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Frontal instability in a sheared basic state

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Frontal instability in a sheared basic state

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**FRONTAL INSTABILITY
IN A SHEARED BASIC STATE**

A Thesis

Presented to

The Faculty of the Department of Meteorology

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

By

William Douglas Heise

May 1992

APPROVED FOR THE DEPARTMENT OF METEOROLOGY

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Dr. Peter F. Lester

APPROVED FOR THE UNIVERSITY

ABSTRACT

FRONTAL INSTABILITY IN A SHEARED BASIC STATE

by William D. Heise

A two layer frontal model is used to investigate the stability of frontal scale waves in the presence of potential vorticity (PV) anomalies which result from horizontal shear. Two unstable, moderately ageostrophic modes are found. One is characterized by barotropic instability while the other is a shallow mode characterized by a mixed baroclinic/barotropic instability. The along-front scale of the most unstable mode in both cases matches the cross-front scale of the anomaly.

The barotropic mode requires some PV anomaly in both layers whereas the mixed mode requires only a lower layer anomaly. The maximum growth rate of the barotropic mode depends on the strength of the anomaly; however, it is independent of the PV anomaly-scale and the Richardson number of the flow. A moderate anomaly in the basic state relative vorticity of $\bar{\zeta} = f$ produces a doubling time of less than half a day for this mode.

The mixed mode is most unstable for anomalies on the scale of the Rossby radius of deformation. A relative vorticity anomaly of $\bar{\zeta} = f$ on this scale quadruples the growth rate of the most unstable mode for the unsheared case. The short doubling times, less than 0.4 days in some cases, have implications for rapid cyclogenesis.

DEDICATION

This work is dedicated to my good friend

Kamala Rose

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LIST OF SYMBOLS

The following is a list of the symbols used in the main body of the text. Symbols used in the appendices are defined therein.

a	shear depth parameter
b_o, c_o, e_o	constants in equation for frontal surface
f	Coriolis parameter
g	gravitational acceleration
$\bar{h}$	mean height of frontal interface
h_1	distance from surface to interface ($h_1 = \bar{h}$)
h_2	distance from interface to upper boundary
i	$\sqrt{-1}$
k	wave number in the along front direction
p, P	pressure
q, PV	potential vorticity
t	time
u, v, w	along-front, cross-front, and vertical wind
x, y, z	along-front, cross-front, and vertical coordinate
A	dimensionless amplitude of the vorticity anomaly

B	number of shear waves or vorticity anomaly peaks
C	constant in mean wind formulation
D	growth rate: doublings per day
F	non-uniform part of frontal slope
H	fixed height of model domain
L_o	Rossby radius of deformation
L_x	along-front wavelength of disturbance
L_y	cross-front length of model domain
L_s	cross-front wavelength of shear wave
L_w	scale length relating cross and along front wave numbers
N	resolution of finite difference grid
Ri	Richardson number
Ri_{BS}	Ri where length scale for baroclinic and barotropic instability are the same
Ro	Rossby number
Ro_{max}	Ro for most unstable solution
Ro_{BC}	Ro appropriate for baroclinic instability
Ro_s	Ro_{max} for S-mode
S	parameter to control cross-front extent of shear
U	basic state along-front wind
W_i	energy conversions, where $i = \{2,3,4,5,6\}$
i, j	orthogonal unit vectors
$\mathbf{V}$	horizontal wind vector
$\hat{\alpha}$	arbitrary perturbation amplitude

β	meridional variation of Coriolis parameter
δ	slope of linear frontal surface
δ_1, δ_2	Kronecker deltas in the frontal slope equation
ε	relative perturbation amplitude
ζ	relative vorticity
η	absolute vorticity
θ	potential temperature
ρ	density
σ	complex eigenfrequency
τ	complex phase speed related to σ
Λ	scale between upper and lower layer wind vectors
ℓ_x	inverse x-wavenumber of the disturbance
ℓ_y	inverse y-wavenumber of the disturbance
$\mathbb{C}$	empirical constant relating Ri, S, and Ro _s
$()_j$	lower layer quantity for j = 1, upper for j = 2
$()_\theta$	quantity measured on an isentrope
$()'$	perturbation quantity
$\hat{()}$	complex amplitude
$\overline{()}$	basic state quantity

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Chapter 1

Introduction

1.1 Frontal Scale Waves and Linear Instability

Frontal scale waves are wave-like perturbations with a horizontal scale of $\simeq 500\text{-}1000$ km occurring on a frontal surface. This scale is in the *meso-alpha* range as classified by Orlanski 1975. In comparison, the typical scale of a synoptic disturbance is $\simeq 1500\text{-}4000$ km. *Linear instability* is the spontaneous growth of infinitesimal perturbations which leave the basic state mean flow unchanged. Thus, if a given basic state is found to be linearly unstable, presumably it would, in the real atmosphere, be associated with the spontaneous growth of wave-like disturbances. The growth rates, scales, structures, and energetics of these observed *incipient* disturbances should be similar to the most unstable solutions found for the linear stability problem. Furthermore, sensitivity of the unstable solutions to differing basic states should be reflected in the real atmosphere by the response of the incipient disturbance to different frontal environments (or initial conditions).

Though often difficult to detect due to the coarse grid of the observational network and the presence of higher obscuring cirrus or cirrostratus clouds in frontal regions, frontal scale waves *are* observed in the real atmosphere. A schematic interpretation of these waves as seen from satellite images is shown in Fig. 1. In this instance, the

Fig. 1. Schematic of a baroclinic zone with the initial development of frontal scale waves, as seen from satellite images (from Weber and Wilderotter 1981).

waves occur as part of the initial phase of the so called A-type development (Petterssen et al. 1962) where there is a semi-stationary baroclinic frontal zone to the southeast of a strong jetstream in a longwave trough. The frontal scale waves appear as weak low pressure centers along the frontal zone where comma shaped areas of convective clouds and precipitation occur.

Frontal waves at a further stage of development are shown in Fig. 2a. This is the 1 December 1982 mean sea level pressure analysis over the North Atlantic. Three frontal scale waves with wavelengths of about 1000 km are seen developing along a

Fig. 2. (a) Analysis of mean sea level pressure over the North Atlantic 1 December 1982 at 12h UTC, produced by the Meteorological Office. Interval between isobars is 4 mb. (b) Analysis of mean sea level pressure for 18h UTC on 20 October 1989. Interval between isobars is 8 mb. (c) Analysis of potential vorticity at 18h UTC on 20 October 1989 the 900 mb level. Contours every 0.25 PVU with values greater than 0.75 PVU in dotted shading (from Joly and Thorpe 1990).

frontal zone trailing behind a more developed storm. There is no evidence of these waves in the corresponding upper-air analyses (not shown) indicating the waves are shallow at this stage. Two more frontal scale waves can be seen near Great Britain on the surface pressure analysis of 20 October 1989 in Fig. 2b. The corresponding 900 mb potential vorticity field is shown in Fig. 2c illustrating the high potential vorticity values (PV anomalies) that occur along the frontal zone (0.25 PVU is considered average). A similar pattern of PV anomalies was found for the early or frontal wave stage of the Great Storm of 15 October 1987 (see Joly and Thorpe 1990 Fig. 2).

The observed time scale of a rapidly developing frontal wave which became the 'Fastnet storm' of August 1979 was described by Joly and Thorpe (1990) who estimated it had a doubling time of approximately $1/3$ day in its incipient stage. Comparing this with the doubling time of approximately 1.4 days predicted by quasi-geostrophic theory for a typical synoptic scale disturbance, it becomes clear that the classical Charney (1947) and Eady (1949) theories of baroclinic instability not only fail to account for the scale and the growth rate of frontal scale waves, they also fail to account for the explosive cyclogenesis that sometimes occurs at a later stage in their development.

1.2 Cyclone Development and Frontal Evolution

In order to better establish the context in which to discuss the instability of frontal scale waves, some background on the development of frontal waves into mature storms and the evolution of frontal zones will be presented.

1.2.1 Development of cyclones from frontal waves

The mere existence of frontal waves such as those illustrated in Figs. 1 and 2 does not imply they are going to develop into mature synoptic scale storms. In order for these shallow frontal waves to undergo a larger scale development, they must exist in a synoptic environment which is conducive to storm growth.

One way of accounting for the existence of a surface cyclone which developed from a frontal wave is that the cyclone is simply a manifestation of an in situ frontal instability in its later stages. This is characteristic of the Petterssen Type A categorization of surface cyclogenesis.

The well known Suttcliffe-Petterssen development scenario, or the Petterssen Type B cyclogenesis process (shown schematically in Fig. 3), is another mechanism for frontal wave development.¹ With the development ideas of Suttcliffe and Forsdyke (1950) in mind, Petterssen (1956) stated the following hypothesis:

Cyclone development at sea level occurs when and where an area of appreciable vorticity advection in the upper troposphere becomes superimposed upon a slowly moving or quasi-stationary front at sea level.

Assuming there are one or more pre-existing small scale cyclonic circulations in the frontal region described above, the positive vorticity advection of the advancing upper level disturbance, in becoming juxtaposed vertically above the surface disturbance, will increase the divergence aloft. This increases low level convergence intensifying the surface circulation. In turn, warm air advection is increased ahead of the disturbance while cold advection is increased behind. The resulting deformation of the temperature field increases the amplitude of the upper level wave increasing vorticity aloft and initiating a positive feedback in the cyclone growth. Obviously, this growth does not continue unrestrained. The limiting process is the adiabatic cooling and warming associated with the vertical motions in the cyclone. These tend to cool/warm the regions of warm/cold air advection. However, if there is sufficient moisture in the system, rising air cools moist adiabatically thus easing the limits on growth. Indeed, it is well known that the potential for rapid cyclone development is increased as more moisture is introduced into the system.

¹ Recently, Thorncroft (1988) has applied and described the Suttcliffe-Petterssen development ideas as a *finite amplitude feedback* growth mechanism between upper tropospheric and surface features.

Fig. 3. Schematic of the Sutcliffe-Petterssen 'self-development' scenario. 500 mb contours (heavy solid lines), 1000 mb contours (thin solid lines), and 1000-500 mb thickness (dashed). (From Palmen and Newton 1969).

Schär and Davies (1990) point out that the Type A and the Type B development scenarios are not mutually exclusive. They note that the advection of a deep upper trough toward a front could be the initial perturbation for a frontal instability leading to a Type A development. Alternatively, they suggest that a frontal instability of ap-

propriate scale and fortuitous location could be the site for the upper/lower level baroclinic interaction associated with a Type B development. In either case there is the suggestion that a link between frontal instability and cyclogenesis requires some type of upper level presence whether it is an upper level manifestation of the original frontal instability or a pre-existing finite amplitude wave aloft.

1.2.2 Basic state evolution

Before there can be frontal waves there must be a front; however, the existence of a front is not necessarily a sufficient condition in itself to permit the initiation of frontal waves. The mean flow (or basic state) of the frontal zone must be conducive to frontal instability. At some given time, the basic state of a frontal zone may not have the characteristics necessary for frontal instability to occur, but it may *evolve* at some later time into a configuration which will allow frontal waves to emerge.

Frontal evolution due to frontogenesis in a maturing storm was described by Hoskins and Bretherton (1972) whose theoretical study showed that the deformation process in a growing baroclinic wave will concentrate the thermal gradient thus forming a frontal zone. This causes a secondary transverse ageostrophic circulation (needed to maintain thermal wind balance) which forces the thermal gradient into a discontinuity (or front) at the surface. In reality, this is not a true discontinuity, rather a narrow zone of strong gradients created by thermal turbulent diffusion (Welander 1963). Hoskins and Bretherton also found that latent heating as a result of condensation enhanced the frontogenetical effect on the ageostrophic circulation.

Using a two dimensional semigeostrophic² framework, Thorpe and Emanuel (1985) demonstrated that the release of latent heat associated with condensation due to slantwise convection in the frontal zone for an initially uniform potential vorticity field would cause a strong local maximum in potential vorticity at low levels behind the front. In addition, they postulated that the *PV anomaly* could give rise to secondary barotropic instabilities whose horizontal scale would be that of the scale of the potential vorticity variation. They estimated a horizontal scale of about 80 km. Sinton and Mechoso (1984), using a nonlinear two layer shallow water equation model, showed that during a time evolution of a synoptic scale cyclone, the initial potential vorticity field in the model was redistributed to cause potential vorticity anomalies behind the cold front.

The phenomenon of *tropopause folding*, originally proposed by Reed (1955) and later verified by many observational studies such as Danielsen (1964, 1968) and Shapiro (1980), is another mechanism by which the basic state of a frontal zone may be modified. Transverse circulations (such as those depicted in Fig. 4) associated with the exit regions of jet streaks and upper level frontal zones are capable of deforming the tropopause such that stratospheric air is "folded" into lower levels of the troposphere--sometimes as low as 800 mb. In this way, high potential vorticity stratospheric air may mix with low level, low potential vorticity air to create a potential vorticity anomaly in the frontal region.

The common element in the frontal evolutions described above is the appearance of a large potential vorticity gradient or potential vorticity anomaly near the cold front

² *Semigeostrophic* refers to the advection of the geostrophic wind by the full wind.

Fig. 4. Schematic illustration of transverse circulations conducive to upper-level frontogenesis and tropopause folding. (From Danielsen 1968).

resulting from cyclone development and/or secondary circulations induced by the frontogenesis associated with cyclone development.

