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FAMILY OF CIRCULANT GRAPHS AND ITS EXPANDER PROPERTIES

A Thesis

Presented to

The Faculty of the Department of Mathematics

San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Vinh Kha Nguyen

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The Designated Thesis Committee Approves the Thesis Titled

FAMILY OF CIRCULANT GRAPHS AND ITS EXPANDER PROPERTIES

by

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APPROVED FOR THE DEPARTMENT OF MATHEMATICS

SAN JOSÉ STATE UNIVERSITY

May 2010

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ABSTRACT

FAMILY OF CIRCULANT GRAPHS AND ITS EXPANDER PROPERTIES

by Vinh Kha Nguyen

In this thesis, we apply spectral graph theory to show the non-existence of an expander family within the class of circulant graphs. Using the adjacency matrix and its properties, we prove Cheeger's inequalities and determine when the equalities hold. In order to apply Cheeger's inequalities, we compute the spectrum of a general circulant graph and approximate its second largest eigenvalue. Finally, we show that circulant graphs do not contain an expander family.

DEDICATION

To my family for the support and love they gave me.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. So, for taking his time to guide me through my thesis. I also give my thanks to Prof. Ka-Hin Leung for suggesting the use of Dirichlet Theorem in Chapter 5. Last but not least, I would like to thank Dr. Day and Dr. Schmeichel for helping me to find mistakes in my thesis and pushing me to further enhance it.

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CHAPTER 1

INTRODUCTION

In this chapter, we introduce the notion of an expander family. The chapter is divided into two parts. The first part focuses on fundamental definitions of graph theory. The second part emphasizes the adjacency matrix and its properties.

1.1 Basic Definitions and Examples

1.1.1 Basic Definitions

We begin by recalling some basic definitions. Readers should refer to [Wes01] for more information.

Definition 1.1.1 (graph). A **graph** G is an abstract set consisting of a finite **vertex set** $V(G)$ and an **edge set** $E(G)$.

In other words, a graph consists of objects and links between them. An object is called a **vertex**, and a link is called an **edge**. We draw a graph by placing an edge $e = uv$ between two vertices u and v. Two vertices are called **endpoints** of an edge. When vertex u has an edge to vertex v, we say u is **adjacent** to v, or u is a **neighbor** of v.

Definition 1.1.2 (loop). A **loop** is an edge that connects a vertex to itself.

Definition 1.1.3 (**multiple edges**). **Multiple edges** are edges having the same pair of endpoints.

Definition 1.1.4 (**simple graph**). A **simple graph** is a graph having no loops or multiple edges.

Remark 1.1.5. In this thesis, we assume that all graphs are simple and have a finite number of vertices.

Definition 1.1.6 (**path**). A (v_1, v_k)-path is a finite sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that v_i is adjacent to v_{i+1}.

The definition of a (u, v)-path is essential to determine the connectedness of a graph G, which plays a very important role in this thesis. G is **connected** if it has a (u, v)-path whenever $u, v \in V(G)$. Otherwise, G is **disconnected**.

Definition 1.1.7 (**degree**). The **degree** of a vertex v, denoted as $d(v)$, is the number of its neighbors or adjacent vertices.

A **d-regular** graph is a graph in which every vertex has degree d. Two simple examples are the **cycle** C_n and the **complete graph** K_n. The cycle C_n is 2-regular because C_n is a graph in which every vertex has degree two. On the other hand, the complete graph K_n is $(n-1)$-regular because K_n is a graph in which every vertex has degree $n-1$.

In every graph G, we can count the number of edges by summing the degrees of all vertices. The resulting formula is a useful tool of graph theory.

Theorem 1.1.8 (**Degree-Sum Formula**).

$$\sum_{v \in V(G)} d(v) = 2|E(G)|$$

Proof. Summing the degrees counts each edge twice because each edge has two vertices as endpoints.

□

Theorem (1.1.8) is sometimes called the "First Theorem of Graph Theory" or the "Handshaking Lemma". It is an easy but far-reaching theorem, and it will be used many times in this thesis.

Remark 1.1.9. Theorem (1.1.8) implies that the number of vertices of a d-regular graph G is always even, when d is an odd integer.

Definition 1.1.10 (subgraph). H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.1.11 (components). The **components** of a graph G are its maximal connected subgraphs.

An **induced subgraph** is a subgraph obtained by deleting a set of vertices and the edges adjacent to them. In particular, when S is a subset of $V(G)$, the induced subgraph $G[S]$ consists of S and all edges whose endpoints are contained in S. The full graph G is itself an induced subgraph, as are the individual vertices. A subgraph, however, may not be an induced subgraph. For example, a path with four vertices is a subgraph of C_4, but it is not an induced subgraph of C_4.

The order of a graph is $|V(G)|$, that is, the number of vertices of the graph G. For every $S \subseteq V(G)$, let $\overline{S} = V(G) - S$ and define ∂S to be the set of edges of G connecting S to $\overline{S}$. We can now define the expander parameter of a graph.

Definition 1.1.12 (expander parameter). The **expander parameter** of a graph G of order n ($n \geq 2$) is defined as follows

$$h(G) = \min_{S:1 \leq |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}.$$

Remark 1.1.13. $h(G) = 0$ if and only if G is disconnected.

When the value of $h(G)$ is positive, G is called an **expander graph**. An expander graph can be viewed as a graph in which every subset S of the vertex set $V(G)$ expands quickly. In other words, S has many edges connected to $\overline{S}$. We are not interested in the expander parameter of a single graph G but an entire family of graphs $\{G_i\}$.

Definition 1.1.14 (**expander family**)**.** A family of graphs $\{G_i\}$ of increasing order is an **expander family** if there exist an integer d and some constant $\epsilon > 0$ such that:

- G_i is d-regular for all i, i.e., $\{G_i\}$ is uniformly regular.
- $h(G_i) > \epsilon$ for all i.

The concept of an expander family was first introduced by Bassalygo and Pinsker in 1973 while they did research on communication networks. These Russian mathematicians proved the existence of such families using probabilistic arguments. They showed that almost every random d-regular graph is an expander, although they did not know how to construct an expander family explicitly [BP73].

The original motivation for finding expander families was to build economical robust networks for telephone and computer communication. Over the past three decades, expander families have been developed into a powerful tool with wide applications in many areas such as fast distributed routing algorithms [PU89], LDPC codes [UW87], and storage schemes [SS96], to name a few. In telecommunication, expander families can be used to construct efficient error-correcting codes with non-zero rates of transmission, which provide great protection against noise [HLW06]. In cryptology, optimal expander families, for example the Ramanujan graphs, are used to construct collision resistant hash functions. These cryptographic hash functions

have been implemented in many information security applications, notably in digital signatures and password verifications [CGL08]. Infinite expander families are relevant to evolving technology, although constructing an applicable one is not an easy task.

1.1.2 Examples

We present two examples of $\{K_n\}$ and $\{C_n\}$ to demonstrate ways that a family of graphs can fail to be an expander family.

Example 1.1.15. $\{K_n\}$ is not an expander family.

Proof. Let $S \subseteq V(K_n)$ where $1 \leq |S| \leq \frac{n}{2}$. Since every vertex of K_n has degree $n-1$, a vertex in S is connected to all the vertices in $\overline{S}$. This implies $|\partial S| = |S| \cdot |\overline{S}|$, so

$$h(K_n) = \min_{S:|S|\leq\frac{n}{2}} \frac{|\partial S|}{|S|} = \min_{S:|S|\leq\frac{n}{2}} \frac{|S| \cdot |\overline{S}|}{|S|} = \min_{S:|S|\leq\frac{n}{2}} |\overline{S}|.$$

$\left|\overline{S}\right|$ is smallest when $|S|$ is biggest, and thus taking $|S| = \frac{n}{2}$ yields

- $h(K_n) = \frac{n}{2}$ when n is even.
- $h(K_n) = \left\lceil \frac{n}{2} \right\rceil$, the smallest integer not less than $\frac{n}{2}$, when n is odd.

Hence $h(K_n) = \left\lceil \frac{n}{2} \right\rceil > 0$. Unfortunately, $\{K_n\}$ is not an expander family because this family does not have uniform regularity, that is, there is no finite d such that every K_n is d-regular.

□

Example 1.1.16. $\{C_n\}$ is not an expander family.

Proof. Let $S \subseteq V(C_n)$ where $1 \leq |S| \leq \frac{n}{2}$. Since every vertex of C_n has degree 2, a simple observation shows:

(1) $|\partial S| = 2$ if $C_n[S]$ is connected.

(2) $|\partial S| > 2$ if $C_n[S]$ is not connected.

(1) and (2) yield

$$h(C_n) = \min_{S:|S|\leq\frac{n}{2}} \frac{|\partial S|}{|S|} = \min_{S:|S|\leq\frac{n}{2}} \frac{2}{|S|}.$$

As $|S|$ gets bigger, $\dfrac{2}{|S|}$ becomes smaller. This means

- $h(C_n) = \frac{4}{n}$ if n is even.
- $h(C_n) = \frac{4}{n-1}$ if n is odd.

Thus $\lim_{n\to\infty} h(C_n) = 0$, hence $\{C_n\}$ is not an expander family even though it has uniform regularity.

□

1.1.3 The Combinatorial Problem

We assume henceforth that every graph G of order n is d-regular. As n increases, it is extremely difficult to find the expander parameter $h(G)$ because there are overwhelmingly many $S \subseteq V(G)$ for which $1 \leq |S| \leq \dfrac{n}{2}$ to consider. For example, let G be a 5-regular graph on 20 vertices. Then there are

$$\binom{20}{1} + \binom{20}{2} + \cdots + \binom{20}{10} \approx \left(\frac{1}{2}\right) 2^{20} = 2^{19}$$

such subsets S, and to calculate $h(G)$ we would need to minimize $\dfrac{|\partial S|}{|S|}$ over all such subsets S. Finding the expander parameter this way is computationally tedious, if even possible. The complication arises exponentially as $|V(G)|$ increases. Hence we will examine the adjacency matrix of G for a faster way to compute $h(G)$.

1.2 Adjacency Matrix and Its Properties

There exist special matrices that fully represent a graph G. One such matrix is the adjacency matrix, which is very useful because its eigenvalues yield many

properties of G such as connectivity and regularity.

Definition 1.2.1 (**adjacency matrix**). An **adjacency matrix** $A(G)$ of a graph G is

$$A(G) = [a_{ij}] \text{ where } a_{ij} = \begin{cases} 0 & \text{if vertices } i \text{ and } j \text{ are not adjacent.} \\ 1 & \text{else.} \end{cases}$$

When $i \neq j, a_{ij} = a_{ji}$ because vertices i and j either form an edge or not. As a result, the matrix $A(G)$ is **symmetric**, i.e., $A(G) = A(G)^T$ where $A(G)^T$ is the **transpose** of $A(G)$ in which column i of $A(G)^T$ is row i of $A(G)$. The main diagonal entries a_{ii} of $A(G)$ are always zero because we assumed earlier that G is a simple graph.

Remark 1.2.2. G is a d-regular graph if and only if every row sum of $A(G)$ is d.

Let $A = A(G)$, then A has an **eigenvalue** λ and a nonzero **eigenvector** x if $Ax = \lambda x$. The eigenvalues of A can be obtained by solving the equation $det(\lambda I - A) = 0$. Explicitly, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are roots of the **characteristic polynomial**

$$p(A, \lambda) = det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - \lambda_i).$$

The set of distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$ is called the **spectrum** of A, written as $Sp(A)$. For more information, readers may refer to [HJ85]. We now state some important properties of the adjacency matrix that will be used to prove several results in the forthcoming chapters.

Definition 1.2.3 (**orthonormal eigenvectors**). The vectors $x_1, x_2, \ldots, x_n$ are **orthonormal** if

- $x_i^T x_j = 0$ for all pairs $1 \leq i < j \leq n$.
- $x_i^T x_i = 1$ for all $i = 1, 2, \ldots, n$.

Theorem 1.2.4 (**Spectral Theorem**). *An $n \times n$ real symmetric matrix has n real eigenvalues counting multiplicities and n orthonormal eigenvectors.*

Proof. See [Wes01] pp.456.

□

By Theorem (1.2.4), we can order the eigenvalues of $A(G)$ as follows:

$$\lambda_{min} = \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1 = \lambda_{max}. \tag{1.1}$$

Let $M_n \in \mathbb{C}^{n\times n}$ be the set of $n \times n$ matrices with complex entries and let $*$ denote the conjugate transpose operation. A matrix $A \in M_n$ is called **Hermitian** if $A^* = A$. It is true that all eigenvalues of a Hermitian matrix are real. A symmetric real matrix, such as an adjacency matrix, is Hermitian. The following theorem, which was discovered by two British physicists, characterizes the eigenvalues of an adjacency matrix $A(G)$.

Theorem 1.2.5 (**Rayleigh-Ritz**). *Let $A \in M_n \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, and let the eigenvalues of A be ordered as in (1.1). Then*

$$\lambda_n x^* x \leq x^* A x \leq \lambda_1 x^* x \quad \forall x \in \mathbb{C}^n$$
$$\lambda_{max} = \lambda_1 = \max_{x\neq 0} \frac{x^* A x}{x^* x} = \max_{x^* x = 1} x^* A x$$
$$\lambda_{min} = \lambda_n = \min_{x\neq 0} \frac{x^* A x}{x^* x} = \min_{x^* x = 1} x^* A x$$

Proof. See [HJ85] pp.176-177.

