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Abstract reflection groups

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Abstract reflection groups

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ABSTRACT REFLECTION GROUPS

A Master's Thesis

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San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

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ABSTRACT

ABSTRACT REFLECTION GROUPS

by Kazumi O. Tsuchiyose

In this paper we will study the Coxeter groups, also called finite reflection groups. They are a subclass of the groups each of whose generators satisfies $s^2 = 1$.

In chapter 1 we study dihedral groups, i.e., those with two distinct generators $s \neq t$ satisfying $s^2 = t^2 = 1$, and establish a representation of such a group as a group of transformations generated by two reflections.

In chapter 2 we define Coxeter group and generalize the theory of chapter 1, including a solution to the word problem for such groups.

Associated with each Coxeter group is a symmetric matrix M and an associated bilinear form. In chapter 3 we classify those Coxeter groups whose bilinear forms are positive definite. There are ten families of such Coxeter groups.

In chapter 4 we compute the order of the groups in those ten classes.

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Kazumi Tsuchiyose

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Chapter 1 Dihedral Groups

Suppose that W is a group and S is a set of generators each element of which has order two. For any pair of elements (s,t) of S , let $m(s,t)$ denote the order of st in W . Note that $m(s,t) = m(t,s)$, and $m(s,t)$ is a positive integer or ∞ .

The infinite dihedral group D_∞ is the group defined by the presentation

$$D_\infty = \langle s, t; s^2 = t^2 = 1 \rangle$$

and a finite dihedral group is the group defined by the presentation

$$D_{2m} = \langle s, t; s^2 = t^2 = (st)^m = 1 \rangle.$$

We need to define the concept of "word," which should be distinguished from the element $w = stst\cdots \in W$. By a word in the generating set $S = \{s,t\}$ or S-word, we mean a sequence $\mathbf{s} = (s, t, \dots, -)$ of elements of S . The length of w , denoted by $l(w)$, is the smallest integer $d \geq 0$ such that w is the product of d elements of S . Note that the length of the identity element is defined to be zero, i.e., $l(w) = 0$ implies $w = 1$. The word $(s, t, \dots, -)$ is called reduced if the corresponding element $w = \underbrace{stst\cdots}_d$ has length $l(w) = d$, i.e., if it can't be represented by a shorter word. We will also say that such a reduced word is a reduced decomposition

of w .

First of all, we want to know if two S -words s and t represent the same element of D_{2m} . We may confine our attention to the case where s and t are alternating words, i.e., where they have no consecutive s 's or t 's. We may also assume, by the definition of length, that both words have the same length, and it is at most m .

Proposition 1.1 The two alternating words of length m represent the same element of D_{2m} ; all the other pairs of distinct alternating words of length $< m$ represent different group elements.

Proof: Given D_{2m} where $(st)^m = 1$. First of all, if $l(w) \geq 2m$, then $(st)^m$ or $(ts)^m$ can be replaced by identity. If $l(w) > m$, then we can shorten the length of word $< m$ as follows.

Assuming $w = \underbrace{stst \dots stst}_{m} \underbrace{\dots}_{k}$,

then $w = \underbrace{tsts \dots tsst}_{m} \underbrace{\dots}_{k}$

$= \underbrace{tsts \dots ts}_{k} \hat{s} \hat{s} \underbrace{st \dots st}_{k}$, where $\hat{s}$ means "delete s ,"

so we see that w can be reduced. So $l(w) = m+k-2k = m-k < m$, which is a contradiction. Thus we have to show the case where $l(w) = l(w') < m$ for some $w, w' \in D_{2m}$. Suppose the

sequence of letters in w and w' are different but $w = w'$ as elements of D_{2m} , then

$$w = stst\cdots-, \text{ and}$$

$$w' = tsts\cdots-.$$

If $w = w'$, then $w^{-1}w' = 1$, but $1 = w^{-1}w' = (st)^m$. This is a contradiction to $l(w^{-1}w') < 2m$. Hence $w \neq w'$. ■

Now we can list all the elements in D_{2m} . They are alternating words of length $\leq m$, starting with either s or t and having length of either odd or even;

$$s, st, sts, stst, \dots, (st)^m = 1$$

$$t, ts, tst, tsts, \dots, (ts)^m = 1.$$

The elements $s, t \in S$ are called fundamental reflections, and $w \in D_{2m}$ whose length is odd are called reflections.

We will show next, a linear representation of dihedral group D_{2m} , as transformations of real vector space $V = \mathbb{R}^2$.

First, we choose two lines through the origin which intersect at an angle of π/m for some integer $m \geq 2$, say L_s, L_t . Let σ_s, σ_t be corresponding reflections along the line. The root vectors or roots are defined to be two unit vectors r_s, r_t perpendicular to the lines of reflections. So if L_s is the y -axis, then they are $r_s = (1, 0), r_t = (-\cos\pi/m, \sin\pi/m)$ and σ_s, σ_t are functions:

$$\sigma_s(x) = x - \frac{2(r_s, x)r_s}{(r_s, r_s)},$$

$$\sigma_t(x) = x - \frac{2(r_t, x)r_t}{(r_t, r_t)}.$$

Such a function σ_s is called a linear reflection; for $\sigma_s(r_s) = -r_s$, $\sigma_t(r_t) = -r_t$ and each is the identity on the lines L_s , L_t respectively.

Now for $m < \infty$, consider the linear action of D_{2m} on V with s acting as σ_s , and with t acting as σ_t , we will see this induces a homomorphism ϕ from D_{2m} into $GL_2(\mathbb{R})$, the general linear group of a 2 dimensional vector space over $\mathbb{R}$. First, $\sigma_s^2(x) = \sigma_t^2(x) = x$ implies that σ_s and σ_t have order 2. Secondly, matrix representation will be obtained by simple computation;

$$\sigma_s(r_s) = -r_s,$$

$$\sigma_t(r_t) = -r_t,$$

$$\sigma_s(r_t) = r_t - \frac{2(r_s, r_t)r_s}{(r_s, r_s)} = r_t + (2\cos\pi/m)r_s,$$

$$\sigma_t(r_s) = r_s + (2\cos\pi/m)r_t.$$

Hence if we use the bases $\{r_s, r_t\}$ for V then,

$$\sigma_s = \begin{bmatrix} -1 & 2\cos\pi/m \\ 0 & 1 \end{bmatrix}, \quad \sigma_t = \begin{bmatrix} 1 & 0 \\ 2\cos\pi/m & -1 \end{bmatrix},$$

and

$$\sigma_s\sigma_t = \begin{bmatrix} 4\cos^2\pi/m - 1 & -2\cos\pi/m \\ 2\cos\pi/m & -1 \end{bmatrix}$$

which is similar to

$$\begin{bmatrix} e^{2\pi i/m} & 0 \\ 0 & e^{-2\pi i/m} \end{bmatrix}.$$

Therefore the order of $\sigma_s \sigma_t$ is m . It implies that $1 = (\sigma_s \sigma_t)^m = [\phi(s)\phi(t)]^m = [\phi(st)]^m = \phi(st)^m = \phi(1)$. Hence this homomorphism of D_{2m} is well defined isomorphism and this linear representation of D_{2m} is faithful.

For $m = \infty$, use $\cos \pi/m = 1$.

So,

$$\sigma_s = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \sigma_t = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix},$$

$$\sigma_s \sigma_t = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}.$$

Hence then $\sigma_s^2 = \sigma_t^2 = 1$ and $\sigma_s \sigma_t$ is similar to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$;

also $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, so $\sigma_s \sigma_t$ has infinite order and

ϕ is a faithful isomorphism.

Finally, we can treat D_{2m} as a group of symmetries of a regular m -gon together with its interior in $\mathbb{R}^2$. This treatment gives us a very simple geometric picture which will be important in all subsequent discussion of reflection group.

We will illustrate the case $m = 4$ in Figure 1 and 2. The square is divided into 8 equal sections, which are called chambers. The chamber bounded by two lines L_s and L_t is called the fundamental chamber and it corresponds to the identity in D_{2m} . Group action on the chambers is simply-transitive. Since $s, t \in S$ correspond to the fundamental reflection along L_s , L_t respectively, each chamber will be expressed as in Figure 2.

Figure 1

Figure 2

Figure 3 below will show the case $m = 3$.

Figure 3

Two chambers C and C' with corresponding group elements $w \neq w'$ are called adjacent if $w' = ws$ or $w' = wt$. To be more specific, we say w and w' are s -adjacent if $w' = ws$, and t -adjacent if $w' = wt$. In the Figure 2, stC , sC are t -adjacent, and tsC , tC are s -adjacent.

A gallery is a sequence of chambers $\Gamma = (C_0, \dots, C_d)$ such that consecutive chambers C_{i-1} , and C_i ($i=1, \dots, d$) are adjacent. The integer d is called the length of Γ . For example, consider $\Gamma: C, sC, stC, stsC, ststC$. We say that Γ is a "gallery from C to $ststC$ " or " Γ connects C to $ststC$." If $C_{i-1} = C_i$ for some i , then we say that Γ stutters. In that case, the length of the gallery is greater than the number of distinct chambers in the gallery.

We can also see that there is a unique line determined by the common boundary of two adjacent chambers. So in the example just given, there are 5 chambers in the gallery, and there are 4 lines to separate chambers. We call these lines walls for reflections. For example, L_s is the wall for reflection s , and we say "reflection s along the wall L_s ." There is 1-1 correspondence between the set of walls H and the set of reflections. The reflections may be identified with the group element conjugate to s and t . When $m = 4$, this 1-1 correspondence between walls and reflections can be illustrated as in Figure 4.

Figure 4

Lastly, we define the combinatorial distance between two chambers. It is the minimal length of a gallery connecting those two chambers. For any two chambers C, D , let $d(C, D)$ be the number of walls $H \in \mathbf{H}$ which separate C from D . If a gallery $\Gamma: C = C_0, \dots, C_d = D$ is non-stuttering, then we have a well defined sequence of walls $H_1, \dots, H_d \in \mathbf{H}$ such that C_{i-1} and C_i are adjacent along H_i . In the example of the gallery $\Gamma: C, sC, stC, stsC, ststC$ when $m \geq 3$, we have $d(C, ststC) = 5$. So corresponding walls crossed by a given gallery are $H_s, sH_t, stH_s, stsH_t$.

Next, we move on to the concept of "half-spaces." For each $H \in \mathbf{H}$, partition the set of chambers into two disjoint sets $U_+(H)$ and $U_-(H)$, where $U_+(H)$ is the set of the chambers on the side of H containing the fundamental chamber C . The set of half-spaces determined by the walls is in 1-1 correspondence with the set $\mathbf{H} \times \{\pm 1\}$. Consider the group

action on this set defined as follows. For each half-space (H, ϵ) and $w \in D_{2n}$,

$$w(H, \epsilon) = (wH, \pm \epsilon)$$

where the sign is + if and only if $wC \in U_+(wH)$, i.e., the sign is + if and only if wC and C are on the same side of the wall wH . Figure 5 contains both walls and chambers when $m = 4$.

Figure 5

We can illustrate the group action on the half-spaces by looking at Figure 5. For example, let $w = s$, then $wH = sH$ and $s(H_s, +1) = (sH_s, -1)$. $(H_s, +1)$ implies the right side of the wall H_s containing C and $s(H_s, +1) = (sH_s, -1)$ implies its reflection along the wall H_s , i.e., $(H_s, -1)$ is the left side of wall H_s . Let $w = st$, then $(H_s, +1) = s(tH_s, +1) = (stH_s, +1)$. $(H_s, +1)$ implies the half-space illustrated in

Figure 6 and $t(H_s, +1)$ is the reflection of the half-space over the wall H_t , so $t(H_s, +1) = (tH_s, +1)$ and $s(tH_s, +1) = (stH_s, +1)$ are indeed the same half-space as $(tH_s, +1)$, (Figure 7).

Figure 6

Figure 7

We will introduce three more conditions which the dihedral group satisfies. The first condition is called the deletion condition.

(D) If $w = s_1 \cdots s_d$ with $d > l(w)$, then there are indices such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d$.

In the case of the dihedral group, this condition becomes:

(D) If $w = \underbrace{st \cdots}_{d}$ with $d > l(w)$ for some $w \in D_\infty$ or D_{2m} ,

then there are two cases;

- 1) If w contains either (s,s) or (t,t) , then the pair will be deleted.
- 2) If w contains no such pair and w has length more than m , then by the proof of the solution of the word problem, we can create and delete at least one pair (s,s) or (t,t) .

The following condition is called the exchange condition.

(E) Given $w \in W$, $s \in S$, and any reduced decomposition $w = s_1 \cdots s_d$ of w , either $l(sw) = d+1$ or else there is an index i such that

$$w = ss_1 \cdots \hat{s}_i \cdots s_d.$$

To see that this holds for a dihedral group, let $w \in D_\infty$ or D_{2m} and $s \in S$ such that a reduced decomposition of w is either $w = stst \cdots -$ or $w = tsts \cdots -$. Apply $s \in S$ to w . Then in the first case, $sw = \hat{s}stst \cdots - = \hat{s}tst \cdots -$, so $s(sw) = s\hat{s}tst \cdots -$, and $w = s\hat{s}tst \cdots -$. In the second case, $sw = ststs \cdots -$ and $l(sw) = l(w)+1$. Hence (E) holds.

Lastly, the third condition is called the folding condition.

(F) Given $w \in W$ and $s, t \in S$ such that $l(sw) = l(w)+1$ and $l(wt) = l(w)+1$, either $l(swt) = l(w)+2$ or else $swt = w$.

To see this holds for a dihedral group, let $w \in D_\infty$ or D_{2m} , and suppose that $l(sw) = l(w)+1$ and $l(wt) = l(w)+1$. Then $l(w) \leq m-1$ and there are two cases.

- 1) If $s \neq t$, w starts with t and ends with s . So $w = (ts)^k$ for some $k \in \mathbb{Z}^+$ with $k < m/2$, and $swt =$

$s(ts)^kt$. Now if $l(w) = m-1$, m must be odd and swt has $m+1$ letters:

$$\begin{aligned} swt &= s(ts)^kt = (st)^kst = (st \cdots ts)t \\ &= (ts \cdots st)t = (ts)^{kt^2} \\ &= (ts)^k = w. \end{aligned}$$

But if $0 \leq l(w) < m-1$, $l(swt) = l(w)+2 \leq m$.

2) If $s = t$, then w can not start with neither s nor t .

So w must be identity to satisfy $l(sw) = l(w)+1$ and

$l(wt) = l(w)+1$. Hence $w = 1$, and $sws = ss = 1 = w$.

Hence (F) holds.

Next, we want to demonstrate "folding" along a wall.

Let C be the set of chambers, then there is a function ϕ :

$C \rightarrow C$ defined as follows. Fix $s \in S$, then for any element w in D_o or D_{2m} , there are two cases;

$$1) \quad l(sw) = l(w)+1$$

$$2) \quad l(sw) = l(w)-1.$$

Therefore folding function ϕ should be defined by

$$\phi(wC) = \begin{cases} wC & \text{if } l(sw) = l(w)+1 \\ swC & \text{if } l(sw) = l(w)-1. \end{cases}$$

Looking at Figure 5, the half-space $(H_s, -1)$ will be folded over $(H_s, +1)$ by ϕ , and $(H_s, +1)$ will be fixed by ϕ . Also ϕ preserves the adjacent relationship between two chambers. For example, from Figure 5;

$$\Gamma : sC, stC, stsC, ststC$$

$$\phi(\Gamma) : C, tC, tsC, tstC.$$

Each chamber in the half-space $(H_s, +1)$ is the image under ϕ of exactly one chamber in its complement. Moreover, ϕ is a retraction onto its own half-space $(H_s, +1)$ if $l(sw) = l(w) + 1$ and it retracts onto $(H_s, -1)$ if $l(sw) = l(w) - 1$. That is if $\Gamma: C, tC, tsC, tstC$, then $\Gamma = \phi(\Gamma)$.

Chapter 2 Abstract Reflection Groups

2.1 Definition

In this chapter, we will let W be a group, possibly infinite, generated by a subset S consisting of elements of order 2. We will define a class of groups W which are called Coxeter groups or reflection groups.

Define a special coset to be a coset $w\langle S' \rangle$ with $w \in W$ and $S' \subseteq S$. Also define $\Sigma = \Sigma(W, S)$ to be the poset of special cosets, ordered by the opposite of the inclusion relation: $B \leq A$ in Σ if and only if $B \supseteq A$ as subsets of W , in which case we say that B is a face of A . Now we call an element of W a reflection if it is conjugate to an element of S . Let H be an abstract set in 1-1 correspondence with the set of reflections, this correspondence being denoted by $H \leftrightarrow s = s_H$ for $H \in H$. The elements of H will be called walls. Since W acts by conjugation on the set of reflections, we may use our 1-1 correspondence to define an action of W on H . By definition, then, we have

$$s_{wH} = ws_Hw^{-1} \quad \text{for } w \in W \text{ and } H \in H.$$

The maximal elements of Σ will be called chambers. They are the minimal special cosets, i.e., the singletons $\{w\} \subseteq W$. Note that W acts on Σ by left translation and that the orbit of any chamber is the set of all the chambers, so

this action is simply-transitive on the chambers. We will set $C = \{1\}$ and call it the fundamental chamber. Thus a typical chamber $\{w\}$ can be written as wC .

The elements $A \in \Sigma$ which are not chambers and which are maximal among the non-chambers are said to have codimension 1. They are the two-element special cosets $w\langle s \rangle = \{w, ws\}$. Such an element will be called a face because it can be identified with the intersection of precisely two chambers, namely, $\{w\}$ and $\{ws\}$.

Two chambers $\{w\}, \{w'\}$ will be called adjacent if they have a common codimension 1 face. If $w \neq w'$, this is the same as saying that $w' = ws$ for some $s \in S$. We can define the notions of gallery and combinatorial distance exactly as in Chapter 1. It follows that a non-stuttering gallery $(C_i)_{0 \leq i \leq d}$ with $C_0 = C$ is in 1-1 correspondence with a sequence $s_1, \dots, s_d$ of elements of S , for which $s_{i-1} \neq s_i$ for all i , the correspondence being given by $C_i \leftrightarrow w_i$, where $w_i = s_1 \cdots s_i$. Let H_i be the wall such that $S_i = S_{H_i}$; then the walls $w_{i-1}H_i$ ($i=1, \dots, d$) will be called the walls crossed by the given gallery. Observe that $wH_i = H_{ws_i w^{-1}}$ because $(ws_i w^{-1})(wH_i) = ws_i H_i = wH_i$.

It should be clear that $d(C, wC)$ is the smallest integer d such that w can be expressed as a word $w = s_1 \cdots s_d$ in the generating set S . As for dihedral groups, this minimal d is called the length of w with respect to S and is denoted by

$l(w)$. Thus we have

$$d(C, wC) = l(w).$$

Let $d(C, wC) = d$ with $w = s_1 \cdots s_d$ reduced, then the corresponding gallery Γ is $C, s_1 C, s_1 s_2 C, \dots, s_1 s_2 \cdots s_d C = wC$. By the definition of $d(C, wC)$ in Chapter 1, it is the minimal length of gallery connecting C to wC . Then by applying reflection s to this gallery, we will get a new gallery Γ' : $sC, ss_1 C, ss_1 s_2 C, \dots, ss_1 s_2 \cdots s_d C = swC$, but the length of gallery connecting sC to swC is still d . Hence $d(sC, swC) = d(C, wC) = d$, and combinatorial distance is invariant under the W -action, so we obtain the following group-theoretic interpretation of the distance function:

$$d(wC, w'C) = d(C, w^{-1}w'C) = l(w^{-1}w').$$

Lastly, we recall half-spaces defined in Chapter 1. For each $H \in \mathcal{H}$, there are (+) side and (-) side by partitioning chambers. The set of "half-spaces" is in 1-1 correspondence with the set $\mathcal{H} \times \{\pm 1\}$.

2.2 Action Condition

The first condition that (W, S) must satisfy to behave like a reflection group is the action condition. We expect to see, as in Chapter 1, the W -action to take half-spaces to half-spaces and hence induce an action of W on the set

$H \times \{\pm 1\}$. The action should have the form

$$w(H, \epsilon) = (wH, \pm \epsilon),$$

where the sign is + if and only if $wC \in U_+(wH)$. In other words, the sign should be + if and only if wC and C are on the "same side" of the wall wH . Applying the action of w^{-1} , the condition for the sign to be + becomes: " C and $w^{-1}C$ are not separated by H ." Thus the action of w on $H \times \{\pm 1\}$ should satisfy:

$$w(H, \epsilon) = (wH, -\epsilon) \leftrightarrow H \text{ separates } C \text{ from } w^{-1}C.$$

Apply this to the case where $w = s \in S$. We expect there to be a unique $H \in \mathcal{H}$ which separates C from sC , namely, the H such that $s = s_H$. Hence, s should act on $H \times \{\pm 1\}$ as the involution Ω_s defined as follows:

$$\Omega_s(H, \epsilon) = \begin{cases} (H, -\epsilon) & \text{if } s = s_H \\ (sH, \epsilon) & \text{otherwise.} \end{cases} \quad (*)$$

Thus, we have arrived at the statement of the action condition:

(A) There is an action of W on the set $\mathcal{H} \times \{\pm 1\}$ such that, for every $s \in S$, s acts as the involution Ω_s defined as (*).

We will call a group W , generated by a subset S consisting of elements of order 2, a Coxeter group, if W satisfies condition (A).

Now if (A) holds, then the set of walls crossed by a

gallery has the following properties.

Theorem 2.2.1 Suppose (W, S) satisfies (A). Then we can associate to each $w \in W$ a finite subset $H(w) \subseteq H$ with the following properties:

- (1) $\text{Card } H(w) = d(C, wC) = l(w)$.
- (2) If Γ is a minimal gallery from C to wC , then the walls crossed by Γ are distinct and are precisely the elements of $H(w)$.
- (3) Let Γ be an arbitrary non-stuttering gallery from C to wC . For any $H \in H$, we have $H \in H(w)$ if and only if H is crossed by Γ an odd number of times.

Proof: Suppose that $w = s_1 \cdots s_d$, when $s_1 \cdots s_d$ is reduced. Let Γ be the corresponding gallery $(w_i C)_{0 \leq i \leq d}$, where $w_i = s_1 \cdots s_i$. Recall for each i , let $H_i = Hs_i$ and $H(w) = \{H_1, w_1 H_2, \dots, w_{d-1} H_d\}$. Let H be any wall. Since $w^{-1} = s_d \cdots s_1$, we can compute $w^{-1}(H, 1)$ by first applying s_1 , then applying s_2 , etc. After applying $s_1, \dots, s_{i-1}$, we will have an element of the form $(w_{i-1}^{-1} H, \epsilon)$ to which we must apply s_i . In view of the definition of Ω_{s_i} , the application of s_i will change ϵ to $-\epsilon$ if and only if $w_{i-1}^{-1} H = H_i$, where H_i is the wall such that $s_i = s_{H_i}$. Hence, $w^{-1}(H, 1) = (w^{-1} H, (-1)^p)$, where p is the number of i such that $H = w_{i-1} H_i$.

If Γ is non-stuttering, we have $H \in H(w)$ if and only if

p is odd. By the definition, $w_{i-1}H_i$ is the i^{th} wall crossed by Γ , so p is the number of times that Γ crosses H . This proves (3). (1) and (2) follow from (3) and the following lemma. ■

Lemma 2.2.2 For any $w \in W$ and any minimal gallery Γ from C to wC , the walls crossed by Γ are distinct; i.e., no wall is crossed more than once by Γ .

