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Configuration theorems in finite affine planes

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CONFIGURATION THEOREMS
IN FINITE AFFINE PLANES

A Thesis

Presented to

The faculty of the Department of Mathematics
San Jose State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

By

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ABSTRACT

CONFIGURATION THEOREMS IN FINITE AFFINE PLANES

by Doug Stone

This thesis introduces the basic properties of finite affine planes, followed by a study of the relationships between the additional axioms normally placed on these planes:

Desargues' Theorem, Pappus' Theorem, the Shear Theorem, Theorem (N), and the special cases known as the Little Desargues Theorem, Little Pappus Theorem, Little Shear Theorem, and Theorem (n). It then shows how the operations of addition and multiplication on the points on a line are naturally introduced into an affine plane followed by a discussion of how each geometric theorem mentioned above corresponds to additional algebraic structure on the set of points on the line. The chapter concludes with the usual proof that Desargues' Theorem implies Pappus' Theorem in finite affine planes by using Wedderburn's Theorem on finite division rings. The final chapter presents a geometric proof that Desargues' Theorem implies Pappus' Theorem in finite affine planes.

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Introduction

In 1905 [4], Hessenberg gave a direct geometric proof that Desargues' Theorem follows from Pappus' Theorem. Though there has been some criticism of Hessenberg's proof and of subsequent proofs by other authors [12], acceptable geometric proofs have been supplied. Although the converse is not true in general, since that time it has been known that Pappus' Theorem is a consequence of Desargues' Theorem in finite affine planes. (In fact, it was later shown that a finite affine plane which satisfies only a few special cases of Desargues' Theorem is a Pappian plane [6],[1],[13].) However, no one had ever given a direct geometric proof of this fact. In this paper, we present a geometric proof recently published by Helga Tecklenburg [14], that a finite Desarguesian affine plane is Pappian and compare it to the standard proof, which depends on an algebraic result known as Wedderburn's Theorem.

In Chapter 1, we give the basic properties of affine planes and examine the major configuration theorems usually studied in these planes: Pappus' Theorem, Theorem (N), Desargues' Theorem, the converse of Desargues' Theorem, the Shear Theorem, and the special cases known as the Little Pappus Theorem, the Little Desargues Theorem, the Little Shear

Theorem, and Theorem (n). In addition, we explain all of the known relationships between the configuration theorems listed above. It will be shown that Pappus' Theorem is equivalent to Theorem (N) and that Desargues' Theorem is equivalent to the Shear Theorem. Because Theorem (N) and the Shear Theorem appear to be very similar configuration theorems, it seems likely that the results stated above may be useful in obtaining a simpler geometric proof that a finite Desarguesian affine plane is Pappian. Many of the configuration theorems studied in Chapter 1 are used in the geometric proof that a finite Desarguesian affine plane is Pappian presented in Chapter 3 and in Chapter 2 we will see how each of these configuration theorems provides a certain algebraic structure to the plane.

Although we will refer to these configuration results as configuration "theorems," the fact is that none of the configuration theorems listed above follow from the axioms of an affine plane. When one or more are adopted as axioms, then several of the others follow as theorems and these interrelationships are shown in Chapter 1. It seems traditional to refer to each of these configuration results as configuration theorems even when they are used as axioms (they are all theorems in affine geometries over fields where they were first discovered), and we do not believe this causes any confusion in the exposition.

The standard proof that a finite Desarguesian affine plane is Pappian relies heavily on algebra and in particular on Wedderburn's Theorem that a finite division ring is a field. In Chapter 2, we show how the operations of addition and multiplication are naturally introduced into an affine plane. We then show how the various configuration theorems, when adopted as axioms, imply certain algebraic results on the set of points on some line in the affine plane. In particular, we show that under the naturally defined addition and multiplication, the points on a line in a Little Desarguesian affine plane form a right quasifield. (In a projective Little Desarguesian plane, the points on a line, under a naturally defined addition and multiplication, form an alternative ring.) We gradually build to the conclusion that the set of points on a line in a Desarguesian affine plane form a division ring and that the set of points on a line in a Pappian affine plane form a field. (Hilbert was the first to realize the equivalence between Pappus' Theorem and the commutativity of multiplication [5],[9].)

Continuing in Chapter 2, we introduce the concept of a vector space in order to develop a Desarguesian affine plane from a division ring and a Pappian affine plane from a field. We show that a vector space over a division ring can be interpreted as a Desarguesian affine plane (by calling

the vectors points and defining lines in the natural way) and that a vector space over a field, interpreted in the same way, is a Pappian affine plane. We also show that if D is the division ring associated with a line in a Desarguesian affine plane and the points are interpreted as vectors, then the Desarguesian affine plane is a vector space over D . This enables us to give an example of a Desarguesian affine plane which is not Pappian, which proves that in general Desargues' Theorem does not imply Pappus' Theorem. We then conclude Chapter 2 with the standard proof that a finite Desarguesian affine plane is Pappian. The proof hinges on the fact that, by Wedderburn's Theorem, the finite division ring associated with a line in the Desarguesian affine plane is a field. Thus, the finite Desarguesian affine plane is a vector space over a field, which is a Pappian affine plane. Although Wedderburn's Theorem is used at the end of Chapter 2, we wait until the beginning of Chapter 3 to give the proof, which is then presented in three parts. We include a complete development of the proof in Chapter 3 so that it can be compared directly to Tecklenburg's geometric proof that a finite Desarguesian affine plane is Pappian.

In Chapter 3, together with the proof of Wedderburn's Theorem, we present and analyze Tecklenburg's recently published geometric proof that a finite Desarguesian affine

plane is Pappian. Although her proof does not explicitly make use of coordinates or of Wedderburn's Theorem, we will show (using the results of the first two chapters) how Tecklenburg's proof can be explained as a geometric interpretation of the key steps in the proof of Wedderburn's Theorem. Tecklenburg's proof establishes geometrically the first two parts of Wedderburn's proof (part 1 is concerned with the order of the division ring, and its center and normalizers; while part 2 is the development of the class equation) and then uses the same number theoretic argument involving cyclotomic polynomials that is used in part three of the proof of Wedderburn's Theorem.

Chapter 1

Configuration Theorems in Affine Planes

In this chapter, we define an affine plane and list some of the properties of finite affine planes. We then examine the relationships between several of the major configuration theorems on affine planes. References used in this chapter include Batten [2], Garner [8], Scherk and Lingenberg [11], and Wlodarski [15]. The relationship of the configuration theorems to the algebraic structure obtained from the affine plane, via the introduction of coordinates, is considered in chapter 3.

A plane is a triple (P, L, I) such that P , L , and I are sets, $P \cap L = \emptyset$, both P and L cannot be empty, and I is contained in $P \times L$. The elements of P are called points and the elements of L are called lines. I denotes the incidence relation on points and lines. The point p is said to be on the line l (and the line l is said to be on the point p) if $(p, l) \in I$. As defined by Stevenson [13], an Affine plane is a plane (P, L, I) such that

1. Every two points lie on exactly one line.
2. If p and L are a given point and line such that p is not on L , then there exists exactly one line M that passes through p and is parallel to L . (M does not intersect L .)

3. There exist three noncollinear points.

The Euclidean plane is an example of an affine plane which has an infinite number of points. An example of an affine plane with only a finite number of points is drawn below in figure 1. An affine plane with a finite number of points is called a finite affine plane.

Figure 1

This plane has four points (labelled a,b,c and d) and six lines. The line on the points a and b will be referred to as the line ab. Notice that the lines ac and bd do not meet. Thus the unique line on the point b parallel to the line ac is the line bd. This example is the smallest affine plane. Figure 2 illustrates the next largest finite affine plane, and is called Young's Geometry. Notice that each line has three points and that each point is on four lines. There are nine points and twelve lines.

Figure 2

Listed below are several properties of finite affine planes.

(See Batten [2] or Garner [8].)

1. If the lines l_1, l_2 are parallel and line $m \neq l_1$ meets l_1 , then m meets l_2 .
2. All lines have the same number of points. The order of a finite affine plane is the number of points on each line.
3. All points are on the same number of lines. If the order of the plane is k , then each point is on $k+1$ lines.
4. If the order of the affine plane is k , then there are k^2 points and k^2+k lines.

The remainder of this chapter is spent establishing all of the known relationships between the following configuration theorems: Pappus' Theorem, Theorem (N), Desargues' Theorem, the converse of Desargues' Theorem, the Shear Theorem, and the special cases known as the Little Pappus Theorem, the Little Desargues Theorem, the Little Shear Theorem and

Theorem (n). These relationships are summarized at the end of the chapter in figure 29.

An affine plane is said to be a Pappian affine plane (or simply Pappian) if the following theorem holds:

(P) The affine Pappus Theorem

Let x, y, z and x', y', z' be sets of three distinct collinear points on distinct lines such that no one of these points is on both lines, and such that $xy' \parallel x'y$ and $x'z \parallel xz'$, then $yz' \parallel y'z$.

Figure 3

Pappus' Theorem is illustrated above in figure 3. A special case of the Pappus Theorem, the Little Pappus Theorem, is given below.

(p) The Little Pappus Theorem:

Let x, y, z and x', y', z' be sets of distinct collinear points on distinct parallel lines and such that $xy' \parallel x'y$ and $x'z \parallel xz'$, then $yz' \parallel y'z$. (See figure 4.)

Figure 4

The other special case of the affine Pappus Theorem is the case where the two distinct lines must intersect, as illustrated in figure 3. That is:

(P') Let x, y, z and x', y', z' be sets of three distinct collinear points on distinct intersecting lines such that no one of these points is on both lines, and such that $xy' \parallel x'y$ and $xz' \parallel x'z$, then $yz' \parallel y'z$.

Our first theorem shows that the Little Pappus Theorem can

be proven from the special case (P') above. Thus the special case (P') is actually equivalent to the affine Pappus Theorem (P).

Theorem 1: (P') implies (p) .

Proof: Suppose that (P') holds while the Little Pappus Theorem (p) does not. Then there exist points x, y, z and x', y', z' such that $xy' \parallel x'y$ and $xz' \parallel x'z$, but with yz' not parallel to $y'z$ as in figure 5. Let y'' be the point on $x'y$ such that $y''z' \parallel y'z$ (y'' exists because xy' and $y'z$ intersect and so $x'y$ and $y'z$ intersect and so $x'y$ and any line parallel to $y'z$ intersect.) Because $y''z' \parallel y'z$ and $y'z \nparallel yz'$, $y''z'$ cannot be parallel to yz' so $y'' \neq y$. Next let z'' be the intersection of xy'' with $x'z$ (z'' exists because xy'' and xz' intersect and so $x'z$ and xy'' intersect.) Because z'' is on

Figure 5

xy'' and $y'' \neq y$, then z'' is not on xy and so $z'' \neq z$. Since $x'y'' \parallel y'x$ and $z'x \parallel x'z''$, then (P') implies that $y'z'' \parallel y''z'$. But since $y''z' \parallel y'z$, we have $y'z'' \parallel y'z$ which is a contradiction ($z'' \neq z$ and z'' is on $x'z$ which is distinct from $y'z$.) Thus $yz' \parallel y'z$ and so (p) holds.

Corollary 1.1: (P') is equivalent to (P) .

A configuration theorem equivalent to the Pappus Theorem is given below [15]:

(N) Let x, y, w, z and x', y', w', z' be sets of points on distinct lines such that no one of these points is on both lines and such that $xx' \parallel zz'$, $x'y \parallel zw'$, and $yy' \parallel w'w$, then $xy' \parallel z'w$. (See figure 6.)

Figure 6

Remark: (N) is vacuously true in an affine plane of order 2.

Theorem 2: (P) is equivalent to (N).

Proof: To see that (N) implies (P), it is enough to realize that (P) is just a special case of (N) where, for example, $y=w$ and $y'=w'$. (See figure 7.)

Figure 7

In order to conclude the proof of theorem 2, we must show that (P) implies (N). Suppose the conditions of (N) are satisfied as in figure 6, so that we need to prove

Figure 8

$xy' \parallel wz'$. Because $x'y \parallel w'z$ and $yy' \parallel ww'$, then by (P), $x'w \parallel y'z$. Then because $xx' \parallel zz'$, we can use (P) again to get $xy' \parallel wz'$ which proves (N). (See figure 8.)

As with Pappus' Theorem, the special case of (N) where the two lines are parallel is worth noting.

(n) Let x, y, w, z and x', y', w', z' be sets of points on distinct parallel lines such that $xx' \parallel zz'$, $x'y \parallel zw'$, and $yy' \parallel w'w$, then $xy' \parallel z'w$. (See figure 9.)

Figure 9

Corollary 2.1: (p) is equivalent to (n).

Proof: The proof is identical to the proof given for Theorem 2 with the substitution of (p) for (P) and (n) for (N).

As we did with Pappus' Theorem, we will now state a special case of (N) where the two lines must intersect:

(N') Let x, y, w, z and x', y', w', z' be sets of points on distinct intersecting lines such that no one of these points is on both lines and such that $xx' \parallel z'z$, $x'y \parallel zw'$, and $yy' \parallel w'w$, then $xy' \parallel z'w$. (See figure 6.)

Corollary 2.2: (P') is equivalent to (N').

Proof: The proof is identical to the proof given for Theorem 2 with the substitution of (P') for (P) and (N') for (N).

Using Theorem 1 and the corollaries to Theorem 2, it is now easy to prove the following:

Theorem 3: (N') implies (n).

Proof: By Corollary 2.2, (N') is equivalent to (P'). By Theorem 1, (P') implies (p). By Corollary 2.1, (p) is equivalent to (n).

Corollary 3.1: (N') is equivalent to (N).

So far we have shown: $(P') \leftrightarrow (P) \leftrightarrow (N) \leftrightarrow (N')$

↓

$(p) \leftrightarrow (n)$

We will discuss other conditions equivalent to Pappus'

Theorem later, but for now we shift our attention to another configuration theorem in the affine plane.

An affine plane is said to be a Desarguesian affine plane (or simply Desarguesian) if the following holds:

(D) The Affine Desargues Theorem: Let x, y, z, x', y', z' be six distinct points such that the distinct lines xx' and yy' and zz' are either parallel (see figure 11) or are concurrent (intersect at a common point) at a seventh point c . If x, y, z and x', y', z' are triangles with $xy \parallel x'y'$ and $yz \parallel y'z'$, then $xz \parallel x'z'$. (See figure 10.)

The triangles x, y, z and x', y', z' are sometimes called central triangles and are said to be perspective from the center point c .

Figure 10

As was the case with our previous configuration theorems, there are two special cases of Desargues' Theorem worth noting.

(d) The Little Desargues Theorem: Let x, y, z, x', y', z' be six distinct points such that the distinct lines xx' , yy' , and zz' are parallel. If $xy \parallel x'y'$ and $yz \parallel y'z'$, then $xz \parallel x'z'$. (See figure 11.)

Figure 11

(D') Let c, x, y, z, x', y', z' be seven distinct points such that c, x, x' and c, y, y' and c, z, z' are collinear and such that x, y, z and x', y', z' are triangles with $xy \parallel x'y'$ and $yz \parallel y'z'$, then $xz \parallel x'z'$.

(See figure 10.)

Because (P') implies (p) and (N') implies (n), it is natural to suspect that (D') might imply (d). We next prove that it is the case that (D') implies (d), and thus (D') is actually

equivalent to (D) [15].

Theorem 4: (D') implies (d).

Proof: Suppose that (D') holds, but (d) does not. Let x, y, z, x', y', z' be six distinct points such that $xx' \parallel yy' \parallel zz'$ and $xy \parallel x'y'$ and $yz \parallel y'z'$ as in figure 11, but suppose that $xz \not\parallel x'z'$.

Let x'' be the point on $x'y'$ such that $x''z' \parallel xz$. Thus $x'' \neq x$ (because $xz \not\parallel x'z'$). Next let c be the intersection of xx'' with yy' and let z'' be the intersection of cz' with yz . (See figure 12.) Since z'' is on cz' , $z \neq z''$ (otherwise zz' would intersect yy' at c , but $zz' \parallel yy'$). Then by (D') on triangles x, y, z'' and x'', y', z' , $xz'' \parallel x''z'$. But $x''z' \parallel xz$, which implies that $xz'' \parallel xz$ which is a contradiction. Thus $xz \parallel x'z'$ and (d) holds.

Figure 12

Corollary 4.1: (D') is equivalent to (D) .

The converse of Desargues' Theorem is also a useful configuration theorem and is actually equivalent to Desargues' Theorem.

(DC) The Converse of the Desargues Theorem: Let x, y, z, x', y', z' be six distinct noncollinear points such that the lines $xx', yy',$ and zz' , are distinct and such that x, y, z and x', y', z' form triangles with $xy \parallel x'y', yz \parallel y'z',$ and $xz \parallel x'z'$, then the lines xx', yy' and zz' are concurrent at some point c or are parallel. (See figures 10 and 11.)

Theorem 5: (D) is equivalent to (DC) .

Proof: To show that (D) implies (DC) , let x, y, z, x', y', z' be six noncollinear points with $xy \parallel x'y', yz \parallel y'z'$ and $xz \parallel x'z'$. It is necessary to examine two cases.

Figure 13

Case 1. The two lines xx' and yy' intersect in a point c .
 (See figure 13.) Let z'' be the intersection point of line cz with line $y'z'$. (The point z'' exists because otherwise $cz \parallel y'z' \parallel yz$ which would be a contradiction.) Then by Desargues' Theorem on triangles x,y,z and x',y',z'' , $xz \parallel x'z''$. Thus z'' is on $x'z'$ and on $y'z'$ so $z''=z'$ and (DC) holds.

Case 2. $xx' \parallel yy'$
 (See figure 14.) Let z'' be the intersection point of the unique line on z parallel to xx' with line $y'z'$. (z'' exists because otherwise a line on z is parallel to $y'z' \parallel yz$ which would be a contradiction.) Then by the Little Desargues Theorem on triangles x,y,z and x',y',z'' , $xz \parallel x'z''$. Thus z'' is on $x'z'$ and on $y'z'$ so $z'=z''$ and (DC) holds.

Figure 14

To prove that (DC) implies (D), let x, y, z, x', y', z' be six points with $xy \parallel x'y'$ and $yz \parallel y'z'$. Again there are two cases.