□

Let $\triangle(G)$ denote the maximum degree of a vertex in G and $\delta(G)$ denote the minimum degree of a vertex in G. Notice that $\triangle(G) = \delta(G)$ if and only if G is regular. The largest eigenvalue of a graph and its multiplicity are related to $\triangle(G)$ as follows.

Theorem 1.2.6. *The eigenvalue of $A(G)$ with largest absolute value is $\triangle(G)$ if and only if some component of G is $\triangle(G)$-regular. The multiplicity of $\triangle(G)$ as an eigenvalue is the number of $\triangle(G)$-regular components.*

Proof. See [Wes01] pp.460-461.

□

Let $\mathbf{1}$ be a vector of 1's; then $A(G)\mathbf{1}$ is a vector in which each entry is a row sum of $A(G)$ respectively. Given a d-regular graph G, then $A(G)\mathbf{1} = d\mathbf{1}$, so d is an eigenvalue of $A(G)$ corresponding to the eigenvector $\mathbf{1}$. Label the eigenvalues of $A(G)$ as in (1.1), then Theorem (1.2.6) implies $\lambda_1 = d$ and $\lambda_2 < d$ if and only if G is connected.

As the number of vertices n increases, the second largest eigenvalue may approach d. We will see that $\lambda_1 - \lambda_2$ plays a crucial role in determining the expander parameter of G. We now introduce another important representation of G. Recall that $d(v)$ is the degree of a vertex v. A Laplacian matrix, named after the well-known French mathematician Pierre-Simon Laplace, is defined as follows.

Definition 1.2.7 (Laplacian matrix). A **Laplacian matrix** $L(G)$ of a graph G is

$$L(G) = [l_{ij}] \text{ where } l_{ij} = \begin{cases} 0 & \text{if } i \text{ and } j \text{ are not adjacent} \\ d(i) & \text{if } i = j \\ -1 & \text{else} \end{cases}$$

Remark 1.2.8. n - $rank(L(G))$ is the number of components of G. In particular, G is connected if and only if $rank(L(G)) = n - 1$.

Using these special matrices and their properties, we give a proof of Cheeger's inequalities in Chapter 2; Cheeger's inequalities estimate the expander parameter of a d-regular graph G. Chapter 3 covers examples achieving the lower equality. Chapter

4 introduces circulant graphs and their spectral properties. We then proceed to show that there is no expander family in the class of circulant graphs in Chapter 5.

CHAPTER 2

CHEEGER'S INEQUALITIES

The purpose of this chapter is to prove Cheeger's inequalities, which estimate the expander parameter of a d-regular connected graph ($d \geq 3$), and to examine when equality occurs. The proof of Cheeger's inequalities is divided into two parts with one section proving the lower bound and the other section proving the upper bound.

2.1 Cheeger's Inequalities and Preliminaries

Jeff Cheeger is an American mathematician distinguished for his excellent research and contributions in the field of differential geometry. One of his well-known discoveries, now called the Cheeger's inequalities, has many profound applications in graph theory and probability theory [Lur99]. Recall that λ_2 is the second largest eigenvalue of the adjacency matrix $A(G)$. The expander parameter $h(G)$ of a d-regular simple connected graph G is estimated by the following theorem.

Theorem 2.1.1 (**Cheeger's inequalities**). *Let G be a d-regular simple connected graph on n vertices. Then*

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}.$$

Remark 2.1.2. If G is a disconnected d-regular graph, then $h(G) = 0$ and $\lambda_2 = d$.

Hence

$$\frac{d-\lambda_2}{2} = h(G) = \sqrt{2d(d-\lambda_2)}.$$

It will be useful to label the vertices of G as $1, 2, \ldots, n$. Let S be any subset of $V(G)$ such that $1 \leq |S| \leq \frac{n}{2}$. Recall that $\partial(S)$ is the set of edges (i, j) where $i \in S$, $j \in \overline{S}$. Since S and $\overline{S}$ are two disjoint subsets of $V(G)$, G can be viewed as follows.

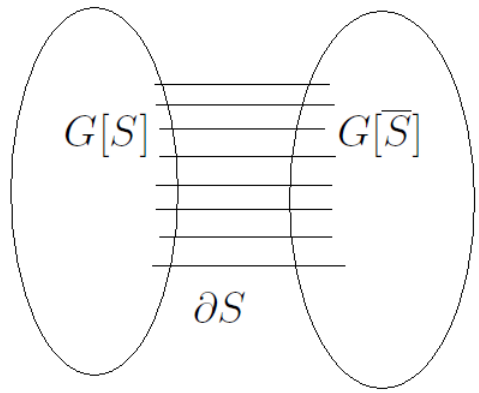

Figure 2.1: A view of the graph G

If $S = \{1, 2, \ldots, |S|\}$ and $\overline{S} = \{|S|+1, \ldots, n\}$, then

$$A(G) = \begin{pmatrix} A(G[S]) & X \\ X^T & A(G[\overline{S}]) \end{pmatrix} = \begin{pmatrix} B & X \\ X^T & C \end{pmatrix}. \tag{2.1}$$

$A(G)$ is a block matrix where B is an $|S| \times |S|$ matrix, C is an $|\overline{S}| \times |\overline{S}|$ matrix, and X and X^T are matrices representing ∂S. Define 1_S to be the $|S| \times 1$ vector of 1's and $1_{\overline{S}}$ to be the $|\overline{S}| \times 1$ vector of 1's. The following results are essential to establish the proof of Cheeger's inequalities.

Lemma 2.1.3. $1_S^T X 1_{\overline{S}} = |\partial S| = 1_{\overline{S}}^T X^T 1_S$

Proof.

$$\begin{aligned}
1_S^T X 1_{\overline{S}} &= \text{sum of all entries in } X \\
&= \text{number of edges from } G[S] \text{ to } G[\overline{S}] \\
&= |\partial S| \\
&= \text{number of edges from } G[\overline{S}] \text{ to } G[S] \\
&= \text{sum of all entries in } X^T \\
&= 1_{\overline{S}}^T X^T 1_S
\end{aligned}$$

□

Lemma 2.1.4. $1_S^T B 1_S = 2\,|E(G[S])|$

Proof.

$$\begin{aligned}
1_S^T B 1_S &= \text{sum of all entries in } B \\
&= \text{total degrees in } G[S] \\
&= 2\,|E(G[S])| \quad \text{by Theorem (1.1.8)}
\end{aligned}$$

□

Lemma 2.1.5. $1_{\overline{S}}^T C 1_{\overline{S}} = 2\left|E(G[\overline{S}])\right|$

Proof.

$$\begin{aligned}
1_{\overline{S}}^T C 1_{\overline{S}} &= \text{sum of all entries in } C \\
&= \text{total degrees in } G[\overline{S}] \\
&= 2\left|E(G[\overline{S}])\right| \quad \text{by Theorem (1.1.8)}
\end{aligned}$$

□

Lemma 2.1.6. $2\,|E(G[S])| + |\partial S| = d|S|$

Proof. Since G is a d-regular graph, each vertex in S has degree d. Hence the lemma follows.

□

Theorem 2.1.7 (**Cauchy-Schwarz inequality**). *If* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$, *then* $|\mathbf{x} \bullet \mathbf{y}| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$.

Proof. See [Wad04] pp.229-230.

□

Remark 2.1.8. Equality holds in Theorem (2.1.7) if and only if $\{\mathbf{x}, \mathbf{y}\}$ is a linearly dependent set of vectors.

2.2 Lower Bound

In this section, we prove the lower bound inequality and examine the equality case. The proof involves many computations related to the block adjacency matrix $A(G)$ as seen in Equation (2.1).

2.2.1 Lower Bound Proof

We want to show $\dfrac{d-\lambda_2}{2} \leq h(G)$ or $d - 2h(G) \leq \lambda_2$. To begin, observe that the definition of an expander parameter yields $d - 2h(G) = d - 2\dfrac{|\partial S|}{|S|}$ for some specific $S \subset V(G)$ with $1 \leq |S| \leq \dfrac{n}{2}$. Notice that $|S| \leq \dfrac{n}{2} \leq |\overline{S}|$ implies $\dfrac{1}{|\overline{S}|} \leq \dfrac{1}{|S|}$, so $d - 2\dfrac{|\partial S|}{|S|} \leq d - \left(\dfrac{|\partial S|}{|S|} + \dfrac{|\partial S|}{|\overline{S}|}\right) = d - |\partial S|\left(\dfrac{1}{|S|} + \dfrac{1}{|\overline{S}|}\right)$. Let $A = A(G)$.

We will proceed by proving several claims that will be useful here. For these, let S be any subset of $V(G)$ with $1 \leq |S| \leq \dfrac{n}{2}$. Define the $n \times 1$ vector v to

be $\begin{pmatrix} 1_S/|S| \\ -1_{\overline{S}}/|\overline{S}| \end{pmatrix}$. Recall from Section (1.3) that $\mathbf{1}$, the $n \times 1$ vector of 1's, is the eigenvector corresponding to the eigenvalue d of A.

Claim 1: v is orthogonal to $\mathbf{1}$ i.e. $v^T\mathbf{1} = 0$.

Proof:

$$v^T\mathbf{1} = v_1 + v_2 + ... + v_n = \sum_{i \in S} v_i + \sum_{j \in \overline{S}} v_j = 1 + (-1) = 0$$

Claim 2: $v^T v = \dfrac{1}{|S|} + \dfrac{1}{|\overline{S}|}$

Proof:

$$v^T v = v_1^2 + v_2^2 + \ldots + v_n^2 = |S|\frac{1}{|S|^2} + |\overline{S}|\frac{1}{|\overline{S}|^2} = \frac{1}{|S|} + \frac{1}{|\overline{S}|}$$

Claim 3: $v^T A v = \left(\dfrac{1}{|S|} + \dfrac{1}{|\overline{S}|}\right) d - \left(\dfrac{1}{|S|} + \dfrac{1}{|\overline{S}|}\right)^2 |\partial S|$

Proof:

$$\begin{aligned}
v^T A v &= \begin{pmatrix} 1_S/|S| \\ -1_{\overline{S}}/|\overline{S}| \end{pmatrix}^T \begin{pmatrix} B & X \\ X^T & C \end{pmatrix} \begin{pmatrix} 1_S/|S| \\ 1_{\overline{S}}/|\overline{S}| \end{pmatrix} \\
&= \frac{1_S^T}{|S|} B \frac{1_S}{|S|} - \frac{1_S^T}{|S|} X \frac{1_{\overline{S}}}{|\overline{S}|} - \frac{1_{\overline{S}}^T}{|\overline{S}|} X^T \frac{1_S^T}{|S|} + \frac{1_{\overline{S}}^T}{|\overline{S}|} C \frac{1_{\overline{S}}}{|\overline{S}|} \\
&= \frac{1}{|S|^2}(1_S^T B 1_S) - \frac{1}{|S||\overline{S}|}(1_S^T X 1_{\overline{S}}) - \frac{1}{|S||\overline{S}|}(1_{\overline{S}}^T X^T 1_S) + \frac{1}{|\overline{S}|^2}(1_{\overline{S}}^T C 1_{\overline{S}}) \\
&= \frac{2\,|E(G[S])|}{|S|^2} - \frac{2|\partial S|}{|S||\overline{S}|} + \frac{2\left|E(G[\overline{S}])\right|}{|S|^2} \quad \text{by Lemma (2.1.3), (2.1.4), (2.1.5)} \\
&= \frac{2|E(G[S])|}{|S|^2} + \frac{2|E(G[\overline{S}])|}{|\overline{S}|^2} + \frac{|\partial S|}{|S|^2} - \frac{|\partial S|}{|S|^2} + \frac{|\partial S|}{|\overline{S}|^2} - \frac{|\partial S|}{|\overline{S}|^2} - \frac{2|\partial S|}{|S||\overline{S}|} \\
&= \frac{2|E(G[S])| + |\partial S|}{|S|^2} + \frac{2|E(G[\overline{S}])| + |\partial S|}{|\overline{S}|^2} - \left(\frac{1}{|S|^2} + \frac{1}{|\overline{S}|^2} + \frac{2}{|S||\overline{S}|}\right)|\partial S| \\
&= \frac{d|S|}{|S|^2} + \frac{d|\overline{S}|}{|\overline{S}|^2} - \left(\frac{1}{|S|} + \frac{1}{|\overline{S}|}\right)^2 |\partial S| \quad \text{by Lemma (2.1.6)} \\
&= \left(\frac{1}{|S|} + \frac{1}{|\overline{S}|}\right) d - \left(\frac{1}{|S|} + \frac{1}{|\overline{S}|}\right)^2 |\partial S|
\end{aligned}$$

Claim 4: $\dfrac{v^T A v}{v^T v} = d - |\partial S| \left(\dfrac{1}{|S|} + \dfrac{1}{|\overline{S}|} \right)$
Proof: apply Claim 2 and Claim 3.