Proof: Let Γ correspond to a sequence of generators $s_1, \dots, s_d$, let $w_i = s_1 \cdots s_i$ and $H_i = H_{s_i}$. Then the minimality of Γ implies that $w = s_1 \cdots s_d$ has length d . Suppose that the walls crossed by Γ are not distinct, that is, we have

$$w_{i-1}H_i = w_{j-1}H_j \text{ for some } i < j.$$

Since $w_{i-1}H_i = H_{w_{i-1}s_i w_{i-1}^{-1}}$ and similarly for the other wall, we have

$$w_{i-1}s_i w_{i-1}^{-1} = w_{j-1}s_j w_{j-1}^{-1},$$

or

$$w_i w_{i-1}^{-1} = w_j w_{j-1}^{-1},$$

$$w_{i-1}^{-1} w_{j-1} = w_i^{-1} w_j.$$

This last equation says

$$s_{i-1} \cdots s_1 s_1 \cdots s_{j-1} = s_i \cdots s_1 s_1 \cdots s_j,$$

and

$$s_i \cdots s_{j-1} = s_{i+1} \cdots s_j,$$

which implies that

$$w = s_1 \cdots s_d = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d,$$

where the hats indicate deleted letters. This contradicts the fact that $d = l(w)$. Thus the walls crossed by Γ are distinct. ■

As a corollary of this proof, we can deduce a consequence of condition (A).

(D) If $w = s_1 \cdots s_d$ with $d > l(w)$, then there are indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d$.

We will call this the "deletion condition." As a corollary of the Theorem 2.2.1 and the proof of the Lemma 2.2.2, we have:

Corollary 2.2.3 If (W, S) satisfies (A), then it satisfies (D).

Proof: Suppose $w = s_1 \cdots s_d$ with $d > l(w)$, and consider the corresponding gallery Γ from C to wC . Then Γ can't cross d distinct walls since $\text{card } H(w) = d(C, wC) = l(w) < d$. Hence, there must be repetition among the walls crossed by Γ . Let this repetition be $w_{i-1}H_i = w_{j-1}H_j$ for some $i < j$, then by the proof of lemma,

$$w = s_1 \cdots s_d = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d.$$

So (D) holds. ■

2.3 Consequence of the Deletion Condition

Now we can state the first consequence of condition (D). It is called the exchange condition:

(E) Given $w \in W$, $s \in S$, and any reduced decomposition $w = s_1 \cdots s_d$ of w , either $l(sw) = d+1$ or else there is an index i such that $w = ss_1 \cdots \hat{s}_i \cdots s_d$. Dually, either $l(ws) = d+1$ or there is an index j such that $w = s_1 \cdots \hat{s}_j \cdots s_d s$.

Theorem 2.3.1 If (W, S) satisfies (D), then (E) holds.

Proof: Let $w \in W$, $s \in S$ and any reduced decomposition $w = s_1 \cdots s_d$ of w , then $l(w) = d$.

Case 1: $l(sw) < d+1$

Then we apply (D) to sw , there exist indices i and j , $i < j$, such that s_i and s_j can be deleted from the word, but since $w = s_1 \cdots s_d$ is reduced, s_i must be s .

$$\begin{aligned} sw &= \hat{s}s_1 \cdots \hat{s}_j \cdots s_d, \\ s(sw) &= ss_1 \cdots \hat{s}_j \cdots s_d, \\ w &= ss_1 \cdots \hat{s}_j \cdots s_d. \end{aligned}$$

Case 2: $l(sw) = d+1$

Then we get reduced decomposition of sw by $sw = ss_1 \cdots s_d$ with length $d+1$.

For the dual case, observe that $w^{-1} = s_d \cdots s_1$ is also reduced, and $l(w) = l(w^{-1}) = d$. By the above, for any $s \in$

S , either $l(sw^{-1}) = d+1$ or there exists index i such that

$$sw^{-1} = \hat{s}s_d \cdots \hat{s}_i \cdots s_1,$$

hence $w^{-1} = s\hat{s}s_d \cdots \hat{s}_i \cdots s_1$,

so $w = s_1 \cdots \hat{s}_i \cdots s_d s$. ■

Next we record a consequence of (E). It will be called the folding condition, because it is closely related to the existence of "foldings" of Σ .

(F) Given $w \in W$ and $s, t \in S$ such that $l(sw) = l(w)+1$ and $l(wt) = l(w)+1$, then either $l(swt) = l(w)+2$ or else $swt = w$.

Theorem 2.3.2 If (W, S) satisfies (E), then (F) holds.

Proof: Let $w \in W$ with reduced decomposition $w = s_1 \cdots s_d$. Since $l(sw) = l(w)+1$ and $l(wt) = l(w)+1$, both $sw = ss_1 \cdots s_d$ and $wt = s_1 \cdots s_d t$ are reduced respectively. Now apply (E) to wt . Then either $l(swt) = l(wt)+1 = d+2$ or there exists an index i such that $swt = \hat{s}s_1 \cdots \hat{s}_i \cdots s_d t$ or $\hat{s}s_1 \cdots s_d \hat{t}$. But sw has a reduced decomposition of $ss_1 \cdots s_d$, so s_i must be t . Hence $swt = \hat{s}s_1 \cdots s_d \hat{t} = s_1 \cdots s_d = w$. ■

Theorem 2.3.3 The condition (F) implies (D), so that (D), (E) and (F) are in fact all equivalent.

Proof: Suppose $w = s_1 \cdots s_d$ with $d > l(w)$. Assuming (F), we

will show by induction on d that we can delete two letters.

I. $d = 2$.

Let $w = s_1 s_2$ satisfying $2 > l(w)$. Let $w_0 = 1$ and observe that

$$l(s_1 w_0) = l(s_1) = 1 = l(w_0) + 1,$$

and $l(w_0 s_2) = l(s_2) = 1 = l(w_0) + 1.$

So (F) says either $l(s_1 s_2) = 2$ or $s_1 s_2 = 1$. Since $l(s_1 s_2) < 2$, $s_1 s_2 = 1$ must hold. (D) is valid in this case.

II. Assume (F) implies (D) for $d-1$.

III. Suppose $w = s_1 \cdots s_d$ with $d > l(w)$. Consider the elements $s_1 \cdots s_{d-1}$ and $s_2 \cdots s_d$. If $l(s_1 \cdots s_{d-1}) < d-1$ or $l(s_2 \cdots s_d) < d-1$, then we are done by the induction hypothesis II. So assume that both have length $d-1$ and let $w' = s_2 \cdots s_{d-1}$. Note $l(w') = d-2$, then s_1 and s_d satisfy the hypothesis of (F), i.e., $l(s_1 w') = l(w') + 1$ and $l(w' s_d) = l(w') + 1$. Thus, either $l(s_1 w' s_d) = l(w') + 2$ or $s_1 w' s_d = w'$. Since $l(w) < d$ and $l(w') + 2 = d$, we must have $s_1 w' s_d = w'$, so $w = w' = \hat{s}_1 s_2 \cdots s_{d-1} \hat{s}_d$ and (D) holds. ■

2.4 Construction of Foldings

As we did in Chapter 1 for the dihedral groups, we want to construct for a fixed $s = s_\mu \in S$, the map ϕ which folds Σ

onto the half-space Φ determined by H that contains the fundamental chamber C .

Proposition 2.4.1 Suppose that (W, S) satisfies the equivalent conditions (D), (E) and (F). Fix $s \in S$, and let $\mathcal{C}$ be the set of chambers of Σ . Then there is a function $\phi = \phi_s: \mathcal{C} \rightarrow \mathcal{C}$ with the following properties:

- (1) ϕ is a retraction onto its image Φ , which consists of the chambers wC such that $l(sw) = l(w)+1$.
- (2) Each chamber in Φ is the image under ϕ of exactly one chamber in the complement Φ' of Φ .
- (3) The action of s on $\mathcal{C}$ interchange the set Φ and Φ' .
- (4) ϕ takes adjacent chambers to adjacent chambers.

Proof: Define ϕ :

$$\phi(wC) = \begin{cases} wC & \text{if } l(sw) = l(w)+1 \\ swC & \text{if } l(sw) = l(w)-1 \end{cases}$$

- (1) (1) is trivial from this definition. Recall that there are two possibilities for an element $w \in W$: either $l(sw) = l(w)-1$ or $l(sw) = l(w)+1$. In the first case, w admits a reduced decomposition starting with s , so there is a minimal gallery of the form $C, sC, \dots, wC$. Therefore H should separate C from wC in this case, and $wC \in \Phi$. In the second case, there is a minimal gallery of the form $C, sC, \dots, swC$. So $swC \in \Phi$, hence its "mirror image" wC is in Φ .

- (2) Suppose $swC = sw'C$ for some $w, w' \in W$ such that $l(sw) = l(sw') = l(w)-1 = l(w')-1$. Since corresponding group element must be equal, $sw = sw'$. Then $w = w'$. Also $\phi^{-1}(swC) = wC$ and $\phi^{-1}(sw'C) = w'C$ imply $wC = w'C$. If $\phi(wC) = wC$ for $l(sw) = l(w)+1$, the claim is trivial.
- (3) By the definition of ϕ , $\phi = U_+(H_s)$ in Section 2.2, i.e., $U_+(H_s) = \{wC: w \in W \text{ such that } w(H_s, 1) = (H_{ws^{-1}}, 1)\}$, so the action of s on $C \in \phi$ will be in $U_-(H_s)$. It implies that any chamber in ϕ will be mapped into ϕ' , and vice versa.

More precise claim of (4) will be that ϕ takes t -adjacent chambers to t -adjacent chambers for all $t \in S$. We may assume that the t -adjacent chambers we start with are distinct, say wC and wtC . We may also assume that $l(wt) = l(w)+1$; for if this fails then we can replace w by wt . Now we have two cases to consider:

- (a) $l(sw) = l(w)+1$

Then $\phi(wC) = wC$. If $l(swt) = l(wt)+1$, then $\phi(wtC) = wtC$, which is t -adjacent to $\phi(wC)$. Otherwise, (F) says that $swt = w$; so we have $\phi(wtC) = swtC = wC$, which is equal to $\phi(wC)$ and hence t -adjacent to it.

- (b) $l(sw) = l(w)-1$

Then w admits a reduced decomposition starting with s , so wt does also. Thus $\phi(wtC) = swtC$, which is t -adjacent to $swC = \phi(wC)$. ■

2.5 The Word Problem

The word problem for (W, S) is the following. Given two S -words, $s = (s_1, \dots, s_d)$ and $t = (t_1, \dots, t_e)$, decide whether they represent the same element of W . By an elementary M-operation on a word we mean an operation of one of the following two types:

- (I) Delete a subword of the form (s, s) .
- (II) Given $s, t \in S$ with $s \neq t$ and $m(s, t) < \infty$, replace an alternating subword $(s, t, \dots)$ of length $m = m(s, t)$ by the alternating word $(t, s, \dots)$ of length m .

We will call a word M-reduced if it can't be shortened by any finite sequence of elementary M-operations. It is not hard to see that we can enumerate all possible words obtainable from a given one by elementary M-operations.

Example: Let W be a symmetric group on four letters. Then, the representation of the group will be

$$W = \langle S_1, S_2, S_3 : S_1^2 = S_2^2 = S_3^2 = (S_1 S_2)^3 = (S_2 S_3)^3 = (S_1 S_3)^2 = 1 \rangle,$$

$$\text{where } S_1 = (1, 2), S_2 = (2, 3), S_3 = (3, 4).$$

There are 16 reduced decomposition for w_0 .

$$\begin{aligned} w_0 &= S_1 S_3 S_2 S_1 S_3 S_2 & (*) \\ &= S_3 S_1 S_2 S_1 S_3 S_2 \\ &= S_3 S_2 S_1 S_2 S_3 S_2 \\ &= S_3 S_2 S_1 S_3 S_2 S_3 \end{aligned}$$

$$\begin{aligned}
w_0 &= s_3 s_2 s_3 s_1 s_2 s_3 \\
&= s_2 s_3 s_2 s_1 s_2 s_3 \\
&= s_2 s_3 s_1 s_2 s_1 s_3 \\
&= s_2 s_3 s_1 s_2 s_3 s_1 \\
&= s_1 s_3 s_2 s_3 s_1 s_2 \\
&= s_1 s_2 s_3 s_2 s_1 s_2 \\
&= s_1 s_2 s_3 s_1 s_2 s_1 \\
&= s_1 s_2 s_1 s_3 s_2 s_1 \\
&= s_2 s_1 s_2 s_3 s_2 s_1 \\
&= s_2 s_1 s_3 s_2 s_3 s_1 \\
&= s_2 s_1 s_3 s_2 s_1 s_3 \\
&= s_3 s_1 s_2 s_3 s_1 s_2
\end{aligned}$$

NOTE: $\equiv$ indicates that elementary M-operation was applied on (*), and $\equiv$ indicates that elementary M-operation was applied on the previous decomposition.

Consequently, the following theorem solves the word problem when (D) holds:

Theorem 2.5.1

- (1) A word is reduced if and only if it is M-reduced.
- (2) If s and t are reduced, then they represent the same element of W if and only if s can be transformed to t by the application of elementary M-operations of type (II).

Proof: We begin with (2). Suppose $s = (s_1, \dots, s_d)$ and $t = (t_1, \dots, t_d)$ are reduced words representing the same element $w \in W$. We will show by induction on $d = l(w)$ that s can be changed to t by operation of type (II).

I. $l(w) = d = 1$.

$w = s_1 = t_1$, where $s = (s_1)$ and $t = (t_1)$. s can be transformed to t .

II. Assume that claim (2) is true for $l(w) = d-1$.

III. $l(w) = d$, let $s = s_1$ and $t = t_1$.

There are two possibilities.

a) $s = t$.

Then $w = ss_2 \cdots s_d = tt_2 \cdots t_d$

$$s_2 \cdots s_d = \hat{s} t t_2 \cdots t_d = t_2 \cdots t_d.$$

Since length of the words on both sides is $d-1$, we are done by the induction hypothesis II.

b) $s \neq t$.

We will show that $m = m(s, t)$ is finite and that w admits a third reduced decomposition u starting with the alternating word $(s, t, s, t, \dots)$ of length m . If we can assume this and let u' must be the word obtained from u by replacing this initial segment of length m by the word $(t, s, t, s, \dots)$ of length m , we can get from s to t by

$$s \xrightarrow{i} u \xrightarrow{ii} u' \xrightarrow{iii} t$$

where (i) and (iii) are given by case (a) and (ii) is an operation of type (II).

Now we have to prove the finiteness of m and the existence of u . We can do this by repeatedly applying the exchange condition. Since w admits a reduced decomposition starting with t , we can find u by exchanging one of the letters in $(s_1, s_2, \dots, s_d)$ for a t in front:

$$w = t \cdots t_d = s \cdots s_d$$

$$tw = t_2 \cdots t_d = ts \cdots s_d.$$

Since $l(tw) = d-1$, by exchange condition there exists i such that

$$w = ts \cdots \hat{s}_i \cdots s_d.$$

However $s \neq s_i$, for we could then cancel $s_2 \cdots s_d$ and conclude $s = t$. Thus, it must be one of the others, and obtain a reduced decomposition of w starting with (t, s) . If $m = 2$, we are done because the (t, s) can be replaced by (s, t) to get u . Now the subword $(s, \dots, \hat{s}_i, \dots, s_d)$ has length $d-1$, so it can be transformed to $t' = (t_2 \cdots t_d)$ by the induction hypothesis.

$$\begin{aligned} w &= s \cdots s_d \leftrightarrow s \\ &= ts \cdots \hat{s}_i \cdots s_d \leftrightarrow u \\ &= t(t_2 \cdots t_d) \leftrightarrow t. \end{aligned}$$

If $m \geq 3$, we continue. Since w admits a reduced decomposition starting with s , we can find one by starting with the decomposition $w = ts \cdots -$:

$$w = tss_2 \cdots \hat{s}_i \cdots s_d = s \cdots s_d,$$

$$sw = stss_2 \cdots \hat{s}_i \cdots s_d = \hat{s}s_2 \cdots s_d.$$

With the same reason as before, we get some j such that

$$w = stss_2 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d.$$

But again, the exchanged letter s_j can't come from the initial segment of length 2; for this word contradicts our analysis of the word problem in the dihedral group of order $2m$ generated by s and t , so it must come from a later part of the word. Hence $d \geq 3$, and we have a reduced decomposition of w starting with (s, t, s) . If $m = 3$, we are done as follows;

$$\begin{aligned} w &= sts \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d \\ &= tst \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_d \\ &= t(t_2 \cdots t_d) = t. \end{aligned}$$

If $m \geq 4$, do an exchange on this most recent decomposition to get a t in front.

As long as our initial alternating segment has length $< m$, we can continue this process. Since this length must always be $\leq d$, it follows that $m \leq d < \infty$ and that we can find an initial alternating segment of length m , as claimed. This completes the proof of (2).

Turning to (1). We now have to show that the word s can be shortened by M -operations, if it is not reduced. We argue by induction on d .

I. $d = 2$: If $s = (s_1, s_2)$ is not reduced, then the

argument is the same as the proof of (F) implies (D) in Section 2.3. So $s_1 = s_2$ and it can be reduced by M-operation (II).

II. Assume that the claim is true for $d-1$.

III. Suppose that the subword $s' = (s_2, \dots, s_d)$ is not reduced, then we are done by the induction hypothesis II. So assume that s' is reduced and let $w' = s_2 \cdots s_d$. Since, $l(s_1 w') < l(w') + 1$, we can find a reduced decomposition of w' starting with s_1 ;

$$s_1 w' = \hat{s}_1 s_2 \cdots \hat{s}_i \cdots s_d$$

by (D). So,

$$w' = s_1 \cdots \hat{s}_i \cdots s_d.$$

Let $t' = (s_1, t_1, \dots, t_{d-2})$. By part (2) of the theorem, which we have already proved, we can transform s' to t' by M-operations, hence we can transform s to $(s_1, s_1, t_1, \dots, t_{d-2})$. But this can be reduced to $t = (t_1, \dots, t_{d-2})$ by an operation of type (I). Thus we have proved that if s is not reduced then s can be shortened by M-operations. ■

This solution to the word problem gives a complete description of the elements of W .

Consequently, we have the following generalization of Proposition 1.1.

Corollary 2.5.2 If (W, S) satisfies the deletion condition and $m(s, t)$ is the order of st , W admits the presentation $W = \langle S; (st)^{m(s, t)} = 1 \rangle$ where there is one relation for each pair s, t with $m(s, t) < \infty$.

Proof: Let $\tilde{W}$ be the abstract group defined by this presentation, and consider the canonical surjection $\tilde{W} \twoheadrightarrow W$. An element $\tilde{w}$ in the kernel of this map can be represented by a word s which is reducible to the empty word by M -operations. But M -operations do not change the element of $\tilde{W}$ represented by a word s , so $\tilde{w} = 1$. Hence this map is bijective. ■

2.6 Coxeter Groups

Let (W, S) be a group and a set of generators whose order are 2. We have proved that if (W, S) satisfies (D), then it admits a representation in which the relations specify the orders of the pairwise products of the elements of S .

(C) W admits the presentation

$$\langle S; (st)^{m(s, t)} = 1 \rangle$$

where $m(s, t)$ is the order of st and there is one relation

for each pair s, t with $m(s, t) < \infty$.

Theorem 2.6.1 If (W, S) satisfies (C), then (A) holds.

Proof: To verify (A), we need to check that the involutions $\Omega_i = \Omega_{s_i}$ which occur in the statement of (A) satisfy the defining relations for W . Consider the relation $(\Omega_s \Omega_t)^{m(s, t)} = 1$. Let $S' = \{s, t\}$ and let W' be the dihedral group of the order $2m$ generated by S' . So the W' -conjugates of s and t are reflections in W' , and they form a subset of the reflections in W ; hence the set H' of W' -walls is a subset of the set H of W -walls. Suppose that we apply $(\Omega_s \Omega_t)^{m(s, t)}$ to $(H, \epsilon) \in H \times \{\pm 1\}$. Clearly the only thing we have to worry about is the possibility of sign changes in the second factor as we successively apply Ω_i . Since the reflection and the wall have 1-1 correspondence, sign change will occur only if $w'H \in H'$ for some $w' \in W'$ in which case we have $H \in H'$, i.e., $H = H_s$ or H_t . If $w'H' \in H'$ for any $w' \in W'$, then the wall for reflection does not exist in H' so there is no sign change. Thus, we are reduced to showing that $(\Omega_s \Omega_t)^{m(s, t)}$ is the identity on $H' \times \{\pm 1\}$.

Let $H = H_{w'fw'^{-1}} \in H'$ with $w' \in \langle S' \rangle$, $f = s$ or t , and $l(w') = k < m$. Then

$$(\Omega_s \Omega_t)^{m(s, t)} \cdot (H_{w'fw'^{-1}}, \epsilon) = (st)^{m(s, t)} \cdot (H_{w'fw'^{-1}}, \epsilon).$$

Observe that in order to have a sign change, $l(w'fw'^{-1}) =$

$2m-1$ or 1 . There are two cases:

1) w' starts with t ,

2) w' starts with s .

$$\begin{aligned}
 1) \quad & \underbrace{(st \cdots st)}_{2m} \cdot (H_{w' f w'^{-1}}^{-1}, \epsilon) \\
 & \quad \leftarrow k \text{ actions.} \\
 & = \underbrace{(st \cdots f)}_{2m-k} \cdot (H_f, \epsilon) \\
 & = \underbrace{(st \cdots -)}_{2m-k-1} \cdot (H_f, -\epsilon) \\
 & \quad \leftarrow (m-1) \text{ actions, where } l(u) = m. \\
 & = \underbrace{(st \cdots f')}_m \cdot (H_{u f u^{-1}}^{-1}, -\epsilon) \\
 & = \underbrace{(st \cdots f')}_m \cdot (H_{f'}, -\epsilon) \\
 & = \underbrace{(st \cdots -)}_m \cdot (H_{f'}, \epsilon) \\
 & = \underbrace{(H_{u' f' u'^{-1}})}_{2m-2k-1} \cdot (H_{f'}, \epsilon) \\
 & \quad \Leftarrow \text{If } l(w_1) = 2k+1 \text{ and } l(w_2) = 2m-2k-1, \\
 & \quad \text{then } l(w_2 w_1^{-1}) = 2m \text{ so } w_1 = w_2. \\
 & = \underbrace{(H_{w' f w'^{-1}})}_{2k+1} \cdot (H_{f'}, \epsilon)
 \end{aligned}$$

$$\begin{aligned}
 2) \quad & \underbrace{(st \cdots st)}_{2m} \cdot (H_{w' f w'^{-1}}^{-1}, \epsilon) \\
 & \quad \leftarrow (m-k-1) \text{ actions, where } l(u) = m-k-1. \\
 & = \underbrace{(st \cdots f')}_{m+k+1} \cdot (H_{u f u^{-1}}^{-1}, \epsilon) \\
 & = \underbrace{(st \cdots f')}_{m+k+1} \cdot (H_{f'}, \epsilon) \\
 & = \underbrace{(st \cdots -)}_{m+k} \cdot (H_{f'}, -\epsilon)
 \end{aligned}$$

$\leftarrow (m-1)$ actions, where $l(u') = m-1$.

$$\begin{aligned}
 &= (st \cdots f'') \cdot (H_{\substack{u' f' u' \\ \substack{\downarrow \\ k+1} \downarrow} \substack{\downarrow \\ 2m-1} \downarrow}, -\epsilon) \\
 &= (st \cdots f'') \cdot (H_{f''}, -\epsilon) \\
 &= (st \cdots -) \cdot (H_{f''}, \epsilon) \\
 &= (H_{\substack{u'' f'' u'' \\ \substack{\downarrow \\ 2k+1} \downarrow}}, \epsilon)
 \end{aligned}$$

Therefore, $(\Omega_s \Omega_t)^{m(s,t)}$ is the identity action on $H' \times \{\pm 1\}$ and the theorem is proved. ■

2.7 Linear Representation of Coxeter Groups

Define a Coxeter matrix to be

$$M = (m_{ij})_{i,j \in I}$$

for an arbitrary index set I with $m_{ij} \in \mathbb{Z} \cup \{\infty\}$, each $m_{ii} = 1$ and $2 \leq m_{ij} = m_{ji} \leq \infty$ for $i \neq j$. By the condition (C), the Coxeter group W has the presentation

$$W = \langle (s_i)_{i \in I}; (s_i s_j)^{m(s_i, s_j)} = 1 \rangle,$$

where the relation occurs only if $m_{ij} < \infty$. Then we will see that the natural way to define a Coxeter group corresponding to such a matrix, and conversely.