Case 1. The lines xx' , yy' and zz' are concurrent at a point c . (See figure 15.) Let z'' be the point on the line $y'z'$ such that $x'z'' \parallel xz$. Then by (DC) on triangles x, y, z and x', y', z'' , c is on zz'' (zz'' is concurrent with xx' and yy' which intersect at c). But c is on zz' , so z'' is on cz' and on $y'z'$, thus $z' = z''$ and (D) holds.

Figure 15

Case 2. The lines xx' , yy' and zz' are parallel. (See figure 16.) Let z'' be the point of intersection of the unique line on x' parallel to xz with the line $y'z'$. (The point z'' exists because otherwise $xz \parallel y'z' \parallel yz$ which contradicts the fact that x, y, z is a triangle.) Then by (DC) on triangles x, y, z and x', y', z'' , $zz'' \parallel yy' \parallel xx'$. So z'' is on zz' and on

$z'=z''$ and (D) holds.

Figure 16

As with (P) and (D), we give two special cases of (DC).

(DC') Let x, y, z, x', y', z' be six distinct points such that xx', yy', zz' are three distinct lines and such that x, y, z and x', y', z' are triangles whose corresponding sides are parallel ($xy \parallel x'y', yz \parallel y'z',$ and $xz \parallel x'z'$) and such that xx' intersects yy' at some point c (see figure 10). Then the lines $xx', yy',$ and zz' are concurrent at c .

(dc) Let x, y, z, x', y', z' be six distinct points such that xx', yy', zz' are three distinct lines and such that x, y, z and x', y', z' are triangles whose corresponding sides are parallel ($xy \parallel x'y', yz \parallel y'z',$ and $xz \parallel x'z'$) and such that $xx' \parallel yy'$ (see figure 11). Then the lines $xx', yy',$ and zz' are

parallel.

Notice that in the proof that (D) implies (DC), case 1 actually used (D') to prove (DC') and case 2 used (d) to prove (dc). Similarly, in the proof that (DC) implies (D), case 1 used (DC') to prove (D') and case 2 used (dc) to prove (d). Thus we have the following corollaries:

Corollary 5.1: (D') is equivalent to (DC').

Corollary 5.2: (d) is equivalent to (dc).

The last configuration theorem that we shall examine in this chapter is called the Shear Theorem. The Shear Theorem is interesting in that it closely resembles (N) but is actually equivalent to Desargues' Theorem. As we have with the other configuration theorems, we will give the Shear Theorem along with two special cases.

(S) The Shear Theorem: Let x, y, w, z and x', y', w', z' be sets of points on distinct lines such that no one of these points is on both lines and such that $xx' \parallel ww'$, $x'y \parallel w'z$, and $yy' \parallel zz'$, then $xy' \parallel wz'$. (See figure 17.)

(s) The Little Shear Theorem: Let x, y, w, z and x', y', w', z' be sets of points on distinct parallel lines such that

Figure 17

$xx' \parallel ww'$, $x'y \parallel w'z$, and $yy' \parallel zz'$, then $xy' \parallel wz'$. (See figure 18.)

Figure 18

(S') Let x, y, w, z and x', y', w', z' be sets of points on distinct intersecting lines such that no one of these points is on both lines and such that $xx' \parallel ww'$, $x'y \parallel w'z$, and $yy' \parallel zz'$, then $xy' \parallel wz'$. (See figure 17.)

Before giving Scherk and Lingenberg's clever proof that (D) is equivalent to (S), we will give an alternate partial proof that Desargues' Theorem implies the Shear Theorem, followed by a complete proof of this implication.

(D) implies (S), first proof:

Let u be the point of intersection of the lines xx' and yy' , and v be the point of intersection of the lines zz' and ww' . (See figure 19.) Then by the converse of Desargues' Theorem on triangles u, x', y and v, w', z , the lines $y'z'$, uv and xw are concurrent at some point c . Then by Desargues' Theorem on triangles y', u, x and z', v, w , $xy' \parallel wz'$ and so the Shear Theorem is proven.

Figure 19

The problem with the above proof is that the points u and v need not exist! Ironically it is the apparently simpler special case where $xx' \parallel yy' \parallel ww' \parallel zz'$ that causes problems. (See figure 20.) The proof given above does not work in this case (even if one works in a projective plane.)

Figure 20

By examining two special cases, we can complete the proof, but the following proof is simpler. The key step is choosing an additional point.

(D) implies (S), second proof:

Choose the point v on $w'z$ distinct from w' and z and let cv intersect $x'y$ in u . (See figure 21.) We will now use Desargues' Theorem three times in order to prove $xy' \parallel wz'$. Consider the triangles y', u, y and z', v, z . The lines $y'z', uv$, and yz are concurrent at c and $yu \parallel zv$ and $yy' \parallel zz'$. Thus (D) implies $y'u \parallel z'v$. Next, consider the triangles x, u, x' and

w, v, w' . Again the two triangles are perspective from the point c and $ux' \parallel vw'$ and $xx' \parallel ww'$, so by (D) we have $xu \parallel wv$. If v is on wz' , then $uy' \parallel vz' \parallel vw \parallel ux$, so u is on xy' . Thus $vw \parallel ux$ would imply that $xy' \parallel wz'$. If v is not on wz' , then consider the triangles x, u, y' and w, v, z' . So far, we have shown that $y'u \parallel z'v$ and $xu \parallel wv$. Because the two triangles are perspective from c , (D) implies that $xy' \parallel wz'$.

Figure 21

If in the proof above, $xy \parallel x'y'$, then we would let u be the point on $x'y$ such that $uv \parallel xy \parallel x'y'$. Then the result would follow from three applications of (d).

We now give Scherk and Lingenberg's proof that (D) is equivalent to (S) [11]. Although their proof that (D) implies (S) requires four applications of Desargues' Theorem (because they are not as particular about the point they introduce), we give it here mainly because figure 22 is

interesting. The proof of the reverse implication is clever and slightly more complicated.

Theorem 6: Desargues' Theorem is equivalent to the Shear Theorem.

Proof: To show that (D) implies (S), suppose x, y, w, z and x', y', w', z' are sets of points on distinct lines which intersect at a point c and such that $xx' \parallel ww'$, $x'y \parallel w'z$, and $yy' \parallel zz'$. Let u be any point not on either line. Let v be the point of intersection of the unique line on z parallel to the line uy with the line cu (see figure 22). Then by (D) on triangles y, y', u and z, z', v , $uy' \parallel vz'$. By (D) on triangles u, x', y and v, w', z , $ux' \parallel vw'$. By (D) on triangles x, x', u and w, w', v , $ux \parallel vw$. Then finally by (D) on triangles x, y', u and w, z', v , $xy' \parallel wz'$ and so the Shear Theorem holds.

Figure 22

To prove (S) implies (D), assume (S) holds and suppose (D) does not. Then there exist triangles x,y,z and x',y',z' such that the lines xx' , yy' , and zz' meet at some point c and $xy \parallel x'y'$ and $yz \parallel y'z'$, but $xz \not\parallel x'z'$.

1. The first part of the proof is to show that the lines xz and $x'z'$ meet at a point r on the line yy' . Suppose r is not on line yy' (see figure 23). Let c' be the point of intersection of the unique line on r parallel to xx' with the line yy' . Let r' be the point of intersection of the unique line on r parallel to xy with the line yy' . (Notice that $c' \neq r'$ because $xx' \not\parallel xy$.) Finally let u and v be the points of intersection of the unique line on c' parallel to zz' with the lines $x'z'$ and xz respectively. We will now use the Shear Theorem twice which will lead to a contradiction which will imply that r is on yy' .

Figure 23

Notice that $zc \parallel vc'$, $cx \parallel c'r$, and $xy \parallel rr'$. Thus, by the Shear Theorem, $zy \parallel vr'$. Similarly, since $z'c \parallel uc'$, $cx' \parallel c'r$, and $x'y' \parallel rr'$, then by the Shear Theorem $z'y' \parallel ur'$. So $vr' \parallel zy \parallel z'y' \parallel ur'$. Thus the points u, v , and r' are collinear. But u, v and c' are also collinear and $c' \neq r'$ so we must have $u=v$. Because u is on $x'z'$ and v is on xz , $u=v$ implies $u=v=r$ (because r is the intersection of xz with $x'z'$). This leads to $xy \parallel r'r = r'v \parallel yz$, but this is a contradiction (x, y, z is a triangle so $xy \nparallel yz$). So we know that if xz and $x'z'$ intersect, then the point of intersection r is on the line yy' . (See figure 24.)

2. We can now complete the proof. The unique line on r parallel to yz intersects zz' at r_1 . The unique line on r

Figure 24

parallel to xy intersects xx' at r_2 . Because x, y, z are not collinear, r, r_1, r_2 are not collinear. If $r_1 r_2 \parallel xz$, then by 1. they would have to intersect on yy' . But since xz intersects yy' at r this would imply that r, r_1, r_2 would be collinear. Since this is not the case, it must be that $r_1 r_2 \not\parallel xz$. Similarly, if $r_1 r_2 \parallel x'z'$, then they would intersect on yy' at r . This is not possible, so $r_1 r_2 \not\parallel x'z'$. Thus $xz \parallel x'z'$ and Desargues' Theorem holds.

There are a few things worth mentioning about the proof that (D) is equivalent to (S). In the proof that (D) implies (S), it was actually proven that (D) implies (S'). But if one wanted to prove (D) implies (s), the only necessary change in the proof given would be to let v be the intersection of the line on z parallel to yu with the line on u parallel to xy . In fact, the four applications of Desargues' Theorem would now be four applications of the Little Desargues Theorem. This gives us the following corollary:

Corollary 6.1: (d) implies (s).

However, in looking at the proof that (S) implies (D), if we attempted to adjust the proof to prove (S) implies (d) instead of (D'), the proof would fail. The point c' (and therefore the points u and v) would not exist if $xx' \parallel yy'$. The proof that (S) implies (D) still holds though because

(S) implies (D') which implies (d). But we do not get the hoped for corollary that (s) implies (d). In fact, according to Scherk and Lingenberg, it is not known whether or not (s) implies (d). It also seems to be unknown whether or not (S') implies (s). (We have found no mention of this implication in the literature and have been unable to prove it.)

But some of the relationships between the Little Pappus, Little Desargues, and Little Shear Theorems are known. We will show below that the Little Pappus Theorem implies the Little Shear Theorem and that the Little Desargues Theorem implies the Little Pappus Theorem. This will give another proof that the Little Desargues Theorem implies the Little Shear Theorem (Corollary 6.1 above).

Theorem 7: (p) implies (s).

Proof: Let x, y, w, z and x', y', w', z' be sets of points on distinct parallel lines with $xx' \parallel ww'$, $x'y \parallel w'z$, and $yy' \parallel zz'$. We need to show that $xy' \parallel wz'$. Choose the points u on line $x'y'$ and v on line xy so that $yu \parallel vw' \parallel xy'$. (See figure 25.) Then since $yx' \parallel zw'$ and $yu \parallel vw'$, by (p) $zu \parallel vx'$. And since $xy' \parallel vw'$ and $xx' \parallel ww'$, then by (p) $vx' \parallel wy'$. Thus $zu \parallel wy'$. Together with $yy' \parallel zz'$, (p) gives $yu \parallel wz'$. Thus $xy' \parallel wz'$ and (s) holds.

Figure 25

Unfortunately, according to Scherk and Lingenberg [11] it is not known whether or not the converse of Theorem 7 is true.

Theorem 8: (d) implies (p).

Proof: Let x, y, z and x', y', z' be sets of distinct points on distinct parallel lines with $xy' \parallel x'y$, and $y'z \parallel yz'$. We need to show that $xz' \parallel x'z$. Let u be the point of intersection of the line on z' parallel to xy' with the line on x' parallel to $y'z$. (See figure 26.) Then by (d) on triangles x', y, z and u, z', y' , $x'z \parallel uy'$. Similarly by (d) on triangles z', x, y and u, y', x' , $uy' \parallel z'x$. Thus $x'z \parallel z'x$, so (p) holds.

As with Theorem 7, it is not known if the converse of Theorem 8 is true or not (Scherk and Lingenberg [11].)

The proof of Theorem 7 can be reinterpreted to give a famous result first proven by Hessenberg in 1905:

Figure 26

Theorem 9: Pappus' Theorem implies Desargues' Theorem.

First Proof: In the proof of Theorem 7, let the two lines xy and $x'y'$ intersect at some other point c . Then instead of using (p) four times, use (P) four times. This gives (P) implies (S) and since (S) is equivalent to (D) by Theorem 6, then (P) implies (D).

There are many proofs of Theorem 9 and we will give two more: first a direct proof and then an indirect (and possibly original) proof.

Second proof that Pappus' Theorem implies Desargues'

Theorem: Let x, y, z and x', y', z' be two distinct triangles

with the lines xx' , yy' , and zz' concurrent at the point c and such that

$$xy \parallel x'y', \text{ and } xz \parallel x'z'. \quad (1)$$

Then, using (P), we must show that $yz \parallel y'z'$.

If both yz and $y'z'$ are parallel to xx' , then we would be done. So we may assume without loss of generality that $yz \not\parallel xx'$.

Let v denote the intersection point of the line xz with the line through c parallel to yz . Let u be the intersection point of the line yy' with the line through z parallel to xx' ($u \neq y$ because $yz \not\parallel xx'$). (See figure 27.)

Figure 27

Then $cv \parallel yz$, and $cx \parallel uz$, so by (P) $uv \parallel yx$. (2)

The line vx' intersects the line zu at some point w ($vx' \not\parallel uz$

because $vx' \parallel xx' \parallel uz$.) Then $vz \parallel x'z'$ and $x'c \parallel wz$ so by (P),

$$cv \parallel z'w. \quad (3)$$

We consider two cases:

Case 1: Suppose that the point y' is on the line vx' . Then by (2), $vx' = x'y' \parallel xy \parallel vu$, so $vx' = vu$. Since vu intersects zw at both u and w , then $u = w$. (Otherwise $uv = zw$, so v, x', w, z, u would be collinear. Since x is on vz and c is on xx' and y is on cu , this gives x, y, z collinear which is not possible since they are assumed to form a triangle.)

Since $vx' = vu$ intersects yy' at both y' and u , we have $u = w = y'$. (Otherwise $vx' = yy'$, so v, x', w, c, u, y, y' would be collinear. Since x is on cx' and z is on uw , this gives x, y, z collinear which is a contradiction.) Then by (3), $yz \parallel cv \parallel z'w = y'z'$. So if y' is on the line vx' , then $yz \parallel y'z'$, so (D) holds.

Case 2: Suppose y' is not on vx' . Then $cx' \parallel uw$ and $x'y' \parallel uv$ (by (2)), so by (P), $cv \parallel y'w$. This together with (3) implies that $wy' \parallel wz'$. Thus $y'z' = wy' \parallel cv \parallel yz$ and so (D) holds.

The third proof of Theorem 9 is a direct result of the following lemma.

Lemma: (N) implies (S).

Proof: Let x, y, w, z and x', y', w', z' be sets of points on distinct lines such that no one of these points is on both lines and such that $xx' \parallel ww'$, $x'y \parallel w'z$, and $yy' \parallel zz'$. We need to show that $xy' \parallel wz'$.

Let u be the intersection point of the line $x'y'$ with the line on w parallel to the line $x'y$. Let v be the intersection point of the line xy with the line on u parallel to the line yy' . (See figure 28.)

Figure 28

Then because $xx' \parallel ww'$, $x'y \parallel wu$, and $yy' \parallel uv$, by (N) $xy' \parallel vw'$. Similarly, because $wv = ww'$, $w'z \parallel wu$, and $zz' \parallel uv$, by (N) $wz' \parallel vw$. (This is actually an application of Pappus' Theorem which is equivalent to (N) by Theorem 2.) Thus $xy' \parallel wz'$, so (S) holds.

The proof of the Lemma holds whether or not the two lines xy and $x'y'$ intersect or not. If $xy \parallel x'y'$, then only (n) is needed. This gives the following corollary.

Corollary to the Lemma: (n) implies (s).

Third proof that Pappus' Theorem implies Desargues' Theorem:

(P) is equivalent to (N) by Theorem 2. (N) implies (S) by the Lemma. (S) is equivalent to (D) by Theorem 6. Thus (P) implies (D).

The converse of Theorem 9 is, in general, not true. An example of a Desarguesian affine plane which is not Pappian will be given in the next chapter.

The fact that Desargues' Theorem does imply Pappus' Theorem in the special case of a finite plane will be covered in the remainder of the paper. The next chapter will give an algebraic proof and the last chapter will analyze Tecklenburg's geometric proof that a finite Desarguesian plane is a Pappian plane.

Figure 29 summarizes the relationships between the configuration theorems that we have examined. On the top level is the Pappus Theorem which is equivalent to (N). On

the next level is Desargues' Theorem which is equivalent to its converse and to the Shear Theorem. Finally, on the lowest level are the "Little" Theorems.

Figure 29

$$\begin{array}{c}
 (P') \leftrightarrow (P) \leftrightarrow (N) \leftrightarrow (N') \\
 \downarrow \\
 (D') \leftrightarrow (D) \leftrightarrow (DC) \leftrightarrow (S) \\
 \downarrow \\
 (d) \rightarrow (p) \leftrightarrow (n) \rightarrow (s)
 \end{array}$$

It is interesting to note that (P) implies (D) and (d) implies (p), yet in general (D) does not imply (P) and it is not known whether or not (p) implies (d).

Chapter 2

Geometric Algebra

In this chapter, we establish the relationships between the geometric properties of affine planes and the corresponding algebraic properties of addition and multiplication defined on the points of a line. The first step is to introduce the operation of addition, denoted $+$, onto the set of points on an arbitrary line in an affine plane.

Let l be a line in the given affine plane and let o be a point on the line. Given any two points a and b on l choose a point x not on l and the point y such that $xy \parallel l$ and $ay \parallel ox$. (See figure 1.) Then $a+b$ is defined to be the point c on l such that $yc \parallel xb$.

Figure 1

Notice that for any two points on l , the point $c=a+b$ exists since $xb \parallel l$ so the line on y parallel to xb must intersect l . This is another way of saying that the operation of addition is closed on the set of points on the line l .

It is important to prove that the definition of $+$ is independent of the choice of the point x . First, let x' be any point, other than x , on the line l . (See figure 2.) Let y' be the intersection point of the line xy with the line on the point a parallel to ox' . Then since $bx \parallel cy$, $xo \parallel ya$, and $ox' \parallel ay'$, the Little Shear Theorem gives $x'b \parallel y'c$. Thus $a+b=c$ for any choice of x' on xy .

