1990

Computer extended perturbation series in fluid mechanics

Abed Alhamid Alfar
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Computer extended perturbation series in fluid mechanics

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San Jose State University, 1990

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**COMPUTER EXTENDED PERTURBATION
SERIES IN FLUID MECHANICS**

A Thesis

Presented to

**The Faculty of the
Department of Mathematics**

San Jose State University

In partial Fulfillment

of the requirements for the degree

Master of Science

By

Abed Alhamid Alfar

December, 1990

Approved for the Department of Mathematics

Dr. Yue-Kuen Kwok

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Dr. Leslie Foster

Approved for the University

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A.Y. ALFAR

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ABSTRACT
COMPUTER EXTENSION OF PERTURBATION SERIES
IN FLUID MECHANICS

by Abed Y. Alfar

The incompressible potential flow over a perturbed body is used as a model to shed light into the nature of the nearest singularity of a perturbation series. Solutions for various physical quantities are developed as a formal perturbation expansion in powers of a parameter up to a finite number of terms. The tedious computation is relegated to a computer. The calculations reveal that the location and structure of the nearest singularity of perturbation series do change with the field point at which the series is evaluated. Also the radius of convergence and behavior of the perturbation series do depend significantly on the choice of the physical quantities. The results may provide salutary information for the analysis of computer extended perturbation series.

CHAPTER I

INTRODUCTION

In recent decades, there has been an increasing trend for researchers in various disciplines to solve their problems by making use of the technique of extending a perturbation series to high order using a computer (1)-(3). An excellent review on the computer extension of perturbation series in fluid mechanics was given by Van Dyke (4). A three-step algorithm is involved in such a semi-numerical technique. The first step is the extension of the perturbation series to as many terms as possible by delegating the tedious labor of computation to a computer. Usually the structures and locations of singularities of the perturbation series have strong physical significance to the problem. When more terms of the perturbation series are available, it becomes plausible in the second step of the algorithm to analyse the series to reveal its locations and structures of the singularities. The locations of the singularities may suggest some appropriate transformations so as to map the nonphysical singularities. In the last step, these transformations are applied to improve the accuracy and the range of convergence of the perturbation series.

It is the author's experiences on this technique that the field point chosen for the evaluation of the perturbation series does affect the location and structure of the nearest singularity of the perturbation series. However, this phenomenon has not been well addressed in most papers on the analysis of computer extended series. In the calculation of incompressible potential flow past a sinusoidal wall of finite amplitude, Whitney (5) discovered that the nature of the nearest singularity of his perturbation series did depend on the field point at which the perturbation series was evaluated. He reasoned that since his problem was truly elliptic, one should expect that the nature and the position of the nearest

singularity does not change with the field point at which the series is evaluated. However, his analysis of the series demonstrated that his postulate was not correct. He also noticed that the rate of convergence and structures of singularities of the perturbation series of different physical quantities could behave very differently. The perturbation series of some physical quantities showed clear structures of algebraic singularities.

This thesis addresses the question of the nature of nearest singularities of perturbation series resulting from different field points chosen for evaluation. To ease the computational efforts, I will solve a simple regular perturbation problem so that the extension of the series can be easily performed. The perturbation series of various physical quantities are evaluated at different field points and their structures of nearest singularities are analysed.

CHAPTER II

INCOMPRESSIBLE FLOW OVER AN ELLIPSE

This problem solves for the incompressible potential flow over a distorted cylinder of shape given by

$$r = 1 - \varepsilon \sin^2 \theta \quad 2.1$$

Where ε is a small parameter. Figure 1(next page) shows the shape of the body for various values of ε . For the ease of applying the boundary conditions, it would be better to formulate the problem in terms of the stream function. The governing equation is the Laplace equation

$$\nabla^2 \Psi = 0 \quad 2.2$$

and subject to the following boundary conditions:

(i) Far stream conditions:

$$\frac{\partial \Psi}{\partial y} \rightarrow 1 \text{ and } \frac{\partial \Psi}{\partial x} \rightarrow 0 \text{ as } r \rightarrow \infty,$$

(ii) At the surface of the body:

$$\Psi(1 - \varepsilon \sin^2 \theta, \theta) = 0 \quad 2.3$$

Notice that the perturbation parameter ε appears in the boundary condition. This problem is proposed in Van Dyke's classical text of perturbation methods in fluid mechanics [6].

We seek the perturbation expansion of stream function in powers of ε as

$$\Psi(r, \theta) = \sum_{k=0}^{\infty} \varepsilon^k \Psi_k(r, \theta) \quad 2.4$$

Using the Taylor series expansion of the boundary condition

$$\Psi(1 - \varepsilon \sin^2 \theta, \theta) = \Psi(1, \theta) + (-\varepsilon \sin^2 \theta) \Psi_r(1, \theta) + \frac{(-\varepsilon \sin^2 \theta)^2}{2!} \Psi_{rr}(1, \theta) + \dots$$

and in general is written in the form

$$\Psi(1-\varepsilon \sin^2 \theta, \theta) = \left[\sum_{m=0}^{\infty} \frac{(-\varepsilon \sin^2 \theta)^m}{m!} \frac{\partial^m}{\partial r^m} \Psi(r, \theta) \right]_{r=1} = 0$$

Figure 1: The Shape of the body

Substituting the summation expansion of $\Psi(r, \theta)$ and expanding the nth summation ,
we get

$$\begin{aligned} & \left[\Psi_0 + \sum_1^{\infty} \epsilon^k \Psi_k \right] + (-\epsilon \sin^2 \theta) \left[\frac{\partial \Psi_0}{\partial r} + \sum_1^{\infty} \epsilon^k \frac{\partial \Psi_k}{\partial r} \right] \\ & + \frac{(-\epsilon \sin^2 \theta)^2}{2!} \left(\frac{\partial^2 \Psi_0}{\partial r^2} + \left[\sum_1^{\infty} \epsilon^k \frac{\partial^2 \Psi_k}{\partial r^2} \right] \right) \\ & + \dots + \frac{(-\epsilon \sin^2 \theta)^n}{n!} \left(\frac{\partial^n \Psi_0}{\partial r^n} + \left[\sum_1^{\infty} \epsilon^k \frac{\partial^n \Psi_k}{\partial r^n} \right] \right) = 0 \end{aligned}$$

Where $\Psi_k = \Psi_k(1, \theta)$

By equating like power terms of ϵ , we obtain

$$O(1) \Rightarrow \Psi_0(1, \theta) = 0$$

$$O(\epsilon) \Rightarrow \Psi_1(1, \theta) - \sin^2 \theta \Psi_{0r}(1, \theta) = 0$$

$$O(\epsilon^2) \Rightarrow \Psi_2(1, \theta) - \sin^2 \theta \Psi_{1r}(1, \theta) + \frac{(-\sin^2 \theta)^2}{2!} \Psi_{0rr}(1, \theta) = 0$$

$\vdots$

$$O(\epsilon^n) \rightarrow \Psi_k(1, \theta) = - \sum_{n=1}^k \frac{(-\sin^2 \theta)^n}{n!} \frac{\partial^n \Psi_{k-n}}{\partial r^n}$$

In polar coordinates

$$\nabla^2 \Psi = \Psi_{rr} + \frac{1}{r} \Psi_r + \frac{1}{r^2} \Psi_{\theta\theta} = 0$$

$$\text{With } \nabla^2 \Psi_k = 0$$

The basic solution $\Psi_0(r, \theta)$ represents the solution of the stream function past the unit circle,
which is given by

$$\Psi_0(r, \theta) = (r - 1/r) \sin \theta$$

Computing the first two terms:

$$\Psi_0 = \left(r - \frac{1}{r}\right) \sin\theta, \quad \frac{\partial \Psi_0}{\partial r} = \left(1 + \frac{1}{r^2}\right) \sin\theta$$

$$\frac{\partial^2 \Psi_0}{\partial r^2} = \frac{-2}{r^3} \sin\theta, \quad \frac{\partial^2 \Psi_0}{\partial \theta^2} = -\left(r - \frac{1}{r}\right) \sin\theta$$

From O(ϵ) equation we have

$$\begin{aligned} \Psi_1(1, \theta) &= \sin^2\theta \Psi_{0r}(1, \theta) \\ &= 2 \sin^2\theta \sin\theta, \quad \text{using } \left(\sin^2\theta \sin\theta = \frac{3}{4} \sin\theta - \frac{1}{4} \sin 3\theta\right) \\ \Psi_1(1, \theta) &= \left(\frac{3}{2} \sin\theta - \frac{1}{2} \sin 3\theta\right) \\ \Psi_1(r, \theta) &= \left(\frac{3}{2} \frac{\sin\theta}{r} - \frac{1}{2} \frac{\sin 3\theta}{r^3}\right) \end{aligned} \quad 2.6$$

To find $\Psi_2(r, \theta)$, we proceed similarly :

$$\frac{\partial \Psi_1}{\partial r}(1, \theta) = \frac{-3}{2} \sin\theta + \frac{3}{2} \sin 3\theta$$

$$\frac{\partial^2 \Psi_1}{\partial r^2}(1, \theta) = 3 \sin\theta - 6 \sin 3\theta$$

$$\begin{aligned} O(\epsilon^2) \Rightarrow \Psi_2(1, \theta) &= \sin^2\theta \Psi_{1r}(1, \theta) - \frac{\sin^4\theta}{2} \Psi_{0rr}(1, \theta) \\ &= \sin^2\theta \left(\frac{-3}{2} \sin\theta + \frac{3}{2} \sin 3\theta\right) - \frac{\sin^4\theta}{2} (-2\sin\theta) \\ &= \frac{-3}{2} (\sin^2\theta \sin\theta - \sin^2\theta \sin 3\theta) + \sin^4\theta \sin\theta \\ &= \frac{-3}{2} \left(\frac{3}{4} \sin\theta - \frac{1}{4} \sin 3\theta + \frac{-1}{4} \sin\theta + \frac{1}{2} \sin 3\theta - \frac{1}{4} \sin 5\theta\right) \\ &\quad + \left(\frac{5}{8} \sin\theta - \frac{5}{16} \sin 3\theta + \frac{1}{16} \sin 5\theta\right) \end{aligned}$$

After some simplifications , we get

$$\Psi_2 (1,\theta) = - \frac{7}{8} \sin\theta + \frac{13}{16}\sin 3\theta - \frac{5}{16}\sin 5\theta$$

giving

$$\Psi_2 (r,\theta) = - \frac{7}{8} \frac{\sin\theta}{r} + \frac{13}{16} \frac{\sin 3\theta}{r^3} - \frac{5}{16} \frac{\sin 5\theta}{r^5} \quad 2.7$$

Similarly , it can be shown that :

$$\Psi_3 (r,\theta) = - \frac{1}{16} \frac{\sin\theta}{r} - \frac{9}{16} \frac{\sin 3\theta}{r^3} + \frac{21}{32} \frac{\sin 5\theta}{r^5} - \frac{7}{32} \frac{\sin 7\theta}{r^7} \quad 2.8$$

CHAPTER III

DERIVATION OF THE GENERAL SOLUTION

We can expand the stream function Ψ as

$$\Psi = \Psi_0 + \varepsilon \Psi_1 + \varepsilon^2 \Psi_2 + \varepsilon^3 \Psi_3 + \dots$$

where Ψ_k is of the form

$$\Psi_k = A_{k0} \frac{\sin \theta}{r} + A_{k1} \frac{\sin 3\theta}{r^3} + \dots + A_{kk} \frac{\sin(2k+1)\theta}{r^{2k+1}} \quad k=1,2,\dots$$

$$\Psi_0 = r \sin \theta + A_{00} \frac{\sin \theta}{r} = r \sin \theta - \frac{\sin \theta}{r}$$

The general form of Ψ is

$$\begin{aligned} \Psi = & r \sin \theta + A_{00} \frac{\sin \theta}{r} + \varepsilon \left(A_{10} \frac{\sin \theta}{r} + A_{11} \frac{\sin 3\theta}{r^3} \right) \\ & + \varepsilon^2 \left(A_{20} \frac{\sin \theta}{r} + A_{21} \frac{\sin 3\theta}{r^3} + A_{22} \frac{\sin 5\theta}{r^5} \right) \\ & + \varepsilon^3 \left(A_{30} \frac{\sin \theta}{r} + A_{31} \frac{\sin 3\theta}{r^3} + A_{32} \frac{\sin 5\theta}{r^5} + A_{33} \frac{\sin 7\theta}{r^7} \right) + \dots \end{aligned} \quad 3.1$$

Collecting like power terms in $\frac{\sin(2j+1)\theta}{r^{2j+1}}$, $j = 0,1,2,\dots$

$$\begin{aligned} \Psi = & r \sin \theta + \frac{\sin \theta}{r} (A_{00} + \varepsilon A_{10} + \varepsilon^2 A_{20} + \varepsilon^3 A_{30} + \dots) \\ & + \frac{\sin 3\theta}{r^3} (\varepsilon A_{11} + \varepsilon^2 A_{21} + \varepsilon^3 A_{31} + \dots) \\ & + \frac{\sin 5\theta}{r^5} (\varepsilon^2 A_{22} + \varepsilon^3 A_{32} + \varepsilon^4 A_{42} + \dots) + \dots \end{aligned}$$

The coefficient of $\frac{\sin(2j+1)\theta}{r^{2j+1}}$ is $\sum_{l=j} \varepsilon^l A_{lj}$, $j=0,1,2,\dots$

To enforce the boundary condition $\Psi(1-\varepsilon \sin^2 \theta, \theta) = 0$, we put

$r = 1 - \varepsilon \sin^2 \theta$ in the above expansion.

