -i_3-1$. Then, by Lemma 6.1.6,

$$\begin{aligned}
e_{21}^{-i_2-1}e_{31}^{-i_3-1}\cdot(x_1^{i_1}x_2^{i_2}x_3^{i_3}) &= e_{21}^{j}e_{31}^{l}\cdot(x_1^{i_1}x_2^{i_2}x_3^{i_3}) \\
&= \frac{i_1!}{(i_1-(l+j))!}x_1^{i_1-(l+j)}x_2^{i_2+j}x_3^{i_3+l} \\
&= \frac{i_1!}{2!}x_1^{2}x_2^{-1}x_3^{-1},
\end{aligned}$$

since $i_1-(l+j)=2$, $i_2+j=-1$, $i_3+l=-1$, and $j+l=-(i_2+i_3)-2=i_1-2$. □

6.2 Irreducibility of L_{12}^{+}, L_{13}^{+}, and L_{23}^{+}.

Lemma 6.2.1. *Let* f *be a nonzero element of* L_{12}^{+}*, i.e.,*

$f = \sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3} + P_0$*, with*

$$J_f \subset \{(i_1,i_2,i_3)|i_1+i_2+i_3=0; i_3<0; i_1,i_2\geq 0\},$$

and choose

$$\begin{aligned} j_2 : &= \max\{i_2 | (i_1, i_2, i_3) \in J_f\}, \\ j_3 : &= \min\{i_3 | (i_1, j_2, i_3) \in J_f\}, \text{ and} \\ j_1 : &= -j_2 - j_3 \end{aligned}$$

Then $e_{31}^{-j_3-1} e_{12}^{j_2} \cdot f = c x_1 x_3^{-1} + P_0$, *where* $c := (j_1 + j_2)! j_2! \alpha_{j_1, j_2, j_3} \neq 0$.

Proof.

$$\begin{aligned} e_{31}^{-j_3-1} e_{12}^{j_2} \cdot f &= e_{31}^{-j_3-1} e_{12}^{j_2} \cdot \left(\sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0 \right) \\ &= \sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} e_{31}^{-j_3-1} e_{12}^{j_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) + P_0 \end{aligned}$$

We will find $e_{31}^{-j_3-1} e_{12}^{j_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3})$ for three different triples (i_1, i_2, i_3).

Case 1: $i_2 = j_2, i_3 = j_3$. *(In particular,* $i_1 = j_1$.*)*

$$\begin{aligned} e_{31}^{-j_3-1} e_{12}^{j_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) &= e_{31}^{-j_3-1} e_{12}^{j_2} \cdot (x_1^{j_1} x_2^{j_2} x_3^{j_3}) \\ &= (j_1 + j_2)! j_2! x_1 x_3^{-1}, \end{aligned}$$

by Lemma 6.1.5. We note that $(j_1 + j_2)! j_2! > 0$.

Case 2: $i_2 = j_2, j_3 < i_3$ *(or* $-j_3 - 1 \geq -i_3$*).*

Let $l := j_2 = i_2$, and $j := -j_3 - 1$. Then $l \leq i_2$, and

$j = j_1 + j_2 - 1 = -j_3 - 1 \geq -i_3 = i_1 + i_2 = i_1 + l$.

If $j > i_1 + l$, then

$$e_{31}^{j} e_{12}^{l} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) = 0,$$

by Lemma 6.1.4.

If $j = i_1 + l$, then

$$\begin{aligned} e_{31}^{j} e_{12}^{l} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) &= \frac{(i_1+l)!}{(i_1+l-j)!} \frac{i_2!}{(i_2-l)!} x_1^0 x_2^0 x_3^0 \\ &= (i_1+l)! j_2! \\ &= b_{i_1}, \end{aligned}$$

by Lemma 6.1.4, where $b_{i_1} = (i_1+l)!j_2! > 0$. (In particular $i_1 + l = i_1 + i_2 = -i_3 \geq 0$.)

Case 3: $i_2 < j_2$.

Here $e_{31}^{-j_3-1} e_{12}^{j_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) = 0$ by Lemma 6.1.4.