Figure 2

Alternatively, let x' be any point not on l or xy as illustrated in figure 3. Let y' be the intersection point of the line on x' parallel to l with the line on a parallel to ox' . If x' is on ox , then y' is on ay , and so

$xx' = ox \parallel ay = yy'$. If x' is not on ox , then the Little Desargues Theorem on triangles x', x, o and y', y, a implies that $xx' \parallel yy'$. If x' is on xb , then $yy' \parallel xx' \parallel xb \parallel yc$. Thus y' is on yc and so $x'b \parallel y'c$. Otherwise, if x' is not on xb , then the Little Desargues Theorem on triangles x', x, b and y', y, c gives $x'b \parallel y'c$. Thus $a+b$ is the point c regardless of which x' , not on l , is chosen. So far it has been shown that $+$ is a well-defined binary operation on the set of points on the line l .

Figure 3

The next step is to show that the set of points on a line, together with the binary operation $+$, form an algebraic structure called a group. Since we must check each property, we repeat the definition of a group.

Definition: A group is a nonempty set of elements, together with a closed binary operation $+$, such that:

a. The operation $+$ is associative: $a+(b+c)=(a+b)+c$ for any elements a, b , and c .

b. There is an element 0 (called the identity) such that $0+a=a+0=a$ for every element a in the set.

c. For every element b there is an element c (called the inverse of b) such that $b+c=c+b=0$.

The group is said to be an abelian group if the operation $+$ is commutative: $a+b=b+a$ for any elements a and b .

Theorem 1: The set of points on a line with the binary operation $+$ (defined above) in an affine plane which satisfies the Little Desargues Theorem forms an abelian group.

Comment: In showing that $+$ is a well-defined binary operation, the Little Shear Theorem was used as well as the Little Desargues Theorem. Because the Little Desargues Theorem implies the Little Shear Theorem (Corollary 1 to Theorem 6 in the previous chapter), the use of the Little Shear Theorem was acceptable.

Proof of Theorem 1: Using the Little Desargues Theorem and the properties of an affine plane the following will be shown:

1. $+$ is commutative.
2. $+$ is associative.

3. o is the identity.
4. Any point b has an inverse.

Proof of (1): Fix a point o on some line l . Let a and b be two points on l . Choose a point x not on l . Let y be the point such that $xy \parallel l$ and $ya \parallel xo$. Let $c = a + b$, so $yc \parallel xb$. Let z be the point of xy such that $bz \parallel ox \parallel ay$ (see figure 4).

Figure 4

Since $ay \parallel bz$, and $xb \parallel yc$, by the Little Pappus Theorem $xa \parallel zc$. Thus $a + b = c = b + a$ for any points a and b on l and since the Little Pappus Theorem is a consequence of the Little Desargues Theorem (Theorem 8 of the previous chapter), $+$ is commutative.

Proof of (2): Let o be a point on l as above. Let a, b , and c be three points on l . Choose a point x not on l . Let y be the intersection point of the line on x parallel to l with

the line on a parallel to ox . Let z be the intersection point of xy with the line on b parallel to ox . (See figure 5.) Then $d=a+b$ is the intersection point of l with the line on y parallel to xb and $e=b+c$ is the intersection point of l with the line on z parallel to xb . Let w be the intersection of xy with the line on $a+b$ parallel to ox . Thus $f=(a+b)+c$ is the intersection of l with the line on w parallel to xc . In order to show $a+(b+c)=(a+b)+c$, we must show that $xe \parallel yf$. But because $ez \parallel fw$ and $zb \parallel wd$ and $bx \parallel dy$, the Shear Theorem gives us $xe \parallel yf$. So addition is associative.

Figure 5

Proof of (3): (See figure 6.) As above, fix o and pick x not on l . Then for any point a on l clearly $o+a=a$. Thus by commutativity $o+a=a+o=a$ and so o is the identity.

Proof of (4): (See figure 6.) Let o, x, a be points as above. Let y be the point such that $xy \parallel l$ and $oy \parallel ax$. Let b be the point on l such that $yb \parallel xo$. Thus $b+a=o$. So by commutativity $a+b=b+a=o$, and thus b is the inverse of a (the inverse of a is usually denoted by $-a$.)

Figure 6

A second binary operation called multiplication, and denoted by $*$, can also be defined on the points of a line in an affine plane satisfying the Little Desargues Theorem.

Let l be a line in the plane. Fix two distinct points, i and o , on l and fix a point i' not on the line l . For any two points a and b on l , let y be the point on oi' such that $ay \parallel ii'$. Then define $a*b$ to be the intersection point of l with the line on y parallel to $i'b$. (See figure 7.)

The point $a*b$ exists because $i'b$ intersects l (so any line parallel to $i'b$ will also intersect l .) Thus $*$ is closed on

the set of points on l . Notice that this multiplication is defined on the points of l with respect to the points o, i , and i' , whereas addition was defined on the points of l with respect to only the point o . If the full Desargues Theorem is assumed, then it is possible to define a multiplication which depends only on the points o and i on l . But before assuming that our affine plane is Desarguesian, we would like to see how much algebraic structure can be obtained from a Little Desarguesian plane (an affine plane satisfying the Little Desargues Theorem.)

Figure 7

Definition: A loop (see Blumenthal [4]) is a nonempty set of elements, together with a binary operation $*$, such that:

- a. There is an element 1 (called the identity) such that $1*a=a*1=a$ for every element a in the set.
- b. $a*b=c$ uniquely determines any one of the elements a, b , or c when the other two are given.

A loop is a slightly weaker structure than a group, mainly because the binary operation need not be associative in a loop.

Theorem 2: In an affine plane which satisfies the Little Desargues Theorem, the set of points on a line, excluding the point o , form a loop under the multiplication operation $*$ defined above.

Proof: Using the Little Desargues Theorem and the properties of an affine plane, the following must be established:

- a. $i*a=a*i=a$ for all points a on the line l .
- b. If two of the three points a, b , and c are given (neither of which are o) then the equation $a*b=c$ uniquely determines the third.

Clearly (see figure 8) $a*1=a=1*a$.

Figure 8

If a and b are given points on l , then the point $a*b$ is uniquely determined by the definition of $*$. If a and c are given points on l , then the point b such that $a*b=c$ is determined in the following manner: let y be the point on the line oi' such that $ay \parallel ii'$. Then b is the point on l such that $i'b \parallel yc$. (See figure 7.) Thus b is uniquely determined.

Similarly if b and c are two given points on l and $a*b=c$, then a is uniquely determined by the following construction: let y be the point on oi' such that $yc \parallel i'b$. (See figure 7.) Then a is the point on l such that $ya \parallel ii'$.

More properties of the affine plane than just the Little Desargues Theorem are needed in order to show that $*$ is associative and commutative. But before exploring those properties, we will see how far the Little Desargues Theorem can take us.

Definition: A right quasifield is a set together with two operations, $+$ and $*$, such that:

- a. The set under $+$ is an abelian group.
- b. The set, excluding the additive identity 0 , under $*$ is a loop.
- c. The right distributive law holds: $(b+c)*a=b*a+c*a$ for any elements a, b , and c .

A left quasifield is defined similarly with the left

distributive law required instead of the right distributive law. A right (or left) quasifield is sometimes also called a right (or left) Veblen-Wedderburn system [9]. The smallest example of a Veblen-Wedderburn plane (a plane coordinatized over a Veblen-Wedderburn system) can be found in Garner [9, p.33].

Theorem 3: In an affine plane which satisfies the Little Desargues Theorem, the set of points on a line forms a right quasifield under $+$ and $*$.

Proof: Because of Theorem 1 and Theorem 2, the only thing to show is the right distributive law.

Let o and i be two points on the line l and fix a point i' not on l . Because $y*o=o*y=o$ for any point y on l , let a, b , and c be any points distinct from o on l . (See figure 9.) Let b' be the point on oi' such that $bb' \parallel ii'$. Similarly, let c' be the point on oi' such that $cc' \parallel ii'$. Let u and v be the points on oi such that $b'u \parallel c'v \parallel i'a$. Thus, $u=b*a$ and $v=c*a$.

Let x be the intersection point of the line on c' parallel to oi with the line on b parallel to oi' . Let y be the point on oi such that $xy \parallel cc'$. Thus, $y=b+c$.

Let y' be the intersection point of the line xy with the line oi' (so $yy' \parallel ii'$). Let w be the point on oi such that

$y'w \parallel i'a$. Thus, $w = y*a = (b+c)*a$.

Let z be the point on $c'x$ such that $uz \parallel oi'$. Let w' be the point on oi such that $zw' \parallel c'v$. Thus, $w = b*a + c*a$.

The objective is to show that $w = w'$.

Figure 9

In order to show $w = w'$, examine the triangles b', b, u and y', x, z . $b'y' = oi' \parallel bx \parallel uz$, and $bb' \parallel xy'$ (because x, y, y' are collinear and $yy' \parallel ii' \parallel bb'$), and $xz \parallel oi = bu$. Thus by the Little Desargues Theorem, $b'u \parallel y'z$. Since $b'u \parallel i'a \parallel y'w$, then $y'w \parallel y'z$. Thus z is on the line $y'w$ and since w' is on oi and $zw' \parallel c'v \parallel i'a \parallel y'w = zw$, then $w = w'$. So $(b+c)*a = b*a + c*a$ and therefore the right distributive law holds. (The proof goes

through the same way for the special cases $b=c$ or $a=b$.)

So the points of a line in an affine plane which satisfies the Little Desargues Theorem, can be given the structure of a right quasifield using the algebraic operations $+$ and $*$ geometrically defined in this chapter. In order to go farther, more conditions are needed on the plane. Before assuming that the plane is Desarguesian though, let us suppose that the plane satisfies another special case of Desargues' Theorem [6].

(DP) Let x, y, z and x', y', z' be triangles such that $xx', yy',$ and zz' are concurrent at some point c and such that $xz \parallel x'z'$, and $xy \parallel x'y' \parallel zz'$. Then $yz \parallel y'z'$. (See figure 10.)

Figure 10

The converse of (DP) is also a useful theorem.

(CD) Let x, y, z and x', y', z' be triangles such that xx', yy' and zz' are distinct lines and such that $xy \parallel x'y' \parallel zz'$, and $xz \parallel x'z'$, and $yz \parallel y'z'$. (See figure 10.) Then the lines xx' , yy' , and zz' are concurrent at some point c .

Theorem 4: (DP) is equivalent to (CD).

Proof: The proof is identical to case 1 of the proof of Theorem 5 in the previous chapter.

Adding the conditions (DP), and therefore (CD), to the Little Desarguesian affine plane gives the left distributive law. (See Blumenthal [4] and Forder [6].)

Theorem 5: The left distributive law, using the operations $+$ and $*$, holds on the set of points on a line in an affine plane satisfying (DP).

Proof: Given the points a, b , and c on a line l , it must be shown that $a*(b+c) = a*b + a*c$.

Let o and i be two points on the line l and fix a point i' not on l . Because $o*y = y*o = o$ for any point y on l , let a, b , and c be any points on l distinct from o . (See figure 11.) Let x be the point on oi' such that $ax \parallel ii'$. Let b' be the point on l such that $xb' \parallel i'b$. Thus $b' = a*b$.

Let d be the intersection point of the line on i' parallel to l with the line on b parallel to oi' . (d exists because oi and oi' intersect which implies that a line parallel to oi must intersect a line parallel to oi' .) Let f be the point on l such that $df \parallel i'c$. Thus $f=b+c$.

Let d' be the intersection point of the line on x parallel to l with the line on b' parallel to oi' . Let f' be the point on l such that $d'f' \parallel xc'$. So $f' = b' + c' = a * b + a * c$.

In order to show that $a*b+a*c=a*(b+c)$, it is only necessary to show that $i'f\|xf'$.

Figure 11

Because $i'b \parallel xb'$, $i'd \parallel xd'$, and $bd \parallel b'd'$, (CD) can be used on triangles i',b,d , and x,b',d' , to show that the points o,d , and d' are collinear. Then by (DP) on triangles i',d,f , and x,d',f' , it follows that $i'f \parallel xf'$. Thus $a*b+a*c=a*(b+c)$.

In order to get a weaker form of the associative laws, we will use another special case of Desargues' Theorem. (See Forder [6].)

(H) If x,y,z and x',y',z' are triangles such that x is on $y'z'$ and two pairs of corresponding sides are parallel, then so is the third pair. (See figure 12.)

Figure 12

The converse of Theorem (H) is also useful.

(CH) If x, y, z, x', y', z' are distinct points such that x, y, z and x', y', z' are triangles with x on $y'z'$ and $xy \parallel x'y'$, $yz \parallel y'z'$, and $xz \parallel x'z'$, then the lines xx' , yy' , and zz' are concurrent at some point c or $xx' \parallel yy' \parallel zz'$. (See figure 12.)

Theorem 6: (H) is equivalent to (CH).

Proof: The proof is nearly identical to the proof of Theorem 5 in the previous chapter.

Definition: An alternative ring is a set, together with the operations $+$ and $*$, such that:

- a. The set under $+$ is an abelian group.
- b. The set, without the additive identity o , under $*$ is a loop.
- c. Both the left and right distributive laws hold.
- d. $a*(a*b)=(a*a)*b$ and $(a*b)*b=a*(b*b)$ for any elements a and b . These two properties are called the left and right alternative laws, respectively. If only the left (right) alternative law holds, then the structure is called a left (right) alternative ring.

It is relatively easy, using (H) and (CH), to show that the left alternative law holds.

Lemma 1: The left alternative law, $a*(a*b)=(a*a)*b$, follows from (H) and (CH).

Proof: Let o and i be distinct points on the line l and let i' be any point not on l . Let a and b be any points distinct from o on the line l ($a*o=o*a=o$ for any point a , so if $a=o$ the result is trivial.) Let x be the point on oi' such that $ax \parallel ii'$. Let c be the point on l such that $xc \parallel i'b$. Thus $c=a*b$.

Let d be the point on l such that $xd \parallel i'c$. Thus $d=a*(a*b)$. Let y be the point on oi' such that $dy \parallel i'b$. Let e be the point on l such that $ey \parallel ii'$. (See figure 13.) If $i'a \parallel xe$, then $e=a*a$ and thus $a*(a*b)=d=(a*a)*b$ and the lemma would be proven. So we must show that $i'a \parallel xe$.

Figure 13

Suppose first that $i'c \parallel ax$.

Let u be the intersection point of $i'c$ with ax . Let v be the intersection point of xd with ey . Then x is on dv and the corresponding sides of triangles y,d,v and x,c,u are all parallel. Thus by (CH), the line uv intersects l at o . But this means that the triangles i',u,a and x,v,e are perspective from the point o . Also, x is on au and $i'u \parallel xv$, and $au \parallel ev$. Thus, by (H) $i'a \parallel xe$.

If $i'c \parallel ax \parallel ii'$, then $i=c=a*b$. As before choose the point x on oi' such that $ax \parallel ii'$. (See figure 14.) So $a=a*i=a*(a*b)$. Let y be the point on oi' such that $ay \parallel ix$ and choose the point d on l such that $yd \parallel xa$. Then $d=a*a$, so $(a*a)*b=a=a*(a*b)$.

Figure 14

So the points on a line in an affine plane satisfying (d), (DP), and (H) form a left alternative ring under $+$ and $*$. It

is possible to show that our left alternative ring is also a right alternative ring, and therefore simply an alternative ring (See Kleinfeld, 1953; Skornyakov, 1951; San Soucie, 1955; this information is in Stevenson's book [13].) It is also possible to achieve the desired result by introducing the inversive laws, stated below.

The left inversive law: If $a^{-1} * a = 1$, then $a^{-1} * (a * b) = b$.

The right inversive law: If $a * a^{-1} = 1$, then $(b * a) * a^{-1} = b$.

A division algebra is a set which is an abelian group under $+$, with a closed well-defined multiplication $*$ having unique inverses for all nonzero points and both distributive laws (see Forder [6].) Thus Theorem 5 could be restated as follows: in a Little Desarguesian affine plane which satisfies (DP), the set of points on a line forms a division algebra under $+$ and $*$. It is known that if the left and right inversive laws hold in a division algebra, then the division algebra is an alternative ring. (Reidemeister's proof in Forder's article, which is not difficult, consists of a page of algebraic manipulations; see also Moufang, 1935, and Hall, 1959, this information is in Stevenson's book [13].)

So assuming (H), we will show that the left and right inversive laws hold, and therefore the points on a line in

an affine plane in which (d), (DP), and (H) are assumed, form an alternative ring under + and *.

The right inversive law, $(b*a)*a^{-1}=b$ is trivially true (see figure 15).

Figure 15

The proof of the left inversive law is very similar to the proof of Lemma 1. Let c and i be points on the line l and let i' be a point not on l . Let a and b be any two points on l distinct from o (a cannot be o and if $b=o$, then the result is trivial). Let x be the point on oi' such that $xi \parallel ai'$. Then a^{-1} is the point on l such that $xa^{-1} \parallel ii'$. Let y be the point on oi' such that $ay \parallel ii'$. Let c be the point on l such that $cy \parallel bi'$. Thus $c=a*b$. (See figure 16.)

So to prove the left inversive law, $a^{-1}*(a*b)=b$, it is only necessary to show that $xb \parallel i'c$. As in Lemma 1, we will first use (CH) and then use (H). By (CH) on triangles y, v, a and

Figure 16

i', u, i , we know that o, u, v are collinear. Then by (H) on triangles i, v, c and x, u, b , we get $i'c \parallel xb$.

The argument above, together with Theorem 5, gives us the following theorem.

Theorem 7: Suppose A is an affine plane satisfying the Little Desargues Theorem and the two special cases of Desargues' Theorem: (DP) and (H). Then the set of points on a line forms an alternative ring under the operations $*$ and $+$.

In order to obtain the associative law of multiplication in our division algebra, it is necessary to assume Desargues' Theorem. A division algebra in which multiplication is known

to be associative is called a division ring.

Definition: A division ring is a set of elements, together with operations $+$ and $*$, such that:

1. The set under $+$ is an abelian group.
2. The set, without the additive identity 0 , is a group under $*$.
3. The distributive laws hold.

If multiplication is commutative, the division ring is called a field.

To conclude our development of algebraic structure in an affine plane, we will show that adopting Desargues' Theorem gives a division ring on l and that adopting Pappus' Theorem gives a field on l .

Theorem 8: The set of points on a line in a Desarguesian affine plane forms a division ring under the operations $+$ and $*$.

