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The Dold-Lefschetz fixed point theorem / Gary M. Wilson

Gary M. Wilson
San Jose State University

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Wilson, Gary Martin, M.S.

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300 N. Zeeb Rd.
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THE DOLD-LEFSCHETZ FIXED POINT THEOREM

A Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

Gary M. Wilson

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APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Dr. Richard Kubelka

Dr. Max Agoston

Dr. Marilyn Blockus

APPROVED FOR THE UNIVERSITY

ABSTRACT

THE DOLD LEFSCHETZ FIXED POINT THEOREM

Gary M. Wilson

Let Y be a Euclidean neighborhood retract and $f: Y \rightarrow Y$ a continuous function. We are interested in the existence of any points that remain fixed by f . By definition there exists a Euclidean neighborhood $U \subset \mathbf{R}^n$ and maps $r: U \rightarrow Y$ and $i: Y \rightarrow U$ such that $ri = 1_Y$. We will define what will be called the *fixed point index* of f , which is related to the homology group $H_n(U, U - F_g)$ where F_g is the fixed point set of the composition $g = ifr$. We will also give an alternate definition of the Lefschetz number of a map and show the equivalence of this definition with the traditional definition. Finally, we will show that for any map from a Euclidean neighborhood retract to itself, the fixed point index and the Lefschetz number agree.

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TABLE OF CONTENTS

<u>Section</u>	<u>Page</u>
Introduction	1
Chapter 1: Introduction to Simplicial Homology	
1.1 Introduction	4
1.2 Simplicial Complexes	7
1.3 Chain Complexes	13
1.4 Orientation of a Simplicial Complex	15
1.5 The Homology Functor	17
1.6 Singular Homology	21
Chapter 2: Sequences in Homology	
2.1 Relative Homology	24
2.2 Excision	28
2.3 The Mayer–Vietoris Sequence	30
2.4 The Universal Coefficient Theorem	32
2.5 Homology and Product Spaces	36
Chapter 3: The Fixed Point Index	
3.1 The Fundamental Class	41
3.2 The Fixed Point Index	44
3.3 Properties of the Fixed Point Index	45
3.4 The Lefschetz Number	57
3.5 Euclidean Neighborhood Retracts	64
3.6 The Dold–Lefschetz Fixed Point Theorem	66
Bibliography	73

Introduction

An interesting question involving continuous maps from a topological space to itself is whether there exist points which remain fixed. Much work has been done in this area and some fascinating results have been discovered concerning spaces which have the fixed point property, that is, the property that any continuous function from the space to itself will have a fixed point. This thesis is concerned with establishing still more tools for investigating this type of phenomenon. We will define the *fixed point index* of a map and show that for certain spaces, namely, Euclidean neighborhood retracts, the fixed point index is equivalent to the Lefschetz number of that map. This work is due to Dold and no claim is made that this thesis presents original discoveries.

Since the fixed point index is based on homology, a background in homology is necessary before an understanding of this new concept can be accomplished. In Chapter 1 we briefly introduce the basics of homology theory. Many definitions are presented and some theorems are stated. This chapter also contains some examples which serve to elaborate upon and demonstrate the concepts discussed. In most cases the examples are chosen so that the results will be useful in later sections.

Chapter 1 introduces homology by describing a functor from the category of

simplicial complexes to the category of graded abelian groups. This functorial description allows for the application of results from category theory to homology in many useful instances. Sections 1.2 through 1.4 are concerned with the concept of the simplicial complex, while section 1.5 introduces the homology functor itself. In section 1.6 we describe a different homology functor called singular homology. We will use singular homology throughout Chapters 2 and 3.

Having introduced simplicial homology theory in Chapter 1, we continue to elaborate upon the concepts of homology theory in Chapter 2. Although the theorems presented are not proven, many examples are given to help demonstrate the results of the theorems. These examples will also be useful in the sections that follow. Sections 2.1 and 2.2 are concerned with the important concepts of relative homology and excision, respectively. The results of these sections will become increasingly important in Chapter 3. In section 2.3 we introduce an exact sequence of homology groups known as the Mayer-Vietoris sequence. Although this sequence will not be used extensively, its inclusion is justified by its importance in homology theory. Section 2.4 defines the homology group with coefficients in an arbitrary abelian group. Some necessary principles of group theory are discussed. In section 2.5 we define the homology group of a product space.

Chapter 3 develops the concept of the fixed point index and introduces the Dold-Lefschetz fixed point theorem. In this chapter more effort is made to prove the claims that are made. However, scope limitations warrant the exclusion of some

proofs. Sections 3.1 and 3.2 introduce the fixed point index and section 3.3 develops some of its properties. In section 3.4 we give an alternate definition of the Lefschetz number and show its equivalence with the traditional definition. Section 3.5 defines the notion of a Euclidean neighborhood retract and finally in section 3.6 we state and prove the Dold–Lefschetz fixed point theorem.

Chapter 1: Introduction To Simplicial Homology

§1.1 *Introduction*

Like point-set topology, algebraic topology is concerned with geometric objects. Of particular interest are the properties of geometric objects that remain unchanged under continuous deformations, as well as properties of continuous functions between such objects. Unlike point-set topology, algebraic topology attempts to study such properties by associating an algebraic structure, for instance a graded abelian group, to each topological space, and by using group theoretic methods to gain information about the abelian group which can then be used to infer information about the underlying topological space. Unfortunately, in the transformation from topological structure to algebraic structure, not all the information about the topology is preserved. However, as we shall see, we can often gain valuable information from the group structure which will allow us to make some statement about the topology of the space.

If we would like to associate an abelian group with a topological space so that the algebraic structure somehow describes the topology of the space, what information would we like to preserve? Certainly, if two spaces are equivalent

(homeomorphic), then we would want their associated groups to be equivalent (isomorphic). Similarly, if there exists a continuous function from one space to another, we would want a homomorphism between the two groups. What we have briefly described is a *functor* from the category of topological spaces to the category of graded abelian groups. In the rest of this chapter, we will develop a functor of this sort, called *homology*, by introducing the categories of simplicial complexes and chain complexes, associating a chain complex to a topological space, and then constructing a functor from the category of chain complexes to the category of graded abelian groups. Before we proceed, we must understand what is meant by a functor.

Definition: 1.1.1 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* T from $\mathcal{C}$ to $\mathcal{D}$ (denoted $T: \mathcal{C} \rightarrow \mathcal{D}$) is a pair of functions: an *object function* that assigns to each object C of $\mathcal{C}$ an object $T(C)$ of $\mathcal{D}$, and a *morphism function* which assigns to each morphism $f: C \rightarrow C'$ of $\mathcal{C}$ a morphism

$$T(f): T(C) \rightarrow T(C')$$

of $\mathcal{D}$ such that

- (i) $T(1_C) = 1_{T(C)}$ for every identity morphism 1_C of $\mathcal{C}$.
- (ii) $T(g \circ f) = T(g) \circ T(f)$ for any two morphisms f, g of $\mathcal{C}$ whose composite $g \circ f$ is defined [Hungerford, 465].

We are interested in the category of topological spaces and continuous functions. In this category, two objects X and Y are equivalent when there exists a

morphism from X to Y whose inverse is also a morphism in the category. That is, the two spaces are *homeomorphic* if there exists a one-to-one, onto, continuous function $f: X \rightarrow Y$ whose inverse $g: Y \rightarrow X$ is also continuous. By i) and ii), for any covariant functor T from the category $\mathcal{C}$ of topological spaces to the category $\mathcal{D}$ of graded abelian groups, we get $1_{\mathcal{D}} = T(1_{\mathcal{C}}) = T(f \circ g) = T(f) \circ T(g)$. Similarly, $1_{\mathcal{D}} = T(1_{\mathcal{C}}) = T(g \circ f) = T(g) \circ T(f)$. This guarantees that the image of a homeomorphism under the homology functor will be an isomorphism of graded abelian groups.

To gain some motivation for the development of the homology functor, let us consider two fundamental examples: the 2-sphere and the torus. Since these two spaces are not homeomorphic, we would hope that homology would be able to distinguish between them. The most fundamental difference between the 2-sphere and the torus is that the torus has a two 1-dimensional “holes” in it.

If homology can somehow count the holes in each space then it will be able to detect this fundamental difference. Several methods of doing this have been developed which give rise to different homology functors; however, the axioms of

homology guarantee that two different homology theories will yield the same results on any space on which they both operate [Spanier, 201]. Since we will not be able to discuss all the theories, we will concentrate on what is considered the most geometric homology theory, *simplicial homology*.

§1.2 *Simplicial Complexes*

In order to develop simplicial homology, we must first look at its basic “building block,” the *geometric simplex*. Recall that points $a_0, \dots, a_p$ in $\mathbf{R}^n$ are *independent* if the set of vectors $\{a_1 - a_0, a_2 - a_0, \dots, a_p - a_0\}$ is a linearly independent set in $\mathbf{R}^n$.

Definition : 1.2.1 Let $\{a_0, \dots, a_p\}$ be a set of independent points in $\mathbf{R}^n$. The *p-dimensional geometric simplex* or *p-simplex*, σ^p , spanned by $\{a_0, \dots, a_p\}$ is the set of all points x in $\mathbf{R}^n$ for which there exist nonnegative real numbers $\lambda_0, \dots, \lambda_p$ such that

$$x = \sum_{i=0}^p \lambda_i a_i \quad \text{and} \quad \sum_{i=0}^p \lambda_i = 1.$$

The numbers $\lambda_0, \dots, \lambda_p$ are called the *barycentric coordinates* of the point x with respect to λ . The points $a_0, \dots, a_p$ are the *vertices* of σ^p [Croom, 8].

Equivalently, we could define the *p-dimensional simplex* spanned by the independent points $a_0, \dots, a_p$ as the “smallest” convex set containing these points. That is, if any convex set U contains all of the points $a_0, \dots, a_p$, then $\sigma^p \subset U$. It can be shown that any simplex in $\mathbf{R}^n$ is a compact subspace of $\mathbf{R}^n$.

Example : 1.2.1 It is easy to see that a 0-dimensional simplex consists of a single point, a 1-dimensional simplex is the line segment joining the vertices, and a 2-dimensional simplex is the triangle (including interior) formed by the three vertices. A 3-simplex is the tetrahedron formed by joining all the vertices by line segments, and the k -simplex is the analogue of a tetrahedron for $k > 3$.

By restricting ourselves to geometric simplexes we would be excluding many complicated and interesting spaces. So we extend the concept of the simplex as a tool for describing topological spaces by piecing them together to build more complicated spaces.

Definition : 1.2.2 A simplex σ^k is a *face* of a simplex σ^p , $k \leq p$, if every vertex of σ^k is a vertex of σ^p [Croom, 9].

The proper faces of a simplex are simplexes of lower dimension. So the proper faces of a 2-simplex (triangle) consist of the 1-simplexes which make up the sides of the triangle, as well as the 0-simplexes that make up the vertices.

Definition : 1.2.3 A *finite simplicial complex* or *geometric complex* K in $\mathbf{R}^m$ is a finite collection of simplexes of $\mathbf{R}^m$ satisfying:

- (i) if σ^n is in K and σ^p is a face of σ^n , then σ^p is in K
- (ii) if σ_1 is in K and σ_2 is in K , then either $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 .

The *geometric carrier* of K is the subspace, $|K|$ of $\mathbf{R}^n$, formed by taking the union of all the simplexes in K . The *dimension* of the complex is the highest of

the dimensions of its simplexes [Hilton, 87].

Now we can examine spaces which are homeomorphic to spaces that can be built by piecing together geometric simplexes. Such spaces are called *triangulable spaces* or *polyhedra* and form a category with continuous functions. A *triangulation* of a space X is a pair (K, f) , where K is a simplicial complex and $f: |K| \rightarrow X$ is a homeomorphism.

Example : 1.2.2 Consider the unit circle S^1 . S^1 is homeomorphic to a triangle without the interior, which can be built by joining three 1-simplexes. Thus we may use the geometric carrier of the complex on the right to discover properties of S^1 .

Example : 1.2.3 Now let's look at the "barbell" formed by joining two unit disks by a line segment (see the following page). It is homeomorphic to the complex on the right. Notice that, in this example, the dimension of the complex is 2 since the interiors of the triangles are included.

Now that we have described the objects in the category of simplicial complexes, we continue by describing the morphisms in this category.

Definition : 1.2.4 A *vertex transformation* $\alpha: K_1 \rightarrow K_2$ is a function α from the vertices of K_1 to the vertices of K_2 . We say that a vertex transformation is *admissible* if whenever $a_0, \dots, a_p$ span a simplex of K_1 , then $\alpha(a_0), \dots, \alpha(a_p)$ span a simplex of K_2 . [Hilton and Wylie, 35].

Notice that although a vertex transformation will map simplexes to simplexes, there is no guarantee that the image of a simplex σ is of the same dimension as σ ; the dimension may indeed be lower. Thus, a map which sends a 1-simplex to a 0-simplex is a vertex transformation.

Before we go on, let us extend the notion of vertex transformation to the class of polyhedra. This is done in a natural way by extending a simplicial map to a map between the geometric carriers of the complexes.

Definition : 1.2.5 Let K and L be simplicial complexes and let $\phi: K \rightarrow L$ be a vertex transformation map. Then ϕ is extended to a function $\phi': |K| \rightarrow |L|$ as

follows:

If $x \in |K|$, then there is a simplex σ^p with vertices $a_0, \dots, a_p$ in K such that $x \in \sigma^p$. It follows that

$$x = \sum_{i=0}^p \lambda_i a_i$$

where λ_i are the barycentric coordinates of x .

Define $\phi'(x) = \sum_{i=0}^p \lambda_i \phi(a_i)$.

This extended function is called a *simplicial map* from $|K|$ to $|L|$ [Croom, 42].

One can show that if x is contained in an n -simplex, σ^n , and also in an m -simplex, σ^m , then ϕ' will give the same value for x regardless of the choice of barycentric representation. Thus ϕ' is a well defined function. Therefore, it is not difficult to show that, with simplicial maps as morphisms, the class of simplicial complexes forms a category.

At first, it may seem that the simplicial map is a very special kind of map and that we would be restricting our scope considerably by working only with functions which can be represented by simplicial maps. However, as we will see shortly, the simplicial map is a much broader descriptive tool than it appears to be.

Since we are interested in describing continuous functions between polyhedra, which behave “nicely” in some neighborhood of any given point in the domain, we would like to know that if we chop up the polyhedron fine enough we can describe or approximate any continuous function by a simplicial map. Indeed, this is the case. This “chopping up” process, called *barycentric subdivision*, consists of adding

vertices to the complex and constructing additional simplexes. The result is a simplicial complex which yields a polyhedron homeomorphic to the original, but formed by joining a larger number of smaller simplexes.

Although every simplicial map is a continuous function, not every continuous function between polyhedra is a simplicial map. However, the following theorem, called the *simplicial approximation theorem*, guarantees that any continuous function between polyhedra is “close” to a simplicial map. First, recall that two continuous functions $f, g: X \rightarrow Y$ are *homotopic* if there exists a continuous map $F: X \times I \rightarrow Y$ such that for $x \in X$

$$F(x, 0) = f(x)$$

$$F(x, 1) = g(x).$$

Two spaces X and Y are called *homotopy equivalent* if there exist continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identities on X and Y , respectively.

Theorem : 1.2.1 Let X and Y be polyhedra with triangulations (K, k) and (L, l) , respectively, and let $f: |K| \rightarrow |L|$ be a continuous function. Then there exists a barycentric subdivision K' of K and a continuous function $g: |K| \rightarrow |L|$ such that $g: |K'| \rightarrow |L|$ is a simplicial map and g is homotopic to f [Croom, 49].

The simplicial approximation theorem guarantees that given any continuous function between polyhedra, we need only subdivide the triangulation of the domain fine enough and we will be able to find a simplicial map that is homotopic to the

original map. Therefore, if we are only interested in homotopy invariants, it will suffice to examine simplicial complexes and simplicial maps.