1.3 Previous Studies

In the early stages of the theories surrounding the formation and growth of extratropical cyclones, Bjerknes (1919) and Bjerknes and Solberg (1923) envisaged the midlatitude cyclone as a disturbance growing on the interface between polar and tropical air masses deriving its kinetic energy from the release of potential energy as the cold polar air displaces the warmer tropical air. Such disturbances are said to grow *baroclinically* while those that draw their energy from the kinetic energy of the basic state are said to grow either from *shearing* or *barotropic* processes. Motivated by the conceptual model above, Solberg (1928) developed a linearized shallow water (incompressible) numerical model on an f -plane with rigid boundaries parallel to the interface. This model was originally developed to study frontal scale waves

but instead found only larger baroclinically growing synoptic scale waves with a horizontal scale of about 2000-3000 km. Kotschin (1932) improved on the two layer model design by introducing a sloping interface which intersected rigid upper and lower boundaries. Even with these modifications, Kotschin was unable to find any unstable solutions for $Ri < 2$, nor was he able to produce unstable solutions in the frontal scale range. Orlanski (1968) used the model of Kotschin to investigate the range $0 \leq Ro \leq 3$, $0 \leq Ri \leq 5$. Here, although unstable synoptic scale waves at all Ri were found, smaller scale modes were only discovered for $Ri < 2$. In addition to the numerical results from this study, Orlanski introduced an energy analysis which attempted to provide information regarding the different growth mechanisms at work in the two layer model. Orlanski (1969) used the two layer model from the previous study, modified with effects of large-scale shear and bottom topography (but without a potential vorticity anomaly), to investigate the instability of ocean currents ($20 \leq Ri \leq 120$). Discovering inconsistencies in the energetics used by Orlanski (1968), Mechoso and Sinton (1983) modified the energy formulation of the two layer model in such a way as to clarify which instability mechanism or combination of mechanisms (kinetic and/or available potential energy) was contributing to the growth of the unstable waves.

As of this writing, no linear formulation of the two layer model has found unstable frontal scale waves for realistic synoptic scale Richardson numbers, i.e., $Ri \gtrsim 5$; however, Sinton and Mechoso (1984) employed Orlanski's (1968) model in a nonlinear form to study the evolution of synoptic scale waves. In association with the formation of the potential vorticity anomalies behind the cold front (mentioned earlier), they found strong cross-front shear and shallow, smaller scale, secondary disturbances

(frontal scale waves) which grew primarily by the barotropic energy conversion. Poor horizontal resolution in the model limited confidence in these findings.

Recently there have been several multi-level model studies on frontal instability. Moore and Peltier (1987), using a multi-level primitive equation model for the perturbation, investigated the linear stability of a uniform potential vorticity field in a semigeostrophic basic state. The most unstable wave was in the synoptic scale and had no significant barotropic growth mechanism. The most unstable frontal scale waves which were found grew baroclinically and had growth rates only about 2/3 as large as the most unstable synoptic scale wave. Schär and Davies (1990) used both nonlinear quasi-geostrophic and semigeostrophic formulations with a uniform potential vorticity basic state which included potential temperature anomalies. They discovered the most unstable mode grew by a combination of barotropic and baroclinic effects. The characteristic horizontal scale of these waves was on the order of 1000 km with a doubling time of about 1.5 days. Joly and Thorpe (1990) examined the linear stability of potential vorticity anomalies using a semigeostrophic multi-level model. They found that they needed a large amplitude, small scale potential vorticity anomaly (absolute vorticity $\approx 3f$, ~ 160 km wide) to produce frontal waves with a scale of about 800 km. These waves were significantly destabilized when the PV anomaly was not confined near the lower boundary, i.e., when there was an interior maximum. The waves grew by barotropic conversion, and the maximum growth rate had a doubling time of about 0.7 days.

1.4 Goals of the Current Study

The primary goal of this study is to examine the nature of frontal scale waves using a linear two layer model. The following questions will be addressed: For a given basic state, are unstable waves able to exist? If so, which waves are the most unstable? What are their growth rates and energy characteristics, and how sensitive are they to changes in the basic state?

It is not the goal of this study to examine the possible future nonlinear development of these waves into mature storms, nor is it the purpose of this work to explain how the basic state in which these waves are embedded evolved initially. The flow-chart shown in Fig. 5. illustrates where this study fits into the overall picture of storm development and frontogenesis. The dashed line represents the current study.

A secondary goal of this study is to determine whether a two layer model, which until now has only been able to yield unstable synoptic scale modes for realistic synoptic scale conditions (i.e., $Ri \gtrsim 5$), is finally capable of producing an unstable *frontal scale wave* for these same conditions. The following chapter will present and describe the model used in this study. Chapter 3 will address the experiments and results, and a summary and discussion will appear in Chapter 4.

Fig. 5. Frontal instability flowchart. The elements within the dashed line represent the current study.

Chapter 2

Model Description

2.1 Overview of the Two Layer Model

The model used in this study is the same as that described by Orlanski (1968: hereafter referred to as IFW) modified to include a basic state wind sheared in the cross-front³ direction. The schematic of the model shown in Fig. 6 illustrates one possible configuration of the frontal surface after shear has been added. This interface is a zero order discontinuity representing the material boundary between two air masses of differing densities, ρ_1 in the lower layer and ρ_2 in the upper layer. Therefore, for a given density difference, anywhere the interface becomes steeper the available potential energy of the system increases.

The linearized two layer model is resolved in three spatial dimensions, x , y , and z . In the x (or along-front) direction there is a spectral representation consisting of one wavelength L_x , that of the perturbation (see section 2.5). Cross-front details are resolved by a finite difference grid in the y -direction, and vertical details are resolved by the two fluid layers whose boundary is the material (or frontal) surface mentioned

³ *Cross-front* in this paper may be regarded as the meridional (or y) direction, and the term *along-front* can be used synonymously with the zonal (or x) direction. This implies an east-west orientation of the surface front (chosen for its mathematical convenience); however, since the model is developed on an f -plane, the results apply equally for any orientation of the front.

Fig. 6. Schematic of the two layer model used in the current study. The y-axis is in the cross-front direction increasing toward colder (denser) fluid. The x-axis is in the along-front direction. The dashed line and crosshatched plane represent the interface position in IFW when $(U_2 - U_1)$ is y-independent. Other terms are defined in the text. Insert shows typical location of the model domain with respect to the synoptic environment.

earlier. The interface $\bar{h}$ between the two layers serves as both a material *and* a coordinate surface. Therefore, vertical advection and/or motion is implicit in the horizontal motions within the layers. In this sense, the frontal surface is analogous to an isentropic surface; thus, as the interface becomes steeper, the baroclinity increases. Additionally, this isentropic analogy means that relatively higher interface heights correspond to relatively colder or denser regions of the fluid while lower heights correspond to warmer or less dense fluid regions. Thus, the deviation of h_0 in the frontal interface $\bar{h}$ represents a relatively colder region just to the cold air side of the

surface front. This is similar to the situation where a "tongue" of cold subsiding air has advected in behind the cold front of a mature synoptic scale cyclone. In satellite images this is often associated with the dry slot (Smigielski and Ellrod 1985).

Three of the fundamental parameters in the two level model are the slope of the front (or slope of the interface) $\frac{d\bar{h}}{dy}$, the Richardson number Ri , and the Rossby number Ro , defined as follows:

$$\frac{d\bar{h}}{dy} = \frac{f\bar{\rho}(U_2 - U_1)}{g(\rho_1 - \rho_2)} \quad (2.1)$$

$$Ri = \frac{gH(\rho_1 - \rho_2)}{\bar{\rho}(U_2 - U_1)^2} = \frac{\text{Static Stability}}{(KE)_{vs}} \quad (2.2)$$

$$Ro = \frac{(U_2 - U_1)k}{2f} = \frac{\text{Inertial Acceleration}}{\text{Coriolis Acceleration}} \quad (2.3)$$

where $k = 2\pi/L_y$, $\bar{\rho} = \frac{(\rho_1 + \rho_2)}{2}$, and $(KE)_{vs}$ is the kinetic energy of the vertical shear across the interface. The top of the model H is fixed at 10 km. Eq. (2.1) results from U_1 and U_2 , the along-front basic state flow in each layer, being in thermal wind balance. It is simply the adaptation of Margules' Formula to the two layer model. The Richardson number is generally related to the cross-front scale L_y of the model. This is seen more clearly by combining (2.1) and (2.2) to give

$$\frac{d\bar{h}}{dy} = \frac{fH}{Ri(U_2 - U_1)} = \frac{H}{L_y} \quad (2.4a)$$

or

$$L_y = \frac{Ri (U_2 - U_1)}{f}. \quad (2.4b)$$

A small value of Ri is associated with low static stability; this implies a steeper front and, since H is fixed, a shorter y -domain (L_y smaller). Large values of Ri , on the other hand, imply higher static stability, a gentler frontal slope, and longer y -domain (L_y larger). The Rossby number is related through the along-front wave number k to the along-front scale L_x of the unstable mode being investigated (see section 2.5). The relationship between Ro and k is such that, when Ro is small, L_x is large, and the flow is quasi-geostrophic; i.e., the Coriolis acceleration is large compared to the inertial acceleration, and the flow is approximately in geostrophic balance. Conversely, when Ro is large, L_x is small, and the flow is ageostrophic. As in IFW, in this study U_1 and U_2 are taken before the addition of shear to be -10 and 10 m s^{-1} , respectively, throughout the domain; gravity g is 10 m s^{-2} ; and the Coriolis parameter f is taken to be 10^{-4} s^{-1} .

2.2 Shear-controlling Parameters

The following is a description of the parameters used to manipulate the basic state shear. The basic state wind varies as the cosine of the cross-front y -coordinate. For this reason the non-uniformity in the mean wind will often be referred to as a "shear wave." The shear wave is initially imposed on the height of the interface by adding it to the linear slope of the frontal surface. Wind and vorticity fields are then obtained by taking the derivatives of the height field. The *shear parameters* presented below are designed to alter the characteristics of the basic state vorticity; this

in turn modifies the basic state wind shear. Through the thermal wind relation, the frontal slope is also modified.

2.2.1 Amplitude of the shear wave, A

The dimensionless amplitude A of the shear wave is defined as follows:

$$A = \frac{h_o}{h_{crit}}, \quad 0 \leq A \leq 1 \quad (2.5)$$

where h_o is the deviation in height units of the interface from its linear profile, and h_{crit} (see Appendix A) is the deviation necessary to create a relative vorticity anomaly of $\bar{\zeta} = f$. Due to symmetric instability, negative values of absolute vorticity $\bar{\eta}$ are not observed in the atmosphere on the wave scales of interest in this study; therefore, the condition $\bar{\eta} = \bar{\zeta} + f \geq 0$ is imposed. If a value of $A \leq 1$ is chosen, it follows that $|\bar{\zeta}_{max}| \leq f$, where $\bar{\zeta}_{max}$ is the maximum value of mean relative vorticity derived from the height field. In addition, the mean absolute vorticity $\bar{\eta}$ will be in the range $0 \leq \bar{\eta} \leq 2f$. So, in essence, the value of A determines the magnitude of the mean absolute vorticity anomaly. In this study, A is always chosen such that $|\bar{\zeta}| \leq f$ (i.e., $0 \leq \bar{\eta} \leq 2f$). Other studies (such as Joly and Thorpe 1990) have used positive $\bar{\eta}$ anomalies as large as $3f$.

2.2.2 Vertical depth of the shear, a

In this model, the following relation applies:

$$\frac{dU_2}{dy} = a \frac{dU_1}{dy}, \quad 0 \leq a \leq 1 \quad (2.6)$$

Fig. 7. The local Richardson number of the sheared two level model as described in the text.

where the parameter a is referred to as the “depth parameter” since its effect on the basic state is to increase the vertical depth of the shear by adding shear to the upper layer wind based on the lower layer wind pattern. From Eq. (2.1) it is clear that the y -variation of the mean vertical shear ($U_2 - U_1$) determines the slope, and thus the height, of the interface. As a approaches 1, the frontal surface becomes linear (the dashed line in Fig. 6) since $\frac{d}{dy}(U_2 - U_1)$ approaches zero. This is illustrated in Fig. 7 which is a plot of the inverse local Richardson number as a function of the cross-front direction. Thus, a peak in this curve represents large baroclinity. Having a depth parameter of zero imposes the greatest deviation of the interface from its linear pro-

file. This implies regions of varying baroclinity in the region of the shear. As the depth parameter is increased, however, the curvature of the height field is reduced and Ri approaches a constant value for the entire domain.

2.2.3 Cross-front extent of shear, S

This parameter controls the wavelength L_s of the shear wave as follows:

$$S = \frac{L_y}{L_s}, \quad S \geq 1 \quad (2.7)$$

The effect increasing S is to confine the basic state shear to a narrower region behind the surface front. This reduces the scale of the shear without altering any of the other shear characteristics.

2.2.4 Number of "shear waves," B

This parameter controls the number of peaks which may occur in the vorticity field by specifying how many cosine waves to add to the linear slope of the frontal surface. If $B = 3$, for example, there are 3 consecutive shear waves and therefore three peaks in the derived relative vorticity field. This parameter allows the additional vorticity anomalies to occur away from the surface front. The value of B is constrained by S such that $B \leq S - 1$. Thus, there cannot be more shear waves of length L_s than will fit into the domain of length L_y .

