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Small gaps between almost primes, the parity problem, and some conjectures of Erdős on consecutive integers II

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General Section

Small gaps between almost primes, the parity problem, and some conjectures of Erdős on consecutive integers II

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ABSTRACT

We show that for any positive integer n, there is some fixed A such that $d(x) = d(x+n) = A$ infinitely often where $d(x)$ denotes the number of divisors of x. In fact, we establish the stronger result that both x and $x+n$ have the same fixed exponent pattern for infinitely many x. Here the exponent pattern of an integer $x > 1$ is the multiset of nonzero exponents which appear in the prime factorization of x.

1. Introduction

This paper is intended as a sequel to [GGPY11] written by four of the coauthors here. In the paper, they proved a stronger form of the Erdős-Mirksy conjecture mentioned in [EM52] which states that there are infinitely many positive integers x such that $d(x) = d(x+1)$ where $d(x)$ denotes the number of divisors of x. This conjecture was first proven by Heath-Brown in 1984 [HB84], but the method did not reveal the nature of the set of

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values $d(x)$ for such x. In particular, one could not conclude that there was any particular value A for which $d(x) = d(x+1) = A$ infinitely often. In [GGPY11], the authors showed that

$$d(x) = d(x+1) = 24 \text{ for infinitely many positive integers } x. \tag{0.1}$$

Similar results were proven for other related arithmetic functions which count numbers of prime divisors. The goal of this paper is to establish results for an arbitrary shift n, i.e., $d(x) = d(x+n) = A$ infinitely often for some fixed A.

2. Notation and preliminaries

For our purposes, a *linear form* is an expression $L(m) = am + b$ where a and b are integers and $a > 0$. We view L both as a polynomial and as a function in m. We say L is *reduced* if $\gcd(a, b) = 1$. If $K(m) = cm + d$ is another linear form, then a *relation* between L and K is an equation of the form $|c_L \cdot L - c_K \cdot K| = n$ where c_L, c_K, n are all positive integers. We call c_L, c_K the *relation coefficients* and we call n the *relation value*. We define the *determinant* of L and K as $\det(L, K) = |ad - bc|$.

For a prime p, a k-tuple of linear forms $L_1, L_2, \ldots, L_k$ is called *p-admissible* if there is an integer t_p such that

$$L_1(t_p)L_2(t_p)\cdots L_k(t_p) \not\equiv 0 \pmod{p}$$

We say that a k-tuple of linear forms is *admissible* if it is p-admissible for every prime p. Note that a k-tuple of linear forms is admissible iff all the forms are reduced and the tuple is p-admissible for every prime $p \leq k$.

An *E_r number* is a positive integer that is the product of r distinct primes. Several of the coauthors here proved the following result on E_2-numbers in admissible triples in [GGPY09]. Later, Frank Thorne [Tho08] obtained a generalization for E_r-numbers with $r \geq 3$.

Theorem 1. *Let C be any constant. If L_1, L_2, L_3 is an admissible triple of linear forms, then there are two among them, say L_j and L_k such that both $L_j(x)$ and $L_k(x)$ are E_2-numbers with both prime factors larger than C for infinitely many x.*

The results obtained in this paper will use Theorem 1 above in combination with Theorem 2 below, a special case of which was proven in the previous paper [GGPY11]. We provide a proof here of the general version since it contains important ideas relevant for the rest of the paper.

Theorem 2 *(Adjoining Primes). Assume that $L_i = a_i m + b_i$ for $i = 1, \ldots, k$ gives an admissible k-tuple with relations $|c_{i,j}L_i - c_{j,i}L_j| = n_{i,j}$. We can always "adjoin" prime factors to the relation coefficients without changing the relation values: for every choice of* *positive integers* $r_1, r_2, \ldots, r_k$ *such that* $\gcd(r_i, a_i) = \gcd(r_i, \det(L_i, L_j)) = \gcd(r_i, r_j) = 1$ *whenever* $i \neq j$*, there is an admissible* k*-tuple of linear forms* $K_1, K_2, \ldots, K_k$ *with relations* $|c_{i,j} r_i K_i - c_{j,i} r_j K_j| = n_{i,j}$.