§1.3 Chain Complexes

Now we turn our attention away from topology and concentrate on the algebraic aspects of simplicial homology theory. Here, we introduce the category of chain complexes and define the homology functor from this category to the category of graded abelian groups. We then extend this functor to a functor from the category of polyhedra by constructing a chain complex for each polyhedron.

Definition : 1.3.1 A *chain complex* $C = \{C_p, \partial_p\}$ is a collection of abelian groups C_p , one for each integer p , and of homomorphisms $\partial_p: C_p \rightarrow C_{p-1}$ such that $\partial_p \partial_{p+1} = 0$ for each p [Hilton and Wylie, 95].

The homomorphism ∂_p is called the p^{th} *boundary map* or *boundary operator* of the complex. It follows from the definition that given any three consecutive groups C_{p+1}, C_p, C_{p-1} in the sequence

$$C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1},$$

we have $Im \partial_{p+1} \subset Ker \partial_p$. The image of ∂_{p+1} , written $B_p(C)$, is called the p^{th} *boundary group*, and the kernel of ∂_p is written as $Z_p(C)$ and referred to as the p^{th} *cycle group*.

Since the definition of ∂_p guarantees that the p^{th} boundary group is a subgroup of the p^{th} cycle group (and all groups are abelian, hence $B_p(C)$ is normal

in $Z_p(C)$), we can form the quotient group $Z_p(C)/B_p(C)$. This abelian group, written $H_p(C)$, is called the p^{th} *homology group* and the graded abelian group $H_*(C) = \{H_p(C)\}_{p \in \mathbb{Z}}$ is called the *homology of the complex*. Recall that a graded abelian group G is a set of subgroups $G_0, G_1, G_2, \dots$ such that $G_0 \oplus G_1 \oplus G_2 \oplus \dots = G$.

To complete the development of the homology functor, we define the morphisms in the category of chain complexes, and determine the image of the morphism function of the homology functor.

Definition : 1.3.2 A *chain map* ϕ from a chain complex $C = \{C_p, \partial_p\}$ to a chain complex $D = \{D_p, \delta_p\}$ is a set of homomorphisms $\phi_p: C_p \rightarrow D_p$ such that $\phi_{p-1}\partial_p = \delta_p\phi_p$ [Hilton and Wylie, 96].

In other words, a chain map between complexes is a set of homomorphisms, one at each level, which commute with the boundary operators. It is not difficult to show that a chain map between two complexes will induce homomorphisms between the respective homology groups.

Theorem : 1.3.1 A chain map $\phi: C \rightarrow D$ induces a graded homomorphism

$$\phi_*: H_*(C) \rightarrow H_*(D)$$

[Hilton and Wylie, 98].

The construction of the homomorphism ϕ_* is quite natural. Let $x \in H_p(C)$. Then $x = [z] = z + B_p$ and we define $\phi_*(x) = \phi_*([z]) = [\phi(z)]$. Since the commutativity of the boundary maps ensures that the image of a cycle group in C

is contained in a cycle group in D , and the image of a boundary group in C is contained in a boundary group in D , ϕ_* is a well-defined function.

The object function of the homology functor from the category of chain complexes to the category of graded abelian groups takes a chain complex C to its homology $H_*(C)$. The morphism function sends a chain map $\phi: C \rightarrow D$ to the homomorphism $\phi_*: H_*(C) \rightarrow H_*(D)$. One can show that the object function and morphism function defined above give a functor from the category of chain complexes to the category of graded abelian groups.

Now let us turn our attention back to geometry to see how to construct a chain complex from a simplicial complex.

§1.4 *Orientation of a Simplicial Complex*

In order to construct a chain complex from a given polyhedron X , we must construct a sequence of abelian groups and define boundary operators between these groups. First, we must give an orientation to each simplex in a complex. We do this by looking at the permutations of the vertices of a simplex.

Definition: 1.4.1 An *orientation* of a p -simplex is an equivalence class of the orderings of the vertices of the simplex. An *oriented geometric simplex* is a simplex together with an orientation. Let $\langle a_0, \dots, a_p \rangle$ denote the oriented simplex with vertexes $\{a_0, \dots, a_p\}$ and orientation given by the ordering. Also let $-\langle a_0, \dots, a_p \rangle$ denote the simplex with opposite orientation.

By assigning each simplex σ^p an orientation in this way, we can get a group

structure associated with the complex. That is, given any abelian group G (or ring or field) we can form a group which is the direct sum of copies of G , one copy for each oriented p -simplex with that orientation. We think of the positively oriented simplexes as generators of this free abelian group. Thus, a chain can be formed by taking a finite linear combination of simplexes in K with coefficients in G . The group we have just described, written $C_p(K; G)$, is called the *p -dimensional chain group*, and the elements of G are called the *coefficients* of the complex. In most cases, we will use the integers as the coefficient group, although we may find it convenient to use the elements of another ring or a field as coefficients. For ease in notation we write $C_p(K)$ for the chain group $C_p(K; \mathbb{Z})$.

The groups in the chain complex of a simplicial complex K will be the p -dimensional chain groups $C_p(K)$. However, to follow the definition of a chain complex precisely, we must define a chain group C_i for every $i \in \mathbb{Z}$. For a simplicial complex of dimension r , define the chain group C_p by

$$C_p = \begin{cases} 0 & \text{if } p < 0 \\ C_p(K) & \text{if } 0 \leq p \leq r \\ 0 & \text{if } p > r. \end{cases}$$

Now we have a sequence of abelian groups, and need only define the boundary operators to have a chain complex. Let σ be a p -simplex and let the ordering $\langle a_0, \dots, a_p \rangle$ determine the positively oriented simplex. Also assume that an ordering has been chosen for each face of σ . If we exclude one vertex from the set $\{a_0, \dots, a_p\}$, we get a set of vertices which generate a face of σ^p . Therefore, the oriented simplex $\sigma_i^{p-1} = \langle a_0, \dots, \hat{a}_i, \dots, a_p \rangle$, formed by excluding the i^{th} vertex of

σ^p , is an element of $C_{p-1}(K)$. Define the map $\partial_p: C_p \rightarrow C_{p-1}$ by

$$\partial_p(\sigma_p) = \sum_{i=0}^p (-1)^i \langle a_0, \dots, \hat{a}_i, \dots, a_p \rangle$$

and extend this map to a homomorphism by linearity. Thus, the image of a p -chain $\sum g_i \sigma_i^p$ is the $(p-1)$ -chain $\sum g_i \partial_p(\sigma_i^p)$.

One can show that for any p , $\partial_p \partial_p = 0$. Therefore, given any polyhedron X , we can get a chain complex

$$\dots \xrightarrow{\partial_{p+2}} C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} \dots$$

where (K, f) is a triangulation of X .

§1.5 *The Homology Functor*

The kernel of ∂_p , called the p^{th} cycle group $Z_p(K)$, contains all p -chains which make one complete circuit or cycle. That is, the simplexes in the chain are joined in such a way that the result of “adding” up the boundaries of all the simplexes is zero. The image of ∂_{p+1} , called the p^{th} boundary group $B_p(K)$, contains all p -chains which are boundaries of $(p+1)$ -chains. Thus, the homology group $H_p(K) = Z_p(K) / B_p(K)$ can be thought of as all the p -cycles which are not the boundary of any $(p+1)$ -chain; that is, as all the boundaryless p -chains which are not boundaries. If a cycle is not a boundary, then it must enclose a “hole” in the complex. Therefore, the homology group, $H_p(K)$, counts the p -dimensional holes in K .

Now we are ready to define the homology functor from the category of simplicial complexes to the category of graded abelian groups. Let K be a simplicial complex. Then we can construct a chain complex $C = \{C_p(K), \partial_p\}$, and form the homology group $H_*(C)$. This gives the object function in the functor. Since the morphisms in the category of simplicial complexes are simplicial maps we need to define the image of a simplicial map under the homology functor. It can be shown that a simplicial map between simplicial complexes induces a chain map between their respective chain complexes. This chain map then induces a homomorphism between the homology groups.

Since a polyhedron X can have many triangulations, simplicial homology does not give a functor from the category of polyhedra to the category of graded abelian groups. However, although it is difficult, one can show that two triangulations of a space X will give isomorphic homology groups. This seems intuitively plausible when we consider that the relationship between cycles and boundaries should reflect an intrinsic geometry of the space, which is independent of a triangulation. In addition, it can be shown that two maps that are homotopic to one another will give equal homomorphisms between homology groups [Hilton and Wylie, 118]. Therefore, any simplicial map that is in the same homotopy class as a given continuous function between polyhedra will be sufficient to define the induced homomorphism between homology groups. The fact that the induced homomorphism in homology depends only on the homotopy class of the continuous function implies, among

other things, that homology will be unable to distinguish between spaces of the same homotopy type. For example, since a single point is homotopy equivalent to a closed ball in $\mathbf{R}^n$, their homologies are the same.

Example : 1.5.1 We have said that the p^{th} homology group counts the number of p -dimensional holes in the space. Let us see what this means at dimension zero. First of all, since $C_{-1} = 0$, ∂_0 is the zero map, and so $Z_0 = C_0$. Therefore, $H_0 = C_0 / B_0$. We claim that this group is a direct sum of copies of the coefficient group, one copy for each path component of the space. Notice that two 0-chains are in the same equivalence class if their difference is the boundary of some 1-chain. Thus, a 0-chain is equivalent to zero if it is the boundary of a 1-chain. Since a 1-chain either has no boundary or has a boundary consisting of two vertices, a single vertex, a_0 , cannot be the boundary of any 1-chain. Therefore, there is at least one coset, $a_0 + B_0$, which is different from $0 + B_0$. Thus a single vertex gives a homology group isomorphic to one copy of the coefficient group and generated by the coset $a_0 + B_0$.

Let x_0 and x_1 be two vertices in the same path component. Then there exists some path in the space which joins the two vertices. Subdivide the complex so that this path is approximated by a chain of 1-simplexes (recall that homology is invariant of triangulation). Then this chain of 1-simplexes will be a chain whose boundary is the difference between the endpoints x_0 and x_1 . Thus x_0 and x_1 are in the same equivalence class. Alternately, if two points are in different path

components, then their difference could not possibly be the boundary of a 1-chain, for then they would be the endpoints of some path. Therefore, we choose one vertex from each path component of the space as a generator of the coefficient group, and a direct sum of these will give us the homology at dimension zero.

Example : 1.5.2 Consider the unit circle S^1 . As we have seen, S^1 can be triangulated giving the complex K shown below.

The oriented complex K consists of the three 1-simplexes, labelled $\sigma_1, \sigma_2, \sigma_3$ and oriented as shown, along with the three 0-simplexes, labelled τ_1, τ_2, τ_3 and given a positive orientation as we move counter-clockwise. Since $C_p(S^1) = 0$ for all $p \geq 2$, $H_p(S^1) = 0$ for all $p \geq 2$. Also since $\partial_2 \equiv 0$, $B_1(S^1) = \text{Im } \partial_2 = 0$, and we have $H_1(S^1) \approx Z_1(S^1)$. We will use the integers as the coefficient group. If $z \in Z(S^1)$, then z is of the form $z = a\sigma_1 + b\sigma_2 + c\sigma_3$ for some $a, b, c \in \mathbb{Z}$, and we have $0 = \partial_1(z) = a\partial_1(\sigma_1) + b\partial_1(\sigma_2) + c\partial_1(\sigma_3) = a(\tau_2 - \tau_1) + b(\tau_3 - \tau_2) + c(\tau_1 - \tau_3) = (c-a)\tau_1 + (a-b)\tau_2 + (b-c)\tau_3$. Therefore, if $z \in Z_1(S^1)$, then we have $a = b = c$, and z is of the form $z = n(\sigma_1 + \sigma_2 + \sigma_3)$, for some $n \in \mathbb{Z}$. This gives $H_1(S^1) \approx Z_1(S^1) \approx \mathbb{Z}$. Example 1.5.1 and the fact that S^1 is path connected, tells us that the zero homology

of S^1 is also isomorphic to $\mathbb{Z}$. Therefore, the homology of S^1 is given by

$$H_p(S^1) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z} & \text{if } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using a similar constructive argument one can show that the homology of the n -sphere, S^n , is given by

$$H_p(S^n) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z} & \text{if } p = n \\ 0 & \text{otherwise.} \end{cases}$$

§1.6 Singular Homology

Before we go on, let us briefly introduce another type of homology theory, called *singular homology*, which will be increasingly useful in the pages that follow.

Definition : 1.5.1 Let σ^p be a p -dimensional geometric simplex and let X be any topological space. A *p -dimensional singular simplex* is a continuous function $f_p: \sigma^p \rightarrow X$ [Hilton and Wylie, 314].

Thus, a singular simplex is any continuous function from a geometric simplex into the space X . Even for a reasonable space X , there can be an enormous number of p -dimensional singular simplexes. By orienting the singular simplexes we can form the direct sum of copies of a coefficient group G , one copy for each positively oriented p -dimensional singular simplex. This group, written $\Delta_p(X)$, is called the *p -dimensional singular chain group*. As with simplicial homology, we define the p -dimensional chain group as $\Delta_p(X) = 0$ for all $p < 0$.

We need only define the boundary operators $\partial_p: \Delta_p(X) \rightarrow \Delta_{p-1}(X)$ to have a chain complex. Let $\sigma^p = \langle a_0, \dots, a_p \rangle$ be a geometric p -simplex. Then by omitting

one vertex we get $\sigma_i^{p-1} = \langle a_0, \dots, \hat{a}_i, \dots, a_p \rangle$, a geometric $(p-1)$ -simplex. There is a unique linear map $d_i: \sigma_i^{p-1} \rightarrow \sigma^p$ given by inclusion. If $f_p: \sigma^p \rightarrow X$ is a generator of $\Delta_p(X)$, then the composition $f_p \circ d_i$ is a map from σ_{p-1} into X , which is a generator of the $(p-1)$ -dimensional chain group $\Delta_{p-1}(X)$. The maps $f_p \circ d_i$ could be thought of as the restrictions of f_p to the $(p-1)$ -faces of σ_p . Therefore, we define the boundary operator $\partial_p: \Delta_p(X) \rightarrow \Delta_{p-1}(X)$ by $f_p \rightarrow \sum_{i=0}^p (-1)^i (f_p \circ d_i)$, and extend to a homomorphism by linearity.

It can be shown that $\partial_{p-1} \partial_p = 0$ [Hilton and Wylie, 315], and so given any topological space X , we get a chain complex $\{\Delta_p(X), \partial_p\}$. The homology functor will then give a graded abelian group $H_*(X)$, called the *singular homology group*.

Since the generators of the group $\Delta_p(X)$ are continuous functions of the form $f_p: \sigma^p \rightarrow X$, there is an enormous number of generators for this group, as well as for the cycle group $Z_p(\Delta_p(X))$. However, the boundary groups $B_p(\Delta_p(X))$ will also be quite large, giving quotient groups, $H_p(\Delta_p(X))$, of modest size. In fact, one of the most useful properties of general homology theory is that, given any polyhedron, the singular homology group is isomorphic to the homology group generated simplicially. One of the advantages of singular homology is its scope. Because the singular simplexes are continuous functions, which may map into arbitrary topological spaces, singular homology may be applied to any space. In fact, it can be shown that singular homology is a functor from the category of topological spaces with continuous functions to the category of graded abelian groups with homomor-

phisms. Singular homology can also be very useful in establishing theorems about homology in general.

Chapter 2: Sequences in Homology

At the end of Chapter 1, we computed the homology group of the unit circle by giving a triangulation and computing the cycle groups. For many spaces, such direct computation can be a complicated and cumbersome process. In this chapter, we develop some alternate methods of computing homology groups by investigating the algebraic properties of these groups and constructing some useful exact sequences among them.

§2.1 *Relative Homology*

Often we will want to compute the homology of a space X when we know a great deal about some subspace of X . We would like to use the information about the subspace to gain information about X .

Definition : 2.1.2 Let K be a complex. A *subcomplex* L of K is a complex with the property that each simplex of L is a simplex of K [Croom, 140].