2.3 Description of the Basic State

2.3.1 The frontal surface

As stated earlier, the fundamental difference between the IFW model and the one used in this study is the introduction of cross-front shear through a non-uniform-sloping frontal surface (Fig. 6). The non-uniform front is composed of the sum of a linear slope and a non-linear slope. The linear part of the slope is the same as that used in IFW given by Margules' Formula (2.1) with $\frac{d}{dy}(U_2 - U_1) = 0$. For this case, the angle δ of the uniform sloping frontal surface is given by

$$\tan \delta = m = \frac{d\bar{h}}{dy} = \frac{H}{L} = \frac{f\bar{\rho}(U_2 - U_1)}{g(\rho_1 - \rho_2)}. \quad (2.8)$$

The non-uniform part of the frontal surface is obtained by adding a combination of a cosine curve and a bell curve to the uniform part. The wavelength L_s of the cosine is given by (2.7). A bell curve is matched at the point $y = (B - 1/4)L_s$, so the non-uniform part "decays" into the uniform part. The equation for the non-uniform frontal surface as a function of y is given below:

$$\bar{h}(y) = my + (1 - a)F(y), \quad (2.9)$$

where

$$\begin{aligned} F(y) = & \left[h_o \left(1 - \cos \frac{2\pi y}{L_s} \right) \right] \delta_1 \\ & + \left[b_o \frac{w^2}{w^2 + (y - y_o)^2} + c_o y + e_o \right] \delta_2, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} [\delta_1, \delta_2] &= [0, 1] & \text{for } y > (B - \frac{1}{4})L_s \\ [\delta_1, \delta_2] &= [1, 0] & \text{for } y \leq (B - \frac{1}{4})L_s. \end{aligned}$$

The quantities b_o , c_o , and e_o are functions of h_o and are used to match the bell curve to the cosine curve such that F , $\frac{dF}{dy}$, $\frac{d^2F}{dy^2}$, and $\frac{d^3F}{dy^3}$ are continuous at the point $y = (B - \frac{1}{4})L_s$. The variable w is the characteristic half-width of the bell curve, and y_o is the offset in the y -direction. Both w and y_o are determined by the geometry of the frontal region. A derivation of the constants in Eq. (2.10) is given in Appendix A.

2.3.2 The mean along-front wind

As a first step, U_2 is held constant. Then rearranging Margules' equation (2.1) and using (2.6) and (2.9), the following expression for U_1 may be obtained:

$$U_1 = U_2 - C \left(m + \frac{dF}{dy} \right), \quad (2.11)$$

where $C = \frac{g}{f} \left(\frac{\rho_1 - \rho_2}{\bar{\rho}} \right)$. It follows that, since $m + \frac{dF}{dy}$ is not necessarily constant, U_1 will vary accordingly in the y -direction, thus horizontal shear is introduced (in the lower layer only). The variation of U_1 as a function of the cross-front direction is shown in Fig. 8 for $Ri = 10$. The derivatives $\frac{dU_1}{dy}$ and $\frac{d^2U_1}{dy^2}$ are obtained through centered differences.

U_2 is then found by rearranging (2.1):

Fig. 8. Lower layer basic state along-front wind for the parameter values shown. Parameters are described in text.

$$U_2 = U_1 + C \frac{d\bar{h}}{dy}. \quad (2.12)$$

To show that the choice of the depth parameter a is consistent with maintaining thermal wind balance across the frontal surface $\bar{h}$, the y -derivative of (2.11) and the second y -derivative of (2.9) can be taken. It can then easily be shown that (2.6) results. Furthermore, from (2.6) it follows immediately that

$$\frac{d^2 U_2}{dy^2} = a \frac{d^2 U_1}{dy^2}. \quad (2.13)$$

It is seen from Eq. (2.9) that, as the depth parameter a approaches 1, the frontal surface becomes linear, and the shear exists equally in both layers.

2.3.3 Relative vorticity and vorticity gradient

The mean relative vorticity for each layer is defined as follows:

$$\bar{\zeta}_1 = -\frac{dU_1}{dy} \quad \text{and} \quad \bar{\zeta}_2 = -\frac{dU_2}{dy}. \quad (2.14)$$

For the case where $a = 0$, if the amplitude parameter A is chosen to be 1, then the maximum value of $\bar{\zeta}_1$ is f . This occurs at the surface front as shown in Fig. 9a. The vorticity gradient $\frac{d\bar{\zeta}_1}{dy}$ for the same basic state conditions is shown in Fig. 9b. It illustrates that the well known *necessary* condition for barotropic instability is met (i.e., $\beta - \frac{d^2U}{dy^2}$ changes sign in the domain).⁴

2.3.4 Potential vorticity

Potential vorticity $q = PV$ is useful as a hydrodynamic tracer since a parcel of air that is not subject to diabatic heating will conserve its potential vorticity as it moves along. Thus, generally,

$$\frac{dq}{dt} = 0, \quad \text{where } q = -\frac{\partial\theta}{\partial p}(f + \zeta_\theta). \quad (2.15)$$

⁴ In this model $\beta = df/dy = 0$.

Fig. 9. Basic state (a) relative vorticity and (b) vorticity gradient for the parameters shown.

The quantity θ is the potential temperature, ζ_θ is the relative vorticity measured on an isentropic surface, and f is the Coriolis parameter. In the shallow water model, potential vorticity for each layer is similarly defined as

$$q_1 = \frac{f + \zeta_1}{h} \quad \text{and} \quad q_2 = \frac{f + \zeta_2}{H - h}, \quad (2.16)$$

where again $\frac{dq_j}{dt} = 0$. We may now define a *potential vorticity anomaly* as any local maxima or minima in the mean potential vorticity field. The mean potential vorticity used in this study is shown in Fig. 10a. It was shown by Charney and Stern (1962) that a quasi-geostrophic fluid could become dynamically unstable if the mean quasi-geostrophic pseudo potential vorticity⁵ gradient changes sign in the domain, even if the basic state potential temperature distribution on the boundaries is uniform. The potential vorticity gradient for a typical basic state used in this study is shown in Fig. 10b. It illustrates that both positive and negative values exist in the frontal domain.

2.4 Model Equations

The nonlinear or full shallow water equations for the two layer model with a *rigid lid* are (from IFW, Eqs. 2.2 through 2.4):

$$\left[\frac{\partial}{\partial t} + (\mathbf{v}_j \cdot \nabla) + f \mathbf{k} \times \right] \mathbf{v}_j = - \frac{1}{\rho_j} \nabla P_j, \quad j = 1, 2 \quad (2.17)$$

$$P_1 = P_2 + (\rho_1 - \rho_2)gh + \rho_2 gH, \quad (2.18)$$

⁵ In the quasi-geostrophic equations, the quasi-geostrophic pseudo potential vorticity is the analog of potential vorticity in the primitive equations.

Fig. 10. As in Fig. 9, but for (a) mean potential vorticity and (b) mean potential vorticity gradient.

$$\frac{\partial h}{\partial t} = -\nabla \cdot h\mathbf{V}_1 = \nabla \cdot (H - h)\mathbf{V}_2, \quad (2.19)$$

where a subscript of 1 refers to the lower layer, 2 refers to the upper layer, and $\mathbf{k}$ is the vertical unit vector. These equations are modified slightly for the free surface model which is used with the eigenvalue approach (see Appendix B). Allowing a y -dependent, steady state basic flow to be perturbed by a wave-like disturbance of small relative amplitude $\varepsilon \ll 1$ gives the following expansion of the model variables [adapted from IFW (3.1)]:

$$\begin{aligned} u_j &= U_j(y) + \varepsilon \text{Re}[\hat{u}_j(y)e^{i(kx+\sigma t)}] + O(\varepsilon^2) + \dots \\ v_j &= 0 + \varepsilon \text{Re}[\hat{v}_j(y)e^{i(kx+\sigma t)}] + O(\varepsilon^2) + \dots \\ P_j &= \bar{P}_j(y) + \varepsilon \text{Re}[\hat{P}_j(y)e^{i(kx+\sigma t)}] + O(\varepsilon^2) + \dots \\ h &= \bar{h}(y) + \varepsilon \text{Re}[\hat{h}(y)e^{i(kx+\sigma t)}] + O(\varepsilon^2) + \dots \end{aligned} \quad (2.20)$$

where $U_j(y)$, $\bar{P}_j(y)$, and $\bar{h}(y)$ are the y -dependent basic state, and $\hat{u}_j(y)$, $\hat{v}_j(y)$, $\hat{P}_j(y)$, and $\hat{h}(y)$ are the complex y -dependent perturbation amplitudes of the perturbation variables u'_j , v'_j , P'_j , and h' .

Substituting equations (2.20) into (2.17) through (2.19) and collecting the first order $[O(\varepsilon)]$ terms gives the following equations [adapted from (3.2) through (3.6) in IFW]:

$$i(\sigma + kU_j)\hat{u}_j + \left(\frac{dU_j}{dy} - f\right)\hat{v}_j = -\frac{ik}{\rho_j}\hat{P}_j \quad (2.21)$$

$$i(\sigma + kU_j) \hat{v}_j + f \hat{u}_j = -\frac{1}{\rho_j} \frac{d\hat{p}_j}{dy} \quad (2.22)$$

$$\hat{p}_1 - \hat{p}_2 = (\rho_1 - \rho_2) g \hat{h} \quad (2.23)$$

$$i(\sigma + kU_1) \hat{h} = -\bar{h} \left(ik\hat{u}_1 + \frac{d\hat{v}_1}{dy} \right) - \frac{d\bar{h}}{dy} \hat{v}_1 \quad (2.24)$$

$$i(\sigma + kU_2) \hat{h} = (H - \bar{h}) \left(ik\hat{u}_2 + \frac{d\hat{v}_2}{dy} \right) - \frac{d\bar{h}}{dy} \hat{v}_2 \quad (2.25)$$

Equations (2.21) and (2.22) can be used to express $\hat{v}_j$ and $\hat{u}_j$ as functions of $\hat{p}_j$ and $\frac{d\hat{p}_j}{dy}$ giving the following equations [adapted from (3.7) and (3.8) in IFW]:

$$\hat{v}_j = \frac{i \left[f k \hat{p}_j - (\sigma + kU_j) \frac{d\hat{p}_j}{dy} \right]}{\rho_j \left[f^2 - f \frac{dU_j}{dy} - (\sigma + kU_j)^2 \right]} \quad (2.26)$$

$$\hat{u}_j = \frac{k(\sigma + kU_j) \hat{p}_j + \left(\frac{dU_j}{dy} - f \right) \frac{d\hat{p}_j}{dy}}{\rho_j \left[f^2 - f \frac{dU_j}{dy} - (\sigma + kU_j)^2 \right]} \quad (2.27)$$

The Boussinesq approximation is applied by replacing ρ_j with $\bar{\rho}$ in (2.24) and (2.25). Then substituting $\frac{d}{dy}$ (2.26) and $ik \times$ (2.27) into equations (2.24) and (2.25), while using (2.23), gives the following two equations [adapted from IFW (3.9) and (3.10)] for the perturbation pressure amplitudes in the two layer domain $0 \leq y \leq L_y$:

$$\begin{aligned}
& \bar{h} \frac{d^2 \hat{P}_1}{dy^2} + \frac{d\bar{h}}{dy} \frac{d\hat{P}_1}{dy} - \left[k^2 \bar{h} + \frac{d\bar{h}}{dy} \frac{kf}{(\sigma + kU_1)} \right] \hat{P}_1 \\
& + \frac{\bar{h} \left[f \frac{d^2 U_1}{dy^2} + 2k(\sigma + kU_1) \frac{dU_1}{dy} \right] \left[\frac{d\hat{P}_1}{dy} - \frac{kf}{(\sigma + kU_1)} \hat{P}_1 \right]}{\left[f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2 \right]} \\
& - \frac{\bar{p}}{g(\rho_1 - \rho_2)} \left[f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2 \right] (\hat{P}_1 - \hat{P}_2) = 0
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
& (\bar{h} - H) \frac{d^2 \hat{P}_2}{dy^2} + \frac{d\bar{h}}{dy} \frac{d\hat{P}_2}{dy} - \left[k^2 (\bar{h} - H) + \frac{d\bar{h}}{dy} \frac{kf}{(\sigma + kU_2)} \right] \hat{P}_2 \\
& + \frac{(\bar{h} - H) \left[f \frac{d^2 U_2}{dy^2} + 2k(\sigma + kU_2) \frac{dU_2}{dy} \right] \left[\frac{d\hat{P}_2}{dy} - \frac{kf}{(\sigma + kU_2)} \hat{P}_2 \right]}{\left[f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2 \right]} \\
& - \frac{\bar{p}}{g(\rho_1 - \rho_2)} \left[f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2 \right] (\hat{P}_1 - \hat{P}_2) = 0.
\end{aligned} \tag{2.29}$$

By restricting the cross-front basic state shear to the two layer domain, boundary conditions identical to IFW (3.16) and (3.17) result:⁸

$$\frac{d\hat{P}_1}{dy} = -k\hat{P}_1 \quad \text{at } y = L_y \tag{2.30}$$

⁸ See Appendix C for alternative boundary conditions.

$$\frac{d\hat{P}_2}{dy} = k\hat{P}_2 \quad \text{at } y = 0 \quad (2.31)$$

Thus, exponential spatial decay of the perturbation is ensured outside of the two layer region. The kinematic boundary conditions

$$(w'_1)_{z=\bar{h}} = \frac{dh'}{dt} = -\bar{h}\nabla \cdot \mathbf{V}'_1 = 0 \quad \text{at } y = 0 \quad (2.32)$$

$$(w'_2)_{z=\bar{h}} = \frac{d}{dt}(h' - H) = (H - \bar{h})\nabla \cdot \mathbf{V}'_2 = 0 \quad \text{at } y = L_y \quad (2.33)$$

when applied to (2.28) at $y = 0$ and (2.29) at $y = L_y$ result in the following two boundary conditions [adapted from (7.4) and (7.5) in IFW]:

$$\begin{aligned} & \frac{d\bar{h}}{dy} \frac{d\hat{P}_1}{dy} - \frac{d\bar{h}}{dy} \frac{kf}{(\sigma + kU_1)} \hat{P}_1 \\ & - \frac{\bar{\rho}}{g(\rho_1 - \rho_2)} \left[f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2 \right] (\hat{P}_1 - \hat{P}_2) = 0 \end{aligned} \quad (2.34)$$

at $y = 0$

$$\begin{aligned} & \frac{d\bar{h}}{dy} \frac{d\hat{P}_2}{dy} - \frac{d\bar{h}}{dy} \frac{kf}{(\sigma + kU_2)} \hat{P}_2 \\ & - \frac{\bar{\rho}}{g(\rho_1 - \rho_2)} \left[f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2 \right] (\hat{P}_1 - \hat{P}_2) = 0 \end{aligned} \quad (2.35)$$

at $y = L_y$

Equations (2.28) and (2.29) with boundary conditions (2.31) and (2.34) at $y = 0$, and (2.30) and (2.35) at $y = L_y$ are then solved for σ , $\hat{P}_1(y)$, and $\hat{P}_2(y)$.