Summarizing the results of the three cases we have:

$$\begin{aligned} e_{31}^{-j_3-1} e_{12}^{j_2} \cdot f &= \sum_{(i_1,i_2,i_3) \in J_f} \alpha_{i_1,i_2,i_3} e_{31}^{-j_3-1} e_{12}^{j_2} \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) + P_0 \\ &= (j_1+j_2)! j_2! \alpha_{j_1,j_2,j_3} x_1 x_3^{-1} + \sum_{i_2=j_2, i_3 \neq j_3} \alpha_{i_1,i_2,i_3} b_{i_1} + P_0 \\ &= c x_1 x_3^{-1} + b + P_0 \\ &= c x_1 x_3^{-1} + P_0, \end{aligned}$$

where $c = (j_1+j_2)!j_2!\alpha_{j_1,j_2,j_3}$ and $b := \sum_{i_2=j_2, i_3 \neq j_3} \alpha_{i_1,i_2,i_3} b_{i_1}$. Note that $c \neq 0$, since $\alpha_{j_1,j_2,j_3} \neq 0$. □

Lemma 6.2.2. *Suppose that* $i_1 + i_2 + i_3 = 0$ *and* $i_1, i_2 \geq 0, i_3 < 0$, *then*

$$x_1^{i_1} x_2^{i_2} x_3^{i_3} = \frac{(-1)^{i_1+i_2-1} i_1!}{(i_1+i_2-1)!(i_1+i_2)!} e_{21}^{i_2} e_{13}^{i_1+i_2-1} \cdot (x_1 x_3^{-1}).$$

Proof.

$$
\begin{aligned}
&\frac{(-1)^{i_1+i_2-1}i_1!}{(i_1+i_2-1)!(i_1+i_2)!}e_{21}^{i_2}e_{13}^{i_1+i_2-1}\cdot(x_1x_3^{-1})\\
&=\frac{(-1)^{i_1+i_2-1}i_1!}{(i_1+i_2-1)!(i_1+i_2)!}e_{21}^{i_2}\cdot\left(x_1^{i_1+i_2-1}\frac{\partial^{i_1+i_2-1}}{\partial x_3^{i_1+i_2-1}}x_1x_3^{-1}\right)\\
&=\frac{(-1)^{i_1+i_2-1}i_1!}{(i_1+i_2-1)!(i_1+i_2)!}e_{21}^{i_2}\cdot((-1)^{i_1+i_2-1}(i_1+i_2-1)!x_1^{i_1+i_2}x_3^{-1-(i_1+i_2-1)})\\
&=\frac{(-1)^{2(i_1+i_2-1)}i_1!}{(i_1+i_2)!}e_{21}^{i_2}\cdot(x_1^{i_1+i_2}x_3^{-(i_1+i_2)})\\
&=\frac{(-1)^{2(i_1+i_2-1)}i_1!}{(i_1+i_2)!}x_2^{i_2}\frac{\partial^{i_2}}{\partial x_1^{i_2}}x_1^{i_1+i_2}x_3^{i_3}\\
&=\frac{i_1!}{(i_1+i_2)!}(i_1+i_2)\cdot\ldots\cdot(i_1+1)x_1^{i_1+i_2-i_2}x_2^{i_2}x_3^{i_3}\\
&=\frac{i_1!}{(i_1+i_2)!}\frac{(i_1+i_2)!}{i_1!}x_1^{i_1}x_2^{i_2}x_3^{i_3}\\
&=x_1^{i_1}x_2^{i_2}x_3^{i_3}
\end{aligned}
$$

□

Lemma 6.2.3. *Suppose that* $f=\sum_{(i_1,i_2,i_3)\in J_f}\alpha_{i_1,i_2,i_3}x_1^{i_1}x_2^{i_2}x_3^{i_3}+P_0$ *is an element in* L_{12}^{+}. *Then*

$$f=\left(\sum_{(i_1,i_2,i_3)\in J_f}\alpha_{i_1,i_2,i_3}b_{i_1,i_2,i_3}e_{21}^{i_2}e_{13}^{i_1+i_2-1}\right)\cdot(x_1x_3^{-1}+P_0),$$

where $b_{i_1,i_2,i_3}=\dfrac{(-1)^{i_1+i_2-1}i_1!}{(i_1+i_2-1)!(i_1+i_2)!}$.