Theorem 2.7.1 Every Coxeter matrix M is the Coxeter matrix

of a Coxeter group W_M .

Proof: Let M be a Coxeter matrix and define

$$W = \langle (s_i)_{i \in I}; (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where the relation occurs only if $m_{ij} < \infty$. We need to show two claims: 1) the images in W of the s_i are distinct and non-trivial, 2) $s_i s_j$ has order m_{ij} in W . First, let V be a vector space with a basis $(e_i)_{i \in I}$. Let B be the bilinear form on V such that $B(e_i, e_j) = -\cos \pi / m_{ij}$, and let $\sigma_i: V \rightarrow V$ be the reflection defined by the formula:

$$\sigma_i(X) = X - 2B(e_i, X)e_i.$$

It takes e_i to $-e_i$ and it fixes the "hyperplane."

$$H_i = e_i^\perp = \{x \in V: B(e_i, x) = 0\}.$$

For any $i \neq j$ with $m_{ij} < \infty$, the bilinear form B is non-degenerate on the subspace $V_1 = \text{Re}_i \oplus \text{Re}_j \subseteq V$. It follows that we have a decomposition $V = V_1 \oplus V_0$, where $V_0 = V_1^\perp = H_i \cap H_j = \{v \in V: B(e_i, v) = B(e_j, v) = 0\}$. This decomposition is invariant under σ_i and σ_j , which generate a dihedral group, acting in the canonical way on V_1 and acting trivially on V_0 . Consider the linear action of W on V with s_i acting as σ_i . Since σ_i are distinct and non-trivial, it follows that corresponding s_i are distinct and non-trivial. Hence the first part of the claim is done. For the second part of the claim, we can refer back to pp4-6 in Chapter 1. ■

As a result of this theorem, we can make up examples of "abstract reflection groups" by writing an arbitrary Coxeter matrix. Now we go on to the second question.

We already have a vector space V on which W is acting from Theorem 2.7.1. The following theorem is due to Tits.

Theorem 2.7.2 The canonical linear representation of W defined by the formula $\sigma_s(x) = x - 2B(e_i, x)e_i$ for $s \in S$ and $x \in V$ is faithful.

Proof: Let $S = \{s_i, i \in I\}$ and V^* be the dual of V . Then, W acts on V^* with s_i acting as σ_i^* . Let $\langle x, \mu \rangle = \mu(x)$ for $x \in V, \mu \in V^*$, be the canonical bilinear pairing between V and V^* . Let C be the "fundamental chamber" in V^* defined by $\{L \in V^*: \langle e_i, L \rangle > 0 \text{ for all } i \in I\}$.

To prove W acts faithfully on V , we will prove that $wC \cap C = \emptyset$ for $1 \neq w \in W$ by action of W on V^* . This follows from the following lemma. ■

Lemma 2.7.3 Fix $s = s_i \in S$ and let $U_+(s)$ and $U_-(s)$ be the open half-spaces in V^* defined, respectively, by $\langle e_i, - \rangle > 0$ and $\langle e_i, - \rangle < 0$ when this is the inner product defined in Theorem 2.7.2. Then for any $w \in W$ we have $wC \subseteq U_+(s)$ if $l(sw) = l(w) + 1$ and $wC \subseteq U_-(s)$ if $l(sw) = l(w) - 1$.

Proof: We use induction on $l(w)$.

I. $l(w) = 1$.

- 1) If $s \in S$ is such that $l(sw) = l(w)+1 = 2$, $C \subseteq U_+(s)$ by definition of C . We know that $s \neq w$ in this case, so let $w = s_j$ acting on V^* as σ_j^* . So the linear functionals in wC is σ_j^*L .

$$\begin{aligned} \langle e_i, \sigma_j^*L \rangle &= \sigma_j^*L(e_i) \\ &= L(\sigma_j(e_i)) \\ &= L(e_i + 2\cos\pi/m_{ij} \cdot e_j) \\ &= L(e_i) + 2\cos\pi/m_{ij} \cdot L(e_j) > 0. \end{aligned}$$

Since $i \neq j$, $m_{ij} \geq 2$. So $\cos\pi/m_{ij} \geq 0$, $L(e_i) > 0$, and $L(e_j) > 0$.

- 2) If $s \in S$ is such that $l(sw) = l(w)-1 = 0$, then $sw = 1$, so $s = w$ and $wC = sC$. $wC = sC \subseteq sU_+(s) = U_-(s)$, since

$$\begin{aligned} \langle e_i, \sigma_i^*L \rangle &= \sigma_i^*L(e_i) \\ &= L(\sigma_i(e_i)) \\ &= L(-e_i) \\ &= -L(e_i) < 0. \end{aligned}$$

Hence, $s_iU_+(s_i) = U_-(s_i)$.

II. Assume that this lemma is true for $l(w) = d-1$.

III. $l(w) = d$.

- 1) If $l(sw) < l(w) = d$, then $l(sw) \leq d-1$. So we can apply induction hypothesis II to sw . Since $l(ssw) = l(w) = l(sw)+1$, $swC \subseteq U_+(s)$ and $U_-(s)$.
- 2) If $l(sw) > l(w) = d$ and $w \neq 1$, then there exists $t \neq s$

in S such that $l(tw) < l(w)$. Let $W' = \langle s, t \rangle$. We can write $w = w'w''$ with $w' \in W'$, maximal factor of W' and w'' doesn't start with s nor t . Assume that w is reduced, then

$$l(w) = l(w') + l(w'') \text{ and}$$

$$l(sw'') > l(w''), l(tw'') > l(w'').$$

By the induction hypothesis II,

$$w''C \subseteq U_+(s) \text{ and } w''C \subseteq U_+(t).$$

So $w''C \subseteq U_+(s) \cap U_+(t) = C'$;

$$C' = \{L \in V^*: (e_i, L) > 0 \text{ and } (e_j, L) > 0\}.$$

C' is the fundamental chamber for the dual of the canonical representation of W' . Since $l(sw') > l(w')$ from $l(sw) = l(sw'w'') > l(w'w'')$, $w'C' \subseteq U_+(s)$ by induction hypotheses.

Hence $wC = w'w''C \subseteq w'C' \subseteq U_+(s)$, since $w''C \subseteq C'$. ■

Before we state the relationship between a finite Coxeter group and its associated bilinear form, we will prove two propositions.

If W is any subgroup of $O(V)$, the group of orthogonal transformations of V , let

$$V_0 = V_0(W) = \cap \{V_T : T \in W\},$$

where V_T is the subspace $\{x \in V : Tx = x\}$. Then V_0 is a subspace of V , $T|_{V_0}$ is the identity transformation on V_0 for

every $T \in W$. In particular, $TV_0 = V_0$ for all $T \in W$; so $T(V_0^\perp) = V_0^\perp$ for all $T \in W$. If V is represented as $V = V_0 \oplus V_0^\perp$, then every $T \in W$ can be represented as $1 \oplus T'$, where $T' = T|_{V_0^\perp}$. The group

$$W' = \{T' : T \in W\} \leq O(V_0^\perp)$$

is isomorphic with W since the action of W is trivial on the first summand and essential on the second, and $V_0(W') = 0$. A subgroup W of $O(V)$ with $V_0(W) = 0$ will be called effective.

Proposition 2.7.4 Suppose that $W \leq O(V)$ is generated by reflections along roots $r_1, \dots, r_k$. Then W is effective if and only if $\{r_1, \dots, r_k\}$ contains a basis for V .

Proof: Set $J = \cap \{r_i^\perp : 1 \leq i \leq k\}$. Since the reflection along r_i acts as the identity transformation on $r_i^\perp$ and each $T \in W$ is a product of the generating reflections, we have $T|_J = 1_J$ for all $T \in W$. Thus $J \subseteq V_0(W)$. On the other hand, if $x \in V_0$, then each generating reflection leaves x invariant, so $x \in r_i^\perp$ for each i . Thus $x \in J$ and $J = V_0(W)$. Consequently, W is effective if and only if $J = 0$, or $J^\perp = V$. But

$$J^\perp = (\cap_{i=1}^k r_i^\perp)^\perp = \sum_{i=1}^k (r_i^\perp)^\perp = \sum_{i=1}^k a_i r_i$$

In other words, the set $\{r_1, \dots, r_k\}$ spans $J^\perp$, since $r_i^{\perp\perp}$ is

the subspace spanned by r_i . Thus W is effective if and only if $\{r_1, \dots, r_k\}$ spans V . ■

Proposition 2.7.5 Suppose W is effective. If $(e_i, e_j) \leq 0$ for each $i \neq j$ ($i, j \leq k$), then $e_1, \dots, e_k$ are linearly independent.

Proof: Assume $e_1, \dots, e_k$ are not linearly independent. Then there is a non-trivial linear relation among them with non-negative coefficients. For let $a_1 e_1 + \dots + a_k e_k = 0$ be an arbitrary linear relation. If the non-zero a_i all have the same sign, the claim follows at once. Otherwise we can rewrite the relation in the form

$$\sum_{i \in I} a_i e_i = \sum_{j \in J} a_j e_j$$

with I and J disjoint non-empty subsets of $\{1, \dots, k\}$ and all coefficients $a_i, a_j > 0$. Then the inner product of the left-hand side of this equation with right-hand side is ≤ 0 . But this is the inner product of a vector with itself, so that vector must be 0. Thus both sides of the equation are 0, which implies that all a_i are 0 and this is a contradiction to the existence of a non-trivial non-negative linear relation among the e_i . So the claim is proved. ■

Theorem 2.7.6 Any finite Coxeter group has a linear

representation as reflections on an inner product space.
Moreover B is positive definite.

Proof: Assume that the generating set S has n generators.
Let $B = B(e_i, e_j) = -\cos\pi/m_{ij}$ with a basis $(e_i)_{i \in I}$ of n -dimensional vector space V , and m_{ij} as an order of $s_i s_j$. As in the proof of Theorem 2.7.1, we have a decomposition $V = V_1 \oplus V_0$ where $V_0 = V_1^\perp = H_i \cap H_j = \{x \in V: B(e_i, x) = B(e_j, x) = 0\}$. This decomposition is invariant under σ_i and σ_j , which generate a dihedral group acting in the canonical way on V_1 , and acting trivially on V_0 . Hence, if $W' = \langle s_i, s_j \rangle$, then $(W', V_1) \approx D_{2m}$.

Let μ_i and μ_j be unit normals of hyperplanes H_i, H_j respectively. From Chapter 1, if they are pointing toward the fundamental chamber C , then

$$\begin{aligned} (\mu_i, \mu_j) &= \cos(2\pi - \pi/m_{ij}) \\ &= -\cos\pi/m_{ij} = B(e_i, e_j) \leq 0. \end{aligned}$$

By Proposition 2.7.5, the set of unit normals of n hyperplanes, $\{\mu_1, \dots, \mu_n\}$, is a linearly independent set in V whose dimension is n . Also they form a basis by Proposition 2.7.4. Then for $v \in V$,

$$\begin{aligned} B(v, v) &= B\left(\sum_i^n a_i e_i, \sum_j^n a_j e_j\right) \\ &= \sum_{i,j}^n a_i a_j \cdot B(e_i, e_j) \end{aligned}$$

$$\begin{aligned}
B(v, v) &= \sum_{i,j}^n a_i a_j \cdot (\mu_i, \mu_j) \\
&= (\sum_i^n a_i \mu_i, \sum_j^n a_j \mu_j) \\
&= (v', v') > 0, \text{ for } v' \neq 0.
\end{aligned}$$

So B is positive definite since it is an inner product. ■

In Chapter 3, we will show that there are ten families of groups with B positive definite. In Chapter 4, we will compute the order of those groups which are classified as Coxeter groups, and hence positive definite Coxeter groups are finite.

Chapter 3 Classification of Positive Definite Irreducible Coxeter Groups

In this chapter, we will classify the Coxeter groups whose bilinear forms are positive definite.

3.1 Coxeter Graphs

A marked graph is a finite set of points (called nodes) such that any two distinct points may or may not be joined by a line (called branch), with the following property; if there is a branch joining the i^{th} and j^{th} nodes, then that branch is marked, or labelled, with a real number $P_{ij} > 2$. If G is a marked graph for which every mark P_{ij} is an integer, then G is called a Coxeter graph.

If G is a marked graph, then we usually suppress the label on any branch for which $P_{ij} = 3$.

If G is a marked graph with n nodes, we will associate with G a quadratic form $Q = Q_G$ on $\mathbb{R}^n$ as follows:

$$Q(x_1, \dots, x_n) = \sum_{i,j} \beta_{ij} x_i x_j$$

where $A = (\beta_{ij})$ is the symmetric matrix with $\beta_{ii} = 1$, $\beta_{ij} = -\cos\pi/P_{ij}$ if there is a branch joining the i^{th} and j^{th} nodes of G , and $\beta_{ij} = 0$ otherwise. If there is no branch joining the i^{th} and j^{th} nodes, $P_{ij} = 2$ and $P_{ii} = 1$ for all i . A marked graph will be called positive definite if and only if its

associated quadratic form is positive definite. This quadratic form $Q = Q_G$ is indeed the symmetric bilinear form B for W with P_{ij} corresponding to m_{ij} , order of $s_i s_j$ for $s_i, s_j \in S$. Therefore, we can have 1-1 correspondence between a Coxeter graph G and a Coxeter group W , where each node of G denotes a generator of W , each marked branch denotes the pairwise relationship.

3.2 Coxeter Graphs of Coxeter Groups

If we have a Coxeter group (W, S) with its associated bilinear form being positive definite, we will write (W, V) where V is a vector space with $\dim V = n$ on which group W is acting faithfully. We will call (W, V) reducible if the direct product $W = W' \times W''$ can be realized as a Coxeter group with B , positive definite, acting on the orthogonal direct sum $V = V' \oplus V''$, with V' and V'' non-trivial for (W', V') and (W'', V'') . It is called irreducible otherwise.

Two distinct nodes a and b in a marked graph G are said to be connected in G if and only if there are nodes $a_1, \dots, a_k$ in G such that $a_1 = a$, a_1 and a_2 are joined by a branch, a_2 and a_3 are joined by a branch, $\dots$, a_{k-1} and a_k are joined by a branch, $a_k = b$. If every two distinct nodes of G are connected in G , then G is said to be connected. For example, the graph in Figure 3.1 is connected.

Figure 3.1

Theorem 3.2.1 The Coxeter graph of a Coxeter group W is connected if and only if W is irreducible.

Proof: If W is reducible, then we can write $W = W' \times W''$ such that $W' = \langle S' \rangle$, $W'' = \langle S'' \rangle$, and $S = S' \cup S''$. So if $w \in W$, then $w = w' \times w''$ for some $w' \in W'$, $w'' \in W''$. Let $w = s_i s_j$ where $s_i \in S'$, $s_j \in S''$, then $w = s_i s_j = s_j s_i$. It implies that $(s_i s_j)^2 = 1$, hence $p_{ij} = 2$ and there is no branch between i^{th} and j^{th} nodes. Conversely, if G is not connected, then $p_{ij} = 2$ for all i and j , $i \neq j$. So $(s_i s_j)^2 = 1$ for any s_i, s_j . It implies that s_i commutes with s_j . Let $W' = \langle S' \rangle$, $W'' = \langle S'' \rangle$, and $W = \langle S' \cup S'' : (s_i s_j)^2 = 1 \text{ for any pair of } i, j \text{ such that } s_i \in S' \text{ and } s_j \in S'' \rangle$. Since $i \neq j$, $W' \cap W'' = \{1\}$. Also $W = W' W''$. Therefore $W = W' \times W''$ and W is reducible. ■

Consider first the Coxeter graphs shown in Figure 3.2. The subscripts on the names of the graphs indicate the number of nodes. The graph H_2^n is clearly the Coxeter graph of dihedral group D_{2m} , so it is positive definite. Note that D_{2m} , when $m = 2$, is a reducible group and that its Coxeter graph is simply two nodes with no branch joining them.

$A_n, n > 1$: $\circ, \circ-\circ, \circ-\circ-\circ, \circ-\circ-\circ-\circ, \dots$

$B_n, n \geq 2$: $\overset{4}{\circ-\circ}, \overset{4}{\circ-\circ-\circ}, \overset{4}{\circ-\circ-\circ-\circ}, \dots$

$D_n, n \geq 4$: $\begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ-\circ \end{array}, \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ-\circ-\circ \end{array}, \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ-\circ-\circ-\circ \end{array}, \dots$

$H_2^n, n \geq 5, n \neq 6$: $\overset{5}{\circ-\circ}, \overset{7}{\circ-\circ}, \overset{8}{\circ-\circ}, \overset{9}{\circ-\circ}, \dots$

G_2 : $\overset{6}{\circ-\circ}$

I_3 : $\overset{5}{\circ-\circ-\circ}$

I_4 : $\overset{5}{\circ-\circ-\circ-\circ}$

F_4 : $\overset{4}{\circ-\circ-\circ-\circ}$

E_6 : $\circ-\circ-\overset{\circ}{\underset{\circ}{\circ}}-\circ-\circ$

E_7 : $\circ-\circ-\overset{\circ}{\underset{\circ}{\circ}}-\circ-\circ-\circ$

E_8 : $\circ-\circ-\circ-\overset{\circ}{\underset{\circ}{\circ}}-\circ-\circ-\circ$

Figure 3.2

To avoid repetitions we have not included the graphs of D_{2m} , when $m = 3, 4$ and 6 among the graph H_2^n , since their graphs are listed as A_2, B_2 , and G_2 . From the historical reason, we usually single out D_{2m} , when $m = 6$, as G_2 .

If $1 \leq k \leq n$, then k^{th} principal minor of an $n \times n$ matrix A is the determinant of the $k \times k$ matrix obtained by deleting the last $n-k$ rows and columns of A .

For a real symmetric matrix (or its associated quadratic form) to be positive definite, it is necessary and sufficient that all its principal minors be positive (p398, [6]). We shall apply this criterion in order to show that the remaining graphs in Figure 3.2 are positive definite.

For each Coxeter graph G , the matrix of the associated quadratic form can be written down directly from the graph. So, we will denote the matrix by the same name as the graph in each case.

For example, the matrix of A_n is

$$\begin{bmatrix} 1 & -1/2 & 0 & 0 & \dots & & & \\ -1/2 & 1 & -1/2 & 0 & \dots & & & \\ 0 & -1/2 & 1 & -1/2 & \dots & & & \\ 0 & 0 & -1/2 & 1 & \dots & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & & \dots & 1 & -1/2 & 0 & 0 \\ & & & & & \dots & -1/2 & 1 & -1/2 & 0 \\ & & & & & & \dots & 0 & -1/2 & 1 & -1/2 \\ & & & & & & & \dots & 0 & 0 & -1/2 & 1 \end{bmatrix}$$

The k^{th} principal minor of A_n is simply $\det A_k$, so to prove that all A_n are positive definite it will suffice to prove that $\det A_n > 0$ for all n . Similar remarks apply in all other cases.

A marked graph G' is called a subgraph of a marked graph G if G' can be obtained from G by removing some of the nodes (and any adjoining branches), or by decreasing the marks on some of the branches, or both. Thus, for example,

I_3 is a subgraph of I_4 , and A_n is a subgraph of B_n for each $n \geq 2$.

The following theorem will prove most useful in discussing positive definiteness of marked graphs.

Theorem 3.2.2 Suppose G is a marked graph having a node a_1 that is adjacent to only one other node a_2 . Denote the subgraph $G \setminus \{a_1\}$ by G_1 , i.e., the graph removing node a_1 and branch between nodes a_1 and a_2 , and the subgraph $G \setminus \{a_1, a_2\}$ by G_2 , i.e., removing nodes a_1 and a_2 , branches between a_1 and a_2 , and all branches emanating from a_2 , and write p for mark p_{12} . Then

$$\det G = \det G_1 - (\cos^2 \pi/p) \det G_2.$$

$$G : \quad a_1 \xrightarrow[p]{} a_2 \xrightarrow{\quad} G_2$$

Proof: The matrix G has the form

$$\begin{bmatrix} 1 & -\cos \pi/p & 0 \\ -\cos \pi/p & 1 & * \\ 0 & * & G_2 \end{bmatrix}$$

with

$$G_1 = \begin{bmatrix} 1 & * \\ * & G_2 \end{bmatrix}$$

If $\tilde{G}$ denote the matrix obtained from G by deleting the first row and the second column then clearly $\det \tilde{G} =$

$(-\cos\pi/p)\det G_2$. If we expand $\det G$ by the first row, we find

$$\begin{aligned}\det G &= 1 \cdot \begin{vmatrix} 1 & * \\ * & G_2 \end{vmatrix} - (-\cos\pi/p) \begin{vmatrix} -\cos\pi/p & * \\ 0 & G_2 \end{vmatrix} \\ &= \det G_1 + (\cos\pi/p) \cdot \det \tilde{G} \\ &= \det G_1 - (\cos^2\pi/p) \cdot \det G_2. \blacksquare\end{aligned}$$

Since G_1 and G_2 have fewer vertices than G , Theorem 3.2.2 will enable us to apply induction in order to compute determinants.

We begin by discussing A_n . Note that $\det A_1 = 1 = 2/2$ and $\det A_2 = 1 - (-1/2)^2 = 3/4 = 3/2^2$. By Theorem 3.2.2, we have

$$\det A_n = \det A_{n-1} - (1/4)\det A_{n-2} \quad (1)$$

for $n \geq 2$. This recurrence relation can be solved as follows; the characteristic equation of the recurrence relation (1) is $x^2 = x - 1/4$. The solution for this equation is $x = 1/2$ with multiplicity 2. So the general solution is

$$\det A_n = c_1(1/2)^n + c_2n(1/2)^n \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

The initial conditions are

$$\det A_1 = 1 = c_1(1/2) + c_2(1/2),$$

so $c_1 + c_2 = 2$, and

$$\det A_2 = 3/4 = c_1(1/2)^2 + 2c_2(1/2)^2,$$

so $c_1 + 2c_2 = 3$. Then we obtain $c_1 = c_2 = 1$. Hence the

recursion formula for $\det A_n$ is

$$\det A_n = (n + 1)/2^n \quad \text{for all } n.$$

Moreover $\det A_n > 0$ for all n . Now we can conclude that every A_n is positive definite.