Proof. Let x be a solution of the congruences $L_i(x) \equiv r_i \pmod{r_i^2}$ for $1 \leq i \leq k$. Such an x exists by the Chinese Remainder Theorem since $\gcd(a_i, r_i) = \gcd(r_i, r_j) = 1$. This x is unique modulo $r = (r_1 r_2 \cdots r_k)^2$. Now define a new k-tuple via $K_i(m) = L_i(rm + x)/r_i$. By construction, we have $|c_{i,j} r_i K_i - c_{j,i} r_j K_j| = n_{i,j}$, so we only need to check that this new k-tuple is admissible. We will show that the new k-tuple is p-admissible for every prime p. There are two cases.

Case 1: Suppose that $p|r$. Since $\gcd(r_i, r_j) = 1$ for $i \neq j$, we have that $p|r_\ell$ for exactly one index ℓ. Now

$$K_\ell(0) = L_\ell(x)/r_\ell \equiv 1 \pmod{r_\ell}$$

so $K_\ell(0) \equiv 1 \not\equiv 0 \pmod p$. We claim that also $K_i(0) \not\equiv 0 \pmod p$ when $i \neq \ell$. Suppose, by way of contradiction, that $K_i(0) \equiv 0 \pmod p$ for some $i \neq \ell$. Then $L_i(x) \equiv 0 \pmod p$ since $r_i \not\equiv 0 \pmod p$, but $L_\ell(x) \equiv r_\ell \equiv 0 \pmod p$, so

$$\det(L_\ell, L_i) = |a_i b_\ell - a_\ell b_i| = |a_i L_\ell(x) - a_\ell L_i(x)| \equiv 0 \pmod p,$$

but this contradicts the assumption that $\gcd(r_\ell, \det(L_\ell, L_i)) = 1$. Thus $K_1(0) \cdots K_k(0) \not\equiv 0 \pmod p$.

Case 2: Now suppose $p \nmid r$. Since $L_1, \ldots, L_k$ is admissible, there is an integer t_p such that $L_1(t_p) \cdots L_k(t_p) \not\equiv 0 \pmod p$. Choose τ_p such that $r\tau_p + x \equiv t_p \pmod p$. Then $L_i(r\tau_p + x) \equiv L_i(t_p) \not\equiv 0 \pmod p$ and $r_i \not\equiv 0 \pmod p$ for all i, so

$$K_1(\tau_p) \cdots K_k(\tau_p) = \frac{L_1(r\tau_p + x)}{r_1} \cdots \frac{L_k(r\tau_p + x)}{r_k} \not\equiv 0 \pmod p. \quad \square$$

Let n be a positive integer and write its prime factorization as $n = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ where the p_i are distinct primes with $k_i > 0$. Then the *exponent pattern* of n is the multiset $\{k_1, k_2, \ldots, k_j\}$ where order does not matter but repetitions are allowed. The values of many important arithmetic functions depend only on the exponent pattern of the input; such functions include:

$$d(x) = \#\text{ of divisors of } x$$

$$\Omega(x) = \#\text{ of prime factors (counted with multiplicity) of } x$$

$$\omega(x) = \#\text{ of distinct prime factors of } x$$

$$\mu(x) = \text{Möbius function} = (-1)^{\omega(x)} \text{ if } n \text{ is squarefree, zero otherwise}$$

$$\lambda(x) = \text{Liouville function} = (-1)^{\Omega(x)}$$

Thus if both x and $x+n$ have the same exponent pattern, then $d(x) = d(x+n)$, $\Omega(x) = \Omega(x+n)$, $\omega(x) = \omega(x+n)$, etc. In establishing the strong form of the Erdős-Mirsky Conjecture (0.1), the authors in [GGPY11] actually proved the following result.

Theorem 3. *There are infinitely many positive integers x such that both x and $x+1$ have exponent pattern* $\{2,1,1,1\}$.

We will show that for any shift n, there are infinitely many positive integers x such that both x and $x+n$ have a fixed small exponent pattern. A key tool for doing this is contained in the next remark.

Remark 4. Suppose we have an admissible triple of forms L_i with relations $|c_{i,j}L_i - c_{j,i}L_j| = n$. For a given form L_i in the triple, we call $c_{i,j}$ and $c_{i,k}$ where $\{i,j,k\} = \{1,2,3\}$ the *pair of relation coefficients for L_i in the triple.* Suppose these pairs of relation coefficients for each form in the triple have matching exponent patterns, i.e., $c_{i,j}$ and $c_{i,k}$ have the same exponent pattern with any choices of i,j,k such that $\{i,j,k\} = \{1,2,3\}$. We then can choose pairwise coprime integers having any desired exponent pattern which are relatively prime to all linear coefficients and determinants (since determinants of distinct reduced forms are always nonzero). In particular, we can adjoin integers to the relation coefficients so that the new triple has the property that all of its relation coefficients have any given exponent pattern $\mathscr{P}$ which contains the exponent patterns of every $c_{i,j}$. Hence by Theorem 1, we would then get infinitely many positive integers x such that both x and $x+n$ have exponent pattern $\mathscr{P} \cup \{1,1\}$. The proofs of Theorems 5 and 7 below will rely heavily on this idea.