Now let S be the specific set such that $h(G) = \dfrac{|\partial S|}{|S|}$. As shown in the first paragraph of the proof, $d - 2h(G) \leq d - |\partial S| \left(\dfrac{1}{|S|} + \dfrac{1}{|\overline{S}|} \right)$. By Claim 4, $d - 2h(G) \leq \dfrac{v^T A v}{v^T v}$. It remains to show $\dfrac{v^T A v}{v^T v} \leq \lambda_2$. Because A is Hermitian, $\lambda_2 = \max_{f: f \perp \mathbf{1}} \dfrac{f^T A f}{f^T f}$ by Theorem (1.2.5). Since $v \perp 1$ by Claim 1, we have $\dfrac{v^T A v}{v^T v} \leq \lambda_2$.

2.2.2 Lower Bound Equality

The preceding proof offers a tight lower bound estimate of the expander parameter $h(G)$. This estimation can be strict equality. The following result characterizes exactly when equality occurs.

Theorem 2.2.1. $\dfrac{d - \lambda_2}{2} = h(G)$ *if and only if* n *is even,* $d + \lambda_2$ *is even, and there exists* S_0 *such that* $|S_0| = \dfrac{n}{2}$ *and* $G[S_0]$, $G[\overline{S}_0]$ *are* $\dfrac{d + \lambda_2}{2}$*-regular.*

Proof. First, we prove the necessary condition. Assume $\dfrac{d - \lambda_2}{2} = h(G)$. Let $S_0 \subset V(G)$ such that $1 \leq |S_0| \leq \dfrac{n}{2}$ and $h(G) = \dfrac{|\partial S_0|}{|S_0|}$. Reorder $V(G)$ so that $S_0 = \{1, 2, \ldots, |S_0|\}$ and $\overline{S_0} = \{|S_0| + 1, \ldots, n\}$. Define the $n \times 1$ vector $f_0 \perp \mathbf{1}$ to be $f_0 = \begin{pmatrix} 1_{S_0}/|S_0| \\ -1_{\overline{S_0}}/|\overline{S_0}| \end{pmatrix}$. Based on the lower bound proof of Cheeger's inequalities,

$$
\begin{aligned}
\lambda_2 &= \max_{f \perp \mathbf{1}} \frac{f^T A f}{f^T f} \\
&\geq \frac{f_0^T A f_0}{f_0^T f_0} && (1) \\
&= d - |\partial S_0| \left(\frac{1}{|S_0|} + \frac{1}{|\overline{S_0}|} \right) \\
&= d - |\partial S_0| \frac{n}{|S_0||\overline{S_0}|} \\
&\geq d - 2\frac{|\partial S_0|}{|S_0|} \quad \text{because } \frac{n}{|\overline{S_0}|} \leq 2 && (2) \\
&= d - 2h(G).
\end{aligned}
$$

Since we assumed $\lambda_2 = d - 2h(G)$, inequalities (1) and (2) must become equalities. Equality in (1) implies f_0 is an eigenvector of A corresponding to λ_2. Equality in (2) implies $|S_0| = |\overline{S_0}| = \frac{n}{2}$, so n is even. It follows that $f_0 = \frac{2}{n}\begin{pmatrix} \mathbf{1} \\ \mathbf{-1} \end{pmatrix}$. We will use this vector f_0 to show there exists S_0 such that $G[S_0]$ and $G[\overline{S_0}]$ are both $\frac{d+\lambda_2}{2}$-regular. Since f_0 is an eigenvector of A corresponding to λ_2, setting $Af_0 = \lambda_2 f_0$ yields

(1) $B\mathbf{1} - X\mathbf{1} = \lambda_2 \mathbf{1}$.

(2) $X^T\mathbf{1} - C\mathbf{1} = -\lambda_2 \mathbf{1}$.

Recall that d is an eigenvalue of A corresponding to the eigenvector $\mathbf{1}$ because G is d-regular. Setting $A\mathbf{1} = d\mathbf{1}$ yields

(3) $B\mathbf{1} + X\mathbf{1} = d\mathbf{1}$.

(4) $X^T\mathbf{1} + C\mathbf{1} = d\mathbf{1}$.

(1) + (3) $\Rightarrow$ $2\mathrm{B}\mathbf{1} = (\lambda_2 + d)\mathbf{1}$ $\Rightarrow$ $B\mathbf{1} = \frac{d+\lambda_2}{2}\mathbf{1}$ $\Rightarrow$ $\mathrm{G[S_0]}$ is $\frac{d+\lambda_2}{2}$-regular.

(4) - (2) $\Rightarrow$ $2\mathrm{C}\mathbf{1} = (\lambda_2 + d)\mathbf{1}$ $\Rightarrow$ $C\mathbf{1} = \frac{d+\lambda_2}{2}\mathbf{1}$ $\Rightarrow$ $\mathrm{G}[\overline{S_0}]$ is $\frac{d+\lambda_2}{2}$-regular.

The proof of the necessary condition is done. We now prove the sufficient

condition. Suppose n is even, $d+\lambda_2$ is even, and there exists S_0 such that $|S_0| = \frac{n}{2}$ and $G[S_0]$, $G[\overline{S_0}]$ are $\frac{d+\lambda_2}{2}$-regular. Reorder $V(G)$ so that $S_0 = \{1, 2, \ldots, |S_0|\}$ and $\overline{S_0} = \{|S_0|+1, \ldots, n\}$. $G[S_0]$ is $\frac{d+\lambda_2}{2}$-regular implies $B\mathbf{1} = \left(\frac{d+\lambda_2}{2}\right)\mathbf{1}$. Because n is even and $|S_0| = \frac{n}{2}$, B, C, X, and X^T are all square matrices of equal sizes. Since G is d-regular, d is an eigenvalue of A corresponding to the eigenvector $\mathbf{1}$, so

$$\begin{pmatrix} B & X \\ X^T & C \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} = d \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}.$$

Block matrix multiplication yields

$$\begin{aligned} B\mathbf{1} + X\mathbf{1} &= d\mathbf{1} \\ \left(\frac{d+\lambda_2}{2}\right)\mathbf{1} + X\mathbf{1} &= d\mathbf{1} \\ X\mathbf{1} &= \left(\frac{d-\lambda_2}{2}\right)\mathbf{1} \\ \mathbf{1}^T X\mathbf{1} &= \left(\frac{d-\lambda_2}{2}\right)\mathbf{1}^T\mathbf{1} \\ |\partial S_0| &= \left(\frac{d-\lambda_2}{2}\right)\left(\frac{n}{2}\right). \end{aligned}$$

By the lower bound of Cheeger's inequalities and the definition of an expander parameter, $\frac{d-\lambda_2}{2} \leq h(G) \leq \frac{|\partial S_0|}{|S_0|} = \frac{d-\lambda_2}{2}$. Thus $h(G) = \frac{d-\lambda_2}{2}$.

□

Remark 2.2.2. The preceding theorem shows that when equality holds, $G[S_0]$ and $G[\overline{S_0}]$ have equal cardinality and regularity. However, these two induced subgraphs of G need not be isomorphic as we will see in an upcoming example in Chapter 3.

2.3 Upper Bound

In this section, we prove the upper bound of Cheeger's inequalities and comment on the equality case. The upper bound proof is much more difficult than the lower

bound proof because it involves many intricacies.

2.3.1 Upper Bound Proof

Let g be a nonzero eigenvector of A corresponding to λ_2. Since A is symmetric, g is orthogonal to the eigenvector $\mathbf{1}$, that is, $g^T\mathbf{1} = 0$. Because g is nonzero, g can't be strictly positive or negative. Therefore we can order $V(G)$ so that $g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix}$ where $g^+ = [g_1\ g_2 \dots g_r]^T$ has $g_1 \geq g_2 \geq \dots g_r > 0$ and $g^- = [g_{r+1}\ g_{r+2} \dots g_n]^T$ has $g_n \leq \dots \leq g_{r+2} \leq g_{r+1} \leq 0$.

Partition A as $\begin{pmatrix} B & X \\ X^T & C \end{pmatrix}$ so that B is $r \times r$ and C is $(n-r) \times (n-r)$. Since $-g$ is also an eigenvector corresponding to λ_2, we may also assume $1 \leq r \leq \frac{n}{2}$. Let $f = \begin{pmatrix} g^+ \\ 0 \end{pmatrix}$ be an $n \times 1$ vector. We want

$$h(G) \leq \sqrt{2d(d-\lambda_2)}$$

$$h^2(G) \leq 2d(d-\lambda_2)$$

$$\frac{h^2(G)}{2d} \leq d - \lambda_2.$$

It suffices to show $\frac{h^2(G)}{2d} \leq \frac{f^T L f}{f^T f} \leq d - \lambda_2$, where L is the Laplacian matrix defined in Chapter 1. We divide the proof into two parts:

(1) $\frac{f^T L f}{f^T f} \leq d - \lambda_2$.

(2) $\frac{h^2(G)}{2d} \leq \frac{f^T L f}{f^T f}$.

To prove inequality (1), notice that $L = dI - A$. This yields $f^T L f = f^T(dI)f -$

$f^TAf = df^Tf - f^TAf$, so it suffices to show $\lambda_2 \leq \dfrac{f^TAf}{f^Tf}$. Consider

$$f^TAf = \begin{pmatrix} g^+ \\ 0 \end{pmatrix}^T \begin{pmatrix} B & X \\ X^T & C \end{pmatrix} \begin{pmatrix} g^+ \\ 0 \end{pmatrix} = {g^+}^T Bg^+.$$

Claim 1: $Bg^+ \geq \lambda_2 g^+$

Proof:

$$Ag = \lambda_2 g \quad \text{implies} \quad \begin{pmatrix} B & X \\ X^T & C \end{pmatrix} \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \lambda_2 \begin{pmatrix} g^+ \\ g^- \end{pmatrix}.$$

Multiplying out the first row of the block matrix yields $Bg^+ + Xg^- = \lambda_2 g^+$. Since $X \geq 0$ and $g^- \leq 0$, $Xg^- \leq 0$. Thus Claim 1 holds.

Applying Claim 1, $f^TAf = (g^+)^TBg^+ \geq (g^+)^T\lambda_2 g^+ = \lambda_2(g^+)^Tg^+ = \lambda_2(f^Tf)$. Therefore, $\lambda_2 \leq \dfrac{f^TAf}{f^Tf}$ proving inequality (1). Next, we prove inequality (2). We want to show

$$\begin{aligned} \frac{h^2(G)}{2d} &\leq \frac{f^TLf}{f^Tf} \\ h(G)\sqrt{f^Tf} &\leq \sqrt{2d(f^TLf)} \\ h(G)\frac{f^Tf}{\sqrt{f^Tf}} &\leq \sqrt{2d(f^TLf)} \\ h(G)f^Tf &\leq \sqrt{2d(f^TLf)(f^Tf)}. \end{aligned}$$

Define $B_f = \sum_{(x,y)\in E(G)} |f_x^2 - f_y^2|$. We split the proof into two parts:

(2a) $h(G)f^Tf \leq B_f$.

(2b) $B_f \leq \sqrt{2d(f^TLf)(f^Tf)}$.

Let $[i] = \{1, \ldots, i\}$, and $[\bar{i}] = \{i+1, \ldots, n\}$ where $1 \leq i \leq r \leq \dfrac{n}{2}$. For a fixed i, define $|E_i|$ to be the number of edges from $[i]$ to $[\bar{i}]$. In other words, $|E_i|$ is the number of edges $(x, y) \in E(G)$ such that $x \leq i < i+1 \leq y$. Note that $|E_i|$ corresponds to $|\partial S|$ in the previous sections.

Claim 2: $|E_i| \geq h(G)i$

Proof: $h(G) = \min_{|S| \leq \frac{n}{2}} \dfrac{|\partial S|}{|S|} \leq \dfrac{|E_i|}{i}$. Thus $|E_i| \geq h(G)i$.

Using Claim 2 and the definition of $|E_i|$, we get

$$
\begin{aligned}
B_f &= \sum_{(x,y)\in E(G)} |f_x^2 - f_y^2| \\
&= \sum_{\substack{(x,y)\in E(G)\\ x<y}} (f_x^2 - f_y^2) \quad \text{by the definition of vector } f \\
&= \sum_{\substack{(x,y)\in E(G)\\ x\leq y}} [(f_x^2 - f_{x+1}^2) + (f_{x+1}^2 - f_{x+2}^2) + \cdots + (f_{y-1}^2 - f_y^2)] \\
&= \sum_{\substack{(x,y)\in E(G)\\ x\leq y}} \sum_{i=x}^{y-1} f_i^2 - f_{i+1}^2 \\
&= (\text{the number of edges } (x,y) \in E(G) \text{ such that } x \leq 1 < 2 \leq y)(f_1^2 - f_2^2) + \ldots \\
&+ (\text{the number of edges } (x,y) \in E(G) \text{ such that } x \leq n-1 < n = y)(f_{n-1}^2 - f_n^2) \\
&= |E_1|(f_1^2 - f_2^2) + \cdots + |E_r|(f_r^2) \quad \text{because } f_i = 0 \text{ for } i > r \\
&\geq h(G)(f_1^2 - f_2^2) + \cdots + h(G)r(f_r^2) \quad \text{by Claim 2} \\
&= h(G)[(f_1^2 - f_2^2) + 2(f_2^2 - f_3^2) + \cdots + r(f_r^2)] \\
&= h(G)(f_1^2 + \cdots + f_r^2) \\
&= h(G)(f^T f).
\end{aligned}
$$

The proof of $(2a)$ is complete. To prove $(2b)$, we use the following results.