Using the expansion form

$$\begin{aligned} \frac{1}{(1-\varepsilon \sin^2 \theta)^{2j+1}} = & 1 + \varepsilon (2j+1) \sin^2 \theta + \varepsilon^2 \frac{(2j+1)(2j+2)}{2!} \sin^4 \theta \\ & + \varepsilon^3 \frac{(2j+1)(2j+2)(2j+3)}{3!} \sin^6 \theta \\ & + \dots + \varepsilon^k \frac{(2j+1)(2j+2)\dots(2j+k)}{k!} \sin^{2k} \theta + \dots \end{aligned} \quad 3.2$$

For the j th term, the expansion of Ψ has the form:

$$\begin{aligned} \sin(2j+1)\theta \left[1 + \varepsilon (2j+1) \sin^2 \theta + \dots + \varepsilon^k \frac{(j+1)(2j+2)\dots(2j+k)}{k!} \sin^{2k} \theta + \dots \right] \\ \times [\varepsilon^j A_{jj} + \varepsilon^{j+1} A_{j+1,j} + \varepsilon^{j+2} A_{j+2,j} + \dots] \end{aligned}$$

If we write

$$(1-\varepsilon \sin^2 \theta)^{-k} = \sum_{j=0}^{\infty} C_{jk} (\varepsilon \sin^2 \theta)^j \quad 3.3a$$

Where

$$C_{ij} = \frac{j(j+1)\dots(j+i-1)}{i!} = \frac{(j+i-1)!}{i!(j-1)!} \text{ if } j \neq 0, \quad 1 \text{ if } j=0 \quad 3.3b$$

Then on the boundary, $r = 1 - \varepsilon \sin^2 \theta$

$$\begin{aligned} \Psi = & (1 - \varepsilon \sin^2 \theta) \sin \theta \\ & + \sin \theta (C_{00} + \varepsilon C_{01} \sin^2 \theta + \varepsilon^2 C_{02} \sin^4 \theta + \dots) (A_{00} + \varepsilon A_{10} + \varepsilon^2 A_{20} + \dots) \\ & + \sin 3\theta (C_{10} + \varepsilon C_{11} \sin^2 \theta + \varepsilon^2 C_{12} \sin^4 \theta + \dots) (\varepsilon A_{11} + \varepsilon^2 A_{21} + \varepsilon^3 A_{31} + \dots) \\ & + \sin 5\theta (C_{20} + \varepsilon C_{21} \sin^2 \theta + \varepsilon^2 C_{22} \sin^4 \theta + \dots) (\varepsilon^2 A_{22} + \varepsilon^3 A_{32} + \dots) \\ & + \dots = 0 \end{aligned}$$

Collecting like power terms in ε

$$O(1): 1 + C_{00} A_{00} = 0 \text{ giving } A_{00} = -1$$

$$O(\varepsilon): -\sin^3 \theta + \sin \theta (C_{00} A_{10} + C_{01} A_{00} \sin^2 \theta) + \sin 3\theta (C_{10} A_{11}) = 0$$

$$A_{10} \sin \theta + A_{11} \sin 3\theta = 2 \sin^3 \theta = 2 \left(\frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta \right) = \frac{3}{2} \sin \theta - \frac{1}{2} \sin 3\theta$$

$$\therefore A_{10} = \frac{3}{2}, \quad A_{11} = -\frac{1}{2} \text{ giving}$$

$$\Psi_1 = \frac{3}{2} \frac{\sin \theta}{r} - \frac{1}{2} \frac{\sin 3\theta}{r^3}$$

$$O(\epsilon^2) : \sin\theta (C_{00} A_{20} + C_{01} A_{10} \sin^2\theta + C_{02} A_{00} \sin^4\theta) \\ + \sin 3\theta (C_{10} A_{21} + C_{11} A_{11} \sin^2\theta) + \sin 5\theta (C_{20} A_{22}) = 0$$

The unknowns are A_{20} , A_{21} and A_{22} , which are found by equating the coefficients of $\sin\theta$, $\sin 3\theta$ and $\sin 5\theta$ to be zero.

Suppose we write

$$\sin(2j+1)\theta \sin^{2k}\theta = \sum_{l=0}^{j+k} B_l^{j,k} \sin(2l+1)\theta \quad 3.4$$

(see Appendix A to show how to find $B_l^{j,k}$)

Consider the general ϵ^n term: ($n \geq 2$)

$$\sin\theta (C_{00} A_{n0} + C_{01} A_{n-1,0} \sin^4\theta + \dots + C_{0n} A_{00} \sin^{2n}\theta) \\ + \sin 3\theta (C_{10} A_{n1} + C_{11} A_{n-1,1} \sin^2\theta + \dots + C_{1,n-1} A_{11} \sin^{2n-2}\theta) \\ + \sin 5\theta (C_{20} A_{n2} + C_{21} A_{n-1,2} \sin^2\theta + \dots + C_{2,n-2} A_{22} \sin^{2n-4}\theta) \\ + \dots + \sin(2n+1)\theta (C_{n0} A_{nn}) = 0$$

Our goal is to find $A_{n0}, A_{n1}, A_{n2}, \dots, A_{nn}$.

The above expansion can be rewritten in the more general form:

$$A_{n0} (C_{00} \sin\theta) + A_{n1} (C_{10} \sin 3\theta) + A_{n2} (C_{20} \sin 5\theta) + \dots + A_{nn} (C_{n0} \sin(2n+1)\theta) \\ = - \sum_{j=0}^{n-1} \left(\sum_{k=1}^{n-j} C_{jk} A_{n-k,j} (\sin^{2k}\theta) \sin(2j+1)\theta \right) \\ = - \sum_{j=0}^{n-1} \sum_{k=1}^{n-j} \left(\sum_{l=0}^{j+k} C_{jk} A_{n-k,j} (B_l^{j,k} \sin(2l+1)\theta) \right) \quad 3.5$$

At this stage we have all the necessary information for the algorithm to determine the coefficients in the general expansion of the stream function. The algorithm requires two subroutines to compute the B's and C's in the above general form as needed to compute the coefficients $A(i,j)$ in terms of which the stream function can be directly computed. The

FORTTRAN program for this procedure is shown in Appendix C.

To estimate the number of reliable significant figures , the computation was done using double and quadruple precision arithmetics. Based on the comparison of the two results, it was evident to claim at least nine significant figures accuracy for the coefficients $A(i,j)$.

The number of terms which can be found hardly reaches beyond thirty due to erosion of computation by roundoff errors. The number of terms that can be extended has been sufficient for subsequent series analysis.

CHAPTER IV

SERIES SOLUTIONS OF SOME OTHER PHYSICAL QUANTITIES

Once the coefficients A_{kj} have been computed, the stream function series expansion can be found directly as follows:

$$\Psi(r, \theta) = \sum_{k=0}^{\infty} \varepsilon^k \Psi_k(r, \theta) \quad , \text{Where}$$

$$\Psi(r, \theta) = \Psi_0(r, \theta) + \sum_{k=1}^{\infty} \sum_{j=0}^k \frac{A_{kj} \sin(2j+1)\theta}{r^{2j+1}} \varepsilon^k \quad 4.1$$

$$\text{With } \Psi_0(r, \theta) = r \sin \theta + A_{00} \frac{\sin \theta}{r}$$

Besides the stream function, there are also a few other important physical quantities associated with the flow problem, such as the velocity potential, complex potential and pressure coefficient. The stream function is just the harmonic conjugate of the velocity potential and the complex potential $w(x, y)$ is known to be

$$w(z) = w(x, y) = \Phi(x, y) + i \Psi(x, y), \quad z = x + i y$$

The pressure coefficient provides information about the pressure distribution around the body, and it is given by

$$C_p = 1 - \frac{dw}{dz} \frac{\overline{dw}}{dz}$$

Once the perturbation series of the stream function is known, the corresponding perturbation series of the velocity potential and pressure coefficient can be computed. To not lose sight of the purpose at hand, the derivations of the general forms of these physical quantities at the surface of the body and at a flow point are presented in Appendix B.

The results of these derivations are as follows:

The velocity potential at any point in the flow is given by:

$$\Phi(r, \theta) = \Phi_0(r, \theta) - \sum_{k=1}^{\infty} \left[\sum_{j=0}^k \frac{A_{kj} \cos(2j+1)\theta}{r^{2j+1}} \right] \epsilon^k \quad 4.2$$

where $\Phi_0(r, \theta) = r \cos \theta - A_{00} \frac{\cos \theta}{r}$

The velocity potential at the surface of the body is given by:

$$\Phi_0(r, \theta) = 2 \cos \theta$$

$$\begin{aligned} \Phi_1(r, \theta) = & -\cos \theta (C_{00} A_{10} + \sin^2 \theta + C_{01} A_{00} \sin^2 \theta) \\ & - \cos 3\theta (C_{10} A_{11}) \end{aligned}$$

and in general

$$\Phi_n(r, \theta) = - \left(\sum_{j=0}^n \sum_{k=0}^{n-j} C_{jk} A_{n-k,j} \cos(2j+1)\theta \sin^{2k}\theta \right) \quad 4.3$$

The pressure coefficient at any field point is given by:

$$C_p = 1 - (u^2 + v^2) \quad 4.4$$

where u, v are the horizontal and vertical speeds, respectively.

The expansions of horizontal and vertical speeds are:

$$u = \left(1 + \frac{A_{00} \cos 2\theta}{r^2} \right) + \sum_{k=1}^{\infty} u_k \epsilon^k \quad 4.5a$$

$$\text{where } u_k = \sum_{j=0}^k \frac{(2j+1) A_{kj} \cos 2(j+1)\theta}{r^{2(j+1)}} \quad 4.5b$$

$$v = \frac{A_{00} \sin 2\theta}{r^2} + \sum_{k=1}^{\infty} v_k \epsilon^k \quad 4.6a$$

$$\text{where } v_k = \sum_{j=0}^k \frac{(2j+1) A_{kj} \sin 2(j+1)\theta}{r^{2(j+1)}} \quad 4.6b$$

The pressure coefficient at the surface is computed as follows:

We define

$$U_{jk} = \frac{(2j+2)(2j+3)\cdots(2j+k+1)}{k!} \quad k \neq 0$$

$$= 1 \quad k=0$$
4.7

Then the surface pressure series is given by:

$$C_p = [-2 A_{00}^2 U_{00}^2 \cos^2 2\theta - A_{00}^2 U_{00}^2] + \sum_{n=1}^{\infty} C_{p,n} \varepsilon^n$$

and so we have

$$C_{p,0} = - A_{00}^2 U_{00}^2 (1+2 \cos^2 2\theta)$$
4.8a

$$C_{p,n} = - \sum_{k=0}^n [u_k u_{n-k} + v_k v_{n-k}]$$
4.8b

Using these general forms , the series coefficients of these physical quantities can directly be computed. The values of $r=1$ and $r=2$ with angles zero, $\pi/8$, $\pi/4$, $3\pi/8$, and $\pi/2$ have been selected as our field points . The coefficients of these series are presented in the following tables.

Table 1: Stream Function Series Coefficients at $r=1$

n	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	0.00000000D+00	0.00000000D+00	0.00000000D+00	0.00000000D+00
1	0.11208537D+00	0.70710670D+00	0.15771610D+01	0.20000000D+01
2	0.12709175D+00	0.17677674D+00	-0.99973632D+00	-0.20000000D+01
3	-0.21015999D-01	-0.75130087D+00	-0.29571767D+00	0.13750000D+01
4	-0.10996690D+00	0.55242583D-01	0.85734392D+00	-0.15000000D+01
5	-0.31969400D-01	0.59385921D+00	-0.11642822D+01	0.16640625D+01
6	0.71071085D-01	-0.32800345D+00	0.13071678D+01	-0.16484375D+01
7	0.59181017D-01	-0.25308086D+00	-0.67564018D+00	0.20878906D+01
8	-0.23323167D-01	0.55311759D+00	-0.18932386D+00	-0.23730469D+01
9	-0.52844005D-01	-0.13918544D+00	0.12244017D+01	0.27298584D+01
10	-0.59613019D-02	-0.53821737D+00	-0.22652304D+01	-0.34654541D+01
11	0.35676294D-01	0.51794396D+00	0.25025433D+01	0.41158886D+01
12	0.17663910D-01	0.20713582D+00	-0.17891361D+01	-0.50665894D+01
13	-0.22044403D-01	-0.76999863D+00	-0.26539189D-01	0.63967041D+01
14	-0.24560697D-01	0.33437456D+00	0.27600110D+01	0.79227854D+01
15	0.66969963D-02	0.67407852D+00	-0.54122715D+01	0.99963833D+01
16	0.23659916D-01	-0.89979325D+00	0.67667990D+01	-0.12724717D+02
17	0.70254135D-02	-0.10483592D+00	-0.55218489D+01	0.16151318D+02
18	-0.14100210D-01	0.12021294D+01	0.86273535D+00	-0.20658339D+02
19	-0.10011156D-01	-0.81480563D+00	0.67427666D+01	0.26580370D+02
20	0.11297919D-01	-0.87395705D+00	-0.15083137D+02	-0.34173271D+02
21	-0.11785616D-02	0.15821029D+01	0.20231514D+02	0.43961558D+02
22	-0.11124313D+00	-0.13330712D+00	-0.17328145D+02	-0.55747004D+02
23	-0.37861594D-01	0.10497282D+01	0.50990428D+01	0.65073587D+02
24	0.10950102D+01	-0.12535820D+02	-0.14114605D+02	-0.45304854D+02
25	0.13027764D+01	-0.34077884D+01		
26	-0.79369088D+01			

Table 2: Stream Function Series Coefficients at $r=2$

n	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	0.57402514D+00	0.10606602D+01	0.13858193D+01	0.15000002D+01
1	0.22927009D+00	0.48613589D+00	0.71682737D+00	0.81250000D+00
2	-0.82614743D-01	-0.23063833D+00	-0.43932644D+00	-0.54882812D+00
3	-0.58626386D-01	-0.85108313D-01	-0.11390701D-01	0.61279297D-01
4	0.87109513D-02	0.32897468D-01	0.12098516D-01	-0.36483765D-01
5	0.21849069D-01	0.27196519D-01	0.28659963D-01	0.55921555D-01
6	0.14226388D-01	0.27267766D-01	0.47600478D-01	0.41305870D-01
7	0.46837024D-02	0.14588801D-01	0.19633235D-01	0.28383922D-01
8	-0.35292368D-02	-0.61450393D-02	-0.65088798D-02	-0.88896865D-02
9	-0.73072972D-02	-0.16303948D-01	-0.25106193D-01	-0.28297401D-01
10	-0.58224208D-02	-0.13822536D-01	-0.22700587D-01	-0.26919482D-01
11	-0.18380950D-02	-0.45095075D-02	-0.71282991D-02	-0.82143183D-02
12	0.13127129D-02	0.38351957D-02	0.75557555D-02	0.95279957D-02
13	0.22338472D-02	0.63788537D-02	0.12023910D-01	0.14905293D-01
14	0.15630144D-02	0.39419462D-02	0.66029857D-02	0.78238000D-02
15	0.62075675D-03	0.44289873D-03	-0.10796951D-02	-0.21561600D-02
16	0.15965482D-03	-0.11085428D-02	-0.42212416D-02	-0.61346411D-02
17	0.87609720D-04	-0.51462451D-03	-0.18385110D-02	-0.26051397D-02
18	0.25475768D-04	0.58001635D-03	0.21060254D-02	0.31245114D-02
19	-0.22155370D-03	0.57074217D-03	0.29911135D-02	0.46144980D-02
20	-0.53774354D-03	-0.40532156D-03	0.10051855D-02	0.19630487D-02
21	-0.33029781D-04	-0.36222626D-03	-0.30006180D-02	-0.50470470D-02
22	0.30740236D-03	-0.16701415D-02	-0.12905343D-02	0.13122780D-02
23	-0.17551205D-02	-0.53043559D-03	-0.54930117D-02	-0.19725057D-01
24	0.17156680D-02	0.64931281D-03	0.30899402D-02	0.44184863D-01
25	0.95566945D-02	0.16685631D-01	0.34267963D-01	-0.77540692D-01
26	-0.30905306D-01	0.62618500D-01	-0.14938733D+00	0.13876243D+00
27	0.38602070D-01	0.12037411D+00	0.43958681D+00	-0.21103831D+00
28	-0.90773717D-02	-0.12606713D+00	-0.10412782D+01	0.25001618D+00
29	0.94832366D-01			