3. Shifts which are even or not divisible by 15

Theorem 5. *Let n be a positive integer with $2|n$ or $15 \nmid n$. Then there are infinitely many positive integers x such that both x and $x+n$ have exponent pattern* $\{2,1,1,1,1\}$.

Proof. Consider the following triple of linear forms: $L_1 = 2m+n, L_2 = 3m+n$, and $L_3 = 5m+2n$. We have the relations

$$3L_1 - 2L_2 = n$$
$$5L_1 - 2L_3 = n$$
$$3L_3 - 5L_2 = n$$

Now define $g_i = \gcd(i,n)$ and reduce the linear forms: take $\widetilde{L}_1 = L_1/g_2$, $\widetilde{L}_2 = L_2/g_3$, and $\widetilde{L}_3 = L_3/g_5$. Then the relations become

$$3 \cdot g_2\widetilde{L}_1 - 2 \cdot g_3\widetilde{L}_2 = n$$
$$5 \cdot g_2\widetilde{L}_1 - 2 \cdot g_5\widetilde{L}_3 = n$$

$$3 \cdot g_5 \widetilde{L}_3 - 5 \cdot g_3 \widetilde{L}_2 = n$$

Case 1: Suppose n is even and write $n = 2n_2$. Then $g_2 = 2$, so $\widetilde{L}_1 = m + n_2$, $\widetilde{L}_2 = (3/g_3)m + 2(n_2/g_3)$, and $\widetilde{L}_3 = (5/g_5)m + 4(n_2/g_5)$.

Subcase 1a: Suppose $2 \mid n_2$. Then

$$\widetilde{L}_1(1)\widetilde{L}_2(1)\widetilde{L}_3(1) \equiv 1^3 \not\equiv 0 \pmod 2,$$

so the triple $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ is 2-admissible. Now we check this triple is also 3-admissible (and therefore admissible).

- If $3 \nmid n_2$, then

$$\widetilde{L}_1(0)\widetilde{L}_2(0)\widetilde{L}_3(0) \equiv n_2(-n_2)(n_2/g_5) \not\equiv 0 \pmod 3.$$

- If $3 \mid n_2$, then $g_3 = 3$, so $\widetilde{L}_1 \equiv m \equiv \pm\widetilde{L}_3 \pmod 3$. Now choose $m_0 \in \{1, -1\}$ such that $\widetilde{L}_2(m_0) \not\equiv 0 \pmod 3$. Then

$$\widetilde{L}_1(m_0)\widetilde{L}_2(m_0)\widetilde{L}_3(m_0) \equiv m_0 \cdot \widetilde{L}_2(m_0) \cdot (\pm m_0) \not\equiv 0 \pmod 3.$$

Here the relation coefficients match in pairs for a given form in the triple and all have exponent patterns contained in $\{1, 1\}$, so by appeal to Remark 4 we have a slightly stronger result, namely, there are infinitely many positive integers x such that both x and $x + n$ have exponent pattern $\{1, 1, 1, 1\}$.

Subcase 1b: Suppose now $2 \nmid n_2$. Let

$$\begin{aligned} K_1 &= \widetilde{L}_1(4m + n_2)/2 = 2m + n_2 \\ K_2 &= \widetilde{L}_2(4m + n_2) = 4 \cdot \frac{3}{g_3} m + 5 \cdot \frac{n_2}{g_3} \\ K_3 &= \widetilde{L}_3(4m + n_2) = 4 \cdot \frac{5}{g_5} m + 9 \cdot \frac{n_2}{g_5} \end{aligned}$$

Our relations thus become

$$\begin{aligned} 2^2 \cdot 3K_1 - 2 \cdot g_3 K_2 &= n \\ 2^2 \cdot 5K_1 - 2 \cdot g_5 K_3 &= n \\ 3 \cdot g_5 K_3 - 5 \cdot g_3 K_2 &= n \end{aligned}$$