Proof. Let $b_{i_1,i_2,i_3} := \dfrac{(-1)^{i_1+i_2-1} i_1!}{(i_1+i_2-1)!(i_1+i_2)!}$. Then we have that

$$\begin{aligned}
&\left(\sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} b_{i_1,i_2,i_3} e_{21}^{i_2} e_{13}^{i_1+i_2-1}\right) \cdot (x_1 x_3^{-1} + P_0) \\
&= \left(\sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} b_{i_1,i_2,i_3} e_{21}^{i_2} e_{13}^{i_1+i_2-1}\right) \cdot (x_1 x_3^{-1}) + P_0 \\
&= \sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} (b_{i_1,i_2,i_3} e_{21}^{i_2} e_{13}^{i_1+i_2-1}) \cdot (x_1 x_3^{-1}) + P_0 \\
&= \sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0 \\
&= f,
\end{aligned}$$

by Lemma 6.2.2. □

Proposition 6.2.4. *L_{12}^+, L_{13}^+, and L_{23}^+ are irreducible representations.*

Proof. Suppose $M \subset L_{12}^+$ is a nontrivial subrepresentation, and that $g \in M$ is a nonzero element. Then $g \in L_{12}^+$. Therefore, by Lemma 6.2.1: $e_{31}^{-j_3-1} e_{12}^{j_2} \cdot g = c x_1 x_3^{-1} + P_0$, for some $j_3 < 0, j_2 \geq 0$, and for some nonzero $c \in \mathbb{C}$. So $x_1 x_3^{-1} + P_0 \in M$. Now, suppose that $f \in L_{12}^+$. Then, by Lemma 6.2.3, $f = \left(\sum_{(i_1,i_2,i_3)\in J_f} \alpha_{i_1,i_2,i_3} b_{i_1,i_2,i_3} e_{21}^{i_2} e_{13}^{i_1+i_2-1}\right) \cdot (x_1 x_3^{-1} + P_0)$. Therefore, since $x_1 x_3^{-1} + P_0 \in M$, we have $f \in M$. We have shown that $L_{12}^+ \subset M$. Hence, $M = L_{12}^+$. A similar argument shows that L_{13}^+ and L_{23}^+ are irreducible. □

6.3 Irreducibility of L_1^+, L_2^+, and L_3^+.

Lemma 6.3.1. *Let* $f = \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+$ *be a nonzero element of* L_1^+, *where*

$$I_f \subset \{(i_1,i_2,i_3) \in \mathbb{Z}^3 | i_1 + i_2 + i_3 = 0 \text{ and } i_2, i_3 < 0\}.$$

Then if

$$\begin{aligned} j_1 :&= \max\{i_1 | (i_1, i_2, i_3) \in I_f\}, \\ j_3 :&= \min\{i_3 | (j_1, i_2, i_3) \in I_f\}, \\ j_2 :&= -j_1 - j_3 \\ l :&= -j_3 - 1, \ \textit{and} \\ j :&= j_1 + j_3 - 1 \end{aligned}$$

we have

$$\left(\frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^j e_{31}^l\right) \cdot f = x_1^2 x_2^{-1} x_3^{-1} + P_0^+$$

Proof.

$$\begin{aligned} \left(\frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^j e_{31}^l\right) \cdot f &= \frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^j e_{31}^l \cdot \left(\sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \right) \\ &= \frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^j e_{31}^l \cdot \left(\sum_{(i_1,i_2,i_3)\in I_f} (\beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+) \right) \\ &= \sum_{(i_1,i_2,i_3)\in I_f} \frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^j e_{31}^l \cdot (\beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+) \\ &= \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^j e_{31}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+) \\ &= \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} (e_{21}^j e_{31}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) + P_0^+) \end{aligned}$$

It suffices to focus on $e_{21}^j e_{31}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) + P_0^+$

Case 1: $i_1 < l + j = j_1 - 2$.

It follows from Lemma 6.1.6 that $e_{21}^j e_{31}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) + P_0^+ = 0$

Case 2: $i_1 \geq l + j = j_1 - 2$.

By Lemma 6.1.6, we have:

$$e_{21}^j e_{31}^l \cdot (x_1^{i_1} x_2^{i_2} x_3^{i_3}) + P_0^+ = \frac{i_1!}{(i_1 - (l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l} + P_0^+.$$

We will now study the exponents of the Laurent polynomial $x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l}$ for different values of i_3 and i_1. We first note that the x_1-exponent, $i_1 - (l+j)$, is non-negative, since $i_1 \geq l + j$.