Proof: The Little Desargues Theorem and (DP) are special cases of the Desargues Theorem. By Theorem 5, we know that the points on a line form a division algebra. Thus, we need only show that $*$ is associative.

Let o and i be two points on a line l . Let i' be a point not

on l . Let a, b , and c be points on l distinct from o and i . (If a, b , or c is either o or i , then the associative law holds trivially.) The following argument works whether or not a, b , and c are distinct ($a=b$ gives the left alternative law, while $b=c$ gives the right alternative law.) Let x be the point on oi' such that $ax \parallel ii'$. Let d be the point on l such that $xd \parallel i'b$. Thus $d=a*b$.

Let y be the point on oi' such that $dy \parallel ii'$. Let e be the point on l such that $ye \parallel i'c$. Thus $e=(a*b)*c$.

Let z be the point on oi' such that $bz \parallel ii'$. Let f be the point on l such that $zf \parallel i'c$. Thus $f=b*c$. (See figure 17.)

Figure 17

Then in order to show $a*(b*c)=(a*b)*c$, it is only necessary to show that $xe \parallel i'f$. But this is just an application of the Shear Theorem, which by Theorem 6 in the previous chapter is

equivalent to Desargues' Theorem. Because $fz \parallel ey$, $zb \parallel yd$, and $bi' \parallel dx$, the Shear Theorem implies that $fi' \parallel ex$. Thus multiplication is associative and so the set of points on the line forms a division ring.

Note the similarity between the proof of the associativity of multiplication and the proof of the associativity of addition which uses the Little Shear Theorem. For the associativity of addition we only needed the Little Desargues Theorem but for the associativity of multiplication we needed the full Desargues Theorem. In proving that addition is commutative (page 3, part 1 of Theorem 1), we used the Little Pappus Theorem. Now, in order to prove that multiplication is commutative, we need to assume the Pappus Theorem.

Theorem 9: The set of points on a line in a Pappian affine plane form a field under the operations of $+$ and $*$.

Proof: By Theorem 9 of the previous chapter we know that Pappus' Theorem implies Desargues' Theorem. Thus by Theorem 8, the only property to prove is the commutativity of multiplication.

Let o and i be two points on a line l and let i' be a point not on l . Let a and b be points on l distinct from o , i and

each other (if a or b was either o or i or if $a=b$, then the commutative law obviously holds.) Let x be the point on oi' such that $ax \parallel ii'$. Let c be the point on l such that $xc \parallel i'b$. Thus $c=a*b$.

Let y be the point on oi' such that $by \parallel ii'$. If $yc \parallel i'a$, then $b*a=c=a*b$. (See figure 18.)

Figure 18

Because $ax \parallel yb$ and $xc \parallel bi'$, Pappus' Theorem implies $yc \parallel i'a$. Thus, the set of points on the line is a field under $+$ and $*$.

We have constructed a division ring from a Desarguesian affine plane and a field from a Pappian affine plane. Given

a Desarguesian affine plane A , we will identify A with the affine plane developed from the division ring associated with A . Given a Pappian affine plane B , we will identify B with the affine plane developed from the field associated with B .

But first we need to prove that the multiplication $*$ defined earlier is actually independent of the point i' . The argument that we use to show that $+$ depends only on o and i is the same argument that we will give to show that $*$ depends only on o and i , except that now we will use the Shear and Desargues Theorems instead of the Little Shear and Little Desargues Theorems.

Let l be a line and o and i be two points on l . Fix a point i' not on l . For any points a and b on l , let y be the point on oi' such that $ay \parallel ii'$. Then the point $a*b$ is the point on l such that $(a*b)y \parallel i'b$. Next pick another point x on oi' and find the point $a*b$ with respect to o, i and x . (See figure 19.) Then by the Shear Theorem, $y'c \parallel xb$, and so $a*b$ is the same point with respect to o, i, i' and with respect to o, i, x for any point x on the line oi' .

Next define $a*b$ with respect to o, i, i' as above and pick a point x not on oi' or l . Find the point $a*b$ with respect to o, i , and x . (See figure 20.) Then by Desargues' Theorem on

Figure 19

triangles x, i', i and y', y, a , $xi' \parallel y'y$. Then by Desargues' Theorem on triangles x, i', b and y', y, c , $xb \parallel y'c$ and so $a*b$ is the same point with respect to o and i regardless of the choice of i' .

Figure 20

We would like to develop an affine plane from the division ring associated with our original Desarguesian affine plane. In doing so, it is useful to know that the division ring obtained from the line l with respect to the points o and i is essentially the same as the division ring obtained from some other line m on o with respect to o and i' . (It is actually true that the same division ring is obtained with respect to any two points on any line in a Desarguesian plane [3].)

Definition: The division rings D and D' are said to be isomorphic if there exists a bijective map $\phi: D \rightarrow D'$ such that: $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(a*b) = \phi(a) * \phi(b)$ for all elements a and b in D . The map ϕ is called the isomorphism between D and D' . Division rings which are isomorphic to each other are essentially the same in that the map ϕ just relabels the elements so that the division rings are identical.

Theorem 10: In a Desarguesian affine plane, the division ring D developed from the line l with respect to the points o and i is isomorphic to the division ring D' developed from the line m with respect to the points o and i' where o is the intersection of l and m .

Proof: Let $\phi: D \rightarrow D'$ such that $\phi(o) = o$, $\phi(i) = i'$ and $\phi(a) = a'$ where a' is the point on m such that $aa' \parallel ii'$. The point a'

is uniquely determined and given a point b' on m there exists a point b on l such that $bb' \parallel ii'$. Thus ϕ is a bijection from D onto D' .

Let a and b be points on the line l and a' and b' be the corresponding points $\phi(a)$ and $\phi(b)$ on m . Let the line on b' parallel to l intersect the line on a parallel to m at the point x . (See figure 21.) Then the point c on l such that $xc \parallel b'b$ is the point $a+b$. The point d on m such that $dx \parallel aa'$ is the point $b'+a' = a'+b'$ (by commutativity of $+$). Then $xd \parallel aa' \parallel bb' \parallel xc$, so x is on dc which is parallel to ii' . Thus $\phi(a+b) = \phi(a) + \phi(b)$.

Figure 21

Next, let o, i, a, b, i', a', b' be as before. Let c be the point on l such that $a'c \parallel i'b$. Thus $c=a*b$.

Let d be the point on m such that $ad \parallel ib'$. Thus $d = a' * b'$.
 In order to show that $\phi(a * b) = \phi(a) * \phi(b)$, we need to show that $cd \parallel ii'$. (See figure 22.) But this is true by the Shear Theorem:
 $bi' \parallel ca'$, $i'i \parallel a'a$, and $ib' \parallel ad$, implies $cd \parallel bb' \parallel ii'$. Thus ϕ is an isomorphism from D onto D' .

Figure 22

In order to develop an affine plane from the division ring D associated with our original Desarguesian affine plane, we will first develop a vector space over the division ring D .

Definition: A vector space V over a division ring D is a set of elements, called vectors (and denoted by capital letters), together with the operations of vector addition (denoted by $A+B$ for vectors A and B) and scalar multiplication (denoted by aB for an element a in D and a

vector B) such that:

1. The set of vectors together with the operation of vector addition is an abelian group.
2. $(a*b)X=a(bX)$ for all a,b in D and all vectors X .
3. $a(X+Y)=aX+aY$ for all a in D and all vectors X,Y .
4. $(a+b)X=aX+bX$ for all a,b in D and all vectors X .
5. $1X=X$ for all vectors X , where 1 is the multiplicative identity in D .

We are now ready to define our vector space V over the division ring D obtained from our original Desarguesian affine plane A . Let the set of vectors be the set of points in A . We will denote the vectors (points) by capital letters and the elements from D in lowercase letters.

Definition of vector addition $+$: For any points X and Y collinear with O , we use the same addition defined earlier on the points of a line. If the vectors X,Y , and O are noncollinear points in the affine plane, then $X+Y$ is the intersection of the line on X parallel to the line OY with the line on Y parallel to OX . (See figure 23.)

There is a natural correspondence between the elements of the division ring D and the points on the line l .

Formalizing this relationship, define $\beta:D \rightarrow l$ such that $\beta(0)=o$, $\beta(1)=I$, and for each $a \in D$, there is a unique A on l

Figure 23

such that $\beta(a)=A$. β is another example of an isomorphism. Let $\phi:l \rightarrow m$ be the map from the line l to the line m such that $\phi(o)=o$, $\phi(I_l)=I_m$ and $\phi(A)=A'$ where A' is the point on m such that $AA' \parallel II'$. ϕ is essentially the map used in the proof of Theorem 10. Then let $\phi_m:D \rightarrow m$ be the composition of β with ϕ . (See figure 24.)

Figure 24

Definition of scalar multiplication: for a in D and P a vector (point), define aP to be equal to the point $\phi_{op}(a) * P$. (See figure 25.) aP is always on the line OP .

Figure 25

Theorem 11: Using the points of a Desarguesian affine plane as the set of vectors, V is a vector space over D with addition and scalar multiplication as defined above [3]. (D is the division ring obtained from a line of the Desarguesian affine plane.)

Proof: First, we will prove that the set of vectors under vector addition forms an abelian group.

Obviously, $+$ is well-defined and commutative (the intersection of the line on X parallel to OY with the line on Y parallel to OX is the point of intersection of the line on Y parallel to OX with the line on X parallel to OY .)

Also, $O+P=P+O=P$.

Given any point P , the inverse of P is just the point $-P$ on the line OP .

The associative law is the only property remaining to be proved. Suppose first that no three of P, Q, S, O are collinear. Then determine the points $P+Q$, $(P+Q)+S$, and $Q+S$ from the definition of $+$. (See figure 26.)

Figure 26

Then by the Little Desargues Theorem on triangles $O, Q, P+Q$, and $S, Q+S, (P+Q)+S$, the line $Q(P+Q)$ is parallel to the line $(Q+S)((P+Q)+S)$. Then by the Little Desargues Theorem on triangles $P, P+Q, (P+Q)+S$, and $O, Q, Q+S$, the line $P((P+Q)+S)$ is parallel to the line $O(Q+S)$. Thus $(P+Q)+S = P+(Q+S)$.

The proof above works for the case where P, Q, S are collinear. We need only examine the cases where O, P, Q are collinear and where O, P, S are collinear.

Suppose O, P, Q are collinear. Determine the points $Q+S$ and $P+(Q+S)$. (See figure 27.) Then find $P+Q$ using the points $Q+S$ and $P+(Q+S)$. This shows that $(P+Q)+S=P+(Q+S)$

Figure 27

Finally, suppose O, P, S are collinear. Determine the points $P+Q$, $Q+S$, and $(P+Q)+S$ as in figure 28.

Figure 28

Since $P(P+Q) \parallel S(Q+S)$ and $O(P+Q) \parallel S((P+Q)+S)$ and $OP \parallel Q(P+Q)$, then by the Little Pappus Theorem $O(Q+S) \parallel P((P+Q)+S)$. Thus $P+(Q+S)=(P+Q)+S$ so the associative law holds and so the set of vectors is an abelian group under vector addition.

Next we must show that properties 2-5 in the definition of a vector space hold. Properties 2, 4, and 5 all hold because the operations involve only addition and multiplication on the line OP . So $(ab)P=a(b*P)$ by the associativity of multiplication on the line OP . $(a+b)P=aP+bP$ holds by the right distributive law on the line OP . Similarly $1P=P$ because 1 is mapped to the multiplicative identity on OP .

Finally, we must show that $a(P+Q)=aP+aQ$. The proof follows that given by Bennett [3]. We can assume that O, P, Q are not collinear (otherwise, the result would follow by the left distributive law on OP). Let $OP=l_1, OQ=l_2$, and $O(P+Q)=l_3$. (See figure 29.) Let $\phi_{ii}(a)=A_i$ for $i=1,2,3$. By multiplying A_1*P via I_3 and A_3 , we get $I_3P \parallel A_3aP$. Similarly, multiplying A_2Q via I_3 and A_3 , we get $I_3Q \parallel A_3aQ$. Multiplying $A_3*(P+Q)$ via P and aP we get $P(P+Q) \parallel aP(a(P+Q))$. By Desargues' Theorem on triangles P, I_3, Q and aP, A_3, aQ we get $PQ \parallel aP(aQ)$. Again using Desargues' Theorem, this time on triangles $P, Q, P+Q$ and $aP, aQ, a(P+Q)$, we obtain the result that $Q(P+Q) \parallel aQ(a(P+Q))$. Thus $OP \parallel aQ(a(P+Q))$. This statement, together with the fact that $OQ \parallel aP(a(P+Q))$ shows that $aP+aQ=a(P+Q)$. Thus, V is a vector

space over D .

Figure 29

We now claim that any two dimensional vector space over a division ring can be viewed as a Desarguesian affine plane and that any vector space over a field can be interpreted as a Pappian plane. The points of the affine plane are the vectors of the vector space. The lines of the affine plane

will be sets of points of the form: $l=A+\langle D \rangle$ where A and $D \neq 0$ are vectors (points) and where $\langle D \rangle = \{xD \mid x \text{ is in the division ring } F\}$. (D is used to suggest "direction vector".) (See figure 30.)

Figure 30

The following lemma will be useful.

Lemma 2: given the lines $l=A+\langle D \rangle$ and $m=B+\langle E \rangle$

1. if $\langle D \rangle = \langle E \rangle$, then $l \parallel m$ or $l=m$.
2. if $\langle D \rangle \neq \langle E \rangle$, then l and m intersect in exactly one point.

Proof of 1: If $\langle D \rangle = \langle E \rangle$, and if there is a point C on both l and m , then $C=A+xD=B+x'E=B+yD$ for some x, x', y in F . Let P be another point on l . So $P=A+zD$ for some z in F . But $P=A+xD-xD+zD=C+(z-x)D=B+yD+(z-x)D=B+(y+z-x)D$. So P is on m . So either the lines are equal or they are parallel.

Proof of 2: If $\langle D \rangle \neq \langle E \rangle$, then the vectors D and E are

independent. So D and E form a basis for the vector space V which means that any vector can be expressed as a linear combination of D and E in one and only one way (Friedberg, Insel and Spence [7].) So there are unique scalars x and y such that $A-B=xD+yE$. So $A-xD=B+yE$ is the one point on both line l and line m .

Theorem 12: Any vector space over a division ring, interpreted as above, is a Desarguesian affine plane.

Proof: We must show that the vector space satisfies the axioms of an affine plane and that Desargues' Theorem holds.

1. For any two points P and Q (so $Q-P \neq 0$), the line $l=P+\langle Q-P \rangle$ contains both P and Q ($P=P+0(Q-P)$, $Q=P+1(Q-P)$). By part 2 of Lemma 2, if line $m=B+\langle E \rangle$ is also on P and Q , then $\langle E \rangle = \langle D \rangle$. But then by part 1 of Lemma 2, either $l \parallel m$ or $l=m$. Thus it must be the case that $l=m$. So two points determine a line.

2. If the point P is not on the line $l=A+\langle D \rangle$, then the line $m=P+\langle D \rangle$ is on P and is parallel to l . If n is another line on P and parallel to l , then $n=B+\langle D \rangle$ (same direction vector D by Lemma 1) for some vector B . But then by Lemma 1, either $m \parallel n$ or $m=n$, so $m=n$. So there is exactly one line on P parallel to l .

3. Let A and B be any two linearly independent vectors (the vector space is two dimensional, so A and B exist). Then A, B , and O are not collinear. (Otherwise $O = A + x(B - A) = (1 - x)A + xB$. But A and B are independent so $1 - x = 0 = x$, which is impossible.)

Thus, the vector space is an affine plane. Next, we show that Desargues' Theorem holds.

Let P_1, P_2, P_3 and Q_1, Q_2, Q_3 be triangles perspective from a point C such that $P_1P_2 \parallel Q_1Q_2$ and $P_2P_3 \parallel Q_2Q_3$. We must show that $P_1P_3 \parallel Q_1Q_3$.

Q_1 is on the line $CP_1 = C + \langle P_1 - C \rangle$, so there is a nonzero d such that $Q_1 = C + d(P_1 - C)$.

Because Q_2 is on $CP_2 = C + \langle P_2 - C \rangle$, $Q_2 = C + x(P_2 - C)$ for some scalar x .

Because $Q_1Q_2 \parallel P_1P_2$, the direction vector of Q_1Q_2 must be a nonzero multiple of the direction vector of P_1P_2 . Thus

$Q_2 = C + d(P_2 - C)$. (This choice of Q_2 gives the following direction vector for Q_1Q_2 : $Q_2 - Q_1 = d(P_2 - P_1)$.)

Similarly, $Q_3 = C + d(P_3 - C)$ satisfies the two conditions: Q_3 is on CP_3 and $Q_2Q_3 \parallel P_2P_3$. Thus, a direction vector for Q_1Q_3 is $Q_3 - Q_1 = d(P_3 - P_1)$. Thus $Q_1Q_3 \parallel P_1P_3$ and Desargues' Theorem holds. So a vector space over a division ring is a Desarguesian affine plane.

Theorem 13: A vector space V over a division ring F is a Pappian affine plane (with points and lines as defined previously) if and only if F is a field.

Proof: We know from Theorem 12 that V is a Desarguesian affine plane. We now show that Pappus' Theorem holds when F is a field.

Let C, P, Q, R and C, P', Q', R' be distinct sets of collinear points, such that $PQ' \parallel QR'$, and $QP' \parallel RQ'$. Then we must show that $PP' \parallel RR'$ if and only if F is a field.

Q is on the line CP so $Q = C + d(P - C)$ for some scalar d . R' is on line CQ' and $QR' \parallel PQ'$, so $R' = C + d(Q' - C)$. (This ensures that the direction vector for QR' will be a multiple of the direction vector for PQ' .) R is on CQ so $R = C + f(Q - C)$ for some scalar f . Thus $Q' = C + f(P' - C)$. (This ensures that Q' is on CP' and $Q'R \parallel P'Q$.) Then $R - R' = C + f(Q - C) - (C + d(Q' - C)) = fQ - dQ' - (f - d)C$. Substituting for Q and Q' yields: $R - R' = df(P - C) - fd(P' - C)$. Thus $R - R' = df(P - P')$ if and only if multiplication is commutative. Which implies that $PP' \parallel RR'$ if and only if F is a field.

The fact that Desargues' Theorem does not imply Pappus' Theorem was mentioned in the last chapter. Theorems 12 and 13 enable us to give a proof by counterexample.