Claim 3: $f^T A f = \sum_{(x,y)\in E(G)} 2 f_x f_y$

Proof: $f^T A f = \sum_{i,j=1}^{n} f_i a_{ij} f_j = \sum_{(x,y)\in E(G)} 2 f_x f_y$ because $a_{ij} = a_{ji} = 1$ if $(i,j) \in E(G)$, and 0 otherwise.

Claim 4: $d(f^T f) = \sum_{(x,y)\in E(G)} f_x^2 + f_y^2$

Proof: $d(f^T f) = d(f_1^2 + \cdots + f_n^2) = \sum_{(x,y)\in E(G)} (f_x^2 + f_y^2)$ because G is d-regular.

Claim 5: $f^T L f = \sum_{(x,y)\in E} (f_x - f_y)^2$

Proof:

$$\begin{aligned} f^T L f &= f^T (dI - A) f \\ &= d(f^T f) - f^T A f \\ &= \sum_{(x,y)\in E} (f_x^2 + f_y^2) - \sum_{(x,y)\in E} 2 f_x f_y \quad \text{by Claim 3 and Claim 4} \\ &= \sum_{(x,y)\in E} (f_x - f_y)^2. \end{aligned}$$

We now prove $(2b)$.

$$\begin{aligned} B_f &= \sum_{(x,y)\in E} |f_x^2 - f_y^2| \\ &= \sum_{(x,y)\in E} |(f_x + f_y)(f_x - f_y)| \\ &= \sum_{(x,y)\in E} |f_x + f_y||f_x - f_y| \\ &\leq \sqrt{\sum_{(x,y)\in E} (f_x + f_y)^2} \sqrt{\sum_{(x,y)\in E} (f_x - f_y)^2} \quad \text{by Theorem (2.1.7)} \\ &\leq \sqrt{\sum_{(x,y)\in E} 2(f_x^2 + f_y^2)} \sqrt{(f^T L f)} \quad \text{by Claim 5} \\ &= \sqrt{2d(f^T f)(f^T L f)} \quad \text{by Claim 4.} \end{aligned}$$

The proof of the upper bound inequality is complete.

2.3.2 Upper Bound Equality

In this section, we show that the upper inequality can not be equality.

Theorem 2.3.1. *$h(G) = \sqrt{2d(d - \lambda_2)}$ never holds when G is connected.*

Proof. Assume $h(G) = \sqrt{2d(d - \lambda_2)}$. Then every inequality in the upper bound proof must become equality. In particular, we have $(f_x + f_y)^2 = 2(f_x^2 + f_y^2)$, so $f_x = f_y$ for

all $(x, y) \in E(G)$. Since G is connected, there is an x, y-path for every $x, y \in V(G)$. This means the $n \times 1$ vector f is constant, i.e., $f = [a\ a \dots a]^T$ for some real value a. However, $f = [g_1 \dots g_r\ 0 \dots 0]^T$ where $g_1 \geq \cdots \geq g_r > 0$ and $1 \leq r \leq \dfrac{n}{2}$ implies $0 \neq a = 0$, a contradiction.

$\square$

CHAPTER 3

CONNECTED GRAPHS ACHIEVING LOWER EQUALITY

In this chapter, we provide three connected graphs that achieve the lower equality of Theorem (2.1.1). They are the regular complete multipartite graph $K_{2p,\ldots,2p}$, the hypercube Q_n, and the specially constructed 4-regular graph of order 12 (see Figure 3.2). To find the expander parameter of a graph G, the number of edges connecting $G[S]$ to $G[\overline{S}]$ is needed. Sometimes this number, $|\partial S|$, can easily be computed using known information about G.

Lemma 3.0.2. *Let G be a d-regular graph on n vertices, and $S \subset V(G)$ with $|S| = |\overline{S}| = \frac{n}{2}$. If $G[S]$ and $G[\overline{S}]$ are k-regular, then $|\partial S| = \frac{n(d-k)}{2}$.*

Proof. Assume $G[S]$ is k-regular. By Theorem (1.1.8), we have

$$\begin{aligned}
\sum_{v \in V(G)} d(v) &= 2|E(G)| \\
nd &= 2(|E(G[S])| + |\partial S| + |E(G[\overline{S}])|) \\
nd &= 2|E(G[S])| + 2|\partial S| + 2|E(G[\overline{S}])| \\
nd &= \frac{n}{2}k + 2|\partial S| + \frac{n}{2}k \\
nd &= nk + 2|\partial S| \\
\frac{n(d-k)}{2} &= |\partial S|
\end{aligned}$$

□

3.1 Complete Multipartite Graph

Definition 3.1.1 (**multipartite graph**). A **multipartite graph** $G_{n_1,n_2,\ldots,n_k}$ is a graph in which the set of vertices $V(G)$ is divided into subsets, called parts, with orders $|S_1| = n_1, |S_2| = n_2, \ldots, |S_k| = n_k \geq 1$ such that no two vertices in the same part have an edge connecting them.

Definition 3.1.2 (**complete multipartite graph**). A **complete multipartite graph** $K_{n_1,n_2,\ldots,n_k}$ is a multipartite graph such that any two vertices that are not in the same part have an edge connecting them.

Consider the regular complete multipartite graph $G = K_{2p,\ldots,2p}$ with $t \geq 2$ parts and $p \geq 1$. Then $|V(K_{2p,\ldots,2p})| = 2pt$, and $K_{2p,\ldots,2p}$ is $2p(t-1)$-regular. Cheeger's inequalities and the definition of an expander parameter yield $\dfrac{2p(t-1)-\lambda_2}{2} \leq h(G) \leq \dfrac{|\partial S|}{|S|}$ for any $S \subset V(G)$ with $|S| \leq \dfrac{n}{2}$. We will calculate λ_2 and show that there exists a set $S \subset V(K_{2p,\ldots,2p})$ such that $\dfrac{|\partial S|}{|S|} = \dfrac{2p(t-1)-\lambda_2}{2}$. The following result is essential to find $\lambda_2(K_{2p,\ldots,2p})$.

Lemma 3.1.3. *Let G be a d-regular graph of order n. If $Sp(A(\bar{G})) = \{n-d-1 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_n\}$, then $Sp(A(G)) = \{d \geq -\alpha_n - 1 \geq -\alpha_{n-1} - 1 \geq \ldots \geq -\alpha_2 - 1\}$.*

Proof. Assume $n-d-1, \alpha_2, \ldots, \alpha_n$ are eigenvalues of $A(\bar{G})$ such that $n-d-1 \geq \alpha_2 \geq \alpha_3 \geq \ldots \geq \alpha_n$. Since $A(\bar{G})$ is real symmetric, let $\{x_1 = \mathbf{1}, x_2, \ldots, x_n\}$ denote an orthonormal set of eigenvectors corresponding to the eigenvalues $n-d-1, \alpha_2, \ldots, \alpha_n$. Consider the identity matrix I. Note that $A(G) + A(\bar{G}) = A(K_n) = \mathbf{1}\mathbf{1}^T - I$, so

$$A(G)x_i + A(\bar{G})x_i = \mathbf{1}\mathbf{1}^T x_i - Ix_i \quad \text{for } i \neq 1 \text{ i.e. } x_i \neq \mathbf{1}$$

$$A(G)x_i + \alpha_i x_i = 0 - x_i \quad \text{by orthonormality}$$

$$A(G)x_i = (-\alpha_i - 1)x_i.$$

□

Recall that K_n is the complete graph on $n \geq 1$ vertices. Let $(\alpha)^{(n-1)}$ denote an eigenvalue α with multiplicity $(n-1)$. The spectrum of $A(K_n)$ is computed as follows.

Lemma 3.1.4. $Sp(A(K_n)) = \{n-1, (-1)^{(n-1)}\}$

Proof. Let J be the matrix of all 1's. Then $A(K_n) = J - I$, and so $Sp(A(K_n)) = Sp(J - I) = Sp(J) - \{(1)^{(n)}\} = \{n, (0)^{(n-1)}\} - \{(1)^{(n)}\}$. Thus $Sp(A(K_n)) = \{n - 1, (-1)^{(n-1)}\}$.

□

Lemmas (3.1.3) and (3.1.4) enable us to calculate the second largest eigenvalue of $K_{2p,\ldots,2p}$.

Lemma 3.1.5. *If* $G = K_{2p,\ldots,2p}$, *then* $\lambda_2(A(G)) = 0$.

Proof. Let $G = K_{2p,\ldots,2p}$, then

$$A(G) = \begin{pmatrix} 0 & J & \ldots & J \\ J & 0 & \ldots & J \\ \vdots & \vdots & \ddots & \vdots \\ J & J & \ldots & 0 \end{pmatrix} \quad \Rightarrow \quad A(\bar{G}) = \begin{pmatrix} A(K_{2p}) & 0 & \ldots & 0 \\ 0 & A(K_{2p}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A(K_{2p}) \end{pmatrix}.$$

$Sp(A(\bar{G})) = t$ copies of $Sp(A(K_{2p})) = t$ copies of $\{2p-1, (-1)^{(2p-1)}\}$ by Lemma (3.1.4). Thus $\lambda_2(A(G)) = -(-1) - 1 = 0$ by Lemma (3.1.3).

□

$\lambda_2(K_{2p,\ldots,2p}) = 0$ implies $p(t-1) \leq h(K_{2p,\ldots,2p})$. We refer to Theorem (2.2.1) to pick an appropriate subset of the vertex set $V(G)$. We pick S such that $|S| = pt = |\overline{S}|$ and $G[S] = K_{p,\ldots,p} = G[\overline{S}]$. Since both $G[S]$ and $G[\overline{S}]$ are $p(t-1)$-regular, Lemma (3.0.2) implies

$$|\partial S| = \frac{2pt[2p(t-1) - p(t-1)]}{2} = pt(p(t-1)).$$

Thus $\dfrac{|\partial S|}{|S|} = \dfrac{pt(p(t-1))}{pt} = p(t-1)$. Hence $h(K_{2p,\ldots,2p}) = p(t-1)$, and the lower bound equality holds for $K_{2p,\ldots,2p}$.

3.2 Hypercube Graph

We begin this section by introducing the notions of Kronecker product and Kronecker sum of matrices, which will be used to find the spectrum of the hypercube graph.

Definition 3.2.1 (Kronecker product). Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the **Kronecker product** $A \otimes B$ is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix} \in \mathbb{R}^{mp\times nq}. \tag{3.1}$$

Definition 3.2.2 (Kronecker sum). Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$, then the **Kronecker sum** $A \oplus B$ is the $mn \times mn$ matrix $(I_m \otimes A) + (B \otimes I_n)$ where I_k is the $k \times k$ identity matrix.

From the definition of Kronecker sum, the spectrum of $(A \oplus B)$ can be written as a linear combination of the spectrum of matrix A and the spectrum of matrix B.

Lemma 3.2.3. $Sp(A \oplus B) = Sp(A) + Sp(B)$

Proof. See [Gra81] pp.30.

□

Both [Lau04] and [Gra81] offer many insights on Kronecker product and Kronecker sum with applications in engineering and statistics. Interested readers should refer to these sources for more information. We continue this section with the definition of a hypercube graph.

Definition 3.2.4 (**hypercube graph**)**.** A **hypercube graph** Q_n is a graph on 2^n vertices of the form $(i_1, \ldots, i_n)$ where each $i_j \in \{0, 1\}$, and two vertices are adjacent if they differ in exactly one coordinate.

Example 3.2.5. We give the example of Q_2 to illustrate the definition. The graph Q_2 has vertex set $\{00, 01, 10, 11\}$. Since two vertices are adjacent if they are differed in exactly one coordinate, Q_2 can be seen as a cycle of length 4, i.e., C_4.

In general, the vertex $(i_1, i_2, \ldots, i_n)$ is adjacent to vertices $(i_1 + 1, i_2 \ldots, i_n)$, $(i_1, i_2 + 1, \ldots, i_n)$, $\ldots$, and $(i_1, i_2, \ldots, i_n + 1)$. Since each vertex of Q_n is adjacent to exactly n vertices, Q_n is n-regular. The number of edges in Q_n can be counted using Theorem (1.1.8), and it is $\dfrac{n2^n}{2} = n2^{n-1}$.

Q_n has a special property that it is constructible from Q_{n-1}. To do so, we first adjoin 0 and 1 to each vertex of Q_{n-1} to get $(i_1, i_2, \ldots, i_{n-1}, 0)$ and $(i_1, i_2, \ldots, i_{n-1}, 1)$ respectively. There now exist two copies of Q_{n-1}. We built Q_n by forming exactly 2^{n-1} edges between vertices $(i_1, i_2, \ldots, i_{n-1}, 0)$ and $(i_1, i_2, \ldots, i_{n-1}, 1)$. Figure (3.1) taken from [Wes01] pp.36 shows the construction of Q_3 from Q_2.

