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Some trees and combinatorial principles in ZFC and L

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SOME TREES AND COMBINATORIAL PRINCIPLES
IN
ZFC AND L

A Thesis
Presented to
The Faculty of the Department of Mathematics
San Jose State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

By
Bernadette E. Moise
December, 1992

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ABSTRACT
SOME TREES AND COMBINATORIAL PRINCIPLES
IN ZFC AND L

by Bernadette E. Moise

This thesis is a study of several tree structures in set theory and some related problems and consistency results. The study is divided into two parts: Part I involves the discussion and construction of κ -Souslin, κ -Aronszajn, and κ -Kurepa trees. It also examines Souslin's problem of characterizing the real numbers and shows how ω_1 -Souslin trees are used to address the equivalency of separability and ccc; a model is presented in which they are equivalent and another in which they are not.

Part II contains the development of the constructible universe, including a discussion of well-founded sets and absoluteness. This all leads up to the proof of the combinatorial principles used in the construction of these trees and establishes the relative consistency of AC, GCH, $\Diamond$, and $\Diamond^+$.

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INTRODUCTION

The purpose of this thesis is to study several tree structures in set theory and some related problems and consistency results. The study is divided into two parts: Part I (Chapters 1 - 6) involves the discussion and construction of κ -Souslin, κ -Aronszajn, and κ -Kurepa trees. Chapter 1 includes initial definitions and a general discussion of these trees and related notions. In Chapter 2 we discuss Souslin's problem and prove that there exists a Souslin line if and only if there exists a Souslin tree. However, it turns out that Souslin's problem is undecidable in ZFC. To demonstrate this, in Chapter 3 we construct an ω_1 -Souslin tree in the constructible universe, L , using the combinatorial principle $\Diamond$; and then in Chapter 4, we discuss a model, using Martin's Axiom, in which such trees do not exist. In Chapter 5, we construct Aronszajn trees in ZFC; and in Chapter 6, using $\Diamond^+$, we construct an ω_1 -Kurepa tree, .

Part II (Chapter 7 - 13) contains the development of the constructible universe and the combinatorial principles used in the construction of these trees. This discussion includes the development of the class of well-founded sets (Chapter 7) and the concept of absoluteness (Chapter 8). We then carefully define L (Chapters 9) and prove first that L is a model of ZF (Chapter 10) and then that the Axiom of Constructibility ($V = L$) holds in L (Chapter 11). We conclude by establishing the relative consistency of AC, GCH, $\Diamond$, and $\Diamond^+$.

Chapter 1

TREES

DEFINITION 1.1: A **linear order** is a pair $\langle A, < \rangle$ where A is set and $<$ is a binary relation on A such that the following properties hold:

- a. $\forall x, y, z \in A (x < y \wedge y < z \rightarrow x < z)$. (transitivity)
- b. $\forall x \in A (\neg(x < x))$. (irreflexivity)
- c. $\forall x, y, z \in A (x = y \vee x < y \vee y < x)$. (trichotomy)

DEFINITION 1.2: A linear order, $\langle A, < \rangle$, is a **well-order** iff

$$\exists x \in A \forall y \in A (x \neq y \rightarrow x < y).$$

DEFINITION 1.3: A **partial order** is a pair, $\langle \mathbb{P}, \leq \rangle$, where the $\mathbb{P}$ is a nonempty set and $\leq$ is a binary relation on $\mathbb{P}$ such that the following properties hold:

- a. $\forall p, q, r \in \mathbb{P} (p \leq q, q \leq r \rightarrow p \leq r)$ (transitivity)
- b. $\forall p \in \mathbb{P} (p \leq p)$ (reflexivity)

$p \leq q$ is read " q **extends** p ". Elements of $\mathbb{P}$ are called **conditions**.

(We will use $\mathbb{P}$ to represent $\langle \mathbb{P}, \leq \rangle$ when the meaning is clear.)