Theorem 14: Desargues' Theorem does not imply Pappus' Theorem.

Proof: Let D be a division ring which is not a field. (The Quaternions are one example of a division ring which is not a field.) Then by Theorem 12, the vector space over D is a Desarguesian affine plane (with points and lines defined as usual). But by Theorem 13, the vector space is not a Pappian affine plane. Thus, we have an example of a Desarguesian plane which is not Pappian.

We now turn our attention back to the vector space created from an affine plane A . The vector space V has the points of the affine plane A as vectors and the elements of the division ring associated with A as scalars. Theorem 12 implies that V is a Desarguesian affine plane. Not surprisingly, V is actually the affine plane A .

Theorem 15: The vector space V associated with the Desarguesian affine plane A (i.e. the vectors of V are the points of A , the scalars of V are the elements of the division ring associated with A , and the operations of vector addition and scalar multiplication are as defined previously) is the original affine plane A (when lines are defined as previously).

Proof: By Theorem 12, V is a Desarguesian affine plane. The points of V are the vectors of V which are the points of A . We will show that the lines of V are the lines of A (a clever proof by Bennett, [3]), i.e.:

For any two distinct points P and Q ,

$$PQ = \{S : S = P + d(Q - P) \text{ for } d \text{ in } D\}.$$

Case 1: Suppose O is on the line PQ . If $S = P + d(Q - P)$, then S is on the line $OP = PQ$ because all addition and multiplication have taken place on that line (i.e. $Q - P$ is on OP , $d(Q - P)$ is on OP , so $P + d(Q - P) = S$ is on OP .)

Conversely, let the point S be on the line PQ . Let $T = (S - P)(Q - P)^{-1}$ where $(Q - P)^{-1}$ is the multiplicative inverse of $(Q - P)$ on the line OP ($Q - P \neq 0$ because $Q \neq P$). So T is on OP which means $T = \phi_{OP}(t)$ for some t in D . So $S - P = T(Q - P) = t(Q - P)$ (by the definition of scalar multiplication.) Thus $S = P + t(Q - P)$.

Case 2: Suppose O is not on the line PQ . Let S be a point on the line PQ . (See figure 31.) $P + (Q - P) = Q$ so the line $O(Q - P) \parallel PQ$, and the line $Q(Q - P) \parallel OP$. Let X be the point on the line $O(Q - P)$ such that $XS \parallel OP$. Thus $OX = O(Q - P) \parallel PQ = PS$, so $S = P + X$.

Because X is on the line $O(Q - P)$, $X(Q - P)^{-1} = T$ for some point T

Figure 31

on the line $O(Q-P)$ because multiplication is closed on the line. As before $T = \phi_{O(Q-P)}(t)$ for some t in D . (For any point in the plane we can assign its corresponding element from the division ring by mapping first to 1 and then to the line $O(Q-P)$.) So $X = T(Q-P) = t(Q-P)$ (by the definition of scalar multiplication), and so $S = P + X = P + t(Q-P)$.

Conversely, let $S = P + t(Q-P)$ for some t in D . Then by vector addition, the line $(t(Q-P))S \parallel OP$, and $PS \parallel O(t(Q-P)) = O(Q-P)$. (See figure 32.) But since $O(Q-P) \parallel PQ$, we have $PS \parallel PQ$. Thus, the point S is on the line PQ . So the lines of V are the lines of A , and the points of V are the points of V , so V is the affine plane A .

Figure 32

Theorem 14 showed that because there are infinite division rings (such as the quaternions) which are not fields, then there are infinite Desarguesian affine planes which are not Pappian. The fact that this never happens for finite planes is investigated in the remainder of this paper. We now give the usual proof of the fact that a finite Desarguesian affine plane is a Pappian affine plane. The following theorem by Wedderburn is crucial and quickly disposes of the argument.

Wedderburn's Theorem: Every finite division ring is a field.

We give the proof of Wedderburn's Theorem in the next chapter where it will be analyzed in detail, emphasizing that the proof consists of three main parts: the class equation, the order of the division ring and its normalizers, and a number theoretic fact about cyclotomic

polynomials. We do this because the long desired geometric proof that a finite Desarguesian affine plane is Pappian, which is given in the next chapter, is best understood as a geometric interpretation of the proof of Wedderburn's Theorem.

Theorem 16: A finite Desarguesian affine plane A is a Pappian affine plane.

Proof: By Theorem 8, we know that there is a division ring D associated with the plane A . Because A is a finite plane, there are only a finite number of points on each line and so the division ring D is a finite division ring. Wedderburn's Theorem implies that D must be a field. By Theorem 11, we can form the vector space V over the field D from the affine plane A . By Theorem 13, we know that V is a Pappian plane. But by Theorem 15, V is the same affine plane as A . Thus, A is Pappian.

We conclude this chapter by mentioning the surprising result that a finite Little Desarguesian projective plane is Pappian (see Stevenson, [5, p.383].) This result depends heavily on Artin's result that a finite alternative ring is a field (see Artin, [1].) We give the corresponding result for affine planes below.

Theorem 17: A finite Little Desarguesian affine plane A which satisfies (DP) and (H) is Pappian.

Proof: Let l and m be any two intersecting lines in A and let x, y, z and x', y', z' be distinct points on l and m respectively such that $xx' \parallel zz'$ and $xy' \parallel yz'$ as in figure 33. Let i be the point on l such that $x'i \parallel y'x$. Then by Theorem 7, the points on l form an alternative ring with i as the multiplicative identity and o = the intersection point of l and m as the additive identity. The point $z = y * x$. Because the plane is finite, the alternative ring is finite and is therefore a field (by Artin's result). Thus $z = x * y$ which implies that $x'y \parallel y'z$ so the Pappus Theorem holds.

Figure 33

Chapter 3

Desargues' Theorem in a Finite Affine Plane Implies Pappus' Theorem: A Geometric Proof

In the previous chapter, we proved the geometric fact that a finite Desarguesian affine plane is Pappian. Our proof, however, relied heavily on algebra. In particular, the use of coordinates and then the use of Wedderburn's Theorem were the crucial steps in our proof. Since 1905, geometers have sought a direct, geometric proof. In this chapter, we will present and analyze a more geometric proof by H.

Tecklenburg, "A proof of the theorem of Pappus in finite Desarguesian affine planes" [14] whose only algebra is the number theory concerning cyclotomic polynomials which will be discussed in part three of the proof of Wedderburn's Theorem. Tecklenburg's proof uses Desargues' Theorem, the converse of Desargues' Theorem and the Shear Theorem; and the Little Pappus and Little Desargues' Theorems.

While Tecklenburg's proof does not use Wedderburn's Theorem, it can best be understood as a geometric interpretation of the proof of Wedderburn's Theorem. For this reason, we now give the proof of Wedderburn's Theorem (see Herstein [15, p.361-363].)

Proof of Wedderburn's Theorem: The proof can be divided into three parts. In part 1, we are concerned with the order of a finite division ring D and the order of its normalizers. In part 2, we develop the class equation for the multiplicative group of nonzero elements of D . In part 3, we use parts 1 and 2 together with a number theoretic result on cyclotomic polynomials in order to conclude that any finite division ring D is a field.

Part 1: Let D be a finite division ring. Let $C = \{c \in D : cx = xc \text{ for all } x \text{ in } D\}$. C is called the center of D and is nonempty since $0, 1 \in C$. If $a \in D$, let $N(a) = \{x \in D : xa = ax\}$. $N(a)$ is called the normalizer of a in D . Both C and $N(a)$ are subdivision rings of D (C is actually a field). Since $N(a)$ is the set of all elements in D which commute with a and C is the set of elements in D which commute with every element in D , C is contained in $N(a)$. Let the order of C be q . The goal of part 1 is to show that the order of D is q^n , the order of $N(a)$ is q^m and that m divides n .

Let $a_1, a_2, \dots, a_k$ be the k elements of a subdivision ring S properly contained in some division ring D . Choose an element b_1 in D , but not in S . Then as i and j vary from 1 to k , we claim that $a_i + b_1 a_j$ are k^2 distinct elements of D . Otherwise, let $a_i + b_1 a_j = a_k + b_1 a_l$. Then $b_1(a_j - a_l) = a_k - a_i$. If $a_j - a_l \neq 0$, then $b_1 = (a_k - a_i)(a_j - a_l)^{-1}$ which implies that $b_1 \in S$ which

contradicts the fact that b_1 is not in S . Thus $a_j = a_1$ and so $a_i = a_k$.

If there is an element b_2 in D but not equal to any of the k^2 elements $a_i + b_1 a_j$, then, by the same argument used above, as i, j and l vary from 1 to k we have k^3 distinct elements of the form: $a_i + b_1 a_j + b_2 a_l$. Because D is finite, there is some n such that every element of D will be represented by $a_i + b_1 a_j + \dots + b_{n-1} a_l$. Thus the order of D is k^n . If we let $S = C$, then we have the order of D equal to q^n for some integer n . If we let $D = N(a)$ and $S = C$, then we have the order of $N(a)$ equal to q^m for some integer m . If we let $S = N(a)$, then we have the order of D equal to $(q^m)^v$ for some integer v . But since the order of D is q^n , we have m divides n .

Part 2: Denote the multiplicative group of nonzero elements of D by D' . Similarly, let C' be the nonzero elements of C and $N(a)'$ be the nonzero elements of the normalizer of a . An element $a \in D'$ is conjugate to an element b of D' if $a = xbx^{-1}$. The set of all elements conjugate to a is called the conjugacy class of a . $ixi^{-1} = x$ for all x in D' where i is the identity, so every element in D' is in a conjugacy class. Next we will show that each element is in only one conjugacy class. Suppose $a = xbx^{-1} = ycy^{-1}$ for $a, b, c, x, y \in D'$. Then $b = x^{-1}ycy^{-1}x = (x^{-1}y)c(x^{-1}y)^{-1}$, so a, b , and c are all in the same conjugacy class. Thus, every element of D' is in one and

only one conjugacy class. In other words, the conjugacy relation partitions D' . (Actually it is easy to show that the conjugacy relation is an equivalence relation.) So we can calculate the order of D' by taking the sum of the orders of the disjoint conjugacy classes. In other words, $|D'| = \sum c_a$ where c_a denotes the number of elements in the conjugate class of a and the sum is taken over one element a in each conjugate class.

We now determine the order of each conjugate class (see Herstein [15, p.84].) With simple algebraic manipulations it is possible to show that two elements in the same right coset of $N(a)'$ in D' yield the same conjugate of a , whereas two elements in different right cosets of $N(a)'$ in D' give rise to different conjugates of a . Thus, the number of elements in each conjugate class is equal to the number of right cosets of $N(a)'$ in D' . The number of right cosets of $N(a)'$ in D' is equal to the Index of $N(a)'$ in D' which is equal to the order of D' divided by the order of $N(a)'$. Thus the class equation states:

$$|D'| = \sum |D'| / |N(a)'|$$

where the sum runs over one element a in each conjugate class. Since $N(a)' = D'$ for each element $a \in C$ (i.e. the conjugate class of a is a), the class equation can be rewritten as:

$$|D'| = |C'| + \sum |D'| / |N(a)'|$$

where the sum runs over one element a in each nontrivial conjugate class. Using part 1, this becomes:

$$q^n - 1 = q - 1 + \sum q^n - 1 / q^{m_a} - 1 \quad (1)$$

where $|N(a)| = q^{m_a} - 1$ and m_a divides n .

Part 3: If $x^n = 1$, x is called an n th root of unity. If $x^n = 1$, and $x^m \neq 1$ for any positive integer $m < n$, then x is called a primitive n th root of unity. $\beta = e^{2\pi i/n}$ is a primitive n th root of unity. The polynomial $\Phi_n(x) = \prod (x - \beta)$ where the product is taken over all of the primitive n th roots of unity is called a cyclotomic polynomial. For example, the first few cyclotomic polynomials are listed below:

$$\begin{aligned} \Phi_1(x) &= x - 1, \quad \Phi_2(x) = x + 1, \quad \Phi_3(x) = x^2 + x + 1, \quad \Phi_4(x) = x^2 + 1, \\ \Phi_5(x) &= x^4 + x^3 + x^2 + x + 1, \quad \Phi_6(x) = x^2 - x + 1. \end{aligned}$$

Notice that these are all monic polynomials with integer coefficients. We will show that all cyclotomic polynomials are monic with integer coefficients.

Each root of the polynomial $\pi_{d|n} \Phi_d(x)$ is a distinct primitive d th root of unity for some d which divides n . Thus, each root is an n th root of unity and so is a root of $x^n - 1$. Now we will show that each root of $x^n - 1$ is distinct and is a root of $\pi_{d|n} \Phi_d(x)$. This will show that $\pi_{d|n} \Phi_d(x) = x^n - 1$.

The roots of $x^n - 1$ are of the form $a_j = e^{2\pi i j/n}$ for $j = 1, \dots, n$. So

each a_j is a primitive $n/(j,n)$ th root of unity (where (j,n) is the greatest common divisor of j and n .) Thus each root of x^n-1 is a root of $\pi_{d|n}\Phi_d(x)$ and so $\pi_{d|n}\Phi_d(x)=x^n-1$ (for example, it is easy to see that $x^6-1=\Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_6(x)$.)

Next, by induction we assume that $\Phi_d(x)$ is a monic polynomial with integer coefficients for all d dividing n and $d \neq n$. Thus $x^n-1=\pi_{d|n}\Phi_d(x)=\Phi_n(x)\pi_{d|n, d \neq n}\Phi_d(x)=\Phi_n(x)g(x)$ where $g(x)$ is a product of monic polynomials with integer coefficients. Thus, $\Phi_n(x)=x^n-1/g(x)$ which, on actual division, tells us that $\Phi_n(x)$ is a monic polynomial with integer coefficients (in the division, we are always using the leading term of $g(x)$ and since $g(x)$ is monic and each term we divide into has an integer coefficient we end up with integer coefficients in the quotient.)

So we have the fact that $\Phi_n(x)$ is a monic polynomial with integer coefficients. Thus for some integer p , $\Phi_n(p)$ is an integer. Also, since $x^n-1=\pi_{d|n}\Phi_d(x)=\Phi_n(x)\pi_{d|n, d \neq n}\Phi_d(x)$, $\Phi_n(x)$ divides x^n-1 and so $\Phi_n(p)$ divides p^n-1 .

Next, we want to show that $\Phi_n(p)$ divides p^n-1/p^m-1 when m divides n . Suppose m divides n . From our previous work, we know that $x^m-1=\pi_{k|m}\Phi_k(x)$. Thus,

$$\begin{aligned} x^n-1 &= \Phi_n(x)\pi_{d|n, d \neq n}\Phi_d(x) = \Phi_n(x)\pi_{k|m}\Phi_k(x)\pi_{d|n, d \neq n, d \neq m}\Phi_d(x) \\ &= \Phi_n(x)(x^m-1)\pi_{d|n, d \neq n, d \neq m}\Phi_d(x). \end{aligned}$$

So $\Phi_n(x)$ divides x^n-1/x^m-1 and so $\Phi_n(p)$ divides p^n-1/p^m-1 when m divides n . Together with (1), and using $q=p$, we have $\Phi_n(p)$ divides $p-1$. Our last step is to show that this implies that $n=1$ and so $|D'|=|C'|$ which implies that $D=C$ is a field.

Suppose $n>1$. The center of any division ring contains the additive and multiplicative identities, 0 and 1. Thus, $p\geq 2$.

Thus,

$$\begin{aligned} |\Phi_n(p)| &= |\pi(p - e^{2\pi i k/n})| \\ &= \pi |p - e^{2\pi i k/n}| > \pi (|p| - |e^{2\pi i k/n}|) \\ &= \pi(p-1) \geq p-1 \end{aligned}$$

(where k and n are relatively prime which makes $e^{2\pi i k/n}$ a primitive n th root of unity.)

Thus $n=1$ and so the theorem is proven.

At each step of Tecklenburg's proof, we will make use of the operations of addition and multiplication, defined geometrically in the previous chapter, in order to interpret the proof algebraically. Tecklenburg's proof consists of 14 lemmas. Lemmas 1 through 5 are used to prove lemma 6 which is essentially the class equation (part 2 of the proof of Wedderburn's Theorem) and lemmas 7 through 12 are used in order to prove lemma 13 which corresponds to part 1 in the proof of Wedderburn's Theorem. In lemma 14, Tecklenburg uses the argument involving cyclotomic polynomials (part 3 in the

proof of Wedderburn's Theorem), in order to conclude the proof.

We begin the proof by picking three noncollinear points o, g, h and defining several maps on the two lines $G=og$ and $H=oh$. First, for any point x on G or H , let $\underline{x}$ be the point on H or G respectively such that $x\underline{x} \parallel gh$ (see figure 1). (The map $_$ is a parallel projection from G or H to H or G .) Naturally $\underline{o}=o$. Let $G'=G \setminus o$ and $H'=H \setminus o$. For the points x on G' and a on H' , let x_a be the point on H such that $xx_a \parallel ga$. Finally, for x on G' and a on H' , Let $\pi_a(x)$ be the point on H such that $\pi_a(x)\underline{x}_a \parallel hx$.

Figure 1

Notice that if the Pappus Theorem holds, then $\pi_a(x)=a$. If, on the other hand, $\pi_a(x)=a$ for every x on G' and for every a

on H' , for all choices of o, g, h , we can conclude that the Pappus Theorem holds in the plane. So the goal is to show that $\pi_a(x)=a$ for all points x on G' and all points a on H' .

For any given a on H distinct from o , we know that there are points x such that $\pi_a(x)=a$. For example, figure 2 shows that $\pi_a(g)=a$ and with the help of the Shear Theorem $\pi_a(\underline{a})=a$.

Figure 2

Let P be the set of points x on G' such that $\pi_a(x)=a$ for every a on H' or in other words $P=\cap_{a \in H'} \pi_a^{-1}(a)$.

So the goal of the proof restated is: we want to prove that $P=G'$.

Before proceeding with the fourteen lemmas, let us use the multiplication defined in the previous chapter to analyze what has been done so far. We let o and h be the additive and multiplicative identities respectively on the line H .