2.5 Solutions

Solutions are of the form $F(y) \exp i(kx + \sigma t)$, where a sinusoidal solution in x is assumed, and σ is the complex frequency of the solution. If σ has an imaginary part, the solution will amplify with time and is said to be unstable; otherwise, the solution will be a stable oscillation. The growth rates of an unstable solution are in terms of *doublings per day* D given by

$$D = \frac{|\sigma_i|}{\ln 2} 8.64 \times 10^4, \quad (2.36)$$

where σ_i is the imaginary part of σ . The effect of D on the amplitude of an arbitrary perturbation quantity $\hat{\alpha}$ is such that $\hat{\alpha}_{24} = 2^D \hat{\alpha}_0$, where the subscripts 0 and 24 refer to a reference times in hours. For example, if $D = 2$, the disturbance amplitude quadruples in one day. In order to simplify the mathematical treatment, in IFW a related phase speed τ was introduced and defined by equations (4.9) through (4.12) in that paper. The same formulation of τ will be used here. Like σ , τ is a complex quantity. The imaginary parts of τ and σ are related by

$$|\sigma_i| = f R_0 |\tau_i|. \quad (2.37)$$

For a given basic state, the solutions of the linear stability problem are obtained through a two step process: (1) An eigenvalue method (described in Appendix B) is employed to find *all* possible unstable solutions using the criterion for instability that $|\tau_i| \geq 10^{-3}$. (2) The unstable "candidates" found in step (1) are used as first guesses in a Newton-Raphson scheme (described in Appendix C) producing a more accurate solution which, based on the same instability criterion, may or may not be unstable.

Once an accurate solution is found, the basic state characteristic parameters can be changed by small increments to find another solution using the previous solution as an initial guess. In this way, the response of the unstable mode to changes in the basic state may be investigated in a pseudo-continuous or parametric manner. For example, given a solution for a set of basic state parameters (i.e., those described in section 2.2), one could increment the Rossby number to obtain a curve of *growth rate* -vs- Ro . This type of graphic is used to illustrate the experiments in the next chapter.

2.6 Energetics

Three basic types of instability exist in the two layer model: *baroclinic*, associated with the potential energy of the mean flow; *Rayleigh* or *Helmholtz*, associated with the kinetic energy of the basic state vertical wind shear across the interface separating the fluid layers; and *barotropic*, associated with the kinetic energy of the basic state horizontal wind shear within each layer. The energy formulation used here is that of Mechoso and Sinton (1983). The energy box diagram resulting from their formulation is shown in Fig. 11a. The potential and kinetic energy of the basic state are represented by P_M and $K_M + K_i$, respectively, and the potential and kinetic energy of the disturbance are represented by P_E and K_E . The reader is referred to Mechoso and Sinton for the equations defining the energy transformations W_i . A brief description will be given here.

W_2 is associated with the mean meridional circulation; however, it is not used in this study since it does not affect the perturbation energy. W_3 represents the baroclinic instability process which increases eddy potential energy by meridionally advecting the perturbation height field. $W_3 > 0$ represents growth of P_E at the expense

Fig. 11. (a) Energy box diagram for the shear formulated two layer model (adapted from Mechoso and Sinton 1983). (b) Schematic of the physical meaning of the Y vector.

of the slope of the interface. $W_4 - W_3$ represents the wave energy flux convergence into the fluid layers from the interface and is associated with the Rayleigh and/or Helmholtz instability. A positive value of $W_4 - W_3$, therefore, represents eddy growth via a reduction of the shear *between* the layers. $W_5 + W_3 - W_4$ represents the transformation from P_E to K_E via the reduction of h' by the mean perturbation vertical velocity $\frac{(w_1' + w_2')}{2}$ (see Eq. (9.29) in IFW). A positive value of $W_5 + W_3 - W_4$ is analogous to warm or light fluid rising and cold or dense fluid sinking, i.e., a direct circulation; a negative value, therefore, is analogous to an indirect circulation. This is the second part of the baroclinic growth mechanism. W_6 represents the barotropic

transformation associated with the shear of the basic state wind within each layer. A positive value represents the down gradient flow of eddy momentum thus strengthening the eddies at the expense of the zonal jet within a given layer.

In the current study, a unit vector $\mathbf{Y}$ is introduced which illustrates the relative importance of the baroclinic and barotropic instability mechanisms at work in the two level model containing horizontal wind shear. The $\mathbf{Y}$ vector is given by

$$\mathbf{Y} = \frac{W_6 \mathbf{i} + W_3 \mathbf{j}}{(W_6^2 + W_3^2)^{1/2}}, \quad (2.38)$$

where $\mathbf{i}$ and $\mathbf{j}$ are orthogonal unit vectors. When pointing straight up with no horizontal component, the unit vector $\mathbf{Y}$ represents baroclinic growth only of the unstable wave being investigated. A horizontal orientation pointing toward the right with no vertical component symbolizes growth by the barotropic process only. If the vector were to point at a 45° angle into the first quadrant, there would be growth by both processes, each contributing equally (see Fig. 11b). In the figures that follow, the $\mathbf{Y}$ vectors are represented by the large arrows on the growth rate curves.

Chapter 3

Experiments and Results

In this chapter, experiments are presented which demonstrate the effects of introducing horizontal shear into the basic state formulation of the two layer model. The synoptic scale baroclinic modes which exist for the non-sheared case are illustrated followed by a presentation of the new modes that appear when shear is added. The growth rates, energetics, and structures of the two most unstable frontal scale modes are investigated for a variety of basic state configurations. First, however, the question of model resolution and its effect on the solution will be addressed.

3.1 Effect of Model Resolution

Computational modes arise due to truncation error and are therefore numerically, but not physically, unstable. If an unstable mode is to be considered physical (i.e., not computational), it should retain its characteristics as the cross-front resolution N in the model is increased. Adding more grid points to the domain should act only to better resolve an actual physical feature of the solution. For lower resolutions in the two layer model (say, $N \leq 50$ grid points in the frontal zone) it is possible for unstable computational modes to emerge which are actually manifestations of this low resolution. IFW illustrates a mode whose *nature* is not clear (shown in Fig. 11 and the last

entries of Appendix C in that paper). This mode resulted from a simulation using a resolution of 40 points in the frontal domain. It is successfully reproduced using the current model at the same resolution ($N=40$), $Ri=5$, and shear amplitude parameter $A=0$ (see Fig. 12). It was found here that as the model resolution is increased to $N=1000$, the mode virtually disappears. Another run (not shown) using a resolution of $N=10,000$, $Ri=5$, and $Ro=1$ indicates that the mode is asymptotically approaching neutrality as the resolution is increased.

Another question of solution dependence on resolution arises in the work of Moore and Peltier (1987). The frontal scale mode which they labeled C0 has an eigenvalue that is somewhat sensitive to their grid resolution. Eigenvalues of their larger, better resolved, synoptic scale modes were not nearly as sensitive to their grid resolution. For Moore and Peltier, higher resolution runs, though desirable, were not possible given their available resources.

In the current study, a resolution of 1000 points in the cross-front direction is routinely used in the Newton-Raphson program, and checks are performed occasionally using resolutions of 2000 to 5000 points in the frontal domain. The eigenvalue program is not able to perform adequately for resolutions of this magnitude; however, all results coming out of the eigenvalue program are either verified or rejected by the Newton-Raphson program. Therefore, confidence in the solution is maintained.

3.2 Transitions of B and E Modes

In the previous works of both Orlanski (1968) and Sinton and Mechoso (1983) using non-sheared formulations of the two layer model, only two unstable modes were

Fig. 12. Spurious mode of Orlanski (1968: solid line, as described in text) and its disappearance as resolution N in the frontal zone is increased (other lines).

found for $Ri = 10$: the B or *Bjerknes* mode and the E or *Eady* mode. These modes are reproduced and shown in Fig. 13 for a shear amplitude parameter of $A = 0$. The B mode shown in Fig. 14 actually represents two modes: the upper B mode (associated with the upper layer) and the lower B mode (associated with the lower layer). A characteristic feature of the upper B mode is that the maximum amplitude of its perturbation pressure field occurs in the upper layer where the interface intersects the rigid upper boundary. The lower B mode is a "mirror image" of the upper B mode, and thus its maximum perturbation pressure occurs in the lower layer at the surface front. These modes will now be modified by gradually adding shear (increasing A) for

RI=10 A=0

Fig. 13. Reproduction of the growth rates for the synoptic scale B and E modes from Sinton and Mechoso (1984) using the current model with a shear amplitude parameter value of $A = 0$. Y vectors are located above the most unstable Rossby number.

the following parameter settings: $Ri = 10$, depth parameter $a = 1$, and shear wavelength parameter $S = 4$ (giving a 500 km shear wave). Since the effect of increasing the shear amplitude is to modify the basic state near the surface front, one expects that, even though the shear exists equally in both layers ($a = 1$), the lower layer B mode will be more affected.

The effect of increasing the shear amplitude A on the B and E modes is illustrated in Figs. 14 through 16. The upper B mode (Fig. 14) is essentially unaffected by the introduction of shear near the surface front. On the other hand, the lower B mode (Fig. 15) is stabilized over the entire range of Ro . The clockwise rotation of the Y

Fig. 14. Upper B mode for shear amplitude values of 0, 0.5, and 1.0 over the full range of Ro for this mode in terms of *doublings per day*. Again, the position of the $\mathbf{Y}$ vector over the abscissa indicates the most unstable Rossby number of the mode. Parameters held constant are: $Ri = 10$, $a = 1$, and $S = 4$.

vector in Fig. 15 illustrates how the barotropic conversion W_b becomes more important as the amplitude is increased.

The result of increasing the shear amplitude parameter on the E1 mode is shown in Fig. 16. Again, the $\mathbf{Y}$ vector shows a tilt toward the barotropic process. The growth rate curves illustrate that increasing the shear amplitude has a stabilizing effect on the E1 mode for Ro below about 0.11, but it has a destabilizing effect for Ro above that value. The increased instability, however, is not significant.

In summary, the overall response of the B and E modes to increases in frontal scale shear ranges from a moderate stabilizing effect on the lower B mode to a weak destabilizing effect on the E1 mode. This relatively small response can be attributed

Fig. 15. As in Fig. 14 except for the lower B mode.

Fig. 16. As in Fig. 14 except for the E1 mode.

to the fact that these modes are large scale (in the lower range of Ro) and primarily baroclinically driven. The scale of the horizontal shear being introduced here is not large enough to interact with the large scale baroclinic processes in these modes. Therefore, it is expected that the effect of introducing frontal scale horizontal shear will show up more clearly at higher Rossby numbers. A different behavior might be expected when the scale of the shear is *similar* to the scale of these baroclinic modes.

3.3 New modes for $Ri = 10$

An investigation is undertaken to explore what modes may exist over a range of Rossby numbers for a given set of shear parameters. The most unstable modes which occur for $Ri = 10$, $A = 1$, $a = 1$, and $S = 4$ are shown in Fig. 17. The upper and lower B modes and E1 mode are seen to the left. In addition to these, there are three new modes with values of $D > 0.8 \text{ day}^{-1}$. The $\mathbf{Y}$ vectors indicate that the large scale modes occurring at lower Rossby numbers are growing by primarily baroclinic processes (vertical arrows) while, as expected, the higher Rossby modes have a very significant barotropic component (horizontal arrows). The strong barotropic nature of these higher Ro modes allows them to combat the stabilizing effect of the short wave cutoff⁷ in baroclinic instability. The mode represented by the solid line is by far the most unstable having a growth rate twice that of the other modes. This mode will be the focus of more detailed investigation and from this point on will be referred to

⁷ The *short wave cutoff* occurs when high vertical velocities associated with short waves augment the adiabatic cooling in regions of warm air advection. This results in eddy potential energy being converted into eddy kinetic energy faster than the potential energy of the basic state can restore the eddy potential energy. Thus, the wave stabilizes (see Fig. 9.1 in Holton 1979).