Table 3: Velocity Potential Series Coefficients at $r=1$

n	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
0	0.20000002D+01	0.18477591D+01	0.14142137D+01	0.76536685D+00
1	-0.10000000D+01	-0.11944776D+01	-0.14142136D+01	-0.10359649D+01
2	0.37500000D+00	0.37787572D+00	0.97227174D+00	0.13742125D+01
3	0.18750000D+00	0.32203924D+00	0.26516514D+00	-0.11857725D+01
4	0.78125000D-02	0.51711492D-01	-0.60214557D+00	0.87549471D+00
5	-0.82031250D-01	-0.16290075D+00	0.16572798D+00	-0.23448547D+00
6	-0.78125000D-01	-0.14385084D+00	0.33905225D+00	-0.64411901D+00
7	-0.27343750D-01	-0.37641090D-03	-0.54379533D+00	0.12653656D+01
8	0.19165039D-01	0.73389006D-01	-0.73973692D-01	-0.16803940D+01
9	0.35827637D-01	0.37316825D-01	0.63002589D+00	0.15180328D+01
10	0.25747299D-01	-0.10945090D-01	-0.28042522D+00	-0.50802586D+00
11	0.63657761D-02	-0.53929085D-02	-0.38727112D+00	-0.95045357D+00
12	-0.70840120D-02	0.19273470D-01	0.65993785D+00	0.26399518D+01
13	-0.98516345D-02	0.11107844D-01	-0.67925138D-01	-0.39105912D+01
14	-0.60017705D-02	-0.18682245D-01	-0.78403867D+00	0.38004299D+01
15	-0.17763078D-02	-0.25414176D-01	0.63743227D+00	-0.18831916D+01
16	-0.29424112D-03	-0.22632530D-02	0.45386128D+00	-0.18637926D+01
17	-0.73516858D-03	0.15773063D-01	-0.11122387D+01	0.67023880D+01
18	-0.10757757D-02	0.71644193D-02	0.31645479D+00	-0.10720673D+02
19	-0.37921929D-03	-0.91591336D-02	0.11318920D+01	0.11500732D+02
20	0.27453641D-04	-0.20360366D-02	-0.12777977D+01	-0.69853711D+01
21	-0.29360436D-02	0.35237490D-01	-0.37579327D+00	-0.35244333D+01
22	-0.16612454D-01	-0.69663900D-02	0.12445790D+01	0.16828484D+02
23	-0.46501727D-01	-0.36014756D+00	0.94190982D+00	-0.20205669D+02
24	-0.18017961D+00	-0.29393484D+00	0.68150293D+01	-0.21638244D+02
25	-0.50228146D+00	0.30134906D+01	-0.70789649D+02	
26	-0.92866464D+00	0.48320092D+01		
27	-0.19237308D+01			
28	-0.22704825D+01			
29	-0.44794084D+01			
30	-0.42359583D+01			

Table 4: Velocity Potential Series Coefficients at $r=2$

n	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
0	0.25000003D+01	0.23096989D+01	0.17677671D+01	0.95670857D+00
1	-0.68750000D+00	-0.66899194D+00	-0.57452427D+00	-0.34475504D+00
2	0.34570313D+00	0.36159387D+00	0.37426941D+00	0.27027778D+00
3	0.82763672D-01	0.62047771D-01	-0.11911702D-01	-0.72602171D-01
4	-0.17288208D-01	-0.21112949D-01	-0.73908751D-02	0.36178235D-01
5	-0.45907974D-01	-0.38481147D-01	-0.19603343D-01	-0.25252369D-01
6	-0.41970104D-01	-0.38777569D-01	-0.36249767D-01	-0.18062340D-01
7	-0.18386472D-01	-0.19362307D-01	-0.18421748D-01	-0.11289878D-01
8	0.86828053D-02	0.73889807D-02	0.58202188D-02	0.21476073D-02
9	0.23120994D-01	0.22779483D-01	0.20058362D-01	0.13026151D-01
10	0.19853668D-01	0.20093297D-01	0.18724484D-01	0.11924593D-01
11	0.62733786D-02	0.63974909D-02	0.59604890D-02	0.37851176D-02
12	-0.58194837D-02	-0.61432394D-02	-0.65243548D-02	-0.48020174D-02
13	-0.94165983D-02	-0.99122928D-02	-0.10238870D-01	-0.72844265D-02
14	-0.56202133D-02	-0.57628508D-02	-0.55059308D-02	-0.36048393D-02
15	-0.22535322D-03	0.17447252D-03	0.12359366D-02	0.15782323D-02
16	0.20771433D-02	0.26830254D-02	0.39617170D-02	0.35830233D-02
17	0.88092163D-03	0.11801091D-02	0.17322062D-02	0.15049388D-02
18	-0.11731457D-02	-0.13812547D-02	-0.19362364D-02	-0.18188511D-02
19	-0.13323609D-02	-0.17727594D-02	-0.28506817D-02	-0.27961967D-02
20	0.16893399D-03	-0.16819744D-03	-0.11197699D-02	-0.14427015D-02
21	0.16037101D-02	0.17653050D-02	0.26685628D-02	0.32201828D-02
22	0.37300719D-03	0.10878088D-02	0.15890740D-02	-0.18105543D-02
23	0.57724838D-02	0.53796592D-02	0.41964412D-02	0.13681867D-01
24	-0.80269575D-02	-0.85553164D-02	-0.48461249D-02	-0.30400495D-01
25	-0.14624351D-01	-0.10769869D-01	-0.11172475D-01	0.56124475D-01
26	0.71510070D-01	0.69189180D-01	0.62325411D-01	-0.92985665D-01
27	-0.15993099D+00	-0.17624906D+00	-0.20322049D+00	0.80805138D-01
28	0.27510568D+00	0.32418003D+00	0.48816556D+00	0.13780788D+00
29	-0.74243073D+00	-0.76378572D+00	-0.10621268D+01	-0.94109110D+00
30	0.25362394D+01	0.24923285D+01	0.25790141D+01	0.31767772D+01

Table 5: Pressure Series Coefficients at $r=1$

n	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	-.200000013D+01	-.100000000D+01	-.199999997D+01	-.300000048D+01
1	-.621320295D+00	-.476399595D-07	0.362132034D+01	0.600000036D+01
2	0.362436799D+00	0.312499998D+01	-.211243692D+01	-.975000058D+01
3	0.660152803D+00	-.349999943D+01	-.384765271D+01	0.128750008D+02
4	-.173819579D+00	-.183593830D+01	0.110956945D+02	-.180156261D+02
5	-.655956244D+00	0.644531200D+01	-.175823248D+02	0.242421889D+02
6	-.698116485D-01	-.274511487D+01	0.191567259D+02	-.319687519D+02
7	0.543209595D+00	-.642822402D+01	-.104504371D+02	0.435000026D+02
8	0.246684588D+00	0.902575382D+01	-.935544778D+01	-.579694859D+02
9	-.391875544D+00	0.129395064D+01	0.376118754D+02	0.774371384D+02
10	-.358563454D+00	-.133144243D+02	-.649375557D+02	-.104622958D+03
11	0.227474231D+00	0.928477884D+01	0.746026428D+02	0.140198020D+03
12	0.424077081D+00	0.106180404D+02	-.497192970D+02	-.188645687D+03
13	-.323524093D-01	-.216184847D+02	-.203024762D+02	0.254744236D+03
14	-.402874382D+00	0.302660042D+01	0.128524469D+03	-.343335646D+03
15	-.147639302D+00	0.272931668D+02	-.240985005D+03	0.463802044D+03
16	0.293979652D+00	-.265968992D+02	0.297777885D+03	-.627589558D+03
17	0.254017555D+00	-.157189093D+02	-.226959807D+03	0.849157669D+03
18	-.154781993D+00	0.499944079D+02	-.233491628D+02	
19	-.275109715D+00	-.194876102D+02	0.440896774D+03	
20	0.113857269D+00	-.532007188D+02	-.909220539D+03	
21	0.245160641D+00	0.684529250D+02		
22	-.169574197D+01	0.129345649D+02		
23	-.288495329D+01	-.800150341D+01		
24	0.187951087D+02	-.307591166D+03		
25	0.502341509D+02	-.731315803D+03		

Table 6: Pressure Series Coefficients at $r=2$

n	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	-.312500034D+00	-.625000000D-01	-.312499991D+00	-.562500119D+00
1	-.187987860D+00	-.189047005D-07	0.375487859D+00	0.585937514D+00
2	0.109659580D+00	0.103759774D+00	-.219034586D+00	-.494384777D+00
3	0.198268483D-01	-.116210928D+00	-.396022330D-01	0.183868413D+00
4	-.178489574D-01	0.309657957D-01	0.519866827D-01	-.116682055D+00
5	-.641126557D-02	0.188469916D-01	-.218737622D-01	0.817120095D-01
6	-.735742235D-02	-.102649595D-01	0.450728571D-01	0.475225981D-02
7	-.684240285D-02	0.733200111D-03	0.186543391D-02	0.347883565D-01
8	0.936535399D-03	-.675535120D-03	-.800214123D-03	-.117263709D-01
9	0.659995981D-02	0.476026171D-04	-.130443820D-01	-.181788038D-01
10	0.611301460D-02	0.263384711D-02	-.124335988D-01	-.230281036D-01
11	0.198553120D-02	0.634197860D-03	-.327991658D-02	-.547077247D-02
12	-.187872069D-02	-.212125305D-02	0.344865415D-02	0.833029057D-02
13	-.308804623D-02	-.269499863D-02	0.623378443D-02	0.134584778D-01
14	-.178712657D-02	-.831477947D-03	0.341331348D-02	0.621767954D-02
15	0.136177505D-03	0.179862395D-02	-.304877072D-03	-.329807082D-02
16	0.970181896D-03	0.270038505D-02	-.188658678D-02	-.706194697D-02
17	0.474499032D-03	0.112946918D-02	-.891462766D-03	-.288896617D-02
18	-.402661725D-03	-.129270960D-02	0.838176973D-03	0.365103897D-02
19	-.594627217D-03	-.222244269D-02	0.116029795D-02	0.567106329D-02
20	-.994429295D-04	-.166430501D-02	0.353150592D-03	0.300111082D-02
21	0.255176016D-03	0.234950022D-02	-.806399080D-03	-.662113659D-02
22	0.120340761D-02	0.616038248D-04	-.375187198D-02	0.523839386D-02
23	0.622819515D-03	-.143665960D-03	0.131452363D-01	-.359149143D-01
24	-.274431291D-02	0.102569984D-01	-.557723563D-01	0.970324518D-01
25	-.112660174D-02	-.283608218D-01		
26	0.248604594D-01	0.588058825D-01		
27	-.700549962D-01			
28	0.997627420D-01			

Table 7 : Potential Series Coefficients at the surface

n	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
0	0.20000002D+01	0.18477591D+01	0.14142137D+01	0.76536685D+00
1	-0.23125000D+01	-0.26211581D+01	-0.31660532D+01	-0.32743468D+01
2	0.37500000D+00	0.27465814D+00	-0.67207936D-01	-0.60868593D+00
3	0.18750000D+00	0.20837922D+00	-0.16310605D-01	-0.10135099D+01
4	0.78125000D-02	0.70337757D-01	0.97633937D-01	-0.10773580D+01
5	-0.82031250D-01	-0.58559413D-01	0.20346419D+00	-0.20942756D+00
6	-0.78125000D-01	-0.10483529D+00	0.21324832D+00	0.21507468D+01
7	-0.27343750D-01	-0.63552920D-01	0.89071132D-01	0.59468541D+01
8	0.19165039D-01	0.85272141D-02	-0.11801409D+00	0.96575879D+01
9	0.35827637D-01	0.50056719D-01	-0.28546255D+00	0.96643863D+01
10	0.25747299D-01	0.42513813D-01	-0.28856767D+00	0.86436685D+00
11	0.63657761D-02	0.10497264D-01	-0.87029461D-01	
12	-0.70840120D-02	-0.12366589D-01	0.22335505D+00	
13	-0.98516345D-02	-0.13472721D-01	0.44040457D+00	
14	-0.60017705D-02	-0.34821437D-02	0.37411712D+00	
15	-0.17763078D-02	0.20381679D-02	-0.27626517D-02	
16	-0.29424112D-03	-0.12353851D-02	-0.47912659D+00	
17	-0.73516858D-03	-0.62170436D-02	-0.69587356D+00	
18	-0.10757757D-02	-0.56985593D-02	-0.39217489D+00	
19	-0.37921929D-03	-0.32496172D-03	0.33707021D+00	
20	0.27453641D-04	0.81367633D-02	0.10007444D+01	
21	-0.29360436D-02	0.37182214D-01	0.10690971D+01	
22	-0.16612454D-01	0.73476189D-01	0.31631542D+00	
23	-0.46501727D-01	-0.15577450D+00	-0.17995249D+01	
24	-0.18017961D+00	-0.10928870D+01	-0.36055106D+01	
25	-0.50228146D+00	-0.87816193D+00	0.58218268D+01	
26	-0.92866464D+00			
27	-0.19237308D+01			
28	-0.22704825D+01			

Table 8: Pressure Series Coefficients at surface

n	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	-.200000013D+01	-.100000000D+01	-.199999997D+01	-.300000048D+01
1	-.711655914D+00	-.123370050D+01	-.111719335D+01	-.385391926D+01
2	-.177224505D+00	0.132936654D+00	0.101888149D+01	-.181975805D+01
3	0.396807740D+00	0.228362220D+01	0.861100821D+01	0.271036869D+02
4	0.445644376D+00	0.373959855D+01	0.243631274D+02	0.154127403D+02
5	0.525898227D-01	0.277339472D+01	0.454939908D+02	0.554897634D+02
6	-.291591490D+00	-.101369090D+01	0.561264358D+02	0.157136398D+02
7	-.273396243D+00	-.561202765D+01	0.223139632D+02	0.366485604D+02
8	-.909978969D-02	-.730100435D+01	-.981588783D+02	0.691693181D+02
9	0.185027931D+00	-.328174337D+01	-.315328026D+03	0.935824703D+02
10	0.149123697D+00	0.533074913D+01	-.530902229D+03	0.320252099D+02
11	-.122735062D-01	0.127618631D+02	-.461025897D+03	
12	-.105181529D+00	0.117726801D+02	0.338512279D+03	
13	-.593207633D-01	-.375488186D+00	0.217578999D+02	
14	0.368264222D-01	-.174201824D+02	0.453772022D+02	
15	0.668878126D-01	-.258460731D+02	0.519387182D+02	
16	0.123718167D-01	-.140909028D+02	-.912528487D+02	
17	-.510950133D-01	0.159448928D+02		
18	-.534588975D-01	0.446565387D+02		
19	0.960497316D-02	0.447371700D+02		
20	0.155712426D+00	0.338779233D+01		
21	0.310948225D+00			
22	-.105319056D+01			
23	-.685154222D+01			
24	-.435665919D+01			

CHAPTER V

ANALYSIS OF THE SERIES

The works by Van Dyke[4], Hunter and Baker[7] and Hunter and Guerrieri[8] popularize the following methods of estimating the radius of convergence of a perturbation series.