DEFINITION 1.4: $C \subseteq \mathbb{P}$ is a **chain** iff it is linearly ordered by $\leq$.

DEFINITION 1.5: In $\langle \mathbb{P}, \leq \rangle$, $\forall p, q \in \mathbb{P}$,

- a. $p \sim q$, read " p and q are compatible," if $\exists r \in \mathbb{P} (r \geq p \wedge r \geq q)$.

b. $p \perp q$, read "p and q are incompatible," if

$$\neg (\exists r \in \mathbb{P} (p \leq r \wedge q \leq r)).$$

DEFINITION 1.6: $A \subseteq \mathbb{P}$ is an **antichain** in $\mathbb{P}$ iff $\forall p, q \in A (p \neq q \rightarrow p \perp q)$.

DEFINITION 1.7: $\langle \mathbb{P}, \leq \rangle$ satisfies the **countable chain condition (ccc)** iff every antichain in $\mathbb{P}$ is countable.

Examples

1. Any countable partial ordering satisfies ccc since $\mathbb{P}$ has only countably many elements.
2. Any linear ordering, and in particular $\langle \mathbb{R}, \leq_{\mathbb{R}} \rangle$, satisfies ccc because every pair of elements is compatible.
3. $\langle \text{the set of nonempty open intervals in } \mathbb{R}, \subseteq \rangle$ satisfies ccc because two intervals are incompatible only if they are disjoint and there are only countably many disjoint open intervals in $\mathbb{R}$.
4. $\langle {}^{<\omega}\omega, \subseteq \rangle$ satisfies ccc because ${}^{<\omega}\omega$, the set of finite sequences of natural numbers, is countable.
5. $\langle {}^{<\omega_1}\omega_1, \subseteq \rangle$ does not satisfy ccc. Although all the sequences of this set are of countable lengths, there are uncountable antichains: the set

$$\text{of functions } A = \{f_\alpha : \alpha < \omega_1\}, \text{ where } f_\alpha(\beta) = \begin{cases} 0 & \text{if } \beta < \alpha \\ 1 & \text{if } \beta = \alpha \end{cases}$$

with $\text{dom}(f_\alpha) = \{\beta : \beta \leq \alpha\}$, is uncountable, and no two elements are compatible.

DEFINITION 1.8: A **tree** is a partial order $\langle T, < \rangle$ such that $\forall x \in T$, the set $\{y \in T : y < x\}$ is well ordered by $<$.

DEFINITION 1.9: If $\langle T, < \rangle$ is a well ordering, then **type** $(T, <)$ is the unique ordinal α such that $\langle T, < \rangle \cong \langle \alpha, \in \rangle$.

DEFINITION 1.10: Let T be a tree.

- a. If $x \in T$, then the **height of x** is defined by:

$$\text{ht}(x) = \text{ht}(x, T) = \text{type}(\{y \in T : y < x\}, <).$$

- b. The α^{th} **level** of T is defined by: $\text{Lev}_\alpha(T) = \{x \in T : \text{ht}(x) = \alpha\}$.

- c. The **height** of T , denoted $\text{ht}(T)$, is the least α such that

$$\text{Lev}_\alpha(T) = \emptyset.$$

$$\text{NOTE: } \text{ht}(T) = \sup\{\text{ht}(x) + 1 : x \in T\}.$$

- d. $T^* \subset T$ is a **subtree** of T iff $\forall x \in T^*$ if $y \in T$ with $y < x$, then $y \in T^*$.

$$\text{NOTE: If } T^* \text{ is a subtree of } T, \text{ then } \forall x \in T^* (\text{ht}(x, T) = \text{ht}(x, T^*)).$$

- e. A **root** of tree T is any element on $\text{Lev}_0(T)$.