Since Q_1 is isomorphic to K_2, $A(Q_1) = A(K_2)$. Recursively, $A(Q_n)$ can be expressed as a Kronecker sum of $A(Q_{n-1})$ and $A(K_2)$ [Har88]. We use this fact and Lemma (3.2.3) to compute the spectrum of Q_n. We prove the following result by inducting on n.

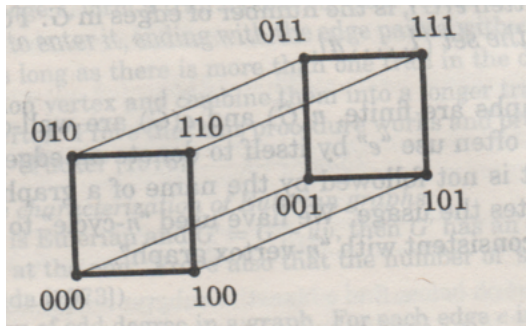

Figure 3.1: Q_3 is formed by connecting 2 copies of Q_2.

Lemma 3.2.6. $Sp(A(Q_n)) = \{(n)^{\binom{n}{0}}, (n-2)^{\binom{n}{1}}, \ldots, (-n)^{\binom{n}{n}}\}$

Proof. Basic step:

$n = 1 \quad \Rightarrow \quad \mathrm{Sp(A}(Q_1)) = Sp(A(K_2)) = \{1, -1\}$

$n = 1 \quad \Rightarrow \quad \{(1)^{\binom{1}{0}}, (-1)^{\binom{1}{1}}\} = \{1, -1\}$

Inductive step:

Assume $Sp(A(Q_n)) = \{(n)^{\binom{n}{0}}, (n-2)^{\binom{n}{1}}, \ldots, (-n)^{\binom{n}{n}}\}$. Consider the hypercube graph Q_{n+1}. Using the recursive expression, we have $Sp(A(Q_{n+1})) = Sp(A(Q_n) \oplus A(K_2))$. The calculation is done as follows.

$$\begin{aligned}
Sp(A(Q_n) \oplus A(K_2)) &= Sp(A(Q_n)) + Sp(A(K_2)) \quad \text{by Lemma (3.2.3)} \\
&= \left\{(n)^{\binom{n}{0}}, (n-2)^{\binom{n}{1}}, \ldots, (-n)^{\binom{n}{n}}\right\} + \{1, -1\} \\
&= \left\{(n+1)^{\binom{n}{0}}, (n-1)^{\binom{n}{1}}, \ldots, (-n+1)^{\binom{n}{n}}, (n-1)^{\binom{n}{0}}, \ldots, (-n-1)^{\binom{n}{n}}\right\} \\
&= \left\{(n+1)^{\binom{n+1}{0}}, (n-1)^{\binom{n}{1}+\binom{n}{1}}, \ldots, (-n+1)^{\binom{n}{n}+\binom{n}{n-1}}, (-n-1)^{\binom{n+1}{n+1}}\right\} \\
&= \left\{(n+1)^{\binom{n+1}{0}}, (n-1)^{\binom{n+1}{1}}, \ldots, (-n+1)^{\binom{n+1}{n}}, (-n-1)^{\binom{n+1}{n+1}}\right\}.
\end{aligned}$$

□

Lemma (3.2.6) implies $\lambda_2(A(Q_n)) = n-2$. Cheeger's inequalities give $1 = \dfrac{n-\lambda_2(A(Q_n))}{2} \leq h(Q_n)$. It remains to show $h(Q_n) \leq \dfrac{|\partial S|}{|S|} = 1$ for a certain $S \subset V(Q_n)$. Since Q_n is formed by two copies of Q_{n-1}, we pick $S \subset V(Q_n)$ such that $|S| = 2^{n-1} = |\overline{S}|$ and $G[S] = Q_{n-1} = G[\overline{S}]$. By the definition of an expander parameter, $h(Q_n) \leq \dfrac{|\partial S|}{|S|} = \dfrac{2^{n-1}}{2^{n-1}} = 1$. Hence $h(Q_n) = 1$, and the lower bound equality holds for Q_n.

Both $K_{2p,\ldots,2p}$ and Q_n satisfy the lower bound equality. When we pick such S in each graph, the induced subgraphs on S and $\overline{S}$ are isomorphic. However, resulting graphs $G[S]$ and $G[\overline{S}]$ need not to be isomorphic as we will see in the next example.

3.2.1 A Constructed 4-regular Graph

We construct a 4-regular graph G in Figure (3.2) to demonstrate that the lower bound equality of Cheeger's inequalities may hold without having isomorphism between $G[S]$ and $G[\overline{S}]$. Here $d = 4$, $|V(G)| = 12$, and $(i, i^*) \in E(G)$ for $i = 1, 2, 3, 4, 5, 6$. The following result will be used to show the non-isomorphism of the two induced subgraphs.

Lemma 3.2.7. *Let $S \subset V(G)$. If $|S| = 6$ and $G[S]$ is 3-regular, then $S = \{1, 2, 3, 4, 5, 6\}$ or $S = \{1^*, 2^*, 3^*, 4^*, 5^*, 6^*\}$.*

Proof. Assume $|S| = 6$ and $G[S]$ is 3-regular. Let $T \subseteq \{1, 2, 3, 4, 5, 6\}$ and $T^* \subseteq \{1^*, 2^*, 3^*, 4^*, 5^*, 6^*\}$. Let $S = T \cup T^*$. Since $|S| = 6$, we have the following cases:

(1) $|T| = 0$ and $|T^*| = 6$.

(2) $|T| = 1$ and $|T^*| = 5$.

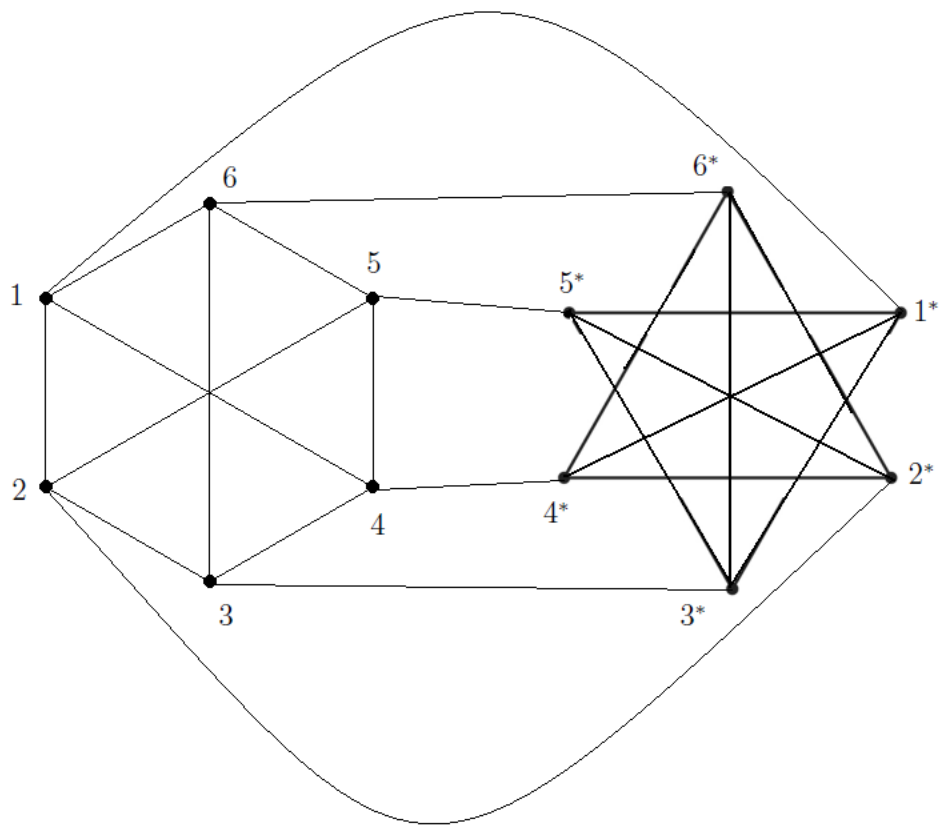

Figure 3.2: 4-regular graph on 12 vertices

(3) $|T| = 2$ and $|T^*| = 4$.

(4) $|T| = 3$ and $|T^*| = 3$.

(5) $|T| = 4$ and $|T^*| = 2$.

(6) $|T| = 5$ and $|T^*| = 1$.

(7) $|T| = 6$ and $|T^*| = 0$.

Cases (1) and (7) are what we want. Cases (2) and (6) contradict our regularity assumption because the maximum degree of a vertex in T or T^* in $G[S]$ is 1. Cases (3) and (5) also contradict our regularity assumption because the maximum degree of a vertex in T or T^* in $G[S]$ is 2.

In case (4), since $G[S]$ is 3-regular, $G[T]$ and $G[T^*]$ must be 2-regular because $(i, i^*) \in E(G)$ for $i = 1, 2, 3, 4, 5, 6$. However, $G[\{1, 2, 3, 4, 5, 6\}]$ has no C_3. Thus case (4) fails. Therefore only cases (1) and (7) are possible, and we must choose $S = \{1, 2, 3, 4, 5, 6\}$ or $S = \{1^*, 2^*, 3^*, 4^*, 5^*, 6^*\}$.

□

Using Matlab, the second largest eigenvalue of $A(G)$ is 2. Cheeger's inequalities imply $1 = \dfrac{4-2}{2} = \dfrac{d-\lambda_2}{2} \leq h(G)$. To show that $h(G) = 1$, it suffices to pick a $S \subset V(G)$ such that $\dfrac{|\partial S|}{|S|} = 1$. Choosing $S = \{1, 2, 3, 4, 5, 6\}$ yields 3-regular induced subgraphs $G[S]$ and $G[\overline{S}]$, and $|\partial S| = 6$. Thus $\dfrac{|\partial S|}{|S|} = \dfrac{6}{6} = 1$. Hence $h(G) = 1$, and the lower bound equality holds for the constructed 4-regular graph in Figure (3.2).

By Lemma (3.2.7), $G[S]$ and $G[\overline{S}]$ are the only 3-regular induced subgraphs of G. Based on Figure (3.2), $G[S]$ is not isomorphic to $G[\overline{S}]$ because one has C_3, but the other doesn't.

CHAPTER 4

CIRCULANT GRAPHS

In this chapter, we introduce the family of circulant graphs. We are interested in this particular family because it has many important applications in engineering and computer science. Its expander parameter, if not zero, will provide useful information to applications of expander graphs. We begin with a preliminary section on definitions. Using a special permutation matrix Z, we then compute the spectrum of a circulant graph and discuss some properties related to the computation of its expander parameter.

4.1 Preliminaries

Definition 4.1.1 (**circulant matrix**)**.** Every $n \times n$ matrix C of the form

$$C = \begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{pmatrix}$$

is called a **circulant matrix**.

The matrix C is completely determined by its first row because other rows are votations of the first row. Notice that C is symmetric if $c_{n-i} = c_i$ for $i = 1, 2, \dots, n-1$.

Moreover, C is an adjacency matrix if $c_0 = 0$ and $c_{n-i} = c_i \in \{0, 1\}$. The class of circulant matrices belongs to a larger class of matrices called **Toeplitz**. Readers who wish to learn more about these matrices and their applications in engineering can check out a review document at `http://www-ee.stanford.edu/ gray/toeplitz.pdf`.

Definition 4.1.2 (**circulant graph**)**.** A **circulant graph** is a graph which has a circulant adjacency matrix.

Examples of circulant graphs are the cycle C_n, the complete graph K_n, and the complete bipartite graph $K_{n,n}$ [Ski90]. Since circulant graphs are recognizable through their adjacency matrices, they form a strong link between graph theory and matrix theory. A graph is called an **integral graph** if it has integral spectrum. Within the class of circulant graphs, integral circulant graphs have many significant applications in telecommunication networks and distributed computing [Kar].

Circulant graphs are always regular. Let $C_{d,n}$ denote a d-regular circulant graph on n vertices. The circulant adjacency matrix of $C_{d,n}$, $A(C_{d,n})$, is easy to formulate, but calculating its spectrum is not straight forward.

4.2 Spectrum of A Circulant Graph

The spectrum of $A(C_{d,n})$ requires clever tricks to obtain. Notice that $A(C_{d,n})$ can be written as a linear combination of powers of the following $n \times n$ matrix

$$Z = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Z is a special permutation matrix with the following properties:

(1) $Z^T = Z^{-1}$ because Z is a permutation matrix.

(2)

$$Z^2 = \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{pmatrix}$$

Technically, Z^2 is a cyclic shifting of columns of Z to the right one time. If we continue to shift them $n-1$ times, we get $Z^n = I$.

(3) $Z^{n-k}Z^k = I$ for $k = 1, 2, \dots, n-1$.

(4) The characteristic polynomial of Z is

$p(x) = det(xI - Z) = x(x^{n-1}) + (-1)^{n-1}(-1)(-1)^{n-1} = x^n - 1.$

(5) $Sp(Z) = \{e^{\frac{2k\pi\mathbf{i}}{n}} : 1 \leq k \leq n\}$ where $\mathbf{i} = \sqrt{-1}$.

