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Gaussian coordinates and the large scale universe

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San Jose State University, 1992

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**GAUSSIAN COORDINATES AND THE
LARGE SCALE UNIVERSE**

**A Thesis
Presented to
the Faculty of the Department of Mathematics
San Jose State University**

**In Partial Fulfillment
of the Requirements for the Degree
Master of Science**

**By
Bernard S. Donovan
May, 1992**

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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ABSTRACT

GAUSSIAN COORDINATES AND THE LARGE SCALE UNIVERSE

by Bernard S. Donovan

In this paper, special coordinates called *Gaussian coordinates* are introduced to a $(k+1)$ -dimensional Riemannian manifold. With respect to these coordinates, the metric tensor takes a form where g_{00} is constant and $g_{0i} = 0$. (Here $i = 1, \dots, k$.) Gaussian coordinates are then used to find a form of the metric tensor for a General Relativistic problem called the *Large Scale Universe*. This metric tensor is referred to as the *Robertson-Walker metric* and has a form where $g_{00} = c^2$ and $g_{ii} = f$. Here c denotes the velocity of light and

$$f = \frac{R(t)^2}{\left[1 + \left(\frac{k}{4}\right)((x^1)^2 + (x^2)^2 + (x^3)^2)\right]^2},$$

where t, x^1, x^2, x^3 are coordinates of a space-time manifold representing the large scale universe and R is the radius of the universe. As a final objective, the Robertson-Walker metric is used to find a particular form of Einstein's field equations with respect to Friedmann's model of the universe. These equations, called Friedmann's equations, describe a relationship between the mass-energy of the universe and the large scale space-time geometry of the universe.

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Introduction

Around the turn of the 19th century Carl Friedrich Gauss, an already renowned German theorist, began working on 2-dimensional nonlinear spaces called *surfaces*. Gauss assumed that a surface was embedded in a 3-dimensional Euclidean space, $\mathbb{R}^3$, and that its local geometry was Euclidean. Coordinates of a surface, called parameters, were mapped from $\mathbb{R}^2$ to the coordinates of $\mathbb{R}^3$ on the surface. The mapping was given by a set of three equations called parametric equations. That is, consider a surface S and let x^1, x^2, x^3 be rectangular cartesian coordinates of the embedding space $\mathbb{R}^3$. It was assumed by Gauss that there existed a mapping $x = \{x^1, x^2, x^3\}$ from a region on $\mathbb{R}^2$ to a region on $S \subset \mathbb{R}^3$ such that,

$$\begin{aligned} x^1 &= f_1(u^1, u^2) \\ x^2 &= f_2(u^1, u^2) \\ x^3 &= f_3(u^1, u^2). \end{aligned}$$

(Here, u^1, u^2 are coordinates in $\mathbb{R}^2$.) Gauss also required that the mapping have nonvanishing continuous derivatives up to order 3 and that

$$\text{rank}\left(\frac{\partial x^i}{\partial u^j}\right) = 2.$$

From the mapping x , Gauss obtained the quadratic form

$$ds^2 = \frac{\partial x}{\partial u^i} \cdot \frac{\partial x}{\partial u^j} du^i du^j = g_{ij} du^i du^j.$$

This quadratic form is referred to as the *first fundamental form* and g_{ij} is called the *fundamental metric*. The fundamental metric can be described as a symmetric, positive definite, twice covariant tensor. (See Chapter 1, section 2.) It assigns an infinitesimal quantity to the points P and $P+(dx^1, dx^2, dx^3)$ on a surface. The square root of this quantity is said to be the infinitesimal distance between these points. Riemann, a pupil of Gauss, extended these notions from surfaces to k -dimensional nonlinear spaces. He accomplished this by assuming that a fundamental metric was defined at each point P of such a space. To represent a k -dimensional nonlinear space, Riemann introduced the concept of a *manifold*. Of major importance in this treatment of nonlinear spaces was the fact that it did not require the notion of a larger Euclidean space of which the nonlinear space was a subspace. Riemann's thoughts constituted a separate geometry now called *Riemannian Geometry*. (For further reading on the history of surface theory and Riemannian geometry see [5], [8] and [18].)

In the early 1900s, Einstein incorporated Riemannian geometry into the General Theory of Relativity. Here, the fundamental metric g_{ij} is used to express a "relativistic" notion of infinitesimal distance at a point on a manifold representing space-time. It is found in General

Relativity that the fundamental metric must satisfy a complicated system of partial differential equations called Einstein's field equations. Given a specified problem, it is common practice first to propose a form of the fundamental metric based on physical observations and then to find a particular set of the field equations for which this proposed fundamental metric is a solution. The resulting field equations are then used to describe the relationship of mass-energy and space-time geometry for the specified problem.

The primary objectives of this paper are:

- 1) To introduce special coordinates called *Gaussian coordinates* on a $(k+1)$ -dimensional differentiable manifold. With respect to these coordinates, g_{00} is constant and $g_{0i} = 0$, where $i = 1, 2, \dots, k$ (Chapter 2, section 3).

- 2) To use Gaussian coordinates in finding a proposed form of the fundamental metric, $g_{\alpha\beta}$, for a General Relativistic problem called *The Large Scale Universe*. (In Chapter 3, section 3, a large scale universe will be described. In Chapter 3, section 4, $g_{\alpha\beta}$ will be used to establish Einstein's field equations for a large scale universe called *Friedmann's Model of the Universe*.)

Chapter 1: The Fundamentals

Differentiable Manifolds

We begin this section with the following definition:

Definition (1.1.1) [12]: A *k-dimensional manifold* is a Hausdorff topological space such that for every point in the space there exists an open set containing the point that is homeomorphic to an open set in $\mathbb{R}^k$.

Let M be a k -dimensional manifold. Then from above there can be associated with M a system of open sets U_α and corresponding homeomorphisms Φ_α . Let this system be denoted by the set $A = \{(U_\alpha, \Phi_\alpha)\}$. Such a system is often referred to as an *atlas*, where each element (U_α, Φ_α) is called a *chart*. The set $\{U_\alpha\}$ is an open covering of M and it is on the open sets of such a cover that all of the mathematics in this paper will be performed. This process is referred to as a local treatment of the manifold.

Suppose that a point P is contained in two of the open sets U_α and U_β . Let Φ_α and Φ_β be the corresponding homeomorphisms into $\mathbb{R}^k$ and let $W = U_\alpha \cap U_\beta$. The restriction of Φ_α to the overlap W will be denoted by $\Phi_\alpha|_W$. Note that the mapping $\Phi_\alpha \circ (\Phi_\beta|_W)^{-1}$ is from an open set in $\mathbb{R}^k$ onto an open set in $\mathbb{R}^k$. Let x^i and y^j be coordinates of the

open sets $\Phi_\alpha(W) = O_\alpha$ and $\Phi_\beta(W) = O_\beta$, respectively, in $\mathbb{R}^k$. (See [12] for notation.) Then, there is an induced mapping,

$$x^i = x^i(y^1, \dots, y^k) ,$$

from the coordinates of O_β to the coordinates of O_α , in $\mathbb{R}^k$, where $i=1, 2, \dots, k$. This mapping is referred to as the *induced mapping* and is said to be of class C^r if it has nonvanishing continuous derivatives up to the order r for all points on the overlap. The notion of an induced mapping motivates the following definition:

Definition (1.1.2) [18]: Let M be a k -dimensional manifold. If all the induced mappings from a particular atlas A of M are of class C^r (where $r \geq 1$) and,

$$\det\left(\frac{\partial x^i}{\partial y^j} \right) \neq 0,$$

for all points in all of the overlaps, then M is said to be a *k -dimensional differentiable manifold of class C^r* .

Assume there are two atlases, $A = \{(U_\alpha, \Phi_\alpha)\}$ and $\bar{A} = \{(\bar{U}_\beta, \bar{\Phi}_\beta)\}$, on a k -dimensional differentiable manifold M and let $W = U_\alpha \cap \bar{U}_\beta$. Let x^i and $\bar{x}^j$ be coordinates of the open sets $\Phi_\alpha(W) = O_\alpha$ and $\bar{\Phi}_\beta(W) = \bar{O}_\beta$, respectively, in $\mathbb{R}^k$, where $i, j = 1, 2, \dots, k$. The induced mapping from $\bar{O}_\beta$ to O_α is then given by,

$$x^i = x^i(\bar{x}^j) . \tag{1.1.1}$$

If for all overlaps $W = U_\alpha \cap \bar{U}_\beta$ the induced mappings are of at least class C^1 and,

$$\det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) > 0, \quad (1.1.2)$$

then the mapping (1.1.1) is called an *allowable coordinate transformation* from the coordinates $\bar{x}^j$ to the coordinates x^i [18]. The coordinates $\bar{x}^j$ and x^i are said to be coordinates of the manifold M with respect to the atlases $\bar{A}$ and A . In the sections to follow, we will simply make reference to coordinates of a manifold and it will be understood that the coordinates are given with respect to some atlas on the manifold (i.e., we will say, "Consider a manifold M referred to coordinates x^i "). In particular, it will be assumed that the coordinates are given with respect to some chart of the atlas. It is important for the reader to understand that this assumption implies that the coordinates will refer to M locally. Also, it is assumed in this paper that there exists a collection of atlases through which M can be expressed and that any two of them are related by an allowable coordinate transformation.

Consider a manifold M as described in the above discussion. In Riemannian geometry, a *curve* C in M is defined to be a continuous mapping of at least class C^1 from a real number interval $I = [t_0, t_b]$ into M [18], that is, $C: [t_0, t_b] \rightarrow M$. In this paper it will be assumed

that there exists a partitioning $[t_0, \dots, t_i, t_{i+1}, \dots, t_b]$ of I such that for each i $C([t_i, t_{i+1}])$ lies entirely on some chart of an atlas of M . This assumption will allow us to refer to the curve C by real-valued functions, $x^i(t)$.

The Metric Tensor and Tensor Quantities

To begin, consider a curve $x^i(t)$ on a k -dimensional differentiable manifold M , where M is of class C^r . Furthermore, consider a point $P = x^i(t_1)$ on this curve. Let ξ^j be a vector field tangent to the curve $x^i(t)$, that is, let

$$\xi^j(x^i(t)) = \frac{dx^j(t)}{dt} . \quad (1.2.1)$$

Each point on the curve is mapped to a tangent vector by (1.2.1). For example,

$$\xi^i(P) = \left. \frac{dx^i}{dt} \right|_{t=t_1}$$

represents the components of a tangent vector on the curve $x^i(t)$ at the point P on M . This tangent vector is located in a vector space called the *tangent space*. The tangent space at P is defined as follows:

Definition (1.2.1): The set containing all tangent vectors of curves at a point P on a differentiable manifold M is called the *tangent space* of M at P and is denoted by $T_P M$.

Let $\{\mathbf{n}_i\}$ be a basis for T_pM . The dual space of the vector space T_pM is defined to be the vector space of all linear functionals over T_pM having a basis $\{\mathbf{n}^j\}$ such that for some specified inner product, $\star$, we have

$$\mathbf{n}^j \star \mathbf{n}_i = \delta_i^j.$$

Let T_pM^* denote the dual vector space of T_pM . An n -fold of these vector spaces is defined to be a cartesian product of the form,

$$\begin{aligned} & T_pM \times \dots r \text{ times} \dots \times T_pM \times T_pM^* \times \dots s \text{ times} \dots \times T_pM^* \\ &= [T_pM]^r \times [T_pM^*]^s, \end{aligned}$$

where $r + s = n$. (Here, r and s have been arbitrarily chosen.) An n -linear functional is defined to be a multilinear functional from an n -fold of vector spaces into a scalar field.

Definition (1.2.2) [25]: Let M be a k -dimensional differentiable manifold of class C^m (where $m \geq 1$) and let L be an n -linear real-valued functional from the n -fold $[T_pM]^r \times [T_pM^*]^s$ into $\mathbb{R}$, that is,

$$L: [T_pM]^r \times [T_pM^*]^s \rightarrow \mathbb{R}.$$

Then L is defined to be an r -times covariant and s -times contravariant tensor of order n and of type (r,s) on a k -dimensional differentiable manifold of class C^m .

The above definition is the algebraic interpretation of a tensor on a k -dimensional differentiable manifold. In tensor terms, the elements

of $T_p M$ and $T_p M^*$ are defined to be *contravariant vectors* and *covariant vectors*, respectively. Contravariant and covariant vectors are first order tensors of type (0,1) and type (1,0), respectively. That is, consider a contravariant vector ξ expressed by the linear combination $\xi^i n_i$. (Unless noted otherwise, in this paper, a repeated index as a superscript and as a subscript will indicate a sum on that index. This convention is referred to as *Einstein's summation convention*.) Let $w = w_j n^j$ denote an arbitrary covariant vector. Define $\xi(w) = \xi^i w_j (n_i \star n^j)$. Then we have $\xi^i w_j \delta_i^j = \xi^i w_i$. Thus $\xi(w)$ is a first order tensor of type (1,0). By defining $w(\xi) = \xi(w)$, we can then view contravariant vectors as first order tensors of type (0,1).

Again, let $\xi^i n_i$ and $w_j n^j$ denote arbitrary covariant and contravariant vectors, respectively. Let L be an n -th order tensor of type (r,s) . From the multilinearity property of L we have

$$\begin{aligned} & L(\xi^{i_1} n_{i_1}, \dots, \xi^{i_r} n_{i_r}, w_{j_1} n^{j_1}, \dots, w_{j_s} n^{j_s}) \\ &= L(n_{i_1}, \dots, n_{i_r}, n^{j_1}, \dots, n^{j_s}) \xi^{i_1} \dots \xi^{i_r} w_{j_1} \dots w_{j_s} \\ &= L_{i_1 \dots i_r}^{j_1 \dots j_s} \xi^{i_1} \dots \xi^{i_r} w_{j_1} \dots w_{j_s} \end{aligned}$$

where,

$$L_{i_1 \dots i_r}^{j_1 \dots j_s} = L(n_{i_1}, \dots, n_{i_r}, n^{j_1}, \dots, n^{j_s}) . \quad (1.2.2)$$

The expression (1.2.2) identifies the *components* of the tensor. In the classical expression of Riemannian geometry, tensors are given by their components. As described above, a repeated index in both an

upper and a lower component of a tensor implies a sum on that index. Setting an upper index equal to a lower index is a unary operation on a tensor called a *contraction*. A contraction of a tensor of order n and of type (r,s) gives a tensor of order $n-2$ and of type $(r-1,s-1)$.

Tensors, as expressed through their components, obey the following algebraic rules:

1) The sum of two tensors of the same type is defined to be the sum of their components. That is, let R and S be tensors of type (r,s) . Then the sum of these tensors is given by

$$Q_{k_1 \dots k_r}^{q_1 \dots q_s} = R_{k_1 \dots k_r}^{q_1 \dots q_s} + S_{k_1 \dots k_r}^{q_1 \dots q_s}$$

where $Q_{k_1 \dots k_r}^{q_1 \dots q_s}$ is a tensor of the same type.

2) The multiplication (or outer product) of two tensors, say

$$R_{j_1 \dots j_n}^{i_1 \dots i_m} \text{ and } S_{k_1 \dots k_r}^{q_1 \dots q_s},$$

is defined to be

$$Q_{j_1 \dots j_n k_1 \dots k_r}^{i_1 \dots i_m q_1 \dots q_s} = R_{j_1 \dots j_n}^{i_1 \dots i_m} S_{k_1 \dots k_r}^{q_1 \dots q_s}$$

where $Q_{j_1 \dots j_n k_1 \dots k_r}^{i_1 \dots i_m q_1 \dots q_s}$ is a tensor of type $(n+r, m+s)$.

We have defined a tensor at a point P on a k -dimensional manifold, that is, it has been defined over an n -fold of the vector spaces $T_P M$ and $T_P M^*$. The next definition considers a function that maps each point on a manifold to a tensor.