EXAMPLES:

1. (Trivial) Let T be any set and $<$ be the empty relation. Then $\text{ht}(x) = 0$
 $\forall x \in T$ ($\text{ht}(T) = \sup_{x \in T} \{\text{ht}(x) + 1\} = 1$), provided that $T \neq \emptyset$.
2. (Trivial) Let T be any ordinal δ with $< = \in$. Then $\text{ht}(\alpha, \delta) = \alpha \forall \alpha \in \delta$,
and $\text{ht}(\delta) = \delta$.
3. $T = {}^{<\delta}I = \bigcup_{\alpha < \delta} {}^\alpha I$ is the **complete I -ary tree**. The elements of ${}^\alpha I$ are
sequences from I of length α . We define $s < t$ in ${}^{<\delta}I$ by $s \subset t$ iff t

extends s . If $\alpha < \delta$, then $\text{Lev}_\alpha(<^\delta I) = {}^\alpha I$ and $\text{ht}(<^\delta I) = \delta$. In particular, for $T = \langle {}^{<\omega} 2, \subseteq \rangle$, $\text{ht}(x) = \text{dom}(x) \ \forall x \in {}^{<\omega} 2$; $\text{Lev}_\alpha(T) = {}^\alpha 2$; and $\text{ht}(T) = \omega_1$. $T_1^* = {}^\alpha 2$ and $T_2^* = \{x \in {}^{<\omega} 2 : \forall \beta (\beta < \text{ht}(x) \wedge \beta \text{ is even} \rightarrow x(\beta) = 0)\}$ are subtrees of T .

DEFINITION 1.11: Let T be a tree. Then we define the following:

- a. A **chain** is a linearly ordered subset of T .
NOTE: A chain may or may not go to the top of the tree.
- b. A **branch** is a maximal chain.
NOTE: A branch, also, may or may not go to the top of the tree.
- c. A **path** is a branch which goes to the top of the tree.
- d. A **κ -chain** (κ -branch, κ -sequence, κ -level, κ -antichain, etc.) is a chain (branch, sequence, level, antichain, etc.) of length κ .
- e. An **antichain** is a subset of T composed of pairwise incompatible elements. (i.e. $A \subset T$ such that $\forall x, y \in A (x \neq y \rightarrow x \perp y \wedge y \perp x)$).

DEFINITION 1.12: For any regular cardinal κ , a **κ -tree** is a tree of height κ such that $\forall \alpha < \kappa (|\text{Lev}_\alpha(T)| < \kappa)$.

NOTE: Since each level of such a tree is of size $< \kappa$, these trees are narrow.

EXAMPLES:

1. $T = \langle {}^{<\omega} 2, \subseteq \rangle$ is not an ω_1 -tree because there are uncountable size levels in this tree: $\text{Lev}_\omega(T) = {}^\omega 2$, the set of all functions $f: \omega \rightarrow 2$, has 2^ω many such functions, and $2^\omega \geq \omega_1$.

2. $T = \langle {}^{<\omega} 2, \subseteq \rangle$ is an ω -tree because $\forall n \in \omega$ there are 2^n such functions; each level is finite.

Before going into the discussion of specific types of trees, we will give several definitions which will be needed for the discussion. Most of the definitions of specific types of large cardinals appear in Appendix B. The definitions for Mahlo, weakly compact, and ineffable cardinals appear in Appendix C. The combinatorial principle $\square_\kappa$ is stated here because it is not used elsewhere in this paper. The combinatorial principles which are actually used to prove notions in this paper are discussed in greater detail in later chapters.

DEFINITION 1.13: If S is a set of sentences, then S is **consistent**, written $\text{Con}(S)$, iff no formula ϕ exists such that S proves both ϕ and $\neg\phi$.

DEFINITION 1.14: Sets of sentences S and T are **relatively consistent** iff the theory in which we are working can prove $\text{Con}(S) \Leftrightarrow \text{Con}(T)$ where our theory can neither prove nor disprove either $\text{Con}(S)$ or $\text{Con}(T)$.

NOTE: This definition becomes necessary because Gödel proved that it is impossible to prove that a theory is consistent from within the theory itself. However, it is often possible to prove one set of axioms is consistent with another set of axioms within the theory.