Then x_a is just the point $\underline{x} * a$ and $\pi_a(x)$ is just the point $(\underline{x} * a) * \underline{x}^{-1}$ (see figure 3). (So x_a is right multiplication of $\underline{x}$ by a and $\pi_a(x)$ is conjugation of a by $\underline{x}$.) So to say that $\pi_a(x) = a$ is equivalent to saying that $(\underline{x} * a) * \underline{x}^{-1} = a$ which is equivalent to $\underline{x} * a = a * \underline{x}$ (i.e. $\underline{x}$ and a commute). So the set P then becomes the set of points x on G' corresponding to the points $\underline{x}$ on H' which commute with every point a on H' . Thus, interpreted algebraically, P is the image, under $_$, of the center of the nonzero elements in the division ring associated with the line H .

Figure 3

So to show that $P = G'$ is to show that the center is the division ring (and thus the division ring is a field.) Also, the set $\pi_a^{-1}(a)$ is the set of points x on G' such that $(\underline{x} * a) * \underline{x}^{-1} = a$, or in other words, the set of points $\underline{x}$ of H' which commute with a . Thus, algebraically, $\pi_a^{-1}(a)$ is the

image under π_a of the normalizer of a in the group of nonzero elements on the line H . Thus, defining

$P = \bigcap_{a \in H} \pi_a^{-1}(a)$ makes sense from an algebraic point of view as the center is the intersection of all normalizers.

Lemma 1: $\pi_a(x) = \pi_b(x)$ if and only if $a = b$ for a, b on H' and x on G' .

Proof: The map $\pi_a: G' \rightarrow H'$ is well-defined, so naturally if $a = b$ then $\pi_a(x) = \pi_b(x)$. If $\pi_a(x) = \pi_b(x)$, then one can work backwards (see figure 1) to get $\underline{x}_a = \underline{x}_b$, and $x_a = x_b$. Thus $a = b$ is the unique point on H such that $xx_a \parallel ga$.

Algebraically, lemma 1 says simply that $(\underline{x} * a) * \underline{x}^{-1} = (\underline{x} * b) * \underline{x}^{-1}$ if and only if $a = b$. Lemma 1 shows that conjugation is a one-to-one function (and thus a bijection since $|G| = |H|$.)

Lemma 2: $\pi_a(x) = a$ if and only if $\pi_x(a) = \underline{x}$ for every a on H' , and x on G' .

Proof: Assuming $\pi_a(x) = a$, we will first use the Shear Theorem to show that $\underline{a}_x = x_a$. (See figure 4.) Because $\pi_a(x) = a$, $hx \parallel ax_a$. Together with $gh \parallel aa$, and $xx \parallel x_a x_a$ the Shear Theorem implies that $gx \parallel \underline{a}_x$. This proves that $\underline{a}_x = x_a$. Now we need only show that $ha \parallel \underline{xx}_a$. Since $hg \parallel \underline{xx}$, $ga \parallel \underline{xx}_a$, and $aa \parallel x_a x_a$, another application of the Shear Theorem gives $ha \parallel \underline{xx}_a$ and so

$\pi_{\underline{x}}(\underline{a}) = \underline{x}$. The reverse implication holds by substituting $\underline{x}$ for $\underline{a}$ and $\underline{a}$ for $\underline{x}$ (and the fact that $_$ maps $\underline{x}$ to $\underline{x}$.)

Figure 4

From the algebraic point of view, we have just proven that if $(\underline{x} * a) * \underline{x}^{-1} = a$, then $(a * \underline{x}) * a^{-1} = \underline{x}$. In the proof above, we simply multiplied on the right by $\underline{x}$ and then by a^{-1} .

Lemma 3: Let $\underline{P}$ be the image of the set P under the map π .
Then $\underline{P} = \{a \text{ on } H' \text{ such that } \pi_a(G') = a\}$.

Proof: By definition $\underline{P} = \{x \text{ on } H' \text{ such that } x \text{ is in } P\}$
 $= \{x \text{ on } H' \text{ such that } \pi_a(x) = a \text{ for every point } a \text{ on the line } H'\}$.

But now lemma 2 implies,

$$\underline{P} = \{ \underline{x} \text{ on } H' \text{ such that } \pi_x(\underline{a}) = \underline{x} \text{ for every point } \underline{a} \text{ on } G' \}.$$

Finally, replacing $\underline{x}$ with $\underline{a}$ gives us:

$$\underline{P} = \{a \text{ on } H' \text{ such that } \pi_a(G') = a\}.$$

$\underline{P} \cup \{o\}$, denoted $\underline{P}^0$, is the center of the division ring associated with the line H .

We now define a map $L_a: G \rightarrow G$ which maps x to the point x' such that $ax' \parallel hx$. (See figure 5.) If $a \neq o$, L_a is bijective (unique intersections of nonparallel lines).

Figure 5

Algebraically, $L_a(x)$ is just the point $\underline{a} * x$ on G . Thus L left multiplies the point x on G by the point on G corresponding, under the map $\underline{}$, to the point a on H .

Lemma 4: $\pi_{\pi_a(b)}(c) = \pi_a L_{\underline{c}}(b)$ for a on H' and b, c on G' .

Proof: Let $d = \pi_a(b)$ and $e = L_{\underline{c}}(b)$. We will work backwards to get the proof. If we can show that $hc \parallel \pi_a(e) \underline{c}_d$, then because $hc \parallel \pi_d(c) \underline{c}_d$ we will have shown that $\pi_a(e) = \pi_d(c)$ which is the

Figure 6

Now we are ready to start. We will use the Shear Theorem three times. First, as we mentioned above $hb \parallel_{\underline{ce}}$. We also have $hg \parallel_{\underline{cc}}$ and $gd \parallel_{\underline{cc}_d}$ (see figure 7), so by the Shear Theorem $db \parallel_{\underline{c}_d e}$. This result together with $b_a b \parallel_{\underline{e}_a e}$ and $b_a b_a \parallel_{\underline{e}_a e_a}$, gives us $\underline{db}_a \parallel_{\underline{c}_d e_a}$ by the Shear Theorem. As

Figure 7

discussed above, this gives us

$c_d e_a \parallel c e$ which enables us to use the Shear Theorem one last time to get $h c \parallel \pi_a(e) c_d$. (See figure 6.) Thus $\pi_d(c) = \pi_a(e)$ and so $\pi_{\pi_a(b)}(c) = \pi_a L_c(b)$.

Lemma 4 is crucial in both lemma 5 and lemma 6.

Algebraically, Lemma 4 says that:

$$c * (b * (a * b^{-1}) * c^{-1}) = (c * b) * (a * (c * b)^{-1}).$$

This is precisely what is needed (see part 2 of the proof of Wedderburn's Theorem) in order to show that the conjugacy relation partitions the points on the line into disjoint equivalence classes. (Equivalently, the equation above is exactly what is necessary in order to show the transitivity of conjugacy as an equivalence relation.)

Lemma 5: $|\pi_a^{-1}\pi_a(b)| = |\pi_a^{-1}(a)|$ for a on H' and b on G' .

Proof: First, $a = \pi_a(\pi_a^{-1}(a))$ because

$\pi_a^{-1}(a) = \{x \in G' : \pi_a(x) = a\}$. Thus by lemma's 1 and 4,

$\pi_a(b) = \pi_{\pi_a(\pi_a^{-1}(a))}(b) = \pi_a L_b \pi_a^{-1}(a)$. So $\pi_a^{-1}\pi_a(b) = L_b \pi_a^{-1}(a)$. So

$L_b^{-1}\pi_a^{-1}\pi_a(b) = \pi_a^{-1}(a)$. Since $b \neq 0$, L_b is a bijection. So

$|\pi_a^{-1}\pi_a(b)| = |L_b^{-1}\pi_a^{-1}\pi_a(b)| = |\pi_a^{-1}(a)|$.

The set of points $\pi_a^{-1}\pi_a(b)$ is very important in lemma 6. The key step in the proof of lemma 5 is the identification of $\pi_a^{-1}\pi_a(b)$ with the set of points $L_b \pi_a^{-1}(a)$. Algebraically, the set $L_b \pi_a^{-1}(a)$ is just the set of points in the normalizer of a , left multiplied by b . This is known as a left coset of the normalizer in the multiplicative group of nonzero points on the line. The proof of lemma 5 goes on to show that since L_b is a bijection, each coset contains the same number of elements as the normalizer. As we shall see, this fact is vital in one of the counting arguments used in lemma 6.

Lemma 6: There exists a set of points M on the line $H' \setminus \underline{P}$ such that $|G'| = |P| + \sum_{a \in M} |\pi_a^{-1}(a)|$.

Proof: We will prove the lemma by counting the points on H' one way and then counting the points on G' in another way.

Let a_1 be a point on $H' \setminus \underline{P}$. Then pick a point a_2 such that a_2

is in the set $H' \setminus \underline{P} \cup \pi_{a_1}(G')$. Continuing recursively, choose the points a_k such that:

$$a_k \in H' \setminus (\underline{P} \cup (\cup \pi_{a_i}(G')))$$

With each additional set $\pi_{a_k}(G')$, we get at least the point a_k . Thus since there are finitely many points on the line H , there is some minimal integer m such that:

$$H' = \underline{P} \cup (\cup_{i=1 \text{ to } m} \pi_{a_i}(G')).$$

We will show that H' is partitioned into the sets $\underline{P}$ and $\pi_{a_i}(G')$. That is, we will show that $\underline{P}$ is disjoint from each $\pi_{a_i}(G')$ and that $\pi_{a_i}(G')$ is disjoint from $\pi_{a_j}(G')$ whenever $a_i \neq a_j$.

First, suppose there is a point x in $\underline{P} \cup \pi_{a_i}(G')$ for some a_i . Then, because x is in $\underline{P}$, $\pi_x(G') = x$. But x in $\pi_{a_i}(G')$ implies that for some y on G' , $\pi_{a_i}(y) = x$. Together, these two statements imply that $\pi_{a_i}(y) = \pi_x(y)$. Lemma 1 then implies that $a_i = x$. But since a_i is not in $\underline{P}$ and x is in $\underline{P}$, this is a contradiction. Thus, $\underline{P}$ is disjoint from $\pi_{a_i}(G')$ for any a_i .

Let $i < j$. If $\pi_{a_i}(G')$ and $\pi_{a_j}(G')$ are not disjoint then there are points x and y on G' such that $\pi_{a_j}(y) = \pi_{a_i}(x)$. Also, because L_y is a bijection, there is a point z on G' such

that $L_Y(z)=x$. Thus substitution and lemma 4 give us:

$$\pi_{aj}(Y)=\pi_{ai}(x)=\pi_{ai}L_Y(z)=\pi_{\pi ai(z)}(Y) .$$

Thus, by lemma 1, $a_j=\pi_{ai}(z)$, which is contained in $\pi_{ai}(G')$, but $a_j \in \pi_{aj}(G')$ is a contradiction so $\pi_{ai}(G')$ and $\pi_{aj}(G')$ are disjoint.

Thus, we have counted the points on H' in the following way:

$$|H'| = |\underline{P}| + \sum_{a \in H} |\pi_a(G')| . \quad (1)$$

Figure 8 illustrates the counting technique which produced equation (1).

Figure 8

Next, we will use a different counting technique in order to count the points on G' . We claim that G' can be partitioned into a disjoint union of sets of the form $\pi_a^{-1}\pi_a(x_i)$ for various points x_i and some point a on H' . The set of points $\pi_a^{-1}\pi_a(x)$ is the set of points on G' which are mapped to $\pi_a(x)$ by π_a . Every point on G' is mapped by π_a to exactly one point on H' . Thus, every point of G' is in a set $\pi_a^{-1}\pi_a(x)$ for some $\pi_a(x)$. Also, if $\pi_a(x) \neq \pi_a(y)$, then the sets $\pi_a^{-1}\pi_a(x)$ and $\pi_a^{-1}\pi_a(y)$ will be disjoint. So we have the fact that G' is the disjoint union of the sets $\pi_a^{-1}\pi_a(x)$ where the union is taken over each distinct point $\pi_a(x)$ in $\pi_a(G')$. (See figure 9.)

Figure 9

This gives us: $|G'| = \sum |\pi_a^{-1}\pi_a(x)|$ where the sum is taken over each distinct point $\pi_a(x)$ in $\pi_a(G')$. By lemma 5, this gives

us:

$$|G'| = \sum |\pi_a^{-1}(a)| = |\pi_a(G')| |\pi_a^{-1}(a)| \quad (2)$$

Using (1) and (2), the fact that $|\underline{P}| = |P|$, and that every line has the same number of points, we have the following:

$$|G'| = |H'| = |\underline{P}| + \sum_{a \in H} |\pi_a(G')| = |P| + \sum_{a \in H} |G'| / |\pi_a^{-1}(a)|.$$

As stated earlier, lemma 6 gives us the class equation. Algebraically, equation (1) in the proof of lemma 6 is the result of the fact that conjugacy is an equivalence relation and thus partitions the set of points into a disjoint union of conjugate classes. Each a_i is a representative from a conjugate class. Each element in $\underline{P}$ is its own one element conjugate class. Lemmas 1 through 4 were established in order to translate this simple algebraic idea into geometric terms.

Equation (2) is derived through an intuitively simple geometric argument. Interpreted algebraically, equation (2) in the proof tells us that the number of elements in the conjugate class of a is equal to the index of the normalizer of a in the multiplicative group of nonzero points on the line. The index of the normalizer is equal to the number of left cosets of the normalizer contained in the group.

However, once we know that $|\pi_a^{-1}\pi_a(x)| = |\pi_a^{-1}(a)|$ from lemma 5,

equation (2) follows easily from the geometric concept of a map between the two lines. The algebraic derivation of equation (2) (Herstein [10]) is more complicated and provides less insight.

With lemma 6 completed, we must only determine the orders of G and $\pi_a^{-1}(a)^0$ as powers of the order of P^0 (where $S^0 = S \cup \{o\}$ for any set S .) Then the theorem will follow from the number theoretic result on cyclotomic polynomials.

Note: when we have concluded the proof, we will know that both of the counting arguments used in lemma 6 are forced to be much simpler than they appear now. In other words, because we will prove that $G' = P$, and therefore $H' = \underline{P}$, there do not exist any points a_i as defined above. So in this particular proof, the complicated counting arguments given above serve no purpose other than to achieve a contradiction.

Before getting to lemma 7, we define the following maps:

For the points x on G and y on H , let $[x,y]$ be the intersection point of the line on x parallel to H with the line on y parallel to G . (See figure 10.) This is a coordinatization of the affine plane with respect to the lines G and H . Since there are n points on each of the lines

G and H for some integer n , then each of the n^2 points of the plane has unique coordinates $[x,y]$ for some points x and y on G and H . For the points a and b on G , define the maps σ_a and τ_b by:

$\sigma_a: G \times H' \rightarrow G$ such that $\sigma_a(x,y) = \text{the point } b \text{ on } G \text{ such that } a[x,y] \parallel hb.$

$\tau_b: G \times H \rightarrow G$ such that $\tau_b(x,y) = \text{the point } a \text{ on } G \text{ such that } a[x,y] \parallel hb.$

Note that the domain of σ_a is $G \times H'$. $\sigma_a(x,o)$ cannot be defined as above, because $a[x,o] = G$ (for any point $a \neq o$ or $x \neq o$) and so $\sigma_a(x,o)$ would be the point on G and on a line parallel but not equal to G , which is a contradiction.

Figure 10

From figure 11, we can see that the point $\tau_b(x,y)$ is the point $x + (y * b)$. Also from figure 11, we can interpret the

Table of Parallel Relations:

The following are derived directly from the definitions of the various maps, and are summarized here as an aid to following the proofs of the lemmas.

1. $x\bar{x}\parallel y\bar{y}\parallel hg$ for all points x,y on G' or H' .
2. $x\bar{x}_a\parallel y\bar{y}_a\parallel ga$ for $a\in H'$ and $x,y\in G'$.
3. $\bar{x}_a\pi_a(x)\parallel \bar{x}_b\pi_b(x)\parallel hx$ for $a,b\in H'$ and $x\in G'$.
4. $aL_a(x)\parallel bL_b(x)\parallel hx$ for $a,b\in H'$ and $x\in G$.
5. $y[x,y]\parallel z[x,z]\parallel G$ for $y,z\in H$ and $x\in G$.
6. $x[x,y]\parallel z[z,y]\parallel H$ for $x,z\in G$ and $y\in H$.
7. $h\sigma_a(x,y)\parallel [x,y]a$ for $a,x\in G$ and $y\in H'$.
8. $[x,y]\tau_b(x,y)\parallel [x',y']\tau_b(x',y')\parallel hb$ for $x,x',b\in G$ and $y,y'\in H$.

Figure 11

Lemma 7: $\sigma_a(x,y)=b$ if and only if $\tau_b(x,y)=a$, for a,b,x on G and y on H' .

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Lemma 8 will give us a way of generating additional elements in the set $\pi_a^{-1}(a)$.

Lemma 8: Let $a \in H'$ and $b \in G$ and

$(x, y) \in \pi_a^{-1}(a)^\circ \times \pi_a^{-1}(a)$. Then $b \in \pi_a^{-1}(a)^\circ$ if and only if $\tau_b(x, y) \in \pi_a^{-1}(a)^\circ$.

Proof: If $b=0$, then $\tau_0(x, y)$ is the point on G and on the line on $[x, y]$ parallel to H . Thus, $\tau_0(x, y)=x$ which is given to be in $\pi_a^{-1}(a)^\circ$. So lemma 8 holds in the case that $b=0$.

From now on we assume that $b \neq 0$. Let $c = \tau_b(x, y)$ and $d = L_y(b)$.

By lemma 1, $\pi_a(b)=a$ if and only if $\pi_{\pi_a(b)}(y)=\pi_a(y)$. Because we are given that $y \in \pi_a^{-1}(a)$, we know $\pi_a(y)=a$. By lemma 4, $\pi_{\pi_a(b)}(y)=\pi_a L_y(b)=\pi_a(d)$. So $\pi_a(b)=a$ if and only if $\pi_a(d)=a$. Thus, we only need to show that $c \in \pi_a^{-1}(a)^\circ$ if and only if $d \in \pi_a^{-1}(a)$.

If $x=0$, then $c = \tau_b(0, y)$ is the intersection of G with the line on y parallel to hb . But this is just the point $L_y(b)=d$. So lemma 8 holds for $x=0$. So we may assume that $x \neq 0$.

We will make four applications of the Little Desargues Theorem in order to show that $hd \parallel [x, h]c$ and $ad \parallel [x_a, a]c_a$. We have $y[x, y] \parallel h[x, h] \parallel G$, and $hy \parallel [x, h][x, y]$ and $yd \parallel hb \parallel [x, y]c$ (see figure 12), thus the

Figure 12

Little Desargues Theorem on the triangles h,y,d and $[x,h],[x,y],c$ implies that $hd \parallel [x,h]c$.