Definition (1.2.3): Let M be a k -dimensional differentiable manifold of class C^r referred to coordinates x^i . An *r -times covariant and s -times contravariant tensor field on M* is a function $L(x^i)$ that maps each point P on M to a tensor $L(P)$ of type (r,s) .

Consider a twice covariant tensor field $g(x^i)$ [a second order tensor field of type $(2,0)$] at a point P on a k -dimensional differentiable manifold M of class C^r with respect to the coordinates x^i . Then we have

$$g(P): T_P M \times T_P M \rightarrow \mathbb{R}.$$

Suppose that $g(P)$ has the properties

$$1) \quad g(P)[\xi^i \mathbf{n}_i, \eta^j \mathbf{n}_j] = g(P)[\eta^j \mathbf{n}_j, \xi^i \mathbf{n}_i] \quad (\text{symmetric property})$$

and

$$2) \quad g(P)[\xi^i \mathbf{n}_i, \xi^j \mathbf{n}_j] \geq 0 \text{ and } g(P)[\xi^i \mathbf{n}_i, \xi^j \mathbf{n}_j] = 0 \text{ iff } \xi^i \mathbf{n}_i = 0$$

(positive definite property)

for every $\xi^i \mathbf{n}_i$ and $\eta^j \mathbf{n}_j$ in $T_P M$. At a point P , the fundamental metric is a twice covariant tensor as described above. For this reason, in

Riemannian geometry the components $g_{ij}(P)$ are defined to be the components of the *fundamental metric tensor*, or for short, the *metric tensor* on M at P with respect to the coordinates x^i . The components $g_{ij}(x^s)$ are defined to be the components of the *fundamental metric tensor field*, or, the *metric tensor field* on M with respect to the coordinates x^s . Recall that Riemann assumed a metric tensor to be given at each point on a manifold. Below, we define a manifold with this assumption incorporated.

Definition (1.2.4): Let M be a k -dimensional differentiable manifold of class C^r referred to coordinates x^i . If there exists a metric tensor field

$$g_{ij}(x^s)$$

on M , then M is said to be a *k-dimensional Riemannian manifold*.

These are the spaces that we will be working on in this paper. It should be noted, though, that the positive definite property is relaxed in relativistic applications. Also, from here on whenever a manifold is specified it will be assumed that it is a Riemannian manifold and therefore there is given a notion of "infinitesimal" distance at each point of the manifold. As indicated in the introduction, it is a primary objective of this paper to introduce a set of coordinates on M such that $g_{00}(x^j)$ is constant and $g_{0i}(x^j) = 0$, where $i = 1, 2, \dots, k$, for all points on M .

In the above discussion, we were careful to distinguish between a tensor and a tensor field on a differentiable manifold M . This was to emphasize that our special coordinates (Gaussian coordinates) are to determine a special form of a metric tensor field on M . In more casual discussions such distinctions will not be made. For example, when speaking of a metric tensor on a manifold it will be assumed that reference is being made to a metric tensor field.

Consider a k -dimensional Riemannian manifold M referred to coordinates x^i . Let $x^i(t)$ be a curve on M and $P_1 = x^i(t_1)$ a point on this curve. Then

$$\sqrt{g_{ij}[x^r(t_1)]dx^i dx^j} ,$$

assigns an infinitesimal notion of length to a segment of the curve between the points P_1 and $P_1 + (dx^1, \dots, dx^k)$. The integral,

$$s = \int_{t_1}^{t_2} \sqrt{g_{ij}[x^r(t)]\dot{x}^i \dot{x}^j} dt , \quad (1.2.3)$$

where $\dot{x}^i = dx^i/dt$, is defined to be the *arc length* of the curve $x^i(t)$ between the points $P_1 = x^i(t_1)$ and $P_2 = x^i(t_2)$.

The quadratic form

$$ds^2 = g_{ij}(x^r)dx^i dx^j \quad (1.2.4)$$

is called the *line element* or, the *differential element* of M with respect to the coordinates x^i . The quantity ds^2 is a tensor of type $(0,0)$. Tensors of this type are called *scalar invariants*.

The reader should note that to know the line element is to know the components of the metric tensor and vice versa. As we will see, it is sometimes more convenient to refer to the components of the metric tensor in terms of the line element.

With respect to rectangular cartesian coordinates x^i , the metric tensor field on $\mathbb{R}^k$ is given by δ_{ij} and the line element for this space is given by

$$ds^2 = \delta_{ij} dx^i dx^j. \quad (1.2.5)$$

Riemannian geometry is an area of mathematics that exploits the usage of tensor quantities and tensor algebra on differentiable manifolds. It was this feature of Riemannian geometry that prompted Einstein to use it for his General Theory of Relativity. That is, from tensor analysis it is known that tensors are invariant with respect to coordinate transformations. In formulating physical laws Einstein proposed that they be represented in a form invariant with respect to an underlying group of transformations. (This proposition is called the Principle of Covariance.) Therefore, expressing physical laws in a tensor form was chosen as a preferred practice in general relativity.

So far we have defined four important tensors in this paper. Contravariant vectors, covariant vectors, the scalar invariant (1.2.3)

and of course the metric tensor. The primary objectives will require that we define several other tensors from Riemannian geometry. We begin with a tensor directly related to the metric tensor.

Definition (1.2.5): Let g_{ij} be a metric tensor on a manifold M and let $g = \det(g_{ij})$. Then

$$g^{ij} = \frac{\text{cofactor of } g_{ji} \text{ in } [g_{ij}]}{\det(g_{ij})} \quad (1.2.6)$$

or equivalently [1],

$$g^{ij} = \frac{1}{g} \frac{\partial g}{\partial g_{ij}}, \quad (1.2.7)$$

are defined to be the components of a twice contravariant tensor called the *associated* or *contravariant metric tensor*. Since the matrix $[g^{ij}]$ is the inverse matrix of $[g_{ik}]$, then $g^{ji}g_{ik} = \delta_k^j$.

In Riemannian geometry, the metric tensor and the associated metric tensor can be used to raise and lower the indices of tensors. For example, without loss of generality, consider a tensor of type (1,1), say Q_k^j . The indices of this tensor can be raised or lowered by making the definitions

$$Q^{ij} = g^{ik} Q_k^j$$

and

$$Q_{ks} = g_{sj} Q_k^j.$$

This can be generalized to any index of any type of tensor.

Definition (1.2.6): Let $\xi^i(t)$ be the components of a contravariant vector field (a tangent vector field) along a curve $x^i(t)$ on a Riemannian manifold. The components

$$\xi^i_{,s} = \frac{\partial \xi^i}{\partial x^s} + \Gamma^i_{ks} \xi^k = \xi^i_{,s} + \Gamma^i_{ks} \xi^k \quad (1.2.8)$$

and

$$\frac{D\xi^i(t)}{dt} = \frac{d\xi^i}{dt} + \Gamma^i_{jk} \xi^j \xi^k \quad (1.2.9)$$

are defined to be those of the *covariant derivative* and of the *absolute derivative*, respectively, of the contravariant vector components ξ^i . As indicated above, the comma denotes conventional differentiation and the components

$$\Gamma^i_{kj} \stackrel{\text{def}}{=} \frac{1}{2} g^{iq} (g_{jq,k} + g_{qk,j} - g_{kj,q}) \quad (1.2.10)$$

are called *Christoffel symbols*.

Conventional differentiation on a tangent field does not retain tensor character under a coordinate transformation. The above definitions of covariant and absolute derivatives were obtained through a quest for a linear correction term that would allow for a form of differentiation that transformed tensorially under a coordinate transformation. That is, these quantities are tensors. The correction term found is the Christoffel symbol as defined above and can be interpreted as a linear connection (by the notion of parallel

displacement of a contravariant vector along a tangent vector field) between tangent spaces on a Riemannian manifold [18].

Definition (1.2.7): The components of the *Riemann-Christoffel tensor* are defined by

$$R^h{}_{ijk} = \Gamma^h_{mj} \Gamma^m_{ik} - \Gamma^h_{mk} \Gamma^m_{ji} - \Gamma^h_{ji,k} + \Gamma^h_{ik,j}. \quad (1.2.11)$$

The components of the contraction

$$R_{ij} = R^h{}_{ijh} \quad (1.2.12)$$

are called the components of the *Ricci tensor*. A further contraction,

$$R_i{}^i = g^{ij} R_{ij}, \quad (1.2.13)$$

is referred to as the *Riemann scalar invariant* or, sometimes, the *curvature invariant*.

Typically, indices are not indicated on a scalar invariant, but the convention has been to denote the radius of the universe by the same symbol R . Therefore, the indices have been retained on the curvature invariant to distinguish it from the radius of the universe.

Definition (1.2.8): The components of the *Einstein tensor* are defined to be

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R_k{}^k. \quad (1.2.14)$$

A *flat space* or a *linear space* is a space where coordinates exist such that the components of the metric tensor field are constant.

Note that all terms of the Riemann-Christoffel tensor involve the product of partial derivatives of the metric tensor. [See (1.2.10).] This implies that its components will vanish in a flat space. In Riemannian geometry it is proven that the reverse is also true, that is, if there exist coordinates on a manifold such that the components of the Riemann-Christoffel tensor vanish, then the space is linear or flat [27]. For such reasons (and others) this tensor has been associated with a measure of curvature on the manifold. In General Relativity it is theorized that mass-energy is responsible for space-time curvature. Einstein's field equations establish a relationship between mass-energy and space-time curvature. The Riemann-Christoffel tensor is associated with this relationship through the Einstein tensor. That is, the Einstein tensor is composed of contracted forms of the Riemann-Christoffel tensor (the Ricci tensor and the curvature invariant) and, it is the Einstein tensor that represents space-time geometry in the field equations (see Chapter 3, section 1).

Definition (1.2.9): Let η^i and ξ^k be the components of arbitrary contravariant vectors on a manifold M . The scalar invariant,

$$K = \frac{R_{ijkl}\eta^i\eta^k\xi^j\xi^L}{(g_{iL}g_{jk} - g_{ik}g_{jL})\eta^i\eta^k\xi^j\xi^L}, \quad (1.2.15)$$

is called *Riemannian curvature* on M . In this paper, this quantity will be referred to as *curvature*.

Chapter 2: Special Coordinates

The Riemannian Form (of the metric tensor)

Consider the two surfaces given in the figures below:

Notice that the 2-sphere looks the same at all points and the cone does not. If one were to stand on these surfaces and were to look out in any direction the geometry of the 2-sphere would remain invariant where the geometry of the cone would not. This contrast in geometry motivates the following definition:

Definition (2.1.1): A manifold with a geometry that remains invariant with respect to direction is called an *Isotropic manifold*. (We can think of "direction" as being given locally by a tangent vector.)

In the study of Riemannian geometry, it is found that there are exactly three types of isotropic manifolds for any given dimension [6]. They are categorized as *spherical*, *pseudospherical* and *Euclidean* manifolds. Since the geometry of an isotropic manifold is indistinguishable at all points, the curvature is constant on the manifold. We can distinguish these three types of manifolds by the signs of their curvature. An n -sphere has a positive constant curvature, a pseudosphere has negative constant curvature and curvature vanishes on a Euclidean manifold.

A special form of the metric tensor for manifolds with constant curvature will now be developed. I will detail a development found in [18]. It was Riemann who first derived this form and it is therefore called the *Riemannian form* [10].

Consider a n -dimensional manifold with constant curvature. Suppose that there exist coordinates x^s on M in which the metric tensor is given by

$$g_{ij} = \frac{\delta_{ij}}{F^2},$$

where F is a function of the coordinates x^s and $F \neq 0$. Then the associated metric tensor has the form,

$$g^{ij} = F^2 \delta^{ij},$$

and

$$\frac{\partial g_{ij}}{\partial x^k} = -\delta_{ij} \frac{2}{F^3} \frac{\partial F}{\partial x^k} = -\frac{2\delta_{ij}}{F^2} \frac{\partial \log F}{\partial x^k}.$$

With the above quantities at hand, the Christoffel symbols can be found in terms of $f = \log F$.

First consider the case where all indices of the Christoffel symbols are distinct. Then from (1.2.10) and from directly above we can construct,

$$\Gamma_{kj}^i = -\frac{F^2 \delta^{iq}}{2} \left(\frac{2\delta_{iq}}{F^2} \frac{\partial \log F}{\partial x^k} + \frac{2\delta_{qk}}{F^2} \frac{\partial \log F}{\partial x^j} - \frac{2\delta_{kj}}{F^2} \frac{\partial \log F}{\partial x^q} \right).$$

Multiplying through by the term δ^{iq} in the above expression and substituting in f for $\log F$ yields

$$\Gamma_{kj}^i = -(\delta_j^i f_{,k} + \delta_k^i f_{,j} - \delta_{kj} f_{,i}).$$

Since i, j, k are distinct, then,

$$\delta_j^i = \delta_k^i = \delta_{jk} = 0,$$

and we obtain

$$\Gamma_{kj}^i = 0. \quad (2.1.1a)$$

Now, consider the case where the lower indices of the Christoffel symbols are the same and distinct from the superscript. Then we have

$$\begin{aligned} \Gamma_{kk}^i &= -\delta^{iq}(\delta_{kq} f_{,k} + \delta_{qk} f_{,k} - \delta_{kk} f_{,q}) \\ &= -(\delta_k^i f_{,k} + \delta_k^i f_{,j} - \delta_{kk} f_{,i}). \end{aligned}$$

Because i and k are distinct then $\delta_k^i = 0$. Also, $\delta_{kk} = 1$ and therefore we have

$$\Gamma_{kk}^i = f_{,i} . \quad (2.1.1b)$$

Similarly, it can be found that

$$\Gamma_{ik}^i = -f_{,k} \quad \text{and} \quad \Gamma_{ii}^i = -f_{,i} . \quad (2.1.1c)$$

(Note that in the above Christoffel symbols we are referring to a specific component that has a form where an upper index is the same as a lower index. Therefore, we are not summing on this index.)

Now that the Christoffel symbols are known in terms of f , we can similarly express the components of the Riemann-Christoffel tensor. We begin by considering the case where all indices are distinct. Then

$$R^h_{ijk} = \Gamma_{mj}^h \Gamma_{ik}^m - \Gamma_{mk}^h \Gamma_{ji}^m - \Gamma_{ji,k}^h + \Gamma_{ik,j}^h .$$

But, from (2.2.1a),

$$\Gamma_{ji}^h = \Gamma_{ik}^h = 0,$$

and we have

$$R^h_{ijk} = \Gamma_{mj}^h \Gamma_{ik}^m - \Gamma_{mk}^h \Gamma_{ji}^m . \quad (2.1.2)$$

Note that we are summing on the index m in the above equation. By (2.1.1a), no contribution will be made to this sum when m is distinct from the indices i, j, k, h . When $m=h$ we have

$$\Gamma_{ik}^h = \Gamma_{ji}^h = 0$$

[see (2.1.1a)] and again no contribution is made to this sum. When $m=j$,

$$\Gamma_{ik}^j = \Gamma_{jk}^h = 0,$$

[see (2.1.1a)] and no contribution is made to the sum. Finally, where $m=i$ or $m=k$ the terms in (2.1.2) vanish by (2.1.1a) and no contribution is made to the sum. Therefore, by exhausting all possibilities in the sum (2.1.2) it can be concluded that

$$R_{ijk}^h = 0. \quad (2.1.3)$$

Next we consider the components of the Riemann-Christoffel tensor where $i=k, j=h$ and $i \neq j$. Thus, we have

$$R_{iji}^j = \Gamma_{mj}^j \Gamma_{ii}^m - \Gamma_{mi}^j \Gamma_{ji}^m - \Gamma_{ji,i}^j + \Gamma_{ii,j}^j. \quad (2.1.4)$$

(We are not summing on the indices i and j . See comment on page 22.) When $m=i$ and $m=j$ we have

$$\begin{aligned} \Gamma_{ij}^j \Gamma_{ii}^i - \Gamma_{ii}^j \Gamma_{ji}^i &= (-f_{,i}) (-f_{,i}) - f_{,j} (-f_{,j}) \\ &= (f_{,i})^2 + (f_{,j})^2 \end{aligned}$$

and

$$\Gamma_{jj}^j \Gamma_{ii}^j - \Gamma_{ji}^j \Gamma_{ji}^j = - (f_{,j})^2 - (f_{,i})^2 ,$$

respectively. This implies that the first two terms of (2.1.4) sum to zero when $m=i$ and $m=j$ are combined in the sum on m . Where m is distinct from i and j the second term of (2.1.4) vanishes. Therefore it can be concluded that

$$R_{iji}^j = - f_{,ii} - f_{,jj} + \sum_{m=1}^{k^*} (f_{,m})^2 , \quad (2.1.5)$$

where $*$ indicates $m \neq i$ or $m \neq j$.