(1) Pade approximants

When the sign pattern of the series coefficients is irregular, the method of Pade approximants is commonly used for locating the nearest singularity.

(2) Singularities of complex conjugate pair

If the sign pattern of the series repeats periodically or the existence of complex conjugate singularities is anticipated, then there is a pair of nearest singularities at complex conjugate points in the complex plane of the critical parameter. The method proposed by Hunter and Guerriere [8] provides an algorithm to determine this conjugate pair.

A description and application of both methods follows:

V.1. Algorithm for Pade Approximants:

Given any power series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

We can produce rational fraction approximations of the form

$$\frac{p_0 + p_1 x + \cdots + p_L x^L}{1 + q_1 x + \cdots + q_m x^m} = \frac{P_L(x)}{Q_m(x)} \quad 5.1.1$$

by expanding $A(x) Q_m(x) - P_L(x)$ as a power series in x and equating the coefficients of the first $(L + m + 1)$ powers of x to zero.

This leads to an $m \times m$ set of linear simultaneous algebraic equations for the q_i (the second set below) and then the p_i are obtained directly from the first set

$$\begin{array}{rcl}
 a_0 & & = p_0 \\
 a_1 + a_0 q_1 & & = p_1 \\
 \vdots & & \\
 a_L + a_{L-1} q_1 + \dots + a_0 q_L & = & p_L
 \end{array} \quad (1)$$

$$\begin{array}{rcl}
 a_{L+1} + a_L q_1 + \dots + a_{L-m+1} q_m & = & 0 \\
 \vdots & & \\
 a_{L+m} + a_{L+m-1} q_1 + \dots + a_L q_m & = & 0
 \end{array} \quad (2)$$

With $a_n = 0$ for $n < 0$ and $q_n = 0$ for $n > m$

The coefficients $p_0, \dots, p_L, q_1, \dots, q_m$ are chosen to make the first $L+m+1$ terms of the expansion agree with the corresponding terms of the Taylor series of $A(x)$ about $x=0$. It is apparently obvious that Pade approximants will represent all the singularities of $A(x)$ as simple poles and therefore will be most accurate if the singularity is a simple pole. Pade approximants were first applied to this type of problem by Baker who has also given comprehensive reviews on the theory and applications of Pade approximants. Since we are only concerned with finding the singularities of our finite series, we will only need to solve for the q_i of equation set (2), and then determine the roots of the denominator polynomial and then detecting the root / roots which converge and is closest to the origin.

Rewriting the linear system (2) in matrix form, we have

$$A Q = B \quad (3)$$

and we need to solve for Q .

After having decided the degrees L and m of the numerator and denominator polynomials, the matrix elements of the matrices A and B are found as follows:

$$A_{i,j} = a_{L-(j-i)} \quad i,j = 1, \dots, m \quad 5.1.2a$$

$$B_i = -a_{L+i} \quad i = 1, \dots, m \quad 5.1.2b$$

where a stands for the coefficients of our finite series.

At this stage we can use any standard subroutine package to solve for the coefficients q_i as the solution of our linear system. Afterwards we solve for the roots of the polynomial with these coefficients using another root finding subroutine. The n/n Pade approximants will be found by setting $L=m$, and the closest root to the origin will be detected as L and m are both increased by 1 reaching a maximum of half the order of the highest term of our finite series.

The procedure of this algorithm is applied to our series with a FORTRAN subroutine shown in Appendix C. We arrive at the following results of closest singularities for our series summarized in the following tables:

Table 9: Pade approximants of the Stream function series at $r=1$

N/N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
2/2	(0.335, ± 1.082)	(-.155, ± 0.995)	(-0.559, ± 1.1130)	0.95478
3/3	(0.285, ± 1.098)	(30.2006, ± 1.036)	-1.04196	-0.98348
4/4	(0.293, ± 1.007)	(-0.2496, ± 0.9261)	(0.832, ± 0.6131)	-0.83459
5/5	(0.310, ± 1.124)	(-0.2697, ± 0.9299)	(-0.778, ± 0.4475)	-0.83780
6/6	(0.303, ± 1.113)	(-0.273, ± 0.9041)	(-0.7105, ± 0.4935)	-0.79615
7/7	(0.295, ± 1.099)	(-0.289, ± 0.874)	(-0.686, ± 0.4638)	-0.78163
8/8	(0.289, ± 1.095)	(-0.303, ± 0.840)	(-0.688, ± 0.4610)	-0.77811
9/9	(0.280, ± 1.099)	(-0.299, ± 0.8304)	(-0.6697, ± 0.4714)	-0.75995
10/10	(0.280, ± 1.096)	(-0.299, ± 0.8303)	(-0.657, ± 0.4497)	-0.75964
11/11	(0.267, ± 1.024)	(-0.299, ± 0.8304)	(-0.657, ± 0.4498)	-0.75396
12/12	(0.277, ± 1.090)	(-0.347, ± 0.8334)	(-0.657, ± 0.4481)	-0.76087
13/13	(0.280, ± 1.097)	(-0.293, ± 0.8297)	(-0.657, ± 0.448)	-0.74681

Table 10: Pade approximants of Stream function series at $r=2$

N/N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
2/2	(0.343, ± 2.181)	(0.1226, ± 2.522)	(-6.337, ± 6.07)	-3.811
3/3	0.7889174	0.41259	0.6136	(1.212, ± 0.734)
4/4	(0.8290, ± 0.9294)	(0.8138, ± 0.9526)	(0.8386, ± 0.947)	(1.126, ± 0.427)
5/5	(0.8293, ± 0.9296)	(0.831, ± 0.8942)	(0.771, ± 0.862)	(0.750, ± 0.864)
6/6	(0.8420, ± 0.975)	(0.827, ± 0.8929)	(0.769, ± 0.868)	(0.749, ± 0.867)
7/7	(0.7651, ± 0.5302)	(0.821, ± 0.9268)	(0.757, ± 0.965)	(0.608, ± 0.933)
8/8	(0.9095, ± 0.7369)	(0.659, ± 0.9409)	(0.606, ± 0.908)	(0.599, ± 0.900)
9/9	(0.910, ± 0.7414)	(0.659, ± 0.9412)	(0.603, ± 0.911)	(0.598, ± 0.901)
10/10	(0.914, ± 0.6800)	(0.678, ± 0.9654)	(0.6129, ± 0.922)	(0.604, ± 0.911)
11/11	(0.9198, ± 0.7265)	(0.658, ± 0.9444)	(0.608, ± 0.916)	(0.601, ± 0.906)
12/12	(0.9124, ± 0.7157)	(0.6742, ± 0.9357)	(0.608, ± 0.912)	(0.601, ± 0.905)

Table 11: Pade approximants of velocity potential series at $r=1$

N/N	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
2/2		(1.08, ± 1.826)	(-0.452, ± 1.344)	(-1.78, ± 0.288)
3/3	(1.21, ± 1.15)	(0.575, ± 1.152)	0.2103	(-0.804, ± 0.311)
4/4	(0.958, ± 1.092)	(0.575, ± 1.152)	(-0.141, ± 0.476)	(-0.833, ± 0.467)
5/5	(0.881, ± 0.991)	(0.586, ± 1.165)	(-0.293, ± 0.932)	(-0.809, ± 0.434)
6/6	(0.881, ± 0.992)	(0.337, ± 1.125)	(-0.311, ± 0.908)	(-0.673, ± 0.467)
7/7	(0.885, ± 0.986)	(0.313, ± 1.109)	(-0.321, ± 0.874)	(-0.690, ± 0.466)
8/8	(0.891, ± 0.983)	(0.273, ± 1.037)	(-0.292, ± 0.844)	(-0.677, ± 0.460)
9/9	(0.924, ± 0.981)	(0.264, ± 1.040)	(-0.299, ± 0.842)	(-0.675, ± 0.451)
10/10	(0.914, ± 0.977)	(0.284, ± 1.035)	(-0.285, ± 0.843)	(-0.658, ± 0.441)
11/11	(0.919, ± 0.980)	(0.281, ± 1.047)	(-0.294, ± 0.843)	(-0.661, ± 0.438)
12/12	(0.910, ± 0.982)	(0.282, ± 1.036)	(-0.295, ± 0.836)	(-0.659, ± 0.440)

Table 12: Pade approximants of velocity potential series at $r=2$

N/N	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
2/2	-3.81	(0.650, ± 2.52)	(0.362, ± 3.855)	(1.49, ± 6.601)
3/3	(1.21, ± 0.734)	(1.55, ± 0.754)	1.152	0.5386
4/4	0.8895	(0.828, ± 0.790)	(0.776, ± 0.789)	(0.744, ± 0.857)
5/5	(0.750, ± 0.864)	(0.785, ± 0.885)	(0.771, ± 0.883)	(0.750, ± 0.867)
6/6	(0.749, ± 0.867)	(0.750, ± 0.953)	(0.764, ± 0.905)	(0.751, ± 0.867)
7/7	(0.608, ± 0.933)	(0.651, ± 0.919)	(0.783, ± 0.672)	(0.748, ± 0.861)
8/8	(0.599, ± 0.900)	(0.641, ± 0.923)	(0.626, ± 0.912)	(0.611, ± 0.907)
9/9	(0.598, ± 0.901)	(0.640, ± 0.924)	(0.657, ± 0.912)	(0.611, ± 0.901)
10/10	(0.604, ± 0.911)	(0.653, ± 0.921)	(0.664, ± 0.928)	(0.611, ± 0.916)
11/11	(0.601, ± 0.906)	(0.642, ± 0.927)	(0.633, ± 0.917)	(0.609, ± 0.911)
12/12	(0.601, ± 0.905)	(0.648, ± 0.930)	(0.661, ± 0.927)	(0.601, ± 0.913)

Table 13: Pade approximants of Surface Pressure series

N/N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
2/2	(0.521, ± 1.027)	(0.521, ± 0.573)	(0.364, ± 0.288)	(0.268, ± 0.184)
3/3	(0.521, ± 1.027)	(0.517, ± 0.580)	(0.393, ± 0.344)	(0.329, ± 0.218)
4/4	(0.522, ± 1.026)	(0.530, ± 0.649)	(0.432, ± 0.382)	(0.354, ± 0.221)
5/5	(0.528, ± 1.059)	(0.530, ± 0.647)	(0.434, ± 0.380)	(0.355, ± 0.223)
6/6	(0.528, ± 1.045)	(0.527, ± 0.642)	(0.434, ± 0.380)	
7/7	(0.521, ± 1.040)	(0.527, ± 0.642)	(0.433, ± 0.381)	
8/8	(0.498, ± 0.983)	(0.526, ± 0.641)	(0.436, ± 0.381)	
9/9	(0.533, ± 1.048)	(0.525, ± 0.641)		
10/10	(0.505, ± 0.991)	(0.525, ± 0.642)		
11/11	(0.509, ± 1.076)			
12/12	(0.525, ± 1.016)			

Table 14: Pade approximants of pressure coefficient series at $r=1$

N/N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
2/2	(0.702, ± 0.776)	(0.02, ± 1.009)	(-0.686, ± 0.717)	-1.043
3/3	(0.742, ± 0.745)	(0.0, ± 1.038)	(-0.734, ± 0.740)	-1.044
4/4	(0.737, ± 0.729)	(0.0, ± 1.039)	(-0.737, ± 0.736)	-1.013
5/5	(0.736, ± 0.727)	(0.0, ± 1.039)	(-0.730, ± 0.727)	-1.015
6/6	(0.721, ± 0.721)	(0.0, ± 1.010)	(-0.726, ± 0.720)	-1.025
7/7	(0.719, ± 0.716)	(0.0, ± 1.014)	(-0.718, ± 0.716)	-1.022
8/8	(0.717, ± 0.716)	(0.0, ± 1.014)	(-0.718, ± 0.715)	-1.006
9/9	(0.718, ± 0.714)	(0.0, ± 1.014)	(-0.717, ± 0.713)	
10/10	(0.716, ± 0.715)	(0.0, ± 1.006)	(-0.714, ± 0.714)	
11/11	(0.715, ± 0.715)	(0.0, ± 1.007)		
12/12	(0.714, ± 0.713)	(0.0, ± 1.007)		
13/13	(0.712, ± 0.712)			