Since L is a complex, we can compute the chain group $C_p(L)$. By giving each simplex in K not lying in L a zero coefficient, we can think of any chain in $C_p(L)$ as a chain in $C_p(K)$. Thus, we have that $C_p(L)$ is normal in $C_p(K)$, and we can form the quotient group $C_p(K) / C_p(L)$. This group, written $C_p(K/L)$ or $C_p(K, L)$, is called

the p^{th} relative chain group of K modulo L . The boundary map $\partial_p: C_p(K, L) \rightarrow C_{p-1}(K, L)$ is defined by $\partial_p([c_p]) = \partial_p(c_p + C_p(L)) = \partial_p(c_p) + C_{p-1}(L)$. It is easily verified that this gives a well-defined homomorphism, and that $\partial_{p-1}\partial_p = 0$. Therefore, we have a chain complex $\{C_p(K, L), \partial_p\}$, and the homology functor gives a graded abelian group $H_*(K, L)$, called the *relative homology group of K modulo L* .

Let L be a subcomplex of K , and consider the inclusion map $i: C_p(L) \rightarrow C_p(K)$. If $c_p = \sum a_i \sigma_i$ is a chain in $C_p(L)$, then $(\partial_p \circ i)(c_p) = \partial_p(i(\sum a_i \sigma_i)) = \partial_p(\sum a_i \sigma_i) = \sum a_i \partial_p(\sigma_i) = i(\sum a_i \partial_p(\sigma_i)) = (i \circ \partial_p)(c_p)$. Therefore, i is a chain map which induces a homomorphism $i_*: H_*(L) \rightarrow H_*(K)$.

If $c_p \in C_p(K)$, then $c_p + C_p(L) = [c_p] \in C_p(K, L)$. Define a map $\alpha: C_p(K) \rightarrow C_p(K, L)$ by $\alpha(c_p) = [c_p]$. If c_p is a chain in $C_p(K)$ then $\partial_p(c_p)$ is a chain in $C_{p-1}(K)$. Therefore, $[\partial_p(c_p)] = \partial_p(c_p) + C_{p-1}(L)$, and we have $(\alpha \circ \partial_p)(c_p) = \partial_p(c_p) + C_{p-1}(L) = \partial_p([c_p]) = (\partial_p \circ \alpha)(c_p)$. So again we have a chain map, which induces a homomorphism $\alpha_*: H_*(K) \rightarrow H_*(K, L)$.

The elements of the chain group $C_p(K, L) = C_p(K) / C_p(L)$ are cosets of the form $c_p + C_p(L)$ where $c_p \in C_p(K)$. Thus, the elements of the homology group $H_p(K, L)$ are cosets of the form $(c_p + C_p(L)) + B_p(K, L)$, where $c_p + C_p(L) \in Z_p(K, L)$. Therefore, if $c_p + C_p(L) \in Z_p(K, L)$, then $0 = \partial_p(c_p + C_p(L)) = \partial_p(c_p) + C_{p-1}(L)$, and we have that $\partial_p(c_p)$ is a chain in $C_{p-1}(L)$. But $\partial_{p-1}\partial_p(c_p) = 0$, which implies that $[\partial_p(c_p)] = \partial_p(c_p) + B_{p-1}(L)$ is an element of $H_{p-1}(L)$. Thus, we

have a map $\beta_*: H_p(K, L) \rightarrow H_{p-1}(L)$ defined by $[c_p + C_p(L)] \mapsto [\partial_p(c_p)]$. One can easily show that this assignment is independent of the choice of representative and gives a well defined homomorphism.

The homomorphisms i_* , α_* and β_* let us form a sequence of groups, and the following theorem guarantees that this is an exact sequence. Recall that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is called exact if $\text{Im} f = \text{Ker} g$.

Theorem : 2.1.1 Let K be a complex and L a subcomplex of K , then the sequence

$$\cdots \longrightarrow H_p(L) \xrightarrow{i_*} H_p(K) \xrightarrow{\alpha_*} H_p(K, L) \xrightarrow{\beta_*} H_{p-1}(L) \longrightarrow \cdots$$

where i_* , α_* , and β_* are as described above, is an exact sequence [Hilton and Wylie, 81].

This sequence, called the *homology sequence of the pair* (K, L) , is very useful in computing the homology of certain spaces. For instance, if enough is known about the homology of K and the homology of the subcomplex L , then this information can be used to find the homology of the pair (K, L) .

Example : 2.1.2 Let us compute the homology of the pair $(\mathbf{R}^n, \mathbf{R}^n - 0)$. Since $\mathbf{R}^n$ is a contractible space, that is, it is homotopic to a single point, it has the same homology as a single point. Therefore $H_p(\mathbf{R}^n) = 0$ for $p \neq 0$ and $H_0(\mathbf{R}^n) = \mathbf{Z}$.

To see what the space $\mathbf{R}^n - 0$ looks like, we will consider the case $n = 2$ where we have the punctured plane. We can no longer contract this punctured plane to a point, but we can continuously deform this space into the unit circle. This

can be done formally by defining a map $f: (\mathbb{R}^2 - 0) \rightarrow S^1$ by $x \mapsto \frac{x}{|x|}$, and a map $g: S^1 \rightarrow (\mathbb{R}^2 - 0)$ by $y \mapsto y$. The composition $g \circ f$ is homotopic to the identity, while the composition $f \circ g$ is the identity. Thus, $(\mathbb{R}^2 - 0)$ and S^1 are homotopy equivalent and have the same homology.

An analogous argument will show that $(\mathbb{R}^n - 0)$ is homotopy equivalent to S^{n-1} , and so has homology given by

$$H_p(\mathbb{R}^n - 0) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z} & \text{if } p = n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now we can use the relative homology sequence to compute the homology of $(\mathbb{R}^n, \mathbb{R}^n - 0)$. The sequence gives

$$\cdots \longrightarrow H_p(\mathbb{R}^n - 0) \longrightarrow H_p(\mathbb{R}^n) \longrightarrow H_p(\mathbb{R}^n, \mathbb{R}^n - 0) \longrightarrow H_{p-1}(\mathbb{R}^n - 0) \longrightarrow H_{p-1}(\mathbb{R}^n) \longrightarrow \cdots$$

For $p \neq n$ and $p \neq 0$, we have $H_p(\mathbb{R}^n) = H_{p-1}(\mathbb{R}^n - 0) = 0$, which gives $H_p(\mathbb{R}^n, \mathbb{R}^n - 0) = 0$. When $p = n$ we get the sequence

$$0 \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and we have an isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \approx \mathbb{Z}$. Unfortunately, when $p = 0$, the sequence becomes

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_0(\mathbb{R}^n, \mathbb{R}^n - 0) \longrightarrow 0$$

and this is not enough information to determine $H_0(\mathbb{R}^n, \mathbb{R}^n - 0)$ precisely. However, we can get more information from further examination of the maps in the sequence

$$\mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{\alpha_*} H_0(\mathbb{R}^n, \mathbb{R}^n - 0) \xrightarrow{\beta_*} 0.$$

First of all, we have a map $i_*: \mathbb{Z} \rightarrow \mathbb{Z}$ induced by the inclusion $(\mathbb{R}^n - 0) \rightarrow \mathbb{R}^n$. One can show that $i_*(1) = 1$, and so i_* is onto. Since the sequence is exact, we have that the kernel of the map $\alpha_*: \mathbb{Z} \rightarrow H_0(\mathbb{R}^n, \mathbb{R}^n - 0)$ is all of $\mathbb{Z}$, and α_* sends everything to zero. Therefore, the image of α_* is zero, which means that the kernel of β_* is also zero. We have a map $\beta_*: H_0(\mathbb{R}^n, \mathbb{R}^n - 0) \rightarrow 0$ which is one-to-one, and clearly onto, hence an isomorphism. The homology of $(\mathbb{R}^n, \mathbb{R}^n - 0)$ is given by

$$H_p(\mathbb{R}^n, \mathbb{R}^n - 0) = \begin{cases} \mathbb{Z} & \text{if } p = n \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

This example illustrates that even if we know two of the three families of homology groups in the homology exact sequence, it is often necessary to know about the connecting homomorphisms as well before we can deduce the third family of homology groups.

§2.2 *Excision*

One of the axioms of homology is the *excision axiom*, which basically says that a piece may be cut away, or excised, from a pair without affecting the relative homology of the pair.

Excision Axiom: For any pair (X, A) , if U is an open subset of X such that $\overline{U} \subset \text{int} A$, then there is an isomorphism $H_*(X - U, A - U) \approx H_*(X, A)$ induced by inclusion. [Spanier, 200].

This is an especially nice property because it often allows us to take a pair, which may be complicated, and cut away the complicated part. The resulting

pair may have relative homology which is easily computed, and this will give the homology of the original pair.

Let us see what this means in simplicial homology theory. An *admissible pair* in simplicial homology, that is, a pair to which the homology functor may be applied, consists of a complex and a subcomplex.

Theorem : 2.2.1 Let K be a simplicial complex, and let L_1 and L_2 be subcomplexes of K such that $L_1 \cup L_2 = K$. Then

$$H_p(L_2, L_1 \cap L_2) \approx H_p(K, L_1)$$

[Hilton and Wylie, 79].

This theorem, called the *simplicial excision theorem*, requires only that L_1 and L_2 be subcomplexes whose union is the entire complex. Actually, except for the uninteresting case where L_1 is sitting inside of L_2 , this requires both L_1 and L_2 to be quite “large” in K . Then we excise $L_1 - (L_1 \cap L_2)$, the portion of the complex lying in L_1 which does not intersect L_2 . It turns out that these requirements on L_1 and L_2 are enough to show the equivalence of this theorem with the excision axiom presented earlier.

As we have said, singular homology can be very useful in establishing theorems about homology in general. As an example, consider a less restrictive form of the excision axiom. In the following theorem, which can be proved using singular homology, the set U is not required to be an open set.

Theorem : 2.2.2 Let X be a topological space and let $U \subset A \subset X$ be such

that $\overline{U} \subset \text{int}A$. Then there is an isomorphism $H_*(X - U, A - U) \approx H_*(X, A)$ [Spanier, 189].

Example : 2.2.1 Let $K \subset V \subset \mathbb{R}^n$, where K is compact and V is open. Also let S^n be the unit n -sphere. We can think of S^n as $\mathbb{R}^n$ with a point at infinity. That is, $S^n \approx \mathbb{R}^n \cup \{\infty\}$. This means that $K \subset V \subset S^n$. Let $U = S^n - V = S^n \cap (V^c \cup \{\infty\})$. Since K is compact, it is closed in $\mathbb{R}^n$ and its complement is open. Similarly, since V is open, we have V^c is closed. Then $K \subset V$ implies $V^c \subset K^c$, and we have $\overline{V^c} = V^c \subset K^c = \text{int}K^c$. Therefore, $\overline{U} = \overline{S^n \cap (V^c \cup \{\infty\})} \subset \overline{S^n} \cap \overline{V^c \cup \{\infty\}} \subset S^n \cap K^c \cup \{\infty\} = \text{int}S^n \cap \text{int}(K^c \cup \{\infty\}) \subset \text{int}(S^n \cap (K^c \cup \{\infty\})) = \text{int}(S^n - K)$, and we may excise the set U from the pair $(S^n, S^n - K)$.

Since $S^n - U = V$ and $(S^n - K) - U = U - K$ the excision theorem gives

$$H_*(S^n, S^n - K) \approx H_*(S^n - U, (S^n - K) - U) = H_*(V, V - K).$$

§2.3 The Mayer-Vietoris Sequence

Suppose we have a complicated space X which can be written as the union of two subspaces, whose respective homologies are known. It would be useful to have a way of computing the homology of the space X from the homologies of the subspaces. In this section we construct a sequence using the union and intersection of two complexes, and apply this to the situation we have just described.

Let K be a simplicial complex and let L_1 and L_2 be subcomplexes of K . Then the intersection $L_1 \cap L_2$ is a subcomplex of K as well. If $c_p \in C_p(L_1 \cap L_2)$, then

c_p is a linear combination of simplexes in $L_1 \cap L_2$. We can think of c_p as linear combination of simplexes in L_1 and also as a linear combination of simplexes in L_2 .

Let

$$i_1: H_p(L_1 \cap L_2) \rightarrow H_p(L_1) \quad \text{and}$$

$$i_2: H_p(L_1 \cap L_2) \rightarrow H_p(L_2).$$

be maps induced by inclusion. Then we may define a map $i_*: H_p(L_1 \cap L_2) \rightarrow H_p(L_1) \oplus H_p(L_2)$ by $[c_p] \mapsto (i_1([c_p]), -i_2([c_p]))$, for all $[c_p] \in H_p(L_1 \cap L_2)$.

Let (c_p, c'_p) be an element of $C_p(L_1) \oplus C_p(L_2)$. This means that c_p is a linear combination of simplexes in L_1 , and c'_p is a linear combination of simplexes in L_2 . Therefore, $c_p + c'_p$ is a linear combination of simplexes in $L_1 \cup L_2$. Again we get a homomorphism $j_*: H_*(L_1) \oplus H_*(L_2) \rightarrow H_*(L_1 \cup L_2)$ induced by inclusion.

To complete the sequence we construct a homomorphism from $H_p(L_1 \cap L_2)$ to $H_{p-1}(L_1 \cup L_2)$. From excision we know that $H_p(L_1 \cup L_2, L_2) \approx H_p(L_1, L_1 \cap L_2)$. If we borrow the maps α_* and β_* from the relative homology sequence, we can get a composition of homomorphisms

$$H_p(L_1 \cup L_2) \xrightarrow{\alpha_*} H_p(L_1 \cup L_2, L_2) \approx H_p(L_1, L_1 \cap L_2) \xrightarrow{\beta_*} H_{p-1}(L_1 \cap L_2).$$

Therefore, we have a homomorphism

$$d_* = \beta_* \circ \alpha_*: H_p(L_1 \cup L_2) \rightarrow H_{p-1}(L_1 \cap L_2).$$

Theorem : 2.3.1 Let K be a complex and let L_1 and L_2 be subcomplexes of K .

Then the sequence

$$\cdots \rightarrow H_p(L_1 \cap L_2) \xrightarrow{i_*} H_p(L_1) \oplus H_p(L_2) \xrightarrow{j_*} H_p(L_1 \cup L_2) \xrightarrow{d_*} H_{p-1}(L_1 \cap L_2) \rightarrow \cdots$$

is an exact sequence, where i_* , j_* , and d_* are as described above [Dold-1, 48].

§2.4 *The Universal Coefficient Theorem*

Although we stated in §1.4 that the elements of any abelian group could be used as coefficients in a chain complex, we have so far used only integers as coefficients. In this section, we justify the exclusive use of the integers as coefficients by developing an exact sequence which gives a relationship between the homology of a complex with integer coefficients and the homology of a complex with coefficients in an arbitrary group. First, we describe some fundamental properties of abelian groups which will be useful in the development of this exact sequence.

Definition : 2.4.1 Let R be a ring. A (left) R -module is an additive group A , together with a function $R \times A \rightarrow A$ (the image of (r, a) being denoted by ra) such that for all $r, s \in R$ and $a, b \in A$:

$$\text{i)} \quad r(a + b) = ra + rb$$

$$\text{ii)} \quad (r + s)a = ra + sa$$

$$\text{iii)} \quad r(sa) = (rs)a$$

If R has an identity element 1_R and

$$\text{iv)} \quad 1_R a = a \text{ for all } a \in A,$$

then A is said to be a *unitary R -module*. A (unitary) *right R -module* is defined similarly, via a function $A \times R \rightarrow A$, denoted by $(a, r) \mapsto ar$, and satisfying the obvious analogues of i) — iv). [Hungerford, 169].

Let G be any additive abelian group and define a function $f: \mathbb{Z} \times G \rightarrow G$ by $(n, g) \mapsto ng$ for all $n \in \mathbb{Z}, g \in G$. It can be easily shown that this satisfies Definition 2.4.1, and so any additive abelian group is a $\mathbb{Z}$ -module. Since for any chain complex $C = \{C_p, \partial_p\}$, C_p is an additive abelian group, we can think of the chain groups as $\mathbb{Z}$ -modules.