Case 2.1: $i_3 > j_3$. We have the x_3-exponent, $i_3 + l \geq 0$, since $i_3 + l = i_3 + (-j_3 - 1) > j_3 + (-j_3 - 1) = -1$ and i_3, l are integers. Therefore $\frac{i_1!}{(i_1 - (l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l} \in P_0^+$ and $\frac{i_1!}{(i_1 - (l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l} + P_0^+ = 0 + P_0^+$.

Case 2.2: $i_3 = j_3$ *and* $i_1 < j_1$. We see that the x_3-exponent, $i_3 + l = -1$ since $i_3 + l = j_3 + l = j_3 + (-j_3 - 1) = -1$. However, the x_2-exponent, $i_2 + j \geq 0$, since $i_2 + j = -i_1 - i_3 + j = -i_1 - j_3 + (j_1 + j_3 - 1) = j_1 - i_1 - 1 \geq 0$, since $j_1 - i_1 \geq 1$. Therefore $\frac{i_1!}{(i_1 - (l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l} \in P_0^+$ and $\frac{i_1!}{(i_1 - (l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l} + P_0^+ = 0 + P_0^+ = 0$.

Case 2.3: $i_3 = j_3$ *and* $i_1 = j_1$. In particular, $i_2 = -i_1 - i_3 = -j_1 - j_3 = j_2$. Then the x_3-, and x_2-exponents are both equal to -1, or equivalently $i_3 + l = -1$ and $i_2 + j = -1$, since $i_3 + l = j_3 + l = j_3 + (-j_3 - 1) = -1$, and $i_2 + j = -i_1 - i_3 + j = -j_1 - j_3 + (j_1 + j_3 - 1) = -1$. In addition, the x_1-exponent equals 2, or equivalently $i_1 - (l + j) = 2$, since $i_3 + l = -1$ and $i_2 + j = -1$ imply that $i_1 - (l + j) = -i_2 - i_3 - j - l = 1 + 1$. Therefore $\frac{i_1!}{(i_1 - (l+j))!} x_1^{i_1-(l+j)} x_2^{i_2+j} x_3^{i_3+l} + P_0^+ = \frac{j_1!}{2!} x_1^2 x_2^{-1} x_3^{-1} + P_0^+$.

Summarizing all cases, we verify:

$$e_{21}^{j}e_{31}^{l}\cdot(x_1^{i_1}x_2^{i_2}x_3^{i_3}+P_0^+)=\begin{cases}\frac{j_1!}{2}x_1^2x_2^{-1}x_3^{-1}+P_0^+, & \text{if } i_1=j_1, i_3=j_3\\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$\begin{aligned}\frac{1}{\beta_{j_1,j_2,j_3}}\frac{2}{j_1!}e_{21}^{j}e_{31}^{l}\cdot f &= \sum_{(i_1,i_2,i_3)\in I_f}\beta_{i_1,i_2,i_3}\left(\frac{1}{\beta_{j_1,j_2,j_3}}\frac{2}{j_1!}e_{21}^{j}e_{31}^{l}x_1^{i_1}x_2^{i_2}x_3^{i_3}\right)+P_0^+\\ &= \beta_{j_1,j_2,j_3}\frac{1}{\beta_{j_1,j_2,j_3}}\frac{2}{j_1!}\frac{j_1!}{2}x_1^2x_2^{-1}x_3^{-1}+P_0^+\\ &= x_1^2x_2^{-1}x_3^{-1}+P_0^+.\end{aligned}$$

□

Lemma 6.3.2. *Let* $f=x_1^{i_1}x_2^{i_2}x_3^{i_3}+P_0^+$ *be an element of* L_1^+*. Then*

$$f=\frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!}e_{12}^{-i_2-1}e_{13}^{-i_3-1}\cdot(x_1^2x_2^{-1}x_3^{-1}+P_0^+)$$

Proof.