Table 15: Pade approximants of pressure coefficient series at $r=2$

N/N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
2/2	(2.878, ± 2.921)	(0.035, ± 3.968)	(-2.806, ± 2.871)	-4.146
3/3	(2.922, ± 2.963)	(0.002, ± 4.156)	(-2.934, ± 2.959)	-4.144
4/4	(2.941, ± 2.938)	(0.001, ± 4.161)	(-2.945, ± 2.939)	-4.137
5/5	(2.926, ± 2.918)	(0.000, ± 4.158)	(-2.922, ± 2.912)	-4.120
6/6	(2.896, ± 2.883)	(0.000, ± 4.031)	(-2.900, ± 2.881)	-4.155
7/7	(2.875, ± 2.869)	(0.000, ± 4.058)	(-2.877, ± 2.872)	-4.110
8/8	(2.878, ± 2.862)	(0.000, ± 4.056)	(-2.879, ± 2.859)	-4.124
9/9	(2.870, ± 2.865)	(0.000, ± 4.059)	(-2.871, ± 2.864)	-4.178
10/10	(2.871, ± 2.858)	(0.024, ± 4.052)	(-2.872, ± 2.859)	-4.020
11/11	(2.873, ± 2.855)	(0.023, ± 4.051)	(-2.872, ± 2.859)	-4.020
12/12	(2.871, ± 2.854)	(0.020, ± 4.052)	(-2.870, ± 2.857)	-4.020

Table 16: Pade approximants of Surface Potential series

N/N	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
2/2	0.030	(1.37, ± 3.99)	(5.01, ± 0.820)	-0.1514
3/3	0.068	(0.956, ± 0.955)	(0.823, ± 0.955)	(0.566, ± 0.550)
4/4	0.101	(0.881, ± 0.992)	(0.726, ± 0.973)	(0.612, ± 0.652)
5/5	0.137	(0.881, ± 0.992)	(0.731, ± 0.992)	(0.621, ± 0.651)
6/6	0.171	(0.885, ± 0.967)	(0.560, ± 1.039)	(0.621, ± 0.651)
7/7	0.213	(0.981, ± 0.984)	(0.575, ± 1.020)	(0.621, ± 0.651)
8/8	0.247	(0.924, ± 1.033)	(0.575, ± 1.019)	(0.585, ± 0.669)
9/9	0.250	(0.911, ± 1.041)	(0.575, ± 1.019)	
10/10	0.298	(0.928, ± 1.086)	(0.581, ± 1.022)	
11/11	0.311	(0.924, ± 1.012)	(0.577, ± 1.014)	
12/12	0.335			
13/13	0.337			

Now that we have the Pade approximants of our series, I shall apply Hunter's method for a closest pair of complex singularities to the series which have a periodically repeating sign pattern . Applying this method will serve two purposes: first, the Hunter algorithm will be checked for efficiency as we apply it on series not having analytic form for their terms(our series) ; second, to check our results for the Pade approximants as the closest singularity.

V.2. Hunter's Algorithm:

Given a finite number of terms in the expansion of a function

$$p(t) = \sum_{n=0}^N p_n t^n \quad 5.2.1$$

the p_n terms are the coefficients of the Taylor series expansion of the function about the origin. The circle of convergence of the Taylor series is limited by the singularity of $p(t)$ closest to the origin which governs the asymptotic behavior for large n .

The Algorithm assumes a specific, though fairly general structure of the singularities, based up on the consequent asymptotic formulas for the Taylor series coefficients as given by

Darboux's theorem, in such a way that successive levels of iteration are more exact.

The results of the derivations by Hunter lead to the following recurrence relations to be used in solving a nonlinear system of equations for the unknowns R (distance from the origin), the cosine of the angle of the singularities, and v (the exponent of the singularities)

$$S_n^0(R, \cos\theta, v) = p_n \quad 5.2.2a$$

$$\begin{aligned} S_n^{m+1}(R, \cos\theta, v) = & R^2 S_n^m(R, \cos\theta, v) \\ & - \frac{2(n+v-2m-1)}{n} R \cos\theta S_{n-1}^m(R, \cos\theta, v) \\ & + \frac{(n+2v-3m-2)(n-m-1)}{n(n-1)} S_{n-2}^m(R, \cos\theta, v) \end{aligned} \quad 5.2.2b$$

Where $S_n^m(R, \cos\theta, v)$ is the m th degree polynomial in R , $\cos\theta$ and v .

and the system of equations to be solved is

$$S_n^m(R, \cos\theta, v) = S_{n-N}^m(R, \cos\theta, v) = S_{n-2N}^m(R, \cos\theta, v) = 0 \quad 5.2.2c$$

Hunter also claims that the highest order Taylor series coefficient that is involved in these equations (when a limited number of coefficients are available) is the highest order term where n is as large as possible and the errors are least when $N=1$.

In order to be able to use the Newton-Raphson method for solving a nonlinear system of equations, we need to find the expressions of the partial derivatives. These derivatives also satisfy the following recurrence relations:

$$\frac{\partial S_n^0}{\partial R} = \frac{\partial S_n^0}{\partial \cos \theta} = \frac{\partial S_n^0}{\partial v} = 0 \quad 5.2.3a$$

$$\begin{aligned} \frac{\partial S_n^{m+1}}{\partial R} = & 2R S_n^m(R, \cos \theta, v) + R^2 \frac{\partial S_n^m}{\partial R}(R, \cos \theta, v) \\ & - \frac{2(n+v-2m-1)}{n} \cos \theta S_{n-1}^m(R, \cos \theta, v) \\ & - \frac{2(n+v-2m-1)}{n} R \cos \theta \frac{\partial S_{n-1}^m}{\partial R}(R, \cos \theta, v) \\ & + \frac{(n+2v-3m-2)(n-m-1)}{n(n-1)} \frac{\partial S_{n-2}^m}{\partial R}(R, \cos \theta, v) \end{aligned} \quad 5.2.3b$$

$$\begin{aligned} \frac{\partial S_n^{m+1}}{\partial \cos \theta} = & R^2 \frac{\partial S_n^m}{\partial \cos \theta}(R, \cos \theta, v) - \frac{2(n+v-2m-1)}{n} R S_{n-1}^m(R, \cos \theta, v) \\ & - \frac{2(n+v-2m-1)}{n} R \cos \theta \frac{\partial S_{n-1}^m}{\partial \cos \theta}(R, \cos \theta, v) \\ & + \frac{(n+2v-3m-2)(n-m-1)}{n(n-1)} \frac{\partial S_{n-2}^m}{\partial \cos \theta}(R, \cos \theta, v) \end{aligned} \quad 5.2.3c$$

$$\begin{aligned} \frac{\partial S_n^{m+1}}{\partial v} = & R^2 \frac{\partial S_n^m}{\partial v}(R, \cos \theta, v) - \frac{2R \cos \theta}{n} S_{n-1}^m(R, \cos \theta, v) \\ & - \frac{2(n+v-2m-1) R \cos \theta}{n} \frac{\partial S_{n-1}^m}{\partial v}(R, \cos \theta, v) \\ & + \frac{2(n-m-1)}{n(n-1)} S_{n-2}^m(R, \cos \theta, v) \\ & + \frac{(n+2v-3m-2)(n-m-1)}{n(n-1)} \frac{\partial S_n^m}{\partial v}(R, \cos \theta, v) \end{aligned} \quad 5.2.3d$$

Let n be the largest available index of the coefficients from the series. Then we need to solve the following three equations

$$S_n^m = S_{n-1}^m = S_{n-2}^m = 0 \quad 5.2.4$$

The description of the Newton-Raphson Algorithm for solving nonlinear equations is fully described in Appendix A.

Defining $\text{PART1}(R, \cos\theta, v) = \frac{\partial S_n^{m+1}}{\partial R}$

$$\text{PART2}(R, \cos\theta, v) = \frac{\partial S_n^{m+1}}{\partial \cos\theta}$$

$$\text{PART3}(R, \cos\theta, v) = \frac{\partial S_n^{m+1}}{\partial v}$$

$$\beta_1 = S_n^{m+1}(R, \cos\theta, v), \quad \beta_2 = S_{n-1}^{m+1}(R, \cos\theta, v), \quad \beta_3 = S_{n-2}^{m+1}(R, \cos\theta, v)$$

$$\alpha_{11} = \frac{\partial S_n^{m+1}}{\partial R} \quad \alpha_{12} = \frac{\partial S_n^{m+1}}{\partial \cos\theta} \quad \alpha_{13} = \frac{\partial S_n^{m+1}}{\partial v}$$

$$\alpha_{21} = \frac{\partial S_{n-1}^{m+1}}{\partial R} \quad \alpha_{22} = \frac{\partial S_{n-1}^{m+1}}{\partial \cos\theta} \quad \alpha_{23} = \frac{\partial S_{n-1}^{m+1}}{\partial v}$$

$$\alpha_{31} = \frac{\partial S_{n-2}^{m+1}}{\partial R} \quad \alpha_{32} = \frac{\partial S_{n-2}^{m+1}}{\partial \cos\theta} \quad \alpha_{33} = \frac{\partial S_{n-2}^{m+1}}{\partial v}$$

then , the problem of solving our three nonlinear equations is directly applied as shown in the Subroutines of Appendix C. This process is applied to the series of our physical quantities. Observe that not all the series could be analyzed using this technique as mentioned by Hunter.

The following tables summarize the results for those series which have a sufficient number of terms , show a convergence of the required parameters for our closest singularities , and

also agree with our findings for the singularities using Pade approximants.

The first three columns of the following tables are the three unknowns we wish to find, and the last two columns are the x and y coordinates of the closest singularity while n is the number of iterations performed.

Table 17: Hunter results for the pressure series at $r=1$, $ANG=\pi/8$

n	R	$\cos\theta$	v	x	y
1	.269788	.361529	.573365	.097536	.251539
2	.225402	.712314	.790651	.160557	.158201
3	.387470	.381214	.720907	.147709	.358210
4	.370054	.397008	.763965	.146914	.339641
5	.377458	.393201	.764698	.148417	.347055
6	1.003459	.707275	.654525	.709722	.709384
7	1.003459	.707275	.654525	.709722	.709384

Table 18: Hunter results of the pressure series at $r=1$, $ANG=\pi/4$

n	R	$\cos\theta$	v	x	y
1	.580876	.179518	-4.99761	.104277	.571440
2	.578483	.247068	-4.79111	.142925	.560549
3	.631320	.237183	-4.89605	.149739	.613305
4	1.002954	.000002	-.642611	.000002	1.002954
5	1.002954	.000002	-.642611	.000002	1.002954
6	1.002954	.000002	-.642611	.000002	1.002954

Table 19: Hunter results of the pressure series at $r=1$, $ANG=3\pi/8$

n	R	$\cos\theta$	v	x	y
1	.651645	-.02385286	1.667015	-.015543	.6514598
2	.568828	.13965422	1.682707	.079439	.5632543
3	.580876	.17951820	1.997613	.104277	.5714400
4	.578483	.24706886	1.791110	.142925	.5605497
5	.631320	.23718383	1.896055	.149739	.6133057
6	1.059290	-.62885707	1.483294	-.666142	.8236205
7	1.059290	-.62885707	1.483294	-.666142	.8236205
8	1.002884	-.70724694	.641270	-.709286	.7090055
9	1.002884	-.70724694	.641270	-.709286	.7090055
10	1.002884	-.70724694	.641270	-.709286	.7090055

Table 20: Hunter results for the pressure series at $r=2$, $ANG=\pi/8$

n	R	$\cos\theta$	v	x	y
1	6.325104	.9894237	1.931010	6.25820	.91748
2	6.222508	.9557231	1.852134	5.94699	1.83108
3	5.780692	.9000178	1.762014	5.20272	2.51953
4	5.644480	.8881007	1.621045	5.01286	2.59447
5	5.448059	.8667510	1.952147	4.72210	2.71716
6	5.280583	.8422250	1.954217	4.45186	2.86900
7	4.488982	.8210354	1.582104	3.68561	2.56265
8	4.404981	.7074590	.999449	3.11634	3.11323
9	4.028967	.7074590	.999449	2.85032	2.84748
10	4.028967	.7074590	.999449	2.85032	2.84748

Table 21: Hunter results of the pressure series at $r=2$, $ANG=\pi/4$

n	R	$\cos\theta$	v	x	y
1	5.8884152	-.3510247	-3.232810	-2.066972	5.513712
2	5.6841028	-.3410025	-3.219438	-1.938292	5.343408
3	5.0983112	-.3201548	-3.111024	-1.595364	4.829636
4	4.6158804	-.2741296	-2.984213	-1.265348	5.922876
5	4.3768416	-.2548735	-2.785101	-1.115540	4.232292
6	4.0291924	-.0000155	-2.145014	-.0000600	4.029192
7	4.0291924	-.0000155	-2.145014	-.0000600	4.029192
8	4.0291924	-.0000155	-2.145014	-.0000600	4.029192

Table 22: Hunter results of the pressure series at $r=2$, $ANG=3\pi/8$

n	R	$\cos\theta$	v	x	y
1	4.850344	-.333610	-1.719231	-1.61812	4.57244
2	4.477784	-.400892	-1.719231	-1.79510	4.10220
3	4.208004	-.521864	-1.712563	-2.19600	3.58954
4	4.029088	-.7074185	-4.800364	-2.85025	2.84774
5	4.029088	-.7074185	-4.800364	-2.85025	2.84774
6	4.029088	-.7074185	-4.800364	-2.85025	2.84774

These results verify our Pade findings for the pressure coefficient series at $r=1$ and at $r=2$. All other series do not show convergence using the Hunter algorithm because of an insufficient number of terms for asymptotic analysis.

Using our closest complex conjugate singularities, we can apply an Euler transformation and analyse the resulting series for any real physical singularities as follows.

V.3. Euler Transformation

Assuming we have a truncated series

$$f(z) = \sum_{n=1}^N a_n z^n$$

and that the closest singularity to the origin of this series is a pair of complex conjugates , then these nonphysical singularities lie closer to the origin than the physically interesting one.

We first apply an Euler transformation to the series where this transformation is chosen so that in the complex plane of the transformed variable the interfering singularities lie beyond the circle defined by the singularity being studied.

Consider the Euler Transformation defined by

$$w = (1+a) z / (1+az) \quad 5.3.1$$

where a is a real constant.

Baker considered the effect of such Euler transformations and showed that they left certain Pade approximants invariant. In this Euler transformation , the origin is taken as a fixed point and the physical singularity the other fixed point.

Once the complex singularities are mapped outside the unit circle , we can determine the real closest singularity to the origin.

