July 2014

Antimagic-type labelings

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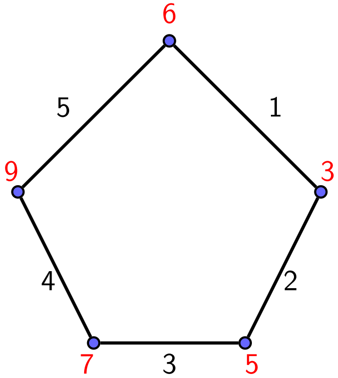
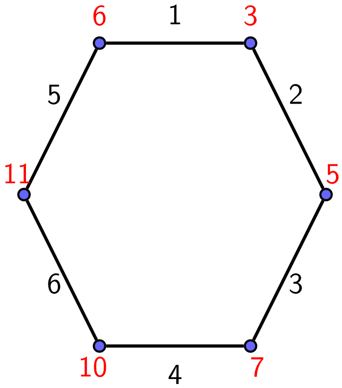
Rocky Mountain-Great Plains Graduate
Research Workshop in Combinatorics
Summer 2014

Definition

A graph with m edges is called antimagic if its edges can be labeled with $1, \ldots, m$ such that the sums of the labels of the edges incident to each vertex are distinct.




6
1
3
5
2
11
5
6
3
10
4
7

Observation

- (Hartsfield and Ringel [1990]) Cycles are antimagic.
- (Hartsfield and Ringel [1990]) Paths of length at least 2 are antimagic.
- (Hartsfield and Ringel [1990]) Complete graphs are antimagic.
- (Hartsfield and Ringel [1990]) Wheels are antimagic.

Some non-antimagic graphs

- Any graph having a K_2-component is not antimagic.

- Any graph having at least two isolated vertices is not antimagic.

- Theorem (Wang, Liu, and Li [2012]) mP_3 with $m \geq 2$ is not antimagic.

Conjectures

- Conjecture (Hartsfield and Ringel [1990]) Every tree except K_2 is antimagic.

- Conjecture (Hartsfield and Ringel [1990]) Every connected graph of order at least 3 is antimagic.

Results

- Theorem (Alon, Kaplan, Lev, Roditty, and Yuster [2004])
  There exists an absolute constant C such that every graph with n vertices and minimum degree at least $C \; log \; n$ is antimagic.

- Theorem (Alon, Kaplan, Lev, Roditty, and Yuster [2004])
  If G has n vertices, $n \geq 4$, and $\Delta(G) \geq n - 2$, then G is antimagic.

- Theorem (Alon, Kaplan, Lev, Roditty, and Yuster [2004])
  All complete partite graphs, but K_2, are antimagic.

Results

- Theorem (Yilma [2011]) n-vertex graphs of order at least 9 with maximum degree at least $n - 3$ are antimagic.

- Theorem (Cranston [2009]) Every regular bipartite graph with degree at least 2 is anitmagic.

- Theorem (Cranston, Liang, and Zhu [2013]) Regular graphs of odd degree, but K_2, are antimagic.

- Theorem (Eccles [2014+]) If a graph has no isolated edges or vertices and has average degree at least 4468, then it is antimagic.

- Conjecture (Eccles [2014+]) If a graph has no isolated edges or vertices and has average degree at least $\sqrt{2}$, then it is antimagic.

Let k be a positive integer and G be a graph.

Definition

We say that G is k-antimagic if there is an injection $f : E \longrightarrow \{1, \ldots, |E| + k\}$ such that for any two distinct vertices u and v, $\sum_{e \in \Gamma(v)} f(e) \neq \sum_{e \in \Gamma(u)} f(e)$.

Definition

We say G is weighted-k-antimagic if for any vertex weight function $g : E \longrightarrow \mathbb{N}$ there is an injection $f : E \longrightarrow \{1, \ldots, |E| + k\}$ such that for any two distinct vertices u and v,
$g(v) + \sum_{e \in \Gamma(v)} f(e) \neq g(u) + \sum_{e \in \Gamma(u)} f(e)$.

facts

- (Wong and Zhu) Not all graphs are weighted-0-antimagic.
- (Wong and Zhu) Not all graphs are weighted-1-antimagic.

Open Problems

- Question (Wong and Zhu [2011]) Is there a constant k such that every connected graph $G \neq K_2$ is weighted-k-antimagic?
- Question (Wong and Zhu [2011]) Is it true that every connected graph $G \neq K_2$ is weighted-2-antimagic?
- Question (Wong and Zhu [2011]) Is there a connected graph G with an odd number of vertices which is not weighted-1-antimagic?

Theorem (Combinatorial Nullstellensatz) (Alon [1999])
Let f be a polynomial of degree t in m variables over a field $\mathbb{F}$. If there is a monomial $\Pi x_i^{t_i}$ in f with $\sum t_i = t$ whose coefficient is nonzero in $\mathbb{F}$, then f is nonzero at some point of ΠS_i, where each S_i is a set of $t_i + 1$ distinct values in $\mathbb{F}$.

- It is an easy exercise using the Combinatorial Nullstellensatz to prove that every n-vertex connected graph with $n \geq 3$ is weighted-$(2n - 3)$-antimagic.

- Theorem (Wong and Zhu [2011]) every n-vertex connected graph with $n \geq 3$ is weighted-$\lfloor \frac{3n}{2} \rfloor$-antimagic.

- Theorem (Wong and Zhu [2011]) If G has a universal vertex and $G - newK_2$, then G is weighted-2-antimagic.

- Theorem (Wong and Zhu [2011]) If G has a prime number of vertices and has a Hamilton path, then G is weighted-1-antimagic.