Definition : 2.4.2 Let A be a right R -module and let B be a left R -module. The *tensor product of A and B* , denoted by $A \otimes B$, is the abelian group generated by symbols of the form $a \otimes b$, satisfying the following relations for all $a, a' \in A$, $b, b' \in B$, and $r \in R$:

- i) $(a + a') \otimes b = a \otimes b + a' \otimes b$
- ii) $a \otimes (b + b') = a \otimes b + a \otimes b'$
- iii) $ar \otimes b = a \otimes rb$

[Spanier, 7].

Before we see how tensor products relate to homology theory, let us explore some of the properties of the tensor product. One such property is that the tensor product distributes over direct sums.

Theorem : 2.4.1 Let R be a ring, A and $\{A_i | i \in I\}$ right R -modules, B and $\{B_j | j \in J\}$ left R -modules. Then there are group isomorphisms:

$$\begin{aligned} \left(\sum_{i \in I} A_i \right) \otimes B &\approx \sum_{i \in I} (A_i \otimes B) \\ A \otimes \left(\sum_{j \in J} B_j \right) &\approx \sum_{j \in J} (A \otimes B_j) \quad [\text{Hungerford, 213}]. \end{aligned}$$

If R is any commutative ring, then it may be considered an R -module. Also, if A is a right R -module and B is a left R -module, then $A \otimes B$ has a right R -module structure given by the function $(a \otimes b)r \mapsto a \otimes rb$. Let R be a commutative ring with identity, A a right unitary R -module, and define a map $\alpha: A \otimes R \rightarrow A$ by $a \otimes r \mapsto ar \in A$. Also, define a map $\beta: A \rightarrow A \otimes R$ by $a \mapsto a \otimes 1$. It can be easily shown that α and β are R -module homomorphisms and $\alpha \circ \beta = \beta \circ \alpha = 1$. Thus we have an isomorphism which gives the following theorem.

Theorem : 2.4.2 If R is a commutative ring with unity, A is a unitary right R -module, and B is a unitary left R -module, then there are R -module isomorphisms

$$A \otimes R \approx A \quad \text{and} \quad R \otimes B \approx B$$

[Hungerford, 212].

Let K be a simplicial complex and let $C = \{C_p(K), \partial_p\}$ be the chain complex of K with integer coefficients. Also, let G be any additive abelian group and $C' = \{C_p(K; G), \partial_p\}$ the chain complex of K with the same orientation, but with coefficients in G . Then $C_p(K)$ is a direct sum of copies of $\mathbb{Z}$, one copy for each p -simplex. If K contains n p -simplexes, then $C_p(K) = \oplus_{i=1}^n \mathbb{Z}$. Thus, the tensor product $C_p(K) \otimes G$ is given by $(\oplus_{i=1}^n \mathbb{Z}) \otimes G = \oplus_{i=1}^n (\mathbb{Z} \otimes G)$. However, since G is a $\mathbb{Z}$ -module, we have $\mathbb{Z} \otimes G = G$, which gives $C_p(K) \otimes G = \oplus_{i=1}^n G$. Since K has n p -simplexes, the chain group $C_p(K; G)$ is just a direct sum of n copies of G . Therefore, $C_p(K) \otimes G \approx C_p(K; G)$, and we may think of the chain group

with coefficients in G as the tensor product of G with the chain group with integer coefficients. In fact, this would be a suitable definition for the chain group with coefficients in an arbitrary abelian group. If $\{C_p(K), \delta_p\}$ is a chain complex, then $\{C_p(K) \otimes G, \partial_p \otimes 1\}$ is also chain complex and we can apply the homology functor yielding the homology group $H_*(C_p(K) \otimes G)$. As we see by the following theorem, called the *universal coefficient theorem*, the homology of $C_p(K) \otimes G$ is not exactly equal to the tensor product of $H_*(C_p)$ with G , but we still have a very useful relationship.

Theorem : 2.4.3 Let C be a free chain complex, i.e. a complex in which C_p is free for all p , and let G be a $\mathbb{Z}$ -module. There is a short exact sequence

$$0 \longrightarrow H_p(C) \otimes G \longrightarrow H_p(C \otimes G) \longrightarrow H_{p-1}(C) * G \longrightarrow 0$$

and this sequence splits, i.e., we have $H_p(C \otimes G) \approx H_p(C) \otimes G \oplus H_{p-1}(C) * G$ [Spanier, 222].

The operation $*$ is called the *torsion product* and, basically, measures the torsion in the two groups. In particular, if A or B is torsion free, then $A * B = 0$ [Spanier, 221].

The universal coefficient theorem allows us to compute the homology of a complex with coefficients in an arbitrary group from the chain complex with integer coefficients. Thus, we need only compute the integral homology of a complex and convert to another coefficient group as the need arises.

Example : 2.4.1 Let K be a complex and $C = \{C_p(K), \partial_p\}$, the chain

complex of K with integer coefficients. Since the group of rational numbers $\mathbf{Q}$ is a $\mathbf{Z}$ -module, we use the universal coefficient theorem to compute the homology of K with coefficients in $\mathbf{Q}$. Applying the theorem, we get

$$0 \longrightarrow H_p(C) \otimes \mathbf{Q} \longrightarrow H_p(C \otimes \mathbf{Q}) \longrightarrow H_{p-1}(C) * \mathbf{Q} \longrightarrow 0$$

or

$$0 \longrightarrow H_p(K) \otimes \mathbf{Q} \longrightarrow H_p(K; \mathbf{Q}) \longrightarrow H_{p-1}(K) * \mathbf{Q} \longrightarrow 0.$$

However, since $\mathbf{Q}$ is torsion free, $H_{p-1}(K) * \mathbf{Q} = 0$ and we have $H_p(K) \otimes \mathbf{Q} \approx H_p(K; \mathbf{Q})$. On the other hand, if K is a finite complex then $H_p(K)$ is a finitely generated abelian group. Therefore, $H_p(K) = \mathbf{Z} \oplus \mathbf{Z} \oplus \cdots \oplus \mathbf{Z} \oplus \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_m}$, and we have $H_p(K) \otimes \mathbf{Q} = (\mathbf{Z} \otimes \mathbf{Q}) \oplus (\mathbf{Z} \otimes \mathbf{Q}) \oplus \cdots \oplus (\mathbf{Z} \otimes \mathbf{Q}) \oplus (\mathbf{Z}_{n_1} \otimes \mathbf{Q}) \oplus \cdots \oplus (\mathbf{Z}_{n_m} \otimes \mathbf{Q}) = \mathbf{Q} \oplus \mathbf{Q} \oplus \cdots \oplus \mathbf{Q} \oplus 0 \oplus \cdots \oplus 0 = \mathbf{Q} \oplus \mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$. Then $H_p(K; \mathbf{Q})$ is isomorphic to the sum of n copies of $\mathbf{Q}$ where n is the rank of the free part of $H_p(K)$. This is useful since $H_p(K; \mathbf{Q})$ is now a vector space and the algebra is greatly simplified.

§2.5 Homology and Product Spaces

In topology we often encounter product spaces, and we would like to compute the homology of such spaces. As we will see, the homology of a product space is related to the tensor product of the respective homologies of the spaces forming the product. First, we must define what is meant by the tensor product of a chain complex.

Definition: 2.5.1 Let $C = \{C_p, \partial_p\}$ and $C' = \{C'_p, \partial'_p\}$ be two chain complexes.

The tensor product $C \otimes C'$ is the chain complex D such that

$$D_p = \bigoplus_{m+n=p} C_m \otimes C'_n$$

with boundary operator given by

$$\partial''_p(c_m \otimes c'_n) = \partial_m(c_m) \otimes c'_n + (-1)^m c_m \otimes \partial'_n(c'_n)$$

and extended to a homomorphism by linearity. [Hilton and Wylie, 209].

The reason for including the factor $(-1)^m$ in the definition of the boundary operator is to ensure that $\partial''_{p-1}\partial''_p = 0$ so that we have a chain complex.

Using singular homology, one can show that the chain complex $\Delta(X \times Y)$ is equivalent to the chain complex $\Delta(X) \otimes \Delta(Y)$. First, we need to clarify the notion of equivalence of chain complexes. If $\phi, \psi: C \rightarrow D$ are chain maps, then a *chain homotopy* between ϕ and ψ is a set of homomorphisms $d_p: C_p \rightarrow D_{p+1}$ such that $\phi - \psi = \partial_{p+1} \circ d_p + d_{p-1} \circ \partial_p$ [Hilton and Wylie, 105]. A chain map $\phi: C \rightarrow D$ is said to be a *chain equivalence* if there exists a chain map $\psi: D \rightarrow C$ such that the compositions $\psi \circ \phi$ and $\phi \circ \psi$ are chain homotopic to the identities on C and D , respectively [Hilton and Wylie, 107]. Two complexes C and D are said to be *chain equivalent* if there exists a map $\phi: C \rightarrow D$ which is a chain equivalence. It is not difficult to verify that chain equivalent complexes have isomorphic homology groups. The following theorem, called the *Eilenberg-Zilber theorem*, establishes the equivalence of the chain group of a product of two spaces and the tensor product of their respective chain groups.

Theorem : 2.5.1 On the category of ordered pairs of topological spaces X and Y there is a natural chain equivalence of the functor $\Delta(X \times Y)$ with the functor $\Delta(X) \otimes \Delta(Y)$ [Spanier, 232].

Therefore, to talk about the homology of the product space $X \times Y$, we need only talk about the homology of $\Delta(X) \otimes \Delta(Y)$. The final task in determining the homology of a product space is in finding a way to compute the homology of the chain complex $C \otimes C'$, where C and C' are chain complexes.

Theorem : 2.5.2 Let C and C' be chain complexes, with C' free. Then there is a split exact sequence

$$0 \rightarrow \sum_{m+n=p} [H_m(C) \otimes H_n(C')] \rightarrow H_p(C \otimes C') \rightarrow \sum_{m+n=p-1} [H_m(C) * H_n(C')] \rightarrow 0$$

[Spanier, 228].

An equivalent form of this theorem, called the *Künneth formula*, says that

$$H_p(C \otimes C') = \sum_{m+n=p} [H_m(C) \otimes H_n(C')] \oplus \sum_{m+n=p-1} [H_m(C) * H_n(C')].$$

Since, given topological spaces X and Y , we know that $\Delta(X \times Y) = \Delta(X) \otimes \Delta(Y)$, we have

$$H_p(X \times Y) = \sum_{m+n=p} [H_m(\Delta(X)) \otimes H_n(\Delta(Y))] \oplus \sum_{m+n=p-1} [H_m(\Delta(X)) * H_n(\Delta(Y))]$$

or

$$H_p(X \times Y) = \sum_{m+n=p} [H_m(X) \otimes H_n(Y)] \oplus \sum_{m+n=p-1} [H_m(X) * H_n(Y)].$$

Example : 2.5.1 Let X and Y be topological spaces, and $H_*(X; \mathbf{Q})$ and $H_*(Y; \mathbf{Q})$ their respective homologies with coefficients in $\mathbf{Q}$. Then for each m and n , $H_m(X; \mathbf{Q})$ and $H_n(Y; \mathbf{Q})$ are either zero or direct sums of copies of $\mathbf{Q}$. Thus, $H_m(X; \mathbf{Q}) * H_n(Y; \mathbf{Q})$ is a direct sum of copies of $\mathbf{Q} * \mathbf{Q}$ (the torsion product distributes over direct sums). Since $\mathbf{Q} * \mathbf{Q} = 0$, the Künneth formula becomes

$$H_p(X \times Y; \mathbf{Q}) = \sum_{m+n=p} [H_m(X; \mathbf{Q}) \otimes H_n(Y; \mathbf{Q})].$$

More precisely, in rational homology, the homology group of a product is exactly the graded tensor product of the homologies of X and Y .

Example : 2.5.2 Let us compute the homology of the unit torus. Since the unit torus is homeomorphic to the product space $S^1 \times S^1$, the Künneth formula will be useful. Recall that the homology of S^1 with integer coefficients is given by

$$H_p(S^1) = \begin{cases} \mathbf{Z} & \text{if } p = 0 \\ \mathbf{Z} & \text{if } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the only contribution to $H_m(S^1) * H_n(S^1)$ will come from $\mathbf{Z} * \mathbf{Z}$. However, since $\mathbf{Z}$ is torsion free, $\mathbf{Z} * \mathbf{Z} = 0$, and we have

$$\sum_{m+n=p-1} [H_m(S^1) * H_n(S^1)] = 0$$

for all p . The Künneth formula reduces to

$$H_p(S^1 \times S^1) = \sum_{m+n=p} [H_m(S^1) \otimes H_n(S^1)].$$

Clearly, $H_p(S^1 \times S^1) = 0$ for all $p > 2$. From the formula $H_2(S^1 \times S^1) = [H_0(S^1) \otimes H_2(S^1)] \oplus [H_1(S^1) \otimes H_1(S^1)] \oplus [H_2(S^1) \otimes H_0(S^1)] = (\mathbf{Z} \otimes 0) \oplus (\mathbf{Z} \otimes \mathbf{Z}) \oplus (0 \otimes \mathbf{Z}) = \mathbf{Z}$.

Also, $H_1(S^1 \times S^1) = [H_0(S^1) \otimes H_1(S^1)] \oplus [H_1(S^1) \otimes H_0(S^1)] = (\mathbf{Z} \otimes \mathbf{Z}) \oplus (\mathbf{Z} \otimes \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. To get $H_0(S^1 \times S^1)$ we need only look at $H_0(S^1) \otimes H_0(S^1) = \mathbf{Z} \otimes \mathbf{Z} = \mathbf{Z}$.

Therefore, the homology of the unit torus, T , is given by

$$H_p(T) = \begin{cases} \mathbf{Z} & \text{if } p = 0 \text{ or } p = 2 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 3: The Fixed Point Index

Now that we have an initial understanding of homology theory and some of its properties, let us see how homology is applied to fixed point theory. In this chapter, we use homology to define the fixed point index of a continuous map. We then define the Lefschetz number of a map, and show that the Lefschetz number and the fixed point index agree for any continuous function from a Euclidean neighborhood retract into itself.

§3.1 *The Fundamental Class*

Let $K \subset V \subset \mathbb{R}^n$, where K is compact and V is open. In Example 1.5.2 we saw that for $S^n = \mathbb{R}^n \cup \{\infty\}$ and $n \geq 1$, $H_n(S^n) = \mathbb{Z}$. Therefore, part of the homology exact sequence of the pair $(S^n, S^n - K)$ gives

$$\mathbb{Z} = H_n(S^n) \xrightarrow{\alpha_*} H_n(S^n, S^n - K).$$

Example 2.2.1 showed that by excising the set $S^n - V$ from the pair

$(S^n, S^n - K)$, we get $H_n(S^n, S^n - K) \approx H_n(V, V - K)$, and the sequence yields the map $\alpha'_*: \mathbb{Z} \rightarrow H_n(V, V - K)$. Let $o \in H_n(S^n) = \mathbb{Z}$ be a class which generates the group $H_n(S^n)$. We are interested in the image of o under α'_* .

Definition : 3.1.1 Let $K \subset V \subset \mathbb{R}^n$ where K is compact and V is open. If o is a generator of $H_n(S^n) = \mathbb{Z}$, then the *fundamental class* $o_K \in H_n(V, V - K)$ is the image of o under the composite map

$$\mathbb{Z} = H_n(S^n) \rightarrow H_n(S^n, S^n - K) \approx H_n(V, V - K)$$

[Dold-2, 1].

Suppose P is a point in the compact set $K \subset V \subset \mathbb{R}^n$. Let U be an open ball around P such that $U \subset V$. Since P is compact, $H_n(V, V - P) \approx H_n(S^n, S^n - P) \approx H_n(U, U - P)$. But U is an open ball, so there exists a homeomorphism $f: U \rightarrow \mathbb{R}^n$ such that $f(P) = 0$. Therefore, f restricts to a homeomorphism $f: U - P \rightarrow \mathbb{R}^n - 0$, and we have $H_*(U, U - P) \approx H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$. Thus, $H_n(V, V - P) = H_n(U, U - P) = H_n(\mathbb{R}^n, \mathbb{R}^n - 0) = \mathbb{Z}$. The fundamental class is uniquely characterized by the property that it maps to 1 under $H_n(V, V - K) \rightarrow H_n(V, V - P) = \mathbb{Z}$, for all $P \in K$. [Dold-2, 1].