Following a similar process as above it can be established that,

$$R_{jjk}^h = f_{,k} f_{,i} + f_{,ik} , \quad (2.1.6)$$

$$R_{ijk}^j = - f_{,k} f_{,j} - f_{,jk} \quad (2.1.7)$$

and

$$R_{hjk}^h = R_{hhk}^h = 0 . \quad (2.1.8)$$

(Note, we are not summing on the indices h, i, j in the above quantities.)

On the other hand, F. Schur proved that for a manifold with constant curvature, K , the components of the Riemann-Christoffel tensor must have the form

$$R_{ijk}^h = K(\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

Recalling that $g_{ij} = \frac{\delta_{ij}}{F^2}$ yields

$$R^h_{ijk} = \frac{K}{F^2} (\delta_k^h \delta_{ij} - \delta_j^h \delta_{ik}).$$

We can then write

$$R^j_{iji} = \frac{K}{F^2} (\delta_i^j \delta_{ij} - \delta_j^j \delta_{ii}).$$

Since i and j are distinct then $(\delta_i^j \delta_{ij} - \delta_j^j \delta_{ii}) = -1$, which yields

$$R^j_{iji} = -\frac{K}{F^2}.$$

From (2.1.5), the above implies that

$$R^j_{iji} = f_{i,i} + f_{j,j} - \sum_{m=1}^{k^*} (f_{i,m})^2 = \frac{K}{F^2}. \quad (2.1.9)$$

For R^j_{ijk} we have

$$R^j_{ijk} = \frac{K}{F^2} (\delta_k^j \delta_{ij} - \delta_j^j \delta_{ik}).$$

Since i , j and k are distinct then $(\delta_k^j \delta_{ij} - \delta_j^j \delta_{ik}) = 0$ and we obtain

$$R^j_{ijk} = f_{k,f,i} + f_{i,k} = 0. \quad (2.1.10)$$

Replacing f with $\log F$ in (2.1.10) gives us

$$\begin{aligned} & \frac{\partial \log F}{\partial x^k} \frac{\partial \log F}{\partial x^i} + \frac{\partial^2 \log F}{\partial x^k \partial x^i} = 0 \\ \Rightarrow & \frac{1}{F} \frac{\partial F}{\partial x^k} \frac{1}{F} \frac{\partial F}{\partial x^i} + \frac{\partial}{\partial x^k} \left[\frac{1}{F} \frac{\partial F}{\partial x^i} \right] = 0 \end{aligned}$$

$$\Rightarrow \frac{1}{F^2} \frac{\partial F}{\partial x^k} \frac{\partial F}{\partial x^i} - \frac{1}{F^2} \frac{\partial F}{\partial x^k} \frac{\partial F}{\partial x^i} + \frac{1}{F} \frac{\partial^2 F}{\partial x^k \partial x^i} = 0$$

$$\Rightarrow \frac{1}{F} \frac{\partial^2 F}{\partial x^k \partial x^i} = 0$$

or

$$\frac{1}{F} (F_{,ik}) = 0 \quad (2.1.11)$$

Similarly, replacing f with $\text{Log}F$ in (2.1.9) gives

$$\frac{\partial^2 \log F}{\partial x^i \partial x^i} + \frac{\partial^2 \log F}{\partial x^j \partial x^j} - \sum_{m=1}^{k^*} (\log F_{,m})^2 = \frac{K}{F^2}$$

$$\Rightarrow \frac{\partial}{\partial x^i} \left[\frac{1}{F} \frac{\partial F}{\partial x^i} \right] + \frac{\partial}{\partial x^j} \left[\frac{1}{F} \frac{\partial F}{\partial x^j} \right] - \sum_{m=1}^{k^*} [1/F (\partial F / \partial x^m)]^2 = \frac{K}{F^2}$$

$$\Rightarrow - \frac{1}{F^2} \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^i} + \frac{1}{F} \frac{\partial^2 F}{\partial x^i \partial x^i} - \frac{1}{F^2} \frac{\partial F}{\partial x^j} \frac{\partial F}{\partial x^j} + \frac{1}{F} \frac{\partial^2 F}{\partial x^j \partial x^j} - \frac{1}{F^2} \sum_{m=1}^{k^*} (\partial F / \partial x^m)^2 = \frac{K}{F^2}$$

$$\Rightarrow - \frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^i} + F \frac{\partial^2 F}{\partial x^i \partial x^i} - \frac{\partial F}{\partial x^j} \frac{\partial F}{\partial x^j} + F \frac{\partial^2 F}{\partial x^j \partial x^j} - \sum_{m=1}^{k^*} (\partial F / \partial x^m)^2 = K .$$

Recall that "*" denotes $m \neq i$ and $m \neq j$. Note that the first and third terms in the above equation can be incorporated into the sum of the fifth term yielding

$$F(F_{j,j} + F_{i,i}) - \sum_{m=1}^k (F_{m,m})^2 = K . \quad (2.1.12)$$

Equation (2.1.11) implies that F must take the form

$$F = G^1 + G^2 + \dots + G^n,$$

where G^i is a function of x^i alone. Substituting this equation into (2.1.12) yields

$$(G^1 + G^2 + \dots + G^n)(G^j_{j,j} + G^i_{i,i}) - \sum_{m=1}^n (G^m_{m,m})^2 = K . \quad (2.1.13)$$

(No sum on i or j in the above equation.) Then, forming a second equation from (2.1.13) with the index k instead of j and subtracting it from (2.1.13) gives us

$$G^j_{j,j} = G^k_{k,k} .$$

Since G^j is a function of x^j only and G^k is a function of x^k only, then it must be the case that $G^j_{j,j}$ and $G^k_{k,k}$ are equal to a constant.

Therefore, G^m must be of the form

$$c(x^m)^2 + b_m x^m + a_m ,$$

where c , b_m and a_m are constants. With this in mind, equation (2.1.13) then becomes

$$\begin{aligned}
& \left[\sum_{m=1}^n c(x^m)^2 + \sum_{m=1}^n b_m(x^m) + \sum_{m=1}^n a_m \right] (2c + 2c) - \sum_{m=1}^n (2cx^m + b_m)^2 = K \\
\Rightarrow & \sum_{m=1}^n 4c^2(x^m)^2 + \sum_{m=1}^n 4cb_mx^m + \sum_{m=1}^n 4ca_m \\
& - \sum_{m=1}^n 4c^2(x^m)^2 - \sum_{m=1}^n 4cb_mx^m - \sum_{m=1}^n (b_m)^2 = K
\end{aligned}$$

and therefore the coefficients c , a_m , and b_m are related by

$$K = \sum_{m=1}^n (4ca_m - b_m^2).$$

Finally, choosing $c = K/4$, $b_m = 0$ and $a_m = 1/n$ yields the Riemannian form, that is,

$$\begin{aligned}
g_{ij} &= \frac{\delta_{ij}}{F^2} = \frac{\delta_{ij}}{\left(\left(\frac{1}{n} + \frac{K}{4}(x^1)^2 \right) + \dots + \left(\frac{1}{n} + \frac{K}{4}(x^n)^2 \right) \right)^2} \\
&= \frac{\delta_{ij}}{\left[1 + \left(\frac{K}{4} \right) \delta_{st} x^s x^t \right]^2}.
\end{aligned}$$

Note that in a Euclidean space $K = 0$ and g_{ij} reduces to the Euclidean metric tensor with respect to rectangular cartesian coordinates. For this reason the coordinates x^s are associated with rectangular cartesian coordinates mapped locally to M . Also, for this reason, the

Riemannian form is often referred to as the *cartesian form* of a manifold with constant curvature.

Let M be a 3-dimensional manifold with constant curvature. Suppose that this manifold is referred to coordinates x^1, x^2, x^3 such that the Riemannian form is given on M . Then we have

$$g_{ij} = \frac{\delta_{ij}}{\left[1 + \left(\frac{K}{4} \right) ((x^1)^2 + (x^2)^2 + (x^3)^2) \right]^2} .$$

Following the work of Atwater [2], we can introduce an isotropic scale change to the Riemannian form by replacing the coordinates x^1, x^2, x^3 with $\bar{x}^1 = x^1/R, \bar{x}^2 = x^2/R, \bar{x}^3 = x^3/R$ and by letting $k = KR^2$. This yields

$$g_{ij} = \frac{R^2 \delta_{ij}}{\left[1 + \left(\frac{k}{4} \right) ((\bar{x}^1)^2 + (\bar{x}^2)^2 + (\bar{x}^3)^2) \right]^2} . \quad (2.1.14)$$

(In Chapter 3, R will be used to represent the radius of the universe.) For $k = 0$ this metric tensor is that of $\mathbb{R}^3$ with respect to rectangular cartesian coordinates. For $k = 1$ it is associated [22] with the metric tensor for a 3-sphere referred to (locally) rectangular cartesian coordinates. Lastly, for $k = -1$ it is associated [22] with the metric tensor for a three dimensional pseudosphere referred to (locally) rectangular cartesian coordinates.

In the next section Gaussian coordinates are defined and a metric tensor form similar to the metric tensor of a large scale universe (to be found) is obtained. The Riemannian form, (2.1.14), will play an important role in obtaining this metric tensor.

Gaussian Coordinates

Consider a $(k+1)$ -dimensional manifold M referred to coordinates $x^0, x^1, \dots, x^k$. Holding all but one coordinate constant produces a curve on M . This curve is called a *coordinate curve* for the coordinate that varies and the set of all coordinate curves of a particular coordinate is called a *family of coordinate curves*. A family of coordinate curves for the coordinate x^i is denoted by x^i -curves. For every point P on M there passes through P exactly one member from each family of coordinate curves, that is, a family of coordinate curves must be nonintersecting and must cover the manifold (actually, they must cover the open sets of an atlas with respect to the manifold).

Since the metric tensor components are dependent on the coordinates of the manifold, its components will be dependent on the families of coordinate curves representing the coordinates. The following discussion will identify a special relationship between families of coordinate curves.

Recall from the introduction that on a surface S we have

$$g_{ij} = \frac{\partial x}{\partial u^i} \cdot \frac{\partial x}{\partial u^j},$$

where " $\cdot$ " denotes the dot product and $i, j = 0, 1$. Note that $\partial x(P)/\partial u^i$ is a tangent vector along that member of the u^i -curves that passes through the point P on S . Two curves passing through a point P on a surface are said to be orthogonal if the dot product of their tangent vectors is zero at P . Therefore, the u^0 curve and the u^1 curve that pass through a point P are orthogonal at P if and only if

$$g_{01}(P) = \frac{\partial x(P)}{\partial u^0} \cdot \frac{\partial x(P)}{\partial u^1} = 0.$$

If for all points on S we have $g_{01} = 0$, then we say that the coordinates u^0, u^1 are *orthogonal coordinates* on S .

The notion of orthogonal coordinates as expressed through the metric tensor is extended to a $(k+1)$ -dimensional manifold in the following definition.

Definition (2.2.1): Let M be a $(k+1)$ -dimensional Riemannian manifold referred to coordinates $x^0, x^1, x^2, \dots, x^k$. Suppose that for some point P on M we have $g_{0i}(P) = 0$, where $i = 1, 2, 3, \dots, k$. Then we will say that the coordinate x^0 is *orthogonal* to the coordinates $x^1, x^2, \dots, x^k$ at P or, equivalently, that the x^0 -curves are orthogonal to the x^i -curves at P for all i . If this is true for all points on M , then we will refer to the coordinate x^0 as being orthogonal to the coordinates $x^1, x^2, \dots, x^k$ on M .

Now let $x^i(t)$ be a curve on M and let $P_1 = x^i(t_1)$ and $P_2 = x^i(t_2)$ be points along this curve. From the calculus of variations [27], it can be shown that in order for $x^i(t)$ to be a curve of minimal length between these points it is a necessary condition that it be a solution of the equations,

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = \frac{d \ln f}{d t} \dot{x}^i, \quad (2.2.1)$$

where $f = \frac{ds}{dt} = (g_{ij} \dot{x}^i \dot{x}^j)^{1/2}$ and $\dot{x}^i = \frac{dx^i}{dt}$. If $t = s$, then $(g_{ij} \dot{x}^i \dot{x}^j)^{1/2} = 1$ and the above equations reduce to

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0. \quad (2.2.2)$$

Definition (2.2.2): A curve that satisfies equations (2.2.1) is called a *geodesic curve* and the equations themselves are referred to as the *geodesic differential equations*.

(Geodesic curves on a manifold are analogous to straight lines in Euclidean space. That is, just as we associate the length of a straight line with the shortest distance between two points in Euclidean space, the shortest distance between two points on a manifold, in general, is associated with the length of a geodesic curve joining the points.)

Finally, with the notions of geodesic curves and orthogonal coordinates established, we now describe Gaussian coordinates. In this description, Greek letters will refer to the indices 0, 1, 2, ..., k,

while Roman letters will be restricted to the indices 1, 2,..., k. The reader is forewarned that this description, and those results to follow, are complicated and may require more than one reading. A similar description of Gaussian coordinates can be found in [1].

To begin, let M be a $(k+1)$ -dimensional manifold where, for any given point P on M , there exists an open set N_P containing P such that a set of nonintersecting geodesics can be found covering N_P . Let the coordinate x^0 vary along this set of geodesic curves on M with respect to N_P . Assume that coordinates $x^0, x^1, \dots, x^k$ are admitted on M such that g_{00} is constant for all points. Note that whenever the coordinate x^0 is held constant a k -dimensional manifold is produced. We will refer to such manifolds as the x^0 -submanifolds of M with respect to N_P and we will denote any particular one, say $x^0 = a$, by M^a . Suppose that for some constant b it is known that the coordinate x^0 is orthogonal to the coordinates $x^1, \dots, x^k$ on the submanifold $x^0 = b$ of M , or equivalently, that $g_{0i} = 0$ on $N_P \cap M^b$ for $i = 1, 2, \dots, k$. Then, in this setting, the coordinates $x^0, x^1, \dots, x^k$ on M are called *Gaussian coordinates*. (It is important to note that whenever a manifold is referred to Gaussian coordinates it will be assumed that the above setting exists on the manifold.) It can be shown that with respect to Gaussian coordinates, the x^0 coordinate is orthogonal to the coordinates $x^1, \dots, x^k$ for all x^0 -submanifolds of M , or equivalently, it can be shown that $g_{0i} = 0$ for all points on M , where $i = 1, 2, \dots, k$.