If $z_0 = (x_0, y_0)$ is to be mapped to w_0 , we define the optimum transformation to be the one which maximizes

$$|w_0|^2 = \frac{(1+a)^2 z z_0^*}{(1+ a z) (1+ a z_0^*)} \quad 5.3.2$$

The condition $\frac{\partial}{\partial a} |w_0|^2 = 0$ gives

$$a = \frac{(x_0-1)}{(x_0(1-x_0)-y_0^2)} \quad 5.3.3$$

We now simplify to get the coefficients of the transformed series proceeding as follows:

Given

$$f(\varepsilon) = \sum_{n=0}^N b_n \varepsilon^n \quad 5.3.4$$

Using the transformation $w = \frac{(1+a)\varepsilon}{1+a\varepsilon}$, we get

$$\varepsilon = \frac{w}{(1+a)-aw} = \frac{w}{(1+a)\left[1 - \frac{aw}{1+a}\right]}$$

Then

$$F(w) = \sum_{n=0}^N b_n \frac{w^n}{(1+a)^n} \left[1 - \frac{aw}{1+a}\right]^{-n} \quad \text{Where } |aw| \ll (1+a)$$

If we define:

$$C_{ij} = \frac{j(j+1)\cdots(j+i-1)}{i!} \quad \begin{matrix} j \neq 0 \\ j = 0 \end{matrix}$$

then $F(w)$ can be simplified as follows

$$\begin{aligned} F(w) &= \sum_{n=0}^N \frac{b_n}{(1+a)^n} w^n \left[1 + n \frac{aw}{1+a} + \frac{n(n+1)}{2!} \left(\frac{aw}{1+a}\right)^2 + \cdots \right] \\ &= \sum_{n=0}^N \frac{b_n}{(1+a)^n} w^n \left(C_{0n} + C_{1n} \frac{aw}{1+a} + C_{2n} \left(\frac{aw}{1+a}\right)^2 + \cdots \right) \\ &= \sum_{n=0}^N \frac{b_n}{(1+a)^n} w^n \left[\sum_{k=0}^{\infty} C_{kn} \left(\frac{aw}{1+a}\right)^k \right] \end{aligned}$$

giving

$$F(w) = \sum_{n=0}^N \sum_{k=0}^{\infty} C_{kn} \frac{b_n a^k}{(1+a)^{n+k}} w^{n+k} \quad 5.3.5$$

This Euler transformation is now applied to the series of our physical quantities. Since the purpose at hand is to determine the real closest physical singularity, only those series which

give a real singularity after the first Euler transformation are presented here. Some of the series lead to another closest pair of complex singularities even after transformation. The following tables summarize the findings:

Table 23: Euler transformed stream function series coefficients at $r=1$

N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	0.0000000D+00	0.0000000D+00	0.0000000D+00	0.0000000D+00
1	0.0000000D+00	0.0000000D+00	0.0000000D+00	0.0000000D+00
2	0.1793366D-01	0.1400870D+00	0.2885438D+00	0.4897959D+00
3	0.3276079D-01	0.2134405D+00	0.3938823D+00	0.3498542D+00
4	0.4405877D-01	0.2323870D+00	0.3950606D+00	0.1817993D+00
5	0.5177400D-01	0.2144192D+00	0.3448111D+00	0.6675790D-01
6	0.5611882D-01	0.1762910D+00	0.2754078D+00	0.2528443D-02
7	0.5748204D-01	0.1311855D+00	0.2049077D+00	-0.2619365D-01
8	0.5635452D-01	0.8794735D-01	0.1422531D+00	-0.3314898D-01
9	0.5326958D-01	0.5149400D-01	0.9093903D-01	-0.2824046D-01
10	0.4875766D-01	0.2374173D-01	0.5144325D-01	-0.1805028D-01
11	0.4331376D-01	0.4622175D-02	0.2273376D-01	- 0.6734292D-02
12	0.3737593D-01 -	0.7039435D-02	0.3152245D-02	0.3252506D-02
13	0.3131311D-01 -	0.1287271D-01 -	0.9100586D-02	0.1064581D-01
14	0.2542028D-01 -	0.1453988D-01 -	0.1571976D-01	0.1501506D-01
15	0.1991936D-01 -	0.1350309D-01 -	0.1818863D-01	0.1649843D-01
16	0.1496410D-01 -	0.1092773D-01	0.1774654D-01	0.1559226D-01
17	0.1064771D-01 -	0.7671504D-02 -	0.1539446D-01	0.1297824D-01
18	0.7012036D-02 -	0.4319350D-02 -	0.1191802D-01	0.9384001D-02
19	0.4057276D-02 -	0.1236967D-02	-0.7916586D-02	0.5478274D-02
20	0.1751667D-02	0.1373844D-02	-0.3832873D-02	0.1801444D-02
21	0.4049800D-04	0.3426445D-02	0.1948798D-04	- 0.1269702D-02
22	-0.1145819D-02	0.4910403D-02	0.3431459D-02	-0.3527393D-02

23	-0.1884857D-02	0.5862363D-02	0.6277334D-02	-0.4915244D-02
24	-0.2256520D-02	0.6345371D-02	0.8496551D-02	-0.5495646D-02

Table 24: Euler transformed stream function series coefficients at $r=2$

N	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
0	0.5740251D+00	0.1060660D+01	0.1385819D+01	0.1500000D+01
1	0.8750387D-01	0.3526629D+00	0.5513870D+00	0.6072261D+00
2	0.4296103D-01	0.2251520D+00	0.3911159D+00	0.4415796D+00
3	0.2017679D-02	0.7656716D-01	0.1605711D+00	0.1793039D+00
4	-0.7203551D-02	0.6248072D-02	0.3965491D-01	0.4617543D-01
5	-0.3427006D-02	-0.1151237D-01	-0.4050871D-02	-0.2515075D-02
6	0.4813188D-03	-0.9029208D-02	-0.1230495D-01	-0.1281329D-01
7	0.1932123D-02	-0.2550526D-02	-0.8048044D-02	-0.9028837D-02
8	0.1762306D-02	0.2391634D-02	-0.1365285D-02	-0.2013752D-02
9	0.9404774D-03	0.4720821D-02	0.3885321D-02	0.3849675D-02
10	0.7885127D-04	0.4844501D-02	0.6610821D-02	0.7138050D-02
11	-0.4856665D-03	0.3585471D-02	0.6955723D-02	0.7825345D-02
12	-0.6560573D-03	0.1783184D-02	0.5603277D-02	0.6548274D-02
13	-0.5240384D-03	0.9897742D-04	0.3393233D-02	0.4186805D-02
14	-0.2644221D-03	-0.1075081D-02	0.1082527D-02	0.1586286D-02
15	-0.2856089D-04	-0.1618188D-02	-0.7926555D-03	-0.6168522D-03
16	0.1104146D-03	-0.1615246D-02	-0.1964323D-02	-0.2071091D-02
17	0.1530568D-03	-0.1260013D-02	-0.2406238D-02	-0.2699864D-02
18	0.1366657D-03	-0.7633529D-03	-0.2260874D-02	-0.2631991D-02
19	0.9915771D-04	-0.2906284D-03	-0.1750757D-02	-0.2106034D-02
20	0.6160183D-04	0.6430724D-04	-0.1100303D-02	-0.1379368D-02

Table 25: Euler transformed surface pressure coefficient series

N	$\Pi/8$	$\Pi/4$
0	-0.2000000D+01	-0.1000000D+01
1	-0.9764566D+00	-0.5000000D+00
2	-0.7417518D+00	-0.6250000D+00
3	-0.6328135D+00	-0.3046876D+00
4	-0.5394327D+00	0.4492188D-01
5	-0.4237422D+00	0.2194824D+00
6	-0.2842131D+00	0.2130738D+00
7	-0.1354842D+00	0.1153737D+00
8	0.3352428D-02	0.1675561D-01
9	0.1155033D+00	-0.3697101D-01
10	0.1899845D+00	-0.4433661D-01
11	0.2228495D+00	-0.2639619D-01
12	0.2167974D+00	-0.5163925D-02
13	0.1797113D+00	0.7351444D-02
14	0.1226256D+00	0.9664063D-02
15	0.5756386D-01	0.6016196D-02
16	-0.4400069D-02	0.1325065D-02
17	-0.5460495D-01	-0.1559972D-02
18	-0.8765601D-01	-0.2165369D-02
19	-0.1016513D+00	-0.1384027D-02
20	-0.9785435D-01	-0.3249609D-03
21	-0.7995673D-01	0.3438894D-03
22	-0.5311029D-01	0.4946111D-03
23	-0.2290946D-01	0.3216713D-03
24	0.5517577D-02	
25	0.2819283D-01	
26	0.4268513D-01	

The Pade approximants of the series which give physical singularities are summarized in the following table

Table 26: Pade approximants of Euler transformed series

N/N	$\Psi(1,\pi/8)$	$\Psi(1,3\pi/8)$	$C_{p,s}(\pi/8)$	$C_{p,s}(\pi/4)$
2/2	1.188720	-0.8312513	0.907841	0.893555
3/3	1.0138	0.999505	0.978413	0.961972
4/4	0.972178	1.003677	1.094111	0.873000
5/5	0.998941	0.997928	0.995287	0.992457
6/6	0.999588	1.000164	1.001208	1.004941
7/7	0.999407	1.000242	0.999235	1.000142
8/8	1.000371	1.000360	0.999364	0.999449
9/9	1.000080	1.000345	1.000704	0.999403
10/10	1.000482	1.000822	0.999319	0.999215
11/11	1.000930	0.999159	0.999789	0.999618
12/12	1.000196	1.001035	0.999788	1.000308

N/N	$\Psi(2,\pi/8)$	$\Psi(2,3\pi/8)$
2/2	1.326039	1.178951
3/3	1.216627	1.114830
4/4	0.966615	1.100217
5/5	0.998129	1.082333
6/6	0.999586	1.066165
7/7	0.998857	1.000537
8/8	1.000311	0.999953
9/9	0.999965	0.999317
10/10	0.955218	0.997347

CHAPTER VI

SUMMARY AND CONCLUSIONS

The series $\Psi(1,\pi/2)$, $C_p(1,\pi/2)$ and $C_p(2,\pi/2)$ show nearest singularities of -0.74,-1.0 and -4.0 consecutively, which are nonphysical as they lie on the negative real axis of the ϵ -complex plane.

The pressure coefficient series at both $r=1$ and $r=2$ show a very interesting phenomenon of the closest singularities being on the circle of radius r^2 (at the point $r^2 e^{i\theta}$) which has a physical meaning I was unable to determine. The series $\Phi_s(0)$ has a real singularity converging towards 0.333 but not very convincingly. Beyond the value 0.333, the flow faces a point of inflexion of the body as seen in Figure 1. All other series exhibit a complex conjugate pair of singularities which varies depending on the point of evaluation with no apparent trend. The following table summarizes the Pade approximant findings (the points are the ones towards which the indicated series converges)

Table 27: Summary of the Pade approximants of all tested series

ANG	$\Pi/8$	$\Pi/4$	$3\Pi/8$	$\Pi/2$
$\Psi(1,\theta)$	(0.280,±1.097)	(-0.293,±0.829)	(-0.657,±0.448)	-0.74681
$\Psi(2,\theta)$	(0.912,±0.715)	(0.674,±0.935)	(0.608,±0.912)	(0.601,±0.905)
$C_{p,s}(\theta)$	(0.528,±1.059)	(0.530,±0.647)	(0.434,±0.380)	(0.355,±0.223)
$C_p(1,\theta)$	(0.717,±0.716)	(0.000,±1.014)	(-0.718,±0.715)	-1.006
$C_p(2,\theta)$	(2.871,±2.854)	(0.020,±4.052)	(-2.870,±2.857)	-4.020
ANG	0	$\Pi/8$	$\Pi/4$	$3\Pi/8$
$\Phi_s(\theta)$	0.337	(0.924,±1.012)	(0.575,±1.014)	(0.585,±0.669)
$\Phi(1,\theta)$	(0.910,±0.982)	(0.282,±1.036)	(-0.295,±0.836)	(-0.659,±0.440)
$\Phi(2,\theta)$	(0.601,±0.905)	(0.648,±0.930)	(0.661,±0.927)	(0.601,±0.913)

The Hunter method is elegant but troublesome in the sense that the initial guess of the parameters of the singularities has to be as close as possible to where they really are to see a converging trend. Also, the series must have a sufficient number of terms since the procedure is based on asymptotic considerations. This method can only be useful if some other method has been used to provide an initial guess for our parameters and a large number of terms is available. The application of this method to our series showed that only the pressure coefficient series give pairs of closest singularities which agree with our Pade approximant results.

After performing an Euler transformation on our series and analyzing the new series, some series show another pair of closest complex singularities, while the six series $\Psi(1,\pi/8)$, $\Psi(1,3\pi/8)$, $\Psi(2,\pi/8)$, $\Psi(2,3\pi/8)$, $C_{p,s}(\pi/8)$ and $C_{p,s}(\pi/4)$ show the real singularity $\epsilon = 1.0$, an expected limiting value of ϵ since the body is not defined for ϵ greater than 1.

The calculations reveal that the location and structure of the nearest singularity of a perturbation series do change with the field point at which the series is evaluated. Also the radius of convergence and behavior of the perturbation series do depend significantly on the choice of physical parameters. These results settle the queries addressed in Whitley's paper and provide salutary information for the analysis of computer extended perturbation series.

APPENDICES

APPENDIX A

GENERAL FORMS AND PROCEDURES

(A1) Computation of the coefficients $B(l,j,k)$

To compute the coefficients $B_l^{j,k}$, we proceed as follows:

Using the well known formulas

$$\sin^{2n}\theta = \frac{(-1)^n}{2^{2n-1}} \left[\sum_{i=0}^{n-1} (-1)^i \binom{2n}{i} \cos 2(n-i)\theta + (-1)^n \left(\frac{1}{2}\right) \binom{2n}{n} \right]$$

$$\cos nx \sin mx = \frac{1}{2} (\sin (n+m)x - \sin (n-m)x)$$

We arrive at the general form

$$\begin{aligned} \sin(2k+1)\theta \sin^{2n}\theta &= \frac{(-1)^n}{2^{2n}} \left[\sum_{j=0}^{n-1} (-1)^j \binom{2n}{j} \{ [\sin[2(k+n-j)+1]\theta] + \sin[2(k-n+j)+1]\theta \} \right] \\ &\quad + \frac{1}{2^{2n}} \binom{2n}{n} \sin(2k+1)\theta \end{aligned}$$

Which can be written as

$$\sin(2k+1)\theta \sin^{2n}\theta = \sum_{l=0}^{k+n} B_l^{k,n} \sin(2l+1)\theta$$

With $B_l^{k,n}$ having the obvious notation which can directly be implemented as a computer procedure as in Appendix C.