DEFINITION 1.15: Let μ be an infinite ordinal. Then $C \subseteq \mu$ is **closed** iff $\forall \alpha < \mu (C \cap \alpha \text{ is cofinal in } \alpha \Rightarrow \alpha \in C)$ (i.e. C contains all of its limit points).

DEFINITION 1.16: $C \subseteq \mu$ is **unbounded** iff C is cofinal in μ .

DEFINITION 1.17: $C \subseteq \mu$ is **c.u.b.** iff C is closed and unbounded.

DEFINITION 1.18: $\square_\kappa$ is the following statement: there exists a sequence $\langle C_\lambda : \text{limit } \lambda < \kappa^+ \rangle$ such that:

- a. C_λ is c.u.b in λ ;
- b. if $\text{cf}(\lambda) < \kappa$, then $|C_\lambda| < \kappa$; and
- c. if γ is a limit point of C_λ , then $C_\lambda = C_\gamma \cap \gamma$.

NOTE: $\square_\kappa$ is actually a combinatorial principle which can be proved to hold for all infinite cardinals κ if $V = L$. The proof of $\square_\kappa$ can be found in Devlin's Constructibility, IV.5.

We will begin our discussion of specific types of trees with Souslin trees.

DEFINITION 1.19: A κ -**Souslin tree** is a tree of size κ which has no κ -chains nor κ -antichains. We will refer to ω_1 -Souslin trees as Souslin trees.

NOTE: κ -Souslin trees have nodes arbitrarily high in κ , but no chains go all the way to the top.

LEMMA 1.20: If $\langle T, < \rangle$ is a κ -Souslin tree for κ regular, then $\langle T, < \rangle$ is a κ -tree.

PROOF: T is a κ -Souslin tree implies that $|T| = \kappa$, but no chains nor

antichains are of length κ . To show that $\text{ht}(T) = \kappa$, assume that $\text{ht}(T) < \kappa$. Also, we have that $\forall \alpha < \text{ht}(T) (|\text{Lev}_\alpha(T)| < \kappa)$ and $T = \bigcup_{\alpha < \text{ht}(T)} \text{Lev}_\alpha(T)$. Now κ is regular implies that $|\bigcup_{\alpha < \text{ht}(T)} \text{Lev}_\alpha(T)| < \text{ht}(T)$ which contradicts our hypothesis that $|T| = \kappa$. Therefore, $\text{ht}(T) = \kappa$. $\square$

LEMMA 1.21 (König): If $\langle T, < \rangle$ is an ω -tree, then T has a chain of length ω .

PROOF (by induction on $\alpha \in \omega$): Pick $x_0 \in \text{Lev}_0(T)$. Since T is an ω -tree, $\text{ht}(T) = \omega$. Then there are ω many levels above $\text{Lev}_0(T)$ and infinitely many nodes. Since every node in the tree is an extension of some x_0 , there exists x_0 's with infinitely many extensions; pick one such x_0 . Then, $\{y \in T : y \geq x_0\}$ is infinite. Next, recursively define chain $\langle x_n : n \in \omega \rangle$ by:

$x_n \in \text{Lev}_n(T)$, $x_{n+1} > x_n$, and $\{y \in T : y \geq x_{n+1}\}$ is infinite (this is possible since $|T| = \omega$ and $|T|_n$ is finite $\forall n \in \omega$). Now assume that there exists an $m \in \omega$ such that m is the highest level which our chain reaches. Then

$|\bigcup_{n=0}^m \text{Lev}_n(T)| < \omega$ since each level is finite and m is finite. But this contradicts

that $|\bigcup_{n=0}^m \text{Lev}_n(T)| = |T| = \omega$. Therefore, there is no highest level which implies

our chain reaches infinitely many levels. Hence, the length of this chain must be ω . $\square$

COROLLARY 1.22: There are no ω -Souslin trees.