To show this, we begin by recalling that since the x^0 -curves are geodesics then by Definition 2.2.2 they will satisfy the geodesic differential equations

$$\frac{d^2 x^\sigma}{ds^2} + \Gamma_{\alpha\beta}^\sigma \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0.$$

Along these geodesics we have

$$\frac{dx^i}{ds} = 0,$$

for $i = 1, 2, \dots, k$. Therefore, the geodesic differential equations,

$$\frac{d^2 x^i}{ds^2} + \Gamma_{\alpha\beta}^i \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0,$$

where $\alpha, \beta = 0, 1, \dots, k$, will reduce to

$$\Gamma_{00}^i \frac{dx^0}{ds} \frac{dx^0}{ds} = 0.$$

Recall from the first section of chapter one that all curves on M are assumed to be of class C^1 . Therefore, since $dx^i/ds = 0$, for all i , then

$$\frac{dx^0}{ds} \neq 0,$$

and we have

$$\frac{1}{2} g^{i\sigma} (g_{0\sigma,0} + g_{\sigma 0,0} - g_{00,\sigma}) = \Gamma_{00}^i = 0.$$

Since g_{00} is constant, then $g_{00,\sigma} = 0$ and the above reduces to

$$\frac{1}{2} g^{i\sigma}(g_{0\sigma,0} + g_{\sigma 0,0}) = 0$$

or

$$g^{i\sigma}(g_{0\sigma,0}) = 0,$$

where $\sigma = 0, 1, \dots, k$. Noting that the above equations vanish when $\sigma = 0$ allows us to write

$$g^{iq}(g_{q0,0}) = 0,$$

where $q = 1, 2, \dots, k$. Multiplying through by g_{ri} gives us

$$g_{r0,0} = 0,$$

where $r = 1, 2, \dots, k$. This implies that the components g_{0i} will be constant for all points on N_p . Therefore, since $g_{0i} = 0$ on $N_p \cap M^b$, then g_{0i} will remain zero for all points on N_p , for all i . This is equivalent to saying that the coordinate x^0 will remain orthogonal to the coordinates $x^1, \dots, x^k$ for all of the x^0 -submanifolds of M .

We can summarize the above results in the following theorem:

Theorem (2.2.1): Let M be a $(k+1)$ -dimensional manifold referred to Gaussian coordinates $x^0, x^1, \dots, x^k$. Then for some constant c we have $g_{00} = c$, and the line element for M takes the form

$$ds^2 = c(dx^0)^2 + g_{ij}(x^0, x^i)dx^i dx^j,$$

where $i, j = 1, 2, \dots, k$. (It is important to note that $g_{ij}(x^0, x^i)$ can be viewed as assigning a metric tensor field to each x^0 -submanifold.)

Example (2.2.1): Consider a 2-sphere S referred to the coordinate system:

$$\begin{aligned} x_1 &= a \sin u^0 \cos u^1 \\ x_2 &= a \sin u^0 \sin u^1 \\ x_3 &= a \cos u^0 \end{aligned} \quad (2.2.5)$$

where $t_1 < u^0 < t_2$ and $\tau_1 < u^1 < \tau_2$. With respect to the above coordinate system the u^0 -curves are called *great circles* and the u^1 -curves are called *latitudinal circles* on S (see Figure 3). In surface theory it is found that the u^0 -curves (the great circles) are geodesic curves on S . In this setting, the u^1 -curves (the latitudinal circles) are u^0 -submanifolds on S . Consider the latitudinal circle $u^0 = \pi/4$ on S .

Along this latitudinal circle we have

$$\begin{aligned} \frac{\partial x}{\partial u^0} &= (a \cos u^0 \cos u^1, a \cos u^0 \sin u^1, -a \sin u^0) \\ &= (a/\sqrt{2} \cos u^1, a/\sqrt{2} \sin u^1, -a/\sqrt{2}) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial x}{\partial u^1} &= (-a \sin u^0 \sin u^1, a \sin u^0 \cos u^1, 0) \\ &= (-a/\sqrt{2} \sin u^1, a/\sqrt{2} \cos u^1, 0), \end{aligned}$$

Then

$$\begin{aligned} g_{01} &= \frac{\partial x}{\partial u^0} \cdot \frac{\partial x}{\partial u^1} = (a/\sqrt{2} \cos u^1, a/\sqrt{2} \sin u^1, -a/\sqrt{2}) \\ &\quad \cdot (-a/\sqrt{2} \sin u^1, a/\sqrt{2} \cos u^1, 0) \\ &= 0. \end{aligned}$$

This implies that the coordinate u^0 is orthogonal to the coordinate u^1 for the latitudinal circle $u^0 = \pi/4$ (see Figure 3). Also, note that

$$\begin{aligned}
 g_{00} &= \frac{\partial x}{\partial u^0} \cdot \frac{\partial x}{\partial u^0} = (a \cos u^0 \cos u^1, a \cos u^0 \sin u^1, -a \sin u^0) \\
 &\quad \cdot (a \cos u^0 \cos u^1, a \cos u^0 \sin u^1, -a \sin u^0) \\
 &= a^2 \cos^2 u^0 \cos^2 u^1 + a^2 \cos^2 u^0 \sin^2 u^1 + a^2 \sin^2 u^0 \\
 &= a^2 \cos^2 u^0 + a^2 \sin^2 u^0 = a^2.
 \end{aligned}$$

Thus g_{00} is a constant. Therefore from above, and by the description of Gaussian coordinate given prior to Theorem 2.2.1, the coordinates u^0, u^1 are Gaussian coordinates. Then by Theorem 2.2.1, the coordinate u^0 will be orthogonal to the coordinate u^1 for all of the latitudinal circles on S and the metric tensor, with respect to these coordinates, will take the form

$$g_{ij} = \begin{pmatrix} a^2 & 0 \\ 0 & g_{11} \end{pmatrix}$$

on S . This can be verified by the simple computation

$$\begin{aligned}
 g_{01} &= \frac{\partial x}{\partial u^0} \cdot \frac{\partial x}{\partial u^1} = -a \cos u^0 \cos u^1 \sin u^0 \sin u^1 \\
 &\quad + a \cos u^0 \cos u^1 \sin u^0 \sin u^1 \\
 &= 0.
 \end{aligned}$$

[The above illustration describes the setting for Example 2.2.1. The curves, $u^1 = c_1$ and $u^1 = c_2$, are arbitrary great circles on S . In Example 2.2.1 it is known that they are orthogonal to the latitudinal circle $u^0 = \pi/4$. Theorem 2.2.1 implies that since u^0, u^1 are Gaussian coordinates then $u^1 = c_1$ and $u^1 = c_2$ will remain orthogonal to all latitudinal circles on S such as $u^0 = d$ depicted above.]

Example (2.2.2): Let M be a 4-dimensional Riemannian manifold referred to Gaussian coordinates x^0, x^1, x^2, x^3 . Then for some constant c we have $g_{00} = c$. Suppose that the 3-dimensional x^0 -submanifolds of M are isotropic manifolds and that for any particular one, say $x^0 = a$, $g_{ij}(a, x^1, x^2, x^3)$ is the Riemannian form (see 2.1.14), where $i, j = 1, 2, 3$. Then by Theorem 2.2.1 the metric tensor for M will be

$$g_{\alpha\beta} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & f \end{pmatrix},$$

where

$$f = \frac{R^2}{[1 + \left(\frac{k}{4}\right)((x^1)^2 + (x^2)^2 + (x^3)^2)]^2}.$$

It is a modified form of the metric tensor given in the above example that is used as the metric tensor for a large scale universe. This will be developed in section three of Chapter 3.

Chapter 3: Application of Gaussian Coordinates to the Large Scale Universe

The Metric Scheme of General Relativity

General Relativity was developed during the years 1905-1915. It previously had been found in the development of Special Relativity that a three dimensional domain was insufficient for describing physical activities in space. It was necessary to include time as a fourth dimension. This pairing of space with time is referred to as *space-time*. General Relativity theorizes that the mass-energy content of space is responsible for space-time curvature. Empty space or unoccupied space is flat, or has zero space-time curvature. The objective of this section is to describe briefly the scheme for assigning a "relativistic" notion of distance to a space-time structure. We will begin with a flat space-time domain and then generalize to a curved space-time domain. (The problem at hand is really the same problem as that of generalizing a notion of distance to a nonlinear space in manifold theory. But here, we will specify a particular form of a metric tensor field on a four dimensional Riemannian manifold.)

An unaccelerated frame of reference in an empty space is referred to as an *inertial frame of reference*. Lorentz derived a set of transformations for which Maxwell's equations remained invariant

with respect to inertial frames of reference. The *Lorentz Transformations* were in agreement with Einstein's postulate on the constancy of the velocity of light's propagation, while Galileo's transformations, which were at the foundation of classical mechanics, were not. Therefore, the Lorentz transformations were retained as the correct ones.

Consider a flat physical space referred to rectangular cartesian coordinates x, y, z and consider a pulse of light generated from a point source at the origin. The wavefront of this pulse will spread out in an isotropic manner. Newtonian physics describes the velocity of this wavefront by $c = \sqrt{(dx^2+dy^2+dz^2)}/dt$. From this the equation

$$0 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (3.1.1)$$

follows. Because the quantity

$$c^2dt^2 - dx^2 - dy^2 - dz^2 \quad (3.1.2)$$

is an invariant under the Lorentz transformations and has a quadratic form similar to that of the line element in the Euclidean space R^4 (with respect to rectangular cartesian coordinates), it is an attractive candidate for a relativistic notion of distance in empty space. But, this quantity will not satisfy the positive definite condition given for a metric tensor. Where this quantity is positive it is referred to as *spacelike*. It is referred to as *timelike* if it is negative and where this quantity vanishes, it is referred to as *lightlike*. Insisting that the quantity (3.1.2) be positive definite, we

then must restrict our observations to where this quantity is spacelike.

With this restriction in mind, we express quantity (3.1.2) in the form

$$(cdt, dx, dy, dz) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (cdt, dx, dy, dz)^t. \quad (3.1.3)$$

If t, x, y, z are the coordinates of a real vector space V representing a flat space-time domain and P is the point (t, x, y, z) , then the square root of the quantity (3.1.3) is described, in relativity, as a "relativistic" notion of infinitesimal distance assigned to the points P and $P+(dt, dx, dy, dz)$. The 4×4 matrix is referred to as the matrix of components of the *Lorentz metric tensor*. The signature, $(+1, -1, -1, -1)$, of the Lorentz metric tensor is called a *hyperbolic signature*.

Suppose that $\bar{t}, \bar{x}, \bar{y}, \bar{z}$ are also coordinates for V and let $\bar{g}_{ij}$ be the metric tensor of this space with respect to these coordinates. If $\bar{g}_{ij}$ has a hyperbolic signature and if there exists an allowable coordinate transformation from the unbarred coordinate system to the barred coordinate system, then $\bar{g}_{ij}$ will be referred to as a *Lorentz-like metric tensor* in this paper.

Definition (3.1.1): Let R^4 be a four dimensional real vector space and let g_{ij} be a Lorentz metric tensor or a Lorentz-like metric tensor. Then the pair (R^4, g_{ij}) is said to be a *Minkowski space-time*.

Definition (3.1.2): Let M be a four dimensional differentiable manifold. If there exists a metric tensor field g_{ij} on M such that for every point P on M the pair $[T_P M, g_{ij}(P)]$ is a Minkowski space-time, then M is said to be a *Riemannian space-time manifold*. (Note that a Riemannian space-time manifold is a Riemannian manifold as defined in Chapter 1. Therefore, all concepts developed in Chapters 1 and 2 will carry over to the present chapter.)

Minkowski space-time is the mathematical representation of Special Relativity. Basically, it is a metric scheme for a linear or flat space-time domain. It is assumed in General Relativity that the metric scheme for a curved space-time domain can be represented by a 4-dimensional Riemannian space-time manifold. The fact that Minkowski space-time is a special case of a Riemannian space-time manifold qualifies a particular form for the metric tensor in general. That is, in General Relativity, the metric tensor of a specified problem is developed in such a way that it reduces to a Lorentz metric tensor or a Lorentz-like metric tensor in an unoccupied space.

The path generated by a particle in a space-time domain is referred to as the *world line* of the particle. Let M be a space-time manifold given by the coordinates x^0, x^1, x^2, x^3 and let $g_{ij}(x^\alpha)$ be a metric tensor field on M . (On a space-time manifold it is conventional to use Greek letters as indices when referring to both the time coordinate and 3-space coordinates. Roman letters are used when specifically referring to the 3-space coordinates.) Let $x^\alpha(\kappa)$ be a curve

on M representing the world line of some particle and let P be a point on this curve. The quantity

$$\sqrt{g_{\alpha\beta}(P)dx^\alpha dx^\beta}$$

assigns a "relativistic" notion of length to an infinitesimal segment of the world line at the point P . The integral,

$$s = \int_{\kappa_1}^{\kappa_2} \sqrt{g_{\alpha\beta}[x^\sigma(\kappa)]\dot{x}^\alpha \dot{x}^\beta} d\kappa ,$$

where $\dot{x}^\alpha = dx^\alpha/d\kappa$, is called the *relativistic arc length* of the world line between the points $P_1 = x^\sigma(\kappa_1)$ and $P_2 = x^\sigma(\kappa_2)$. The quadratic form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

is referred to as the *world line element* on M with respect to the coordinates x^σ .

Particles that move through space-time under the influence of no external forces (with gravity not included as an external force) are called *free particles*. From a relativistic point of view, a free particle is influenced only by the geometry of space-time. According to Newtonian physics the velocity components of a path generated by a nonaccelerating particle will vanish under conventional differentiation. Recall from Chapter 1 that covariant and absolute differentiation were adopted by Einstein as preferred forms of differentiation in General Relativity. Conforming to Einstein's

preferred forms of differentiation, and carrying over the above Newtonian thought to a space-time domain, then, the world line of a free particle on M should satisfy the equation

$$\frac{d^2 x^\sigma}{d\kappa^2} + \Gamma_{\beta\lambda}^\sigma \frac{dx^\beta}{d\kappa} \frac{dx^\lambda}{d\kappa} = 0, \quad (3.1.4)$$

where κ is an arbitrary parameter. Note that by Definition 2.2.2, the world line of a free particle will be represented by a geodesic curve when κ is arc length.

As indicated in the introduction, Einstein used a system of partial differential equations to govern the relationship between space-time geometry and the mass-energy content of space. This system of equations is referred to as *Einstein's Field Equations*. They are given by

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R_\sigma{}^\sigma = -\frac{8\pi G}{c^2} T_{\alpha\beta}, \quad (3.1.5)$$

where

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2} \right) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} - \frac{p}{c^2} g^{\alpha\beta} + \epsilon_0 (F^{\alpha\sigma} F_\sigma{}^\beta + \frac{1}{4} g^{\alpha\beta} F^{\nu\mu} F_{\nu\mu}) \quad (3.1.6)$$

The term

$$\epsilon_0 (F^{\alpha\sigma} F_\sigma{}^\beta + \frac{1}{4} g^{\alpha\beta} F^{\nu\mu} F_{\nu\mu})$$

represents the energy contribution of the electromagnetic field and p and ρ denote the pressure and the mass density of the universe, respectively. Of course G is the gravitational constant. For the objectives of this paper, the energy contribution of the electromagnetic field will be considered insignificant to the large scale geometry of the universe, therefore, reducing (3.1.6) to

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2} \right) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} - \frac{p}{c^2} g^{\alpha\beta}. \quad (3.1.7)$$

The left side of Einstein's field equations (3.1.5) should be recognized as the components of the Einstein tensor given in Chapter 1.

In this section, we have described the metric scheme for a curved space-time domain as a metric tensor field on a space-time manifold and have given a system of partial differential equations (Einstein's field equations) for which this metric tensor field is the solution. With respect to the large scale universe, it is for this metric tensor field that a proposed form will be found. The field equations for the large scale universe will then be constructed from this proposed form. That is, the quantities $R_{\alpha\beta}$ and $R_{\sigma}{}^{\sigma}$ will be found from this proposed form and then substituted into Einstein's field equations.