(A2) Newton-Raphson Method For Nonlinear Equation Systems:

Given $f_i(x_1, \dots, x_N) = 0$ or $f(\mathbf{x}) = 0$

Expanding in the neighborhood of $\mathbf{x}$ as a Taylor series and neglecting terms of order $\delta \mathbf{x}^2$ or higher

$$f_i(\mathbf{x} + \delta \mathbf{x}) \cong f_i(\mathbf{x}) + \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \delta x_j = 0$$

$$\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \delta x_j = -f_i(\mathbf{x})$$

and in general
$$\sum_{j=1}^n \alpha_{ij} \delta x_j = \beta_i \dots\dots\dots(1)$$

The matrix equation (1) of the linear equations set is solved for δx_j . The corrections are then added to the solution vector

$$x_i^{\text{new}} = x_i^{\text{old}} + \delta x_i \quad i=1,2,\dots,N$$

The process is iterated for convergence.

APPENDIX B

DERIVATION OF GENERAL FORMULAS

(B1) Derivations of the velocity potential series:

Now suppose

$$\Psi(r, \theta) = r \sin \theta + A_{00} \frac{\sin \theta}{r} + \varepsilon \left(A_{10} \frac{\sin \theta}{r} + A_{11} \frac{\sin 3\theta}{r^3} \right) + \varepsilon^2 \left(A_{20} \frac{\sin \theta}{r} + A_{21} \frac{\sin 3\theta}{r^3} + A_{22} \frac{\sin 5\theta}{r^5} \right) + \dots$$

is known, then the potential function Φ (the harmonic conjugate) is given by

$$\Phi(r, \theta) = r \cos \theta - A_{00} \frac{\cos \theta}{r} - \varepsilon \left(A_{10} \frac{\cos \theta}{r} + A_{11} \frac{\cos 3\theta}{r^3} \right) - \varepsilon^2 \left(A_{20} \frac{\cos \theta}{r} + A_{21} \frac{\cos 3\theta}{r^3} + A_{22} \frac{\cos 5\theta}{r^5} \right) + \dots$$

To find the potential on the surface, we proceed as follows

$$\begin{aligned} \Phi &= r \cos \theta - \frac{\cos \theta}{r} (A_{00} + \varepsilon A_{10} + \varepsilon^2 A_{20} + \dots) - \frac{\cos 3\theta}{r^3} (\varepsilon A_{11} + \varepsilon^2 A_{21} + \dots) - \dots \\ &= (1 - \varepsilon \sin^2 \theta) \cos \theta - \cos \theta (C_{00} + \varepsilon C_{01} \sin^2 \theta + \varepsilon^2 C_{02} \sin^4 \theta + \dots) ((A_{00} + \varepsilon A_{10} + \dots) \\ &\quad - \cos 3\theta (C_{10} + \varepsilon C_{11} \sin^2 \theta + C_{12} \varepsilon^2 \sin^4 \theta + \dots) (\varepsilon A_{11} + \varepsilon^2 A_{21} + \dots) - \dots \\ &\quad - \cos(2j+1)\theta (C_{j0} + C_{j1} \varepsilon \sin^2 \theta + \dots + C_{jk} \varepsilon^k \sin^{2k} \theta + \dots) (\varepsilon^j A_{jj} + \varepsilon^{j+1} A_{j+1,j} + \dots)) \end{aligned}$$

Note that the constants $C(i,j)$ and $A(i,j)$ are the same as defined in the text of the thesis.

Simplifying further

$$\begin{aligned} \Phi &= \cos \theta - \varepsilon \sin^2 \theta \cos \theta \\ &\quad - \cos \theta (C_{00} + \varepsilon C_{01} \sin^2 \theta + \varepsilon^2 C_{02} \sin^4 \theta + \dots) (A_{00} + \varepsilon A_{10} + \varepsilon^2 A_{20} + \dots) \\ &\quad - \cos 3\theta (C_{10} + \varepsilon C_{11} \sin^2 \theta + \varepsilon^2 C_{12} \sin^4 \theta + \dots) (\varepsilon A_{11} + \varepsilon^2 A_{21} + \dots) \\ &\quad - \cos 5\theta (C_{20} + \varepsilon C_{21} \sin^2 \theta + \varepsilon^2 C_{22} \sin^4 \theta + \dots) (\varepsilon^2 A_{22} + \varepsilon^3 A_{32} + \dots) \\ &\quad - \cos(2j+1)\theta (C_{j0} + C_{j1} \varepsilon \sin^2 \theta + \dots) (\varepsilon^j A_{jj} + \varepsilon^{j+1} A_{j+1,j} + \dots) \end{aligned}$$

then,

$$O(1) : \quad \cos\theta (1 - C_{00} A_{00}) = 2 \cos\theta$$

$$O(\varepsilon) : \quad - \cos\theta (C_{00} A_{10} + \sin^2\theta + C_{01} A_{00} \sin^2\theta) \\ - \cos 3\theta (C_{10} A_{11})$$

$$O(\varepsilon^2) : \quad - \cos\theta (C_{00} A_{20} + C_{01} A_{10} \sin^2\theta + C_{02} A_{00} \sin^4\theta) \\ - \cos 3\theta (C_{10} A_{21} + A_{11} C_{11} \sin^2\theta) \\ - \cos 5\theta (C_{02} A_2)$$

And in general we would have

$$n \geq 2$$

$$O(\varepsilon^n) : - \cos\theta (C_{00} A_{n0} + C_{01} A_{n-1,0} \sin^2\theta + C_{02} A_{n-2,0} \sin^4\theta + \dots + C_{0n} A_{00} \sin^{2n}\theta) \\ - \cos 3\theta (C_{10} A_{n1} + C_{11} A_{n-1,1} \sin^4\theta + \dots + C_{1,n-1} A_{01} \sin^{2n-2}\theta) \\ \vdots \\ - \cos (2n+1)\theta (C_{n0} A_{nn})$$

So the series elements of the potential function Φ at the surface are given by

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + \varepsilon^2 \Phi_2 + \dots$$

$$\Phi_0 = 2 \cos\theta$$

$$\Phi_1 = - \cos\theta (C_{00} A_{10} + \sin^2\theta + C_{01} A_{00} \sin^2\theta) \\ - \cos 3\theta (C_{10} A_{11})$$

$$\Phi_n = - \left(\sum_{j=0}^n \sum_{k=0}^{n-j} C_{jk} A_{n-k,j} \cos(2j+1)\theta \sin^{2k}\theta \right)$$

(B2) Derivations of the pressure coefficient series:

The complex potential is defined to be

$$w = \Phi + i \Psi$$

so that it is given by

$$w = z - \frac{A_{00}}{z} - \varepsilon \left(\frac{A_{10}}{z} + \frac{A_{11}}{z^3} \right) - \varepsilon^2 \left(\frac{A_{20}}{z} + \frac{A_{21}}{z^3} + \frac{A_{22}}{z^5} \right) + \dots$$

and the complex velocity $V = u - i v = \frac{dw}{dz}$

$$V = 1 + \frac{A_{00}}{z^2} + \varepsilon \left(\frac{A_{10}}{z^2} + \frac{3A_{11}}{z^4} \right) + \varepsilon^2 \left(\frac{A_{20}}{z^2} + \frac{3A_{21}}{z^4} + \frac{5A_{22}}{z^6} \right) + \dots$$

Where u, v are the horizontal and vertical speeds consecutively.

To find the velocity, we substitute

$$z = r e^{i\theta}$$

$$\text{so that } \frac{1}{z^k} = r^{-k} (\cos k\theta - i \sin k\theta)$$

which gives the following expressions for the vertical and horizontal speeds:

$$u = \left(1 + \frac{A_{00} \cos 2\theta}{r^2} \right) + \sum_{k=1}^{\infty} u_k \varepsilon^k \quad \text{where} \quad u_k = \sum_{j=0}^k \frac{(2j+1) A_{kj} \cos 2(j+1)\theta}{r^{2(j+1)}}$$

$$v = \frac{A_{00} \sin 2\theta}{r^2} + \sum_{k=1}^{\infty} v_k \varepsilon^k \quad \text{where} \quad v_k = \sum_{j=0}^k \frac{(2j+1) A_{kj} \sin 2(j+1)\theta}{r^{2(j+1)}}$$

The pressure coefficient is defined as follows:

$$C_p = 1 - (u^2 + v^2)$$

which upon substitution would give:

$$C_p(r, \theta) = - \left(2 \frac{A_{00}^2 \cos^2 2\theta}{r^2} + \frac{A_{00}^2}{r^4} \right) - \sum_{k=1}^{\infty} \sum_{j=0}^k (u_j u_{k-j} + v_j v_{k-j}) \varepsilon^k$$

The above form can be used to compute the pressure coefficient at any field point

Now for the surface pressure coefficient we proceed as follows:

$$\text{Using } \binom{-k}{L} = \frac{-k \cdot (-k+1) \cdots (-k-L+1)}{L!} = (-1)^L k \cdot (k+1) \cdots (k+L-1)/L!$$

$$\begin{aligned} (1-\varepsilon \sin^2\theta)^{-(2j+2)} &= 1 + \sum_{L=1}^{\infty} \varepsilon^L \sin^{2L}\theta \frac{(2j+2)(2j+3) \cdots (2j+2+L-1)}{L!} \\ &= 1 + \varepsilon (2j+2) \sin^2\theta + \varepsilon^2 \frac{(2j+2)(2j+3)}{2!} \sin^4\theta + \cdots \end{aligned}$$

Define

$$U_{jk} = \frac{(2j+2)(2j+3) \cdots (2j+k+1)}{k!} \quad \begin{matrix} k \neq 0 \\ 1 \quad k=0 \end{matrix}$$

giving

$$(1-\varepsilon \sin^2\theta)^{-(2j+2)} = U_{j0} + U_{j1} \varepsilon \sin^2\theta + \varepsilon^2 U_{j2} \sin^4\theta + \cdots$$

Substituting this into the general form of the horizontal speed, we have

$$\begin{aligned} u &= 1 + \cos 2\theta (U_{00} + \varepsilon U_{01} \sin^2\theta + \varepsilon^2 U_{02} \sin^4\theta + \cdots) (A_{00} + A_{10} \varepsilon + A_{20} \varepsilon^2 + \cdots) \\ &\quad + 3 (\cos 4\theta) (U_{10} + \varepsilon U_{11} \sin^2\theta + \cdots) (\varepsilon A_{11} + \varepsilon^2 A_{21} + \varepsilon^3 A_{31} + \cdots) \\ &\quad + 5 \cos 6\theta (U_{20} + \varepsilon U_{21} \sin^2\theta + \cdots) (\varepsilon^2 A_{22} + \varepsilon^3 A_{32} + \cdots) \end{aligned}$$

Collecting like power terms :

$$O(1) : u_0 = 1 + A_{00} U_{00} \cos 2\theta$$

$$O(\varepsilon) : u_1 = \cos 2\theta (U_{01} A_{00} \sin^2\theta + U_{00} A_{10}) + 3 \cos 4\theta U_{10} A_{11}$$

$$\begin{aligned} O(\varepsilon^n) : u_n &= \cos 2\theta (U_{00} A_{n0} + U_{01} A_{n-1,0} \sin^2\theta + \cdots + U_{0n} A_{00} \sin^{2n}\theta) \\ &\quad + 3 \cos 4\theta (U_{10} A_{n1} + U_{11} A_{n-1,1} \sin^2\theta + \cdots + U_{1,n-1} A_{11} \sin^{2n-2}\theta) \\ &\quad + \cdots + (2n+1) \cos 2(n+1)\theta U_{n0} A_{nn} \end{aligned}$$

And in general, we get the following for the horizontal speed at the surface:

$$u_n = \sum_{j=0}^n \sum_{k=0}^{n-j} (2j+1) U_{jk} A_{n-k,j} \cos(2j+2)\theta \sin^{2k}\theta$$

Similarly, for the vertical speed we have:

$$v_n = \sum_{j=0}^n \sum_{k=0}^{n-j} (2j+1) A_{n-k,j} U_{jk} \sin(2j+2)\theta \sin^{2k}\theta$$

And so, the surface pressure coefficient will be given by:

$$\begin{aligned} C_p &= 1 - (u^2 + v^2) = 1 - \left(\left(\sum_{n=0}^{\infty} u_n \varepsilon^n \right)^2 + \left(\sum_{n=0}^{\infty} v_n \varepsilon^n \right)^2 \right) \\ &= 1 - \left[\sum_{n=0}^{\infty} \sum_{k=0}^n u_k u_{n-k} \varepsilon^k + \sum_{n=0}^{\infty} \sum_{k=0}^n v_{n-k} v_k \varepsilon^k \right] \end{aligned}$$

which can be written as follows:

$$C_p = [-2 A_{00}^2 U_{00}^2 \cos^2 2\theta - A_{00}^2 U_{00}^2] + \sum_{n=1}^{\infty} C_{pn} \varepsilon^n$$

So we have

$$C_{p0} = -A_{00}^2 U_{00}^2 (1 + 2 \cos^2 2\theta)$$

$$C_{pn} = - \sum_{k=0}^n [u_k u_{n-k} + v_k v_{n-k}]$$

APPENDIX C

PROGRAMS AND SUBROUTINES

(C1) Complete Program

This program will compute the series coefficients of the stream function and velocity potential at $r=1$

PROGRAM EIIIPSE

- * Program to compute the series coefficients of the stream function
- * and velocity potential at $r=1$

PARAMETER(NORD=30)

DOUBLE PRECISION A(0:NORD,0:NORD),C(0:NORD,0:NORD),B(0:NORD)

DOUBLE PRECISION STRM1(0:NORD,4),POT1(0:NORD,0:3)

COMMON B

M=1;MP=1;A(0,0)=-1.0;A(1,0)=1.5;A(1,1)=-0.5

DO 51 J=0,NORD

C(J,0)=1.0

DO 61 K=1,NORD

C(J,K)=1.0

DO 81 L=1,K

C(J,K)=C(J,K)*DBLE(FLOAT(2*J+L))/DBLE(FLOAT(L))