PROOF: ω is regular so an ω -Souslin tree is an ω -tree (Lemma 1.20).

Furthermore, an ω -tree has a chain of length ω (Lemma 1.21). But ω -Souslin

trees cannot have a chain of length ω by definition. Therefore, there are no ω -Souslin trees. $\square$

THEOREM 1.23: If κ is singular, then there exists a κ -Souslin tree.

PROOF: Let $\text{cf}(\kappa) = \lambda < \kappa$. Let sequence $\langle \alpha_\delta : \delta < \lambda \rangle$ be cofinal in κ where each α_δ is an ordinal in κ . Now take the disjoint union of the α_δ 's which will have the following ordering: if x and y are in the same α_δ , they'll inherit the order from the α_δ ; otherwise, they are incompatible. This partial order has no chain of length κ because the length of each α_δ is $< \kappa$; and there are no antichains of length κ since there are only λ many disjoint α_δ . However, we can still get to the top because each $\langle \alpha_\delta : \delta < \lambda \rangle$ is cofinal in κ . $\square$

Souslin was concerned with characterizing the ordering of the real numbers. Cantor had proved that the reals could be characterized up to isomorphism by the following three properties:

- a. X has no first or last element.
- b. X is connected in the order topology.
- c. X is separable in the order topology.

This characterization, along with Lemma 2.9 that separability implies ccc, lead Souslin to propose the following question: Are the reals characterized up to isomorphism by:

- a. X has no first or last element,
- b. X is connected in the order topology, and
- c. X is ccc (see Appendix B)?

In other words, Souslin asked whether or not separability and ccc are equivalent in context of properties (a) and (b). It turns out that this question is undecidable in ZFC. In fact, this question is still undecidable in both $ZFC + GCH$ and $ZFC + \neg GCH$. Jensen, however, was able to show that $\text{Con}(ZFC) \Leftrightarrow \text{Con}(ZFC + GCH + \text{"there does not exist an } \omega_1\text{-Souslin tree"})$. In Chapter 2, we shall prove that separability and ccc are not equivalent if there exists an ω_1 -Souslin tree. In Chapter 3 we shall present Jensen's proof that in the constructible universe L , there exists an ω_1 -Souslin tree; hence, separability and ccc are not equivalent in L . Solovay and Tennenbaum, on the other hand, showed that $MA + \neg CH$ implies the nonexistence of an ω_1 -Souslin tree; in this case, separability and ccc are equivalent. This will be discussed in Chapter 4.

We would like to mention here several facts about higher Souslin trees which have been established in the last 25 years, though their proofs are beyond the scope of this thesis. Using $\square_\kappa$, Jensen constructed a κ^+ -Souslin tree in L for $\kappa \geq \omega_1$. Gregory showed that $\square_{\omega_1} + GCH$ implies the existence of an ω_2 -Souslin tree. This, along with Jensen's proof that " κ is not a Mahlo cardinal in L " $\Rightarrow \square_\kappa$, led Gregory to the conclusion that

$$\begin{aligned} \text{Con}(ZFC + GCH + \text{"}\omega_2\text{-Souslin Hypothesis"}) &\Rightarrow \\ \text{Con}(ZFC + \text{"there exists a Mahlo cardinal"}). \end{aligned}$$

In particular, " $GCH + \omega_2$ is not Mahlo in L " $\Rightarrow$ "there exists an ω_2 -Souslin tree" (Devlin, Constructibility, pp. 304 - 305). Jensen also showed that in L , there exists a κ -Souslin tree iff κ is not weakly compact. Jensen and Solovay showed that

$$\text{Con}(\text{ZFC} + \text{GCH} + \neg \square_\kappa) \Leftrightarrow \\ \text{Con}(\text{ZFC} + \text{"there exists a Mahlo cardinal"}).$$