The Redshifting of light and Black Body Radiation

The proposed form for the metric tensor of the large scale universe is created from assumptions based on astronomical observations. I will refer to these assumptions as the "cosmological assumptions." For the most part, the cosmological assumptions have been deduced from the analysis of *electromagnetic radiation* (EMR) in the universe. It is the goal of this section to summarize applicable physical characteristics of this phenomenon. It must be understood that we will not be discussing the observations themselves in this section, but, rather, the phenomenon that is being observed. The observations and events that are responsible for the cosmological assumptions will be given in the next section.

EMR is the transmission of energy in the form of a wave having both an electric and magnetic component [15]. Maxwell's equations, developed in the mid 1800s by Scottish theoretical physicist James Clerk Maxwell, disclosed that EMR comes in many forms characterized by wavelength or frequency. Arrangement of these forms of EMR according to wavelength or frequency constitutes the *EMR spectrum* (we will always refer to wavelength instead of frequency). There are several classes of EMR:

Class	Wavelength
Gamma Rays	.00002 nm - .01 nm
X-rays	.001 nm - 10 nm
Ultraviolet	10 nm - 400 nm
Optical	380 nm - 780 nm
Infrared	730 nm - 1 mm
Submillimeter	.1 mm - 1 mm
Microwave	1 mm - .03 m

Table 1

Optical EMR, commonly referred to as light, is perceptible to the eye giving us and other animal forms the ability to see. It occupies a small continuous section of the EMR spectrum and has a coloration of shades ranging from violet (blue) to red, blue being the shorter waves and red the longer waves. In some cases optical EMR may have a shorter wavelength when reaching an observer than it did when it was emitted. This shift in wavelength is called *blueshift*. If the wave is shifted to a longer wavelength then it is referred to as *redshift*. The redshifting of optical EMR will be of primary concern to us. There are two reasons commonly accepted for the redshifting of optical EMR. The first is an outcome from the theory of General Relativity. General Relativity predicts that an observer will detect a redshifting of light that passes through a gravitational field. In

general, this has been an important observation in the world of physics, but it has had little direct consequence in forming the cosmological assumptions. The second reason for a redshift in optical EMR is a Doppler effect. That is, if a source of optical EMR is moving away from an observer, then the observer will detect a redshifting of the EMR on account of the relative motion. Speculation on redshift observations due to a Doppler effect has made a significant contribution to the cosmological assumptions and will be discussed in the next section.

EMR is emitted from all bodies whose temperature is above absolute zero. For a particular body at a given temperature, a specific amount of energy is released with respect to each wavelength. The wavelength that corresponds to the maximal energy level is called the maximal wavelength. As the temperature of the body increases more energy is released through radiation and the maximal wavelength will correspond to shorter waves. The energy distribution curve with respect to wavelength at a given temperature T will be referred to as the *EMR spectrum of the body at a temperature T* . The EMR spectrum of a body is not only dependent on temperature, but also, on the material, shape and size of the body. The details of the EMR spectrum of a body cannot easily be described with so many variables to consider. Therefore, as is often done in physics, the notion of an ideal body is established to simplify the problem. (By ideal it is meant that the EMR spectrum of the ideal body is represented by a particularly nice function which can be

used to model the EMR spectrum of those bodies that have a similar EMR spectrum.) According to Halliday and Resnick [15], the radiation released from a cavity formed in a solid body was found to be dependent only on temperature and could be described by a simple differentiable function. It is the EMR spectrum of cavity radiation that was used to represent the EMR spectrum of an ideal body. It should be noted that the ideal body is, in truth, a hypothetical body. As common practice, cavity radiation will be referred to as *black body radiation*. Radiation found to have an EMR spectrum similar to that of black body radiation will be said to have a *black body spectrum*. The following example will help illustrate many of the notions that have been discussed here on black body radiation.

Example (3.2.1) [4]: It has been found that a fireplace poker has an EMR spectrum nearly that of black body. Suppose that you have placed the tip of a fireplace poker into a hot fire. After a few moments you remove the poker and see no apparent change. Your hand, though, senses that the tip of the poker is hot by the heat radiating from it. This heat is actually energy in the form of EMR being emitted from the poker. You again place the tip of the poker into the fire. Soon you note that the poker begins to glow red. This indicates that the maximal wavelength is corresponding to a red shade of the optical spectrum. (The eye favors the wavelength emitting the most energy and therefore it sees the red shade.) As the temperature of the poker rises the maximal wavelength will

correspond to shorter waves and the glow of the poker will change in color from red to orange to white.

Observations of both optical and blackbody radiation have played critical roles in forming our viewpoint on the structure of the universe. The details of such observations are given in the next section.

The Large Scale Universe and the Cosmological Assumptions

Until now we have spoken freely of a large scale universe but have never given its description. The following chronology will provide us with:

- 1) a description of a large scale universe
and
- 2) the observations, conjectures and events that have led to the assumptions that underlie the proposed form of the metric tensor for a large scale universe (the cosmological assumptions).

This chronology contains both current and historical comments that are directly or indirectly related to the cosmological assumptions.

Although there was earlier speculation on the structure of the universe, I begin with the time period 140 A.D. A "*" will indicate that the material to follow is directly related to the cosmological assumptions.

140 A.D. - Ptolemy, a Greek astronomer, asserted that earth was at the center of the heavens. Early astronomers had only the visible part of the EMR spectrum for observation. They often mixed philosophy with naked eye observations of the night sky.

1543 - Nicholaus Copernicus challenged Ptolemaic views by releasing his journal, "Revolutions of the Heavenly Orbs." In opposition to one of the most powerful institutes of the time, the Roman Catholic Church, he indicated that earth and all other planets revolved around the sun. Most importantly, he disputed the notion of earth having a distinguished position in the universe.

1609 - Galileo introduced the optical refracting telescope (a telescope that uses a lens to collect and focus light) to astronomy. Among many other discoveries, he found that the sun shares the universe with other stars. (Although Galileo is often credited for building the first telescope it was actually constructed a year earlier in Holland by Hans Lippershy [17].)

1668 - Issac Newton constructed the first successful version of a reflecting telescope (a telescope that uses a mirror to collect and focus light).

1799 - William Herschel conducted heat experiments with the dispersion of light. According to Henry C. King [17], Herschel "casted in turn various colors of a solar spectrum on a guinea and observed it through a microscope." He then placed a thermometer at the red section of the solar spectrum and found a greater rise in temperature than when placed at the green section. Herschel then placed a thermometer just outside of the red section and he found that there was an even greater rise in temperature. At that time, Herschel referred to the phenomenon as invisible light. He later coined the term "infrared rays" for the invisible light. Although Herschel might not have completely understood what it was that he was observing, he can certainly be credited as one of the first to explore EMR other than the visible light spectrum.

Today, as indicated in the last section, there are several classifications of the EMR spectrum. Ironically, infrared would be one of the last classifications to be studied by cosmologists. Almost 200 years would pass before astronomers would begin to explore the infrared spectrum of EMR in the universe.

1912 - Vesta M. Slipher recorded the redshifting of light coming from distant celestial objects.

*1917 - William de Sitter and Albert E. Einstein worked on relativistic cosmological models of the universe. To help simplify the problem, Einstein made the assumption that the mass content of

space was completely responsible for 3-space curvature. This assumption was referred to by Einstein as Mach's principle [1]. (This principle is named after E. Mach, who conjectured that all mass in the universe is related.) Although Einstein had little evidence to support such an observation, he assumed that, from a large scale point of view, the distribution of mass in the universe was homogeneous. (By large scale it is meant that local features of the distribution are ignored. This was the first version of a large scale universe.) Mach's principle was then used by Einstein to imply that the physical universe had an isotropic geometry.

Einstein tried to obtain a set of field equations that described a physical universe with one particular type of isotropic geometry. He considered both a set that described a Euclidean geometry and a set that described a spherical geometry for the physical universe. Spherical geometry was attractive to Einstein because it gave a finite universe with no boundaries [22].

In forming the field equations, Einstein noted that under the conditions stated above, they predicted a dynamic universe. Einstein dismissed the dynamic universe on the grounds that an expanding or contracting universe was physically unreasonable. Instead, he pursued a static model of the universe. In order to satisfy a static model of the universe Einstein modified the field equations. In their modified form, the field equations were given by

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R_{\sigma}^{\sigma} + \Lambda g_{\alpha\beta} = - \frac{8\pi G}{c^2} T_{\alpha\beta} \quad (3.3.1)$$

where Λ is called the *cosmological constant*.

1918 - George E. Hale concluded the construction of a 100-inch reflecting telescope—the Hooker telescope—on Mount Wilson in southern California. In 1929 he initiated the construction of a 200-inch telescope at Mount Palomar, also in southern California. Hale died in 1948, ten years before the completion this telescope which bears his name.

1922 - Alexander Friedmann proposed a set of field equations for a cosmological model of the universe in which pressure and the cosmological constant were set equal to zero. These field equations conformed to a dynamic universe where all three isotropic geometries were considered as possibilities for the geometry of the physical universe. Friedmann was the first to incorporate these notions into a mathematical model of the universe [14]. (Details of the Friedmann model are given in section four of this chapter.)

* 1924 - Edwin Hubble photographed sections of the universe with the 100-inch Hooker telescope from the Mount Wilson observatory. His photographic plates revealed, for the first time, the existence of galaxies throughout the universe. (A galaxy is a family of stars, numbering into the billions, related by mutual gravitational attraction.) The universe has since been estimated to contain up to 10^8 visible galaxies [2]. Hubble also found that the number of galaxies counted increased proportionately to a decrease in the

luminosity of the galaxies. This was anticipated by Hubble to be a characteristic of a homogeneous distribution of galaxies in the universe [23]. (In this paper it will be accepted that, from a large scale point of view, there is a homogeneous distribution of the galaxies in the universe. Once again, by large scale it is meant that local features of the distribution are ignored.) Hubble's findings motivated cosmologists to concentrate on a relationship between the distribution of galaxies in the universe and the space-time curvature of the universe. A model that forms such a relationship is referred to as a *Large Scale Universe*. If Mach's principle and Hubble's observations are assumed for a large scale universe, then the resulting model must have a physical universe with an isotropic geometry.

Since the release of Copernicus's journal in 1543, cosmologists had been struggling with the notion of a distinguished position in the universe. In light of the new evidence supporting an isotropic physical universe, it now seemed more reasonable than ever to believe that no distinguished position existed. But truthfully, the assumption of no distinguished points in the universe is more of a philosophy than it is an observation.

1925 - Slipher found that light coming from distant galaxies appeared to have shifted towards the red end of the spectrum.

1927 - George Lemaitre, a Roman Catholic Priest, proposed a theory of an evolutionary universe. Lemaitre referred to the origin of the

universe as the "primeval atom" and suggested that the universe expanded from this primordial universe. He theorized that the primeval atom was extremely hot and made up of subatomic particles. A detailed model was never pursued by Lemaitre.

Lemaitre also found a set of field equations describing a large scale universe. His field equations offered a dynamic universe, which considered all three of the isotropic geometries for the physical universe, without setting the cosmological constant equal to zero.

* 1929 - Edwin Hubble confirmed Slipher's findings on the red shifting of light from distant galaxies. This phenomenon was interpreted, by Hubble, to be a Doppler effect on the light waves due to an expanding universe. Hubble viewed the galaxies to be uniformly receding from each other. Hubble also proposed that the amount of redshifting of light from the receding galaxies was linearly related to their distance from us. (For details see [13] and [23].) This linear relationship is called *Hubble's Law* and the constant of this linear relationship is called *Hubble's Constant*.

It had long been speculated by many that the universe was dynamic in nature. As early as 1785 Herschel had conceptualized the possibility of a dynamic universe. Einstein, de Sitter, Friedmann and Lemaitre had at least considered the possibility of an evolutionary universe. But never before had this notion been so strongly substantiated as by Hubble's conjecture on the redshifting light from distant galaxies. After the 1929 discovery, cosmologists had turned

their thoughts towards a dynamic large scale model of the universe. Among these were included Einstein and de Sitter, who had previously rejected the notion. Today, the universe is viewed as a fluid of mass where the galaxies are the fluid particles. From a space-time point of view the galaxies are considered to be free particles moving outward from each other through space.

1931 - K. G. Jansky investigated radio waves (EMR with the longest wavelengths) coming from space at the Bell Telephone Laboratory (Bell Labs) in New Jersey. He was unable to detect any high frequency radio radiation coming from the sun so Jansky concluded that it was coming from the stars. This was the first time that radio detection equipment was used to observe radio radiation in the universe and is recognized as the beginning of radio astronomy. Radio astronomy was refined throughout the 1940s and, as we will see, it has made tremendous contributions to our interpretation of the structure of the universe.

* 1948 - In the wake of Lemaitre's theory on a primordial universe, George Gamow, Ralph Alpher and Robert Herman proposed a theory detailing the first few million years of the expansion. This theory would come to be known as the "Big Bang" theory. Friedmann's field equations were used to show (in terms of Einstein's equation $E=mc^2$) that, "there must have been a time when the energy density of the radiation was greater than the energy density of the matter [14]." From this it was speculated that immediately following the initial

expansion, the universe was made up of extremely hot radiation. As the universe expanded the radiation cooled enough—to about 3000 K (where K denotes Kelvin)—to form the first elements in the form of hydrogen and helium. At that time, the Gamow team conjectured that radiation with a black body spectrum began to propagate freely through space and has been propagating ever since. Assuming that the radiation was cooling at a constant rate, Gamow, Alpher and Herman calculated that this left over black body radiation would today have a temperature characteristic of about 5 K. Although the Gamow team noted that this radiation could probably still be detected at the present, it never attempted to do so.

* 1965 - Arno Penzias and Robert Wilson were working at the Crawford Hill site of Bell Research Laboratories in New Jersey attempting to convert a horn antenna into a radio telescope capable of detecting weak radio radiation in the universe. For such an achievement, it would be necessary to make the receiver as sensitive as possible by eliminating all undesirable noise. In the process of doing so they ran up against a persistent faint form of radio noise. No matter which direction the horn was pointed in, the radio noise remained consistent in its strength. Therefore, the source of the noise being any one object in the universe was dismissed. Penzias and Wilson then exhausted all other possibilities in trying to identify the source of the irritating noise, including cleaning pigeon droppings

from the horn itself [14], but all attempts failed. Penzias and Wilson were unable to determine the origin of the radio noise.

At the same time a Princeton team was constructing a radio telescope with the intent to explore the universe for the cosmic background radiation (CBR) at a temperature of a few Kelvins predicted by the Gamow team. Among the team members was cosmologist Bob Dicke. According to [14], Penzias heard of the Princeton project and telephoned Bob Dicke to tell him of their findings at the Bell Research Laboratories. The Princeton team went to the Crawhill site soon after and made an energy measurement of one wavelength. It was found to match the EMR spectrum of a black body at a temperature of about 2.7 K. After checking the energy levels of many different wavelengths from many different directions (this process is referred to as mapping the EMR spectrum) Penzias, Wilson and the Princeton team concluded that the noise was that of a uniform cosmic background radiation with a black body spectrum and a temperature characteristic of 2.7 K. This located the maximal wavelength in the microwave classification of the EMR spectrum and for this reason this radiation is often referred to as *cosmic microwave background radiation*.

Not since the times of Hubble had such an important observation been made in the world of cosmology! The Big Bang was no longer a theory of philosophy, but, rather, a theory of science. In the future, cosmologists would continue to conduct experiments testing the results of Penzias, Wilson and the Princeton team. (At the

end of this chronology, the significance of cosmic background radiation to our interpretation of the universe's large scale structure will be outlined.)

1989 - NASA launched a satellite into the earth's orbit on a mission to map cosmic background radiation. (Outside of the earth's atmosphere a truer mapping of cosmic background radiation can be obtained.) This satellite is referred to as the *Cosmic Background Explorer*, or for short, the *COBE*. Information sent back from the COBE has so far confirmed the previous findings of a uniform cosmic background radiation with a black body spectrum of about 2.7 K [20].