Fig. 17. All modes which exist for the following basic state setting: $Ri = 10$, $A = 1$, $a = 1$, and $S = 4$. Growth rates are in terms of *doublings per day*.

as the *S-mode* ("S" for shear). In the non-linear realm of the real atmosphere, for a basic state analogous to that described above, further development of the smaller competing modes on the same scale will be suppressed as the S-mode uses the energy that is available for its own growth. The second most unstable mode (dash-dot line in Fig. 17) will be discussed in the final section of this chapter. Its energetics are marked by both a baroclinic component and a shear component; therefore, it is named the *Baroclinic-Shear* mode or simply the *BS-mode*. As the only mode that extends over the entire range of Ro , this mode is a likely candidate to respond to shear at either the frontal scale or the baroclinic (synoptic) scale.

Six other new modes (not shown) with values of $D < 0.35 \text{ day}^{-1}$ were also found for the parameter values of Fig. 17. It is possible that some of these are harmonics of the more unstable modes. These modes have such small growth rates that future development is not expected.

3.4 Investigation of the S-mode

In the following sub-sections, results will be presented which illustrate how the S-mode from the previous section is affected when there is variation of the shear controlling parameters of section 2.2. In addition, the structure and energetics of this mode will be investigated in more detail. In order to set the perspective for the energetics discussions ahead, one is reminded that the barotropic processes defined by W_6 dominate in the S-mode. This is seen clearly in Fig. 18 which shows the variation of the energy conversions $\frac{W_i}{|W_3|}$ through Ro-space for $Ri = 10$. W_6 is typically 20 to 30 times greater than W_3 .

3.4.1 Variation of the "S" parameter

As shown by Eq. (2.7), the effect of increasing the parameter S is to decrease the cross-front extent of the shear (i.e., reduce the wavelength of the shear wave). The effect of this on the S-mode is shown in Fig. 19. The S parameter is varied between 3 and 5. For $S = 3$, the shear wave occupies $1/3$ of the frontal domain L_y (a relatively large scale phenomena compared to $S = 4$ or 5). The effect is to shift the mode to a lower range of Rossby numbers. Conversely, when the S parameter is increased, the shear is confined within a smaller region and the mode is seen to shift to higher Rossby numbers. Since the Rossby number is inversely proportional to the along-

Fig. 18. Variation with Ro of the S-mode $W_i/|W_3|$ energy conversions for the other parameters shown.

front scale of the disturbance, there is the implication that as the shear is confined to increasingly narrower regions behind the surface front the resultant S-modes will have correspondingly shorter along-front scales. Thus the most unstable S-modes maintain a symmetry between their cross-front and along-front dimensions. This *scale dependency* of the unstable mode on the cross-front extent of the shear is a linear relationship. The linearity can be deduced from Fig. 19 by noticing that $\frac{dRo_{\max}}{dS} \sim \text{constant}$, where $Ro_{\max}$ is Ro for the most unstable wave. In addition, the peak growth rates are independent of S variation since $\frac{dD}{dS} \sim 0$. The $\mathbf{Y}$ vectors demonstrate that the barotropic character of the S-mode is also independent of S .

Fig. 19. Variation of the S parameter and its effect on the S -mode.

3.4.2 Variation of Ri

Up to now, only the case for $Ri = 10$ has been investigated. It is of interest to see how changing the static stability characteristics (Ri) of the basic state will alter the S -mode. The results of increasing the Richardson number from 10 to 50 are shown in Fig. 20. Here again, the $\mathbf{Y}$ vector energetics are relatively unaffected during the parameter variation. Throughout the range of Ri , the S -mode retains its barotropic character (even at low Ro), and the peak growth rates remain essentially constant at just above 2 doublings per day.

Fig. 20. Variation of the Ri parameter and its effect on the S-mode.

The relationship between the most unstable Rossby number Ro_{max} and the Ri parameter may be considered linear since, from Eq. (2.4), the frontal domain L_y , and therefore the scale of the shear L_s , varies directly as the Richardson number (notice that when Ri is increased from 10 to 20, Ro_{max} decreases from about 0.76 to 0.38). This result is similar to that of the S parameter variation where it was shown that the growth rate of the instability is independent of the cross-front extent of the shear.

It may now be stated that the maximum growth rate of this mode is virtually independent of *both* Ri and S. However, the along-front scale of the S-mode is not in-

dependent of these quantities since it adheres to the relation (inferred from Figs. 19 and 20):

$$Ro_{\max} \propto \frac{S}{Ri}. \quad (3.1)$$

Another illustration of the independence of the growth rate from Ri is shown in Fig. 21 which contours the growth rate of the S-mode instability in Ri - Ro space for various shear amplitude values. Given any value of Ri , a Rossby number exists where the growth rate and energy characteristics of the S-mode are virtually identical. The difference is in the cross-front scale of the instability. The scale of the disturbances which are of interest in the current study is on the order of 800 km (i.e., frontal scale, as observed in the atmosphere). In this model, for $S = 4$, the Richardson number which corresponds to this scale is roughly $Ri = 10$. This value of Ri is used to demonstrate the variation of the other parameters.

3.4.3 Variation of the "A" parameter

As stated earlier, the S-mode does not exist in an unsheared basic state. Therefore, since the A parameter determines the magnitude of the shear (or mean vorticity anomaly), it is the ultimate source of the S-mode instability. It is expected that as the amplitude parameter is increased the mode will become destabilized, or, conversely, as the parameter is decreased the mode will die out. This is the behavior shown in Figs. 21a through 21d which illustrate the stabilization of the S-mode for decreasing values of A .

Fig. 21. Ri -vs- Ro for the S-mode with $a=1$ and $S=4$. Contours are of *doublings per day*. Range of ordinate is $4 \leq Ri \leq 25$, and range of abscissa is $0 \leq Ro \leq 2$. (a) $A=1$, (b) $A=0.75$, (c) $A=0.5$, (d) $A=0.25$.

3.4.4 Variation of the "a" parameter

It was pointed out by Joly and Thorpe (1990) that frontal scale instability is increased when the potential vorticity anomaly has a larger vertical extent in the fluid,

Fig. 22. Variation of the depth parameter " a " and its effect on the S-mode.

i.e., when the PV anomaly is not confined to the lower boundary. In the current two layer model, the technique used to produce the analog of an interior potential vorticity anomaly is the implementation of the *depth parameter* (described in section 2.2.2). When the depth parameter a is equal to 1, the cross-front shear (and therefore the PV anomaly) exists equally in both layers. It is for this case where the maximum instability is to be expected. As a is decreased, stabilization is expected. Fig. 22 shows the response of the S-mode to variations in the depth parameter. As a is decreased, the growth rate also decreases, and the S-mode disappears completely for $a \lesssim 0.1$. The positions of the $\mathbf{Y}$ vectors show that $Ro_{\max}$ remains fairly constant at about

$Ro_{\max} = 0.76$, and the barotropic process defined by W_6 dominates the energetics. Furthermore, throughout the parameter variation, it is W_6 from the *upper layer* in the region of the potential vorticity anomaly which contributes the most, regardless of the depth parameter value (compare Figs. 23a and 23b).

3.4.5 Structure of the S-mode

It was mentioned in section 2.1 that the along-front direction of the model is represented spectrally. In order to display the perturbation pressure characteristics of a mode in the x-direction (given the finite difference solution in the y-direction), x is varied from 0 to 2π in the expression $p' = \hat{P}(y) \exp i(kx + \sigma t)$, where $t = 0$ and $\hat{P}(y)$ is the solution from the Newton-Raphson program. This procedure yields a two-dimensional perturbation pressure field. The p' contours of the S-mode are shown in Fig. 24 for $Ro = Ro_{\max}$. The units are non-dimensionalized, and the difference between the minimum and maximum values is arbitrarily set between -100 and 100 with contours every 5 units. This nondimensionalization is used because the interest here is in *relative* magnitudes and positions of the features on the plot, not specific numerical values which are arbitrary for perturbations in linear studies. Included to the right of each pressure plot is the corresponding basic state wind profile $U(y)$ for that layer. The range of the wind scale is from -20 to 20 $m s^{-1}$. Again, what is most important is the *shape* of the curve. Although the frontal domain for $Ri = 10$ is $L_y = 2000$ km, only 800 km is shown since the perturbation amplitudes in each layer essentially go to zero past this point.

The peak pressure perturbation is in the lower layer at the surface front. Beyond 500 km there is virtually no amplitude left in the pressure field. In the upper layer the

Fig. 23. W_6 for each layer for the parameter values shown. (a) $a=1.0$, and (b) $a=0.1$. $|\bar{W}_3|$ is the mean absolute value of W_3 .

Fig. 24. Perturbation pressure field of the S-mode for the parameters shown. (a) is the upper layer and (b) is the lower layer. The panels to the right are the corresponding $U(y)$ fields; the scale ranges from -20 to 20 m/s.

peaks are also found at less than 500 km. Recall that for $S = 4$ the value of L_y is $\frac{L_y}{4}$, or 500 km in this case. This illustrates that the cross-front scale of the perturbation is close to the along-front scale. For this reason the mode is said to be *square*. Furthermore, based on the scale arguments of sections 3.4.1 and 3.4.2. the mode will remain square no matter what value of Ri or S is chosen.

Another feature of the S-mode displayed in Fig. 24 is the relative location of the upper and lower layer low pressure centers. The vertical tilt of the trough axis varies from negative or "west" to "east" along the surface front to the more traditional quasi-geostrophic baroclinic (or east to west) tilt in the interior of the frontal region. Traditional arguments must be applied judiciously since this mode is not only predominantly barotropically driven, but also (as will be shown later) not necessarily quasi-geostrophic.

The source of the barotropic conversion is clearly seen when the wind profile for Fig. 24a is compared to the phase shift in the pressure field. Recall that for quasi-geostrophic barotropic growth in the real atmosphere there should be a southeast to northwest orientation of the ridge/trough axes south of the zonal jet axis and a southwest to northeast orientation north of the jet axis. In the two level model, the zero line of the p' contours may be used to determine the orientation of the ridge/trough axes. The orientation most favorable for barotropic development is located at about 250 km (see Fig. 24a). In addition, $\frac{dU}{dy}$ is positive in this region; in fact, it is a maximum at 250 km (inferred from Fig. 9a and right hand panels of Fig. 24). The condition for barotropic development can be expressed by the inequality $-\overline{u'v'} \frac{dU}{dy} > 0$. Fig. 25 shows the perturbation wind vector $u'I + v'J$. Not only are the vectors at their largest amplitude in the region near 250 km, but the quantity $\overline{u'v'}$

Fig. 25. Perturbation vector wind field $u'_1 + v'_j$ superimposed on the perturbation pressure contours of Fig. 24. The contour interval is increased by a factor of 2 to aid in clarity. The scale Λ between the upper and lower layer vectors is $\Lambda = 0.942$ where $|V_1|_{\max} = \Lambda |V_2|_{\max}$.

(deduced from the direction of the vectors) has a relatively large negative value there. These elements combine to cause a large barotropic conversion in the region near 250 km. Referring back to Fig. 23a, it is seen that, indeed, the greatest contribution to the barotropic development is in the region near 250 km. Furthermore, there is essentially no contribution to development (or decay for that matter) from the lower layer.

Looking again at Fig. 25, it becomes clear that in many regions of the pressure field, especially where the wind vectors are greatest in magnitude, the flow is ageostrophic (i.e., there is cross-isobar flow). The ageostrophy is greatest in the lower layer near the surface front. It is the use of primitive equations which allows ageostrophic flow to occur. Therefore, in a quasi-geostrophic or semigeostrophic model, modes which exhibit this degree of ageostrophic behavior may not appear.

The contour plots in Fig. 26 illustrate the structure of the S-mode when the depth parameter is decreased to $a = 0.1$. The maximum amplitude of the perturbation pressure is still at the surface front; however, in this case a secondary maximum exists in the region of $(\frac{dU}{dy})_{\max}$. In the upper layer, there are no longer two well-defined low pressure centers as in the previous case, nor is there an extreme phase shift; however, the orientation of the ridge/trough axes in the cross-front direction is still favorable for barotropic development. The mean wind profile in the upper layer, though nearly constant for this value of a , still yields a positive value of $\frac{dU}{dy}$ in the region near 250 km. Looking back to Fig. 23b, W_s in the upper layer, although reduced, is still the primary contributor to the barotropic conversion.

The perturbation wind vectors for the $a = 0.1$ case are shown in Fig. 27. Again, the lower layer wind is quite ageostrophic near the surface front. But the upper layer

Fig. 26. As in Fig. 24 except for $a=0.1$.

Fig. 27. As in Fig. 25 except wind vectors are superimposed on the pressure field of Fig. 26. Scale: $\Delta = 1.23$.

wind is now essentially geostrophic with only slight deviations into ageostrophy occurring near the maximum in $\frac{dU}{dy}$. It should be kept in mind that, although the above analysis for $a = 0.1$ is insightful into the behavior of the mode, the most unstable growth rate for this case is only 0.14 doublings per day (i.e., it takes 7 days for the disturbance to double). Therefore, it is unlikely that a mode with these characteristics would undergo development.