Since α'_* is a homomorphism, $\alpha'_*(-o + o) = \alpha'_*(-o) + \alpha'_*(o) = 0$, and we have that $(-o)_K = -o_K$. Therefore, the choice of generator of $H_n(S^n)$ is not important.

For $K \subset V \subset \mathbb{R}^n$ we can think of the fundamental class o_K as the relative n -cycle that makes up the subspace V . That is, for most spaces, o_K is the class, $[z] \in H_n(V, V - K)$, where the complex formed by taking the union of the simplexes in z is homeomorphic to V and the boundary of that complex does not meet K .

Example : 3.1.1 Let $V = D^{o2} \subset \mathbb{R}^2$ be the open unit disk, and let K be the origin. Clearly, $K \subset V$ is compact, and so the class $o_K \in H_2(V, V - K)$ maps

to a generator of $H_2(V, V - P)$ for any $P \in K$. Here we need only consider the identity map, since $H_2(V, V - K) = H_2(V, V - 0) = H_2(V, V - P)$. Therefore, the fundamental class σ_K is exactly a generator of $H_2(V, V - 0) = H_2(D^{\circ 2}, D^{\circ 2} - 0) = \mathbb{Z}$.

Let D^2 be the closed unit disk in $\mathbb{R}^2$. Let $f: D^2 \rightarrow D^{\circ 2}$ be defined by

$$f(\vec{x}) = \begin{cases} \frac{\vec{x}}{2|\vec{x}|} & \text{if } \frac{1}{2} \leq |\vec{x}| \leq 1 \\ \vec{x} & \text{otherwise} \end{cases}$$

and $g: D^{\circ 2} \rightarrow D^2$ be defined by $g(\vec{x}) = \vec{x}$. It is not difficult to show that the compositions $(f \circ g)$ and $(g \circ f)$ are homotopic to the identities on $D^{\circ 2}$ and D^2 , respectively. In fact, one can show that this is a homotopy equivalence which restricts to a homotopy equivalence in $D^{\circ 2} - 0$ and $D^2 - 0$. This gives $H_2(D^2, D^2 - 0) = H_2(D^{\circ 2}, D^{\circ 2} - 0) = \mathbb{Z}$. Let us compute the generator of $H_2(D^2, D^2 - 0)$. Let D^2 be triangulated, as shown below, by joining three 2-simplexes, where the vertex in the middle is the origin.

Since there are no 3-simplexes, $H_2(D^2, D^2 - 0) = Z_2(D^2, D^2 - 0)$. If $z \in Z_2(D^2, D^2 - 0)$ is a relative cycle, then z is a chain in D^2 whose boundary is a chain in $D^2 - 0$. Consider the chain $c_2 = \sigma_1 + \sigma_2 + \sigma_3$. The boundary of c_2 is the chain formed by adding the three 1-simplexes making up outer edge of the

complex. Since this boundary chain does not include the vertex in the center, it is a chain in $D^2 - 0$, and hence, a relative 2-cycle. With a little investigation, one can see that any chain whose boundary does not include the vertex at the center is a multiple of c_2 , and any chain of the form $a\sigma_1 + b\sigma_2 + c\sigma_3$, without $a = b = c$, will have a boundary which includes the vertex in the center. Thus, c_2 generates $H_2(D^2, D^2 - 0)$, and so $\alpha_K = c_2$, which is a chain that represents the entire subspace V .

§3.2 The Fixed Point Index

Let $V \subset \mathbf{R}^n$ be open and suppose $g: V \rightarrow \mathbf{R}^n$ is continuous with fixed point set $F_g = \{x \in V: g(x) = x\}$. If $i: V \rightarrow \mathbf{R}^n$ is the inclusion map, there is a function $(i - g): V \rightarrow \mathbf{R}^n$, given by $(i - g)(x) = x - g(x)$ for all $x \in V$. For any $x \in V$ such that $x \notin F_g$, we have $x \neq g(x)$, and so $(i - g)(x) \neq 0$. Thus $(i - g)$ is a map from $V - F_g$ into $\mathbf{R}^n - 0$, and it induces a homomorphism $(i - g)_*: H_n(V, V - F_g) \rightarrow H_n(\mathbf{R}^n, \mathbf{R}^n - 0)$. By Example 2.1.2, $H_n(\mathbf{R}^n, \mathbf{R}^n - 0) = \mathbf{Z}$ and so $(i - g)_*$ maps into $\mathbf{Z}$.

If F_g is compact, we may define the fundamental class of F_g as in Definition 3.1.1. We also have a composition

$$\mathbf{Z} = H_n(S^n) \longrightarrow H_n(S^n, S^n - F_g) \xrightarrow{(i-g)_*} H_n(\mathbf{R}^n, \mathbf{R}^n - 0) = \mathbf{Z}.$$

The *fixed point index* is defined as the image of the generator of $H_n(S^n)$ under this composition.

Definition : 3.2.1 Let $V \subset \mathbf{R}^n$ be an open set and $g: V \rightarrow \mathbf{R}^n$ a map. Assume the fixed point set, F_g , is compact. Then the *fixed point index* I_g is defined as $I_g = (i-g)_*(o_{F_g}) \in H_n(\mathbf{R}^n, \mathbf{R}^n - 0) = \mathbf{Z}$, where $(i-g)(x) = x - g(x)$ for all $x \in V$ [Dold-2, 1].

§3.3 Properties of the Fixed Point Index

Let $g: V \rightarrow \mathbf{R}^n$ be a continuous map, where V is open in $\mathbf{R}^n$. Also, let $W \subset V$ be an open set such that there exists a compact set K with $F_g \subset K \subset W \subset V$. Since $F_g \subset K$, we have $K^c \subset F_g^c$, giving $(S^n \cap K^c) \subset (S^n \cap F_g^c)$. Since $[x] \mapsto [x]$ under the map $j_*: H_n(S^n, S^n - K) \rightarrow H_n(S^n, S^n - F_g)$ induced by inclusion, we have a commutative diagram

$$\begin{array}{ccccc} \mathbf{Z} = H_n(S^n) & \longrightarrow & H_n(S^n, S^n - K) & \approx & H_n(W, W - K) \\ \downarrow \approx & & \downarrow j_* & & \\ \mathbf{Z} = H_n(S^n) & \longrightarrow & H_n(S^n, S^n - F_g) & \approx & H_n(V, V - F_g). \end{array}$$

Moving down and across, we see that $1 \mapsto o_{F_g}$. However, if we go across the top we get $1 \mapsto o_K$, which means that o_K must be sent to o_{F_g} under j_* .

Since $W \subset V$ and $W - K \subset V - F_g$ we may restrict g to $(W, W - K)$. Consider the commutative diagram below.

$$\begin{array}{ccccc} H_n(S^n, S^n - K) & \xrightarrow{\approx} & H_n(W, W - K) & \xrightarrow{(i-g)_*} & H_n(\mathbf{R}^n, \mathbf{R}^n - 0) \\ \downarrow j_* & & & & \downarrow \approx \\ H_n(S^n, S^n - F_g) & \xrightarrow{\approx} & H_n(V, V - F_g) & \xrightarrow{(i-g)_*} & H_n(\mathbf{R}^n, \mathbf{R}^n - 0) \end{array}$$

Since j_* sends o_K to o_{F_g} , we have that $o_K \mapsto I_g$ by the path going down and then across. But by the path moving across and then down we get that $(i-g)_*(o_K) = I_g$.

This proves the following proposition.

Proposition : 3.3.1 If $g: V \rightarrow \mathbb{R}^n$ is a continuous function with V open and $F_g \subset V$ compact, and if $F_g \subset K \subset W \subset V$, where K is compact and W is open, then $(i - g): (W, W - K) \rightarrow (\mathbb{R}^n, \mathbb{R}^n - 0)$ and $(i - g)_*(o_K) = I_g$.

This is an especially nice property because it allows us to trim down the domain of g and fatten up the compact set around the fixed point set. In particular, if we can find an open neighborhood around F_g , then we may use that as the domain of g when computing I_g . Unfortunately, it is not always possible to find a neighborhood around the fixed point set that is contained in the domain. However, we can at least find a neighborhood around each component of F_g that is contained in V .

Let $V \subset \mathbb{R}^n$ and $g: V \rightarrow \mathbb{R}^n$ be the constant map defined by $g(v) = c$ for all $v \in V$. Also assume that $c \notin V$. Then $F_g = \emptyset$, which is trivially compact, and $V - F_g = V$. Therefore $H_n(V, V - F_g) = H_n(V, V) = 0$, which gives $I_g = 0$. Now suppose $c \in V$. Then $F_g = \{c\}$ which is compact, and we may use Proposition 3.3.1 to restrict the domain of g to an open ball $U \subset V$ around v without changing the fixed point index. From section 3.1, $o_{F_g} \in H_n(U, U - F_g) = \mathbb{Z}$ maps into 1 under $H_n(U, U - F_g) \rightarrow H_n(U, U - P)$ for every point $P \in F_g$. Letting $P = c$ we have that o_{F_g} maps into 1 under the identity isomorphism $H_n(U, U - F_g) \rightarrow H_n(U, U - \{c\})$. Thus, o_{F_g} is a generator of $H_n(U, U - F_g) = \mathbb{Z}$. Since $(i - g): U - F_g \rightarrow \mathbb{R}^n - 0$ defines a homeomorphism, $(i - g)_*: \mathbb{Z} = H_n(U, U - F_g) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - 0) = \mathbb{Z}$ defines an isomorphism, and the generator of $H_n(U, U - F_g)$ maps to the generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - 0)$. Thus, o_{F_g} maps to 1 and by definition $I_g = 1$.

Proposition : 3.3.2 Let $g: V \rightarrow \mathbf{R}^n$ be defined by $g(v) = c$ for all $v \in V$ where V is open. Then $I_g = 0$ if $c \notin V$ and $I_g = 1$ if $c \in V$.

Recall that if L_1 and L_2 are complexes, then part of the Meyer-Vietoris sequence gives

$$H_n(L_1 \cap L_2) \longrightarrow H_n(L_1) \oplus H_n(L_2) \longrightarrow H_n(L_1 \cup L_2) \longrightarrow H_{n-1}(L_1 \cap L_2).$$

However, if L_1 and L_2 are disjoint and $n > 0$ the sequence becomes

$$0 \longrightarrow H_n(L_1) \oplus H_n(L_2) \longrightarrow H_n(L_1 \cup L_2) \longrightarrow 0$$

and we have $H_n(L_1) \oplus H_n(L_2) \approx H_n(L_1 \cup L_2)$. Therefore, the homology of a finite union of disjoint sets is equal to the direct sum of the individual homologies of the sets in the union.

Let $g: V \rightarrow \mathbf{R}^n$ be a continuous map where V is open and F_g is compact. Also, assume V is the finite union of sets V_j $j = 1, \dots, r$ where the fixed point sets $F^j = \{x \in V_j: g(x) = x\} = F_g \cap V_j$ are mutually disjoint. Given F^j , we may choose an open neighborhood W_j such that $F^j \subset W_j \subset V_j$. In fact, we may trim down the neighborhood W_j until we have $W_j \cap W_i = \emptyset$ for $i \neq j$. Thus, since $W = \cup W_j \subset \cup V_j = V$, we have

$$\bigoplus_{j=1,r} H_n(W_j, W_j - F^j) \approx H_n(W, W - F_g),$$

where the isomorphism is given by $(x_1, x_2, \dots, x_r) \mapsto \sum_{j=1,r} x_j$. Also, it is not difficult to see that the fundamental class of $\bigoplus_j H_n(W_j, W_j - F^j)$ is of the form

$(o_{F^1}, o_{F^2}, \dots, o_{F^r})$, where o_{F^j} is the fundamental class in $H_n(W_j, W_j - F^j)$. Hence, the fundamental class in $H_n(W, W - F_g)$ is given by $\sum_j o_{F^j}$.

In the composition

$$\oplus_j H_n(W_j, W_j - F^j) \approx H_n(W, W - F_g) \xrightarrow{(i-g)_*} H_n(\mathbb{R}^n, \mathbb{R}^n - 0),$$

the fundamental class in $\oplus_j H_n(W_j, W_j - F^j)$ is mapped to I_g . However, $(o_{F^1}, o_{F^2}, \dots, o_{F^r})$ is sent to $\sum_j o_{F^j}$ which maps to $(i-g)_*(\sum_j o_{F^j}) = \sum_j I_{g|_{W_j}}$. Proposition 3.3.1 tells us that since $W_j \subset V_j$, $I_{g|_{W_j}} = I_{g|_{V_j}}$, and we have established the following proposition.

Proposition : 3.3.3 Let $V \subset \mathbb{R}^n$ be an open set and $g: V \rightarrow \mathbb{R}^n$ a continuous map. Assume V is represented as a finite union of open sets V_j $j = 1, \dots, r$ such that the fixed point sets $F^j = F_g \cap V_j$ are mutually disjoint. Then $I_g = \sum_j I_{g|_{V_j}}$.

Now we are able to restrict our attention to a neighborhood around the fixed point set. If the fixed point set is not connected, then we may choose mutually disjoint neighborhoods of each component of the fixed point set and compute the index in each neighborhood. Since the fixed point set is assumed to be compact there will be a finite number of components. We then need only add up these indices to get the fixed point index for g .

Now we establish a property of the fixed point index which is related to products of spaces. In section 3.6 we will see how the fixed point index of map in rational homology is related to the Lefschetz number of that map. Since rational homology is easy to compute for products of spaces, it is useful to be able to compute the

fixed point index of the product of two spaces.

Before we go on, we must determine what is meant by the product of topological pairs. Given two pairs (X, A) and (Y, B) we intuitively think of the product $(X, A) \times (Y, B)$ as the union of chains in $X \times Y$ that are completely contained in the complement of $A \times B$. That is, we would think that $(X, A) \times (Y, B) = (X \times Y, A \times B)$. However, as we see in the following example, this definition will not suffice.

Example : 3.3.1 Consider the two spaces X and Y defined by

$X = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq 1; v = 0\}$ and $Y = \{(u, v) \in \mathbb{R}^2 : u = 0; 0 \leq v \leq 1\}$ and let A and B be the subsets defined by $A = \{(1, 0)\} \subset X$ and $B = \{(0, 1)\} \subset Y$.

Then X can be triangulated as a single 1-simplex along the x -axis with the subspace A being the 0-simplex making up the right endpoint. Similarly, the space Y can be triangulated as a single 1-simplex along the y -axis with B as the topmost endpoint. The product $X \times Y$ is a square which can be triangulated as two 2-simplexes, inheriting the subcomplex structure of A and B , as shown below.

Since $A \times B$ is just the vertex making up the top right corner of this square, the pair $(X \times Y, A \times B)$ is the union of all simplexes in the square which do not include

that vertex. This gives the triangulation shown below.

Now let us construct $(X, A) \times (Y, B)$. Since (X, A) is the union of simplexes which do not have A as a vertex, we are left with the single vertex at the origin. Similarly for (Y, B) . Therefore $(X, A) \times (Y, B)$ is not the space shown above, but just the vertex making up the origin. The problem with our fallacious definition is that we did not “cut away” enough of the product space. In particular, we need to consider those points formed from the product of A with Y and the product of B with X . Consider the pair $(X \times Y, X \times B \cup A \times Y)$. Since $A \times Y$ and $X \times B$ are 1-simplexes which make up the right and top edges of the square, respectively, the pair $(X \times Y, X \times B \cup A \times Y)$ is the union of simplexes in $X \times Y$ with no vertexes contained in these edges. Some investigation shows that this leaves the single vertex at the origin which we’ve shown to be $(X, A) \times (Y, B)$.

Definition : 3.3.1 Given two topological pairs (X, A) and (Y, B) we define the *product* $(X, A) \times (Y, B)$ to be the pair $(X \times Y, X \times B \cup Y \times A)$ [Spanier, 234].