$$
\begin{aligned}
&\frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1} \cdot (x_1^2 x_2^{-1} x_3^{-1} + P_0^+) \\
&= \frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1} \cdot (x_1^2 x_2^{-1} x_3^{-1}) + P_0^+ \\
&= \frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} \cdot x_1^{-i_3-1} \frac{\partial^{-i_3-1}}{\partial x_3^{-i_3-1}} x_1^2 x_2^{-1} x_3^{-1} + P_0^+ \\
&= \frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} \cdot (-1)(-2) \cdot \ldots \cdot (i_3+1) x_1^{-i_3+1} x_2^{-1} x_3^{-1-(-i_3-1)} + P_0^+ \\
&= \frac{(-1)^{i_1}(-1)(-2) \cdot \ldots \cdot (i_3+1)}{(-i_2-1)!(-i_3-1)!} x_1^{-i_2-1} \frac{\partial^{-i_2-1}}{\partial x_2^{-i_2-1}} x_1^{-i_3+1} x_2^{-1} x_3^{i_3} + P_0^+ \\
&= \frac{(-1)^{i_1}(-1)^{i_3+1}(-i_3-1)!(-1)(-2) \cdot \ldots \cdot (i_2+1)}{(-i_2-1)!(-i_3-1)!} x_1^{-i_2-i_3} x_2^{-1-(-i_2-1)} x_3^{i_3} + P_0^+ \\
&= \frac{(-1)^{i_1}(-1)^{i_3+1}(-1)^{i_2+1}(-i_2-1)!}{(-i_2-1)!} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \\
&= (-1)^{i_1+i_3+i_2+2} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \\
&= (-1)^2 x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+ \\
&= x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+
\end{aligned}
$$

□

Lemma 6.3.3. *Suppose* $f = \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+$ *is an element of* L_1^+. *Then*

$$
f = \left(\sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1} \right) \cdot (x_1^2 x_2^{-1} x_3^{-1} + P_0^+)
$$

Proof.

$$\left(\sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1}\right) \cdot (x_1^2 x_2^{-1} x_3^{-1} + P_0^+)$$
$$= \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \left(\frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1}\right) \cdot (x_1^2 x_2^{-1} x_3^{-1} + P_0^+)$$
$$= \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \left(\frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1} \cdot (x_1^2 x_2^{-1} x_3^{-1} + P_0^+)\right)$$
$$= \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} (x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+) \text{ by Lemma 6.3.2}$$
$$= \sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} + P_0^+$$
$$= f$$

□

Proposition 6.3.4. *L_1^+, L_2^+, and L_3^+ are irreducible representations.*

Proof. Suppose M is a nontrivial subrepresentation of L_1^+, and that $g \in M$ is a nonzero element. Then $g \in L_1^+$. Therefore, by Lemma 6.3.1:
$\frac{1}{\beta_{j_1,j_2,j_3}} \frac{2}{j_1!} e_{21}^{j} e_{31}^{l} \cdot g = x_1^2 x_2^{-1} x_3^{-1} + P_0^+$, for some $j, l, j_1 > 0$, and $\beta_{j_1,j_2,j_3} \neq 0$, so $x_1^2 x_2^{-1} x_3^{-1} + P_0^+ \in M$. Now, suppose that $f \in L_1^+$. Then, by Lemma 6.3.3,
$f = \left(\sum_{(i_1,i_2,i_3)\in I_f} \beta_{i_1,i_2,i_3} \frac{(-1)^{i_1}}{(-i_2-1)!(-i_3-1)!} e_{12}^{-i_2-1} e_{13}^{-i_3-1}\right) \cdot (x_1^2 x_2^{-1} x_3^{-1} + P_0^+)$.
Therefore, since $x_1^2 x_2^{-1} x_3^{-1} + P_0^+ \in M$, we also have $f \in M$. We have shown that $L_1^+ \subset M$. Hence, $M = L_1^+$. We can show, by a similar argument, that L_2^+, and L_3^+ are irreducible. □

6.4 Indecomposability of $\widetilde{P_0}$

Proposition 6.4.1. *The vector space $\widetilde{P_0}$ is an infinite dimensional indecomposable $\mathfrak{sl}(3)$-representation.*

Proof. From Proposition 4.1.1 and Section 4.1 we have that $\widetilde{P_0}$ is an $\mathfrak{sl}(3)$-representation and $\dim \widetilde{P_0} = \infty$. Next we give a rather schematic proof that $\widetilde{P_0}$ is indecomposable.