We will apply Theorem 3.2.2 to the remaining graphs of Figure 3.2 with the following choices of nodes a_1 and a_2 .

$$B_n, n \geq 2: \quad \begin{array}{c} \text{---} \overset{4}{\text{O}} \text{---} \\ \text{a}_1 \quad \text{a}_2 \end{array} \quad A_{n-2}$$

$$\begin{aligned} \det B_n &= \det A_{n-1} - \cos^2 \pi/4 \cdot \det A_{n-2} \\ &= n/2^{n-1} - (\sqrt{2}/2)^2 \cdot (n-1)/2^{n-2} = 1/2^{n-1} > 0 \end{aligned}$$

G_2 and H_2^n , $n \geq 5$, $n \neq 6$:

$$\det H_2^n = \begin{vmatrix} 1 & -\cos \pi/n \\ -\cos \pi/n & 1 \end{vmatrix} = 1 - \cos^2 \pi/n > 0$$

since $\cos^2 \pi/n < 1$ for $n \geq 5$.

$$D_n, n \geq 4: \quad \begin{array}{c} \text{---} \overset{\text{a}_2}{\text{O}} \text{---} \text{O} \text{---} \dots \text{---} \text{O} \\ \text{O} \text{---} \text{a}_1 \end{array}$$

$$\begin{aligned} \det D_n &= \det A_{n-1} - (1/4) \det A_{n-3} \\ &= n/2^{n-1} - (n-2)/2^{n-1} = 1/2^{n-2} > 0 \end{aligned}$$

$$I_3: \begin{array}{c} \text{---}5\text{---} \\ \text{a}_1 \quad \text{a}_2 \end{array}$$

$$\begin{aligned} \det I_3 &= \det A_2 - (\cos^2 \pi/5) \det A_1 \\ &= 3/4 - (3+\sqrt{5})/8 = (3-\sqrt{5})/8 > 0 \end{aligned}$$

$$I_4: \begin{array}{c} \text{---}5\text{---} \\ \text{a}_1 \quad \text{a}_2 \end{array}$$

$$\text{Let } \cos^2 \pi/5 = (3+\sqrt{5})/8 = \mu^2.$$

$$\begin{aligned} \det I_4 &= \det A_3 - \mu^2 \det A_2 \\ &= 4/8 - \mu^2(3/4) = 1/2 - (3+\sqrt{5})/8 \cdot (3/4) \\ &= (7-3\sqrt{5})/32 > 0 \end{aligned}$$

$$F_4: \begin{array}{c} \text{---}4\text{---} \\ \text{a}_1 \quad \text{a}_2 \end{array}$$

$$\begin{aligned} \det F_4 &= \det B_3 - (1/4) \det A_2 \\ &= 1/2^2 - (1/4) \cdot (3/4) = 1/16 > 0 \end{aligned}$$

$$E_n, \quad n = 6, 7, 8: \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{a}_1 \quad \text{a}_2 \quad \text{---} \end{array}$$

$$\begin{aligned} \det E_n &= \det D_{n-1} - (1/4) \det A_{n-2} \\ &= 1/2^{n-3} - (1/4) \cdot (n-1)/2^{n-2} = (9-n)/2^n > 0 \end{aligned}$$

Thus all the graphs in Figure 3.2 are positive definite.

We also will show that the list in Figure 3.2 contains all the connected positive definite Coxeter graphs.

Consider an auxiliary list of marked graphs in Figure 3.3.

The matrix of each graph will be shown to have determinant

zero, so that none of them are positive definite.

$P_n, n \geq 3:$

$Q_n, n \geq 5:$

$S_n, n \geq 3:$

$T_n, n \geq 4:$

$U_3:$

$Z_4:$

$Y_5:$

$$\cos \pi/q = 3/4$$

$V_5:$

$R_7:$

$R_8:$

$R_9:$

Figure 3.3

In case of P_n , the matrix looks like (3) on top of the next page. The sum of all the rows is the zero vectors so the rows are dependent and $\det P_n = 0$.

$$\begin{bmatrix}
 1 & -1/2 & 0 & \dots & -1/2 \\
 -1/2 & 1 & -1/2 & 0 & \dots & 0 \\
 0 & -1/2 & 1 & -1/2 & \dots & 0 \\
 \vdots & & & & \ddots & \vdots \\
 0 & & \dots & -1/2 & 1 & -1/2 \\
 -1/2 & 0 & \dots & 0 & -1/2 & 1
 \end{bmatrix}, \quad (3)$$

$$Q_n, \quad n \geq 5: \quad \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \end{array} - \circ - \dots - \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \end{array} \begin{array}{l} a_1 \\ a_2 \end{array}$$

$$\begin{aligned}
 \det Q_n &= \det D_{n-1} - (1/4) \det D_{n-3} \\
 &= 1/2^{n-3} - (1/4) \cdot (1/2^{n-5}) = 0
 \end{aligned}$$

$$S_n, \quad n > 3: \quad \begin{array}{c} \circ \xrightarrow{4} \circ - \circ - \dots - \circ \xrightarrow{4} \circ \\ \phantom{\circ \xrightarrow{4} \circ - \circ - \dots - \circ} a_2 \quad a_1 \end{array}$$

$$\begin{aligned}
 \det S_n &= \det B_{n-1} - (\cos^2 \pi/4) \det B_{n-2} \\
 &= 1/2_{n-1} - (1/2) \cdot (1/2^{n-3}) = 0
 \end{aligned}$$

$$T_n, \quad n > 4: \quad \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \end{array} \begin{array}{l} a_2 \\ a_1 \end{array} - \circ - \dots - \circ \xrightarrow{4} \circ$$

$$\begin{aligned}
 \det T_n &= \det B_{n-1} - (1/4) \det B_{n-3} \\
 &= 1/2^{n-2} - (1/2) \cdot (1/2^{n-3}) = 0
 \end{aligned}$$

$$U_3: \begin{array}{ccc} o & \text{---} o & \overset{6}{\text{---}} o \\ a_1 & a_2 & \end{array}$$

$$\begin{aligned} \det U_3 &= \det A_2 - (\cos^2 \pi/6) \det A_1 \\ &= 3/4 - 3/4 = 0 \end{aligned}$$

$$Z_4: \begin{array}{cccc} o & \text{---} o & \overset{q}{\text{---}} o & \text{---} o \\ & & & \end{array} \quad \cos \pi/q = 3/4$$

$$\det Z_4 = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ -1/2 & 1 & -3/4 & 0 \\ 0 & -3/4 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3/4 & 0 \\ -3/4 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} + (1/2) \begin{bmatrix} -1/2 & 0 & 0 \\ -3/4 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{bmatrix} = 0$$

$$Y_5: \begin{array}{ccccc} o & \overset{5}{\text{---}} o & \text{---} o & \text{---} o & \overset{5/2}{\text{---}} o \\ & & & & a_2 \quad a_1 \end{array}$$

$$\begin{aligned} \det Y_5 &= \det I_4 - \beta^2 \det I_3 \quad \text{where } \beta = \cos^2 2\pi/5 \\ &= \frac{7 - 3\sqrt{5}}{32} - \frac{(3 - \sqrt{5})}{8} \frac{(3 - \sqrt{5})}{8} = 0 \end{aligned}$$

$$V_5: \begin{array}{ccccc} o & \text{---} o & \overset{4}{\text{---}} o & \text{---} o & \text{---} o \\ a_1 & a_2 & & & \end{array}$$

$$\det V_5 = \det B_4 - (1/4) \det A_3 = 1/2^3 - (1/2^2) \cdot (4/2^3) = 0$$

$$\begin{aligned}\det R_7 &= \det E_6 - (1/4)\det A_5 \\ &= 3/2^6 - (1/4) \cdot (6/2^5) = 0\end{aligned}$$

$$\begin{aligned}\det R_8 &= \det E_7 - (1/4)\det D_6 \\ &= 2/2^7 - (1/4) \cdot (1/2^4) = 0\end{aligned}$$

$$\begin{aligned}\det R_9 &= \det E_8 - (1/4)\det E_7 \\ &= 1/2^8 - (1/4) \cdot (2/2^7) = 0\end{aligned}$$

A cycle in a marked graph G is a subgraph of the form P_n (Figure 3.3) for some $n \geq 3$. A branch point in G is a node having three or more branches emanating from it.

Theorem 3.2.3 A non empty subgraph G' of a positive definite marked graph G is also positive definite.

Proof: Order the nodes $a_1, a_2, \dots, a_n$ of G in such a way that $a_1, \dots, a_k$ are the nodes of G' . If $A = (a_{ij})$ and $B = (\beta_{ij})$ are the matrices of G and G' respectively, then $a_{ij} \leq \beta_{ij}$ for all i and j between 1 and k , by definition of subgraph. Let Q_G

and Q_G , denote the corresponding quadratic forms. If Q_G is not positive definite, choose $x \neq 0$ in $\mathbb{R}^k$ for which $Q_G(x) \leq 0$. If $x = (x_1, \dots, x_k)$, set $y = (|x_1|, \dots, |x_k|, 0, \dots, 0) \in \mathbb{R}^n$, then $y \neq 0$.

$$0 \geq Q_G(x) = \sum_{i,j}^k \beta_{ij} x_i x_j \quad (2)$$

$$\geq \sum_{i,j}^k \beta_{ij} |x_i| |x_j| \quad (*)$$

$$\geq \sum_{i,j}^n a_{ij} |x_i| |x_j|$$

$$= Q_G(y) > 0.$$

(*) If one of x_i, x_j are negative, then (2) > 0 but

$$\beta_{ij} |x_i| |x_j| < 0.$$

This is a contradiction. Hence, subgraph G' of positive definite marked graph G is positive definite. ■

Theorem 3.2.3 implies that none of the graphs in Figure 3.3 can occur as subgraphs of positive definite Coxeter graphs.

Theorem 3.2.4 If G is connected positive definite Coxeter graph, then G is one of the graphs $A_n, B_n, D_n, H_2^n, G_2, I_3, I_4, F_4, E_6, E_7$, or E_8 .

Proof: 1) G can not have any cycle as subgraphs since P_n is not positive definite.

2) If H_2^n ($0 \xrightarrow{n} 0$) is a subgraph of G for any $n \geq 7$,

then $G = H_2^n$, for otherwise U_3 ($o \overset{6}{-} o - o$) would be a subgraph of G .

- 3) If G_2 ($o \overset{6}{-} o$) is a subgraph of G , then $G = G_2$, for otherwise U_3 would be a subgraph of G .

For the remainder of the proof, we assume that any branch of G is marked 3, 4, or 5.

- 4) If B_2 ($o \overset{4}{-} o$) is a subgraph of G , then it occurs only once; otherwise some S_n ($o \overset{4}{-} o - \dots - o \overset{4}{-} o$) would be a subgraph. Also G can not have a branch point; otherwise some T_n ($\begin{array}{c} o \\ \diagdown \\ o - o - \dots - o \overset{4}{-} o \\ \diagup \\ o \end{array}$) would be a subgraph.

If the branches are extended to both ways from B_2 ($o \overset{4}{-} o$), it must stop at F_4 ($o - o \overset{4}{-} o - o$); otherwise V_5 ($o \overset{4}{-} o - o - o - o$) would be a subgraph of G . Hence if B_2 is a subgraph of G , then G must be either B_n or F_4 .

- 5) If H_2^n ($o \overset{5}{-} o$) is a subgraph of G , then $G = H_2^n$, I_3 ($o \overset{5}{-} o - o$), or I_4 ($o \overset{5}{-} o - o - o$); otherwise Y_5 ($o \overset{5}{-} o - o - \overset{5/2}{o}$) or Z_4 ($o - o \overset{q}{-} o - o$), where $\cos \pi/q = 3/4$, would be a subgraph of G .

6) All branches of G are unmarked, i.e., $p = 3$, then G can have at most one branch point and only three branches can emanate from any branch point; otherwise Q_n ( ) would be a

subgraph of G .

7) If G has no branch point, then $G = A_n$ for some n .

8) If G has one branch point, then $G = D_n, E_6, E_7$, or E_8 ; otherwise R_7, R_8 , or R_9 would be a subgraph of G . ■

As a consequence of Theorem 3.2.4, the Coxeter graph of a positive definite irreducible Coxeter group must appear in Figure 3.2. We will show in the next chapter that, conversely every graph in Figure 3.2 is the graph of a positive definite irreducible Coxeter group.

Chapter 4 Construction of Positive Definite Irreducible Coxeter Groups

Before we construct various positive definite irreducible Coxeter groups, we need to introduce two important concepts. The first one is root systems and the second one is fundamental regions.

4.1 Root Systems

Suppose that we have (W, V) , where W is a group generated by reflections preserving the inner product and V is an inner product vector space on which W is acting. A reflection of V is a linear transformation S that carries each vector to its mirror image with respect to a fixed hyperplane P containing origin, i.e., $Sx = x$ if $x \in P$ and $S(x) = -x$ if $x \in P^\perp$. Suppose that $0 \neq r \in P^\perp$, then transformation S_r is given by

$$S_r(x) = x - \frac{2(x, r)r}{(r, r)}$$

for all $x \in V$, so

$$S_r(x) = x \quad \text{if } x \in P,$$

$$S_r(r) = r - 2r = -r.$$

If r is chosen to be a unit vector in $P^\perp$, then S is simplified to be

$$S_r(x) = x - 2(x,r)r$$

for all $x \in V$. We say that S_r is a reflection through P or the reflection along r . Note that eigenvalues of S_r are ± 1 , and $\langle S_r \rangle$ is a group of orthogonal transformations of V .

If $S \in W$ is a reflection through a hyperplane P , the two unit vectors $\pm r$ that are perpendicular to P , so $S = S_r$, are called roots of W . For example, if the dihedral group H_2^4 is viewed as the symmetric group of the square with vertices $(\pm 1, \pm 1)$ in $\mathbb{R}^2$, it is "natural" to choose the vectors $\{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$ (Figure 4.1) as the set of roots of H_2^4 .

Figure 4.1

Proposition 4.1.1 If r is a root of $W \leq O(V)$ and if $T \in W$, then Tr is also a root of W . In fact, if $T(r) = x$, then $S_x = TS_r T^{-1} \in W$.

Proof: Let $P = r^\perp$ and $P' = TP$. Then P' is a hyperplane, and

$$P' = TP = T(r^\perp) = (Tr)^\perp = x^\perp$$

since $T \in O(V)$. If $y = T(z) \in P'$ with $z \in P$, then

$$TS_r T^{-1}(y) = TS_r T^{-1}(Tz) = TS_r(z) = T(z) = y$$

and also

$$TS_r T^{-1}(x) = TS_r(r) = T(-r) = -T(r) = -x$$

So $S_x = TS_r T^{-1} \in W$. ■

The Proposition 4.1.1 enables us to realize abstract definition of reflection in Section 2.1; i.e., an element of W is a reflection if it is conjugate to an element of generators.

Suppose that $W \leq O(V)$ is generated by a finite set of reflections. We will denote by $\hat{\Delta}$ the set of all roots corresponding to the generating reflections, together with all images of these roots under all transformations in W , i.e., $\hat{\Delta}$ is the set of all roots corresponding to the reflections TST^{-1} , where T ranges over W and S ranges over the generating set of reflections. The set $\hat{\Delta}$ will be called a root system for W .

Proposition 4.1.2 Suppose that $W \leq O(V)$ is generated by a finite set of reflections, and that W is effective. If the root system $\hat{\Delta}$ is finite, then W is finite.

Proof: By the definition of root system, we have $T(\hat{\Delta}) = \hat{\Delta}$ for all $T \in W$. Thus by restricting each $T \in W$ to $\hat{\Delta}$, we may view W as a permutation group on $\hat{\Delta}$. Since W is effective, $\hat{\Delta}$ contains a basis for V by Proposition 2.7.4; so if $T|_{\hat{\Delta}}$ is the identity map on $\hat{\Delta}$, then $T = 1$. But it means that W is

faithful on $\hat{\Delta}$, so W is finite if $\hat{\Delta}$ is finite. ■

We will assume for the remainder of this chapter that W is an effective subgroup of $O(V)$ that is generated by a set of reflections, with root system $\hat{\Delta}$.

Choose a vector $t \in V$ such that $(t, r) \neq 0$ for every root r of W . Then the root system $\hat{\Delta}$ is partitioned into two subsets,

$$\hat{\Delta}_t^+ = \{r \in \hat{\Delta} : (t, r) > 0\}$$

and

$$\hat{\Delta}_t^- = \{r \in \hat{\Delta} : (t, r) < 0\}.$$

Geometrically, $\hat{\Delta}_t^+$ and $\hat{\Delta}_t^-$ are the subsets of $\hat{\Delta}$ lying on the two sides of the hyperplane $t^\perp$. If $r \in \hat{\Delta}$, then also $-r \in \hat{\Delta}$, by definition, and $(t, -r) = -(t, r)$. Thus $r \in \hat{\Delta}_t^+$ if and only if $-r \in \hat{\Delta}_t^-$, and so $|\hat{\Delta}_t^+| = |\hat{\Delta}_t^-|$.

Example: H_2^4

$$\hat{\Delta} = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$$

Let $t = (2, 3)$, then

$$\hat{\Delta}_t^+ = \{(1, 0), (0, 1), (1, 1), (1, -1)\}$$

$$\hat{\Delta}_t^- = \{(-1, 0), (0, -1), (-1, -1), (-1, 1)\}$$

Choose a subset π of $\hat{\Delta}_t^+$ that is minimal with respect to the property that every $r \in \hat{\Delta}_t^+$ is a linear combination, with all coefficients non-negative, of elements of π . Such a minimal subset π is called a t-base for $\hat{\Delta}$. In the example

above, $\pi = \{(0, -1), (1, -1)\}$. Since $\hat{\Delta}_t^- = -\hat{\Delta}_t^+$, every $r \in \hat{\Delta}_t^-$ is a linear combination of elements of π with all coefficients non-positive.

Let π be a fixed t -base for $\hat{\Delta}$. A vector $x \in V$ is t -positive if it is possible to write x as a linear combination of $r_1, \dots, r_m \in \pi$ with all coefficients non-negative. For example, every $r \in \hat{\Delta}_t^+$ is t -positive. Similarly, $x \in V$ is t -negative if it is a non-positive linear combination of $r_1, \dots, r_m \in \pi$. Observe that if x is t -positive, then $(x, t) \geq 0$; and if t -negative, then $(x, t) \leq 0$.

Proposition 4.1.3 If $r_i, r_j \in \pi$ with $i \neq j$, and a_i, a_j are positive real numbers, then the vector $x = a_i r_i - a_j r_j$ is neither t -positive nor t -negative.

Proof: If x were t -positive, we could write

$$x = a_i r_i - a_j r_j = \sum_{k=1}^m \mu_k r_k$$

with all $\mu_k \geq 0$. If $a_i \leq \mu_i$, then

$$0 = (\mu_i - a_i) r_i + (\mu_j + a_j) r_j + \sum \{\mu_k r_k : k \neq i, j\}$$

and so

$$\begin{aligned} 0 &= (t, (\mu_i - a_i) r_i + (\mu_j + a_j) r_j + \sum \{\mu_k r_k : k \neq i, j\}) \\ &= (\mu_i - a_i) \cdot (t, r_i) + (\mu_j + a_j) \cdot (t, r_j) + \sum \{\mu_k \cdot (t, r_k) : k \neq i, j\} \\ &\geq a_j \cdot (t, r_j) > 0 \end{aligned}$$

So this is a contradiction. If $a_i > \mu_i$, then

$$(a_i - \mu_i)r_i = (\mu_j + a_j)r_j + \sum\{\mu_k r_k : k \neq i, j\}.$$

But we may divide by $(a_i - \mu_i)$ and express r_i as a non-negative linear combination of the elements of $\pi \setminus \{r_i\}$, contradicting the minimality of π . Thus x is not t -positive. On the other hand, if x were t -negative then $-x$ would be t -positive, which is impossible by the argument above with i and j interchanged. ■

Proposition 4.1.4 Suppose that $r_i, r_j \in \pi$ are distinct and let S_i denote the reflection along r_i . Then $S_i r_j \in \hat{\Delta}_t^+$, and $(r_i, r_j) \leq 0$.

Proof: Since $S_i r_j \in \hat{\Delta}$, we know that $S_i r_j$ is either t -positive or t -negative. But

$$S_i r_j = r_j - 2(r_j, r_i)r_i,$$

with one coefficient already positive. If $(r_j, r_i) > 0$, then by Proposition 4.1.3 $S_i r_j$ is neither positive nor negative, so $(r_i, r_j) \leq 0$ and $S_i r_j$ is t -positive. ■

Geometrically, $(r_i, r_j) \leq 0$ means that the angle θ , between the vectors r_i, r_j is obtuse, since $(r_i, r_j) = \cos \theta$ if r_i, r_j are unit vectors.

Theorem 4.1.5 If π is a t -base for $\hat{\Delta}$, then π is a

basis for V .

Proof: Since W is effective, $\hat{\Delta}$ spans V by Proposition 2.7.4. Since every $r \in \hat{\Delta}$ is a linear combination of roots in π , V is spanned by π . By Proposition 2.7.5 and 4.1.4, π is linearly independent, so π is a basis. ■

Proposition 4.1.6 There is only one t -base for $\hat{\Delta}$.

Proof: Suppose that π_1 and π_2 are t -bases. Since each root in π_1 is a non-negative linear combination of elements of π_2 , by the definition of π , the change of basis matrix A from the basis π_2 to the basis π_1 has non-negative entries. Likewise, the change of basis matrix $B = A^{-1}$ from π_1 to π_2 has non-negative entries. Denote by $a_1, \dots, a_n$ the rows of A and by $b_1, \dots, b_n$ the columns of B . Since $AB = I$, we have $a_i^t \perp b_j$, $2 \leq i \leq n$, in $\mathbb{R}^n$. There can be at most one index j for which the j^{th} entry in all of $b_2, \dots, b_n$ is zero, for otherwise $b_2, \dots, b_n$ would be linearly dependent, and hence B would be singular. It follows that a_1 has at most one non-zero entry. Similarly, each a_i has at most one non-zero entry. Since A is nonsingular, we conclude that A has exactly one positive entry in each row and in each column, and all other entries zero. Thus each root in π_1 is a positive multiple of a root in π_2 . Since no positive

multiple of a root r is a root except for r itself, A is a permutation matrix and $\pi_1 = \pi_2$. ■

Proposition 4.1.7 Suppose that S_i is the reflection along $r_i \in \pi = \{r_1, \dots, r_n\}$. If $r \in \hat{\Delta}^+$ but $r \neq r_i$, then $S_i r \in \hat{\Delta}^+$.