Let $g: V \rightarrow \mathbb{R}^n$ and $g': V' \rightarrow \mathbb{R}^{n'}$ be continuous maps such that $V \subset \mathbb{R}^n$ and

$V' \subset \mathbf{R}^{n'}$ are open sets. Also assume that F_g and $F_{g'}$, the fixed point sets of g and g' , respectively, are compact. Let $x \in F_g$ and $x' \in F_{g'}$. Then $g(x) = x$ and $g'(x') = x'$ and the map $g \times g': V \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'}$ sends (x, x') to $(g(x), g'(x')) = (x, x')$. Similarly, if $(g \times g')(y, y') = (y, y')$, then $g(y) = y$ and $g'(y') = y'$ and we have $y \in F_g$ and $y' \in F_{g'}$. Thus, it is clear that $F_{g \times g'} = F_g \times F_{g'}$ which implies that if F_g and $F_{g'}$ are compact then $F_{g \times g'}$ is compact.

Since $g \times g'$ maps $V \times V'$ to $\mathbf{R}^n \times \mathbf{R}^{n'}$ we are interested in the map

$$(i \times i' - g \times g'): (V \times V', V \times V' - F_g \times F_{g'}) \rightarrow (\mathbf{R}^n \times \mathbf{R}^{n'}, (\mathbf{R}^n \times \mathbf{R}^{n'}) - 0)$$

Let us look at the product of the pairs $(V, V - F_g)$ and $(V', V' - F_{g'})$. By definition,

$$(V, V - F_g) \times (V', V' - F_{g'}) = (V \times V', (V - F_g) \times V' \cup V \times (V' - F_{g'})).$$

However, it is not difficult to show that

$$[(V - F_g) \times V'] \cup [V \times (V' - F_{g'})] = V \times V' - F_g \times F_{g'}$$

and we have

$$(V, V - F_g) \times (V', V' - F_{g'}) = (V \times V', V \times V' - F_g \times F_{g'})$$

which gives

$$H_{n+n'}[(V, V - F_g) \times (V', V' - F_{g'})] = H_{n+n'}(V \times V', V \times V' - F_g \times F_{g'}).$$

In fact, the fundamental class $o_{F_g \times F_{g'}} \in H_{n+n'}(V \times V', V \times V' - F_g \times F_{g'})$ is equal to $o_{F_g} \otimes o_{F_{g'}} \in H_{n+n'}[(V, V - F_g) \times (V', V' - F_{g'})]$. Also, for

$(x, y) \in (V \times V', V \times V' - F_g \times F_{g'}) = (V, V - F_g) \times (V', V' - F_{g'})$ we have

$$[(i \times i') - (g \times g')](x, y) = (i \times i')(x, y) - (g \times g')(x, y) = (x, y) - (g(x), g'(y)) =$$

$$(x - g(x), y - g'(y)) = [(i - g)(x), (i' - g')(y)] = [(i - g) \times (i' - g')](x, y).$$

Therefore the maps

$$[(i \times i') - (g \times g')]: (V \times V', V \times V' - F_g \times F_{g'}) \rightarrow (\mathbf{R}^n \times \mathbf{R}^{n'}, (\mathbf{R}^n \times \mathbf{R}^{n'}) - 0)$$

and

$$(i - g) \times (i' - g'): [(V, V - F_g) \times (V', V' - F_{g'})] \rightarrow [\mathbf{R}^n \times \mathbf{R}^{n'}, (\mathbf{R}^n \times \mathbf{R}^{n'}) - 0]$$

are identical. This leads to

$$I_{g \times g'} = (i \times i' - g \times g')_* o_{F_g \times F_{g'}} = [(i - g) \times (i' - g')]_* (o_{F_g} \times o_{F_{g'}})$$

$$= [(i - g)_* o_{F_g}] \otimes [(i' - g')_* o_{F_{g'}}] = I_g \cdot I_{g'}$$

establishing the following proposition.

Proposition : 3.3.4 If $g: V \rightarrow \mathbf{R}^n$, $g': V' \rightarrow \mathbf{R}^{n'}$ are maps with V and V' open and having compact fixed point sets F_g and $F_{g'}$, respectively, then the fixed point index of $g \times g': V \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'} = \mathbf{R}^{n+n'}$ is defined and $I_{g \times g'} = I_g \cdot I_{g'}$ [Dold-2, 2].

As we saw in section 1.5, one of the most useful properties of the homology functor is that it is homotopy invariant. Since the fixed point index is defined on

homology we suspect that the fixed point index is also homotopy invariant. This is indeed the case; however, we must require that the fixed points stay away from the boundary of V .

Proposition : 3.3.5 Let $V \subset \mathbb{R}^n$ be open, $g: V \rightarrow \mathbb{R}^n$ be a continuous map with compact fixed point set, and $G: V \times I \rightarrow \mathbb{R}^n$ be a homotopy such that $G(x, 0) = g(x)$ for all $x \in V$. Denote $G(x, t)$ by $g_t: V \rightarrow \mathbb{R}^n$. Also, assume $K = \{x \in V: G(x, t) = x \text{ for some } t\} = \cup F_{g_t}$ is compact. Then $I_{g_0} = I_{g_1}$ [Dold-2, 2].

Since for each $t \in I$, $F_{g_t} \subset K \subset V$, we can take advantage of Proposition 3.3.1. Therefore, $I_{g_t} = (i - g_t)_*(o_K)$ for all $t \in I$. But since G is a homotopy, the maps g_0 and g_1 are homotopic, giving $(i - g_0)_* = (i - g_1)_*$. Thus, $I_{g_0} = (i - g_0)_*(o_K) = (i - g_1)_*(o_K) = I_{g_1}$. The type of homotopy described in Proposition 3.3.5 is called a *deformation* of $g = g_0: V \rightarrow \mathbb{R}^n$.

Now we examine how the fixed point index behaves with regard to compositions of maps. In particular, is the fixed point index of the composition gf the same as that of the composition fg ?

Let $f: U \rightarrow \mathbb{R}^{n'}$ and $g: U' \rightarrow \mathbb{R}^n$ be continuous maps such that $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^{n'}$ are open sets and the sets $f^{-1}(U')$ and $g^{-1}(U)$ are nonempty. Then $V = f^{-1}(U') \subset U$ is open in $\mathbb{R}^n$ and $V' = g^{-1}(U) \subset U'$ is open in $\mathbb{R}^{n'}$. Define maps $fg: V' \rightarrow \mathbb{R}^{n'}$ and $gf: V \rightarrow \mathbb{R}^n$ by composition. Now consider the fixed point sets F_{gf} and F_{fg} . If $v \in F_{gf}$, then $v \in V = f^{-1}(U')$. Therefore, $f(v) = v'$ for

some $v' \in U'$. Define a map $h: F_{gf} \rightarrow F_{fg}$ by $v \mapsto f(v)$ for all $v \in F_{gf}$. To show this is well defined, we must show $f(v) \in F_{fg}$ for all $v \in F_{gf}$. If $v \in F_{gf}$, then $(gf)(v) = g(f(v)) = v$. Therefore, $(fg)(f(v)) = f[g(f(v))] = f(v)$. Similarly, it can be shown that the map $h': F_{fg} \rightarrow F_{gf}$ defined by $v' \mapsto g(v')$ is well defined.

Let $v \in F_{gf}$. Then

$$(h'h)(v) = h'[h(v)] = h'[f(v)] = g[f(v)] = (gf)(v) = v.$$

Similarly, $(hh')(v') = v'$ for all $v' \in F_{fg}$. Thus, h defines a homeomorphism $F_{fg} \approx F_{gf}$.

If the fixed point sets F_{fg} and F_{gf} are compact, then I_{gf} and I_{fg} each exist. Consider the map $\gamma: V \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'}$ defined by $(x, y) \mapsto (g(y), f(x))$. We will show that $I_{gf} = I_\gamma = I_{fg}$. First we must establish the existence of I_γ . Let $h: F_{gf} \rightarrow F_\gamma$ be defined by $x \mapsto (x, f(x))$. Since f is a well defined map and $\gamma(x, f(x)) = (gf(x), f(x)) = (x, f(x))$ we have $h(x) \in F_\gamma$ and so h is also well defined. If $(x, y) \in F_\gamma$, then $\gamma(x, y) = (g(y), f(x)) = (x, y)$, giving $gf(x) = g(y) = x$. Thus, the map $h': F_\gamma \rightarrow F_{gf}$ defined by $(x, y) \mapsto x$ is well defined. Since $f(x) = y$ for all $(x, y) \in F_\gamma$ we have $hh'(x, y) = h(x) = (x, f(x)) = (x, y)$. Also, $h'h(x) = h'(x, f(x)) = x$, which means h and h' define a homeomorphism $F_{gf} \approx F_\gamma$. Since F_{gf} is assumed to be compact we have F_γ is compact and so I_γ exists.

Now deform γ by the collection $\{\gamma_t: V \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'}\}$ defined by

$$\gamma_t(x, y) = [g(y), tf(g(y)) + (1-t)f(x)].$$

Clearly, this is a collection of continuous functions on $V \times V' \times \mathbf{I}$ and since $\gamma = \gamma_0$, it is a deformation of γ . We need only show $\cup F_{\gamma_t}$ is compact to satisfy the conditions of Proposition 3.3.5. To do this we show that $\cup F_{\gamma_t} = F_\gamma \approx F_{fg}$, which is assumed to be compact. Let $(x, y) \in F_\gamma$. Then $f(x) = y$ and $g(y) = g(f(x)) = x$. Therefore,

$$\gamma_t(x, y) = [g(y), tf(g(y)) + (1-t)f(x)] = [x, tf(x) + (1-t)f(x)] = [x, f(x)] = (x, y).$$

Thus, $F_\gamma \subset F_{\gamma_t}$. On the other hand, if $(x, y) \in F_{\gamma_t}$, then $g(y) = x$ which gives

$$y = tf(g(y)) + (1-t)f(x) = tf(x) + (1-t)f(x) = f(x).$$

So $\gamma(x, y) = (g(y), f(x)) = (x, y)$ and we have $F_\gamma = F_{\gamma_t}$ for all $t \in \mathbf{I}$. Therefore, $\cup F_{\gamma_t} = \cup F_\gamma = F_\gamma$ is compact and from Proposition 3.3.5 we get $I_\gamma = I_{\gamma_0} = I_{\gamma_1}$.

Now we expand the domain of γ_1 to form a new map $\delta: \mathbf{R}^n \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'}$ defined by $\delta(x, y) = [g(y), f(g(y))]$. It is not difficult to show that F_δ is compact and since $F_{\gamma_1} \subset F_\delta \subset V \times V' \subset \mathbf{R}^n \times \mathbf{R}^{n'}$ we have $I_\delta = I_{\gamma_1} = I_\gamma$ by Proposition 3.3.1.

Consider the deformation of δ defined by $\{\delta_t: \mathbf{R}^n \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'}\}$ where $\delta_t(x, y) = [(1-t)g(y), f(g(y))]$. One can show that F_{δ_t} is compact by considering the function $h: [0, 1] \times V' \rightarrow \mathbf{R}^n \times \mathbf{R}^{n'}$ defined by $(t, y) \mapsto [(1-t)g(y), y]$. Let $(x', y') = h(t, y)$ for some $(t, y) \in [0, 1] \times F_{fg}$. Then

$$y' = f(g(y)) = y$$

and

$$(x', y') = [(1-t)g(y), y] = [(1-t)g(y), f(g(y))] = \delta_t(x', y').$$

This means that the image of $[0, 1] \times F_{fg}$ under h is contained in F_{δ_t} . It is clear that $F_{\delta_t} \subset h([0, 1] \times F_{fg})$ since $(x, y) \in F_{\delta_t}$ implies $f(g(y)) = y$ and $(1-t)g(y) = x$. Therefore, F_{δ_t} is the image of a compact set $[0, 1] \times F_{fg}$ under the continuous map h , and so is itself compact.

Again, we apply Proposition 3.3.5 to get $I_\gamma = I_{\gamma_1} = I_\delta = I_{\delta_1}$. However, $\delta_1 = c_0 \times fg$ where c_0 is the constant map $c_0: \mathbf{R}^n \rightarrow 0$. Then by Proposition 3.3.2 we have $I_{c_0} = 1$ and applying Proposition 3.3.4 gives $I_\gamma = I_{\delta_1} = I_{c_0} \cdot I_{fg} = I_{fg}$.

A similar argument using the deformations $\{\gamma'_t\}$ of γ defined by $\gamma'_t(x, y) = [tg(f(x)) + (1-t)g(y), f(x)]$ and $\{\delta'_t\}$ of δ defined by $\delta'_t(x, y) = [g(f(x)), (1-t)f(x)]$ will show that $I_{gf} = I_\gamma = I_{fg}$. Thus we have established the following proposition.

Proposition : 3.3.6 If $U \subset \mathbf{R}^n$, $U' \subset \mathbf{R}^{n'}$ are open sets and $f: U \rightarrow \mathbf{R}^{n'}$, $g: U' \rightarrow \mathbf{R}^n$ are continuous maps, then the two composites

$$gf: V = f^{-1}(U') \rightarrow \mathbf{R}^n$$

$$fg: V' = g^{-1}(U) \rightarrow \mathbf{R}^{n'}$$

have homeomorphic fixed point sets, $F_{gf} \approx F_{fg}$. In particular, I_{fg} is defined if and only if I_{gf} is defined. In this case, $I_{gf} = I_{fg}$ [Dold-2, 2].

§3.4 The Lefschetz Number

A useful theory concerning fixed points of maps associates with each map an integer, called the *Lefschetz number*, and determines the existence of fixed points based on the value of this integer. In this section, we introduce the Lefschetz number by its traditional definition, and then give an alternate definition based on the inverse image of a certain module homomorphism. We would like to use this alternate definition in the sections that follow, so we will show that the two definitions are equivalent.

Given a map $h: |K| \rightarrow |K|$, the Lefschetz number of h is based on what is called the *trace* of a homomorphism.

Definition : 3.4.1 Let G be a free abelian group with basis $e_1, \dots, e_n$ and $\phi: G \rightarrow G$ a homomorphism. The *trace* of ϕ is given by $\text{tr} \phi = \text{tr} A$, where A is the matrix of ϕ relative to the given basis [Munkres, 122].

Recall that the trace of a square matrix A , $\text{tr} A$, is the sum of the diagonal elements of A . Using facts about $\text{tr} A$ from linear algebra, one can show that $\text{tr} \phi$ is independent of the choice of basis for G .

Let K be a complex and $\phi = \{\phi_p: C_p(K) \rightarrow C_p(K)\}$ be a chain map. Since $C_p(K)$ is a free abelian group and ϕ_p is a homomorphism, we may define the trace of ϕ_p , written $\text{tr}(\phi, C_p(K))$. Although we cannot be sure that $H_p(K)$ is free, we can define the trace of $\phi_*: H_p(K) \rightarrow H_p(K)$ by considering only the free part of $H_p(K)$. If $T_p(K)$ is the torsion subgroup of $H_p(K)$, then the quotient group $H_p(K) / T_p(K)$

is free, and ϕ_* will give a homomorphism $\phi'_*: H_p(K)/T_p(K) \rightarrow H_p(K)/T_p(K)$. Thus, we can define the trace of ϕ_* , written $tr(\phi_*, H_p(K))$, as the trace of ϕ'_* . The following theorem, called the *Hopf trace theorem*, gives a very useful relationship between $tr(\phi, C_p(K))$ and $tr(\phi_*, H_p(K))$.

Theorem : 3.4.1 Let K be a finite complex and let $\phi = \{\phi_p: C_p(K) \rightarrow C_p(K)\}$ be a chain map. Then

$$\sum_p (-1)^p tr(\phi, C_p(K)) = \sum_p (-1)^p tr(\phi_*, H_p(K))$$

[Munkres, 122].

The reason this theorem is so valuable will become clear once we see how the Lefschetz number is defined.