Together with $d_a[x,d_a] \parallel G$, and $hd_a \parallel [x,h][x,d_a]$, the Little Desargues Theorem on the triangles d_a,h,d and $[x,d_a],[x,h],c$ implies that $dd_a \parallel c[x,d_a]$.

This last statement implies that the point $[x,d_a]$ is on the line cc_a (because $cc_a \parallel dd_a \parallel c[x,d_a]$) which implies that $c_a[x,d_a] \parallel x_a x$. Since $x_a[x_a,d_a] \parallel x[x,d_a] \parallel H$, and $[x,d_a][x_a,d_a] \parallel G$, then the Little Desargues Theorem on the triangles x_a,x,x_a and $c_a,[x,d_a],[x_a,d_a]$, implies that $x_ax_a \parallel c_a[x_a,d_a]$.

But, since $c_a c_a \parallel x_a x_a$, we know that $[x_a,d_a]c_a \parallel x_a x_a \parallel d_a d_a$ (because $[x_a,d_a]$ is on line $c_a c_a$.) Since $d_a[x_a,d_a] \parallel a[x_a,a] \parallel G$ and $[x_a,a][x_a,d_a] \parallel H$, by the Little Desargues Theorem on the triangles d_a,a,d_a and $[x_a,d_a],[x_a,a],c_a$, we have $d_a a \parallel c_a[x_a,a]$.

We need one more piece of information before showing that $c \in \pi_a^{-1}(a)^\circ$ if and only if $d \in \pi_a^{-1}(a)$. Since $x \in \pi_a^{-1}(a)$, we have $\pi_a(x) = a$ which implies that $ax_a \parallel hx$. Since $x[x,h] \parallel x_a[x_a,a]$ and $h[x,h] \parallel a[x_a,a]$, the converse of Desargues Theorem on triangles $a,x_a,[x_a,a]$ and $h,x,[x,h]$ implies that $o,[x,h],[x_a,a]$ are collinear.

If $d \in \pi_a^{-1}(a)$, then consider the triangles $h,[x,h],c$ and

$a, [x_a, a], c_a$. Because $o, [x, h]$, and $[x_a, a]$ are collinear, the triangles are perspective from the point o . Because $\pi_a(d)=a$, $hd \parallel ad_a$, $hd \parallel [x, h]c$, and $ad_a \parallel [x_a, a]c_a$, we have $[x, h]c \parallel ad_a \parallel [x_a, a]c_a$. Together with $h[x, h] \parallel a[x_a, a]$, Desargues' Theorem implies $hc \parallel ac_a$. This implies that either $c=o$ or $\pi_a(c)=a$, so $c \in \pi_a^{-1}(a)^\circ$.

If $c \in \pi_a^{-1}(a)^\circ$, then consider again the triangles $[x, h], h, c$ and $[x_a, a], a, c_a$. Because $o, [x, h]$ and $[x_a, a]$ are collinear, the triangles are perspective from the point o . If $c=o$, the results of the previous page: $hd \parallel [x, h]c$, $d_a a \parallel c_a [x_a, a]$, and $o, [x, h], [x_a, a]$ collinear; still hold. So $hd \parallel [x, h]o = [x_a, a]o \parallel ad_a$, which implies that $a = \pi_a(d)$.

So we can assume that $c \neq o$. Because $\pi_a(c)=a$, $hc \parallel ac_a$. Together with $[x, h]h \parallel [x_a, a]a$, Desargues' Theorem implies that $[x, h]c \parallel [x_a, a]c_a$. Thus, $hd \parallel [x, h]c \parallel [x_a, a]c_a \parallel ad_a$ which implies that $a = \pi_a(d)$. So $d \in \pi_a^{-1}(a)$.

Remark 1: Algebraically, lemma 8 states that if $b, x, y \in \pi_a^{-1}(a)$, then $x + (y * b) \in \pi_a^{-1}(a)$. The first part of the proof, where we established that $\pi_a(b)=a$ if and only if $\pi_a(d)=a$, is interesting in that algebraically this implies that a commutes with b if and only if a commutes with d . Remember $d = L_y(b)$ where $y \in \pi_a^{-1}(a)$. Thus $d = y * b$ and we know that y commutes with a . So the first part of the proof, which was

an easy consequence of previous lemmas, implies that $\pi_a^{-1}(a)$ is closed under multiplication.

Remark 2: The actual steps in the proof of part 1 are:

$$\underline{b}\underline{a}\underline{b}^{-1}=\underline{a} \text{ iff } \underline{a}=\underline{y}\underline{a}\underline{y}^{-1}=\underline{y}(\underline{b}\underline{a}\underline{b}^{-1})\underline{y}^{-1}=(\underline{y}\underline{b})\underline{a}(\underline{y}\underline{b})^{-1}=\underline{d}\underline{a}\underline{d}^{-1}.$$

For the rest of the proof, the results $\underline{h}\underline{d}\|[\underline{x},\underline{h}]\underline{c}$ and $\underline{d}\underline{d}_a\|[\underline{c}[\underline{x},\underline{d}_a]$ both are algebraically equivalent to $\underline{x}+\underline{d}=\underline{c}$, but performing the addition using the lines $\underline{h}[\underline{x},\underline{h}]$ and $\underline{d}_a[\underline{x},\underline{d}_a]$ respectively. $\underline{x}_a\underline{x}_a\|[\underline{c}_a[\underline{x}_a,\underline{d}_a]$ is equivalent to $\underline{d}_a+\underline{x}_a=\underline{c}_a$ while $\underline{d}_a\underline{a}\|[\underline{c}_a[\underline{x}_a,\underline{a}]$ is equivalent to $\underline{x}_a+\underline{d}_a=\underline{c}_a$. If $\pi_a(\underline{d})=\underline{a}$ ($\underline{a}$ and $\underline{d}$ commute), then $\underline{o},[\underline{x},\underline{h}],[\underline{x}_a,\underline{a}]$ collinear gives: $\underline{x}\underline{z}=\underline{c}$ for some point $\underline{z}$ and so $\underline{z}=\underline{x}^{-1}\underline{c}$; and $(\underline{x}_a)\underline{z}=\underline{c}_a$. Substituting gives $(\underline{x}_a)\underline{x}^{-1}\underline{c}=\underline{c}_a$ and since $\underline{x}$ commutes with $\underline{a}$, this implies that $\underline{a}$ commutes with $\underline{c}$ (which is equivalent to $\underline{a}$ commutes with $\underline{c}$ by isomorphism $_$). If $\pi_a(\underline{c})=\underline{a}$, the collinearity of $\underline{o},[\underline{x},\underline{h}],[\underline{x}_a,\underline{a}]$ gives us $\underline{a}\underline{d}=\underline{d}\underline{a}$. (Also, the final Pappus configurations corresponding to $\underline{a}\underline{d}=\underline{d}\underline{a}$ and $\underline{a}\underline{c}=\underline{c}\underline{a}$ can be seen.)

As stated earlier, lemma 8 gives us a new way of generating additional elements of $\pi_a^{-1}(a)^\circ$. For example, we know that $\underline{a}$ is in $\pi_a^{-1}(a)^\circ$. Lemma 8 implies that any power of $\underline{a}$ is in $\pi_a^{-1}(a)^\circ$ and that any sums of these powers are in $\pi_a^{-1}(a)^\circ$. Thus, we now know that any polynomial in $\underline{a}$ with integer coefficients is in $\pi_a^{-1}(a)^\circ$ (e.g. $5\underline{a}^3+2\underline{a}^2+\underline{a}$, obviously not all of these polynomials will correspond to distinct points).

Finally, because lemma 8 shows that the operations $+$ and $*$ are closed in $\pi_a^{-1}(a)^\circ$, $\pi_a^{-1}(a)^\circ$ is a division ring.

Geometrically (see figure 13), if x, y, b are any three points of $\pi_a^{-1}(a)^\circ$, then lemma 8 tells us that the line on $[x, y]$ parallel to hb , intersects the line G in a point of $\pi_a^{-1}(a)^\circ$.

Figure 13

In other words, the map τ_b is closed on the set $\pi_a^{-1}(a)^0$, which brings us to lemma 9.

Lemma 9:

- a. $\sigma_b(\pi_a^{-1}(a)^\circ \times \pi_a^{-1}(\underline{a}))$ is contained in $\pi_a^{-1}(a)^\circ$ and $\tau_b(\pi_a^{-1}(a)^\circ \times \pi_a^{-1}(\underline{a}))$ is contained in $\pi_a^{-1}(a)^\circ$ for $a \in H'$ and $b \in \pi_a^{-1}(a)^\circ$.
- b. $\sigma_b(P^\circ \times \underline{P})$ is contained in P° and $\tau_b(P^\circ \times \underline{P}^\circ)$ is contained in P° for $b \in P^\circ$.

Proof of a: By lemma 8, $b \in \pi_a^{-1}(a)^\circ$ implies that $\tau_b(x, y) \in \pi_a^{-1}(a)^\circ$ for all $(x, y) \in \pi_a^{-1}(a)^\circ \times \pi_a^{-1}(\underline{a})$. Since $\tau_b(x, 0) = x$, this gives us $\tau_b(\pi_a^{-1}(a)^\circ \times \pi_a^{-1}(\underline{a}))$ is a subset of $\pi_a^{-1}(a)^\circ$.

If $\sigma_b(x, y) = z$ for $b \in \pi_a^{-1}(a)^\circ$ and for $(x, y) \in \pi_a^{-1}(a)^\circ \times \pi_a^{-1}(\underline{a})$, then by lemma 7, $\tau_z(x, y) = b$ which, by lemma 8, is in $\pi_a^{-1}(a)^\circ$ if and only if $z \in \pi_a^{-1}(a)^\circ$. So $b \in \pi_a^{-1}(a)^\circ$ implies that $z = \sigma_b(x, y) \in \pi_a^{-1}(a)^\circ$. Thus, $\sigma_b(\pi_a^{-1}(a)^\circ \times \pi_a^{-1}(\underline{a}))$ is contained in $\pi_a^{-1}(a)^\circ$.

Proof of b: Because $P^\circ = (\cap_{a \in H'} \pi_a^{-1}(a)) \cup \{0\}$, and part a. holds for every a on H' , we have $\sigma_b(P^\circ \times \underline{P})$ is contained in P° and $\tau_b(P^\circ \times \underline{P}^\circ)$ is contained in P° for $b \in P^\circ$.

Lemma 9a implies that every line on h parallel to $[x, y]b$, for any points $x, y, b \in \pi_a^{-1}(a)^\circ$ with $y \neq 0$, intersects G in a point of $\pi_a^{-1}(a)$. (See figure 14.) Lemma 9b implies that P° is similarly closed under σ and τ . Thus, both $\pi_a^{-1}(a)^\circ$ and P° are division rings.

Figure 14

Lemma 10: Let $b \in G$ and $(u, v), (x, y) \in G \times H$ with $v \neq y$ and $\tau_b(u, v) = \tau_b(x, y)$. Then $\sigma_u(x, \sigma_y(v, h)) = b$.

Proof: If $v = o$ (see figure 15), then $\sigma_y(o, h) = y$ (the point on G and on the line on h parallel to $[o, h]y = hy$). Also, since $\tau_b(x, y) = \tau_b(u, v) = \tau_b(u, o) = u$, lemma 7 implies that $\sigma_u(x, y) = b$. Thus, the lemma is true for $v = o$.

figure 15

Suppose $v \neq o$. Let $z = \sigma_y(\underline{v}, y)$. Since $v \neq y$, $[\underline{v}, y]y$ is not parallel to H and so $z \neq o$. (See figure 17.) It suffices to show

$$[u, v][x, y] \parallel [u, x, z] \quad (1)$$

for this implies $c = \sigma_u(x, z) =$ the point on G such that $hc \parallel [u, x, z] \parallel [u, v][x, y]$. But because $\tau_b(x, y) = \tau_b(u, v)$, $[u, v][x, y] \parallel [u, v]\tau_b(u, v)$ which is by definition parallel to hb . Thus if (1) holds, $b = c$ so the lemma holds.

Consider the triangles z, h, z and $y, [v, h], [v, z]$. (See figure 16.) $G = zy \parallel h[v, h] \parallel z[v, z]$ and $H = hz \parallel [v, h][v, z]$ and $z = \sigma_y(\underline{v}, h)$ is by definition the point on G such that $hz \parallel [v, h]y$. Thus the Little Desargues Theorem implies that $zz \parallel y[v, z]$. Since $zz \parallel yy$, we know that $[\underline{v}, z]$ is on yy .

Figure 16

Case 1: Let $x=o$. (See figure 16.) We will first use the Little Pappus Theorem and then get the result we need with the Little Desargues Theorem. Notice that the line on $\underline{y}, [\underline{y}, v], [\underline{y}, \underline{z}]$ is parallel to H and that $v, y, \underline{z}$ are all on H . By the work on the previous page, $v\underline{y} \parallel y[\underline{y}, \underline{z}]$. Since $v[\underline{y}, v] \parallel \underline{z}[\underline{y}, \underline{z}]$, the Little Pappus Theorem implies $y[\underline{y}, v] \parallel \underline{v}\underline{z}$.

Now we consider the triangles $u, \underline{y}, \underline{z}$ and $[u, v], [\underline{y}, v], y$. Since $u[u, v] \parallel \underline{y}[\underline{y}, v] \parallel H = \underline{z}y$ and $[\underline{y}, v][u, v] \parallel G = \underline{y}u$, the result $y[\underline{y}, v] \parallel \underline{v}\underline{z}$ enables us to use the Little Desargues Theorem in order to conclude $u\underline{z} \parallel [u, v]y$. Since $x=o$, $y=[o, y]=[x, y]$ and $\underline{z}=[o, \underline{z}]=[x, \underline{z}]$ we have $[u, v][x, y] \parallel u[x, \underline{z}]$.

Case 2: Let $x \neq o$. (See figure 17.) We will again use the

Figure 17

Little Pappus Theorem and then use the Little Desargues Theorem twice. Notice that the line on $\underline{v}, [\underline{v}, \underline{y}], [\underline{v}, \underline{z}]$ is parallel to H and that o, v, y are on H . By the work on the previous page, $\underline{v}\underline{y} \parallel y[\underline{v}, \underline{z}]$ and $y[\underline{v}, \underline{y}] \parallel G = o\underline{v}$ so by the Little Pappus Theorem $o[\underline{v}, \underline{z}] \parallel v[\underline{v}, \underline{y}]$. This result together with $x[x, y] \parallel \underline{v}[\underline{v}, \underline{y}] \parallel H = o\underline{v}$ and $[\underline{v}, \underline{y}][x, y] \parallel [\underline{v}, \underline{z}][x, \underline{z}]$ enables us to use the Little Desargues Theorem on the triangles $v, [\underline{v}, \underline{y}], [x, y]$ and $o, [\underline{v}, \underline{z}], [x, \underline{z}]$ to get $o[x, \underline{z}] \parallel v[x, y]$. Finally, we can now use the Little Desargues Theorem on the triangles $o, [x, \underline{z}], u$ and $v, [x, y], [u, v]$ to get the desired result. Since $u[u, v] \parallel [x, \underline{z}][x, y] \parallel H = o\underline{v}$ and $v[u, v] \parallel G = o\underline{u}$ and $o[x, \underline{z}] \parallel v[x, y]$ (from above), the Little Desargues Theorem implies $u[x, \underline{z}] \parallel [u, v][x, y]$.

Lemma 10 gives us information about the point b on G such that $hb \parallel l$, for any line l in the plane (in lemma 10 the line $l = [u, v][x, y]$). With suitable restrictions on the line l (i.e. the points $[u, v], [x, y]$), this information will be very useful. The algebraic interpretation of lemma 10 gives us a hint as to how that information will be used. Algebraically, lemma 10 states that if $x + yb = u + \underline{v}b$ with $\underline{y} \neq \underline{v}$, then $b = (\underline{y} - \underline{v})^{-1} * (u - x)$ (interpreting h as the multiplicative identity). But this was the type of argument that we needed in part 1 of the proof of Wedderburn's Theorem when we were trying to show that $x_i + x_j b$ gave us n^2 distinct points for $i, j = 1, \dots, n$ and b not in the span of the x_i 's. Lemma 10 will

be used in an equivalent way in lemma 12.

Lemma 11 will be the last lemma proved directly from the configuration theorems and the proof is a long one. Lemma's 7 through 11 will then be used to prove lemma's 12 and 13. Lemma's 6 and 13 with the number theoretic results will complete the proof in lemma 14.

Lemma 11: Let $b \in G$, $c \in H'$ and $(u, v), (x, y) \in G \times H$. Then $\sigma_w(z, c) = \tau_b(\sigma_u(x, c), \sigma_y(y, c))$ where $w = \tau_b(u, v)$ and $z = \tau_b(x, y)$.

Proof: Let $r = \sigma_u(x, c)$, $s = \sigma_y(y, c)$, $t = \sigma_w(z, c)$,
 i = the point on G such that $ic \parallel hr$
 j = the point on G such that $jc \parallel hs$
 k = the point on G such that $kc \parallel ht$
 l = the point on G such that $l[u, y] \parallel hb$.

The proof is long partly because there are many special cases to consider. We begin with the case where $b = o$ (see figure 18.) We have, $w = \tau_b(u, v) = \tau_o(u, v) = u$ and $z = \tau_b(x, y) = \tau_o(x, y) = x$. Thus, $\tau_b(r, s) = \tau_o(r, s) = r = \sigma_u(x, c) = \sigma_w(z, c)$ so the lemma holds for $b = o$ and from now on we assume $b \neq o$.

We will now prove: $u[x, c] = l[z, c]$. (1)

If $y = o$ (see figure 19), then l is a point on G and on a line containing the point $[u, y] = [u, o] = u$ which is on G . So $l = u$. Also, $z = \tau_b(x, y) = \tau_b(x, o) = x$. Thus (1) is trivially true in the

case $y=0$.

If $u=x$, then $z=\tau_b(u,y)=$ the point on G such that $z[u,y]\parallel hb$, but this is the point 1. Since $u[u,c]\parallel 1[1,c]\parallel H$, (1) holds.