1989- Margaret Geller and John Huchra announced that they have mapped a distinct pattern of galaxies in the universe. It is five hundred million light years long and two hundred million light years in height. Cosmologists refer to it as the *Great Wall*. This large clump of galaxies obviously disputes prior observations of a homogeneous distribution of galaxies in the universe, but, the COBE has not yet found any disturbances in the cosmic background radiation that would support or explain such a feature in the universe. In the June 3, 1991 issue of *Newsweek* [4], a cosmologist from MIT commented on the phenomenon, "It is so far beyond our understanding that theorists dismiss them (the galaxies of the Great Wall) for the time being hoping they're an illusion." So, in order to preserve the assumption of a homogeneous distribution of mass in the universe, we too, must hesitantly dismiss them.

1990- NASA's *Hubble Space Telescope* was placed into the earth's orbit. One of its many objectives will be to study the distribution of galaxies in the universe. It is hoped that the Hubble telescope will be able to view the universe as far back as a billion years after the big bang (i.e., about 14 billion years ago).

1991- Two new large telescopes are expected to be operating by the end of this year: a 10-meter telescope called the *Keck telescope* in Hawaii and a telescope equipped with four 8-meter mirrors in Chile. The latter will be the largest in the world. Also, a Keck twin should be operating by the year 1996 on Mauna Kea [4].

1991 - NASA launched the *Gamma Ray Observatory (GRO)* into space.

As previously mentioned, for the most part, everything we know about the universe comes to us by way of EMR. Cosmologists hope that by continuing to explore all forms of EMR in the universe that, someday, they will be able to explain such a phenomenon as the Great Wall. One problem that has confronted cosmologists is that EMR with shorter wavelengths cannot penetrate the earth's atmosphere as well as EMR with longer wavelengths. Therefore, an expedition has been set in motion to place several satellites in orbit equipped with highly sophisticated instruments capable of analyzing EMR with shorter wavelengths. We have seen the initiation of this expedition by the launching of the COBE and the GRO. We conclude this chronology by indicating those projects currently in progress.

1992 - A European infrared satellite is expected to be launched.

1997 - A satellite used for observing the x-ray classification of EMR should be launched. It is called the *X-ray Astrophysics Facility* [13].

1999 - NASA anticipates the launching of the *Space Infrared Telescope Facility* [13].

21st century - The *Large Deployable Reflector* is anticipated to be launched and will be used to observe submillimeter EMR [13].

The cosmological assumptions to be given will reflect an interpretation of the large scale geometry of the universe based on observations of cosmic background radiation, the redshifting of light from distant galaxies and of a homogeneous distribution of galaxies in the universe. Of these three observations, cosmic background radiation has been the most influential. That is, it supports implications of the other two observations and has distinct implications of its own. Unfortunately, in the above chronology, a large digression would have been necessary to detail all the implications of cosmic background radiation. (Those implications of the redshifting of light from distant galaxies and of a homogeneous distribution of galaxies in the universe, were clearly stated.) Therefore, in the following I have outlined the significance of cosmic background radiation to our interpretation of the universe's large scale structure.

1) As indicated in the chronology, it was theorized by Gamow and others that the expansion of the universe was responsible for cooling the cosmic background radiation from 3,000 K to 2.7 K. This means that as the radiation propagated through space it underwent a waveshift due to the expansion of the universe which lowered the temperature characteristic of the EMR spectrum—i.e., the weak microwaves of CBR detected today began their journey as high energy gamma rays. Therefore, the existence of cosmic background radiation at a temperature of 2.7 K supports an expanding universe as previously postulated by Hubble in his 1929 interpretation of the redshifting of light from distant galaxies.

2) The uniform temperature characteristic of the cosmic background radiation is evidence of an isotropic early physical universe (pre-galaxy). It also implies that the expansion itself must be isotropic, that is, at least up until the present day expansion. If the expansion were not isotropic, then certainly the EMR spectrum would not have shifted in a uniform manner. Therefore, if the early physical universe was isotropic and the expansion was isotropic, then one could speculate that the present physical universe is isotropic.

3) An isotropic expansion also implies that, from a large scale viewpoint, all galaxies are moving uniformly away from each other during the expansion. Here by large scale it is meant that local irregular motions of the galaxies are ignored.

4) The time element of our problem will be concerned only with the large scale geometric features of the physical universe and the expansion of the physical universe. Local activities such as the motion of subatomic particles are ignored. Therefore, the temperature characteristic of cosmic background radiation gives us a universal clock without contradicting Special Relativity. That is, it comes from the time of an early universe (pre-galaxy), it saturates the universe, and its temperature characteristic has been uniformly changing ever since it began propagating. Basically, this is saying that it is plausible to assume universal time, in the Newtonian sense, for the problem at hand [2].

We can summarize the above, and the remarks of the last two sections, in the following assumptions on the geometric structure of the large scale universe:

Cosmological Assumption 1 (CA-1): For any instant of time, the physical universe has an isotropic geometry. Also, any two points in the physical universe are indistinguishable (homogeneity).

Cosmological Assumption 2 (CA-2): The physical universe is isotropically expanding.

Cosmological Assumption 3 (CA-3): Time is universal in the Newtonian sense.

With the cosmological assumptions in hand, we will develop a proposed form of the metric tensor for the large scale universe.

Consider a large scale space-time structure of the universe that satisfies all three of the cosmological assumptions. Let M be a Riemannian space-time manifold representing this space-time structure where for any given point P on M there exists an open set N_P such that a set of nonintersecting geodesic curves can be found covering N_P . Let the time coordinate vary on this set of geodesic curves. Assume that for any given time t the corresponding physical universe is represented by a 3-dimensional Riemannian manifold M^t . (These are the x^0 -submanifolds.) Then, by CA-1, each M^t has an isotropic geometry and it will be assumed that for any given time, the corresponding physical universe has the same isotropic geometry (i.e., this bars the possibility of the physical universe having a spherical geometry today and a pseudospherical geometry tomorrow). Let x^1, x^2, x^3 be coordinates on each M^t such that the metric tensor for these submanifolds is the Riemannian form. That is, for any time t , $g_{ij}(t, x^1, x^2, x^3)$ is the Riemannian form [see (2.1.14)], where $i, j = 1, 2, 3$. Here, R is referred to as the radius of the universe. By CA-2, the radius of the universe will be dependent on time, that is, we will have $R(t)$. (The final comment in this section will allow us to conclude that the radius of the universe is a function of time alone.) Since, by CA-3, time is universal, then g_{00} will be constant. In order to satisfy a Minkowski space-time (this occurs when the physical universe has a Euclidean geometry) we will select

c^2 as this constant. Here, c denotes the velocity of light. Also, it will be assumed that for some instant of time, say $x^0 = b$, the x^0 coordinate is orthogonal to the coordinates x^1, x^2, x^3 on M^b . Thus, the coordinates x^0, x^1, x^2, x^3 are Gaussian coordinates and our discussion prior to Theorem 2.2.1 implies that the coordinate x^0 will remain orthogonal to the coordinates x^1, x^2, x^3 for all x^0 -submanifolds representing the physical universe on N_p . Therefore, by Theorem 2.2.1, and by the assumption that $g_{\alpha\beta}$ has a hyperbolic signature, then, with respect to these Gaussian coordinates the metric tensor on M will take the form

$$g_{\alpha\beta}(x^\alpha) = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \\ 0 & 0 & -f & 0 \\ 0 & 0 & 0 & -f \end{pmatrix}, \quad (3.3.2)$$

where

$$f = \frac{R(t)^2}{[1 + \left(\frac{k}{4}\right)((x^1)^2 + (x^2)^2 + (x^3)^2)]^2}.$$

This form of the metric tensor was first given by Howard Robertson and Arthur Walker in 1935 and for this reason it is called the *Robertson-Walker metric*. In the next section we will find a

particular form of the field equations for which the Robertson-Walker metric is a solution. That is, we will construct a special form of Einstein's field equations from the Robertson-Walker metric.

Before moving on to the next section, I would like to make a few important final comments. With respect to a uniform recession of the galaxies, cosmologists have assumed that for any instant of time the galaxies are at rest in the corresponding physical universe. This assumption implies that the paths of the galaxies will be represented by the x^0 -curves. That is, a path of a galaxy must be represented by a curve of the form $x^1 = \text{constant}$, $x^2 = \text{constant}$, $x^3 = \text{constant}$. Therefore, suppose a galaxy is positioned at a point P in the physical universe for some instant of time, then it will remain at P for any future time period. The scale of distance between the galaxies will change, but according to the coordinates x^1 , x^2 , x^3 on M the galaxies will have remained at rest. In this context, the physical coordinates can be viewed as moving with the galaxies during the expansion. Consequently, Gaussian coordinates are referred to as *comoving coordinates* in application to a large scale universe. With respect to such coordinates we have $dx^1/ds = dx^2/ds = dx^3/ds = 0$ for galactic motion. Therefore, from above, and from (3.1.7), a mass-energy tensor satisfying all three of the cosmological assumptions will take the form

$$T_{\alpha}^{\beta} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{-p}{c^2} & 0 & 0 \\ 0 & 0 & \frac{-p}{c^2} & 0 \\ 0 & 0 & 0 & \frac{-p}{c^2} \end{pmatrix}. \quad (3.3.3)$$

The notion of comoving coordinates also implies that the radius, pressure, and mass density of the universe are not functions of the coordinates x^1, x^2, x^3 , but rather, functions of time alone.

Einstein's Field Equations with Respect to Friedmann's Model

A relativistic large scale model of the universe that

1) satisfies all of the cosmological assumptions

and

2) sets the cosmological constant and the pressure of the universe equal to zero

is called *Friedmann's Model of the Universe*. This model is named after Alexander Friedmann (1888-1925), a Russian mathematician/meteorologist. Friedmann, like Einstein, believed that the geometry of the physical universe was isotropic. Unlike Einstein he saw the universe as being dynamic rather than static. Intuitively,

Friedmann described the universe as a balloon that could either expand or contract as a function of time. (It is important for the reader to note that viewing the universe as a balloon is only a 2-dimensional analogy of the physical universe.) As previously mentioned in the chronology, in 1922 he published a set of field equations for a large scale universe where the cosmological constant and the pressure of the universe were set equal to zero. These field equations considered all three isotropic geometries as possibilities for the geometry of the physical universe. Friedmann was the first to incorporate such notions into a mathematical model of the universe [14]. His demands on the space-time geometry of the large scale universe concur completely with the cosmological assumptions and are compatible with the popular Big Bang theory.

In the following, Friedmann's equations will be constructed by detailing an outline of their development found in [2]. That is, the quantities $R_{\alpha\beta}$ and R_{σ}^{σ} will be found from the Robertson-Walker metric and then substituted into Einstein's field equations (3.3.1),

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R_{\sigma}^{\sigma} + \Lambda g_{\alpha\beta} = - \frac{8\pi G}{c^2} T_{\alpha\beta} .$$

To begin, we assume that we are working on a 4-dimensional space-time manifold M representing a large scale space-time structure of the universe which satisfies all three of the cosmological assumptions. We also assume that Gaussian coordinates (or comoving coordinates) t, x, y, z can be found locally on M such that the metric tensor takes the form given by (3.3.2). That is,

$$ds^2 = c^2 dt^2 - f dx^2 - f dy^2 - f dz^2,$$

where

$$f = \frac{R(t)^2}{[1 + \left(\frac{k}{4}\right)(x^2 + y^2 + z^2)]^2}.$$

Here, we have set $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Let κ be an arbitrary parameter. Dividing through by $d\kappa^2$ yields

$$\frac{ds^2}{d\kappa^2} = c^2 \dot{t}^2 - f \dot{x}^2 - f \dot{y}^2 - f \dot{z}^2,$$

where the dot above the variable denotes differentiation with respect to κ .

Consider the equations

$$\frac{\partial}{\partial x^\mu} (g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta) = \frac{d}{d\kappa} \left(\frac{\partial}{\partial \dot{x}^\mu} (g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta) \right). \quad (3.4.1)$$

Since $g_{\alpha\beta}$ is not a function of $\dot{x}^\alpha$, the above equations can be reduced to

$$\frac{\partial g_{\alpha\beta}}{\partial x^\mu} \dot{x}^\alpha \dot{x}^\beta - 2 \frac{d(g_{\alpha\mu} \dot{x}^\alpha)}{d\kappa} = 0$$

$$\Rightarrow \quad \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \dot{x}^\alpha \dot{x}^\beta - \frac{\partial g_{\alpha\mu}}{\partial x^\beta} \dot{x}^\beta \dot{x}^\alpha - g_{\lambda\mu} \ddot{x}^\lambda = 0$$

$$\begin{aligned}
\Rightarrow & \quad \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \dot{x}^\alpha \dot{x}^\beta - \frac{1}{2} \frac{\partial g_{\alpha\mu}}{\partial x^\beta} \dot{x}^\beta \dot{x}^\alpha - \frac{1}{2} \frac{\partial g_{\alpha\mu}}{\partial x^\beta} \dot{x}^\beta \dot{x}^\alpha - g_{\lambda\mu} \ddot{x}^\lambda = 0 \\
\Rightarrow & \quad g_{\lambda\mu} \ddot{x}^\lambda + \frac{1}{2} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\beta} \dot{x}^\beta \dot{x}^\alpha + \frac{\partial g_{\alpha\mu}}{\partial x^\beta} \dot{x}^\beta \dot{x}^\alpha - \frac{\partial g_{\alpha\beta}}{\partial x^\mu} \dot{x}^\alpha \dot{x}^\beta \right) = 0.
\end{aligned}$$

Multiplying through by $g^{\rho\mu}$ and relabeling indices gives us

$$\ddot{x}^\rho + \Gamma_{\alpha\beta}^\rho \dot{x}^\beta \dot{x}^\alpha = 0. \quad (3.4.2)$$

Therefore we can use equations (3.4.1) to find all nonvanishing Christoffel symbols in terms of f . Equations (3.4.1) are called the *Euler-Lagrange equations*.