Definition : 3.4.2 Let K be a finite complex and $h: |K| \rightarrow |K|$ a continuous map. The number $\Lambda(h) = \sum_p (-1)^p tr(h_*, H_p(K))$ is called the *Lefschetz number* of h [Munkres, 125].

If $h: |K| \rightarrow |K|$ is a continuous map, then h induces a chain map $h_\#: C_p(K) \rightarrow C_p(K)$. In addition, $h_\#$ induces a homomorphism $h_*: H_p(K) \rightarrow H_p(K)$. Since the Hopf trace theorem tells us that the alternating sum of $tr(h_\#, C_p(K))$ is equal to the alternating sum of $tr(h_*, H_p(K))$, the Lefschetz number can be computed at either the chain level or the homology level.

The importance of the Lefschetz number is in determining whether a map $h: |K| \rightarrow |K|$ has any fixed points. The following theorem, called the *Lefschetz*

fixed point theorem, gives the relationship between the Lefschetz number of a map and the existence of points fixed by that map.

Theorem : 3.4.2 Let K be a finite complex and let $h: |K| \rightarrow |K|$ be a continuous map. If $\Lambda(h) \neq 0$, then h has a fixed point [Munkres, 125].

Notice that Theorem 3.4.2 does not state an equivalence. That is, there may be maps with fixed points but having Lefschetz number equal to zero. Theorem 3.4.2, and the Lefschetz fixed point theory in general, have been very valuable in establishing properties of maps on certain spaces such as spheres.

In the sections that follow, we will require an alternate definition of the Lefschetz number. However, we would like to take advantage of the Lefschetz fixed point theorem, so we will show that the two definitions are equivalent. First we define a homomorphism of graded R -modules which will be instrumental in the definition of the Lefschetz number.

Definition : 3.4.3 Let R be a commutative ring with identity and let $M = \{M_i\}_{i \in \mathbb{Z}}$ be a graded R -module and $M^* = \{Hom(M_{-i}, R)\}_{i \in \mathbb{Z}}$ its dual. For every graded R -module N define

$$\Theta = \Theta_{MN}: M^* \otimes N \rightarrow Hom(M, N)$$

by

$$[\Theta(\phi \otimes n)](m) = (-1)^{|\phi||n|} \phi(m) \cdot n$$

where $\phi \in M^*$, $m \in M$, $n \in N$ and $||$ denotes dimension.

Recall that for graded R -modules, $A = \{A_i\}_{i \in \mathbb{Z}}$ and $B = \{B_i\}_{i \in \mathbb{Z}}$, the tensor product $A \otimes B$ is the graded R -module defined by $(A \otimes B)_p = \sum_{i+j=p} A_i \otimes B_j$. Given any $\phi \otimes n \in M^* \otimes N$, $\Theta(\phi \otimes n)$ is a homomorphism from M into N , and one can show that Θ is a well defined homomorphism. Let $a = \sum_{i=1}^s (\phi_i \otimes n_i) \in M^* \otimes N$. Then $\Theta(a) = \sum \Theta(\phi_i \otimes n_i)$ and so $[\Theta(a)](m) = \sum [\Theta(\phi_i \otimes n_i)](m) = \sum (-1)^{|\phi_i||n_i|} \phi_i(m) n_i$. Define a map $\alpha: M \rightarrow R^s$, where R^s is a direct sum of s copies of R , by

$$m \mapsto ((-1)^{|\phi_1||n_1|} \phi_1(m), \dots, (-1)^{|\phi_s||n_s|} \phi_s(m)).$$

Also define a map $\psi: R^s \rightarrow N$ by $(r_1, \dots, r_s) \mapsto \sum r_i n_i$. It is easy to see that $\Theta(a) = \psi \circ \alpha$, and since R^s is free and has a finite base, the image of a under Θ factors through a finitely generated free module.

On other hand, let β be any homomorphism which is the composition of module homomorphisms $\beta: M \xrightarrow{\mu} F \xrightarrow{\nu} N$, where F is a finitely generated free module. If F has rank s , then $F = R^s$, and $\mu: M \rightarrow F$ is given by $\mu = (\mu_1, \dots, \mu_s)$ where $\mu_i: M \rightarrow R$. Similarly, $\nu = \sum \nu_i$ where $\nu_i: R \rightarrow N$. But since R is a ring with identity, $\nu_i(r) = r n_i$, where $\nu_i(1) = n_i$. Now, let $a = \sum (-1)^{|\mu_i||n_i|} (\mu_i \otimes n_i)$, where μ_i and n_i are given as above. Then

$$\begin{aligned} [\Theta(a)](m) &= \sum (-1)^{|\mu_i||n_i|} [\Theta(\mu_i \otimes n_i)](m) \\ &= \sum (-1)^{|\mu_i||n_i|} (-1)^{|\mu_i||n_i|} \mu_i(m) n_i = \sum \mu_i(m) n_i \\ &= (\nu \circ \mu)(m) = \beta(m). \end{aligned}$$

Thus, we have proved the first part of the following proposition.

Proposition : 3.4.1 The image of Θ consists of those homomorphisms β which factor through a finitely generated free module, i.e. composites $\beta: M \rightarrow F \rightarrow N$, where F has a finite base. These homomorphisms are called of *finite rank*.

If N is free, then Θ is monomorphic, that is, $\Theta: M^* \otimes N \approx \text{Hom}_f(M, N)$, where Hom_f denotes the homomorphisms of finite rank [Dold-2, 4].

Let N be a free module and $N^* = \{\text{Hom}(N_{-i}, R)\}_{i \in \mathbb{Z}}$ its dual. Then for any $\phi \in N_i^*$, ϕ is a map from N_{-i} into R . However, ϕ can be extended to a map $\phi: N \rightarrow R$ by the assignment $n \mapsto 0$ for all $n \notin N_{-i}$. Therefore, we may define an evaluation map $\epsilon: N^* \otimes N \rightarrow R$ by $\epsilon(\phi \otimes n) = \phi(n) \in R$. If $\beta: N \rightarrow N$ is of finite rank, then β is in the image of Θ . But since N is free, Θ is an isomorphism and the assignment $\beta \mapsto \Theta^{-1}(\beta)$ is well defined. The evaluation of the inverse image of β is defined to be the Lefschetz number of β .

Definition : 3.4.3 Let N be a graded R -module and N^* its dual. Also let $\epsilon: N^* \otimes N \rightarrow R$ denote the evaluation map, $\epsilon(\phi \otimes n) = \phi(n)$. If N is free and $\beta: N \rightarrow N$ is an endomorphism of finite rank, then $\Theta^{-1}(\beta) \in N^* \otimes N$, and $\Lambda(\beta) = \epsilon[\Theta^{-1}(\beta)] \in R$ is called the *Lefschetz number* of β .

If $f: Y \rightarrow Y$ is a continuous map, then $f_*: H_*(Y; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$ is a module homomorphism. From Example 2.4.1 we know that if Y is a finite complex then $H_*(Y; \mathbb{Q})$ is a finite dimensional free module over $\mathbb{Q}$. This means that f_* is of finite rank, and we may define the Lefschetz number of f .

Definition : 3.4.4 Let Y be a finite complex and $f: Y \rightarrow Y$ a continuous function.

Then the *Lefschetz number* of f , written $\Lambda(f)$, is defined by

$$\Lambda(f) = e[\Theta^{-1}(f_*)]$$

where $f_*: H_*(Y; \mathbb{Q}) \rightarrow H_*(Y; \mathbb{Q})$.

Since f_* is a free module homomorphism, we do not need to mod out by the torsion part of $H_p(Y; \mathbb{Q})$ before computing the trace of f_* . That is, $\text{tr}(f_*, H_p(Y; \mathbb{Q}))$ is exactly the trace of $f_p: H_p(Y; \mathbb{Q}) \rightarrow H_p(Y; \mathbb{Q})$. Therefore, to prove the equivalence of the two definitions of the Lefschetz number for rational homology we need only consider a homomorphism $\beta: V \rightarrow V$, where V is a finite dimensional free module over a field F ; i.e. a finite dimensional vector space. The restriction to rational homology is justified since we will be using rational homology in the sections that follow. Let $V = \{V_i\}_{i \in \mathbb{Z}}$ be a graded finite dimensional vector space over a field F , and let $\{v_j\}_{j=1,r}$ be a basis for the vector space V_i . Also, let $\beta: V \rightarrow V$ be a graded F -module homomorphism. Then $\beta_i: V_i \rightarrow V_i$ is a linear transformation with matrix representation, relative to the given basis, given by $A^i = (a_{jl}^i)$, where $\beta_i(v_j) = \sum_{l=1,r} a_{jl}^i v_l$. Therefore, the traditional definition of the Lefschetz number gives

$$\Lambda(\beta) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}(\beta_i) = \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j=1,r} a_{jj}^i.$$

If $\phi \in V^{-i} = \text{Hom}(V_i, F)$, then we may extend ϕ to a homomorphism $\phi: V \rightarrow F$ by the assignment $v \mapsto 0$ for all $v \notin V_i$. Let $\phi \otimes v \in V^* \otimes V$ where $\phi \in V^{-i}$, $v \in V_j$, and $i \neq j$. Then the evaluation map gives $e(\phi \otimes v) = \phi(v) = 0$. Therefore, $e: V^* \otimes V \rightarrow F$ is the zero map except on the elements of the form $\phi \otimes v$ where

$\phi \in \text{Hom}(V_i, F)$ and $v \in V_i$. By the definition of V^* , this means that $\phi \otimes v$ is of dimension zero. Hence, we may restrict our attention of the evaluation map at dimension zero.

Since β is clearly of finite rank, $\Theta^{-1}(\beta) \in V^* \otimes V$. We are only concerned with the portion of $\Theta^{-1}(\beta)$ which is a finite sum of elements of the form $\phi \otimes v$ where $\phi \in V^{-i}$ and $v \in V_i$. Let us concentrate on the contributions to this sum from a fixed dimension i . If $\{v_j\}_{j=1,r}$ is the basis for V_i chosen previously, then the contribution from dimension i will take the form $\sum(\phi_j \otimes v_j)$. By the definition of Θ ,

$$\begin{aligned} \Theta[\sum(\phi_j \otimes v_j)](v) &= \sum[\Theta(\phi_j \otimes v_j)](v) \\ &= \sum(-1)^{|\phi_j||v_j|} \phi_j(v) v_j = \sum(-1)^{(-i)(i)} \phi_j(v) v_j = \sum(-1)^i \phi_j(v) v_j = \beta_i(v) \end{aligned}$$

for all $v \in V_i$. In particular, $\Theta[\sum(\phi_j \otimes v_j)](v_j) = \sum(-1)^i \phi_j(v_j) v_j = \beta_i(v_j) = \sum_{l=1,r} a_{jl}^i v_l$. But each element of V_i has a unique representation in terms of the basis elements $\{v_j\}$. Therefore, the coefficient of v_j in $\Theta[\sum(\phi_j \otimes v_j)](v_j)$ is the same as the coefficient of v_j in $\beta_i(v_j)$. That is, $(-1)^i \phi_j(v_j) = a_{jj}^i$, or equivalently, $\phi_j(v_j) = (-1)^i a_{jj}^i$.

If we operate the evaluation map on the portion of $\Theta^{-1}(\beta)$ contributed from dimension i , we get

$$e[\sum(\phi_j \otimes v_j)] = \sum e(\phi_j \otimes v_j) = \sum \phi_j(v_j) = \sum(-1)^i a_{jj}^i = (-1)^i \sum a_{jj}^i = (-1)^i \text{tr} \beta_i.$$

Adding up the contributions to $\Theta^{-1}(\beta)$ for each dimension gives $e[\Theta^{-1}(\beta)] = \sum(-1)^i \text{tr}(\beta_i)$, which shows that the two definitions give the same value.

§3.5 Euclidean Neighborhood Retracts

In the next section we state and prove the Dold–Lefschetz fixed point theorem. However, since this theorem applies to Euclidean neighborhood retracts we must first define the notion of a retract.

Definition : 3.5.1 A subspace A of a space X is a *retract* of X if there is a continuous function $r: X \rightarrow A$ such that $r(a) = a$ for each $a \in A$. The function r is called a *retraction* [Schurle, 90].

Intuitively, a retraction of X is a function which “smashes” all of X onto a subspace A . This definition is a bit restrictive because it requires that a space always retract to some subspace. Thus, any Euclidean neighborhood in $\mathbf{R}^n$ must retract to some subspace in $\mathbf{R}^n$. To introduce generality, we extend the notion of a retract so that an arbitrary topological space may be considered a retract of a Euclidean neighborhood. We take advantage of the idea that $r(a) = a$ for all $a \in A$ means that the inclusion map $i: A \rightarrow X$ is a right inverse of $r: X \rightarrow A$.

Definition : 3.5.2 A topological space Y is called a *Euclidean neighborhood retract* if there exists an open set $O \subset \mathbf{R}^n$ and mappings $r: O \rightarrow Y$ and $i: Y \rightarrow O$ such that $ri = 1_Y$ [Dold–2, 3].

Let Y be a Euclidean neighborhood retract and $U \subset Y$, where U is open. Also, let $g: U \rightarrow Y$ be a continuous map such that the fixed point set F_g is compact. Since the definition of the fixed point index given in section 3.2 applies only to maps into $\mathbf{R}^n$, we cannot use this definition directly. However, since Y is a Euclidean

neighborhood retract there exist maps $r: O \rightarrow Y$ and $i: Y \rightarrow O \subset \mathbb{R}^n$ such that $ri = 1_Y$. Let V be the inverse image of U under r ; that is, $V = r^{-1}(U)$. Clearly, $V \subset O \subset \mathbb{R}^n$ is open and the composition igr is a map from V into $\mathbb{R}^n$. Thus, to satisfy Definition 3.2.1 we need only show F_{igr} is compact. To show this, we show that F_{igr} is the image of the compact set F_g under the continuous map $i: Y \rightarrow O$.

If $x \in i(F_g)$, then $x = i(u)$ for some $u \in U$, such that $g(u) = u$. So $x = ig(u)$ and since $ri = 1_Y$, we have $r(x) = ri(u) = u$. Therefore $igr(x) = ig(u) = x$ and so $i(F_g) \subset F_{igr}$. Now, let $x \in F_{igr}$. Then

$$x = (igr)(x)$$

and so

$$gr(x) = (ri)[gr(x)] = r[(igr)(x)] = r(x).$$

Therefore, there exists $u \in U$, namely $u = r(x)$, such that $g(u) = u$ and $x = i(u)$.

This means $x \in i(F_g)$ and we have $F_{igr} = i(F_g)$.

Since F_{igr} is compact we can apply Definition 3.2.1 to get a definition of the fixed point index for an arbitrary Euclidean neighborhood retract.

Definition: 3.5.3 Let Y be a Euclidean neighborhood retract and $U \subset Y$ an open subset. Let $g: U \rightarrow Y$ be a continuous map and assume the fixed point set of g , F_g , is compact. Let $i: Y \rightarrow O$, $r: O \rightarrow Y$ be a Euclidean neighborhood retraction. Then the *fixed point index* of g , I_g , is defined to be the fixed point index of the composition $igr: r^{-1}(U) \rightarrow Y$ [Dold-2, 3].

It is easy to see that the choice of retraction is insignificant since for any other Euclidean neighborhood retraction $i': Y \rightarrow O'$, $r': O' \rightarrow Y$ we have

$$(ir')(i'gr) = i(r'i')gr = igr$$

and

$$(i'gr)(ir') = i'g(ri)r' = i'gr'.$$

Since $ir': O' \rightarrow \mathbf{R}^n$ and $i'gr: r^{-1}(V) \rightarrow \mathbf{R}^{n'}$ and O' and $r^{-1}(V)$ are open, we can apply Proposition 3.3.6. Therefore

$$I_{igr} = I_{(ir')(i'gr)} = I_{(i'gr)(ir')} = I_{i'gr'}.$$

§3.6 The Dold-Lefschetz Fixed Point Theorem

Finally, we are ready to introduce the Dold-Lefschetz fixed point theorem. As we have seen, the Lefschetz number of a map h , $\Lambda(h)$, gives information about the existence of fixed points of h . Unfortunately, it can often be laborious to compute the Lefschetz number. The Dold-Lefschetz fixed point theorem guarantees that the fixed point index, which may be easier to compute in some cases, agrees with the Lefschetz number. Thus, we may apply the results of the Lefschetz fixed point theorem described in section 3.4 to the fixed point index.