For a final comparison, the perturbation vector/contour plots corresponding to $A = 0.25$ and $a = 1$ for the S-mode are shown in Fig. 28. The major characteristics of the mode are the same as before (maximum pressure at the front in the lower layer, upper layer maximum away from the surface front, etc.), but in this case the upper layer pattern bears a strong resemblance to the upper layer pattern for $a = 0.1$ (Fig. 27a). The relative maxima and minima are in generally the same location, and its perturbation wind in (shown in Fig. 28a) is also quite geostrophic. If the A parameter were increased further, presumably the magnitudes of the pressure extremes could be made nearly the same. These results are not entirely surprising since decreasing the shear amplitude parameter A has the same effect on the upper layer mean wind as decreasing the depth parameter a .

3.4.6 Structure of the most unstable mode for $B = 3$

The function of the parameter B (described in section 2.2.4) is to create one or more potential vorticity anomalies at a greater distance behind the surface front. The profiles of the potential vorticity and potential vorticity gradient in the lower layer for $B = 3$ are shown in Fig. 29. Notice that the potential vorticity gradient changes sign six times in the frontal domain.

Fig. 28. As in Fig. 25 except $A=0.25$. Scale: $\Lambda = 1.32$.

Fig. 29. As in Fig. 10 except for $B=3$.

The resultant S-mode for $B=3$ is shown in Fig. 30. Since much more of the frontal domain is affected by this change, the entire domain is displayed. The x-scale is the same as the y-scale. Some remnants of the $B=1$ characteristics are seen in the shapes of the curves in the respective upper and lower layers at $y < 800$ km; however, in this case the maximum perturbation pressure occurs in the *upper* layer and is located near the center of the domain. The peak growth rate has also increased by about 34 percent. The barotropic growth mechanism is supported by the ridge/trough axis orientation and the sign of $\frac{dU}{dy}$ throughout the upper layer domain. The $\mathbf{Y}$ vector in Fig. 30d shows that the energetics are, as expected, barotropic. Fig. 30d also illustrates the effect on this mode of removing the shear from the upper layer. Reducing a causes the line labeled "B" in panel (c) to become more linear. As seen in panel (d), the growth rates of the mode continue to decrease as a is decreased until the mode dies completely between $a = 0.1$ and 0.2 .

3.5 Investigation of the BS-mode

To demonstrate that the emergence of an unstable mode is possible with no shear in the upper layer, investigations are performed with $(a,B)=(0,3)$ and $(a,B)=(0,1)$. The most unstable waves for this set of parameters are shown in Figs. 31 and 32, respectively. Clearly this mode is different from the S-mode. It is a *shallow* frontal mode with essentially no features in the upper layer at all, while in the lower layer most of the amplitude is concentrated at the surface front. In this respect (and in its energetics) it is quite similar to the secondary disturbances found by Sinton and Mechoso (1984). Its growth rate is about 53 percent that of the S-mode, and when the depth parameter is increased (Fig. 31d) the effect on the growth rate is negligible.

Fig. 30. As in Fig. 24 except $B=3$. Panel (a) is lower layer, (b) is upper layer, (c) is the mean wind profiles for the two layers (A and B respectively), and (d) shows the variation of growth rate with the depth parameter a . All other parameters shown at top.

Fig. 31. As in Fig. 30 except for the BS-mode with $B=3$.

Fig. 32. As in Fig. 31 except $B = 1$.

This is expected since the upper layer has little perturbation pressure amplitude. The $\mathbf{Y}$ vector shows this mode to be mostly barotropic; however, it is more baroclinic than the S-mode. This mode is in fact the BS-mode shown in Fig. 17 where $B = 1$. This is apparent from the mode's structure, growth rate, energetics, and low sensitivity to the depth parameter a (see Fig. 32). For the BS-mode at $a = 0$, increasing B somewhat reduces the along-front scale of the most unstable wave from $Ro_{\max} = 0.708$ at $B = 1$ to $Ro_{\max} = 0.873$ at $B = 3$. The effect of increasing a on this mode for $B = 1$ is to slightly increase the along-front scale of the wave by shifting $Ro_{\max} = 0.71$ at $a = 0$ to $Ro_{\max} = 0.69$ at $a = 1$. At any rate, for shear confined to the lower layer, the BS-mode is the most unstable mode.

3.5.1 Variation of the Ri parameter

In order to test the hypothesis that the BS-mode might be responsive to the introduction of shear on the baroclinic (synoptic) scale, the Ri parameter is varied. As found by Simmons (1974), meridional shears on the scale of the Rossby radius of deformation (the appropriate length scale for baroclinic instability) can destabilize quasi-geostrophic, baroclinic modes by introducing barotropic instability.

The Rossby radius of deformation L_o is defined for this model (e.g., Sinton 1984) as:

$$L_o^2 = Ri \frac{(U_2 - U_1)^2}{f^2}. \quad (3.2)$$

For the horizontally two-dimensional "square" waves in this study, a scaling length L_w can be defined as:

$$L_w^2 = \ell_x^2 + \ell_y^2, \quad (3.3)$$

where $\ell_x = \frac{1}{k}$ and $\ell_y = \ell_x$. This gives:

$$L_w^2 = \frac{2}{k^2}. \quad (3.4)$$

From the definition of Ro:

$$Ro = \frac{k(U_2 - U_1)}{2f} = \frac{(U_2 - U_1)}{2\ell_x f}. \quad (3.5)$$

Equations (3.4) and (3.5) then give

$$L_w^2 = \frac{(U_2 - U_1)^2}{2Ro^2 f^2}. \quad (3.6)$$

Thus from equations (3.2) and (3.6)

$$\frac{L_o^2}{L_w^2} = 2 Ri Ro^2. \quad (3.7)$$

At the scale appropriate for baroclinic instability, the scale of the wave equals the Rossby radius of deformation as stated above. Thus, from (3.7)

$$2 Ri Ro_{BC}^2 = 1, \quad (3.8)$$

where Ro_{BC} is defined as the Rossby number appropriate for baroclinic instability. Solving (3.8) for Ro_{BC} gives

$$Ro_{BC} = \frac{1}{\sqrt{2} Ri}. \quad (3.9)$$

An examination of Ro_{max} indicates that Eq. (3.9) holds when baroclinic instability is the primary growth mechanism (see Figs. 13, 14, and 15). For $Ri=10$, Eq. (3.9) gives $Ro_{BC} = 0.2284$ which agrees well with these results. For the shearing modes, Eq. (3.1) can be rewritten as:

$$Ro_S = C \frac{S}{Ri}, \quad (3.10)$$

where Ro_S is Ro_{max} for shearing modes, and C is a constant determined empirically from the results in Figs. 19 and 20. $C \simeq 1.77$ for the BS-mode over the range $10 \leq Ri \leq 40$ (see Fig. 33). Equations (3.9) and (3.10) can now be combined to determine the Ri at which the length scale for shearing instability *and* baroclinic instability are the same ($Ro_S = Ro_{BC}$). Thus, Ri_{BS} is defined as:

$$Ri_{BS} = 2(C S)^2. \quad (3.11)$$

For $S = 4$ and $C = 1.77$ (given above), $Ri_{BS} \simeq 100$. Thus, as Ri increases, the scale of the shear approaches the synoptic scale appropriate for baroclinic instability and the BS mode destabilizes as shown in Fig. 33.

3.5.2 Structure of the most unstable BS-mode

Equation (3.11) is applied for $S = 2$, $a = 0$, and $A = 1$ to yield $Ri_{BS} \simeq 25$. The most unstable wave is found at $Ro = 0.154$. Its growth rate, energetics, and structure are

Fig. 33. Trace of the BS-mode through Ri -space for $A=1$, $a=0$, and $S=4$. All arrows are Y vectors. The numbers above each post are the Ro_{max} for those Ri numbers.

shown in Figs. 34 and 35. Note that its Y vector is at a 45° angle indicating it is growing equally by baroclinic and barotropic processes. The maximum amplitude is found in the lower layer just behind the front, and the upper layer amplitude is approximately 25% of the lower layer. This is significantly more than is the case with shears occurring on a scale smaller than L_s (see Fig 32).

This structure is similar to that found for the lower B mode at the same Ri and Ro for $A=0$, i.e., the unsheared case shown in Fig. 36. Indeed, this lower B or Bjerknes mode results from the BS-mode when the shear-wave amplitude is de-

Fig. 34. Trace of the BS-mode through Ro -space in terms of doublings per day.

creased to zero.⁸ This suggests an alternative name for this mode: the *Bjerknes-Shearing mode* (still the BS-mode). The wind vectors in Figs. 35 and 36, for the sheared and unsheared cases, respectively, indicate that the sheared case is considerably more ageostrophic than the unsheared case. In addition, the growth rate for the most unstable mode (a B mode) in the unsheared case at $Ri=25$ occurs at $Ro=0.165$ and is only about 25.5 percent of the growth rate for the BS-mode in the sheared case discussed above. Comparing the growth rates of the unsheared B

⁸ Another case for $Ri=10$, $A=1$, and $a=0$ shows the most unstable mode for these parameters to be a BS-mode occurring at $Ro=0.231$ and $S=1.09$. The growth rate is 2.619 doublings per day, and the Y vector has an angle of about 72° off the abscissa indicating the growth process is about $1/4$ barotropic and $3/4$ baroclinic.

Fig. 35. Perturbation pressure and wind vectors for the BS-mode for the parameters shown. (a) upper layer (b) lower layer. Scale $\Lambda = 6.42$.

Fig. 36. As in Fig. 35 except $A=0$. Scale $\Lambda = 6.35$.

mode and the sheared BS-mode reveals that the introduction of scale appropriate shear results in a quadrupling of the growth rate.

Chapter 4

Summary and Conclusions

The motivation for this work stemmed from the hypothesis that *frontal scale waves are primarily shear driven*. To address this problem, cross-front shear was introduced into the basic state of the two layer model of Orlanski (1968). The two layer model was chosen for its simplicity, ability to simulate ageostrophic flow, and ability to facilitate a large number of grid points (high resolution) in the frontal domain without requiring extensive computer resources.

The model was first used with the shear turned off ($A=0$) to investigate the sensitivity to cross-front grid resolution of a higher Ro mode from Orlanski (1968). It was determined that this mode was spurious since it disappeared when grid resolution was increased.

With frontal scale shear turned on ($A=1$), the model was applied to the larger scale B and E modes which exist in the non-sheared two layer model. There was little response since these are *baroclinic modes* and thus mostly unaffected by the relatively small scale barotropic processes brought about by the frontal scale shear formulation.

Another motivation for choosing the two layer model was to ascertain whether it is capable of producing the frontal scale unstable modes that Solberg (1928) had ori-

ginally sought to find. With the implementation of the shear formulation it was found that, indeed, frontal scale waves *were* unstable.

It is important to keep a proper perspective when interpreting the results from Chapter 3. It would not be prudent to conclude from this study that, given some basic state conditions in the real atmosphere which are similar to those from a case in this paper, a wave will develop with the exact scale and growth rates as the one produced by this model. There are obviously a great many simplifications in the two layer frontal model, and it is intent of this study that the reader draw the more general conclusions stated in Table 1.

The boundary condition formulation used in this study is somewhat abrupt in that it causes the shear to end immediately outside the frontal domain. For comparison, an alternate boundary condition was implemented which includes a lateral "sponge" zone in the non-frontal regions (see Appendix C). All major results were the same, particularly for $a = 0$ when the upper layer shear is zero and the lower layer shear is negligible where the interface intersects the upper boundary. However, for $a = 1$ the S-mode was destabilized by about 30 percent, and the value of Ro_{max} shifted slightly toward lower Ro . It is expected that altering the boundary conditions will affect some of the specific results. Since many boundary condition formulations could be used with equal justification, the point is again made that the results of this study should be interpreted generally, not specifically.

Finally, the results for the BS-mode have implications for rapid cyclogenesis in frontal zones. An application of this result may help explain why rapid and intense cyclogenesis occurs along fronts in the presence of PV anomalies and moisture. The moisture lowers the effective static stability, thus shortening the scale of the unstable

Table 1: Characteristics of the unstable frontal scale modes

S-mode	BS-mode
Only exists for $a > 0$ (i.e., "deep" anomalies).	Exists for $a \geq 0$.
Growth rate is independent of both the scale of shear and the static stability (Ri).	Growth rate increases as scale of shear approaches Rossby radius of deformation ($L_s \rightarrow L_o$).
Along front scale of disturbance is directly proportional to scale of the shear and inversely proportional to Ri.	Same as S-mode.
Maximum growth rate of the disturbance is ~ 2 doublings day^{-1} for a $\zeta = f$ anomaly and $a = 1$. For a basic state with $Ri = 10$ and $S = 4$: $L_x \sim 800$ km	Quadrupling of non-sheared B mode growth rate, from 0.55 to 2.17 doublings day^{-1} when anomaly of $\zeta = f$ is added scale appropriately: $Ri = 25$, $a = 0$, and $S = 2$; $L_x \sim 3930$ km.
The flow is ageostrophic.	The flow is ageostrophic for $A > 0$.
Energetics are barotropic only. The barotropic conversion is maximum in the upper layer in the region of $(dU_2/dy)_{\text{max}}$.	Energetics are mixed: barotropic-baroclinic. The barotropic conversion is maximum in the lower layer in the region of $(dU_1/dy)_{\text{max}}$.

baroclinic waves to the point where they can interact with the PV anomaly. A suitably scaled PV anomaly might be provided by a tropopause fold and the ensuing injection of high PV air into the frontal zone. Obviously, modelling such a scenario is beyond the capabilities of a simple two layer model, but the instability mechanism revealed by this study may operate in the real atmosphere-- for example the 11/13/81 bomb

off the Northern California coast (see Fig. 2 in Smigielski and Ellrod 1985). A revision of the flow chart from Fig. 5 which incorporates the current findings is considered in Fig. 37. Hopefully the hypothesis posed by these results (represented by the hollow arrows on the flowchart) will be tested by others using more sophisticated tools.