Observing that

$$\frac{ds^2}{d\kappa^2} = g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

and then substituting this into equations (3.4.1) yields

$$\frac{d}{d\kappa} \left[\frac{\partial}{\partial \dot{x}^\alpha} \left(\frac{ds}{d\kappa} \right)^2 \right] = \frac{\partial}{\partial x^\alpha} \left(\frac{ds}{d\kappa} \right)^2.$$

For the time coordinate t we have

$$\frac{d}{d\kappa} \left[\frac{\partial}{\partial \dot{t}} \left(\frac{ds}{d\kappa} \right)^2 \right] - \frac{\partial}{\partial t} \left(\frac{ds}{d\kappa} \right)^2 = 0$$

which implies,

$$2c^2 \ddot{t} = -f_t \dot{x}^2 - f_t \dot{y}^2 - f_t \dot{z}^2.$$

Before we can obtain the Christoffel symbols we must put the above equations into the form of equations (3.4.2). That is,

$$\ddot{t} + \frac{f_t}{2c^2} \dot{x}^2 + \frac{f_t}{2c^2} \dot{y}^2 + \frac{f_t}{2c^2} \dot{z}^2 = 0. \quad (3.4.3a)$$

For the coordinate x we find,

$$\begin{aligned} & \frac{d}{d\kappa} \left[\frac{\partial}{\partial \dot{x}} \left(\frac{ds}{d\kappa} \right)^2 \right] - \frac{\partial}{\partial x} \left(\frac{ds}{d\kappa} \right)^2 = 0 \\ \Rightarrow & \frac{d}{d\kappa} (-2f\dot{x}) = -f_x \dot{x}^2 - f_x \dot{y}^2 - f_x \dot{z}^2. \\ \Rightarrow & -2\ddot{f}x - 2f_t \dot{x} - 2f_x \dot{x}^2 - 2f_y \dot{y}\dot{x} - 2f_z \dot{z}\dot{x} = -f_x \dot{x}^2 - f_x \dot{y}^2 - f_x \dot{z}^2 \\ \Rightarrow & \ddot{x} + \frac{f_t}{f} \dot{x}\dot{t} + \frac{f_x}{2f} \dot{x}^2 + \frac{f_y}{f} \dot{x}\dot{y} + \frac{f_z}{f} \dot{x}\dot{z} - \frac{f_x}{2f} \dot{y}^2 - \frac{f_x}{2f} \dot{z}^2 = 0 \end{aligned} \quad (3.4.3b)$$

Any term that includes a product $\dot{x}^\beta \dot{x}^\lambda$ must be counted as twice that quantity since $\dot{x}^\beta \dot{x}^\lambda = \dot{x}^\lambda \dot{x}^\beta$. This implies that (3.4.2) will take the form

$$\ddot{x} + \frac{f_t}{2f} \dot{x}\dot{t} + \frac{f_t}{2f} \dot{t}\dot{x} + \frac{f_x}{2f} \dot{x}^2 + \frac{f_y}{2f} \dot{x}\dot{y} + \frac{f_y}{2f} \dot{y}\dot{x} + \frac{f_z}{2f} \dot{x}\dot{z} + \frac{f_z}{2f} \dot{z}\dot{x} - \frac{f_x}{2f} \dot{y}^2 - \frac{f_x}{2f} \dot{z}^2 = 0.$$

Proceeding with the same process for the coordinates y and z we find the equations

$$\begin{aligned} \ddot{y} + \frac{f_t}{2f} \dot{y}\dot{t} + \frac{f_t}{2f} \dot{t}\dot{y} + \frac{f_x}{2f} \dot{x}\dot{y} + \frac{f_x}{2f} \dot{y}\dot{x} + \frac{f_y}{2f} \dot{y}^2 + \frac{f_z}{2f} \dot{y}\dot{z} + \frac{f_z}{2f} \dot{z}\dot{y} - \frac{f_y}{2f} \dot{x}^2 - \frac{f_y}{2f} \dot{z}^2 &= 0 \\ \ddot{z} + \frac{f_t}{2f} \dot{z}\dot{t} + \frac{f_t}{2f} \dot{t}\dot{z} + \frac{f_x}{2f} \dot{x}\dot{z} + \frac{f_x}{2f} \dot{z}\dot{x} + \frac{f_y}{2f} \dot{y}\dot{z} + \frac{f_y}{2f} \dot{z}\dot{y} + \frac{f_z}{2f} \dot{z}^2 - \frac{f_z}{2f} \dot{x}^2 - \frac{f_z}{2f} \dot{y}^2 &= 0. \end{aligned}$$

Therefore, from (3.4.3a) and from the last three equations given above we can read off the nonvanishing Christoffel symbols in terms of f . They are,

$$\Gamma_{ti}^0 = \frac{f_t}{2c^2}, \quad \Gamma_{0i}^i = \frac{f_t}{2f}, \quad \Gamma_{ii}^i = \frac{f_i}{2f}, \quad \Gamma_{ij}^i = -\frac{f_i}{2f}, \quad \Gamma_{ij}^j = \frac{f_j}{2f}. \quad (3.4.3c)$$

We can express the Ricci tensor components of the forms R_{00} , R_{ii} and R_{ij} , where $i \neq j$, in terms of f with the above Christoffel symbols. Recall from (1.2.11) that the components of the Ricci tensor are given by

$$R_{\beta\lambda} = R^{\alpha}_{\beta\lambda\alpha} = \Gamma^{\alpha}_{\beta\alpha,\lambda} - \Gamma^{\alpha}_{\lambda\beta,\alpha} + \Gamma^{\alpha}_{\sigma\lambda}\Gamma^{\sigma}_{\beta\alpha} - \Gamma^{\sigma}_{\lambda\beta}\Gamma^{\alpha}_{\sigma\alpha}.$$

Relabeling indices and noting the symmetry in the components of the metric tensor yields

$$\Gamma^{\alpha}_{\sigma\alpha} = \frac{1}{2} g^{\alpha\rho} (g_{\alpha\rho,\sigma} + g_{\rho\sigma,\alpha} - g_{\sigma\alpha,\rho}) = \frac{1}{2} g^{\alpha\rho} g_{\alpha\rho,\sigma}.$$

From (1.2.7) we have

$$\Gamma^{\alpha}_{\sigma\alpha} = -\frac{1}{2} \frac{1}{g} \frac{\partial(-g)}{\partial g_{\alpha\rho}} \frac{\partial g_{\alpha\rho}}{\partial x^{\sigma}},$$

or

$$\Gamma^{\alpha}_{\sigma\alpha} = \ln(\sqrt{-g})_{,\sigma}$$

Since $-g = c^2 f^3$ we then have

$$\begin{aligned}
\Gamma_{\sigma\alpha}^{\alpha} &= \ln(\sqrt{c^2 f^3})_{,\sigma} \\
&= \ln(c)_{,\sigma} + \ln(\sqrt{f^3})_{,\sigma} \\
&= \frac{3}{2} \ln f_{,\sigma} .
\end{aligned}$$

This implies that the components of the Ricci tensor can be put into the form

$$R_{\beta\lambda} = \frac{3}{2} \ln f_{,\beta,\lambda} - \Gamma_{\lambda\beta,\alpha}^{\alpha} + \Gamma_{\sigma\lambda}^{\alpha} \Gamma_{\beta\alpha}^{\sigma} - \Gamma_{\lambda\beta}^{\sigma} \left(\frac{3}{2} \ln f_{,\sigma} \right).$$

Then for the components of the form R_{ii} we have

$$R_{ii} = \frac{3}{2} \ln f_{,i,i} - \Gamma_{ii,\alpha}^{\alpha} + \Gamma_{\sigma i}^{\alpha} \Gamma_{i\alpha}^{\sigma} - \Gamma_{ii}^{\sigma} \frac{3}{2} \ln f_{,\sigma}. \quad (3.4.4)$$

By the product rule and noting that $\ln f_{,i} = f_i/f$, the first term on the right can be expressed as

$$\frac{3}{2} \ln f_{,i,i} = \frac{3}{2} \left(\frac{f_{,ii}}{f} - \left(\frac{f_{,i}}{f} \right)^2 \right) = \frac{3}{2} \frac{f_{,ii}}{f} - \frac{3}{2} \left(\frac{f_{,i}}{f} \right)^2 \quad (3.4.4a)$$

Summing on α in the second term of (3.4.4) gives us

$$\Gamma_{ii,\alpha}^{\alpha} = \Gamma_{ii,0}^0 + \Gamma_{ii,1}^1 + \Gamma_{ii,2}^2 + \Gamma_{ii,3}^3 .$$

Now, expressing the Christoffel symbols in terms of f and writing out the cases for $i = 1, 2, 3$ yields:

$$\Gamma_{11,\alpha}^{\alpha} = \frac{f_{,tt}}{2c^2} + \frac{1}{2} \frac{f_{,11}}{f} - \frac{1}{2} \left(\frac{f_{,1}}{f} \right)^2 - \frac{1}{2} \frac{f_{,22}}{f} + \frac{1}{2} \left(\frac{f_{,2}}{f} \right)^2 - \frac{1}{2} \frac{f_{,33}}{f} + \frac{1}{2} \left(\frac{f_{,3}}{f} \right)^2$$

$$\Gamma_{22,\alpha}^{\alpha} = \frac{f_{tt}}{2c^2} - \frac{1}{2} \frac{f_{11}}{f} + \frac{1}{2} \left(\frac{f_1}{f} \right)^2 + \frac{1}{2} \frac{f_{22}}{f} - \frac{1}{2} \left(\frac{f_2}{f} \right)^2 - \frac{1}{2} \frac{f_{33}}{f} + \frac{1}{2} \left(\frac{f_3}{f} \right)^2$$

$$\Gamma_{33,\alpha}^{\alpha} = \frac{f_{tt}}{2c^2} - \frac{1}{2} \frac{f_{11}}{f} + \frac{1}{2} \left(\frac{f_1}{f} \right)^2 - \frac{1}{2} \frac{f_{22}}{f} + \frac{1}{2} \left(\frac{f_2}{f} \right)^2 + \frac{1}{2} \frac{f_{33}}{f} - \frac{1}{2} \left(\frac{f_3}{f} \right)^2 .$$

Observing the indicated pattern allows us to write

$$\Gamma_{ii,\alpha}^{\alpha} = \frac{f_{tt}}{2c^2} + \frac{f_{ii}}{2f} - \frac{1}{2} \left(\frac{f_i}{f} \right)^2 - \frac{f_{jj}}{2f} + \frac{1}{2} \left(\frac{f_j}{f} \right)^2 - \frac{f_{kk}}{2f} + \frac{1}{2} \left(\frac{f_k}{f} \right)^2 . \quad (3.4.4b)$$

(Note, in the above formula $i, j, k = 1, 2, 3$ and are chosen such that i, j, k are distinct.) For the next term in R_{ii} , we begin by summing on σ and α yielding

$$\begin{aligned} \Gamma_{\sigma i}^{\alpha} \Gamma_{i\alpha}^{\sigma} = & \Gamma_{0i}^0 \Gamma_{i0}^0 + \Gamma_{1i}^0 \Gamma_{i0}^1 + \Gamma_{2i}^0 \Gamma_{i0}^2 + \Gamma_{3i}^0 \Gamma_{i0}^3 \\ & + \Gamma_{0i}^1 \Gamma_{i1}^0 + \Gamma_{1i}^1 \Gamma_{i1}^1 + \Gamma_{2i}^1 \Gamma_{i1}^2 + \Gamma_{3i}^1 \Gamma_{i1}^3 \\ & + \Gamma_{0i}^2 \Gamma_{i2}^0 + \Gamma_{1i}^2 \Gamma_{i2}^1 + \Gamma_{2i}^2 \Gamma_{i2}^2 + \Gamma_{3i}^2 \Gamma_{i2}^3 \\ & + \Gamma_{0i}^3 \Gamma_{i3}^0 + \Gamma_{1i}^3 \Gamma_{i3}^1 + \Gamma_{2i}^3 \Gamma_{i3}^2 + \Gamma_{3i}^3 \Gamma_{i3}^3 . \end{aligned}$$

By setting $i=1$ and expressing the Christoffels in terms of f we obtain

$$\begin{aligned} \Gamma_{\sigma 1}^{\alpha} \Gamma_{1\alpha}^{\sigma} = & 0 + \frac{f_t}{2c^2} \frac{f_t}{2f} + 0 + 0 \\ & + \frac{f_t}{2f} \frac{f_t}{2c^2} + \frac{f_1}{2f} \frac{f_1}{2f} + \frac{f_2}{2f} \left(\frac{-f_2}{2f} \right) + \frac{f_3}{2f} \left(\frac{-f_3}{2f} \right) \\ & + 0 + \frac{-f_2}{2f} \frac{f_2}{2f} + \frac{f_1}{2f} \frac{f_1}{2f} + 0 \end{aligned}$$

$$\begin{aligned}
& + 0 + \frac{-f_3}{2f} \frac{f_3}{2f} + 0 + \frac{f_1}{2f} \frac{f_1}{2f} \\
& = \frac{1}{2c^2} \frac{f_t^2}{f} + \frac{3}{4} \left(\frac{f_1}{f} \right)^2 - \frac{1}{2} \left(\frac{f_2}{f} \right)^2 - \frac{1}{2} \left(\frac{f_3}{f} \right)^2,
\end{aligned}$$

(Note that the 0's in the above sum are the result of vanishing Christoffels.) Similarly, it is found that

$$\begin{aligned}
\Gamma_{2\alpha}^\sigma \Gamma_{\sigma 2}^\alpha &= \frac{1}{2c^2} \frac{f_t^2}{f} + \frac{3}{4} \left(\frac{f_2}{f} \right)^2 - \frac{1}{2} \left(\frac{f_1}{f} \right)^2 - \frac{1}{2} \left(\frac{f_3}{f} \right)^2 \\
\Gamma_{3\alpha}^\sigma \Gamma_{\sigma 3}^\alpha &= \frac{1}{2c^2} \frac{f_t^2}{f} + \frac{3}{4} \left(\frac{f_3}{f} \right)^2 - \frac{1}{2} \left(\frac{f_1}{f} \right)^2 - \frac{1}{2} \left(\frac{f_2}{f} \right)^2
\end{aligned}$$

and therefore it can be concluded that

$$\Gamma_{i\alpha}^\sigma \Gamma_{\sigma i}^\alpha = \frac{1}{2c^2} \frac{f_t^2}{f} + \frac{3}{4} \left(\frac{f_i}{f} \right)^2 - \frac{1}{2} \left(\frac{f_j}{f} \right)^2 - \frac{1}{2} \left(\frac{f_k}{f} \right)^2. \quad (3.4.4c)$$

(See comment on page 76 for choosing i, j, k in the above formula.)

Summing on α and writing out the cases for i in the next term in R_{ii} gives us

$$\begin{aligned}
\Gamma_{11}^\sigma \left(\frac{3}{2} \ln f_{,\sigma} \right) &= \Gamma_{11}^0 \left(\frac{3}{2} \ln f_{,0} \right) + \Gamma_{11}^1 \left(\frac{3}{2} \ln f_{,1} \right) + \Gamma_{11}^2 \left(\frac{3}{2} \ln f_{,2} \right) + \Gamma_{11}^3 \left(\frac{3}{2} \ln f_{,3} \right) \\
\Gamma_{22}^\sigma \left(\frac{3}{2} \ln f_{,\sigma} \right) &= \Gamma_{22}^0 \left(\frac{3}{2} \ln f_{,0} \right) + \Gamma_{22}^1 \left(\frac{3}{2} \ln f_{,1} \right) + \Gamma_{22}^2 \left(\frac{3}{2} \ln f_{,2} \right) + \Gamma_{22}^3 \left(\frac{3}{2} \ln f_{,3} \right) \\
\Gamma_{33}^\sigma \left(\frac{3}{2} \ln f_{,\sigma} \right) &= \Gamma_{33}^0 \left(\frac{3}{2} \ln f_{,0} \right) + \Gamma_{33}^1 \left(\frac{3}{2} \ln f_{,1} \right) + \Gamma_{33}^2 \left(\frac{3}{2} \ln f_{,2} \right) + \Gamma_{33}^3 \left(\frac{3}{2} \ln f_{,3} \right).
\end{aligned}$$

Expressing the Christoffels in terms of f results in

$$\Gamma_{11}^{\sigma}(\frac{3}{2} \ln f_{,\sigma}) = \frac{f_t}{2c^2} (\frac{3}{2} \frac{f_t}{f}) + \frac{f_1}{2f} (\frac{3}{2} \frac{f_1}{f}) - \frac{f_2}{2f} (\frac{3}{2} \frac{f_2}{f}) - \frac{f_3}{2f} (\frac{3}{2} \frac{f_3}{f})$$

$$\Gamma_{22}^{\sigma}(\frac{3}{2} \ln f_{,\sigma}) = \frac{f_t}{2c^2} (\frac{3}{2} \frac{f_t}{f}) - \frac{f_1}{2f} (\frac{3}{2} \frac{f_1}{f}) + \frac{f_2}{2f} (\frac{3}{2} \frac{f_2}{f}) - \frac{f_3}{2f} (\frac{3}{2} \frac{f_3}{f})$$

$$\Gamma_{33}^{\sigma}(\frac{3}{2} \ln f_{,\sigma}) = \frac{f_t}{2c^2} (\frac{3}{2} \frac{f_t}{f}) - \frac{f_1}{2f} (\frac{3}{2} \frac{f_1}{f}) - \frac{f_2}{2f} (\frac{3}{2} \frac{f_2}{f}) + \frac{f_3}{2f} (\frac{3}{2} \frac{f_3}{f}),$$

which implies that

$$\Gamma_{ii}^{\sigma}(\frac{3}{2} \ln f_{,\sigma}) = \frac{3}{4c^2} \frac{f_t^2}{f} + \frac{3}{4} \left(\frac{f_i}{f} \right)^2 - \frac{3}{4} \left(\frac{f_j}{f} \right)^2 - \frac{3}{4} \left(\frac{f_k}{f} \right)^2. \quad (3.4.4d)$$

(See comment on page 76 for choosing i, j, k in the above formula.)