Proof: If $r \in \pi$, then $S_i r \in \hat{\Delta}^+$ by Proposition 4.1.4. If $r \notin \pi$, then $r = \sum_j a_j r_j$ and at least two of the coefficients a_j are positive; so we may assume that $r_i \neq r_1$ and that $a_1 > 0$. Thus

$$S_i r = \sum_j a_j S_i r_j = a_1 r_1 + \sum_{j=2}^n a_j r_j - 2 \left(\sum_{j=1}^n a_j (r_j, r_i) \right) r_i.$$

Since $S_i r \in \hat{\Delta}$, $S_i r$ is either t -positive or t -negative, and it has at least one positive coefficient, a_1 , we conclude that all coefficients are non-negative, and hence that $S_i r \in \hat{\Delta}^+$. ■

The roots $r_1, \dots, r_n$ in the base π are called fundamental roots, because we call reflections $S_1, \dots, S_n$ along the roots $r_1, \dots, r_n$ in the generating set the fundamental reflections of W .

Proposition 4.1.8 Assume W is finite. If $x \in V$, there is a transformation $T \in W$ such that $(Tx, r_i) \geq 0$ for all $r_i \in \pi$.

Proof: Let $x_0 = (1/2)\Sigma\{r: r \in \hat{\Delta}^+\}$. Since W is a finite group, it is possible to choose $T \in W$ for which (Tx, x_0) is maximal. If S_i is the reflection along r_i , then by Proposition 4.1.7 we have

$$\begin{aligned} S_i x_0 &= S_i((1/2)r_i + (1/2)\Sigma\{r \in \hat{\Delta}^+: r \neq r_i\}) \\ &= -(1/2)r_i + (1/2)\Sigma\{r \in \hat{\Delta}^+: r \neq r_i\} \\ &= (1/2)\Sigma\{r: r \in \hat{\Delta}^+\} - r_i = x_0 - r_i. \end{aligned}$$

Thus, by the maximality of (Tx, x_0) ,

$$\begin{aligned} (Tx, x_0) &\geq (S_i Tx, x_0) = (Tx, S_i x_0) = (Tx, x_0 - r_i) \\ &= (Tx, x_0) - (Tx, r_i); \end{aligned}$$

so $(Tx, r_i) > 0$. ■

4.2 Fundamental Regions for Finite Coxeter Groups

Suppose that W is a group generated by a set of reflections and V is the vector space on which W is acting. A subset F of V is called a fundamental region for W in V if and only if

- (1) F is open,
- (2) $F \cap TF = \emptyset$ if $1 \neq T \in W$,

and

- (3) $V = \cup\{(TF)^-: T \in W\}$.

More generally, if X is a subspace of V invariant under W , then a subset F of X is a fundamental region for W in X if

and only if

(1) F is relatively open in X ,

(2) $F \cap TF = \emptyset$ if $1 \neq T \in W$,

and

(3) $X = \cup \{ (TF)^- \cap X : T \in W \}$.

Proposition 4.2.1 Assume that V is a vector space over the infinite field. If $\dim V \geq 1$, then V is not the union of a finite number of proper subspaces.

Proof: If $\dim V = 1$, then 0 is the only proper subspace of V . Assume that the claim holds for spaces of dimension $n-1$, where $n \geq 2$ is the dimension of V . Suppose that $V = V_1 \cup \dots \cup V_m$ with each V_i a proper subspace, and let Y be any subspace of V of dimension $n-1$. Then,

$$Y = Y \cap V = Y \cap (\cup V_i) = (Y \cap V_1) \cup \dots \cup (Y \cap V_m).$$

By the induction hypothesis $Y = Y \cap V_i$ for some i . Since $\dim Y = n-1$, $\dim V_i \leq n-1$, and $Y \subseteq V_i$, we can conclude that $Y = V_i$. We have shown that every subspace Y of dimension $n-1$ occurs as one of the subspaces $V_1, \dots, V_m$. This is a contradiction since V has infinitely many subspaces of dimension $n-1$ (p32, [2]). ■

If $|W| = n$, let $T_0, T_1, \dots, T_{n-1}$ with $T_0 = 1$ be its elements and let $x_i = T_i x_0$, $0 \leq i \leq n-1$, where $x_0 \in V$ is not

fixed by any non identity element of W . So $\text{Stab}(x_0) = 1$ and $|\text{Orb}(x_0)| = |W|$. Now we can write

$$\text{Orb}(x_0) = \{x_0, x_1, \dots, x_{n-1}\}.$$

If $i \neq 0$, the line segment $[x_0 x_i]$ is defined by

$$[x_0 x_i] = \{x_0 + a(x_i - x_0) : 0 \leq a \leq 1\},$$

so $x_i - x_0$ is a vector parallel to $[x_0 x_i]$. The midpoint of $[x_0 x_i]$ is the vector $(1/2) \cdot (x_0 + x_i)$, since

$$d(x_0, (1/2) \cdot (x_0 + x_i)) = d(x_i, (1/2) \cdot (x_0 + x_i)).$$

If the hyperplane $(x_0 - x_i)^\perp$ is denoted by P_i , then the midpoint $(1/2) \cdot (x_0 + x_i)$ is in P_i , since

$$\begin{aligned} ((1/2) \cdot (x_0 + x_i), x_0 - x_i) &= (|x_0|^2 - |x_i|^2)/2 \\ &= (|x_0|^2 - |T_i x_0|^2)/2 \\ &= (|x_0|^2 - |x_0|^2)/2 = 0 \end{aligned}$$

Thus P_i is the "perpendicular bisector" of the line segment $[x_0 x_i]$.

Figure 4.2

If $x \in V$, it is clear that $x \perp (x_0 - x_i)$ if and only if $d(x, x_0) = d(x, x_i)$, using the fact that $x_i = T_i x_0$. It follows that

$$P_i = \{x \in V: d(x, x_0) = d(x, x_i)\}.$$

Denote by L_i the open set

$$\{x \in V: d(x, x_0) < d(x, x_i)\}, \quad 1 \leq i \leq n-1.$$

The set L_i is called a half-space determined by P_i . It can be seen as the set of all points that are on the same side of the hyperplane P_i as x_0 is. Now we let

$$F = \cap \{L_i: 1 \leq i \leq n-1\}.$$

Theorem 4.2.2 If W is finite, the set F is a fundamental region for W in V .

Proof: F is open since each L_i is open. If $T_i \neq 1$, then $T_i F = T_i(\cap L_j)$, or

$$\begin{aligned} T_i F &= T_i(\{x: d(x, x_0) < d(x, x_j), \quad 1 \leq j \leq n-1\}) \\ &= \{T_i x: d(T_i x, T_i x_0) < d(T_i x, T_i T_j x_0), \quad 1 \leq j \leq n-1\} \\ &= \{y: d(y, x_i) < d(y, T_k x_0), \quad 0 \leq k \leq n-1, \quad k \neq i\}, \end{aligned}$$

since $\{T_i T_j: 1 \leq j \leq n-1\} = W \setminus \{T_i\}$. Thus,

$$T_i F = \{x: d(x, x_i) < d(x, x_j), \quad \text{all } j \neq i\}.$$

If $x \in F \cap T_i F$, then $d(x, x_0) < d(x, x_i)$, and also $d(x, x_i) < d(x, x_0)$, a contradiction; so $F \cap T_i F = \emptyset$ for all $T_i \neq 1$.

Finally, if $x \in V$, choose an index i for which $d(x, x_i)$ is minimal, and hence $d(x, x_i) \leq d(x, x_j)$ for all j . Since

$$(T_i F)^- = \{x: d(x, x_i) \leq d(x, x_j), 0 \leq j \leq n-1\},$$

we have $x \in (T_i F)^-$. Thus

$$V = \cup \{(T_i F)^- : 0 \leq i \leq n-1\}$$

and F is a fundamental region. ■

Theorem 4.2.3 If $T \in W$ and $T\pi = \pi$, then $T = 1$.

Proof: Suppose that $T \neq 1$. Let $T = S_1 \cdots S_k$ be a product of fundamental reflections, where k is minimal for some positive integer k . Since $T\pi = \pi$, we have

$$\text{Tr}_k = S_1 \cdots S_k r_k = S_1 \cdots S_{k-1}(-r_k) = -S_1 \cdots S_{k-1} r_k \in \pi,$$

so $S_1 \cdots S_{k-1} r_k \in \hat{\Delta}^-$. Let

$$u_0 = S_1 \cdots S_{k-1} r_k,$$

$$u_1 = S_1 u_0 = S_2 \cdots S_{k-1} r_k,$$

$$u_2 = S_2 u_1 = S_3 \cdots S_{k-1} r_k,$$

.

.

$$u_{k-1} = S_{k-1} u_{k-2} = r_k,$$

and note that $u_0 \in \hat{\Delta}^-$, $u_{k-1} \in \hat{\Delta}^+$. Suppose that $u_0, u_1, \dots, u_{j-1} \in \hat{\Delta}^-$, but $u_j \in \hat{\Delta}^+$; i.e., u_j is the first of the roots u_i to be positive. Since

$$u_j = S_j u_{j-1} \in \hat{\Delta}^+,$$

and

$$S_j u_j = u_{j-1} \in \hat{\Delta}^-,$$

it follows from Proposition 4.1.7 that $u_j = r_j$; so

$$r_j = S_{j+1} \cdots S_{k-1} r_k.$$

But by Proposition 4.1.1, we have

$$S_j = (S_{j+1} \cdots S_{k-1}) S_k (S_{j+1} \cdots S_{k-1})^{-1}.$$

Thus

$$S_j (S_{j+1} \cdots S_{k-1}) = (S_{j+1} \cdots S_{k-1}) S_k,$$

and so

$$\begin{aligned} T = S_1 \cdots S_k &= (S_1 \cdots S_{j-1}) (S_j \cdots S_{k-1}) S_k \\ &= (S_1 \cdots S_{j-1}) (S_{j+1} \cdots S_k) S_k \\ &= S_1 \cdots S_{j-1} S_{j+1} \cdots S_{k-1}, \end{aligned}$$

representing T as a product of $k-2$ fundamental reflections.

This is contradicting the minimality of k . ■

Proposition 4.2.4 If $T \in W$, then $T(\hat{\Delta}_t^+) = \hat{\Delta}_{T(t)}^+$;
consequently, $T(\pi_t) = \pi_{T(t)}$.

Proof: Since every root in $T(\hat{\Delta}_t^+)$ is a non-negative linear combination of roots in $T(\pi_t)$, the second claim follows from the first by Proposition 4.1.6. As for the first claim,

$$\begin{aligned} T(\hat{\Delta}_t^+) &= T\{r \in \hat{\Delta} : (t, r) > 0\} \\ &= \{Tr \in \hat{\Delta} : (t, r) = (Tt, Tr) > 0\} \\ &= \{s \in \hat{\Delta} : (Tt, s) > 0\} = \hat{\Delta}_{T(t)}^+. \blacksquare \end{aligned}$$

Proposition 4.2.5 If $T \in W$ and $T(\hat{\Delta}^+) = \hat{\Delta}^+$, then $T = 1$.

Proof: By proposition 4.2.4, we have

$$\hat{\Delta}_t^+ = T(\hat{\Delta}_t^+) = \hat{\Delta}_{T(t)}^+,$$

so $\pi_t = \pi_{T(t)}$ by Proposition 4.1.6. But $T\pi_t = \pi_t$ by Proposition 4.2.4, so $T = 1$ by Theorem 4.2.3. ■

Let us denote by $\pi^* = \{s_1, \dots, s_n\}$ the dual basis of π in V , so that $(r_i, s_j) = \delta_{ij}$ for all i and j . Let

$$F = \{x \in V: x = \sum_{i=1}^n a_i s_i, a_i \in \mathbb{R}, \text{ all } a_i > 0\}$$

For any $x \in V$, write $x = \sum_{i=1}^n a_i s_i$. Then

$$(x, r_j) = \sum_i a_i (s_i, r_j) = a_j$$

for all j , so $x = \sum_i (x, r_i) s_i$. Hence

$$\begin{aligned} F &= \{x \in V: (x, r_i) > 0, \text{ all } r_i \in \pi\} \\ &= \bigcap_{i=1}^n \{x \in V: (x, r_i) > 0\}. \end{aligned}$$

In other words, F is the intersection of open half-spaces determined by the hyperplane $P_i = r_i^\perp$, $r_i \in \pi$. It follows that F is open.

Theorem 4.2.6 The set F is a fundamental region for the Coxeter group $W \leq O(V)$.

Proof: It is clear that F is open. Suppose that $T \in W$ and $x \in F \cap TF$. Let $R = T^{-1}$, then we have $Rx = T^{-1}x \in F$, since $x \in TF$. Since $x \in F$, $(x, r_i) > 0$ for all i ; and so $(x, r) > 0$

for all $r \in \hat{\Delta}_t^+$. It follows that $\hat{\Delta}_x^+ = \hat{\Delta}_t^+$, and hence $\pi_x = \pi_t$, by Proposition 4.1.6. Also $\pi_{R_x} = \pi_t$. By Proposition 4.2.4, we have

$$\pi_t = \pi_{R_x} = R\pi_x = R\pi_t.$$

Thus $R = T = 1$ by Theorem 4.2.3. Finally, if $y \in V$, then by Proposition 4.1.8 there is a transformation $T \in W$ such that $(Ty, r_i) \geq 0$ for all $r_i \in \pi$, and so $Ty \in F^-$. Thus

$$y \in T^{-1}(F^-) = (T^{-1}F)^-;$$

so

$$V = \bigcup \{ (RF)^- : R \in W \}. \blacksquare$$

4.3 Construction of Positive Definite Irreducible

Coxeter Groups

In this section, we will show the existence of the positive definite irreducible Coxeter groups for each classified Coxeter graph from Chapter 3. For types A_n , B_n , and D_n , we will use symmetric groups to construct the Coxeter group. For the remainder of the graphs, we will directly construct the group from reflections along a set of vectors that will ultimately be a base for the root system.

In order to construct a representation of the Coxeter group W with matrix $A = (m_{ij})$ as reflections on $V = \mathbb{R}^n$, we determine root vectors r so that $s \in W$ acting on V as S_r :

$v \rightarrow v$ defined by

$$S_r(v) = v - \frac{2(r,v)r}{(r,r)}.$$

In order for this to be a representation of W , we need

$(S_{r_i} S_{r_j})^{m_{ij}} = 1$. In the plane spanned by r_i and r_j ,

$$[S_{r_i}] = \begin{bmatrix} -1 & -2(r_i, r_i)/(r_i, r_j) \\ 0 & 1 \end{bmatrix}, \quad [S_{r_j}] = \begin{bmatrix} 1 & 0 \\ -2(r_j, r_i)/(r_j, r_j) & -1 \end{bmatrix}.$$

And,

$$[S_{r_i} S_{r_j}] = \begin{bmatrix} 4(r_i, r_j)^2/(r_i, r_i)(r_j, r_j) - 1 & 2(r_i, r_j)/(r_i, r_i) \\ -2(r_j, r_i)/(r_j, r_j) & -1 \end{bmatrix}.$$

So if $S_{r_i} S_{r_j}$ has order m_{ij} , then it must be conjugate to

$$\begin{bmatrix} e^{2\pi i/m_{ij}} & 0 \\ 0 & e^{2\pi i/m_{ij}} \end{bmatrix}.$$

so that

$$\begin{aligned} \text{tr } [S_{r_i} S_{r_j}] &= \frac{4(r_i, r_j)^2}{(r_i, r_i)(r_j, r_j)} - 2 = 2e^{2\pi i/m_{ij}} \\ &= 2\cos 2\pi/m_{ij}. \end{aligned}$$

It implies that

$$\begin{aligned} \frac{(r_i, r_j)^2}{(r_i, r_i)(r_j, r_j)} &= 1/4(2\cos 2\pi/m_{ij} + 2) \\ &= 1/2(\cos 2\pi/m_{ij} + 1) \\ &= (\cos \pi/m_{ij})^2. \end{aligned}$$

Hence $(r_i, r_j)/|r_i||r_j| = -\cos \pi/m_{ij}$, since B is positive definite.

* Construction of H_2^n and its root system

If the group is a dihedral group H_2^n , we can find its root system easily from a regular n -gon; $\hat{\Delta} = \{(\cos k\pi/n, \sin k\pi/n): k = 0, 1, \dots, 2n-1\}$.

* Construction of A_n and its root system

Now we may view the symmetric group L_{n+1} as a group of linear transformations of R^{n+1} , if we agree that each $T \in L_{n+1}$ is a permutation of the basis vectors $e_1, e_2, \dots, e_{n+1}$. It is known that L_{n+1} is generated by the n successive transpositions

$$S_1 = (e_1 e_2), S_2 = (e_2 e_3), \dots, S_n = (e_n e_{n+1}).$$

If the permutations S_i are considered as linear transformations, it must satisfy

$$\begin{aligned} S_i(e_{i+1} - e_i) &= -(e_{i+1} - e_i), \\ S_i(e_i + e_{i+1}) &= e_i + e_{i+1}, \\ S_i(e_j) &= e_j \quad \text{if } j \neq i, j \neq i+1. \end{aligned}$$

Since $\{e_j: j \neq i, j \neq i+1\} \cup \{e_i + e_{i+1}\}$ spans the hyperplane $P = (e_{i+1} - e_i)^\perp$, we see that each S_i is a reflection and that $r_i = e_{i+1} - e_i$ may be taken as a root of S_i . Thus L_{n+1} , as a group of transformations, is generated by the reflections $S_1, \dots, S_n$ along the roots $r_1, \dots, r_n$.

To obtain the root system $\hat{\Delta}$ of L_{n+1} we must find the roots of all conjugates in L_{n+1} of the generating reflections

S_i . But in a symmetric group, the set of conjugates of any transposition is the set of all transpositions. Thus the set of conjugate reflections is the set of transpositions $(e_i e_j)$, $i \neq j$, and the root system is

$$\hat{\Delta} = \{e_i - e_j : i \neq j, 1 \leq i, j \leq n+1\}.$$

Let us denote by V the subspace of $\mathbb{R}^{n+1}$ spanned by the roots $r_1, \dots, r_n$ of $\underline{L}_{n+1}$, and by $\underline{A}_n$ the group of restrictions to V of the transformations in $\underline{L}_{n+1}$. With the generating set, $\{S_1, \dots, S_n\}$, as transpositions, $\underline{A}_n$ is effective. And notice that $(r_i, r_{i+1})/|r_i||r_{i+1}| = -1/2$ for $1 \leq i \leq n-1$, so the Coxeter graph of $\underline{A}_n$ is A_n . Hence $\underline{A}_n$ is isomorphic with $\underline{L}_{n+1}$ and we have $|\underline{A}_n| = (n+1)!$.

* Construction of $\underline{B}_n$ and its root system

For $\underline{B}_n$, we will consider the group of "signed permutations" in $\mathbb{R}^n$ whose elements permute the basis vectors $e_1, \dots, e_n$, and then replace some of them by their negatives. Firstly consider the symmetric group $\underline{L}_n$ of permutations of $\{e_1, \dots, e_n\}$. Here we denote the generating reflections of $\underline{L}_n$ by $S_2, \dots, S_n$, with respective roots $r_2 = e_2 - e_1, \dots, r_n = e_n - e_{n-1}$. Secondly let a group $\underline{K}_n$ be generated by the n reflections $S_{e_1}, \dots, S_{e_n}$ along the basis vectors $e_1, \dots, e_n$. The effect of S_{e_i} on any $x \in \mathbb{R}^n$ is simply to replace the i^{th} entry of x by its negative. Since the reflections S_{e_i} all commute with one another, $\underline{K}_n$ is abelian and is the direct

product of the two-element subgroups $\{1, Se_i\}$, $1 \leq i \leq n$.

Thus $|\underline{K}_n| = 2^n$.

For each subset J of $\{e_1, \dots, e_n\}$, define $f_J: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by setting

$$f_J(e_i) = \begin{cases} -e_i & \text{if } e_i \in J, \\ e_i & \text{if } e_i \notin J. \end{cases}$$

Clearly, f_J is the product of all Se_i for which $e_i \in J$, with $f_J = 1$ if $J = \emptyset$, and $\underline{K}_n$ consists of all such transformations f_J . The symmetric difference $J + L$ of two sets J and L is defined by

$$J + L = (J \cup L) \setminus (J \cap L). \quad (*)$$

If $f_J, f_L \in \underline{K}_n$, then

$$f_J f_L(e_i) = \begin{cases} f_J(-e_i) = e_i & \text{if } e_i \in L \text{ and } J; e_i \in J \cap L \\ f_J(-e_i) = -e_i & \text{if } e_i \in L \text{ but } e_i \notin J; e_i \in J \cup L \\ f_J(e_i) = -e_i & \text{if } e_i \notin L \text{ but } e_i \in J; e_i \in J \cup L \\ f_J(e_i) = e_i & \text{if } e_i \notin L \text{ and } e_i \notin J; e_i \notin J \cup L \end{cases}$$

On the other hand,

$$f_{J+L}(e_i) = -e_i \quad \text{if } e_i \in J + L$$

$$f_{J+L}(e_i) = e_i \quad \text{if } e_i \notin J + L.$$

Hence by $(*)$, we can conclude $f_J f_L = f_{J+L}$.

Now we will investigate how two groups $\underline{L}_n$ and $\underline{K}_n$ interact. Suppose that $T \in \underline{L}_n$ and $f_J \in \underline{K}_n$. Given a basis vector e_i , write $e_i = T(e_j)$. If $e_j \in J$ or equivalently, if $e_i \in T(J)$, then

$$Tf_J T^{-1}(e_i) = Tf_J T^{-1}T(e_j) = Tf_J(e_j) = T(-e_j) = -T(e_j) = -e_i;$$

and if $e_j \notin J$, or $e_j \notin T(J)$, then

$$Tf_jT^{-1}(e_i) = Tf_j(e_i) = T(e_i) = e_i.$$

Thus

$$Tf_jT^{-1} = f_{T(J)},$$

and $\underline{K}_n$ is normalized by $\underline{L}_n$. It follows that the group $\underline{B}_n$ which is generated by $\underline{K}_n \cup \underline{L}_n$ consists of all products f_jT , where $f_j \in \underline{K}_n$ and $T \in \underline{L}_n$. By considering their effects on the basis vectors $e_1, \dots, e_n$, we see that $\underline{K}_n \cap \underline{L}_n = 1$, from which it follows that each transformation in $\underline{B}_n$ has a unique expression as a product f_jT . Thus the order of $\underline{B}_n$ is the product of the orders of $\underline{K}_n$ and $\underline{L}_n$; i.e.,

$$|\underline{B}_n| = |\underline{K}_n| |\underline{L}_n| = 2^n \cdot n!.$$

Products in $\underline{B}_n$ are given by

$$(f_jT)(f_lT') = f_jTf_lT^{-1}TT' = f_jf_{T^{-1}(l)}TT' = f_{j+T^{-1}(l)} \cdot TT'.$$

If we let $f_i = f_{\langle e_i \rangle} = Se_i$, $1 \leq i \leq n$, then $\underline{B}_n$ is generated by the set of reflections

$$\{f_1, \dots, f_n, S_2, \dots, S_n\}.$$

Among the roots of $\underline{B}_n$ are those of $\underline{L}_n$. In particular, $e_i - e_1$ is a root if $2 \leq i \leq n$. Let T_i denote the reflection with the root $e_i - e_1$, then

$$T_i f_1 T_i^{-1} = f_{T_i \langle e_i \rangle} = f_{\langle e_i \rangle} = f_i.$$

Consequently, $\{f_1, S_2, \dots, S_n\}$ is a generating set of reflections for $\underline{B}_n$. Now rename f_1 with S_1 and let $r_1 = e_1$, a root of S_1 . Since $\{r_1, \dots, r_n\}$ is a basis for $\mathbb{R}^n$, $\underline{B}_n$ is effective.