These properties of Z are essential to determine the spectrum of a circulant graph of order n. Let

$$\omega = e^{\frac{2k\pi\mathbf{i}}{n}} = \cos\frac{2k\pi}{n} + \mathbf{i}\sin\frac{2k\pi}{n} \quad \text{for } 1 \leq k \leq n \quad \text{[SS03].} \tag{4.1}$$

The definition of ω leads to an equally important result which will be applied in the proof of Theorem (4.2.2).

Lemma 4.2.1. $\omega^t + \omega^{n-t} = 2\cos\dfrac{2kt\pi}{n}$ *for* $1 \leq k \leq n$ *and* $1 \leq t \leq n$.

Proof. Let $1 \leq k \leq n$ and $1 \leq t \leq n$, then

$$\begin{aligned}
\omega^t + \omega^{n-t} &= e^{\frac{2kt\pi\mathbf{i}}{n}} + e^{2kt\pi\mathbf{i}} e^{\left(-\frac{2kt\pi\mathbf{i}}{n}\right)} \\
&= e^{\frac{2kt\pi\mathbf{i}}{n}} + e^{-\frac{2kt\pi\mathbf{i}}{n}} \quad \text{because } e^{2\pi\mathbf{i}} = 1 \\
&= \cos\frac{2kt\pi}{n} + \mathbf{i}\sin\frac{2kt\pi}{n} + \cos\frac{2kt\pi}{n} - \mathbf{i}\sin\frac{2kt\pi}{n} \quad \text{by Equation (4.1)} \\
&= 2\cos\frac{2kt\pi}{n}
\end{aligned}$$

□

Using properties of the matrix Z and Lemma (4.2.1), the spectrum of a circulant graph is established in the following theorem.

Theorem 4.2.2. *Let A be an adjacency matrix of a circulant graph on n vertices. If n is odd, then*

$$Sp(A) = \left\{ \sum_{r=1}^{\frac{n-1}{2}} 2c_r \cos\frac{2kr\pi}{n} : 1 \leq k \leq n \right\}.$$

If n is even, then

$$Sp(A) = \left\{ \sum_{r=1}^{\frac{n-2}{2}} 2c_r \cos\frac{2kr\pi}{n} + c_{\frac{n}{2}} \cos k\pi : 1 \leq k \leq n \right\}.$$

Proof. Label the vertices of a circulant graph as $0, 1, \ldots, n-1$. Then the adjacency matrix A is

$$\begin{pmatrix}
0 & c_1 & \ldots & c_{n-2} & c_{n-1} \\
c_{n-1} & 0 & c_1 & & c_{n-2} \\
\vdots & c_{n-1} & 0 & \ddots & \vdots \\
c_2 & & \ddots & \ddots & c_1 \\
c_1 & c_2 & \ldots & c_{n-1} & 0
\end{pmatrix}$$

where $c_i = c_{n-i} = 0$ if vertices i and $n-i$ are not adjacent, and $c_i = c_{n-i} = 1$ if vertices i and $n-i$ are adjacent. We have $A = c_1 Z^1 + c_2 Z^2 + \ldots + c_{n-1} Z^{n-1} = \sum_{r=1}^{n-1} c_r Z^r$.

If n is odd, then $A = c_1(Z + Z^{n-1}) + \ldots + c_{\frac{n-1}{2}}\left(Z^{\frac{n-1}{2}} + Z^{\frac{n+1}{2}}\right)$, and $\omega \in Sp(Z)$ implies $c_1(\omega^1 + \omega^{n-1}) + c_2(\omega^2 + \omega^{n-2}) + \ldots + c_{\frac{n-1}{2}}\left(\omega^{\frac{n-1}{2}} + \omega^{\frac{n+1}{2}}\right) \in Sp(A)$. By Lemma (4.2.1),

$$\begin{aligned} Sp(A) &= \left\{ 2c_1 \cos\frac{2k\pi}{n} + \ldots + 2c_{\frac{n-1}{2}} \cos\frac{2k\left(\frac{n-1}{2}\right)\pi}{n} : 1 \leq k \leq n \right\} \\ &= \left\{ \sum_{r=1}^{\frac{n-1}{2}} 2c_r \cos\frac{2kr\pi}{n} : 1 \leq k \leq n \right\}. \end{aligned}$$

If n is even, then $A = c_1(Z + Z^{n-1}) + \ldots + c_{\frac{n-2}{2}}\left(Z^{\frac{n-2}{2}} + Z^{\frac{n+2}{2}}\right) + c_{\frac{n}{2}}\left(Z^{\frac{n}{2}}\right)$, and $\omega \in Sp(Z)$ implies $c_1(\omega^1 + \omega^{n-1}) + \ldots + c_{\frac{n-2}{2}}\left(\omega^{\frac{n-2}{2}} + \omega^{\frac{n+2}{2}}\right) + c_{\frac{n}{2}}\left(\omega^{\frac{n}{2}}\right) \in Sp(A)$. By Lemma (4.2.1),

$$\begin{aligned} Sp(A) &= \left\{ 2c_1 \cos\frac{2k\pi}{n} + \ldots + 2c_{\frac{n-2}{2}} \cos\frac{2k\left(\frac{n-2}{2}\right)\pi}{n} + c_{\frac{n}{2}} \cos\frac{2k\left(\frac{n}{2}\right)\pi}{n} : 1 \leq k \leq n \right\} \\ &= \left\{ \sum_{r=1}^{\frac{n-2}{2}} 2c_r \cos\frac{2kr\pi}{n} + c_{\frac{n}{2}} \cos k\pi : 1 \leq k \leq n \right\}. \end{aligned}$$

□

The regularity d of a circulant graph affects the number of its vertices n. If d is odd, then n is even. If d is even, then n is either odd or even. Note that the value of d depends on the number of nonzero c_r. The position r of c_r affects the connectivity of a circulant graph. This relation will be explained in the next section.

4.3 Spectral Properties

The spectrum of a circulant graph indicates whether it is connected or not. The connectivity also depends on an algebraic relation between parameters r and n in Theorem (4.2.2). We begin this section by reviewing some number theory concepts. These backgrounds enable us to prove some interesting spectral properties of circulant graphs.

4.3.1 Number Theory Backgrounds

The following fundamental definitions and theorems are based on [Kos07].

Definition 4.3.1 (**divisor**)**.** An integer a is a **divisor** of an integer b if $ax = b$ for some integer x. We write $a|b$.

Definition 4.3.2 (**relatively prime**)**.** Two positive integers, a and b, are **relatively prime** if 1 is the greatest common divisor of a and b; that is, if $\gcd(a, b) = 1$.

Theorem 4.3.3. *Two positive integers, a and b, are relatively prime if and only if there are integers x and y such that $ax + by = 1$.*

Proof. see [Kos07] pp.161.

□

Definition 4.3.4 (**congruence**)**.** An integer a is **congruent** to an integer b modulo m if $m|(a - b)$. In symbols, we write $a \equiv b \pmod{m}$.

Theorem 4.3.5. $a \equiv b \pmod{m}$ *if and only if $a = b + km$ for some integer k.*

Proof. see [Kos07] pp.231.

□

4.3.2 Connectedness of $C_{3,n}$

We want to consider connected $C_{d,n}$, because if it is not connected, then its expander parameter is automatically equal to 0. We first examine the case of $C_{3,n}$.

Theorem 4.3.6. *Let $Sp(A(C_{3,n})) = \left\{2\cos\frac{2rk\pi}{n} + \cos k\pi : 1 \leq k \leq n\right\}$ for a unique r such that $1 \leq r \leq \frac{n}{2} - 1$. Then $C_{3,n}$ is connected if and only if $\gcd(r, \frac{n}{2}) = 1$.*

Proof. Since the regularity of $C_{3,n}$ is odd, n is even. Let $\frac{n}{2} = m$. We divide the proof into two parts.

<u>Part 1</u>: If $\gcd(r, m) = 1$, then $C_{3,n}$ is connected. We prove by contradiction.

Assume $\gcd(r, m) = 1$, but $C_{3,n}$ is disconnected. We will show that this assumption leads to a contradiction. Let $Sp(A(C_{3,n})) = \{\lambda_i : i = 1, 2, \ldots, n\}$. Notice that $\lambda_1 = 3$ when $k = n = 2m$. By Theorem (1.2.6), $C_{3,n}$ is connected if and only if $\lambda_2 = \max\left\{2\cos\frac{2rk\pi}{n} + \cos k\pi : 1 \leq k \leq n-1\right\} < 3$. There are two cases to consider.

Case 1: k is odd.

We have $2\cos\frac{2rk\pi}{n} + \cos k\pi = 2\cos\frac{2rk\pi}{n} - 1 < 3$, this automatically makes $C_{3,n}$ connected.

Case 2: k is even

Let $k = 2t$ for some integer t such that $1 \leq t \leq m-1$. Since $C_{3,n}$ is assumed to be disconnected, $\lambda_2 = \max\left\{2\cos\frac{2rk\pi}{n} + \cos k\pi : 1 \leq k \leq n-1\right\} = 3$, so

$$\begin{aligned}
\cos\frac{2rk\pi}{n} = 1 &\iff \frac{2rk\pi}{n} = 2l\pi \quad \text{for some integer } l \geq 1 \\
&\iff 2rk\pi = 2l\pi n \\
&\iff rk = ln \\
&\iff n|rk \\
&\iff m|rt \quad \text{because } n = 2m \text{ and } k = 2t \\
&\iff rt \equiv 0 \pmod{m} \quad \text{by Theorem (4.3.5).}
\end{aligned}$$

Since $\gcd(r, m) = 1$, r and m do not share common prime factors. This means the least value of t which makes rt a multiple of m is $t = m$, a contradiction to the fact that $1 \leq t \leq m-1$. Therefore, if $\gcd(r, m) = 1$, then $C_{3,n}$ is connected.

<u>Part 2</u>: If $C_{3,n}$ is connected, then $\gcd(r, m) = 1$. We prove by contrapositive.

Assume $\gcd(r, m) \neq 1$. We want to show that $C_{3,n}$ is not connected. By the assumption, $r = ta$ and $m = tb$ for some integers a, b with $t > 1$. Choose $k = 2b$, then $k \leq tb = m < n$. Consider the spectrum of $A(C_{3,n})$

$$\left\{2\cos\frac{2rk\pi}{n} + \cos k\pi : 1 \leq k \leq n\right\}.$$

Replacing $r = ta, m = tb$, and $k = 2b$ yields $2\cos\dfrac{2rk\pi}{n} + \cos k\pi = 3$. Notice that $2\cos\dfrac{2rk\pi}{n} + \cos k\pi = 3$ when $k = n = 2m$. Let $Sp(A(C_{3,n})) = \{\lambda_i : i = 1, 2, \ldots, n\}$, then $\lambda_1 = 3 = \lambda_2$ implies $C_{3,n}$ is not connected by Theorem (1.2.6). Therefore, if $C_{3,n}$ is connected, then $\gcd(r, m) = 1$.

□

Several calculations reveal that the second largest eigenvalue of $A(C_{3,n})$ is always 1 when $C_{3,n}$ is connected. This sparks our interest in finding whether the same result holds for other regular circulant graphs. We discover that if the first row of $A(C_{d,n})$ is $[0\ 0\ 1 \ldots 1\ 0]$, then the second eigenvalue equals to 1.

Theorem 4.3.7. *Let $n \geq 6$ be an even integer. If A is the adjacency matrix of $C_{n-3,n}$ with its first row being $[0\ 0\ 1 \ldots 1\ 0]$, then $\lambda_2(A) = 1$.*

Proof. Suppose A is the adjacency matrix of $C_{n-3,n}$ with its first row being $[0\ 0\ 1 \ldots 1\ 0]$. According to the proof of Theorem (4.2.2), $A = Z^2 + Z^3 + \ldots + Z^{n-2}$. If $\omega = e^{\frac{2\pi k \mathbf{i}}{n}}$ for $1 \leq k \leq n-1$, then

$$
\begin{aligned}
\lambda_2(A) &= \max_{1\le k\le n-1} \omega^2+\omega^3+\ldots+\omega^{n-2} \\
&= \max_{1\le k\le n-1} \omega^2\left(1+\omega+\ldots+\omega^{n-4}\right) \\
&= \max_{1\le k\le n-1} \omega^2\left(\frac{1-\omega^{n-3}}{1-\omega}\right) \quad \text{geometric series sum} \\
&= \max_{1\le k\le n-1} \frac{\omega^2-\omega^{n-1}}{1-\omega} \\
&= \max_{1\le k\le n-1} \frac{\cos\frac{4\pi k}{n}-\cos\frac{2\pi k}{n}}{1-\cos\frac{2\pi k}{n}} \\
&= \max_{1\le k\le n-1} \frac{\cos 2\theta - cos\theta}{1-\cos\theta} \quad \text{for } \theta=\frac{2\pi k}{n} \\
&= \max_{1\le k\le n-1} \frac{2\cos^2\theta-1-\cos\theta}{1-\cos\theta} \\
&= \max_{1\le k\le n-1} \frac{(2\cos\theta+1)(\cos\theta-1)}{1-\cos\theta} \\
&= \max_{1\le k\le n-1} -(2\cos\theta+1).
\end{aligned}
$$

Let $f(k) = -(2\cos\theta+1)$. To find $\lambda_2(A)$, we compute the maximum value of $f(k)$. This computation, however, is the same as finding the minimum value of $\cos\theta$ for a valid k. Notice that $\cos\theta = \cos\frac{2\pi k}{n} = -1$ implies $\frac{2\pi k}{n} = \pi, 3\pi, \ldots$ Since $1 \le k \le n-1$, $k = \dfrac{n}{2}$ is the only value that maximizes $f(k)$. Thus $\lambda_2(A) = f(\frac{n}{2}) = 1$.