Suppose $\widetilde{P_0} = M \oplus N$ for some $\mathfrak{sl}(3)$-representations M and N, and assume that $M, N \neq 0$. Elements in M and N are of the form $\sum_{i_1+i_2+i_3=0} \alpha_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$. Suppose f is a nonzero element of N. Then $f = \sum_{i_1+i_2+i_3=0} \alpha_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$ where $\alpha_{i_1,i_2,i_3} \neq 0$ for all (i_1, i_2, i_3). We will show that applying the action of e_{23}, e_{13}, e_{21}, and finally e_{32} on our element f, we obtain a nonzero constant, also an element of N since N is an $\mathfrak{sl}(3)$-representation.

Step 1.

Let $k = \max(\{i_3 | \alpha_{i_1,i_2,i_3} \neq 0\}, 0)$. If $k > 0$, the action of e_{23}^k on f is as follows:

$$\begin{aligned} e_{23}^k \cdot f &= x_2^k \frac{\partial^k}{\partial x_3^k}(f) \\ &= x_2^k \frac{\partial^k}{\partial x_3^k} \sum_{i_1+i_2+i_3=0} \alpha_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3} \\ &= \sum_{i_1+i_2+i_3=0} \alpha_{i_1,i_2,i_3} x_2^k \frac{\partial^k}{\partial x_3^k} x_1^{i_1} x_2^{i_2} x_3^{i_3}, \end{aligned}$$

and if $k = 0$, $e_{23}^k \cdot f = e_{23}^0 \cdot f = f$, i.e., e_{23}^k does nothing. The action of e_{23}^k on $x_1^{i_1} x_2^{i_2} x_3^{i_3}$ when $k > 0$ can be summarized as follows:

$$e_{23}^k \cdot x_1^{i_1} x_2^{i_2} x_3^{i_3} = \begin{cases} 0 & \text{if } k > i_3 \geq 0, \\ k! x_1^{i_1} x_2^{i_2+k} & \text{if } k = i_3 \geq 0, \\ \dfrac{(-i_3+k-1)!}{(-i_3-1)!}(-1)^k x_1^{i_1} x_2^{i_2+k} x_3^{i_3-k} & \text{if } k > 0 > i_3. \end{cases}$$

After this step all x_3-exponents have become negative or zero, i.e.,

$$\begin{aligned} e_{23}^k \cdot f &= \sum_{j_1+j_2+j_3=0, j_3 \leq 0} \beta_{j_1,j_2,j_3} x_1^{j_1} x_2^{j_2} x_3^{j_3} \\ &= g, \end{aligned}$$

where the nonzero constants β_{j_1,j_2,j_3} are determined by: $\beta_{j_1,j_2,j_3} = \alpha_{j_1,j_2,j_3}$ in the case $k = 0$, and $\beta_{j_1,j_2,j_3} = c_{j_1,j_2-k,j_3+k}\alpha_{j_1,j_2-k,j_3+k}$ where c_{j_1,j_2-k,j_3+k} are constants, if $k > 0$. g is an element of N for which all x_3-exponents are negative or zero.

Step 2.

Let $l = -\min(\{j_2|\beta_{j_1,j_2,j_3} \neq 0\}, 0)$. If $l > 0$, the action of e_{23}^l on g is as follows:

$$\begin{aligned}
e_{23}^l \cdot g &= x_2^l \frac{\partial^l}{\partial x_3^l}(g) \\
&= x_2^l \frac{\partial^l}{\partial x_3^l} \sum_{j_1+j_2+j_3=0, j_3\leq 0} \beta_{j_1,j_2,j_3} x_1^{j_1} x_2^{j_2} x_3^{j_3} \\
&= \sum_{j_1+j_2+j_3=0, j_3\leq 0} \beta_{j_1,j_2,j_3} x_2^l \frac{\partial^l}{\partial x_3^l} x_1^{j_1} x_2^{j_2} x_3^{j_3} \\
&= \sum_{j_1+j_2+j_3=0, j_3\leq 0} \beta_{j_1,j_2,j_3} \frac{(-j_3+l-1)!}{(-j_3-1)!}(-1)^l x_1^{j_1} x_2^{j_2+l} x_3^{j_3-l},
\end{aligned}$$

and if $l = 0$, $e_{23}^l \cdot l = e_{23}^0 \cdot g = g$, i.e., e_{23}^l does nothing. After this step, all x_2-coefficients have become non-negative:

$$\begin{aligned}
e_{23}^l \cdot g &= \sum_{k_1+k_2+k_3=0, k_2\geq 0, k_3\leq 0} \phi_{k_1,k_2,k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3} \\
&= w,
\end{aligned}$$

where the nonzero constants ϕ_{k_1,k_2,k_3} are determined by: $\phi_{k_1,k_2,k_3} = \beta_{k_1,k_2,k_3}$ in the case $l = 0$, or $\phi_{k_1,k_2,k_3} = c_{k_1,k_2-l,k_3+l}\beta_{k_1,k_2-l,k_3+l}$ where c_{k_1,k_2-l,k_3+l} are constants, if $l > 0$. w is an element of N whose x_2-exponents are non-negative, and whose x_3-exponents are negative or zero.

Step 3.

Let $m = -\min(\{k_1 | \phi_{k_1,k_2,k_3} \neq 0\}, 0)$. If $m > 0$, the action of e_{13}^m on w is as follows:

$$\begin{aligned}
e_{13}^m \cdot w &= x_1^m \frac{\partial^m}{\partial x_3^m}(w) \\
&= x_1^m \frac{\partial^m}{\partial x_3^m} \sum_{k_1+k_2+k_3=0, k_1\geq 0, k_3 \leq 0} \phi_{k_1,k_2,k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3} \\
&= \sum_{k_1+k_2+k_3=0, k_1\geq 0, k_3 \leq 0} \phi_{k_1,k_2,k_3} x_1^m \frac{\partial^m}{\partial x_3^m} x_1^{k_1} x_2^{k_2} x_3^{k_3} \\
&= \sum_{k_1+k_2+k_3=0, k_1\geq 0, k_3 \leq 0} \phi_{k_1,k_2,k_3} \frac{(-k_3+m-1)!}{(-k_3-1)!}(-1)^m x_1^{k_1+m} x_2^{k_2} x_3^{k_3-m},
\end{aligned}$$

and if $m = 0$, $e_{13}^m \cdot w = e_{13}^0 \cdot w = w$, i.e., e_{13}^m does nothing. After this step, all x_1-coefficients have become non-negative:

$$\begin{aligned}
e_{13}^m \cdot w &= \sum_{l_1+l_2+l_3=0, l_1\geq 0, l_2 \geq 0, l_3 \leq 0} \theta_{l_1,l_2,l_3} x_1^{l_1} x_2^{l_2} x_3^{l_3} \\
&= u,
\end{aligned}$$

where the nonzero constants θ_{l_1,l_2,l_3} are determined by: $\theta_{l_1,l_2,l_3} = \phi_{l_1,l_2,l_3}$ in the case $m = 0$, or $\theta_{l_1,l_2,l_3} = s_{l_1-m,l_2,l_3+m}\phi_{l_1-m,l_2,l_3+m}$ if $m > 0$. u is an element of N whose x_1- and x_2-exponents are non-negative, and whose x_3-exponents are negative or zero.

Step 4.

Let $n = \max(\{l_1 | \theta_{l_1,l_2,l_3} \neq 0\}, 0)$. If $n > 0$, the action of e_{21}^n on u is as follows:

$$\begin{aligned}
e_{21}^n \cdot u &= x_2^n \frac{\partial^n}{\partial x_1^n}(u) \\
&= x_2^n \frac{\partial^n}{\partial x_1^n} \sum_{l_1+l_2+l_3=0, l_1\geq 0, l_2 \geq 0, l_3 \leq 0} \theta_{l_1,l_2,l_3} x_1^{l_1} x_2^{l_2} x_3^{l_3} \\
&= \sum_{l_1+l_2+l_3=0, l_1\geq 0, l_2 \geq 0, l_3 \leq 0} \theta_{l_1,l_2,l_3} x_2^n \frac{\partial^n}{\partial x_1^n} x_1^{l_1} x_2^{l_2} x_3^{l_3} \\
&= \sum_{l_1+l_2+l_3=0, l_1 = n, l_2 \geq 0, l_3 \leq 0} \theta_{n,-n-l_3,l_3} n! x_2^{l_2+n} x_3^{l_3} \\
&= \sum_{l_1+l_2+l_3=0, l_1 = n, l_2 \geq 0, l_3 \leq 0} \theta_{n,-n-l_3,l_3} n! x_2^{-l_3} x_3^{l_3} \\
&= \sum_{l_1 = n, l_3 \leq 0} \theta_{n,-n-l_3,l_3} n! x_2^{-l_3} x_3^{l_3} \\
&= v,
\end{aligned}$$

where v is a nonzero element of N.