Antimagic labeling of directed graphs

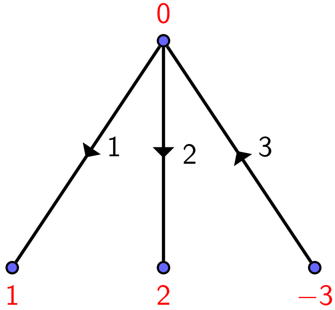
- Definition

  In an edge-labeling of a digraph D, the oriented vertex sum of a vertex v is the sum of labels of all edges entering v minus the sum of labels of all edges leaving it.

- Definition

  An antimagic labeling of a directed graph D with n vertices and m edges is a bijection from the set of edges of D to the integers $\{1, \ldots, m\}$ such that all n oriented vertex sums are pairwise distinct.

Example



Question (Hefetz, Mütze, and Schwartz [2009]) Is every connected digraph on at least four vertices antimagic?

- Theorem (Hefetz, Mütze, and Schwartz [2009]) There exists a constant C such that for every undirected graph on n vertices with minimum degree at least $C \, log \, n$ every orientation is antimagic.
- Theorem (Hefetz, Mütze, and Schwartz [2009]) Every orientation of W_n is antimagic.

- Question (Hefetz, Mütze, and Schwartz [2009]) Given any undirected graph G, does there exist an orientation of G which is antimagic?

- Conjecture (Hefetz, Mütze, and Schwartz [2009]) Every connected undirected graph admits an antimagic orientation.

- Theorem (Hefetz, Mütze, and Schwartz [2009]) Almost every undirected d-regular graph admits an orientation which is antimagic

- Theorem (Hefetz, Mütze, and Schwartz [2009]) Every regular graph of odd degree has an antimagic orientation.

- Theorem (Hefetz, Mütze, and Schwartz [2009]) Every n-vertex regular connected graph of even degree having a matching of size $\lfloor \frac{n}{2} \rfloor$ has an antimagic orientation.

- Theorem (Hefetz, Mütze, and Schwartz [2009]) For every orientation of complete graphs, wheels, and stars with at least 4 vertices, there exists an antimagic labeling.

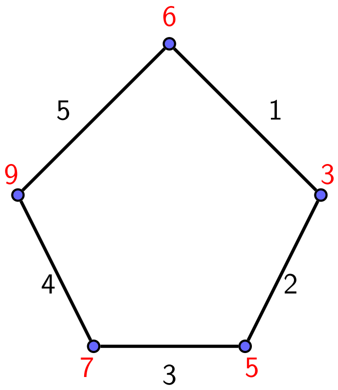
- Theorem (Hefetz, Mütze, and Schwartz [2009]) There exists an absolute constant C such that any n-vertex undirected graph G with minimum degree at least $C \ log \ n$ has an orientation which is antimagic.

Neighbor sum Distinguishing Index

Definitions

- A proper $[k]$-edge colorings of a graph G is called neighbor sum distinguishing if for any pair of adjacent vertices x and y the sum of colors taken on the edges incident to x is different from the sum of colors taken on the edges incident to y.
- The smallest value k for which G has a neighbor sum distinguishing coloring is called Neighbor sum distinguishing index of G and is denoted by $nsdi(G)$.

Example



We have $nsdi(C_5) = 5$.

- Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) $nsdi(P_k) = 3$ for all $k \geq 3$.

- Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) $nsdi(C_m) = 3$ when $3|m$.

- Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) $nsdi(C_m) = 4$ when $3 \not| m$ and $m \neq 5$.

- Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) $nsdi(K_{n,n}) = n + 2$ when $n \geq 2$.

- Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) $nsdi(K_{n,p}) = n$ when $n \geq 2$ and $n > p$.

- Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) Let T be a tree of order at least 3 and maximum degree Δ. We have $nsdi(T) = \Delta$, when vertices of degree Δ in T form an independent set. $nsdi(T) = \Delta + 1$, otherwise.

Conjecture (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013])
$nsdi(G) \leq \Delta(G) + 2$, where G is a connected graph of order at least 3 and $G \neq C_5$.

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Theorem (Flandrin, Marczyk, PrzybyĹo, Saclé, Woźniak [2013]) $nsdi(G) \leq \frac{7\Delta(G)}{2}$ for any graph G with $\Delta(G) \geq 2$.

Theorem (Wang and Yan [2014]) $nsdi(G) \leq \frac{5(\Delta(G)+2)}{2}$ for all graphs having no pendant edges.

Theorem (Dong and Wang [2012])
$nsdi(G) \leq max\{2\Delta(G) + 1, 25\}$ for all planar graphs having no pendant edges.

Theorem (Dong and Wang [2012]) $nsdi(G) \leq max\{2\Delta(G), 19\}$ for all planar graphs G with $mad(G) \leq 5$.

Theorem (Wang, Chen, and Wang [2014])
$nsdi(G) \leq max\{\Delta(G) + 10, 25\}$ for all planar graphs having no pendant edges.

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$$\Delta(G) \leq \chi'(G) \leq ndi(G) \leq nsdi(G)$$

Conjecture (Zhang and Wang [2002]) $ndi(G) \leq \Delta(G) + 2$, where G is a graph of order at least 6 having no pendant edges.

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Theorem (Hatami [2005]) $ndi(G) \leq \Delta(G) + 300$ for for any graph G with $\Delta(G) > 10^{20}$ having no pendant edges.

Theorem (Hatami [2005]) $ndi(G) \leq max\{14, \Delta(G) + 2\}$ for any planar graph G having no pendant edges.

So the Conjecture is valid for all planar graphs having maximum degree at least 12.

Thank you very much!