Theorem : 3.6.1 Let Y be a Euclidean neighborhood retract, and $f: Y \rightarrow Y$ a mapping whose image $f(Y)$ is contained in some compact set $K \subset Y$. Then the fixed point index of f and the Lefschetz number of $(f|_K)_*: H(K; \mathbf{Q}) \rightarrow H(K; \mathbf{Q})$

are both defined and they agree. That is,

$$I_f = \Lambda(f|_K)_*$$

[Dold-2, 5].

First we justify the use of rational homology in the case of the fixed point index. Previously, we used integral homology in defining and describing the fixed point index. Recall that the fixed point index is defined as the image of 1 under a composition of maps in homology. Therefore, we show the connection between homomorphisms in integral homology to homomorphisms in rational homology. If given a map between homology groups, $\phi_*: H_*(X; \mathbb{Z}) \rightarrow H_*(Y; \mathbb{Z})$, we can take tensor products to get a map $\phi_* \otimes id: H_*(X; \mathbb{Z}) \otimes \mathbb{Q} \rightarrow H_*(Y; \mathbb{Z}) \otimes \mathbb{Q}$. But by the universal coefficient theorem, $H_*(X; \mathbb{Z}) \otimes \mathbb{Q} \approx H_*(X; \mathbb{Q})$ and $H_*(Y; \mathbb{Z}) \otimes \mathbb{Q} \approx H_*(Y; \mathbb{Q})$. Thus we get the diagram.

$$\begin{array}{ccc} H_*(X; \mathbb{Z}) & \xrightarrow{\phi_*} & H_*(Y; \mathbb{Z}) \\ \downarrow id \otimes 1 & & \downarrow id \otimes 1 \\ H_*(X; \mathbb{Z}) \otimes \mathbb{Q} & \xrightarrow{\phi_* \otimes id} & H_*(Y; \mathbb{Z}) \otimes \mathbb{Q} \\ \downarrow \approx & & \downarrow \approx \\ H_*(X; \mathbb{Q}) & \xrightarrow{\bar{\phi}_*} & H_*(Y; \mathbb{Q}) \end{array}$$

Here $\bar{\phi}_*$ is the homomorphism induced by the isomorphism $H_*(X; \mathbb{Z}) \otimes \mathbb{Q} \approx H_*(X; \mathbb{Q})$. In particular, for $q \cdot a \in H_*(X; \mathbb{Q})$ where $a \in H_*(X; \mathbb{Z})$ we have $\bar{\phi}_*(q \cdot a) = q \cdot \phi_*(a)$. Let $a \in H_*(X; \mathbb{Z})$. Following the path going down then over gives $a \mapsto a \otimes 1 \mapsto 1 \cdot a \mapsto \bar{\phi}_*(1 \cdot a) = \phi_*(a)$. Moving across the top and then down we get $a \mapsto \phi_*(a) \mapsto \phi_*(a) \otimes 1 \mapsto 1 \cdot \phi_*(a) = \phi_*(a)$. Therefore, the diagram is commutative, guaranteeing that the results developed in integral homology can be

applied to rational homology. More importantly, we are ensured that it is sufficient to work in rational homology.

If the image of Y under f is contained in the compact subset $K \subset Y$ then any point in the fixed point set of f must necessarily be in K . Thus by restricting the domain of f to $f|_K$ we do not lose any information about fixed points.

Since $f: Y \rightarrow Y$ is a Euclidean neighborhood retract, there exist maps

$$j: Y \rightarrow V \text{ and } r: V \rightarrow Y$$

where V is open in $\mathbf{R}^n$ and $rj = id$. If $g = jfr$ then $g(V) = (jfr)(V) = (jf)[r(V)] \subset (jf)(Y) = j[f(Y)]$. But since $f(Y) \subset K$, $j[f(Y)] \subset j(K)$, which gives $g(V) \subset j(K)$. The map $j: Y \rightarrow V$ must be one-to-one because it has a left inverse, and since j is onto its image, we get a homeomorphism by restriction $j|_K: K \rightarrow j(K)$. Therefore $g(V) \subset j(K) \approx K$, and we can think of the image of V under g as sitting inside of K .

Theorem 3.6.1 is concerned with the restriction of f to K . We need to establish the relationship of $f|_K$ to g . Consider the set $(jfr)[(j(K))]$. Since $rj = id$ this set is just $j[f(K)]$. Again, the restriction $j|_{f(K)}: f(K) \rightarrow j[f(K)]$ gives a homeomorphism $j[f(K)] \approx f(K)$. Thus, we can restrict g to the homeomorphic copy of K sitting in V , and the image of that restriction is a space homeomorphic to $f(K)$.

At this time we would like to introduce a lemma that will be instrumental in the rest of this section. In the interest of time we will merely state the lemma, however, the reader should refer to the source for a proof.

Before we go on, we define some maps used in the lemma. Let the diagonal map $\Delta: (V, V - K) \rightarrow (V, V - K) \times V$ be defined by $\Delta(v) = (v, v)$. Define the transposition map $t: V \times K \rightarrow K \times V$ by $t(v, k) = (k, v)$. Also let the difference map $d: (V, V - K) \times K \rightarrow (\mathbf{R}^n, \mathbf{R}^n - 0)$ be given by $d(v, k) = v - k$.

Lemma : 3.6.1 Let $K \subset V \subset \mathbf{R}^n$ where V is open, K compact, and let

$o_K \in H_n(V, V - K)$ denote the fundamental class. Then the composite

$$\begin{aligned} \Phi = \Phi(K, V): H_\lambda(K) &\xrightarrow{o_K \times} H_{\lambda+n}[(V, V - K) \times K] \xrightarrow{(\Delta \times id)_*} H_\lambda[(V, V - K) \times V \times K] \\ &\xrightarrow{(id \times t)_*} H_{\lambda+n}[(V, V - K) \times K \times V] \xrightarrow{(d \times id)_*} H_{\lambda+n}[(\mathbf{R}^n, \mathbf{R}^n - 0) \times V] \\ &\xrightarrow{(o_0 \times)^{-1}} H_\lambda(V) \end{aligned}$$

coincides with the inclusion homomorphism

$$i_*: H_\lambda(K) \rightarrow H_\lambda(V)$$

[Dold-2, 5]

The maps in Lemma 3.6.1 are defined on products of spaces. Thus our use of rational homology will greatly simplify things since as we have shown in Example 2.5.1, $H_*(X \times Y; \mathbf{Q}) = H_*(X) \otimes H_*(Y)$. This gives the following maps in homology.

$$\Delta_*: H_*(V, V - K) \rightarrow H_*(V, V - K) \otimes H_*(K)$$

$$t_*: H_*(V) \otimes H_*(K) \rightarrow H_*(K) \otimes H_*(V)$$

$$d_*: H_*(V, V - K) \otimes H_*(K) \rightarrow H_*(\mathbf{R}^n, \mathbf{R}^n - 0)$$

For each $v \in H_*(V, V - K)$ define $\phi_v: H_*(K) \rightarrow \mathbb{Q}$ by $k \mapsto d_*(v \otimes k)$ where d is the difference map presented above. Clearly ϕ_v is well defined and is a homomorphism, and so $\phi_v \in \text{Hom}(H_*(K); \mathbb{Q})$. Next consider the map $\hat{d}: H_*(V, V - K) \rightarrow \text{Hom}(H_*(K), \mathbb{Q})$ defined by $v \mapsto \phi_v$ and examine the diagram below.

$$\begin{array}{ccccc}
 H_*(V, V - K) & \xrightarrow{\Delta_*} & H_*(V, V - K) \otimes H_*(V) & \xrightarrow{id \otimes g_*} & H_*(V, V - K) \otimes H_*(K) \\
 (i - g)_* \searrow & & d_* \swarrow & & \downarrow d \otimes id \\
 & & H_*(\mathbb{R}^n, \mathbb{R}^n - 0) & \xleftarrow{e} & \text{Hom}(H_*(K), \mathbb{Q}) \otimes H_*(K)
 \end{array}$$

where the evaluation map

$$e: \text{Hom}(H_*(K), \mathbb{Q}) \otimes H_*(K) \rightarrow \mathbb{Q}$$

is defined by $e(\phi \otimes n) = \phi(n)$ (see the definition of the Lefschetz number). We will show the commutativity of this diagram.

Let $v \in H_*(V, V - K)$. Then $(d_*)(id \otimes g_*)\Delta_*(v) = (d_*)(id \otimes g_*)(v \otimes v) = d_*(v \otimes g_*(v)) = v - g_*(v) = (i - g)_*(v)$. This proves the commutativity of the left triangle. To show the commutativity of the right triangle let $v \otimes k \in H_*(V, V - K) \otimes H_*(K)$. Then $e[\hat{d}[id](v \otimes k)] = e(\phi_v \otimes k) = \phi_v(k) = d_*(v \otimes k)$ and we have the desired result.

Let $a_g \in \text{Hom}(H_*(K), \mathbb{Q}) \otimes H_*(K)$ denote the image of the fundamental class o_K under the composition defined by the outside path of the diagram. That is, $a_g = \phi_{o_K} \otimes g_*(o_K)$. Clearly I_g is the image of o_K under the left diagonal path. Recall that for a map β the Lefschetz number is defined as $e[\Theta^{-1}(\beta)]$, where Θ

is the homomorphism described in Definition 3.4.4. Thus if we can show that $\Theta(a_g) = (g|_K)_*$, we can use the commutativity of the diagram to get $e[\Theta^{-1}(g|_K)_*] = \Lambda(g|_K)_* = e(a_g) = I_g$. Since we know that the image of a_g under the evaluation map e is equal to I_g , we need only show that $\Theta(a_g) = (g|_K)_*$ to prove $I_g = \Lambda(g|_K)$.

Let us examine the diagram below.

$$\begin{array}{ccccc}
H_*(V, V-K) \otimes H_*(V) \otimes H_*(K) & \xrightarrow{id \otimes t_*} & H_*(V, V-K) \otimes H_*(K) \otimes H_*(V) & \xrightarrow{d_* \otimes id} & H_*(\mathbb{R}^n, \mathbb{R}^n - 0) \otimes H_*(V) \approx H_*(V) \\
\downarrow d_* \otimes g_* \otimes id & & \downarrow d_* \otimes id \otimes g_* & & \downarrow g_* \\
H_*(K) \otimes H_*(K) \otimes H_*(K) & \xrightarrow{id \otimes t_*} & H_*(K) \otimes H_*(K) \otimes H_*(K) & \xrightarrow{e \otimes id} & \mathbb{Q} \otimes H_*(K) \approx H_*(K)
\end{array}$$

To show the commutativity of the left square, let us look at $[\hat{d} \otimes id \otimes g_*][id \otimes t_*]$. That is, the path going across and then down. Since $(id \otimes t_*)$ takes $(v \otimes v' \otimes k)$ to $(-1)^{|k||v'|}(v \otimes k \otimes v')$, we end up with $(-1)^{|k||v'|}\phi_v \otimes k \otimes g_*(v')$ after completion of the path. On the other hand, moving down the leftmost arrow gives $v \otimes v' \otimes k \mapsto \phi_v \otimes g_*(v') \otimes k$ which is then sent to $(-1)^{|k||g_*(v')|}\phi_v \otimes k \otimes g_*(v')$ by $(id \otimes t_*)$. Thus the left square is commutative.

For the right square, we examine the map $d_* \otimes id$. This map sends $v \otimes k \otimes v'$ to $d_*(v \otimes k) \otimes v'$. The definition of the tensor product of graded groups gives

$$\begin{aligned}
[H_*(\mathbb{R}^n, \mathbb{R}^n - 0) \otimes H_*(V)]_m &= \oplus_{i+j=m} H_i(\mathbb{R}^n, \mathbb{R}^n - 0) \otimes H_j(V) = \\
H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \otimes H_{m-n}(V) &= \mathbb{Q} \otimes H_{m-n}(V).
\end{aligned}$$

However, since $H_{m-n}(V)$ is just a direct sum of copies of $\mathbb{Q}$ we get $\mathbb{Q} \otimes H_{m-n}(V) \approx$

$H_{m-n}(V)$ by the map $(x \otimes y) \mapsto x \cdot y$. Thus, the top path of the rightmost square gives $v \otimes k \otimes v' \mapsto g_*[d_*(v \otimes k) \otimes v'] = g_*[d_*(v \otimes k) \cdot v']$ by the above isomorphism. Following the path going down and then across we get

$$[e \otimes id][\hat{d} \otimes id \otimes g_*](v \otimes k \otimes v') = [e \otimes id](\phi_v \otimes k \otimes g_*(v')) =$$

$$\phi_v(k) \otimes g_*(v') = d_*(v \otimes k) \otimes g_*(v') = g_*[d_*(v \otimes k) \cdot v'].$$

Since $g(V) \subset j(k) \approx K$, we can consider $g(V)$ as sitting inside K . Then we know $g_*(v') \in H_*(K)$. By definition of d we have $d_*(v \otimes k) \in \mathbf{Q}$ and again we take advantage of the isomorphism $\mathbf{Q} \otimes H_*(K) \approx H_*(K)$, giving $d_*(v \otimes k) \otimes g_*(v') = d_*(v \otimes k) \cdot g_*(v') = g_*[d_*(v \otimes k) \cdot v']$. This proves the commutativity of the right square, and since both squares commute we have commutativity of the entire diagram.

Finally we are ready to show that $\Theta(a_g) = (g|_K)_*$. To do this we show that $[\Theta(a_g)](k) = (g|_K)_*(k)$ for all $k \in H_*(K)$. Let $k \in H_*(K)$. Then $\Delta_*(o_K) \otimes k \in H_*(V, V - K) \otimes H_*(V) \otimes H_*(K)$. Here is where we use Lemma 3.6.1. Moving along the top row of the rectangular diagram on the previous page gives the last part of the composition map Φ defined in Lemma 3.6.1. Since the first part of Φ is given by

$$H_\lambda(K) \xrightarrow{o_K \times} H_{\lambda+n}[(V, V - K) \times K]$$

$$\xrightarrow{(\Delta \times id)_*} H_\lambda[(V, V - K) \times V \times K],$$

the operation of the top row of the diagram on $\Delta_* \otimes k$ is merely the completion of Φ on k . Recall that the use of rational homology guarantees that the homology group of the cross product of two spaces is just the tensor product of their individual homology groups. Lemma 3.6.1 tells us that the result of this operation is $i_*(k)$ where i_* is the inclusion map $i_*: H_*(K) \rightarrow H_*(V)$. Therefore the path along the top and down and gives $\Delta(o_K)_* \otimes k \mapsto g_* i_*(k) = (g|_K)_*(k)$ for arbitrary k .

Now, going down first, we follow $\Delta_*(o_K) \otimes k$ to $\phi_{o_K} \otimes g_*(o_K) \otimes k$. However, this is mapped to $(-1)^{|k||o_K|} \phi_{o_K} \otimes k \otimes g_*(o_K)$ by $(id \otimes t_*)$. Notice that

$$[\Theta(a_g)](k) = [\Theta(\phi_{o_K} \otimes g_*(o_K))](k) = (-1)^{|\phi_{o_K}||o_K|} \phi_{o_K}(k) \cdot g_*(o_K).$$

Continuing on the path we get

$$(-1)^{|k||o_K|} e(\phi_{o_K} \otimes k) \cdot g_*(o_K) = (-1)^{|k||o_K|} \phi_{o_K}(k) \cdot g_*(o_K) = [\Theta(a_g)(k)].$$

By the commutativity of the diagram we have that the image of a_g under Θ is $(g|_K)$ which gives

$$\Theta^{-1}(g|_K) = a_g.$$

Since we have already established that $e(a_g) = I_g$ we finally have the desired result, that is,

$$I_g = \Lambda(g|_K).$$

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