81 CONTINUE

61 CONTINUE

51 CONTINUE

DO 11 N=2,NORD

DO 12 I=0,NORD

A(N,I)=0.0

12 CONTINUE

11 CONTINUE


```

*      Computation of the Coefficients A(k,j)
      DO 10 N=2,NORD
      DO 13 J=0,N-1
      DO 14 K=1,N-J
      CALL BCOEFF(J,K)
      DO 15 L=0,J+K
      A(N,L)=A(N,L)-C(J,K)*A(N-K,J)*B(L)
15      CONTINUE
14      CONTINUE
13      CONTINUE
10      CONTINUE
      OPEN(UNIT=7)
      PI=DATAN(1.0D0)*4.0D0
      DO 40 IANG=1,4
      DO 40 K=0,NORD
      STRM1(K,IANGL)=0.0
40      CONTINUE
*      Computation of the stream function series coefficients at r=1
*      and four angular positions(PI/8,PI/4,3PI/8,PI/2)
      DO 50 K=1,NORD
      DO 50 J=0,K
      DO 55 IANG=1,4
      ANG=PI/8.0D0*DBLE(FLOAT(IANG))
      STRM1(K,IANGL)=A(K,J)*DSIN((2.0*DBLE(FLOAT(J))+1.0)*ANG)
      * +STRM1(K,IANGL)
55      CONTINUE
50      CONTINUE
      WRITE(7,300)
300     FORMAT(/,'Stream Function Series Coefficients at r=1')

```

```

WRITE(7,201)
201  FORMAT(5X,'PI/4',10X,'PI/4',10X,'3PI/8',10X,'PI/2')
      DO 60 K=0,NORD
        WRITE(7,200) K,STRM1(K,1),STRM1(K,2),STRM1(K,3),STRM1(K,4)
200  FORMAT(2X,I2,4(3X,D15.8))
60   CONTINUE
      WRITE(7,500)
500  FORMAT(/,'Velocity Potential Series Coefficients at r=1')
*    Computation of the velocity potential series coefficients
*    at r=1 and four angular positions(0,PI/8,PI/4,3PI/8)
      DO 90 IANG=0,3
        ANG=PI/8.0*DBLE(FLOAT(IANG))
        POT1(0,IAN)=((1.0-A(0,0)))*COS(ANG)
        DO 52 K=1,NORD
          POT1(K,IAN)=0.0
52   CONTINUE
        DO 54 K=1,NORD
          DO 54 J=0,K
            POT1(K,IAN)=POT1(K,IAN)
            * -A(K,J)*DCOS((2.0*DBLE(FLOAT(J))+1.0)*ANG)
54   CONTINUE
90   CONTINUE
      WRITE(7,205)
205  FORMAT(20X,'ANG= 0',10X,'PI/8',10X,'PI/4',10X,'3PI/8')
      DO 59 K=0,NORD
        WRITE(7,400) K,POT1(K,0),POT1(K,1),POT1(K,2),POT1(K,3)
400  FORMAT(2X,I3,4(3X,D15.8))
59   CONTINUE
      STOP
      END

```

```

SUBROUTINE BCOEFF(J,K)
PARAMETER(NORD=30)
DOUBLE PRECISION B(0:NORD)
COMMON B
DO 100 L=0,J+K
100  B(L)=0.0
DO 200 M=0,K-1
    L1=J+K-M
    IF(L1.GE.0)THEN
        B(L1)=B(L1)+COETEMP(K,M)
    ELSE
        L1=-(L1+1)
        B(L1)=B(L1)-COETEMP(K,M)
    END IF
    L2=J-K+M
    IF(L2.GE.0)THEN
        B(L2)=B(L2)+COETEMP(K,M)
    ELSE
        L2=-(L2+1)
        B(L2)=B(L2)-COETEMP(K,M)
    END IF
200  CONTINUE
    B(J)=B(J)+COETEMP(K,K)
    RETURN
END

FUNCTION COETEMP(K,M)
COETEMP=DBLE(FLOAT((-1)**(K+M)))/4.0**(K-M)
DO 10 L=1,M

```

```
COETEMP=COETEMP*DBLE(FLOAT(2*K-M+L))/DBLE(FLOAT(4*L))  
10  CONTINUE  
    RETURN  
    END
```

(C2) Subroutines

The following FORTRAN codes are the Subroutines used in series analysis.

```

SUBROUTINE PADE(P,IANG,NP,L,ROOTS,INDX,POLISH,AA,BB,Q)
*   The program segment using Pade Approximants to analyze the m
*   program series. This subroutine will return ROOTS as the roots of
*   the denominator polynomial of the NP/L pade approximant of the
*   series coefficients stored in array P
PARAMETER(NORD=30)
DOUBLE PRECISION P(-NORD:NORD,4),AA(NP,NP),BB(NP)
DOUBLE PRECISION INDX(NP)
COMPLEX Q(NP+1),ROOTS(NP)
LOGICAL POLISH
*   Initialization of the complex array Q
Q(1)=(1.0,0.)
DO 5 I=1,NP
DO 6 J=1,NP
AA(I,J)=P(L-(J-I),IANG)
6   CONTINUE
5   CONTINUE
DO 7 I=1,NP
BB(I)=-P(L+I,IANG)
7   CONTINUE
CALL LUDCMP(AA,NP,NP,INDX,D)
CALL LUBKSB(AA,NP,NP,INDX,BB)
*   LUDCMP and LUBKSB are standard subroutine packages which together
*   will solve the linear system and return the solution in array BB
DO 8 I=1,NP
Q(I+1)=CMPLX(BB(I))

```

```

8      CONTINUE
*      Zroots solves for the roots of the denominator polynomial
      CALL ZROOTS(Q,NP,ROOTS,TRUE.)
      RETURN
      END

*      Computation of the Euler transformed series coefficients once the
*      coefficients of the actual series are computed
*      POT1 is the array where the series coefficients are stored
      DOUBLE PRECISION X0(4),Y0(4),AX0Y0(4)
      DOUBLE PRECISION C2(0:NORD,0:NORD)
      X0(1)=0.04;Y0(1)=0.137;X0(2)=0.017;Y0(2)=0.339
      X0(3)=-0.14;Y0(3)=0.45;X0(4)=-0.75;Y0(4)=0.0
      DO 51 J=0,NORD
        C2(J,0)=1.0
      DO 61 K=1,NORD
        C2(J,K)=1.0
      DO 81 L=1,K
        C2(J,K)=C2(J,K)*DBLE(FLOAT(J+L-1))/DBLE(FLOAT(L))
81      CONTINUE
61      CONTINUE
51      CONTINUE
      DO 111 I=1,4
        AX0Y0(I)=(X0(I)-1.0)/(X0(I)*(1.0-X0(I))-Y0(I)**2)
      REPEAT
      DO 100 I=0,3
      DO 100 K=0,NORD
      DO 100 N=0,NORD
        EULP1(N+K,I)=C2(K,N)*POT1(N,I)*((AX0Y0(I))**K/

```

```

      * (1.0+AX0Y0(I))**(N+K))+EULP1(N+K,I)
100  CONTINUE
      WRITE(7,77)
77   FORMAT('THE EULER TRANSFORMED SERIES',/)
      DO 69 K=0,NORD
      WRITE(7,20) K,EULP1(K,0),EULP1(K,1),EULP1(K,2),EULP1(K,3)
20   FORMAT(2X,I2,4(3X,D15.7))
69   CONTINUE

*     This program segment will use the guesses stored in X to solve the
*     nonlinear system of equations for (R,cos(t),v).STRM1 is the array
*     where the series coefficients are stored.
      PARAMETER(NTRIAL=10)
      DOUBLE PRECISION X1(3),X2(3),X3(3),X4(3)
      X1(1)=0.5D0;X1(2)=0.0D0;X1(3)=1.0D0
      X2(1)=0.87D0;X2(2)=0.3D0;X2(3)=1.0D0
      X3(1)=0.8D0;X3(2)=0.8D0;X3(3)=1.0D0
      X4(1)=0.8D0;X4(2)=-1.0D0;X4(3)=1.0D0
      WRITE(7,77)
77   FORMAT(8X,'R',15X,'COS(t)',15X,'v',14X,'X','Y')
      DO 49 M=0,10
      CALL MNEWT(NTRIAL,X1,TOLX,TOLF,M,I1,STRM1)
      WRITE(7,44) M,X1(1),X1(2),X1(3),X1(1)*X1(2),X1(1)
      * *DSIN(DACOS(X1(2)))
44   FORMAT(I2,3(2X,F15.8),2(2X,F15.8))
49   CONTINUE
      WRITE(7,177)
177  FORMAT(8X,'R',15X,'COS(t)',15X,'v',14X,'X','Y')
      DO 149 M=0,10

```

```

CALL MNEWT(NTRIAL,X2,TOLX,TOLF,M,I2,STRM1)
WRITE (7,144) M,X2(1),X2(2),X2(3),X2(1)*X2(2),X2(1)
*   *DSIN(DACOS(X2(2)))
144  FORMAT(I2,3(2X,F15.8),2(2X,F15.8))
149  CONTINUE
      WRITE(7,277)
277  FORMAT(8X,'R',15X,'COS(t)',15X,'v',14X,'X','Y')
      DO 249 M=0,10
          CALL MNEWT(NTRIAL,X3,TOLX,TOLF,M,I3,STRM1)
          WRITE(7,244) M,X3(1),X3(2),X3(3),X3(1)*X3(2),X3(1)
          *   *DSIN(DACOS(X3(2)))
244  FORMAT(I2,3(2X,F15.8),2(2X,F15.8))
249  CONTINUE
      WRITE(7,377)
377  FORMAT(8X,'R',15X,'COS(t)',15X,'v',14X,'X','Y')
      DO 349 M=0,10
          CALL MNEWT(NTRIAL,X4,TOLX,TOLF,M,I4,STRM1)
          WRITE(7,344) M,X4(1),X4(2),X4(3),X4(1)*X4(2),X4(1)
          *   *DSIN(DACOS(X4(2)))
344  FORMAT(I2,3(2X,F15.8),2(2X,F15.8))
349  CONTINUE

SUBROUTINE MNEWT(NTRIAL,X,TOLX,TOLF,M,IANG,P)
*   This subroutine will solve the nonlinear system using the Newton-Raphson
*   method. TOLX, TOLF are some set tolerances, P is the array where series
*   coefficients are stored, X is the current values for our variables.
PARAMETER(NORD=26)
PARAMETER(NP=3)
DOUBLE PRECISION P(-NORD:NORD,4)

```



```

DOUBLE PRECISION X(NP),ALPHA(NP,NP),BETA(NP),INDX(NP)
D=0.0
DO 13 K=1,NTRIAL
CALL HUNTER(X,ALPHA,BETA,M,P,IANG)
ERRF=0.
DO 11 I=1,NP
ERRF=ERRF+ABS(BETA(I))
11  CONTINUE
IF(ERRF.LE.TOLF)RETURN
CALL LUDCMP(ALPHA,NP,NP,INDX,D)
CALL LUBKSB(ALPHA,NP,NP,INDX,BETA)
*   LUDCMP and LUBKSB are standard subroutine packages which together
*   will solve the linear system and return the solution in array BETA
ERRX=0.
DO 12 I=1,NP
ERRX=ERRX+ABS(BETA(I))
X(I)=X(I)+BETA(I)
12  CONTINUE
IF(ERRX.LE.TOLX)RETURN
13  CONTINUE
RETURN
END

```

```

SUBROUTINE HUNTER(X,ALPHA,BETA,M,P,IANG)
*   This subroutine will take the series coefficients of P, and
generate ALPHA and BETA for the linear system to be solved
PARAMETER(N=3)
PARAMETER(NORD=26)
DOUBLE PRECISION P(-NORD:NORD,4)
DOUBLE PRECISION S(0:NORD,0:21),SPART1(0:NORD,0:21)

```

```

DOUBLE PRECISION SPART2(0:NORD,0:21),SPART3(0:NORD,0:21)
DOUBLE PRECISION ALPHA(N,N),BETA(N),X(N)
DO 1 J=0,NORD
  S(J,0)=P(J,LANG)
  SPART1(J,0)=0.0D0
  SPART2(J,0)=0.0D0
  SPART3(J,0)=0.0D0
1  CONTINUE
  DO 22 I=0,M
    DO 2 J=2,NORD
      S(J,I)=(X(1)**2)*S(J,I)-(2.0*(J+X(3)-2.0*I-1.0)/J)
      * X(1)*X(2)*S(J-1,I)
      * +((J+2.0*X(3)-3.0*I-2.0)/J)
      * *(FLOAT((J-I-1)/(J-1)))*S(J-2,I)
2    CONTINUE
22   CONTINUE
      DO 4 I=0,M
        DO 3 J=2,NORD
          SPART1(J,I+1)=2.0*X(1)*S(J,I)+(X(1)**2)*SPART1(J,I)
          * -(2.0*(J+X(3)-2.0*I-1.0)/J)*X(2)*S(J-1,I)
          * -(2.0*(J+X(3)-2.0*I-1.0)/J)
          * *X(1)*X(2)*SPART1(J-1,I)
          * +((J+2.0*X(3)-3.0*I-2.0)/J)
          * *(FLOAT((J-I-1)/FLOAT(J-1)))*SPART1(J-2,I)
          SPART2(J,I+1)=(X(1)**2)*SPART2(J,I)
          * -(2.0*(J+X(3)-2.0*I-1.0)/J)*X(1)*S(J-1,I)
          * -(2.0*(J+X(3)-2.0*I-1.0)/J)
          * *X(1)*X(2)*SPART2(J-1,I)
          * +((J+2.0*X(3)-3.0*I-2.0)/J)

```

```

*   *(FLOAT((J-I-1)/FLOAT(J-1)))*SPART2(J-2,I)
    SPART3(J,I+1)=(X(1)**2)*SPART3(J,I)
*   -(2.0/J)*X(1)*X(2)*S(J-1,I)
*   -(2.0*(J+X(3)-2.0*I-1.0)/J)
*   *X(1)*X(2)*SPART3(J-1,I)
*   +(2.0*FLOAT((J-I-1)/FLOAT(J-1)))*S(J-2,I)*(1.0/(J))
*   +((J+2.0*X(3)-3.0*I-2.0)/J)
*   *(FLOAT((J-I-1)/FLOAT(J-1)))*SPART3(J-2,I)
3   CONTINUE
4   CONTINUE
    BETA(1)=-S(NORD,M+1)
    BETA(2)=-S(NORD-1,M+1)
    BETA(3)=-S(NORD-2,M+1)
    ALPHA(1,1)=SPART1(NORD,M+1)
    ALPHA(1,2)=SPART2(NORD,M+1)
    ALPHA(1,3)=SPART3(NORD,M+1)
    ALPHA(2,1)=SPART1(NORD-1,M+1)
    ALPHA(2,2)=SPART2(NORD-1,M+1)
    ALPHA(2,3)=SPART3(NORD-1,M+1)
    ALPHA(3,1)=SPART1(NORD-2,M+1)
    ALPHA(3,2)=SPART2(NORD-2,M+1)
    ALPHA(3,3)=SPART3(NORD-2,M+1)
    RETURN
    END

```

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