Fig. 37. Revised frontal instability flowchart.

Appendix A

Frontal Slope Equation

A.1 Critical Amplitude, h_{crit}

The critical amplitude is defined such that $h_o = h_{crit}$ when $\bar{\zeta}_1 + f = 0$. Thus

$$\bar{\zeta}_1 = -\frac{dU_1}{dy} = -f. \quad (A.1)$$

Taking $\frac{d}{dy}$ (2.11), substituting the result into (A.1), and rearranging gives

$$\frac{d^2 F}{dy^2} = \frac{f^2}{g} \left(\frac{\bar{\rho}}{\rho_1 - \rho_2} \right). \quad (A.2)$$

Taking $\frac{d^2}{dy^2}$ (2.10) for $\delta_2 = 0$ and substituting into (A.2) gives

$$h_0 \ell^2 \cos \ell y = \frac{f^2}{g} \left(\frac{\bar{\rho}}{\rho_1 - \rho_2} \right), \quad (A.3)$$

where $\ell = \frac{2\pi}{L_s}$. Noting that the maximum $h_o = h_{crit}$ occurs when $\cos \ell y = 1$, and using equation (2.7), the following expression may be obtained:

$$h_{crit} = \frac{L_y^2 f^2 \left(\frac{\bar{\rho}}{\rho_1 - \rho_2} \right)}{g(2\pi S)^2}. \quad (A.4)$$

Using (2.4b) and (2.2), (A.4) is simplified to

$$h_{crit} = \frac{H Ri}{4\pi^2 S^2}. \quad (A.5)$$

A.2 Coefficients of the Frontal Slope Equation

In order to match the cosine curve to the bell curve with third order continuity at the point $\hat{y} = (B - 1/4)L_s$, the first, second, and third derivatives of (2.9) are evaluated at $y = \hat{y}$ for $\delta_1 = 0$ to give the bell curve matching conditions, and for $\delta_2 = 0$ to give the cosine matching conditions. The two are then set equal giving

$$\bar{h} = m\hat{y} + b_o \frac{w^2}{w^2 + (\hat{y} + y_o)^2} + c_o \hat{y} + e_o = m\hat{y} + h_o, \quad (A.6)$$

$$\bar{h}' = m - b_o \frac{2w^2(\hat{y} - y_o)}{[w^2 + (\hat{y} + y_o)^2]^2} + c_o = m - \ell h_o, \quad (A.7)$$

$$\bar{h}'' = b_o \left\{ \frac{8w^2(\hat{y} - y_o)^2}{[w^2 + (\hat{y} + y_o)^2]^3} - \frac{2w^2}{[w^2 + (\hat{y} + y_o)^2]^2} \right\} = 0, \quad (A.8)$$

$$\bar{h}''' = b_o \frac{24w^2(\hat{y} - y_o) \left[1 - \frac{2(\hat{y} + y_o)^2}{w^2 + (\hat{y} + y_o)^2} \right]}{[w^2 + (\hat{y} + y_o)^2]^3} = \ell^3 h_o, \quad (A.9)$$

where again $\ell = \frac{2\pi}{L_s}$, L_s is given by equation (2.7), and h_o is given by (2.5). The final condition is imposed that the bell curve at L_y is the same as the linear front at that point:

$$\bar{h} = mL_y + b_o \frac{w^2}{w^2 + (L_y + y_o)^2} + c_o L_y + e_o = mL_y \quad (\text{A.10})$$

which simplifies to

$$b_o \frac{w^2}{w^2 + (L_y + y_o)^2} = -(c_o L_y + e_o). \quad (\text{A.11})$$

There are now 5 equations (A.6) through (A.10) in the 5 unknowns w , y_o , b_o , c_o , and e_o . Equation (A.8) is solved for y_o to yield

$$y_o = \hat{y} - \frac{w\sqrt{3}}{3}. \quad (\text{A.12})$$

Solving (A.9) for b_o yields

$$b_o = \frac{16\ell^3 w^3 h_o \sqrt{3}}{81}. \quad (\text{A.13})$$

Using (A.13) in (A.7) and solving for c_o yields

$$c_o = \ell h_o \left(\frac{2\ell^2 w^2}{9} - 1 \right). \quad (\text{A.14})$$

Equations (A.12) through (A.14) can be used in (A.6) to solve for e_o in terms of w as follows:

$$e_o = h_o \left[1 - \frac{4\sqrt{3}}{27} \ell^3 w^3 - \ell \left(\frac{2\ell^2 w^2}{9} - 1 \right) \hat{y} \right]. \quad (\text{A.15})$$

Equations (A.12) in (A.14) and (A.15) may now be used in (A.11) with some algebra to obtain the following quadratic equation for w :

$$w^2 \left[\frac{4\hat{T}}{3} - \frac{2(\hat{T} + 1)^3}{9} \right] + w \left[\frac{2\hat{L}\sqrt{3}}{3} \hat{T} \right] + \hat{L}^2 \hat{T} = 0, \quad (\text{A.16})$$

where $\hat{T} = \hat{L}\ell - 1$ and $\hat{L} = L_y - \hat{y}$. After solving the quadratic equation for w , this value may be substituted into (A.12) through (A.15) to obtain the other constants.

Appendix B

Eigenvalue Method of Solution

The eigenvalue method of solution used in this study is to solve the equations governing flow in the periodic, shallow water, two layer model on an f -plane with a free upper surface and rigid lateral boundaries. It is a modification of the method employed by Sinton (1984). The model equations are:

$$\frac{\partial \mathbf{V}_j}{\partial t} = -\mathbf{V}_j \cdot \nabla \mathbf{V}_j - f \mathbf{k} \times \mathbf{V}_j - \nabla \phi_j, \quad (B.1)$$

$$\frac{\partial h_j}{\partial t} = -\nabla \cdot (h_j \mathbf{V}_j), \quad (B.2)$$

where $j = 1$ for the lower layer and $j = 2$ for the upper layer; $\mathbf{V}_j$, h_j , and ρ_j are the same as those defined in Chapter 2; and $\mathbf{k}$ is the vertical unit vector. The geopotential height ϕ_j for each layer are given by:

$$\phi_1 = g \left(h_1 + \frac{\rho_2}{\rho_1} h_2 \right), \quad (B.3)$$

$$\phi_2 = g(h_1 + h_2). \quad (B.4)$$

should decay exponentially. Therefore, W is chosen to be far enough from the frontal region such that the condition $v'_j = 0$ at the boundaries will not alter the solutions. A suitable decay distance of 3 e-foldings can be given in terms of the x-wavenumber k as $\exp(-kW) = \exp(-3)$ from which $W = 3/k$ results. Where the frontal surface intersects the upper and lower boundaries, the constraint is that those points remain on their respective surfaces such that $\left(\frac{dh'_1}{dt}\right)_B = 0$ and $\left(\frac{dh'_2}{dt}\right)_C = 0$, where

$$\frac{dh'_j}{dt} = -\bar{h}_j \left(\frac{\partial u'_j}{\partial x} + \frac{\partial v'_j}{\partial y} \right) \quad (B.6)$$

Since for the lower layer $\left(\frac{dh'_1}{dt}\right)_B = 0$, $\left(\frac{dh'_1}{dt}\right)_Q$ can be found through linear interpolation as follows

$$\left(\frac{dh'_1}{dt}\right)_Q = \frac{2}{3} \left(\frac{dh'_1}{dt}\right)_B + \frac{1}{3} \left(\frac{dh'_1}{dt}\right)_P = \frac{1}{3} \left(\frac{dh'_1}{dt}\right)_P \quad (B.7)$$

Equation (B.6) in finite difference form can then be solved for $(v'_1)_B$. Satisfying the boundary condition at point C with $(v'_2)_C$ is done in a similar way using points R and S.

The matrix A is banded with *upper band* = *lower band* = 12. The only matrix solving algorithm known (at the time of this writing) which takes advantage of the banded nature of some matrices only applies to band widths no greater than 5. Therefore the general complex matrix solver IMSL EIGCC was used in this study. A deficiency in this algorithm is that numerical precision is reduced as the size of the matrix increases. It was determined empirically that it is rarely possible to use a resolution greater than 64 points in the frontal domain and 8 points in each W region.

Above this, the accuracy of the solution becomes too low and the run times too high. Since there are two (one) variables carried at even grid points, and four (two) variables carried on the odd points in the frontal (non-frontal) region, the dimension of the matrix A is given by $(3[n + m] - 2) \times (3[n + m] - 2)$ where n is the number of grid intervals in the frontal region, and m is the number of grid intervals in one W (non-frontal) region.

Appendix C

Newton-Raphson Method of Solution

The numerical scheme presented here is an adaptation of the Newton-Raphson method used in IFW and presented in Appendix A of that paper. For a fixed set of the parameters, Ri , Ro , A , S , B , and a , the value of $\hat{P}_2$ at the surface front, $y = 0$, is arbitrarily set equal to 1 while $\hat{P}_1$ at $y = 0$ is given an initial guess of D^{v-1} where the superscript v refers to the guess iteration number. Thus since $\hat{P}_2$ at $y = 0$ is always 1:

$$\left(\frac{\hat{P}_1}{\hat{P}_2} \right)_{y=0}^v = \frac{D^v}{1} = D^v \quad (C.1)$$

for all v . Equations (2.28) and (2.29) can be written symbolically as

$$M_1(\hat{P}_1, \hat{P}_2, \sigma)^v = 0, \quad (C.2)$$

$$M_2(\hat{P}_1, \hat{P}_2, \sigma)^v = 0, \quad (C.3)$$

where M_1 and M_2 are second order linear differential operators. At $y = 0$, boundary conditions (2.34) and (2.31), respectively, give

$$\left(\frac{d\hat{P}_1}{dy} \right)_{y=0}^v = \left\{ D \left[\frac{kf}{(\sigma + kU_1)} + K_1 \right] - K_1 \right\}^v \quad (C.4)$$

and

$$\left(\frac{d\hat{P}_2}{dy} \right)_{y=0}^v = k \quad (C.5)$$

where

$$K_1 = \frac{\frac{\bar{\rho}}{g(\rho_1 - \rho_2)} \left[f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2 \right]}{\frac{d\bar{h}}{dy}}$$

and $\hat{P}_1(y=0) = D^v$ and $\hat{P}_2(y=0) = 1$ have been used.

At the intersection of the frontal surface and the rigid upper lid at $y = L_y$, the boundary conditions (2.30) and (2.35), respectively, give

$$F^v = \left\{ \frac{d\hat{P}_1}{dy} + k\hat{P}_1 \right\}_{y=L_y}^v = 0 \quad (C.6)$$

and

$$G^v = \left\{ \frac{d\bar{h}}{dy} \left(\left[\frac{d\hat{P}_2}{dy} - \frac{kf}{(\sigma + kU_2)} \hat{P}_2 \right] - K_2 [\hat{P}_1 - \hat{P}_2] \right) \right\}_{y=L_y}^v = 0 \quad (C.7)$$

where

$$K_2 = \frac{\frac{\bar{\rho}}{g(\rho_1 - \rho_2)} \left[f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2 \right]}{\frac{d\bar{h}}{dy}}$$

The frontal domain, $0 \leq y \leq L_y$, is divided into N equal intervals so that finite difference approximations to $\frac{d\hat{P}_j^v}{dy}$ and $\frac{d^2\hat{P}_j^v}{dy^2}$ can be constructed. Using some initial guess σ^{v-1} and D^{v-1} for σ and $\hat{P}_1(y=0)$ respectively, equations (C.2) and (C.3) with starting boundary conditions (C.4) and (C.5) can be integrated or "shot" forward in y over the N intervals to $y = L_y$. At this point, F^v and G^v are evaluated using the bracketed terms in (C.6) and (C.7). The resulting F^v and G^v will not generally equal zero as they should if the $y = L_y$ boundary conditions are to be met. Thus corrections in σ^v and D^v must be made in order to better satisfy the boundary conditions so that for some iteration, say $v = m$, the values of F^{v-m} and G^{v-m} approach zero. F^v and G^v can be written symbolically as:

$$F^v = F(\sigma^v, D^v) \quad \text{and} \quad G^v = G(\sigma^v, D^v). \quad (\text{C.8})$$

Then a logical correction scheme for F^v and G^v would be:

$$F^{v+1} = F^v + dF^v = 0 \quad \text{and} \quad G^{v+1} = G^v + dG^v = 0. \quad (\text{C.9})$$

$$\text{Thus} \quad dF^v = -F^v \quad \text{and} \quad dG^v = -G^v \quad (\text{C.10})$$

$$\text{where} \quad dF^v = \frac{\partial F^v}{\partial \sigma} \Delta \sigma^v + \frac{\partial F^v}{\partial D} \Delta D^v = -F^v \quad (\text{C.11})$$

and
$$dG^v = \frac{\partial G^v}{\partial \sigma} \Delta \sigma^v + \frac{\partial G^v}{\partial D} \Delta D^v = -G^v \quad (C.12)$$