Since the effect of each transformation in $\underline{B}_n$ on the basis vectors $e_1, \dots, e_n$ is a permutation, followed by some sign changes, the root system $\hat{\Delta}$ of $\underline{B}_n$ is

$$\hat{\Delta} = \{\pm e_i: 1 \leq i \leq n\} \cup \{e_i \pm e_j: i \neq j, 1 \leq i, j \leq n\}.$$

So,

$$e_i = e_1 + (e_2 - e_1) + (e_3 - e_2) + \dots + (e_i - e_{i-1})$$

$$= r_1 + r_2 + \dots + r_i = \sum_{k=1}^i r_k$$

$$e_i - e_j = (e_i - e_{i-1}) + (e_{i-1} - e_{i-2}) + \dots + (e_{j+1} - e_j)$$

$$= \sum_{k=j}^{i-1} r_{k+1} \quad \text{if } i > j,$$

$$e_i + e_j = 2 \sum_{k=1}^j r_k + \sum_{k=j}^{i-1} r_{k+1} \quad \text{if } i > j;$$

so $\{r_1, \dots, r_n\}$ is a base for $\underline{B}_n$. The Coxeter graph of $\underline{B}_n$ is B_n since

$$(r_1, r_2) / |r_1| |r_2| = -\sqrt{2}/2,$$

and for $2 \leq i \leq n-1$

$$(r_i, r_{i+1}) / |r_i| |r_{i+1}| = -1/2.$$

Hence $|\hat{\Delta}| = 2n + 2n(n-1) = 2n^2$, and $|\underline{B}_n| = 2^n \cdot n!$.

* Constriction of $\underline{D}_n$ and its root system

For D_n , we replace $\underline{K}_n$ by its subgroup $\underline{K}_n'$

$$\underline{K}_n' = \{f_j \in \underline{K}_n: |J| \text{ is even}\},$$

whose transformations effect even numbers of sign changes in coordinates of vectors in $\mathbb{R}^n$. $\underline{K}_n'$ is a subgroup of $\underline{K}_n$ since

the product of two even numbers of sign changes is even numbers of sign changes again by

$$|J + L| = |J| + |L| - 2|J \cap L|,$$

for any finite sets J and L .

As above $\underline{K}_n'$ is normalized by $\underline{L}_n$ and the group $\underline{D}_n$ generated by $\underline{K}_n' \cup \underline{L}_n$ consists of all products $f_j T$, $f_j \in \underline{K}_n'$, $T \in \underline{L}_n$. Let $r_1 = e_1 + e_2$ and $r_i = e_i - e_{i-1}$, $2 \leq i \leq n$, and let $S_j \in \underline{D}_n$ be the reflection along r_j , $1 \leq j \leq n$. Note that $\langle S_2, \dots, S_n \rangle = \underline{L}_n$.

Since any even number of sign changes can be obtained by successively changing signs two at a time, $\underline{L}_n$ is generated by the products $S_{e_i} S_{e_j}$, $i \neq j$. Among the transformations in $\underline{D}_n$ are the reflections $S_{e_i + e_j}$, $i \neq j$; i.e.,

$$S_{e_i + e_j}(x_i) = x_j, \quad S_{e_i + e_j}(x_j) = x_i,$$

and

$$S_{e_i + e_j}(x_k) = x_k \quad \text{for all } k \neq i, j.$$

If $i \neq j$, choose $T \in \underline{L}_n$ such that $T e_1 = e_i$ and $T e_2 = e_j$.

Then by Proposition 4.1.1, we have

$$T S_{e_1 + e_2} T^{-1} = S_{e_i + e_j}.$$

Also, if $i \neq j$, then

$$S_{e_i - e_j}(x_i) = -x_j, \quad S_{e_i - e_j}(x_j) = -x_i,$$

and

$$S_{e_i - e_j}(x_k) = x_k, \quad \text{for all } k \neq i, j,$$

so $S_{e_i - e_j} \in \underline{L}_n$. Moreover,

$$S_{e_i - e_j} S_{e_i + e_j}(x_i) = S_{e_i - e_j}(x_j) = -x_i,$$

and

$$S_{e_i - e_j} S_{e_i + e_j}(x_j) = S_{e_i - e_j}(x_i) = -x_j.$$

This implies

$$S_{e_i - e_j} S_{e_i + e_j} = S_{e_i} S_{e_j}.$$

It follows that $\{S_1, S_2, \dots, S_n\}$ is a generating set of reflections for $\underline{D}_n$.

Applying all permutations of $\{e_i\}$ to the roots $\{r_1, \dots, r_n\}$, followed by even numbers of sign changes, we obtain the root system

$$\Delta = \{e_i \pm e_j : i \neq j, 1 \leq i, j \leq n\}.$$

Since

$$e_i - e_j = \sum_{k=j}^{i-1} r_{k+1} \quad \text{if } i > j,$$

$$e_i + e_1 = r_1 + \sum_{k=3}^i r_k \quad \text{if } i \neq 1,$$

$$e_i + e_2 = \sum_{k=1}^i r_k \quad \text{if } i > 2,$$

and

$$e_i + e_j = r_1 + r_2 + 2 \sum_{k=3}^j r_k + \sum_{k=j}^{j-1} r_{k+1} \quad \text{if } 2 < j < i,$$

we can see that $\{r_1, \dots, r_n\}$ is a base for $\underline{D}_n$. Since

$$(r_1, r_2) / |r_1| |r_2| = (e_1 + e_2, e_2 - e_1) = 0,$$

$$(r_1, r_3) / |r_1| |r_3| = (e_1 + e_2, e_3 - e_2) = -1/2,$$

and

$$(r_i, r_{i+1}) / |r_i| |r_{i+1}| = (e_i - e_{i-1}, e_{i+1} - e_i) = -1/2$$

$$\text{for } 2 \leq i \leq n-1,$$

its Coxeter graph is D_n .

The order of D_n can be found by the following.

The function h from B_n to $\{\pm 1\}$ defined by

$$h(f_j T) = (-1)^{|J|}$$

is a homomorphism onto $\{\pm 1\}$ with kernel D_n . Thus $[B_n : D_n] = 2$ and $|D_n| = (1/2) \cdot |B_n| = 2^{n-1} \cdot n!$.

In order to establish the existence of a group for each of the remaining graphs G in Figure 3.2, we will exhibit a set of mutually obtuse vectors with Coxeter graph G . Using them we will construct a Coxeter group W and its root system $\hat{\Delta}$, and show that the original set of vectors is a base for W .

* Construction of E_8 , E_7 , E_6

If the base vectors of A_7 are relabeled as

$$r_i = e_i - e_{i-1}, \quad 2 \leq i \leq 8,$$

we may adjoin r_1 such that $(r_1, r_4) = -1/2$, $(r_1, r_i) = 0$ for $2 \leq i \leq 8$ and $i \neq 4$.

$$\text{Hence } r_1 = (1/2) \left(\sum_{i=1}^3 e_i - \sum_{i=4}^8 e_i \right);$$

then $\{r_1, r_2, \dots, r_8\}$ has graph E_8 . Clearly the subsets $\{r_1, \dots, r_6\}$ and $\{r_1, \dots, r_7\}$ have graphs E_6 and E_7 .

* Construction of F_4

In the case of F_4 , the graph B_3 is a subgraph of F_4 , and

a base for the group $\underline{B}_3$ was discussed previously. We shall "extend" that base to a base having the graph F_4 .

If the base vectors r_1, r_2 , and r_3 of $\underline{B}_3$ are relabeled as

$$r_2 = e_1, \quad r_3 = e_2 - e_1, \quad r_4 = e_3 - e_2$$

and viewed as vectors in $\mathbf{R}^4$, then we want to add a vector r_1 so that $\{r_1, r_2, r_3, r_4\}$ has Coxeter graph F_4 . The requirements for $r_1 = (\mu_1, \mu_2, \mu_3, \mu_4)$ are

$$|r_1|^2 = \sum_{i=1}^4 \mu_i^2 = 1,$$

$$(r_1, r_2) = \mu_1 = -1/2,$$

$$(r_1, r_3) = \mu_2 - \mu_1 = 0,$$

$$(r_1, r_4) = \mu_3 - \mu_2 = 0.$$

Thus $r_1 = (-1/2, -1/2, -1/2, \mu_4)$ with $\mu_4^2 = 1/4$. We choose $\mu_4 = -1/2$, and

$$r_1 = -1/2 \sum_{i=1}^4 e_i.$$

So the resulting graph is F_4 .

* Construction of $\underline{G}_2$

For G_2 , $r_1 = (1, 0)$ and $r_2 = (-\sqrt{3}/2, 1/2)$ since $\underline{G}_2$ is a dihedral group with $n = 6$.

* Construction of $\underline{I}_3$

$\underline{I}_3$ corresponds to the symmetry group of dodecahedron

which has 12 pentagonal faces, or dually the icosahedron which has 20 triangular faces. The base $\{r_1, r_2, r_3\}$ of unit vectors is found in p49, [2].

They are

$$r_1 = \beta(2a+1, 1, -2a)$$

$$r_2 = \beta(-2a-1, 1, 2a)$$

$$r_3 = \beta(2a, -2a-1, 1),$$

where $a = \cos\pi/5$ and $\beta = \cos 2\pi/5$. Then by the relation between a and β ;

$$4a^2 = 2a+1, \quad 4\beta^2 = -2\beta+1, \quad 2a = 2\beta+1, \quad \text{and} \quad 4a\beta = 1,$$

$$(r_1, r_2) = -a = -\cos\pi/5$$

$$(r_1, r_3) = 0$$

$$(r_2, r_3) = -1/2;$$

so $\{r_1, r_2, r_3\}$ has Coxeter graph I_3 .

* Construction of I_4

By the above procedure, we will view r_1, r_2 , and r_3 as vectors in $\mathbb{R}^4$ and extend to a base with graph I_4 by adjoining the vector r_4 such that $(r_1, r_4) = 0$, $(r_2, r_4) = 0$, and $(r_3, r_4) = -1/2$. So $r_4 = \beta(-2a, 0, -2a-1, 1)$. Hence a base for I_4 is

$$r_1 = \beta(2a+1, 1, -2a, 0)$$

$$r_2 = \beta(-2a-1, 1, 2a, 0)$$

$$r_3 = \beta(2a, -2a-1, 1, 0)$$

$$r_4 = \beta(-2a, 0, -2a-1, 1).$$

The bases for all the graphs in Figure 3.2 are tabulated in Table 4.1.

Graph	Base
A_n	$r_i = e_{i+1} - e_i, 1 \leq i \leq n$
B_n	$r_1 = e_1, r_i = e_i - e_{i-1}, 2 \leq i \leq n$
D_n	$r_1 = e_1 + e_2, r_i = e_i - e_{i-1}, 2 \leq i \leq n$
H_2^n	$r_1 = (1,0), r_i = (-\cos\pi/n, \sin\pi/n)$
G_2	$r_1 = (1,0), r_2 = (-\sqrt{3}/2, 1/2)$
I_3	$r_1 = \beta(2a+1, 1, -2a), r_2 = \beta(-2a-1, 1, 2a)$ $r_3 = \beta(2a, -2a-1, 1)$
I_4	$r_1 = \beta(2a+1, 1, -2a, 0), r_2 = \beta(-2a-1, 1, 2a, 0)$ $r_3 = \beta(2a, -2a-1, 1, 0), r_4 = \beta(-2a, 0, -2a-1, 1)$
F_4	$r_1 = -(1/2)\sum_{i=1}^4 e_i, r_2 = e_1, r_3 = e_2 - e_1, r_4 = e_3 - e_2$
E_k	$r_1 = (1/2)(\sum_{i=1}^3 e_i - \sum_{i=4}^8 e_i), r_i = e_i - e_{i-1}, 2 \leq i \leq k$ $k = 6, 7, \text{ and } 8$

Table 4.1

For each graph G from G_2 to E_8 in Table 4.1, let V be the space spanned by the set of vectors $\Gamma = \{r_1, \dots, r_n\}$. In each case Γ is a linearly independent set so it is a basis

for V . Thus there is a vector $t \in V$ such that $(t, r_i) > 0$, $1 \leq i \leq n$; we can choose such scalars so that $(t, r_i) = (\sum a_j r_j, r_i) = \sum a_j (r_j, r_i) > 0$ since $(r_j, r_i) = -\cos \pi / m_{ij} \leq 0$. Let S_i be the reflection in $O(V)$ along r_i , $1 \leq i \leq n$, and let $W = \langle S_1, \dots, S_n \rangle \leq O(V)$.

The following algorithm can be used to obtain the root system $\hat{\Delta}$ of W . Let $\Gamma_0 = \Gamma = \{r_1, \dots, r_n\}$. For each $r_i \in \Gamma_0$ with $(r_1, r_i) < 0$, apply S_1 to r_i to obtain the root $S_1 r_i$. The same procedure is applied in turn with S_1 and r_1 replaced by S_2 and r_2 , then by S_3 and r_3 , . . . , S_n and r_n . Denote the set of roots obtained, including Γ_0 , by Γ_1 . The process is repeated, with S_i applied to each $r \in \Gamma_1 \setminus \Gamma_0$ for which $(r_i, r) < 0$, $1 \leq i \leq n$, and the resulting set of roots is denoted by Γ_2 . We continue this process and obtain

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \subseteq \dots$$

Since S_i is applied to $r \in \Gamma_j$ only if $(r_i, r) < 0$, and since

$$S_i r = r - \frac{2(r_i, r)r_i}{(r_i, r_i)},$$

we see inductively that every root obtained by this process is a non-negative linear combination of $\{r_1, \dots, r_n\}$.

When the algorithm is applied to the bases in Table 4.1, we find that in each case the procedure terminates in a finite number of steps, in the sense that for some k we have $(r_i, r) \geq 0$ for all $r \in \Gamma_k \setminus \Gamma_{k-1}$, $1 \leq i \leq n$. Then we adjoin the

negatives of all roots obtained, setting $\Gamma^* = \Gamma_k \cup -\Gamma_k$.

The next step is to verify that $S_i \Gamma^* = \Gamma^*$, $1 \leq i \leq n$, and hence that $T \Gamma^* = \Gamma^*$ for all $T \in W$ since W is generated by $\{S_1, \dots, S_n\}$. It follows that $\Gamma^* = \hat{\Delta}$, the root system of W , by the definition of $\hat{\Delta}$.

Since $\hat{\Delta}$ contains a basis for V , the group W is effective. Also since the algorithm produces only roots that are non-negative linear combinations of $\{r_1, \dots, r_n\}$, and since the only other roots are negatives of those, we can see that $\{r_1, \dots, r_n\}$ is a t-base for $\hat{\Delta}$.

* Root system of G_2

Now we will apply the algorithm in the case of G_2 . $\Gamma_0 = \{r_1, r_2\}$, where $r_1 = (1, 0)$ and $r_2 = (-\sqrt{3}/2, 1/2)$, we let $W = \langle S_1, S_2 \rangle$. Then $(r_1, r_2) = -\sqrt{3}/2 < 0$, so at the first step,

$$S_1 r_2 = r_2 + \sqrt{3} r_1, \quad S_2 r_1 = r_1 + \sqrt{3} r_2.$$

$$\Gamma_1 \setminus \Gamma_0 = \{\sqrt{3} r_1 + r_2, r_1 + \sqrt{3} r_2\}.$$

At the second step,

$$(r_1, \sqrt{3} r_1 + r_2) = \sqrt{3}/2 > 0, \quad (r_1, r_1 + \sqrt{3} r_2) = -1/2 < 0,$$

$$(r_2, \sqrt{3} r_1 + r_2) = -1/2 < 0, \quad (r_2, r_1 + \sqrt{3} r_2) = \sqrt{3}/2 > 0.$$

And we obtain

$$S_1(r_1 + \sqrt{3} r_2) = -r_1 + \sqrt{3}(\sqrt{3} r_1 + r_2) = 2r_1 + \sqrt{3} r_2,$$

$$S_2(\sqrt{3} r_1 + r_2) = \sqrt{3}(r_1 + \sqrt{3} r_2) - r_2 = \sqrt{3} r_1 + 2r_2.$$

$$\Gamma_2 \setminus \Gamma_1 = \{2r_1 + \sqrt{3} r_2, \sqrt{3} r_1 + 2r_2\}.$$

At the third step,

$$(r_1, 2r_1 + \sqrt{3}r_2) = 1/2 > 0, \quad (r_1, \sqrt{3}r_1 + 2r_2) = 0,$$

$$(r_2, 2r_1 + \sqrt{3}r_2) = 0, \quad (r_2, \sqrt{3}r_1 + 2r_2) = 1/2 > 0.$$

Hence the procedure terminates at the third step, and we get

$$\Gamma^* = \Gamma_2 \cup -\Gamma_2. \quad \text{Also we can count } |\hat{\Delta}| = 12.$$

In order to verify that $S_i \Gamma^* = \Gamma^*$, it suffices to check that $S_i \Gamma_2 \subseteq \Gamma^*$; if $r \in \Gamma_2$, we already know that $S_i r \in \Gamma_2$ if $(r_i, r) < 0$. If $(r_i, r) = 0$, then $S_i r = r$. So it remains only to verify that $S_i r \in \Gamma^*$ if $r \in \Gamma_2$ and $(r_i, r) > 0$. If $r = r_i$, then $S_i r = -r \in \Gamma^*$, so the only cases left are

$$i = 1: r = \sqrt{3}r_1 + r_2, \quad 2r_1 + \sqrt{3}r_2;$$

$$i = 2: r = r_1 + \sqrt{3}r_2, \quad \sqrt{3}r_1 + 2r_2.$$

But in each of these cases $r = S_i r'$ for some $r' \in \Gamma_2$, e.g.,

$$2r_1 + \sqrt{3}r_2 = S_1(r_1 + \sqrt{3}r_2), \text{ so}$$

$$S_i r = S_i^2 r' = r' \in \Gamma_2.$$

Hence $\hat{\Delta} = \Gamma^*$.

* Root system of $\underline{I}_3$

$$\Gamma_0 = \{r_1, r_2, r_3\}, \text{ where } r_1 = \beta(2a+1, 1, -2a),$$

$$r_2 = \beta(-2a-1, 1, 2a), \quad r_3 = \beta(2a, -2a-1, 1).$$

$$(r_1, r_2) = -a < 0, \quad (r_1, r_3) = 0, \quad (r_2, r_3) = -1/2 < 0.$$

$$S_1 r_2 = \beta(2a, 1+2a, -1) = r_{12}, \quad S_2 r_1 = \beta(-2a, 1+2a, 1) = r_{21},$$

$$S_2 r_3 = \beta(-1, -2a, 2a+1) = r_{23}, \quad S_3 r_2 = \beta(-1, -2a, 2a+1).$$

$$\Gamma_1 \setminus \Gamma_0 = \{r_{12}, r_{21}, r_{23}\}.$$

$$(r_1, r_{12}) = a > 0, \quad (r_2, r_{12}) = -\beta < 0, \quad (r_3, r_{12}) = -1/2.$$

$$(r_1, r_{21}) = -\beta < 0, (r_2, r_{21}) = \beta^2(4a^2+8a+1) > 0,$$

$$(r_3, r_{21}) = -a < 0.$$

$$(r_1, r_{23}) = -a < 0, (r_2, r_{23}) = 1/2, (r_3, r_{23}) = 1/2.$$

$$S_2 r_{12} = e_2, S_3 r_{12} = e_1, S_1 r_{23} = e_1.$$

$$S_1 r_{21} = e_2, S_3 r_{21} = \beta(1, -2a, 1+2a) = r_{321}.$$

$$\Gamma_2 \setminus \Gamma_1 = \{e_1, e_2, \beta(1, -2a, 1+2a)\}.$$

$$(r_1, e_1) = a > 0, (r_2, e_1) = -a < 0, (r_3, e_1) = \beta > 0.$$

$$(r_1, e_2) = \beta > 0, (r_2, e_2) = \beta > 0, (r_3, e_2) = \beta > 0.$$

$$(r_1, r_{321}) = 0, (r_2, r_{321}) = 0, (r_3, r_{321}) = a > 0.$$

$$S_2 e_1 = \beta(-1, 2a, 2a+1) = r.$$

$$\Gamma_3 \setminus \Gamma_2 = \{\beta(-1, 2a, 2a+1)\}.$$

$$(r_1, r) = -1/2 < 0, (r_2, r) = a > 0, (r_3, r) = -\beta < 0.$$

$$S_1 r = \beta(2a, 2a+1, 1) = r', S_3 r = e_3.$$

$$\Gamma_4 \setminus \Gamma_3 = \{\beta(2a, 2a+1, 1), e_3\}.$$

$$(r_1, r') = 1/2 > 0, (r_2, r') = 0, (r_3, r') = -\beta < 0.$$

$$(r_1, e_3) = -1/2 < 0, (r_2, e_3) = 2a\beta > 0, (r_3, e_3) = \beta > 0.$$

$$S_3 r' = \beta(2a+1, 1, 2a). S_1 e_3 = \beta(2a+1, 1, 3a) = r''.$$

$$\Gamma_5 \setminus \Gamma_4 = \{\beta(2a+1, 1, 2a)\}.$$

$$(r_1, r'') = 2\beta^2(2a^2+1) > 0, (r_2, r'') = -\beta < 0, (r_3, r'') = \beta < 0.$$

$$S_2 r'' = \beta(1, 2a, 2a+1) = r^\circ.$$

$$\Gamma_6 \setminus \Gamma_5 = \{\beta(1, 2a, 2a+1)\}.$$

$$(r_1, r^\circ) = 0, (r_2, r^\circ) = \beta^2(4a^2+2a-1) > 0, (r_3, r^\circ) = 0.$$

$\hat{\Delta} = \{\pm e_i, i = 1, 2, 3; \beta(\pm(2a+1), \pm 1, \pm 2a), \text{ and all even permutations of coordinates}\}.$

* Root system of F_4

$$\Gamma_0 = \{r_1, r_2, r_3, r_4\},$$

where $r_1 = (-1/2)\sum_{i=1}^4 e_i$, $r_2 = e_1$, $r_3 = e_2 - e_1$, $r_4 = e_3 - e_2$.