Step 5.

Let $p = -\min(\{l_3 | \theta_{n,-n-l_3,l_3} \neq 0\}, 0)$. If $p > 0$, the action of e_{32}^p on v is as follows:

$$\begin{aligned} e_{32}^p \cdot v &= x_3^p \frac{\partial^p}{\partial x_2^p}(v) \\ &= x_3^p \frac{\partial^p}{\partial x_2^p} \sum_{l_1=n, l_3 \leq 0} \theta_{n,-n-l_3,l_3} n! x_2^{-l_3} x_3^{l_3} \\ &= \theta_{n,-n-l_3,l_3} n! x_3^p \frac{\partial^p}{\partial x_2^p} x_2^{-l_3} x_3^{l_3} \\ &= \theta_{n,-n+p,-p} n! p! x_2^{p-p} x_3^{-p+p} \\ &= \theta_{n,-n+p,-p} n! p! x_2^{p-p} \\ &= k, \end{aligned}$$

where k is a constant and k is an element of N. Since k is in N we have $\frac{1}{k}k = 1 \in N$. If $p = 0$, we have $e_{32}^p \cdot v = e_{32}^0 \cdot v = v = \theta_{n,-n,0} n! x_2^0 x_3^0 = k$, where k is a nonzero constant and k is an element of N and since k is in N again we have $\frac{1}{k}k = 1 \in N$.

We have shown that 1 is an element of N. If we pick an arbitrary nonzero element of M and follow steps 1 through 5, we can show that 1 also is an element of M. But $\widetilde{P_0} = M \oplus N$ implying that $M \cap N = \{0\}$. This is a contradiction. We have shown that $\widetilde{P_0}$ is indecomposable. □

6.5 Main Theorem

Main Theorem. *The Laurent polynomial representation $\widetilde{P_0}$ is an infinite dimensional indecomposable representation of $\mathfrak{sl}(3)$ for which*

(i) P_0 is an irreducible subrepresentation of $\widetilde{P_0}$.

(ii) The quotient representation $\bar{P_0} := \widetilde{P_0}/P_0$ contains the direct sum, $L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+$ of three irreducible representations L_{ij}^+.

(iii) The quotient representation $\bar{P_0}/(L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+)$ equals the direct sum

$L_1^+ \oplus L_2^+ \oplus L_3^+$ of the three irreducible representations L_i^+.

In particular, $\widetilde{P_0}$ has length 7, i.e., it has 7 irreducible subquotients.

Proof. $\widetilde{P_0}$ is an $\mathfrak{sl}(3)$-representation by Proposition 4.1.1 and Section 4.1, and Proposition 6.4.1 shows that it is infinite dimensional and irreducible. Furthermore, $\bar{P}_0 = \widetilde{P_0}/P_0 \supset P_0^+/P_0 = L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+$ by Lemma 6.1.2 and since $P_0^+ \subset \widetilde{P_0}$. In addition, Proposition 6.2.4 shows that L_{12}^+, L_{13}^+ and L_{23}^+ are irreducible representations. Since (i) $\bar{P}_0/(P_0^+/P_0) = (\widetilde{P_0}/P_0)/(P_0^+/P_0) \cong \widetilde{P_0}/P_0^+$ by the Third Isomorphism Theorem, (ii) $P_0^+/P_0 = L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+$ by Lemma 6.1.2 and (iii) $\widetilde{P_0}/P_0^+ = L_1^+ \oplus L_2^+ \oplus L_3^+$ by Proposition 6.1.3, it follows that $\bar{P}_0/(L_{12}^+ \oplus L_{13}^+ \oplus L_{23}^+)$ equals the direct sum, $L_1^+ \oplus L_2^+ \oplus L_3^+$. Finally, $\widetilde{P_0}$ has length 7, since it has 7 irreducible subquotients: $P_0, L_{12}^+, L_{13}^+, L_{23}^+ L_1^+, L_2^+$, and L_3^+. □

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