Here the pairs of relation coefficients for each form in the triple have matching exponent patterns. We will check that the triple K_1, K_2, K_3 is admissible. First, we note that each form is still reduced:

$$K_1 = 2m + n_2$$

is reduced since $2 \nmid n_2$.

$$K_2 = 4 \cdot \frac{3}{g_3} m + 5 \cdot \frac{n_2}{g_3}$$

is reduced since the constant term is odd and not divisible by 3 if $g_3 = 1$.

$$K_3 = 4 \cdot \frac{5}{g_5} m + 9 \cdot \frac{n_2}{g_5}$$

is reduced since the constant term is odd and not divisible by 5 if $g_5 = 1$.

Next $K_1 K_2 K_3 \equiv 1 \pmod 2$, so the triple is indeed 2-admissible. Now we check that this triple is 3-admissible.

- If $3 \nmid n_2$, then $g_3 = 1$, so

$$K_1(-n_2) K_2(-n_2) K_3(-n_2) \equiv (-n_2)^2 (n_2/g_5) \not\equiv 0 \pmod 3$$

- If $3 \mid n_2$, then $K_1 K_3 \equiv \pm m^2 \pmod 3$. Choose $m_0 \in \{1, -1\}$ such that $K_2(m_0) \not\equiv 0 \pmod 3$. Then

$$K_1(m_0) K_2(m_0) K_3(m_0) \equiv \pm (m_0)^2 K_2(m_0) \not\equiv 0 \pmod 3.$$

Here the relation coefficients all have exponent patterns contained in $\{2, 1, 1\}$, so adjoining primes again gives us the statement of the theorem.

Case 2: Now suppose n is odd, so $g_2 = 1$ from now on. Our relations for $\widetilde{L}_i$ become

$$\begin{aligned} 3\widetilde{L}_1 - 2 \cdot g_3 \widetilde{L}_2 &= n \\ 5\widetilde{L}_1 - 2 \cdot g_5 \widetilde{L}_3 &= n \\ 3 \cdot g_5 \widetilde{L}_3 - 5 \cdot g_3 \widetilde{L}_2 &= n \end{aligned}$$

If we look at this modulo 2, we get $\widetilde{L}_1 \equiv 1, \widetilde{L}_2 \equiv m + 1, \widetilde{L}_3 \equiv m$. Thus this triple is not 2-admissible here. However, we can restrict $m \pmod 2$ and reduce to get 2-admissible. To do this, we write

$$\begin{aligned} M_1 &= \widetilde{L}_1(2m) = 4m + n \\ M_2 &= \widetilde{L}_2(2m) = 2 \cdot \frac{3}{g_3} m + \frac{n}{g_3} \\ M_3 &= \widetilde{L}_3(2m)/2 = \frac{5}{g_5} m + \frac{n}{g_5}. \end{aligned}$$

The triple M_1, M_2, M_3 has reduced forms and is 2-admissible with relations

$$\begin{aligned} 3M_1 - 2 \cdot g_3 M_2 &= n \\ 5M_1 - 2^2 \cdot g_5 M_3 &= n \\ 2 \cdot 3 \cdot g_5 M_3 - 5 \cdot g_3 M_2 &= n \end{aligned}$$

Note, however, that the relation coefficients for M_3 do not have matching exponent patterns. We can remedy this by restricting and reducing modulo 3.

Subcase 2a: Suppose $3 \nmid n$, so $g_3 = 1$. Take

$$\begin{aligned} N_1 &= M_1(3m+n) = 12m + 5n \\ N_2 &= M_2(3m+n) = 18m + 7n \\ N_3 &= M_3(3m+n)/3 = \frac{5}{g_5} m + 2 \cdot \frac{n}{g_5} \end{aligned}$$

Now we get relations

$$\begin{aligned} 3N_1 - 2N_2 &= n \\ 5N_1 - 2^2 \cdot 3 \cdot g_5 N_3 &= n \\ 2 \cdot 3^2 \cdot g_5 N_3 - 5N_2 &= n \end{aligned}$$

All these forms are reduced and the triple is still 2-admissible since $N_1(1)N_2(1)N_3(1) \equiv 1^3 \not\equiv 0 \pmod 2$. In fact, the triple is 3-admissible too since

$$N_1(0)N_2(0)N_3(0) \equiv (-n)(n)(-n/g_5) \not\equiv 0 \pmod 3.$$

Here the relation coefficients all have exponent patterns contained in $\{2, 1, 1\}$, so adjoining primes again gives us the statement of the theorem. In fact, if we also have $5 \nmid n$ here, then the relation coefficients all have exponent patterns contained in $\{2, 1\}$ so we get infinitely many positive integers x such that x and $x + n$ both have exponent pattern $\{2, 1, 1, 1\}$.