□

Unfortunately, the same result doesn't hold for odd n as described in Table (4.1). In Chapter 5, Theorem (4.3.6) will be used to find explicit values of the second largest eigenvalues of connected 3-regular circulant graphs.

Table 4.1: Value of λ_2 of a particular circulant graph

n	1st row of A	λ_2
6	[0 0 1 1 1 0]	1
7	[0 0 1 1 1 1 0]	0.8019
8	[0 0 1 1 1 1 1 0]	1
9	[0 0 1 1 1 1 1 1 0]	0.8794
10	[0 0 1 1 1 1 1 1 1 0]	1
11	[0 0 1 1 1 1 1 1 1 1 0]	0.9190
12	[0 0 1 1 1 1 1 1 1 1 1 0]	1
$\vdots$	$\vdots$	$\vdots$
50	[0 0 1 . . . 1 0]	1
100	[0 0 1 . . . 1 0]	1

CHAPTER 5

FAMILY OF D-REGULAR CIRCULANT GRAPHS

In this chapter, we compute expander parameters of the family of d-regular circulant graphs. To do so, we calculate the second largest eigenvalues and apply Theorem (2.1.1). We consider values of $d \geq 3$ because computational results for $d = 0, 1$, or 2 are known. When $d = 0$, we have a graph with just one vertex and no edge. When $d = 1$, we have K_2. When $d = 2$, we have C_n; its expander parameter has been calculated in Example (1.1.16).

5.1 $d = 3$

It is not possible to picture a 3-regular circulant graph on n vertices as n gets larger, let alone to compute its expander parameter directly from the definition. $C_{3,n}$'s expander parameter, in a sense, can be estimated using Cheeger's inequalities. To do so, we need to find its second largest eigenvalue. The following lemma is crucial to establish the explicit value of $\lambda_2(C_{3,n})$.

Lemma 5.1.1. *Let r and m be two positive integers. If $\gcd(r, m) = 1$, then $rk_i \not\equiv rk_j \pmod{2m}$ for every $k_i \neq k_j \in \{2, 4, \ldots, 2m-2\}$.*

Proof. Assume $\gcd(r, m) = 1$, and there exist k_1 and k_2 such that $rk_1 \equiv rk_2 \pmod{2m}$. Then $r(k_1 - k_2) \equiv 0 \pmod{2m}$. This implies $2m | r(k_1 - k_2)$. Since k_1 and k_2 are even, $k_1 - k_2 = 2x$ for some integer x. Note that $1 \leq x \leq m - 2$ because

- $\min_{i,j} |k_i - k_j| = 2$.
- $\max_{i,j} |k_i - k_j| = 2m - 2 - 2 = 2(m-2)$.

$2m|r(k_1 - k_2) \iff 2m|2rx \iff m|rx$. Since $\gcd(r,m) = 1$, there exist integers α and β such that $m\alpha + r\beta = 1$. Multiply x to both sides yields $xm\alpha + xr\beta = x$. Since $m|xm\alpha$ and $m|xr\beta$, $m|(xm\alpha + xr\beta)$, and thus $m|x$. However, $m|x$ contradicts the fact that $1 \le x \le m-2$. By contradiction, $rk_1 \not\equiv rk_2 \pmod{2m}$.

□

Recall that $Sp(A(C_{3,n})) = \left\{2\cos\frac{2rk\pi}{n} + \cos k\pi : 1 \le k \le n\right\}$ for a unique r such that $1 \le r \le m-1$ where $m = \frac{n}{2}$. We have $\lambda_1(A(C_{3,n}) = 3$ when $k = n = 2m$, and $\lambda_2(A(C_{3,n}) = \max\left\{2\cos\frac{2rk\pi}{n} + \cos k\pi : 1 \le k \le n-1\right\} < 3$ if and only if $C_{3,n}$ is connected.

Theorem 5.1.2. *If $C_{3,n}$ is connected, then $\lambda_2(A(C_{3,n})) = 2\cos\frac{2\pi}{m} + 1$ where $m = \frac{n}{2}$.*

Proof. Assume $C_{3,n}$ is connected. Connectedness occurs when $\gcd(r,m) = 1$, by Theorem (4.3.6). Now $\gcd(r,m) = 1$ implies the smallest value of $\{2r, 4r, \ldots, (2m-2)r\}$ is 2 by Lemma (5.1.1). The computation of $\lambda_2(A(C_{3,n}))$ is done as follows.

$$\begin{aligned}
\lambda_2(A(C_{3,n})) &= \max_{1\le k\le n-1} 2\cos\frac{2\pi rk}{n} + \cos\pi k \\
&= \max_{1\le k\le 2m-1} 2\cos\frac{2\pi rk}{2m} + \cos\pi k \\
&= \max_{\substack{2\le k\le 2m-2 \\ k \text{ is even}}} 2\cos\frac{rk\pi}{m} + 1 \\
&= 2\cos\frac{2\pi}{m} + 1
\end{aligned}$$

□

Note that $\lim_{n\to\infty} \lambda_2(A(C_{3,n})) = \lim_{m\to\infty} 2\cos\frac{2\pi}{m} + 1 = 2\cos 0 + 1 = 2 + 1 = 3$. We next examine the 4-regular circulant graph $C_{4,n}$.

5.2 d = 4

Because $C_{4,n}$ is 4-regular, n can be either odd or even. For instance, the first row of the circulant adjacency matrix of $C_{4,n}$ is $[0\ 1\ 1\ 0\ 1\ 1]$ for $n = 6$, and either $[0\ 1\ 1\ 0\ 0\ 1\ 1]$ or $[0\ 0\ 1\ 1\ 1\ 1\ 0]$ for $n = 7$. Let A be the circulant adjacency matrix of $C_{4,n}$. Theorem (4.2.2) gives

$$Sp(A) = \left\{ 2\cos\frac{2\pi r_1 k}{n} + 2\cos\frac{2\pi r_2 k}{n} : 1 \leq k \leq n \right\}$$

for some r_1, r_2 such that $1 \leq r_1 < r_2 \leq \dfrac{n-1}{2}$.

The values of $\lambda_2(A)$ are different with respect to different pairs of r_1, r_2. This behavior is shown in Table (5.1) in which we record all possible values of $\lambda_2(A)$ for $n = 11$ and $n = 12$. According to Table (5.1), there exist repeating values of $\lambda_2(A)$ for certain pairs of r_1, r_2, but this occurrence is completely random.

Recall that $C_{4,n}$ is connected if $\lambda_2(A) < 4$, and disconnected if $\lambda_2(A) = 4$. To ignore the connectedness issues, we consider the minimum value of $\lambda_2(A)$ over all pairs of r_1, r_2. Theoretically, this number is less than 4. But can it equal to 4 in the limit as $n \to \infty$?

Let $\min\{\lambda_2(A)\}$ denote the minimum value of $\lambda_2(A)$ over all pairs of r_1, r_2. As the number of vertices n increases, $\min\{\lambda_2(A)\}$ also increases. For example, $\min\{\lambda_2(A)\} = 1$ when $n = 10$, $\min\{\lambda_2(A)\} = 1.6180$ when $n = 15$, and $\min\{\lambda_2(A)\} = 2$ when $n = 18$. We predict that $\min\{\lambda_2(A)\}$ approaches 4 as n goes to infinity. Our prediction is supported by Figure (5.1).

For easy use of notation, we define $\triangle_4(n)$ to be $\min\{\lambda_2(A)\}$, explicitly

$$\triangle_4(n) = \min_{1 \leq r_1 < r_2 \leq \frac{n-1}{2}} \quad \max_{1 \leq k \leq n-1} \quad 2\cos\frac{2\pi r_1 k}{n} + 2\cos\frac{2\pi r_2 k}{n}.$$

Computing the exact value of $\triangle_4(n)$ can be frustratingly tedious. A clever method is to estimate $\triangle_4(n)$ with a function depending only on n. In the next

Table 5.1: $\lambda_2(A)$ for $n = 11, 12$. There are repeating values, but they occur randomly.

n	1st row of A i.e. $[0\ c_1\ c_2 \ldots c_2\ c_1]$	value of $\lambda_2(A)$
11	$c_1 = c_2 = 1$, other $c_i = 0$	2.5131
11	$c_1 = c_3 = 1$, other $c_i = 0$	1.3979
11	$c_1 = c_4 = 1$, other $c_i = 0$	1.3979
11	$c_1 = c_5 = 1$, other $c_i = 0$	2.5131
11	$c_2 = c_5 = 1$, other $c_i = 0$	1.3979
11	$c_3 = c_5 = 1$, other $c_i = 0$	2.5131
11	$c_4 = c_5 = 1$, other $c_i = 0$	1.3979
11	$c_4 = c_3 = 1$, other $c_i = 0$	2.5131
11	$c_4 = c_2 = 1$, other $c_i = 0$	2.5131
11	$c_3 = c_2 = 1$, other $c_i = 0$	1.3979
12	$c_1 = c_2 = 1$, other $c_i = 0$	2.7321
12	$c_1 = c_3 = 1$, other $c_i = 0$	1.7321
12	$c_1 = c_4 = 1$, other $c_i = 0$	2
12	$c_1 = c_5 = 1$, other $c_i = 0$	2
12	$c_2 = c_5 = 1$, other $c_i = 0$	2.7321
12	$c_3 = c_5 = 1$, other $c_i = 0$	1.7321
12	$c_4 = c_5 = 1$, other $c_i = 0$	2
12	$c_4 = c_3 = 1$, other $c_i = 0$	2
12	$c_4 = c_2 = 1$, other $c_i = 0$	4
12	$c_3 = c_2 = 1$, other $c_i = 0$	1

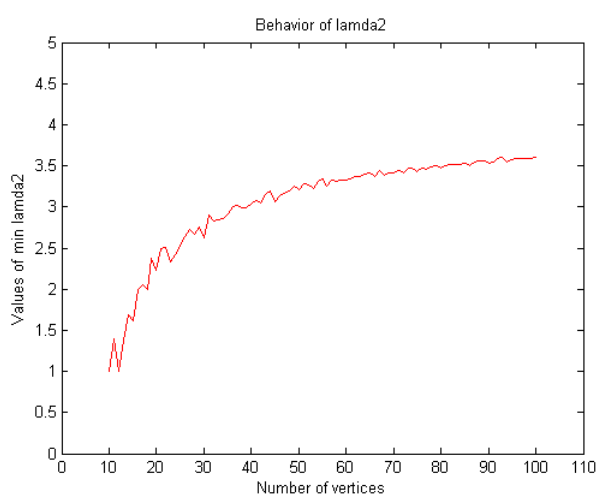

Figure 5.1: A plot of minimum values of $\lambda_2(A)$ for $n = 10, 11, \ldots, 100$. It shows $\min\{\lambda_2(A)\}$ is approaching 4.

section, we will show that $\triangle_4(n) \geq 4 - \dfrac{8\pi^2}{\left(\sqrt{n-1}-1\right)^2}$. Figure (5.2) (left) depicts $4 - \dfrac{8\pi^2}{\left(\sqrt{n-1}-1\right)^2}$ approaching 4 as n goes to infinity. Figure (5.2) (right) compares the estimated values with the real values of $\triangle_4(n)$. Notice that the estimated graph is smoother, and it lies below the real graph of $\triangle_4(n)$.

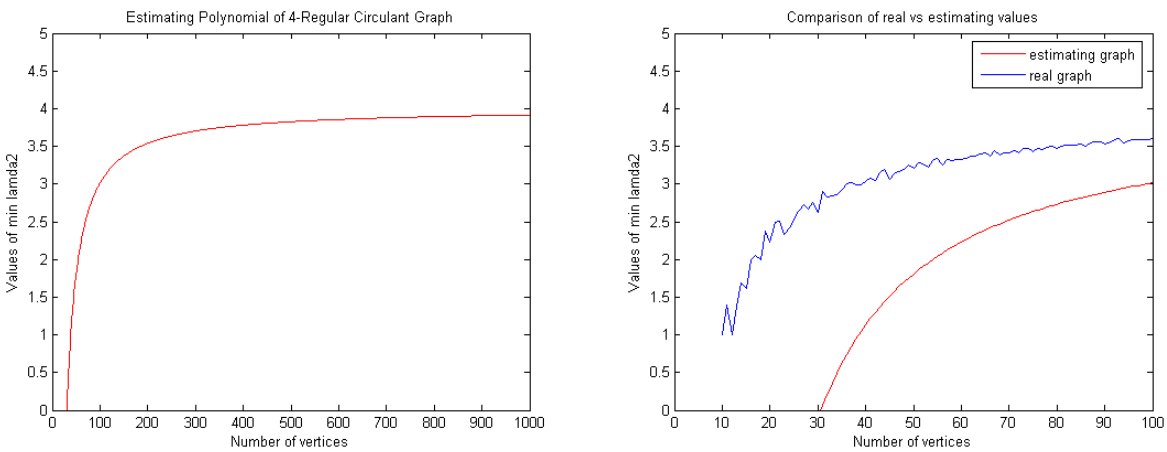

Figure 5.2: The left figure is the plot of the estimated function of $\triangle_4(n)$. The curve is smooth, and it is approaching 4. The right figure shows the plot generated by the estimated function of $\triangle_4(n)$ lying below the plot of the true values of $\triangle_4(n)$.