Now adding and subtracting (3.4.4a), (3.4.4b), (3.4.4c) and (3.4.4d) as indicated by (3.4.4) will yield

$$R_{ii} = \frac{f_{ii}}{2f} + \frac{f_{ii}}{2f} + \frac{f_{kk}}{2f} + \frac{f_{ii}}{2f} - \frac{3}{4} \left(\frac{f_i}{f} \right)^2 - \frac{1}{4} \left(\frac{f_i}{f} \right)^2 - \frac{1}{4} \left(\frac{f_j}{f} \right)^2 - \frac{1}{4} \left(\frac{f_j^2}{f} \right)^2 - \frac{1}{4c^2} \frac{f_t^2}{f} - \frac{f_{tt}}{2c^2}. \quad (3.4.5)$$

By a similar process we can find, in terms of f, the Ricci components,

$$R_{ij} = \frac{f_{ij}}{2f} - \frac{3}{4} \frac{f_i f_j}{f^2}, \quad (3.4.6)$$

where $i \neq j$, and

$$R_{00} = \frac{3}{2} \frac{f_{tt}}{f} - \frac{3}{4} \left(\frac{f_t}{f} \right)^2. \quad (3.4.7)$$

Next we put the Ricci tensor component forms R_{00} , R_{ii} and R_{ij} into terms of the radius R and coordinates t, x^1, x^2, x^3 . Recall that $f = R(t)^2 [1 + (k/4)r^2]^{-2}$, where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. Then, letting $a = (k/4)$ we obtain

$$f_i = -4ax^i R^2 [1 + ar^2]^{-3}$$

$$f_{ii} = 24a^2(x^i)^2 R^2 [1 + ar^2]^{-4} - 4a R^2 [1 + ar^2]^{-3}$$

$$f_{ij} = 24a^2 x^i x^j R^2 [1 + ar^2]^{-4}$$

$$f_t = 2RR_t [1 + ar^2]^{-2}$$

$$f_{tt} = 2R_t^2 [1 + ar^2]^{-2} + 2RR_{tt} [1 + ar^2]^{-2}.$$

Straightforward substitution of the appropriate quantities from above into (3.4.5) yields

$$\begin{aligned} R_{ii} &= 8a^2 r^2 [1 + ar^2]^{-2} - 8a [1 + ar^2]^{-1} - \frac{1}{c^2} (2R_t^2 + RR_{tt}) [1 + ar^2]^{-2} \\ &= 8a [1 + ar^2]^{-1} (ar^2 [1 + ar^2]^{-1} - 1) - \frac{1}{c^2} (2R_t^2 + RR_{tt}) [1 + ar^2]^{-2}. \end{aligned}$$

Noting that

$$ar^2 [1 + ar^2]^{-1} - 1 = \frac{ar^2}{[1 + ar^2]} - \frac{[1 + ar^2]}{[1 + ar^2]} = \frac{-1}{[1 + ar^2]} = -[1 + ar^2]^{-1}$$

and replacing a with $(k/4)$ leads to

$$R_{ii} = \frac{-[2k + \frac{2}{c^2} R_t^2 + \frac{1}{c^2} RR_{tt}]}{[1 + (k/4)r^2]^2}. \quad (3.4.8)$$

Also,

$$R_{ij} = \frac{24a^2x^ix^jR^2[1+ar^2]^{-4}}{2R^2[1+ar^2]^{-2}} - \frac{3}{4} \left(\frac{16a^2x^ix^jR^4[1+ar^2]^{-6}}{R^4[1+ar^2]^{-4}} \right)$$

$$\Rightarrow R_{ij} = 0 \quad (3.4.9)$$

and finally,

$$R_{00} = \frac{3}{2} \left(\frac{2R_t^2[1+ar^2]^{-2} + 2RR_{tt}[1+ar^2]^{-2}}{R^2[1+ar^2]^{-2}} \right)$$

$$- \frac{3}{4} \left(\frac{4R^2R_t^2[1+ar^2]^{-4}}{R^4[1+ar^2]^{-4}} \right)$$

$$= \frac{3R_t^2 + 3RR_{tt}}{R^2} - \frac{3R_t^2}{R^2}$$

$$\Rightarrow R_{00} = \frac{3R_{tt}}{R}. \quad (4.4.10)$$

At this point, we have found the component forms of the Ricci tensor given by (4.4.8), (4.4.9) and (4.4.10). To form Einstein's Field Equations it will also be necessary to find the Riemann scalar invariant. Since

$$g^{\alpha\beta} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & 0 \\ 0 & 0 & -\frac{1}{f} & 0 \\ 0 & 0 & 0 & -\frac{1}{f} \end{pmatrix},$$

the Riemannian scalar invariant is given by the product

$$\begin{aligned} R_{\sigma}^{\sigma} = g^{\sigma\sigma} R_{\sigma\sigma} &= \frac{3}{c^2} \frac{R_{tt}}{R} + 3 \frac{[2k + \frac{2}{c^2} R_t^2 + \frac{1}{c^2} R R_{tt}]}{R^2} \\ &= \frac{6}{c^2 R^2} [c^2 k + R_t^2 + R R_{tt}] . \end{aligned} \quad (3.4.11)$$

Now, substituting (3.4.10) and (3.4.11) into equations (3.3.1) results in

$$\begin{aligned} R_{00} - \frac{1}{2} g_{00} R_{\sigma}^{\sigma} + \Lambda g_{00} &= -\frac{8\pi G}{c^2} T_{00} \\ \Rightarrow \frac{3R_{tt}}{R} - \frac{3}{R^2} [c^2 k + R_t^2 + R R_{tt}] + \Lambda c^2 &= -\frac{8\pi G}{c^2} T_{00} \\ \Rightarrow -\frac{3R_t^2}{R^2} - c^2(\Lambda - \frac{3k}{R^2}) &= \frac{8\pi G}{c^2} T_{00}. \end{aligned} \quad (3.4.12)$$

Substituting (3.4.8) and (3.4.11) into equations (3.3.1) yields

$$\begin{aligned}
R_{ii} - \frac{1}{2} g_{ii} R_{\sigma}^{\sigma} + \Lambda g_{ii} &= -\frac{8\pi G}{c^2} T_{ii} \\
\Rightarrow \frac{-[2k + \frac{2}{c^2} R_t^2 + \frac{1}{c^2} R R_{tt}]}{[1 + (k/4)r^2]^2} \\
+ \frac{1}{2} R(t)^2 [1 + (k/4)r^2]^{-2} \frac{6}{c^2 R^2} [c^2 k + R_t^2 + R R_{tt}] \\
- \Lambda R(t)^2 [1 + (k/4)r^2]^{-2} \\
&= -\frac{8\pi G}{c^2} T_{ii} \\
\Rightarrow \frac{-2kc^2 - 2R_t^2 - R R_{tt} + 3R R_{tt} + 3R_t^2 + 3kc^2 - c^2 R^2 \Lambda}{c^2 [1 + (k/4)r^2]^2} \\
&= -\frac{8\pi G}{c^2} T_{ii} \\
\Rightarrow \frac{c^2 k + R_t^2 + 2R R_{tt} - c^2 R^2 \Lambda}{c^2 [1 + (k/4)r^2]^2} &= -\frac{8\pi G}{c^2} T_{ii}. \quad (3.4.13)
\end{aligned}$$

Now, multiplying through equations (3.4.12) and (3.4.13) by $g^{\alpha\alpha}$ gives us

$$\frac{3R_t^2}{c^2 R^2} + \frac{3k}{R^2} - \Lambda = \frac{8\pi G}{c^2} T_0^0 \quad (3.4.14)$$

and

$$\frac{-2R_{tt}}{c^2 R} - \frac{R_t^2}{c^2 R^2} - \frac{k}{R^2} + \Lambda = -\frac{8\pi G}{c^2} T_i^i. \quad (3.4.15)$$

Then, from (3.3.3) we obtain Einstein's field equations in the form

$$\frac{3R_t^2}{c^2 R^2} + \frac{3k}{R^2} - \Lambda = \frac{8\pi G}{c^2} \rho \quad (3.4.16a)$$

$$\frac{-2R_{tt}}{c^2 R} - \frac{R_t^2}{c^2 R^2} - \frac{k}{R^2} + \Lambda = \frac{8\pi G}{c^2} \frac{p}{c^2}. \quad (3.4.16b)$$

Finally, setting $\Lambda=0$ and $p=0$ we obtain Einstein's field equations with respect to Friedmann's model. That is, we have

$$\frac{3R_t^2}{c^2 R^2} + \frac{3k}{R^2} = -\frac{8\pi G}{c^2} \rho \quad (3.4.17a)$$

$$\frac{-2R_{tt}}{c^2 R} - \frac{R_t^2}{c^2 R^2} - \frac{k}{R^2} = 0. \quad (3.4.17b)$$

These equations are often referred to as *Friedmann's equations*. Equations (3.4.16a), (3.4.16b), (3.4.17a) and (3.4.17b) are used by cosmologists to describe the relationship between mass-energy and space-time geometry for the large scale universe in Friedmann's model. For example, these equations can be used to determine that

the expansion of the universe is decelerating. That is, multiplying (3.4.16b) by 3 and setting $\Lambda = 0$, in both (3.4.16a) and (3.4.16b), and then adding the resulting two equations gives us

$$\frac{-3R_{tt}}{c^2 R} = \frac{4\pi G}{c^2} \left(\rho + \frac{3p}{c^2} \right). \quad (3.4.18)$$

Making the reasonable assumptions that p and ρ are positive, then the right side of equation (3.4.18) will be positive. Thus, since R is positive, it must be that $R_{tt} < 0$. In other words, the universe is decelerating in its expansion.

Also, the radius of the universe can be related to mass-energy by Friedmann's equations. I will give this relationship as found in [1], but first, we must identify those terms in which the radius is expressed. To begin, multiplying (3.4.18) by 2, setting $p = 0$ and then adding the resulting equation to (3.4.17a) gives us the single equation

$$\frac{-2R_{tt}}{c^2 R} - \frac{R_t^2}{c^2 R^2} - \frac{k}{R^2} = 0.$$

or

$$R_t^2 + 2RR_{tt} + c^2 k = 0 \quad (3.4.20a)$$

$$\Rightarrow R_t(R_t^2 + 2RR_{tt}) = -c^2 k R_t \quad (3.4.20b)$$

Following the work of Atwater we note that

$$\frac{d(R_t^2 R)}{dt} = R_t(R_t^2 + 2RR_{tt}),$$

which implies that equation (3.4.20b) can be expressed as

$$\frac{d(R_t^2 R)}{dt} = -c^2 k \frac{dR}{dt}.$$

Integrating both sides results in

$$R_t^2 = \frac{a - kR}{R} c^2, \quad (3.4.21)$$

where a is the constant of integration. Substituting (3.4.21) into (3.4.20a) will give

$$R_{tt} = \frac{-ac^2}{2R^2}, \quad (3.4.22)$$

which implies that a must be positive. Substituting (3.4.22) into (3.4.18) yields

$$a = \left(\frac{4\pi}{3} R^3 \rho \right) \frac{2G}{c^2}.$$

Adler, Bazin and Schiffer argue that since the volume of a 3-sphere is $4/3\pi R^3$, the quantity inside of the parentheses is proportional to the total mass of the universe. Therefore, they set

$$\bar{M} = \left(\frac{4\pi}{3} R^3 \rho \right),$$

that is, $\bar{M}$ is the total mass of the physical universe with respect to the type of isotropic geometry we use to represent it. From this we have

$$a = \bar{M} \frac{2G}{c^2}.$$

Adler, Bazin and Schiffer then use equation (3.4.21) to establish a relationship between the radius of the universe and the quantity "a" given above. Of course, this relationship is with respect to the isotropic geometry representing the physical universe. All three cases have been listed below.

Type of Geometry for 3-space	k	Radius of the Universe
Euclidean	0	$R(t) = \left(\frac{3}{2}c\right)^{2/3} \left(\bar{M} \frac{2G}{c^2}\right)^{1/3} t^{2/3}$
Pseudospherical	-1	$R(t) = \frac{\bar{M}G}{c^2} [\cosh(2t) - 1]$
Spherical	1	$R(t) = \frac{\bar{M}G}{c^2} [1 - \cos(2t)]$

Table 2

For a Euclidean geometry the radius describes a physical universe with a continuous expansion. At the origin the expansion is abrupt. Such an abrupt beginning corresponds to the Big Bang. For a pseudospherical geometry the radius describes a forever monotonically expanding physical universe. Finally, for a spherical

geometry the radius describes a physical universe that expands and contracts periodically. Because this correlates nicely with a gravitational collapse of the physical universe and, because a spherical geometry describes a closed physical universe with no boundaries, many cosmologists have favored a spherical geometry over the past few years for the physical universe.

Conclusion

We have now succeeded in achieving the primary objectives of this paper. Gaussian coordinates were used to establish a form of the metric tensor for a large scale universe called the Robertson-Walker metric and, in return, this metric tensor was used to find a specific form of Einstein's field equations with respect to Friedmann's model of the universe (Friedmann's equations).

Until 1989 Friedmann's equations were thought by cosmologists to most accurately describe the relationship between mass-energy and the space-time geometry of the large scale universe. It was in 1989 that the COBE satellite confirmed the existence of a uniform cosmic background radiation as first observed in 1965 by Penzias, Wilson and the Princeton team. As indicated in the development of the cosmological assumptions, cosmic background radiation supported the previous observations by Hubble of an isotropic and expanding universe and was used as a universal clock in the assumption of a universal time coordinate. The cosmological world, for a short period of time, was ecstatic. All the pieces of the puzzle seemed to be falling together. Ironically, it was in that same year, 1989, that Magaret Geller and John Huchra announced that they had mapped a distinct pattern of galaxies in the universe (the Great Wall). Suddenly, all was in doubt. Many cosmologists have since accepted the Great Wall to be reality. If the Great Wall is reality then cosmologists must rethink the large scale universe as we know

it today, that is, as presented in this paper. Clearly the Robertson-Walker metric will not describe a large scale universe with a nonhomogeneous distribution of mass. Such questions as "If the physical universe does not have an isotropic geometry, then what type of geometry does it have?" and "Why is a nonhomogeneous distribution of mass in the universe not accounted for by cosmic background radiation?" must be addressed before a new form of the metric tensor can be found.

One reason for a sigh of relief is that galactic patterns such as the Great Wall do not necessarily invalidate the application of Gaussian coordinates to a large scale universe. That is, most cosmologists still favor the Big Bang theory and believe that cosmic background radiation is a fossil left over from this event [11]. Thus, if a distribution of mass in the universe is found giving a large scale geometry such that $*g_{ij}$ is a metric tensor for the physical universe, then it may still be permissible to use Theorem 2.2.1 in establishing a metric tensor of the form

$$g_{\alpha\beta}(x^\sigma) = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & & & \\ 0 & (-*g_{ij}) & & \\ 0 & & & \end{pmatrix}.$$

In any case, it is important to understand where cosmologists have been in thought, and are today, so that we can start thinking

towards tomorrow's solutions. Observations such as the Great Wall should not be feared as symbols of failure, but as opportunities to explore and understand one of the greatest mysteries of all, the universe.

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