Subcase 2b: Suppose now $3 \mid n$, so $5 \nmid n$ by our assumption that $15 \nmid n$. We still must factor out a 3 from M_3, but doing so will force us to also factor out a 3 from M_1 which then tells us to also factor out a 5 from M_1 to make its pair of relation coefficients in the triple have matching exponent patterns. Thus we will restrict modulo 15: write $n = 3n_3$ and take

$$\begin{aligned} J_1 &= M_1(15m - 4n)/15 = 4m - n \\ J_2 &= M_2(15m - 4n)/(g_9/3) = 10 \cdot \frac{9}{g_9} m - 23 \cdot \frac{n}{g_9} \\ J_3 &= M_3(15m - 4n)/3 = 25m - 19n_3 \end{aligned}$$

where, as indicated above, $g_9 = \gcd(9, n)$ which is either 3 or 9 in this case. Here we have relations

$$\begin{aligned} 3^2 \cdot 5J_1 - 2 \cdot g_9 J_2 &= n \\ 3 \cdot 5^2 J_1 - 2^2 \cdot 3J_3 &= n \\ 2 \cdot 3^2 J_3 - 5 \cdot g_9 J_2 &= n \end{aligned}$$

All the forms are reduced (since $5 \nmid n$) and the triple is 2-admissible since $J_1(0)J_2(0)J_3(0) \equiv 1^3 \not\equiv 0 \pmod 2$.

Now we check that this triple is 3-admissible.

- If $3 \nmid n_3$, then $g_9 = 3$, so

$$J_1(-n_3)J_2(-n_3)J_3(-n_3) \equiv (-n_3)(n_3)^2 \not\equiv 0 \pmod 3.$$

- If $3 \mid n_3$, then $g_9 = 9$ so $J_1 J_3 \equiv m^2 \pmod 3$. Choose $m_0 \in \{1, -1\}$ such that $J_2(m_0) \not\equiv 0 \pmod 3$. Then

$$J_1(m_0)J_2(m_0)J_3(m_0) \equiv (m_0)^2 J_2(m_0) \not\equiv 0 \pmod 3.$$

Here the relation coefficients all have exponent patterns contained in $\{2, 1, 1\}$ (or even in $\{2, 1\}$ in the case that $9|n$), so adjoining primes again gives us the statement of the theorem. □

Remark 6. If we assume the twin prime conjecture, then for any positive integer n, there are primes p and $p+2$ such that neither divide $15n$. In this case, we can use the following triple: $L_1 = 2m + n$, $L_2 = pm + n(p - 1)/2$, $L_3 = (p + 2)m + n(p + 1)/2$. Building off this triple will show—as in Subcase 2a above—that there are infinitely many positive integers x such that x and $x + n$ both have exponent pattern $\{2, 1, 1, 1\}$. We will not include the details here since we give an unconditional proof of a result for the remaining case not covered by Theorem 5.

4. Shifts which are odd and divisible by 15

Theorem 7. *Let n be a positive integer with $2 \nmid n$ and $15|n$. Then there are infinitely many positive integers x such both x and $x + n$ have exponent pattern $\{3, 2, 1, 1, 1, 1, 1\}$.*

Proof. By considering the admissible triple $m, m+4, m+10$, we find that for any constant C there are infinitely many pairs of E_2 numbers each having prime factors bigger than C and which are a distance of either 4, 6, or 10 apart. In particular, there are odd E_2 numbers q_1, q_2 such that $\gcd(q_i, n) = 1$ for $i = 1, 2$ and $q_2 = q_1 + 2j$ where $j \in \{2, 3, 5\}$. Thus we may write $q_1 = p_{1,1}p_{1,2}$ and $q_2 = p_{2,1}p_{2,2}$ where $p_{1,1}$, $p_{1,2}$, $p_{2,1}$, and $p_{2,2}$ are all distinct primes, none of which divide $2n$. There are integers a, b with a even and b odd such that $-aq_2 + bq_1 = 1$. Write $a = 2a_2$ and define the triple of linear forms