$$(r_1, r_2) = -1/2 < 0, (r_1, r_3) = 0, (r_1, r_4) = 0.$$

$$(r_2, r_3) = -1 < 0, (r_2, r_4) = 0, (r_3, r_4) = -1 < 0.$$

$$S_2 r_1 = r_1 + r_2, S_1 r_2 = r_1 + r_2 = 1/2(1, -1, -1, -1).$$

$$S_3 r_2 = r_2 + r_3 = e_2, S_2 r_3 = e_1 + e_2.$$

$$S_4 r_3 = r_3 + r_4 = e_3 - e_1, S_3 r_4 = e_3 - e_1.$$

$$\Gamma_1 \setminus \Gamma_0 = \{r_1 + r_2, e_2, e_3 - e_1, e_1 + e_2\}.$$

$$(r_1, r_1 + r_2) = 1/2, (r_2, r_1 + r_2) = 1/2, (r_3, r_1 + r_2) = -1 < 0,$$

$$(r_4, r_1 + r_2) = 0.$$

$$S_3(r_1 + r_2) = r_1 + r_2 + r_3 = 1/2(-1, 1, -1, -1).$$

$$(r_1, e_2) = -1/2 < 0, (r_2, e_2) = 0, (r_3, e_2) = 1, (r_4, e_2) = -1.$$

$$S_1 e_2 = r_1 + r_2 + r_3, S_4 e_2 = r_2 + r_3 + r_4 = e_3.$$

$$(r_1, e_3 - e_1) = 0, (r_2, e_3 - e_1) = -1, (r_3, e_3 - e_1) = 1, (r_4, e_3 - e_1) = 1.$$

$$S_2(e_3 - e_1) = e_1 + e_3.$$

$$(r_1, e_1 + e_2) = -1, (r_2, e_1 + e_2) = 1, (r_3, e_1 + e_2) = 0,$$

$$(r_4, e_1 + e_2) = -1.$$

$$S_1(e_1 + e_2) = -e_3 - e_4, S_4(e_1 + e_2) = e_1 + e_3.$$

$$\Gamma_2 \setminus \Gamma_1 = \{r_1 + r_2 + r_3, e_3, e_1 + e_3, -e_3 - e_4\}.$$

$$\begin{aligned}
(r_1, r_1+r_2+r_3) &= 1/2, \quad (r_2, r_1+r_2+r_3) = -1/2 < 0, \\
(r_3, r_1+r_2+r_3) &= 1, \quad (r_4, r_1+r_2+r_3) = -1 < 0. \\
S_2(r_1+r_2+r_3) &= r_1 + 2r_2 + r_3 = 1/2(1, 1, -1, -1), \\
S_4(r_1+r_2+r_3) &= r_1 + r_2 + r_3 + r_4 = 1/2(-1, -1, 1, -1). \\
(r_1, e_3) &= -1/2, \quad (r_2, e_3) = 0, \quad (r_3, e_3) = 0, \quad (r_4, e_3) = 1. \\
S_1 e_3 &= r_1 + r_2 + r_3 + r_4. \\
(r_1, e_3+e_1) &= -1, \quad (r_2, e_3+e_1) = 1, \quad (r_3, e_3+e_1) = -1, \\
(r_4, e_3+e_1) &= 1. \\
S_1(e_3+e_1) &= -e_2 - e_4, \quad S_3(e_3+e_1) = e_3 + e_2. \\
(r_1, -e_3-e_4) &= 1, \quad (r_2, -e_3-e_4) = 0, \quad (r_3, -e_3-e_4) = 0, \\
(r_4, -e_3-e_4) &= -1. \\
S_4(-e_3-e_4) &= -e_2 - e_4. \\
\Gamma_3 \setminus \Gamma_2 &= \{r_1+2r_2+r_3, r_1+r_2+r_3+r_4, -e_2-e_4, e_3+e_2\}.
\end{aligned}$$

$$\begin{aligned}
(r_1, r_1+2r_2+r_3) &= 0, \quad (r_2, r_1+2r_2+r_3) = 1/2, \\
(r_3, r_1+2r_2+r_3) &= 0, \quad (r_4, r_1+2r_2+r_3) = -1 < 0. \\
S_4(r_1+2r_2+r_3) &= r_1 + 2r_2 + r_3 + r_4 = 1/2(+1, -1, 1, -1). \\
(r_1, r_1+r_2+r_3+r_4) &= 1/2, \quad (r_2, r_1+r_2+r_3+r_4) = -1/2 < 0, \\
(r_3, r_1+r_2+r_3+r_4) &= 0, \quad (r_4, r_1+r_2+r_3+r_4) = 1. \\
S_2(r_1+r_2+r_3+r_4) &= r_1 + 2r_2 + r_3 + r_4. \\
(r_1, -e_2-e_4) &= 1, \quad (r_2, -e_2-e_4) = 0, \quad (r_3, -e_2-e_4) = -1, \\
(r_4, -e_2-e_4) &= 1. \\
S_3(-e_2-e_4) &= -e_1 - e_4. \\
(r_1, e_2+e_3) &= -1, \quad (r_2, e_2+e_3) = 0, \quad (r_3, e_2+e_3) = 1, \quad (r_4, e_2+e_3) = 0. \\
S_1(e_2+e_3) &= -e_1 - e_4.
\end{aligned}$$

$$\Gamma_4 \setminus \Gamma_3 = \{r_1 + 2r_2 + r_3 + r_4, -e_1 - e_4\}.$$

$$(r_1, r_1 + 2r_2 + r_3 + r_4) = 0, (r_2, r_1 + 2r_2 + r_3 + r_4) = 1/2,$$

$$(r_3, r_1 + 2r_2 + r_3 + r_4) = -1 < 0, (r_4, r_1 + 2r_2 + r_3 + r_4) = 1.$$

$$S_3(r_1 + 2r_2 + r_3 + r_4) = r_1 + 2r_2 + 2r_3 + r_4 = 1/2(-1, 1, 1, -1).$$

$$(r_1, -e_1 - e_4) = 1, (r_2, -e_1 - e_4) = -1, (r_3, -e_1 - e_4) = 1,$$

$$(r_4, -e_1 - e_4) = 0.$$

$$S_2(-e_1 - e_4) = e_1 - e_4.$$

$$\Gamma_5 \setminus \Gamma_4 = \{r_1 + 2r_2 + 2r_3 + r_4, e_1 - e_4\}.$$

$$(r_1, r_1 + 2r_2 + 2r_3 + r_4) = 0, (r_2, r_1 + 2r_2 + 2r_3 + r_4) = -1/2 < 0,$$

$$(r_3, r_1 + 2r_2 + 2r_3 + r_4) = 1, (r_4, r_1 + 2r_2 + 2r_3 + r_4) = 0.$$

$$S_2(r_1 + 2r_2 + 2r_3 + r_4) = r_1 + 3r_2 + 2r_3 + r_4 = 1/2(1, 1, 1, -1).$$

$$(r_1, e_1 - e_4) = 0, (r_2, e_1 - e_4) = 1, (r_3, e_1 - e_4) = -1, (r_4, e_1 - e_4) = 0.$$

$$S_3(e_1 - e_4) = e_2 - e_4.$$

$$\Gamma_6 \setminus \Gamma_5 = \{r_1 + 3r_2 + 2r_3 + r_4, e_2 - e_4\}.$$

$$(r_1, r_1 + 3r_2 + 2r_3 + r_4) = -1/2 < 0, (r_2, r_1 + 3r_2 + 2r_3 + r_4) = 1/2,$$

$$(r_3, r_1 + 3r_2 + 2r_3 + r_4) = 0, (r_4, r_1 + 3r_2 + 2r_3 + r_4) = 0.$$

$$S_1(r_1 + 3r_2 + 2r_3 + r_4) = 2r_1 + 3r_2 + 2r_3 + r_4 = -e_4.$$

$$(r_1, e_2 - e_4) = 0, (r_2, e_2 - e_4) = 0, (r_3, e_2 - e_4) = 1, (r_4, e_2 - e_4) = -1.$$

$$S_4(e_2 - e_4) = e_3 - e_4.$$

$$\Gamma_7 \setminus \Gamma_6 = \{2r_1 + 3r_2 + 2r_3 + r_4, e_3 - e_4, -e_4\}$$

$$(r_1, 2r_1 + 3r_2 + 2r_3 + r_4) = 1/2, (r_2, 2r_1 + 3r_2 + 2r_3 + r_4) = 0,$$

$$(r_3, 2r_1 + 3r_2 + 2r_3 + r_4) = 0, (r_4, 2r_1 + 3r_2 + 2r_3 + r_4) = 0.$$

$$(r_1, e_3 - e_4) = 0, (r_2, e_3 - e_4) = 0, (r_3, e_3 - e_4) = 0, (r_4, e_3 - e_4) = 1.$$

$$\Delta = \{\pm e_i, 1 \leq i \leq 4; \pm e_i \pm e_j, 1 \leq j < i \leq 4; 1/2 \sum_i \epsilon_i e_i, \\ \epsilon_i = \pm 1\}.$$

* Root system of E_8

The construction can be simplified for E_8 if we modify the algorithm. Let $\underline{E}_8 = \langle S_1, \dots, S_8 \rangle$. Since $\{r_2, \dots, r_8\}$ is a base for $\underline{A}_7$, $\underline{A}_7$ is a subgroup of $\underline{E}_8$, and among the roots of $\underline{E}_8$ are the 28 positive roots $r_{ij} = e_i - e_j$, $1 \leq j < i \leq 8$, of $\underline{A}_7$. Let $S_{ij} = S_{r_{ij}}$ and begin the algorithm with the larger set

$$\Gamma_0 = \{r_{ij}: 1 \leq j < i \leq 8\} \cup \{r_1\}.$$

If $j \leq 3 < i$, then $(r_1, r_{ij}) = -1$ and $S_1 r_{ij} = r_{ij} + r_1$ is a root, the effect being a change of sign in the i^{th} and j^{th} entries of r_1 . But $(r_{km}, r_{ij}) \geq 0$ for $j, m \leq 3 < i, k$. So at the next step we have

$$(r_{km}, r_{ij} + r_1) = \begin{cases} 1 & \text{if } i = k \text{ and } j = m \\ 0 & \text{if } j \neq m, i = k \text{ or } j = m, i \neq k \\ -1 & \text{if } i \neq k \text{ and } j \neq m \end{cases} (*)$$

where $j, m \leq 3 < i, k$.

Hence we need to proceed the algorithm to the last case (*),

$$S_{km}(r_{ij} + r_1) = \begin{cases} S_{km}r_{ij} + S_{km}r_1 \\ r_1 + r_{km} + r_{ij}. \end{cases}$$

This is a root that changes the k^{th} , m^{th} , i^{th} , j^{th} signs in r_1 .

At the third step we have

$$(r_{i,j'}, r_1 + r_{ij} + r_{km}) = \begin{cases} 3 & \text{(i)} \\ 1 & \text{(ii)} \\ -1 & \text{(iii)}; \end{cases}$$

(i) if $i = i' = k$ and $j = j' = m$,

(ii) one subscript is different from the other two, among (i, i', k) and (j, j', m) ,

(iii) if i, i', k are distinct and j, j', m are distinct,

where $j, j', m \leq 3 < i, i', k$.

So $S_{i,j'}(r_1 + r_{ij} + r_{km}) = r_1 + r_{i,j'} + r_{ij} + r_{km}$. This is a root which changes the $i^{\text{th}}, i'^{\text{th}}, j^{\text{th}}, j'^{\text{th}}, k^{\text{th}}, m^{\text{th}}$ signs in r_1 .

Hence we obtained as roots all the vectors that result from permuting the entries of r_1 . In particular, we obtain

$$r = (1/2)(-1, -1, -1, 1, 1, 1, -1, -1),$$

and $(r, r_1) = -1$, so

$$S_1 r = r + r_1 = -e_7 - e_8$$

is a root. If $i \leq 6$, then $(r_{7i}, -e_7 - e_8) = -1$ and we obtain

$$S_{7i} S_1 r = (-e_7 - e_8) + (e_7 - e_i) = -e_i - e_8.$$

Then if $j \neq i, j \neq 8$, we also have $(r_{8j}, -e_i - e_8) = -1$. Hence $S_{8j}(-e_i - e_8) = -e_i - e_j$ for all i, j with $i \neq j$.

Note that we obtain all the vectors $(1/2)\sum_{i=1}^8 \epsilon_i e_i$, where one ϵ_i is 1 and the others are -1. Thus all the roots of E_8 obtained are non-negative linear combinations of $\{r_1, \dots, r_8\}$. Adjoining their negatives, we have the following set Γ^* of 240 roots:

all $\pm e_i \pm e_j$, $i \neq j$, $1 \leq i, j \leq 8$,

all $(1/2)\sum_1^8 \epsilon_i e_i$, $\epsilon_i = \pm 1$, $\prod_1^8 \epsilon_i = -1$.

In order to show that $\Gamma^* = \hat{\Delta}$, the root system of E_8 , it will suffice to show that $S_i \Gamma^* = \Gamma^*$, $1 \leq i \leq 8$. The roots $\pm e_i \pm e_j$ are the roots of D_8 , as constructed above, and D_8 has $S_2, \dots, S_8$ among its reflections, so those roots are invariant under $S_2, \dots, S_8$. As for the effect of S_1 on those vectors, it will suffice to consider $S_1(e_i \pm e_j)$, with $i > j$. Here we call roots of the form $(1/2)\sum \epsilon_i e_i$ 1/2-vectors. It is easily verified that

$$S_1(e_i + e_j) = e_i + e_j \quad \text{if } j \leq 3 < i;$$

$$S_1(e_i + e_j) = e_i + e_j - r_1$$

is a 1/2-vector with one minus sign if $i \leq 3$;

$$S_1(e_i + e_j) = e_i + e_j + r_1$$

is a 1/2-vector with three minus signs if $j > 3$;

$$S_1(e_i - e_j) = e_i - e_j \quad \text{if } i \leq 3 \text{ or } j \geq 4;$$

and $S_1(e_i - e_j) = e_i - e_j + r_1$

is a 1/2-vector with five minus signs if $j \leq 3 < i$.

It remains to check the effects of all S_i on the 1/2-vectors $(1/2)\sum \epsilon_i e_i$. If $i \neq 1$ and r is a 1/2-vector, then either $S_i r = r$ or else $S_i r$ is another 1/2-vector with two sign changes. If we write

$$r_1 = (1/2)\sum \mu_i e_i, \quad r = (1/2)\sum \epsilon_i e_i,$$

then $(r_1, r) = (1/4)\sum \mu_i \epsilon_i$,

and

$$\prod_1^8 (\mu_i \epsilon_i) = (\prod \mu_i) (\prod \epsilon_i) = (-1)^2 = 1;$$

so $\mu_i \epsilon_i = -1$ for an even number of indices i . Assuming that $r \neq \pm r_1$, the number is two, four, or six. If it is four, then $(r_1, r) = 0$ and $S_1 r = r$. If it is two, then $(r_1, r) = 1$ and $S_1 r = r - r_1$. But then r_1 and r agree except at two entries, so $r - r_1$ is $\pm e_i \pm e_j$ for some i and j . Similarly, if $\mu_i \epsilon_i = -1$ for six values of i , then $S_1 r = r + r_1$, and $S_1 r$ is again $\pm e_i \pm e_j$ for some i and j . Thus $\Gamma^* = \hat{\Delta}$, and we may conclude that $\underline{E}_8$ is a Coxeter group with Coxeter graph E_8 .

* Root system of $\underline{E}_7$

If we let $\underline{E}_7 = \langle S_1, \dots, S_7 \rangle$, then $\underline{E}_7 \leq \underline{E}_8$, and the roots of $\underline{E}_7$ are among those of $\underline{E}_8$. The root $u = (1/2) \cdot (1, 1, 1, 1, 1, 1, 1, -1)$ of $\underline{E}_8$ is orthogonal to $r_1, \dots, r_7$, so the roots of $\underline{E}_7$ are precisely the roots of $\underline{E}_8$ that are orthogonal to u . These are the 42 roots $e_i - e_j$: $i, j \neq 8$; the 14 roots $\pm(e_i + e_8)$, $i \neq 8$; and the 70 $1/2$ -vectors having either three negative entries with the eighth positive or five negative entries including the eighth. Thus $|\hat{\Delta}| = 126$ for $\underline{E}_7$.

* Root system of $\underline{E}_6$

Similaly, we let $\underline{E}_6 = \langle S_1, \dots, S_6 \rangle$, then $\underline{E}_6 \leq \underline{E}_7$. So the roots of $\underline{E}_6$ are the 30 roots of $e_i - e_j$: $i, j \neq 7, 8$; 2 roots of $e_7 + e_8$, $-e_7 - e_8$; and the 40 roots of $1/2$ -vectors having

three negative entries with the seventh and the eighth positive or six negative entries including the seventh and the eighth entries.

* Root system of $\underline{I}_4$

For $\underline{I}_4$, with graph I_4 , we will take a different approach. Identify $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ with the real quaternions $x = x_1 + x_2i + x_3j + x_4k \in \mathbb{Q}$. Then we can show that the inner product in $\mathbb{R}^4$ can be expressed in terms of addition, multiplication, and the adjoint operation in $\mathbb{Q}$ by the formula;

$$(x, y) = 1/2(xy^* + yx^*).$$

Also, if $r \in \mathbb{R}^4$ is a unit vector, then

$$\begin{aligned} S_r(x) &= x - 2(x, r)r \\ &= x - 2(1/2) \cdot (xr^* + rx^*)r \\ &= x - xr^*r - rx^*r \\ &= -rx^*r, \quad \text{for all } x \in \mathbb{R}^4. \end{aligned}$$

Let K be a finite subgroup of the multiplicative group of nonzero quaternions, then $|x|^2 = 1$. Otherwise, K would be an infinite group. If $x \in K$, $xx^* = 1 \in K$ implies that x^* is the inverse of x and so $x^* \in K$. If $|K|$ is even, then $x \in \mathbb{Q} \setminus \{1\}$ has a unique inverse to make up pairs. So in order to satisfy $|K| = \text{even}$, $-1 \in \mathbb{R}$ must be in K . Now for each $x \in K$, let $S_x \in O(\mathbb{R}^4)$ be the reflection along x and let $|K|$ be even, then $S_x K = K$ for each $x \in K$; since $S_x(y) = -xy^*x \in K$

for all $y \in K$.

Suppose K is a group generated by the fundamental roots of I_4 which are shown in the table 4.1 on page 87; $K = \langle r_1, r_2, r_3, r_4 \rangle$. Using quaternion's multiplication, we can get $(r_3)^3 = -1 \in K$ so that $S_x(K) = K$ for all $x \in K$ and hence K is the root system for I_4 . Since $(r_1)^5 = (r_3^2)^3 = (r_1 r_3^2)^2 = -1$, $K/\{\pm 1\} = \langle r_1, r_3: (r_1)^5 = (r_3^2)^3 = (r_1 r_3^2)^2 = 1 \rangle$ is known to be the rotation group of a regular icosahedron whose order is 60 (p15,[2]). Hence $|K| = 60 \cdot 2 = 120$.

4.4 Orders of Positive Definite Irreducible

Coxeter Groups

The orders of the groups A_n , B_n , and D_n were computed in Section 4.3, and the order of H_2^n is $2n$. Since G_2 is just the dihedral group H_2^6 , its order is 12. The computation of the remaining positive definite irreducible Coxeter groups requires more preparation.

Proposition 4.4.1 If $\underline{W}$ is a permutation group on a set $\underline{L}$ and $x \in \underline{L}$, then $[\underline{W}:\text{Stab}(x)] = |\text{Orb}(x)|$.

Proof: Let $\underline{K} = \text{Stab}(x)$. If $R, T \in \underline{W}$ and $R\underline{K} = T\underline{K}$, then $T^{-1}R \in \underline{K}$, or $T^{-1}R(x) = x$; so $R(x) = T(x)$. Thus $\phi(T\underline{K}) = T(x)$

defines a mapping ϕ from the set of left cosets of $\underline{K}$ onto the orbit of x . If $R(x) = T(x)$, then $T^{-1}R \in \underline{K}$, and $R\underline{K} = T\underline{K}$. Thus ϕ is also 1-1 and the proposition is proved. ■

Let W be a Coxeter group. As in Section 4.2, we denote by $\{s_1, \dots, s_n\}$ the basis dual to $\pi = \{r_1, \dots, r_n\}$. The open half-space containing r_i determined by the hyperplane $P_i = r_i^\perp$ will be denoted by L_i ; i.e.,

$$L_i = \{x \in V: (x, r_i) > 0\}.$$

Now we will prove Witt's theorem.

Theorem 4.4.2 If $\underline{H}$ is the subgroup of W leaving s_i fixed, then

$$\underline{H} = \langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n \rangle.$$

Proof: There is no loss of generality in assuming that $i = n$. Let $\underline{K} = \langle s_1, \dots, s_{n-1} \rangle$. Then if $s_i \in \underline{K}$,

$$\begin{aligned} s_i(s_n)(v) &= s_n s_i(v) \\ &= s_n \left\{ v - \frac{2(v, r_i) r_i}{(r_i, r_i)} \right\} \\ &= (s_n, v) - \frac{2(v, r_i)}{(r_i, r_i)} s_n(r_i) \\ &= s_n(v); \text{ since } s_n(r_i) = 0 \text{ for } n \neq i. \end{aligned}$$

Hence $s_i s_n = s_n$, and $s_i \in \underline{H}$. So $\underline{K} \leq \underline{H}$. Let X_i be the $(n-1)$ -sphere of radius $|s_n|$ centered at the origin; i.e.,

$$X_1 = \{x \in V: |x| = |s_n|\}.$$

Let X_2 be an $(n-1)$ -sphere of radius d centered at s_n , where d is sufficiently small so that $X_2 \subseteq L_n$. $s_n \in L_n$ since $(s_n, r_n) = 1$, and L_n is an open set.

Let O_1, O_2 be the center of X_1, X_2 respectively. From geometry, $\triangle ABO_2, \triangle CAO_2$ are right triangles and they are similar. So,

$$\frac{AB}{CA} = \frac{BO_2}{AO_2} = \frac{AO_2}{CO_2}.$$

Hence

$$\frac{x}{CA} = \frac{y}{d} = \frac{d}{2|s_n|},$$

and $y = d^2/2|s_n|$.

By Pythagorean theorem,

$$CA^2 + d^2 = (2|s_n|)^2.$$

$$\text{So } CA = \sqrt{4|s_n|^2 - d^2}.$$

$$\text{Therefore, } x = (d/2|s_n|) \cdot \sqrt{4|s_n|^2 - d^2}$$

is the radius of X_2 , centered at

$|s_n| - y = |s_n|(1 - d^2/2|s_n|^2)$. According to Figure 4.3, $X = X_1 \cap X_2$ is an $(n-2)$ -sphere of radius $(d/2|s_n|) \cdot \sqrt{4|s_n|^2 - d^2}$ centered at the point $(1 - d^2/2|s_n|^2)s_n$. Since $\underline{H} \leq \mathcal{O}(V)$ and $\underline{H}$ leaves s_n fixed, the sphere X is invariant under $\underline{H}$.

Choose a point $x_0 \in L_1 \cap L_2 \cap \dots \cap L_{n-1}$ that is not fixed by any nonidentity transformation in $\underline{H}$. Then the Fricke-Klein construction in Section 4.2 can be applied, using the point x_0 , to construct fundamental regions $F(\underline{K})$

Figure 4.3

and $F(\underline{H})$ for $\underline{K}$ and $\underline{H}$ in V . Since $\underline{K} \leq \underline{H}$, we have $F(\underline{H}) \subseteq F(\underline{K})$. The half-space $L_1, \dots, L_{n-1}$ are obtained when the reflections $S_1, \dots, S_{n-1}$ are applied to x_0 , so $F(\underline{K}) \subseteq L_1 \cap \dots \cap L_{n-1}$. As we indicated in Section 4.2, we can obtain fundamental regions for $\underline{K}$ and $\underline{H}$ in X by setting $F_X(\underline{K}) = F(\underline{K}) \cap X$ and $F_X(\underline{H}) = F(\underline{H}) \cap X$. By Theorem 4.2.6 the set $F(W) = L_1 \cap \dots \cap L_n$ is a fundamental region for W in V . Since $X \subseteq L_n$, we have $F_X(\underline{K}) = F(\underline{K}) \cap X \subseteq L_1 \cap \dots \cap L_{n-1} \cap X \subseteq L_1 \cap \dots \cap L_n \subseteq F(W)$. If $F_X(\underline{H}) \neq F_X(\underline{K})$, then we may choose $x \in F_X(\underline{K}) \setminus F_X(\underline{H})$, and there is $y \in F_X(\underline{H})$ and $T \in \underline{H}$ satisfying $Ty = x$. But this is impossible, since x and y are distinct points in the fundamental region $F(W)$. Thus $F_X(\underline{H}) = F_X(\underline{K})$, from which it follows that $|\underline{H}| = |\underline{K}|$, and hence that $\underline{H} = \underline{K}$. ■

Proposition 4.4.3 Suppose that W is irreducible and that its Coxeter graph G has no marks over its branches. Then W is transitive as a permutation group on its root system $\hat{\Delta}$.