5.3 Results

In this section, we show that circulant graphs do not contain an expander family. The following results are essential to develop estimating functions of $\triangle_4(n)$ and in general $\triangle_d(n)$. Using the Pigeonhole Principle, which informally states: "if a flock n pigeons comes to roost in a house with r pigeonholes and $n > r$, then at least one hole contains more than one pigeon", we prove the first result.

Theorem 5.3.1. *Given N real numbers $a_1, a_2, \ldots, a_N$ and a positive integer q, we*

can find an integer t in the range $1 \le t \le q^N$ and integers $x_1, x_2, \ldots, x_N$ such that

$$|ta_i - x_i| \le \frac{1}{q} \quad \textit{for } i = 1, 2, \ldots, N.$$

Proof. Consider the q^N compartments $\left[\frac{i_1}{q}, \frac{i_1+1}{q}\right) \times \ldots \times \left[\frac{i_N}{q}, \frac{i_N+1}{q}\right)$ for $0 \le i_1, \ldots i_N \le q-1$ of the N-dimensional unit cube obtained by partitioning each edge adjacent to the origin into q parts. Also consider (q^N+1) N-dimensional points, $(ua_1 - \lfloor ua_1 \rfloor, \ldots, ua_N - \lfloor ua_N \rfloor)$ where $0 \le u \le q^N$ and $\lfloor ua \rfloor$ denotes the largest integer not greater than ua, inside the N-dimensional unit cube. By the Pigeonhole Principle, there are two points lying in the same compartment. That is, there exist integers u_1 and u_2 such that $0 \le u_1 < u_2 \le q^N$ and $0 \le i_1, \ldots, i_N \le q-1$ satisfying

$$(u_1a_1 - \lfloor u_1a_1 \rfloor, \ldots, u_1a_N - \lfloor u_1a_N \rfloor) \in \left[\frac{i_1}{q}, \frac{i_1+1}{q}\right) \times \ldots \times \left[\frac{i_N}{q}, \frac{i_N+1}{q}\right).$$
$$(u_2a_1 - \lfloor u_2a_1 \rfloor, \ldots, u_2a_N - \lfloor u_2a_N \rfloor) \in \left[\frac{i_1}{q}, \frac{i_1+1}{q}\right) \times \ldots \times \left[\frac{i_N}{q}, \frac{i_N+1}{q}\right).$$

Note that

$$u_1a_1 - \lfloor u_1a_1 \rfloor, u_2a_1 - \lfloor u_2a_1 \rfloor \in \left[\frac{i_1}{q}, \frac{i_1+1}{q}\right)$$
$$\vdots$$
$$u_1a_N - \lfloor u_1a_N \rfloor, u_2a_N - \lfloor u_2a_N \rfloor \in \left[\frac{i_N}{q}, \frac{i_N+1}{q}\right).$$

Thus for $1 \le i \le N$,

$$|(u_2a_i - \lfloor u_2a_i \rfloor) - (u_1a_i - \lfloor u_1a_i \rfloor)| \le \frac{1}{q}$$
$$|(u_2 - u_1)a_i - (\lfloor u_2a_i \rfloor - \lfloor u_1a_i \rfloor)| \le \frac{1}{q}$$
$$|ta_i - x_i| \le \frac{1}{q}$$

where $1 \le t = u_2 - u_1 \le q^N$ and $x_i = \lfloor u_2a_i \rfloor - \lfloor u_1a_i \rfloor \in Z$.

□

Theorem (5.3.1) is a special case of the well-known Dirichlet Theorem from [Tit51] pp.152-153. It can be applied to an even more specific case.

Corollary 5.3.2. *Given positive integers n, t, and $r_1, r_2, \ldots, r_t$, there exist an integer k in the range $1 \leq k \leq n-1$ and integers $x_1, \ldots, x_t$ such that*

$$\left| \frac{kr_i}{n} - x_i \right| \leq \frac{1}{(n-1)^{\frac{1}{t}} - 1} \quad \textit{for } i = 1, 2, \ldots, t.$$

Proof. Let $\frac{r_i}{n} = a_i$ where $i = 1, 2, 3, \ldots, t = N$ and $q = \left\lfloor (n-1)^{\frac{1}{t}} \right\rfloor$. Theorem (5.3.1) implies there exist an integer k in the range $1 \leq k \leq q^t \leq n-1$ and integers $x_1, \ldots, x_t$ such that

$$\begin{aligned} \left| \frac{kr_i}{n} - x_i \right| &\leq \frac{1}{q} \\ &= \frac{1}{\left\lfloor (n-1)^{\frac{1}{t}} \right\rfloor} \\ &\leq \frac{1}{(n-1)^{\frac{1}{t}} - 1}. \end{aligned}$$

□

It remains to approximate the cosine of an angle. The approximation is relevant because the spectrum of a circulant graph is computed in terms of cosines.

Lemma 5.3.3. $\cos\theta \geq 1 - \frac{\theta^2}{2}$ *for any* θ.

Proof. We want to show $\cos\theta - 1 + \frac{\theta^2}{2} \geq 0$. Let $f(\theta) = \cos\theta - 1 + \frac{\theta^2}{2}$. It suffices to prove $f(\theta)$ is an increasing function which has a minimum value at $\theta = 0$. Since $f'(\theta) = \theta - \sin\theta \geq 0$, $f(\theta)$ is an increasing function. Setting $f'(\theta) = 0$ to get $\theta = 0$. Now substitute $\theta = 0$ into $f(\theta)$ to get $f(0) = 0$, and thus $f(\theta)$ has a minimum value at $\theta = 0$.

□

5.3.1 $d = 4$ Revisited

We apply the preceding results to estimate $\cos\left(2\pi\frac{kr_i}{n}\right)$ for $i = 1, 2$.

$$\begin{aligned}
\cos\left(2\pi\frac{kr_i}{n}\right) &= \cos\left(2\pi\left(\frac{kr_i}{n} - x_i\right)\right) \quad \text{where } x_i \text{ is the integer from Corollary (5.3.2)} \\
&\geq 1 - 2\pi^2\left|\frac{kr_i}{n} - x_i\right|^2 \quad \text{by Lemma (5.3.3)} \\
&\geq 1 - 2\pi^2\left(\frac{1}{\sqrt{n-1}-1}\right)^2 \quad \text{by Corollary (5.3.2) with } t = 2.
\end{aligned}$$

We can now estimate $\triangle_4(n)$.

$$\begin{aligned}
\triangle_4(n) &= \min_{1\leq r_1<r_2\leq\frac{n-1}{2}} \quad \max_{1\leq k\leq n-1} \quad 2\cos 2\pi\frac{r_1k}{n} + 2\cos 2\pi\frac{r_2k}{n} \\
&\geq 2\left(1 - 2\pi^2\frac{1}{(\sqrt{n-1}-1)^2}\right) + 2\left(1 - 2\pi^2\frac{1}{(\sqrt{n-1}-1)^2}\right) \\
&= 4 - \frac{8\pi^2}{\left(\sqrt{n-1}-1\right)^2}.
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{1}{\sqrt{n-1}-1} = 0$, $\lim_{n\to\infty}\triangle_4(n) \geq 4$. The largest eigenvalue of a 4-regular circulant graph on n vertices is 4. Thus $\lim_{n\to\infty}\triangle_4(n) = 4$.

Using similar approximation steps, we estimate $\triangle_d(n)$ for both odd and even values of d. We assume henceforth d is a fixed integer i.e. $d < \infty$. When d is odd, there are $\frac{d-1}{2}$ pairs of $\cos\left(2\pi\frac{r_ik}{n}\right)$ for $i = 1, 2, \ldots, \frac{d-1}{2}$, plus the extra term of $\cos k\pi$. When d is even, the calculation is less messy because there are exactly $\frac{d}{2}$ pairs of $\cos\left(2\pi\frac{r_ik}{n}\right)$ for $i = 1, 2, \ldots, \frac{d}{2}$. Given this information, we estimate $\triangle_d(n)$ in the following subsection.

5.3.2 Main Theorem

Theorem 5.3.4.

$$\lim_{n\to\infty}\triangle_d(n) = d$$

Proof. <u>Case 1</u>: d is odd.

$$\begin{aligned}
\triangle_d(n) &= \min_{1\le r_1<\ldots<r_{\frac{d-1}{2}}\le\frac{n-2}{2}} \quad \max_{1\le k\le n-1} \quad 2\cos\left(2\pi\frac{r_1k}{n}\right)+\cdots+2\cos\left(2\pi\frac{r_{\frac{d-1}{2}}k}{n}\right)+\cos k\pi \\
&\ge 2\left(1-2\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d-1}}-1}\right)^2\right)+\cdots+2\left(1-2\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d-1}}-1}\right)^2\right)+1 \\
&= 2\left(\frac{d-1}{2}\right)\left(1-2\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d-1}}-1}\right)^2\right)+1 \\
&= d-2(d-1)\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d-1}}-1}\right)^2.
\end{aligned}$$

Because d is fixed, $lim_{n\to\infty}\left(\frac{1}{(n-1)^{\frac{2}{d-1}}-1}\right)=0$.

<u>Case 2</u>: d is even.

$$\begin{aligned}
\triangle_d(n) &= \min_{1\le r_1<\ldots<r_{\frac{d}{2}}\le\frac{n-1}{2}} \quad \max_{1\le k\le n-1} \quad 2\cos\left(2\pi\frac{r_1k}{n}\right)+\cdots+2\cos\left(2\pi\frac{r_{\frac{d}{2}}k}{n}\right) \\
&\ge 2\left(1-2\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d}}-1}\right)^2\right)+\cdots+2\left(1-2\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d}}-1}\right)^2\right) \\
&= 2\left(\frac{d}{2}\right)\left(1-2\pi^2\left(\frac{1}{(n-1)^{\frac{2}{d}}-1}\right)^2\right) \\
&= d-2\pi^2 d\left(\frac{1}{(n-1)^{\frac{2}{d}}-1}\right)^2.
\end{aligned}$$

Because d is fixed, $lim_{n\to\infty}\left(\frac{1}{(n-1)^{\frac{2}{d}}-1}\right)=0$. Combining the two cases, we have $\lim_{n\to\infty}\triangle_d(n)\ge d$. Since the largest eigenvalue of a d-regular circulant graph is d, $\lim_{n\to\infty}\triangle_d(n)=d$.

□

5.3.3 Conclusion

Cheeger's inequalities and Theorem (5.3.4) yield

$$\lim_{n\to\infty} h(C_{d,n}) \leq \lim_{n\to\infty} \sqrt{2d(d-\lambda_2(A(C_{d,n})))} \leq \lim_{n\to\infty} \sqrt{2d(d-\triangle_d(n))} = 0.$$

Since $h(C_{d,n})$ is nonnegative, $\lim_{n\to\infty} h(C_{d,n}) = 0$. This means circulant graphs do not contain an expander family according to Definition(1.1.12).

There may be a better way to approximate $\lambda_2(A(C_{d,n}))$. Our method eliminates the difficulty in choosing r_i, and it omits the connectedness issues of $C_{d,n}$. Having many inequalities involved in the approximation may weaken our method. In particular, estimated values of $\lambda_2(A(C_{d,n}))$ are very near d or equal to d most of the time. However, we are only interested in the case when n goes to infinity, so a better method is sufficient but not necessary.

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APPENDIX A

MATLAB CODE TO GENERATE FIGURE (5.2)

```
function [tttt,poly]= comparison()
tttt=[];
poly=[];
for n=10:100
    m=floor((n-1)/2);
    a=n-1;
    b=m*(m-1)/2;
    tt=[];
        for i=1:m
        for j=1:m
        if i =j  i¡j
        for k=1:a
        z(k)=2*cos(2*pi*i*k/n) + 2*cos(2*pi*j*k/n);
        end
        for l=1:b
        x(l)=max(z);
        end
        ta=min(x);
```

```
            tt=[tt,ta];
            end
            end
            end
            ttt=min(tt);
            tttt=[tttt,ttt];
end

      for n=10:10000
      t(n) = 4 - ((8*pi^2)/(sqrt(n - 1) - 1)^2);
end
poly = t(1, 10 : 100);
x = [10 : 100];
f(x) = poly;
g(x) = tttt;
plot(x, f(x),' r', x, g(x),' b')
axis([0 100 0 5])
xlabel('Number of vertices')
ylabel('Values of min lamda2')
title('Comparison of real vs estimating values')
legend('estimating graph', 'real graph')
```