Solving the system of equations (C.11) and (C.12) for the unknowns $\Delta \sigma^v$ and ΔD^v gives:

$$\Delta \sigma^v = \left(\frac{\partial F^v}{\partial D} G^v - \frac{\partial G^v}{\partial D} F^v \right) \left[\frac{\partial F^v}{\partial \sigma} \frac{\partial G^v}{\partial D} - \frac{\partial G^v}{\partial \sigma} \frac{\partial F^v}{\partial D} \right]^{-1} \quad (C.13)$$

and

$$\Delta D^v = \left(\frac{\partial F^v}{\partial \sigma} G^v - \frac{\partial G^v}{\partial \sigma} F^v \right) \left[\frac{\partial F^v}{\partial D} \frac{\partial G^v}{\partial \sigma} - \frac{\partial G^v}{\partial D} \frac{\partial F^v}{\partial \sigma} \right]^{-1} \quad (C.14)$$

Corrected values for σ and D then become:

$$\sigma^{v+1} = \sigma^v + \Delta \sigma^v \quad \text{and} \quad D^{v+1} = D^v + \Delta D^v \quad (C.15)$$

In order to find $\frac{\partial F^v}{\partial D}$, $\frac{\partial F^v}{\partial \sigma}$, $\frac{\partial G^v}{\partial D}$, and $\frac{\partial G^v}{\partial \sigma}$ used in (C.13) and (C.14), $\frac{\partial}{\partial D}$ and $\frac{\partial}{\partial \sigma}$ of (C.6) and (C.7) must be taken. Defining

$$\hat{T}_j^v = \frac{\partial \hat{P}_j^v}{\partial D} \quad \text{and} \quad \hat{S}_j^v = \frac{\partial \hat{P}_j^v}{\partial \sigma} \quad (C.16)$$

then gives:

$$\frac{\partial F^v}{\partial D} = \left\{ \frac{d\hat{T}_1}{dy} + k\hat{T}_1 \right\}_{y=L_y}^v \quad (C.17)$$

$$\frac{\partial F^v}{\partial \sigma} = \left\{ \frac{d\hat{S}_1}{dy} + k\hat{S}_1 \right\}_{y=L_y}^v \quad (C.18)$$

$$\frac{\partial G^v}{\partial D} = \left\{ \frac{d\bar{h}}{dy} \left(\left[\frac{d\hat{T}_2}{dy} - \frac{kf}{(\sigma + kU_2)} \hat{T}_2 \right] - K_2 [\hat{T}_1 - \hat{T}_2] \right) \right\}_{y=L_y}^v \quad (C.19)$$

$$\begin{aligned} \frac{\partial G^v}{\partial \sigma} = & \left\{ \frac{d\bar{h}}{dy} \left(\left[\frac{d\hat{S}_2}{dy} - \frac{kf}{(\sigma + kU_2)} \hat{S}_2 \right] - K_2 [\hat{S}_1 - \hat{S}_2] \right) \right. \\ & \left. + \frac{d\bar{h}}{dy} \frac{kf}{(\sigma + kU_2)^2} \hat{P}_2 + \frac{2\bar{p}}{g(\rho_1 - \rho_2)} (\sigma + kU_2) (\hat{P}_1 - \hat{P}_2) \right\}_{y=L_y}^v \end{aligned} \quad (C.20)$$

$\hat{P}_1$ and $\hat{P}_2$ at $y = L_y$ were determined previously by numerically integrating (C.2) through (C.5) in y to $y = L_y$. $\hat{S}_1$, $\hat{S}_2$, $\hat{T}_1$, and $\hat{T}_2$ must also be found at $y = L_y$ in order to evaluate (C.17) through (C.20). To accomplish this, $\frac{\partial}{\partial D}$ and $\frac{\partial}{\partial \sigma}$ of (C.2) through (C.5) must be taken. Before this can be done $\hat{S}_1$, $\hat{S}_2$, $\hat{T}_1$, and $\hat{T}_2$ must be determined at $y = 0$. Since $\hat{P}_2(y = 0) = 1$,

$$\hat{T}_2^v = \hat{S}_2^v(y = 0) = 0. \quad (C.21)$$

Since $\hat{P}_1(y = 0) = D^*$,

$$\hat{T}_1^v = 1 \quad \text{and} \quad \hat{S}_1^v(y = 0) = 0. \quad (C.22)$$

Now taking $\frac{\partial}{\partial D}$ (C.5) with (C.21) and (C.22)

$$\left\{ \frac{d\hat{T}_2}{dy} = 0 \right\}_{y=0}^v \quad (C.23)$$

$\frac{\partial}{\partial \sigma}$ (C.5) gives:

$$\left\{ \frac{d\hat{S}_2}{dy} = 0 \right\}_{y=0}^v \quad (C.24)$$

$\frac{\partial}{\partial D}$ (C.4) gives:

$$\left(\frac{d\hat{T}_1}{dy} \right)_{y=0}^v = \left\{ \left(\frac{d\bar{h}}{dy} \right)^{-1} \frac{kf}{(\sigma + kU_1)} + K_1 \right\}^v \quad (C.25)$$

and finally $\frac{\partial}{\partial \sigma}$ (C.4) gives:

$$\left(\frac{d\hat{S}_1}{dy} \right)_{y=0}^v = \left\{ (1 - D) \left(\frac{d\bar{h}}{dy} \right)^{-1} \left[\frac{2\bar{\rho}}{g(\rho_1 - \rho_2)} (\sigma + kU_1) \right] - D \left[\frac{kf}{(\sigma + kU_1)^2} \right] \right\}^v \quad (C.26)$$

Similarly, taking $\frac{\partial}{\partial D}$ of (C.2) and (C.3) gives:

$$M_1(\hat{T}_1, \hat{T}_2, \sigma)^v = 0 \quad (C.27)$$

$$M_2(\hat{T}_1, \hat{T}_2, \sigma)^v = 0 \quad (C.28)$$

while $\frac{\partial}{\partial \sigma}$ of (C.2) and (C.3) gives:

$$M_1(\hat{S}_1, \hat{S}_2, \sigma)^v + \left\{ \frac{d\bar{h}}{dy} \Phi_1^A + \bar{h} [\Phi_1^B + \Phi_1^C + \Phi_1^{D+}] + \Phi_1^E \right\}^v = 0, \quad (C.29)$$

where

$$\begin{aligned} \Phi_1^A &= \frac{kf}{(\sigma + kU_1)^2} \hat{P}_1 \\ \Phi_1^B &= \frac{\left(\frac{d\hat{P}_1}{dy} - \frac{kf}{(\sigma + kU_1)} \hat{P}_1 \right) 2k \frac{dU_1}{dy}}{f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2} \\ \Phi_1^C &= \frac{2 \left(f \frac{d^2 U_1}{dy^2} + 2k \frac{dU_1}{dy} (\sigma + kU_1) \right) \left(\frac{d\hat{P}_1}{dy} - \frac{kf}{(\sigma + kU_1)} \hat{P}_1 \right) (\sigma + kU_1)}{f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2} \\ \Phi_1^D &= \frac{\left(f \frac{d^2 U_1}{dy^2} + 2k \frac{dU_1}{dy} (\sigma + kU_1) \right) \left(-\frac{kf}{(\sigma + kU_1)^2} \hat{P}_1 \right)}{f^2 - f \frac{dU_1}{dy} - (\sigma + kU_1)^2} \\ \Phi_1^E &= \frac{2\bar{\rho}}{g(\rho_1 - \rho_2)} (\sigma + kU_1) (\hat{P}_1 - \hat{P}_2) \end{aligned}$$

and

$$M_2(\hat{S}_1, \hat{S}_2, \sigma)^v + \left\{ \frac{d\bar{h}}{dy} \Phi_2^A + (\bar{h} - H) [\Phi_2^B + \Phi_2^C + \Phi_2^{D+}] + \Phi_2^E \right\}^v = 0, \quad (C.30)$$

where

$$\begin{aligned}
\Phi_2^A &= \frac{kf}{(\sigma + kU_2)^2} \hat{P}_2 \\
\Phi_2^B &= \frac{\left(\frac{d\hat{P}_2}{dy} - \frac{kf}{(\sigma + kU_2)} \hat{P}_1 \right) 2k \frac{dU_2}{dy}}{f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2} \\
\Phi_2^C &= \frac{2 \left(f \frac{d^2 U_2}{dy^2} + 2k \frac{dU_2}{dy} (\sigma + kU_2) \right) \left(\frac{d\hat{P}_2}{dy} - \frac{kf}{(\sigma + kU_2)} \hat{P}_2 \right) (\sigma + kU_2)}{f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2} \\
\Phi_2^D &= \frac{\left(f \frac{d^2 U_2}{dy^2} + 2k \frac{dU_2}{dy} (\sigma + kU_2) \right) \left(\frac{kf}{(\sigma + kU_2)^2} \hat{P}_2 \right)}{f^2 - f \frac{dU_2}{dy} - (\sigma + kU_2)^2} \\
\Phi_2^E &= \frac{2\bar{\rho}}{g(\rho_1 - \rho_2)} (\sigma + kU_2) (\hat{P}_1 - \hat{P}_2)
\end{aligned}$$

A "sponge" boundary condition, where $\frac{dU_2}{dy}$ and $\frac{dU_1}{dy}$ "decay" to zero by a $\sin^2$ profile, for example, from their values at $y = 0$ and $y = L_y$, respectively, can be implemented. If the point outside the frontal region where $\frac{dU_2}{dy} = 0$ is given by $-y_d$, and the point where $\frac{dU_1}{dy} = 0$ is given by $L_y + y_d$, the following adjustments to the Newton-Raphson method can be invoked:

$$(\hat{P}_2^v)_{y=-y_d} = 1 \quad \text{and} \quad (\hat{S}_2^v = \hat{t}_2^v)_{y=-y_d} = 0.$$

Equation (C.5) now applies at $y = -y_d$ while (C.6) now applies at $y = L_y + y_d$. In the regions $y > L_y$ and $y < 0$ or $L_y < y < L_y + y_d$ and $-y_d < y < 0$ all terms multiplied by

$\frac{d\bar{h}}{dy}$ and $\frac{\bar{\rho}}{g(\rho_1 - \rho_2)}$ drop out. Otherwise the procedure is exactly the same as given before except that $\hat{P}_2$, $\hat{T}_2$, and $\hat{S}_2$ are integrated from $y = -y_d$ to $y = L_y$ and $\hat{P}_1$, $\hat{T}_1$, and $\hat{S}_1$ are integrated from $y = 0$ to $y = L_y + y_d$.

Appendix D

Entry Points for the Numerical Schemes

Tables 2 through 5 give a list of entry points which may be used in the Newton-Raphson scheme. All symbols are defined in List of Symbols. D is the pressure ratio as defined by Eq. C.1 in Appendix C.

Table 2. Entry points for the S-mode: $B = 1$.

R_i	R_o	A	a	S	τ_r	τ_i	D_r	D_i
10.0	0.778	1.00	0.10	4.0	-0.9535	-0.0148	4.8424	1.7687
10.0	0.759	1.00	1.00	4.0	-1.0369	-0.2201	2.7785	0.5788
10.0	0.759	0.25	1.00	4.0	-0.9713	-0.0496	3.7549	1.7535

Table 3. Entry points for the S-mode: $B = 3$.

R_i	R_o	A	a	S	τ_r	τ_i	D_r	D_i
10.0	0.754	1.00	1.00	4.0	-0.9300	-0.2979	2.4765	1.1559

Table 4. Entry points for the BS-mode: $B = 1$.

R_i	R_o	A	a	S	τ_r	τ_j	D_r	D_j
10.0	0.710	1.00	0.00	4.0	0.8726	-0.1213	-45.3030	159.2755
20.0	0.360	1.00	0.00	4.0	0.7528	-0.2345	8.4581	-78.3366
25.0	0.154	1.00	0.00	2.0	0.0221	-1.2780	1.8230	-3.1280
25.0	0.154	0.00	0.00	2.0	-0.6608	-0.2867	4.9310	-8.2510
30.0	0.253	1.00	0.00	4.0	0.5780	-0.3172	16.7346	-27.1560
40.0	0.175	1.00	0.00	4.0	0.2788	-0.4338	8.7562	-7.9005
50.0	0.160	1.00	0.00	4.0	0.2176	-0.8078	4.9931	-6.0855
60.0	0.130	1.00	0.00	4.0	0.1013	-1.1095	3.0615	-4.1899
70.0	0.112	1.00	0.00	4.0	0.0405	-1.3835	2.3214	-3.2944
80.0	0.098	1.00	0.00	4.0	-0.0075	-1.6507	1.9310	-2.6831
90.0	0.088	1.00	0.00	4.0	-0.0376	-1.8931	1.7176	-2.2764
100.0	0.079	1.00	0.00	4.0	-0.0743	-2.1559	1.5663	-1.9309

Table 5. Entry points for the BS-mode: $B = 3$.

R_i	R_o	A	a	S	τ_r	τ_j	D_r	D_j
10.0	0.873	1.00	0.00	4.0	0.8589	-0.1371	-40.3254	86.1727

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