Proof: Since $\hat{\Delta} = \{Tr_i : T \in W, r_i \in \pi\}$, it will suffice to prove that if $r_i, r_j \in \pi$, then $Tr_i = r_j$ for some $T \in W$. Suppose that r_i and r_j correspond to nodes of G that are adjacent, i.e., joined by a branch. Then $(r_i, r_j) / |r_i| |r_j| = -1/2$, and

$$S_i S_j r_i = S_i(r_i + r_j) = -r_i + (r_i + r_j) = r_j.$$

Since roots corresponding to adjacent nodes are W -

equivalent, and since G is connected, by Proposition 3.2.1, it follows that any two roots in π are W -equivalent, and the proof is complete. ■

The proof of Proposition 4.4.3 will also show that $\underline{I}_3$ and $\underline{I}_4$ are transitive on their root systems if we can show that the roots $r_1, r_2 \in \pi$, with $(r_1, r_2) = -\cos\pi/5$, can be interchanged by elements of $\underline{I}_3$ and $\underline{I}_4$, respectively. It can be checked directly that $(S_1 S_2)^2 r_1 = r_2$, since $\langle S_1, S_2 \rangle$ is a dihedral group of order 10, and to verify that $S_1 S_2 S_1 S_2 r_1 = r_2$ by following the arrows in Figure 4.4.

Figure 4.4

Now we know that $\underline{A}_n, \underline{D}_n, \underline{E}_6, \underline{E}_7, \underline{E}_8$, and $\underline{I}_3, \underline{I}_4$ are transitive as a permutation group on its root system $\hat{\Delta}$.

The group $\underline{E}_4$ is not transitive on its root system $\hat{\Delta}$. A modified version of the algorithm used previously to construct all roots from the roots in a base can be used to

determine the orbits of $\hat{\Delta}$. If we begin with a single root in the base of E_4 , say r_1 , and apply to it every S_i for which $(r_i, r_1) \neq 0$, then apply every S_i to each resulting root r for which $(r, r_i) \neq 0$, and continue in this manner, then the set of roots obtained is clearly $\text{Orb}(r_1)$.

* Computation of finding $\text{Orb}(r_1)$ of E_4 :

1)

$$(r_1, r_2) = -1/2, \quad (r_1, r_3) = 0, \quad (r_1, r_4) = 0.$$

$$S_1 r_1 = -r_1 = 1/2(-1, -1, -1, -1),$$

$$S_2 r_1 = r_1 + r_2 = 1/2(1, -1, -1, -1).$$

2)

$$(r_1, r_1+r_2) = 1/2, \quad (r_2, r_1+r_2) = 1/2, \quad (r_3, r_1+r_2) = -1,$$

$$(r_4, r_1+r_2) = 0.$$

$$S_1(r_1+r_2) = r_2 = S_1 S_2 r_1 = e_1,$$

$$S_2(r_1+r_2) = r_1 = S_2 S_2 r_1 = 1/2(-1, -1, -1, -1),$$

$$S_3(r_1+r_2) = r_1 + r_2 + r_3 = S_3 S_2 r_1 = 1/2(-1, 1, -1, -1).$$

3)

$$(r_1, r_2) = -1/2, \quad (r_3, r_2) = -1, \quad (r_4, r_2) = 0.$$

$$S_1 r_2 = r_1 + r_2 = S_1 S_1 S_2 r_1 = S_2 r_1,$$

$$S_2 r_2 = -r_2 = S_2 S_1 S_2 r_1 = -e_1,$$

$$S_3 r_2 = r_2 + r_3 = S_3 S_1 S_2 r_1 = e_2.$$

$$(r_1, r_1+r_2+r_3) = 1/2, \quad (r_2, r_1+r_2+r_3) = -1/2, \quad (r_3, r_1+r_2+r_3) = 1,$$

$$(r_4, r_1+r_2+r_3) = -1.$$

$$S_1(r_1+r_2+r_3) = r_2 + r_3 = S_1 S_3 S_2 r_1,$$

$$S_2(r_1+r_2+r_3) = r_1 + 2r_2 + r_3 = S_2 S_3 S_2 r_1 = 1/2(1, 1, -1, -1),$$

$$S_3(r_1+r_2+r_3) = r_1 + r_2 = S_3S_3S_2r_1 = S_2r_1,$$

$$S_4(r_1+r_2+r_3) = r_1 + r_2 + r_3 + r_4 = S_4S_3S_2r_1 = 1/2(-1, -1, 1, -1).$$

4)

$$(r_1, -r_2) = 1/2, (r_3, -r_2) = 1, (r_4, -r_2) = 0.$$

$$S_1(-r_2) = -r_1 - r_2 = S_1S_2S_1S_2r_1 = 1/2(-1, 1, 1, 1),$$

$$S_2(-r_2) = r_2 = S_2S_2S_1S_2r_1 = S_1S_2r_1,$$

$$S_3(-r_2) = -r_2 - r_3 = S_3S_2S_1S_2r_1 = -e_2.$$

$$(r_1, r_2+r_3) = -1/2, (r_2, r_2+r_3) = 0, (r_3, r_2+r_3) = 1,$$

$$(r_4, r_2+r_3) = -1.$$

$$S_1(r_2+r_3) = r_1 + r_2 + r_3 = S_1S_3S_1S_2r_1,$$

$$S_3(r_2+r_3) = r_2 = S_3S_3S_1S_2r_1 = S_1S_2r_1,$$

$$S_4(r_2+r_3) = r_2 + r_3 + r_4 = S_4S_3S_1S_2r_1.$$

$$(r_1, r_1+2r_2+r_3) = 0, (r_2, r_1+2r_2+r_3) = 1/2, (r_3, r_1+2r_2+r_3) = 0,$$

$$(r_4, r_1+2r_2+r_3) = -1.$$

$$S_2(r_1+2r_2+r_3) = r_1 + r_2 + r_3 = S_2S_2S_3S_2r_1 = S_3S_2r_1,$$

$$S_4(r_1+2r_2+r_3) = r_1 + r_2 + r_3 + r_4 = S_4S_2S_3S_2r_1.$$

$$(r_1, r_1+r_2+r_3+r_4) = 1/2, (r_2, r_1+r_2+r_3+r_4) = -1/2,$$

$$(r_3, r_1+r_2+r_3+r_4) = 0, (r_4, r_1+r_2+r_3+r_4) = 1.$$

$$S_1(r_1+r_2+r_3+r_4) = r_2 + r_3 + r_4 = S_1S_4S_3S_2r_1 = e_3,$$

$$S_2(r_1+r_2+r_3+r_4) = r_1 + 2r_2 + r_3 + r_4 = S_2S_4S_3S_2r_1 \\ = 1/2(1, -1, 1, -1),$$

$$S_4(r_1+r_2+r_3+r_4) = r_1 + r_2 + r_3 = S_4S_4S_3S_2r_1 = S_3S_2r_1.$$

5)

$$(r_1, -r_1-r_2) = -1/2, (r_2, -r_1-r_2) = -1/2, (r_3, -r_1-r_2) = 1,$$

$$(r_4, -r_1-r_2) = 0.$$

$$S_1(-r_1-r_2) = -r_2 = S_2S_1S_2r_1,$$

$$\begin{aligned}
S_2(-r_1-r_2) &= -r_2 = S_2S_1S_2S_1S_2r_1, \\
S_3(-r_1-r_2) &= -r_1 - r_2 - r_3 = S_3S_1S_2S_1S_2r_1 = 1/2(1, -1, 1, 1). \\
(r_1, -r_2-r_3) &= 1/2, \quad (r_2, -r_2-r_3) = 0, \quad (r_3, -r_2-r_3) = -1, \\
(r_4, -r_2-r_3) &= 1. \\
S_1(-r_2-r_3) &= -r_1 - r_2 - r_3 = S_1S_3S_2S_1S_2r_1, \\
S_3(-r_2-r_3) &= -r_2 = S_2S_1S_2r_1, \\
S_4(-r_2-r_3) &= -r_2 - r_3 - r_4 = S_4S_3S_2S_1S_2r_1 = -e_3. \\
(r_1, r_2+r_3+r_4) &= 1/2, \quad (r_2, r_2+r_3+r_4) = 0, \quad (r_3, r_2+r_3+r_4) = 0, \\
(r_4, r_2+r_3+r_4) &= 1. \\
S_1(r_2+r_3+r_4) &= r_1 + r_2 + r_3 + r_4 = S_1S_4S_3S_1S_2r_1, \\
S_4(r_2+r_3+r_4) &= r_2 + r_3 = S_4S_4S_3S_1S_2r_1 = S_3S_1S_2r_1. \\
(r_1, r_1+2r_2+r_3+r_4) &= 0, \quad (r_2, r_1+2r_2+r_3+r_4) = 1/2, \\
(r_3, r_1+2r_2+r_3+r_4) &= -1, \quad (r_4, r_1+2r_2+r_3+r_4) = 1. \\
S_2(r_1+2r_2+r_3+r_4) &= r_1 + r_2 + r_3 + r_4 = S_2S_2S_4S_3S_2r_1 = S_4S_3S_2r_1, \\
S_3(r_1+2r_2+r_3+r_4) &= r_1 + 2r_2 + 2r_3 + r_4 = S_3S_2S_4S_3S_2r_1 \\
&= 1/2(-1, 1, 1, -1), \\
S_4(r_1+2r_2+r_3+r_4) &= r_1 + 2r_2 + r_3 = S_4S_2S_4S_3S_2r_1, \\
(r_1, -r_1-r_2-r_3) &= -1/2, \quad (r_2, -r_1-r_2-r_3) = 1/2, \\
(r_3, -r_1-r_2-r_3) &= -1, \quad (r_4, -r_1-r_2-r_3) = 1. \\
S_1(-r_1-r_2-r_3) &= -r_2 - r_3 = S_1S_3S_1S_2S_1S_2r_1, \\
S_2(-r_1-r_2-r_3) &= -r_1 - 2r_2 - r_3 = S_2S_3S_1S_2S_1S_2r_1 \\
&= 1/2(-1, -1, 1, 1), \\
S_3(-r_1-r_2-r_3) &= -r_1 - r_2 = S_1S_2S_1S_2r_1, \\
S_4(-r_1-r_2-r_3) &= -r_1 - r_2 - r_3 - r_4 = S_4S_3S_1S_2S_1S_2r_1 \\
&= 1/2(1, 1, -1, 1).
\end{aligned}$$

$$(r_1, -r_2 - r_3 - r_4) = 1/2, \quad (r_2, -r_2 - r_3 - r_4) = 0,$$

$$(r_3, -r_2 - r_3 - r_4) = 0, \quad (r_4, -r_2 - r_3 - r_4) = -1.$$

$$\begin{aligned} S_1(-r_2 - r_3 - r_4) &= -r_1 - r_2 - r_3 - r_4 = S_1 S_4 S_3 S_2 S_1 S_2 r_1 \\ &= 1/2(1, 1, -1, 1), \end{aligned}$$

$$S_4(-r_2 - r_3 - r_4) = -r_2 - r_3 = S_3 S_2 S_1 S_2 r_1.$$

$$(r_1, r_1 + 2r_2 + 2r_3 + r_4) = 0, \quad (r_2, r_1 + 2r_2 + 2r_3 + r_4) = -1/2,$$

$$(r_3, r_1 + 2r_2 + 2r_3 + r_4) = 1, \quad (r_4, r_1 + 2r_2 + 2r_3 + r_4) = 0.$$

$$\begin{aligned} S_2(r_1 + 2r_2 + 2r_3 + r_4) &= r_1 + 3r_2 + 2r_3 + r_4 = S_2 S_3 S_2 S_4 S_3 S_2 r_1 \\ &= 1/2(1, 1, 1, -1), \end{aligned}$$

$$\begin{aligned} S_3(r_1 + 2r_2 + 2r_3 + r_4) &= r_1 + 2r_2 + r_3 + r_4 = S_3 S_3 S_2 S_4 S_3 S_2 r_1 \\ &= S_2 S_4 S_3 S_2 r_1. \end{aligned}$$

6)

$$(r_1, -r_1 - 2r_2 - r_3) = 0, \quad (r_2, -r_1 - 2r_2 - r_3) = -1/2,$$

$$(r_3, -r_1 - 2r_2 - r_3) = 0, \quad (r_4, -r_1 - 2r_2 - r_3) = 1.$$

$$S_2(-r_1 - 2r_2 - r_3) = -r_1 - r_2 - r_3 = S_2 S_2 S_3 S_1 S_2 r_2 = S_3 S_1 S_2 r_1,$$

$$S_4(-r_1 - 2r_2 - r_3) = -r_1 - r_2 - r_3 - r_4 = S_4 S_2 S_3 S_1 S_2 S_1 S_2 r_1.$$

$$(r_1, -r_1 - r_2 - r_3 - r_4) = -1/2, \quad (r_2, -r_1 - r_2 - r_3 - r_4) = 1/2,$$

$$(r_3, -r_1 - r_2 - r_3 - r_4) = 0, \quad (r_4, -r_1 - r_2 - r_3 - r_4) = -1.$$

$$S_1(-r_1 - r_2 - r_3 - r_4) = -r_2 - r_3 - r_4 = S_1 S_4 S_3 S_1 S_2 S_1 S_2 r_1,$$

$$\begin{aligned} S_2(-r_1 - r_2 - r_3 - r_4) &= -r_1 - 2r_2 - r_3 - r_4 = S_2 S_4 S_3 S_1 S_2 S_1 S_2 r_1 \\ &= 1/2(-1, 1, -1, 1), \end{aligned}$$

$$\begin{aligned} S_4(-r_1 - r_2 - r_3 - r_4) &= -r_1 - r_2 - r_3 = S_4 S_4 S_3 S_1 S_2 S_1 S_2 r_1 \\ &= S_3 S_1 S_2 S_1 S_2 r_1. \end{aligned}$$

$$(r_1, r_1 + 3r_2 + 2r_3 + r_4) = -1/2, \quad (r_2, r_1 + 3r_2 + 2r_3 + r_4) = 1/2,$$

$$(r_3, r_1 + 3r_2 + 2r_3 + r_4) = 0, \quad (r_4, r_1 + 3r_2 + 2r_3 + r_4) = 0.$$

$$S_1(r_1+3r_2+2r_3+r_4) = 2r_1 + 3r_2 + 2r_3 + r_4 = S_1S_2S_3S_2S_4S_3S_2r_1 \\ = -e_4,$$

$$S_2(r_1+3r_2+2r_3+r_4) = r_1 + 2r_2 + 2r_3 + r_4 = S_2S_2S_3S_2S_4S_3S_2r_1 \\ = S_3S_2S_4S_3S_2r_1.$$

7)

$$(r_1, -r_1-2r_2-r_3-r_4) = 0, (r_2, -r_1-2r_2-r_3-r_4) = 1/2,$$

$$(r_3, -r_1-2r_2-r_3-r_4) = 1, (r_4, -r_1-2r_2-r_3-r_4) = -1.$$

$$S_2(-r_1-2r_2-r_3-r_4) = -r_1 - r_2 - r_3 - r_4 = S_4S_3S_1S_2S_1S_2r_1,$$

$$S_3(-r_1-2r_2-r_3-r_4) = -r_1 - 2r_2 - 2r_3 - r_4 = S_3S_2S_4S_3S_1S_2S_1S_2r_1 \\ = 1/2(1, -1, -1, 1),$$

$$S_4(-r_1-2r_2-r_3-r_4) = -r_1 - 2r_2 - r_3 = S_4S_2S_4S_3S_1S_2S_1S_2r_1.$$

$$(r_1, 2r_1+3r_2+2r_3+r_4) = 1/2, (r_2, 2r_1+3r_2+2r_3+r_4) = 0,$$

$$(r_3, 2r_1+3r_2+2r_3+r_4) = 0, (r_4, 2r_1+3r_2+2r_3+r_4) = 0.$$

$$S_1(2r_1+3r_2+2r_3+r_4) = r_1 + 3r_2 + 2r_3 + r_4 = S_1S_3S_2S_4S_3S_2r_1.$$

$$(r_1, -r_1-2r_2-2r_3-r_4) = 0, (r_2, -r_1-2r_2-2r_3-r_4) = 1/2,$$

$$(r_3, -r_1-2r_2-2r_3-r_4) = -1, (r_4, -r_1-2r_2-2r_3-r_4) = 0.$$

$$S_2(-r_1-2r_2-2r_3-r_4) = -r_1 - 3r_2 - 2r_3 - r_4 = S_2S_3S_2S_4S_3S_1S_2S_1S_2r_1 \\ = 1/2(-1, -1, -1, 1),$$

$$S_3(-r_1-2r_2-2r_3-r_4) = -r_1 - 2r_2 - r_3 - r_4 = S_2S_4S_3S_1S_2S_1S_2r_1.$$

8)

$$(r_1, -r_1-3r_2-2r_3-r_4) = 1/2, (r_2, -r_1-3r_2-2r_3-r_4) = -1/2,$$

$$(r_3, -r_1-3r_2-2r_3-r_4) = 0, (r_4, -r_1-3r_2-2r_3-r_4) = 0.$$

$$S_1(-r_1-3r_2-2r_3-r_4) = -2r_1 - 3r_2 - 2r_3 - r_4 = S_2S_3S_2S_4S_3S_2r_1 \\ = e_4,$$

$$S_2(-r_1-3r_2-2r_3-r_4) = -r_1 - 2r_2 - 2r_3 - r_4 = S_2S_3S_2S_4S_3S_2r_1.$$

$$\begin{aligned}
9) \quad & (r_1, -2r_1 - 3r_2 - 2r_3 - r_4) = -1, \quad (r_2, -2r_1 - 3r_2 - 2r_3 - r_4) = 0, \\
& (r_3, -2r_1 - 3r_2 - 2r_3 - r_4) = 0, \quad (r_4, -2r_1 - 3r_2 - 2r_3 - r_4) = 0. \\
& S_1(-2r_1 - 3r_2 - 2r_3 - r_4) = -r_1 - 3r_2 - 2r_3 - r_4.
\end{aligned}$$

So $\text{Orb}(r_1)$ is the set of 24 roots of the forms $\pm e_i$, $1/2\Sigma \pm e_i$, and $\text{Orb}(r_4)$ is the set of 24 roots of the form $\pm e_i \pm e_j$. Hence $\hat{\Delta}$ has two orbits. Finally we can compute the orders of the remaining groups. In each case we will find a root $r \in \hat{\Delta}$ that is orthogonal to all but one of the roots in the base π . If $r_i \in \pi$ is the fundamental root not orthogonal to r , then r is a scalar multiple of the dual basis vector s_i ; since $(r_i, s_i) = 1$. Thus the stabilizer $\text{Stab}(r)$ is the subgroup of W generated by all fundamental reflections except S_i , by Witt's theorem. In each case $|\text{Stab}(r)|$ and $|\text{Orb}(r)|$ will be known, so Proposition 4.4.1 allows us to compute $|W|$.

For $\underline{E}_4$, the root $r = e_4 - e_3$ is orthogonal to r_1, r_2 , and r_3 , so $r = \mu s_4$ for some scalar μ . And by Witt's theorem, $\text{Stab}(r) = \text{Stab}(s_4) = \langle S_1, S_2, S_3 \rangle$. Since the graph of $\{r_1, r_2, r_3\}$ is just B_3 , we can see that $\text{Stab}(r) = \underline{B}_3$, and so $|\text{Stab}(r)| = 2^3 \cdot 3! = 48$. As we see in the computation above, each orbit of $\hat{\Delta}$ has 24 roots, so by Proposition 4.4.1,

$$|\underline{E}_4| = [\underline{E}_4 : \text{Stab}(r)] \cdot |\text{Stab}(r)| = |\text{Orb}(r)| \cdot |\underline{B}_3| = 24 \cdot 48 = 152.$$

For $\underline{I}_3$, the root $r = \beta(1, 2a, 2a+1)$ is orthogonal to r_1

and r_3 , so $r = \mu s_2$ for some scalar μ . But the graph of $\{r_1, r_3\}$ is simply $A_1 \times A_1$, so $|\text{Stab}(r)| = 2 \cdot 2 = 4$. Since we know that $\underline{I}_3$ is transitive, $|\text{Orb}(r)| = |\hat{\Delta}| = 30$. Hence $|\underline{I}_3| = 4 \cdot 30 = 120$.

Similarly for $\underline{I}_4$, take $r = e_4$. Then the graph of $\{r_1, r_2, r_3\}$ is the same as $\underline{I}_3$, so $|\text{Stab}(r)| = |\underline{I}_3| = 120$. Since $|\text{Orb}(r)| = |\hat{\Delta}| = 120$, $|\underline{I}_4| = 120 \cdot 120 = 14,400$.

In the case of $\underline{E}_6$, take $r = e_7 + e_8$. Then the graph of $\{r_2, \dots, r_6\}$ is the same as A_5 , so $|\text{Stab}(r)| = |\underline{A}_5| = 6!$. Since $\underline{E}_6$ is transitive, $|\text{Orb}(r)| = |\hat{\Delta}| = 72$ and $|\underline{E}_6| = 72 \cdot 6! = 2^7 \cdot 3^4 \cdot 5 = 51,840$.

Similarly for $\underline{E}_7$, take $r = e_1 + e_8$. Then the graph of $\{r_1, r_3, r_4, r_5, r_6, r_7\} = D_6$, so $|\text{Stab}(r)| = |\underline{D}_6| = 2^5 \cdot 6!$. Since $|\text{Orb}(r)| = |\hat{\Delta}| = 126$, $|\underline{E}_7| = 126 \cdot (2^5 \cdot 6!) = 2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2,903,040$.

Lastly for $\underline{E}_8$, take $r = (1/2)\sum_1^7(e_i - e_8)$. Then the graph of $\{r_1, \dots, r_7\}$ is $\underline{E}_7$, so $|\text{Stab}(r)| = |\underline{E}_7|$. Since $|\text{Orb}(r)| = |\hat{\Delta}| = 240$, $|\underline{E}_8| = 240 \cdot (2^{10} \cdot 3^4 \cdot 5 \cdot 7) = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696,729,600$.

We have shown that irreducible Coxeter groups whose associated bilinear forms are positive definite are indeed finite. Table 4.2 will summarize the orders of positive definite irreducible Coxeter groups.

Group	$ W $
A_n	$(n+1) !$
B_n	$2^n \cdot n !$
D_n	$2^{n-1} \cdot n !$
H_2^n	$2n$
F_4	$2^7 \cdot 3^2$
G_2	12
I_3	$2^3 \cdot 3 \cdot 5$
I_4	$2^6 \cdot 3^2 \cdot 5^2$
E_6	$2^7 \cdot 3^4 \cdot 5$
E_7	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

Table 4.2

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