Figure 18

The Little Pappus and Little Desargues Theorems will show that (1) holds in the case $y \neq 0$ and $u \neq x$. (See figure 20.) The line on $[u, y], [x, y], [z, y]$ is parallel to the line G on u, l, z . Also, $u[u, y] \parallel z[z, y]$ and $[u, y]l \parallel hb \parallel [x, y]z$ ($z = \tau_b(x, y)$), so the Little Pappus Theorem implies $u[x, y] \parallel l[z, y]$.

Considering the triangles $u, [x, c], [x, y]$ and $l, [z, c], [z, y]$, we have $[x, y][z, y] \parallel [x, c][z, c] \parallel ul = G$ and $[x, y][x, c] \parallel [z, y][z, c]$. Because $u[x, y] \parallel l[z, y]$, the Little

Desargues Theorem implies (1).

We will now use (1) to prove the lemma for the special case $v=y$. (See figure 21.) If $v=y$, then $s=\sigma_{\underline{y}}(\underline{y},c)=\sigma_{\underline{y}}(\underline{y},c)=o$ and l =the point on G such that $l[u,y]=l[u,v]\parallel hb$, which is just $r_b(u,v)=w$. Because $r=\sigma_u(x,c)$, (1) implies $hr\parallel u[x,c]\parallel l[z,c]=w[z,c]$. But $t=\sigma_u(z,c)$, so $w[z,c]\parallel ht$. Thus $hr\parallel ht$, so $r=t$ and $r_b(r,\underline{s})=r_b(r,o)=r=t$.

Figure 19

Figure 20

Figure 21

Figure 22

Figure 23

Therefore we may assume $v \neq y$. Thus $s = \sigma_y(y, c) \neq o$ which implies $j \neq o$.

We will now prove the following relations:

$$k[i, j] \parallel w[l, j] \quad (2)$$

$$[u, v][l, j] \parallel hb \quad (3)$$

which will enable us to conclude the proof.

Proof of (2): If $c \neq j$ (see figure 20), then consider the triangles $c, i, [i, j]$ and $[z, c], l, [l, j]$. We have $c[z, c] \parallel [i, j][l, j] \parallel il = G$ and $i[i, j] \parallel l[l, j]$. Also, by the definitions of i and r and (1), $ci \parallel hr \parallel u[x, c] \parallel l[z, c]$. Thus, the Little Desargues Theorem implies $c[i, j] \parallel [z, c][i, j]$. (2) then follows by the Little Desargues Theorem on the triangles $k, c, [i, j]$ and $w, [z, c], [l, j]$, because by definition of k and t , $kc \parallel ht \parallel w[z, c]$.

If $c = j$ (see figure 22), the line on $c, [i, c], [z, c], [l, c]$ is parallel to G which contains the points k, i, w, l . By the definition of k and t , $kc \parallel th \parallel w[z, c]$. By the definition of i, r and (1), $ji \parallel hr \parallel [x, c]u \parallel [z, c]l$. Since $i[i, c] \parallel l[l, c]$, the Shear Theorem implies $k[i, c] \parallel w[l, c]$ which is (2) when $c = j$.

Proof of (3): If $y = o$ (see figure 19), then, as above, $l = u$. $s = \sigma_y(y, c) = \sigma_y(o, c)$ which is the point on G such that $hs \parallel cy$. But $cj \parallel hs$, so $j = y$, and so $v = j$. Thus (3) is satisfied.

in case $y=o$. So we may assume that $y \neq o$.

We will use the Little Desargues Theorem on the triangles j, c, i and $v, [y, c], [y, i]$. (See figure 20). The lines $c[y, c]$, $i[y, i]$ and $G=jy$ are parallel. By the definition of j and s , $jc \parallel hs \parallel v[y, c]$. Also, we have $ci = H \parallel [y, c][y, i]$. The Little Desargues Theorem implies $ji \parallel v[y, i]$. Since $ji \parallel vy$, the points $v, y, [y, i]$ are collinear.

If $l \neq o$, notice triangles $v, [y, i], [l, i]$ and y, y, l . The lines $l[l, i]$, $y[y, i]$ and $H=vy$ are parallel, as are the sides $[y, i][l, i]$ and $yl=G$. Since $v, y, [y, i]$ are collinear, we have $v[y, i] \parallel vy \parallel yy$. Thus the Little Desargues Theorem implies $v[l, i] \parallel yl$. We can now use the Little Desargues Theorem on the triangles $[u, v], v, [l, i]$ and $[u, y], y, l$. We have $[u, v][u, y] \parallel [l, i]l \parallel H=vy$ and $y[u, y] \parallel v[u, v]$. So our previous result allows us to use the Little Desargues Theorem in order to conclude $[u, v][l, i] \parallel [u, y]l$. Because $[u, y]l \parallel hb$ (by the definition of l), we have $[u, v][l, i] \parallel hb$.

If $l=o$ (see figure 23), the line on $[y, i], y, [y, v], [y, y]$ is parallel to H , which includes the points j, o, v, y . Since $[y, y]y \parallel [y, v]v$, $yy \parallel v[y, i]$, and $yo=G \parallel [y, i]j$, the Little Shear Theorem implies $[y, y]o \parallel [y, v]j$. Together with $[u, y][u, v] \parallel [y, y][y, v] \parallel H=oj$, and $[y, y][u, y] \parallel [y, v][u, v]$, the Little Desargues Theorem on the triangles $o, [y, y], [u, y]$, and

$i, [y, v], [u, v]$ implies $[o, j][u, v] = j[u, v] \parallel o[u, y] = l[u, y]$. By the definition of l , we have $l[u, y] \parallel hb$, and so (3) holds.

We will use (2) and (3) to complete the proof. (See figure 20.) By the definitions of i and j , we have $ic \parallel rh$, and $cj \parallel hs$. Together with $ji \parallel ss$, the Shear Theorem implies $ij \parallel rs$. By the definition of k , $kc \parallel th$. As above, $ic \parallel rh$ and $cj \parallel hs$. Thus the Shear Theorem implies $kj \parallel ts$. All three pairs of corresponding sides of the triangles $i, j, [i, j]$ and $r, s, [r, s]$ are parallel, so the converse of Desargues' Theorem implies that $o, [r, s], [i, j]$ are collinear. Thus, the triangles $t, s, [r, s]$ and $k, j, [i, j]$ are perspective from the point o . Because $kj \parallel ts$, and $j[i, j] \parallel s[r, s]$, Desargues' Theorem implies $k[i, j] \parallel t[r, s]$. By (2), we have $t[r, s] \parallel w[l, j]$. By (3) and the definition of w , we have $[l, j][u, v] \parallel hb$ and $w[u, v] \parallel hb$, and so $w[l, j] \parallel hb$. Therefore $t[r, s] \parallel hb$, and so $r_b(r, s) = t$ which proves the lemma.

Algebraically, lemma 11 says:

$$c^{-1}[(u+vb) - (x+yb)] = c^{-1}(u-x) + c^{-1}(v-y)b.$$

This equation could arise in the following situation.

Suppose $x+yb+cb' = u+vb$ where $x, y, u, v \in \{x_i\}$'s as in part one of the proof of Wedderburn's Theorem and let b' be an element not in $S = \{x_i + x_j b \text{ for all } i \text{ and } j\}$. Then we could solve for b' , obtaining $b' = c^{-1}[(u+vb) - (x+yb)] = c^{-1}(u-x) + c^{-1}(v-y)b$.

This equation would then contradict the fact that b' is not

in S . Thus $x+yb+cb' \neq u+vb$ which means that each of the points of the form $x_i+x_jb+x_kb'$ is distinct from any element of S whenever $c \neq 0$. (Each special case in the proof of lemma 11 becomes a special case of the equation above.)

Geometrically, lemma 11 will be used in an analogous way. In lemma 12, lemma 11 will be used to show that a certain mapping is one-to-one.

Lemma 12: Let S and T be subsets of G with $g \in S$ and S contained in T such that $\sigma_a(SxS')$ is contained in S and $\tau_b(TxT)$ is contained in T hold for all $a \in S$ and $b \in T$. Then there exists $m \in \mathbb{N}$ with $|T| = |S|^m$.

Proof: Since $g \in S$, $\sigma_g(g, g) \in S$ and $\sigma_g(g, g) = o$, we have $o \in S$. (See figure 24.)

Figure 24

Let $S_1 = S$. For $i=1,2,3\dots$ choose $b_{i-1} \in T \setminus S_{i-1}$ and put $S_i = \tau_{b_{i-1}}(S_{i-1} \times \underline{S})$. (See figure 25.) We are taking the set of points $S_i \times \underline{S}_1$ and mapping them via τ_{b_1} into the set S_2 where b_1 is an element outside of S_1 but in T . We continue mapping the set of points $S_i \times \underline{S}_1$ into the set S_{i+1} by τ_{b_i} for some $b_i \in T \setminus S_i$.

Figure 25

Since $\tau_b(x, o) = x$ and since $o \in \underline{S}$, we have $S_{i-1} = \tau_{b_{i-1}}(S_{i-1}, o)$ is a subset of $\tau_{b_{i-1}}(S_{i-1} \times \underline{S}) = S_i$. Thus $S = S_1$ is a subset of S_2 which is a subset of...

(S_i is properly contained in S_{i+1} because $b_i \in T \setminus S_i$ but $g = h \in \underline{S}$ and so $\tau_{b_i}(o, h) = b_i \in S_{i+1} \setminus S_i$).

Also, since by hypothesis $\tau_b(TxT)$ is a subset of T for all $b \in T$, then $S_i = \tau_{b_{i-1}}(S_{i-1}xS)$ is a subset of T for all i . Because the S_i 's increase in size and are contained in T , there is some minimal integer m such that $S_m = T$. We will show that the map $\tau_{b_i}: S_i x S \rightarrow S_{i+1}$ is one-to-one. This will enable us to count the number of points in the set S_m .

First we need the following result *: $\sigma_w(S_i x S')$ is a subset of S_i for every $w \in S_i$ and for $i=1, \dots, m$ which we will prove by induction. One of the hypotheses of lemma 11 is that $\sigma_w(S_i x S')$ is a subset of S_i for all $w \in S$. As induction hypothesis, suppose $\sigma_w(S_{i-1} x S')$ is a subset of S_{i-1} for all $w \in S_{i-1}$. Let w and z be any points in S_i and c be any point in S' . Because $S_i = \tau_{b_{i-1}}(S_{i-1} x S)$, $w = \tau_{b_{i-1}}(u, v)$ and $z = \tau_{b_{i-1}}(x, y)$ where $(u, v), (x, y) \in S_{i-1} x S$. Then lemma 11 implies $\sigma_w(z, c) = \tau_{b_{i-1}}(\sigma_u(x, c), \sigma_v(y, c))$.

Because $(x, c), (y, c) \in S_{i-1} x S'$ and $u, v \in S_{i-1}$ ($v \in S$ which is a subset of S_{i-1}), $\sigma_u(x, c), \sigma_v(y, c) \in S_{i-1}$ by the induction hypothesis. So $\sigma_w(z, c) \in \tau_{b_{i-1}}(S_{i-1} x S) = S_i$. Since this is true for any $z, w \in S_i$ and $c \in S'$, we have $\sigma_w(S_i x S')$ is contained in S_i for every $w \in S_i$ and for $i=1, \dots, m$.

This result tells us that the projection by σ_w of each set of points $[S_i x S]$ onto the line G is closed in the sense that each point is mapped to a point of S_i . In terms of τ_b , this

We are now ready to prove that the map $\tau_{bi}: S_i \times \underline{S} \rightarrow S_{i+1}$ is one-to-one. Suppose $\tau_{bi}(x, y) = \tau_{bi}(u, v)$ for $(x, y), (u, v) \in S_i \times \underline{S}$.

The diagram shows two intersecting planes, \$S\$ and \$H\$, meeting at a horizontal line representing their intersection. Plane \$S\$ is indicated by a bracket labeled \$S\$ below the intersection line, and plane \$H\$ is indicated by a bracket labeled \$H\$ above it. On plane \$S\$, there are points \$o\$, \$o_w(x,y)\$, \$w\$, \$b\$, and \$\tau_b(x,y)\$ marked along the intersection line. On plane \$H\$, there is a point \$h\$. A vertical line segment connects \$h\$ to \$o_w(x,y)\$. Another line segment connects \$h\$ to \$w\$. A third line segment connects \$h\$ to \$\tau_b(x,y)\$. The point where the line segment \$hw\$ intersects the intersection line is labeled \$[x,y]\$. There are also some small tick marks and additional labels like \$G\$ and \$I\$ near the right end of the intersection line.

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$[x,y]z \parallel_{hb_i} [u,v]z$. Since $y=v$, we have $[x,y]z \parallel [u,y]z$ and so $x=u$ (see figure 27.)

Figure 27

Thus, τ_{bi} maps each point of $S_i \times S$ onto a distinct point of S_{i+1} . We can now count the points in any S_i .

$|S_2| = |\tau_{b_1}(S_1, xS)| = |S_1 xS| = |S|^2$. Suppose as induction hypothesis that $|S_{i-1}| = |S|^{i-1}$. Then

$$|S_i| = |\tau_{bi-1}(S_{i-1} \underline{xS})| = |S_{i-1} \underline{xS}| = |S|^{i-1} |S| = |S|^i. \text{ Thus, } |T| = |S_m| = |S|^m.$$

In lemma 13, we use lemma 12 twice. First, we let $S=\pi_a^{-1}(a)^\circ$ and $T=G$. Then we let $S=P^\circ$ and $T=\pi_a^{-1}(a)^\circ$.

Lemma 13: There exists an integer n with $|G| = |P^0|^n$. For every $a \in H'$ there exists an integer n_a such that n_a divides n and $|\pi_a^{-1}(a)^0| = |P^0|^{n_a}$.

Proof: Let $a \in H'$. Let $S = \pi_a^{-1}(a)^{\circ}$ and $T = G$. So we have $g \in S$, and S is a subset of $T = G$. Lemma 9a gives us $\sigma_b(S \times S')$ is contained in S for any $b \in S$. Obviously, $\tau_b(T \times T)$ is contained in $T = G$ for all $b \in G$. Thus lemma 12 implies that there exists an integer m_a such that $|G| = |\pi_a^{-1}(a)^{\circ}|^{m_a}$.

Next let $S = P^{\circ}$ and $T = \pi_a^{-1}(a)^{\circ}$. So we know that S and T are subsets of G , $g \in S$, and S is a subset of T . Lemma 9b implies that $\sigma_b(S \times S')$ is a subset of S for all $b \in S$. Lemma 9a gives us the fact that $\tau_b(T \times T)$ is contained in T for all $b \in T$. Thus lemma 12 implies that there exists an integer n_a such that $|\pi_a^{-1}(a)^{\circ}| = |P^{\circ}|^{n_a}$.

Let $n = m_a n_a$. Then $|G| = |P^{\circ}|^n$.

Together, lemmas 6 and 13 give us our final lemma.

Lemma 14: $P = G'$.

Proof: Let $p = |P^{\circ}|$. Since $g \in P$, $p \geq 2$. By lemma 6, there exists a subset M of $H' \setminus \underline{P}$ with $|G'| = |P| + \sum_{a \in M} |G'| / |\pi_a^{-1}(a)|$.

By lemma 13, there exists an integer n such that $|G| = p^n$ and for every $a \in M$ there is an integer n_a which divides n and $|\pi_a^{-1}(a)^{\circ}| = p^{n_a}$. Because $a \in H' \setminus \underline{P}$ and by lemma 3 $\underline{P} = \{a \in H' \text{ such that } \pi_a(G') = a\}$, we know $\pi_a(G') = a$. Thus

$|G| \neq |\pi_a^{-1}(a)^o|$. So by lemma 13 again,

$|G| = |\pi_a^{-1}(a)^o|^{m_a}$ with $m_a=1$ and $n=n_a m_a$. Thus $n=n_a$.

Thus:

$$p^n - 1 = p - 1 + \sum_{a \in M} p^n - 1 / p^{n_a} - 1.$$

Then by the same number theoretic argument on cyclotomic polynomials as was used in part 3 of the proof of Wedderburn's Theorem, we conclude that $n=1$. Thus

$|G| = |P^o|$ and so $G'=P$.

As stated earlier, $G'=P$ implies that the Pappus Theorem holds for any four points on the lines oh and og together with h and g . Since the only requirement on o, h, g was that they be noncollinear, the Pappus Theorem holds for the entire plane.

This completes Tecklenburg's proof that a finite Desarguesian plane is Pappian. With the help of the geometric algebra developed in Chapter 2, we have shown that Tecklenburg's proof is a geometric interpretation of the proof of Wedderburn's Theorem. Each of her fourteen lemmas translates into a crucial algebraic step in the proof of Wedderburn's Theorem. The complicated nature of Tecklenburg's proof, as compared to the standard proof that a finite Desarguesian plane is Pappian, serves to illustrate the power of analytic geometry.

Bibliography

1. E. Artin, Geometric Algebra, Interscience, 1957.
2. L. Batten, Combinatorics of Finite Geometries, Cambridge University Press, 1986.
3. K. Bennett, "Coordinatization of Affine Projective Space," Discrete Mathematics 4 (1973) 219-231.
4. L. Blumenthal, A Modern View of Geometry, W.H. Freeman, 1961.
5. H. Coxeter, The Real Projective Plane, McGraw Hill, 1949, p.145.
6. H. Forder, "Coordinates In Geometry," Auckland University College Bulletin, No. 41 (1953) 5-16.
7. S. Freidberg, A. Insel, and L. Spence, Linear Algebra, Prentice Hall, 1989.
8. Garner, An Outline of Projective Geometry, North Holland, 1981.
9. M. Hall, "Finite Projective Planes," American Mathematical Monthly, 62 Part 2, (1955) 18-24.
10. I. Herstein, Topics in Algebra, Xerox College Publishing, 1975.
11. P. Scherk and R. Lingenberg, Rudiments of Plane Affine Geometry, University of Toronto Press, 1975.
12. A. Seidenberg, "Pappus Implies Desargues," American Mathematical Monthly 83 (1976) 190-192.
13. F. Stevenson, Projective Planes, W.H. Freeman, 1972.
14. Tecklenburg, H., "A Proof of the Theorem of Pappus in Finite Desarguesian Affine Planes," J.Geom. 30 (1987), no.2, 172-181.
15. L. Wlodarski, "On the Pappus, Desargues and Related Theorems of Affine Geometry" Bulletin De L'Academie Polonaise Des Sciences, Serie des sciences mathematiques Vol XXVII, No. 11-12, (1979) 869-873.