$$\begin{aligned} L_1 &= q_1 m + a_2 n \\ L_2 &= 2q_2 m + bn \\ L_3 &= 4 \cdot \frac{j}{g} m + (b-a)\frac{n}{g} \end{aligned}$$

where $g = 1$ if $j = 2$ and $g = j$ otherwise. Now we check that this triple is admissible. We only need to check for 2-admissible and 3-admissible since each form is reduced by construction. The triple is 2-admissible since $L_1 \cdot L_2 \cdot L_3 \equiv L_1 \cdot 1 \cdot 1 \pmod 2$. To check the triple is 3-admissible, choose $m_0 \in \{1, -1\}$ with $L_3(m_0) \not\equiv 0 \pmod 3$. Then $L_1(m_0)L_2(m_0)L_3(m_0) \equiv (q_1 m_0)(-q_2 m_0)L_3(m_0) \not\equiv 0 \pmod 3$. Moreover, the triple satisfies the relations

$$\begin{aligned} q_1 L_2 - 2q_2 L_1 &= n \\ gq_1 L_3 - 2^2 j L_1 &= n \\ gq_2 L_3 - 2jL_2 &= n \end{aligned} \tag{7.1}$$

However, the pairs of relation coefficients for L_1, L_2 do not have matching exponent patterns in the triple, so we will need to adjoin primes using Theorem 2. We will break up the proof into cases depending on the value of j, but in both cases we need to note that the pairwise determinants are relatively prime to the integers we want to adjoin:

$$\begin{aligned} \det(L_1, L_2) &= q_1 bn - 2a_2 n q_2 = n \\ \det(L_1, L_3) &= q_1(b-a)\frac{n}{g} - 4a_2 n \cdot \frac{j}{g} = \frac{n}{g} \\ \det(L_2, L_3) &= 2q_2(b-a)\frac{n}{g} - 4bn \cdot \frac{j}{g} = 2 \cdot \frac{n}{g} \end{aligned}$$

Case 1: Suppose $j = 2$, so $g = 1$.

We apply Theorem 2 directly with $r_1 = p_{2,1}^2 p_{2,2}$, $r_2 = p_{1,1}$, and $r_3 = 1$, so we get a new admissible triple of forms K_i which satisfies the following relations:

$$\begin{aligned} |p_{1,1}^2 p_{1,2} K_2 - 2p_{2,1}^3 p_{2,2}^2 K_1| &= n \\ |q_1 K_3 - 2^3 p_{2,1}^2 p_{2,2} K_1| &= n \\ |q_2 K_3 - 2^2 p_{1,1} K_2| &= n. \end{aligned}$$

Here the relation coefficients of K_1 both have exponent pattern $\{3, 2, 1\}$, the relation coefficients of K_2 both have exponent pattern $\{2, 1\}$, and the relation coefficients of K_3 both have exponent pattern $\{1, 1\}$. Thus by another application of Theorem 2 via

Remark 4 we can arrange an admissible triple with common relation value n and all relation coefficients having exponent pattern $\{3, 2, 1, 1, 1\}$ (or even $\{3, 2, 1, 1\}$ in this case).

Case 2: Suppose $j \neq 2$, so $g = j$. We apply Theorem 2 directly with $r_1 = p_{2,1}$, and $r_2 = r_3 = 1$, so we get a new admissible triple of forms K_i which satisfies the following relations:

$$\begin{aligned} |q_1 K_2 - 2p_{2,1}^2 p_{2,2} K_1| &= n \\ |jq_1 K_3 - 2^2 jp_{2,1} K_1| &= n \\ |jq_2 K_3 - 2jK_2| &= n. \end{aligned}$$

Here the relation coefficients of K_1 both have exponent pattern $\{2, 1, 1\}$, the relation coefficients of K_2 both have exponent pattern $\{1, 1\}$, and the relation coefficients of K_3 both have exponent pattern $\{1, 1, 1\}$. Thus by appeal to Theorem 2 via Remark 4 we can arrange an admissible triple with common relation value n and all relation coefficients having exponent pattern $\{3, 2, 1, 1, 1\}$ (or even $\{2, 1, 1, 1\}$ in this case).

Therefore, in either case, there are infinitely many pairs of positive integers both having exponent pattern $\{3, 2, 1, 1, 1, 1, 1\}$ which are a distance of n apart. □

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