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Random walks on wheels

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[GTN XXXIII:6] RANDOM WALKS ON WHEELS†

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Abstract

Suppose two particles occupy distinct vertices of a wheel graph and at each step the two particles move independently to adjacent vertices. In this paper we find the expected number of moves until the particles land on the same vertex.

1. Introduction

Consider a random walk of two particles on a graph. At each time step the particles move independently to adjacent vertices with all possible moves equally likely. This random walk terminates when the particles first collide; that is, when the particles land on the same vertex. For any initial positions of the particles we would like to determine the expected number of moves until the particles collide. For odd cycle graphs the solution appears in [1] and in [2] we give the solution for all cycle graphs.

The *wheel graph* W_n consists of a cycle of length $n-1$ with all the vertices on the cycle connected to a single *hub*. The *pseudo-wheel* consists of a cycle graph on $2n$ vertices, with n additional edges connecting vertices on opposite sides of the cycle. The *de Bruijn graph* is a directed graph on 2^n vertices, where the vertices are labeled with the binary n-tuples. Each node has in-degree and out-degree equal to 2, with directed edges from $(b_{n-1}, b_{n-2}, \ldots, b_0)$ to $(b_{n-2}, \ldots, b_0, c)$, for $c = 0, 1$. The *hypercube* is an undirected graph with 2^n vertices where, as with the de Bruijn graph, the vertices are labeled with the binary n-tuples. There is an edge between vertices that differ in precisely one bit position. In [2] we give the complete solutions for wheels, pseudo-wheels, de Bruijn graphs and hypercubes. Here we summarize the results for the wheels.

2. Wheel Graphs

Let W_n denote the wheel graph of order n. Let x_k be the expected number of time steps until two particles that start at a distance k collide, where the distance is measured around the cycle or *rim* of the wheel. Also, let x_m be the expected number of moves until a collision when one particle starts on the hub and one on the rim of the wheel.

For $n > 1$, the wheels W_{4n+1} have expectations satisfying the $(2n+1) \times (2n+1)$ linear system shown below. It can be verified using Maple — see [2] for details — that:

$$x_{2k+1} = \tfrac{4}{5}x_m + \tfrac{9}{5}$$

$$x_{2k+2} = \left(\tfrac{4}{5}x_m + \tfrac{9}{5}\right)\left(1 - \tfrac{1}{2}\cdot\frac{(7-3\sqrt{5})\,\lambda_1^n\lambda_2^k + (7+3\sqrt{5})\,\lambda_1^k\lambda_2^n}{\lambda_1^n + \lambda_2^n}\right), \text{ and}$$

$$x_m = \tfrac{3}{4}\cdot\frac{(110n - 9\sqrt{5})\,\lambda_1^n + (110n + 9\sqrt{5})\,\lambda_2^n}{(5n + 3\sqrt{5})\,\lambda_1^n + (5n - 3\sqrt{5})\,\lambda_2^n},$$

where $\lambda_1 = \frac{7}{2} + \frac{3}{2}\sqrt{5}$ and $\lambda_2 = \frac{7}{2} - \frac{3}{2}\sqrt{5}$.

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$$\begin{bmatrix} 1 & -\frac{1}{6} & 0 & 0 & \cdots & & & & & & & -\frac{2}{3} \\ -\frac{1}{7} & 1 & -\frac{1}{7} & 0 & & & & & & & & -\frac{4}{7} \\ 0 & -\frac{1}{7} & 1 & -\frac{1}{7} & & & & & & & & -\frac{4}{7} \\ \vdots & & \ddots & \ddots & \ddots & & & & & & & \vdots \\ & & & -\frac{1}{7} & 1 & -\frac{1}{7} & & & & & & -\frac{4}{7} \\ & & & & -\frac{1}{6} & 1 & 0 & & & & & -\frac{2}{3} \\ & & & & & 0 & 1 & -\frac{1}{7} & & & & -\frac{4}{7} \\ & & & & & & -\frac{1}{7} & 1 & -\frac{1}{7} & & & -\frac{4}{7} \\ & & & & & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & & & & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{4}{7} \\ & & & & & & & & & -\frac{2}{7} & 1 & -\frac{4}{7} \\ -\frac{1}{2n} & -\frac{1}{2n} & \cdots & & & & & & \cdots & -\frac{1}{2n} & -\frac{1}{4n} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_5 \\ \vdots \\ x_{2n-3} \\ x_{2n-1} \\ x_2 \\ x_4 \\ \vdots \\ x_{2n-2} \\ x_{2n} \\ x_m \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{9}{7} \\ \frac{9}{7} \\ \vdots \\ \frac{9}{7} \\ \frac{3}{2} \\ \frac{9}{7} \\ \frac{9}{7} \\ \vdots \\ \frac{9}{7} \\ \frac{9}{7} \\ \frac{3}{2} \end{bmatrix}$$

The other wheel graphs yield similar results and in fact the following curious theorem follows immediately from the formulas for x_m (again, see [2] for details).

Theorem: Consider a random walk of two particles on the wheel graph W_n. Let X_n be the random variable for the number of moves until the two particles collide when one particle starts on the hub and one on the rim. Then

$$\lim_{n \to \infty} E(X_n) = \frac{33}{2}. \qquad \blacksquare$$

References

[1] J.L. Palacios; A double random walk on an odd cycle, Problem 10299, *The American Mathematical Monthly*, **103**, 81–82 (1996).

[2] M. Stamp and M. Lee; Random walks on graphs — in preparation.

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