Using forcing, Laver and Shelah proved

$$\text{Con}(\text{ZFC} + \text{"there exists a weakly compact cardinal"}) \Rightarrow \\ \text{Con}(\text{ZFC} + \text{CH} + \text{SH}_{\aleph_2}).$$

The following are many of the questions which are still open concerning higher Souslin trees:

1. Is GCH consistent with the existence of no ω_2 -Souslin trees?
2. Does "there exists no ω_2 -Souslin trees" imply that ω_2 is weakly compact? (We know from Gregory's work that ω_2 must at least be Mahlo.)
3. With which assumptions, if any, is "there exists no $\aleph_\omega^+$ -Souslin trees" consistent?
4. Are $\text{cf}^V(\kappa) > \omega$ and "there exist κ -Souslin tree" and "every constructible κ -Souslin tree is not Souslin in V " consistent?

Next we shall discuss κ -Aronszajn trees.

DEFINITION 1.24: For all regular κ , a κ -**Aronszajn tree** is a κ -tree with no chains of length κ (even though $\text{ht}(T) = \kappa$, there are no κ -chains).

NOTE: κ -Aronszajn trees are generalizations of the κ -Souslin trees: a κ -Aronszajn tree can have a κ -antichain even though all levels are of size $< \kappa$.

THEOREM 1.25: A κ -Souslin tree for κ regular is a κ -Aronszajn.

PROOF: Let κ be regular and T be a κ -Souslin tree. By definition of a κ -Souslin tree, T has no κ -chains; and by Lemma 1.20, T is a κ -tree. Hence, T is a κ -Aronszajn tree. $\square$

NOTES:

1. There are no ω -Aronszajn trees because every ω -tree has a chain of length ω (Lemma 1.21).
2. $\text{Ht}(T) = \sup(\{\text{ht}(x) + 1 : x \in T\})$; therefore, there is no chain whose length equals the height of the tree. However, there does exist a chain of length α for any ordinal $\alpha < \text{ht}(T)$.

It might seem that Lemma 1.21 could be extended to ω_1 -trees: if $\langle T, < \rangle$ is an ω_1 -tree, then T has a chain of length ω_1 . Aronszajn showed the existence of ω_1 -trees with countable levels and no ω_1 -chains, namely ω_1 -Aronszajn trees. The proof that such trees exist is a constructive one; two different constructions, the first by Kunen and the second by Specker, appear in Chapter 5.

Many facts about higher Aronszajn trees have also been established. As discussed by Jech, Specker was able to extend his construction of the ω_1 -Aronszajn tree to κ^+ -Aronszajn trees for κ regular, but he needed GCH to do so (this construction also appears in Chapter 5). Silver showed that

$\text{Con}(\text{ZFC} + \text{"there exists a weakly compact cardinal"}) \Rightarrow$

$\text{Con}(\text{ZGC} + \neg\text{GCH} + \text{"there exists no } \omega_2\text{-Aronszajn trees"})$.

Furthermore, if there exists no ω_2 -Aronszajn trees, then there exists a weakly compact cardinal in L .

DEFINITION 1.26: An ω_1 -Aronszajn tree, T , is **special** iff it is the union of ω many antichains.

The interest in special Aronszajn trees stemmed from interest in cardinal transfer properties. Silver, for example, proved that there exists a sentence with a model $\text{type}(\kappa^*, \kappa)^1$ iff there exists a special κ^+ -Aronszajn tree (Mitchell, p.39). Devlin discusses that the proof of the existence of an ω_1 -Aronszajn tree can be modified to show the existence of a special ω_1 -Aronszajn tree; and then with the added assumptions that κ be regular and that $2^{<\kappa} = \kappa$, this proof can be generalized to show the existence of a special κ^+ -Aronszajn tree. Mitchell showed that "if there is no special κ^+ -Aronszajn trees, then κ^+ is Mahlo in L " (Mitchell, pp. 40 - 41).

The third type of tree that this paper will consider is the Kurepa tree.

DEFINITION 1.27: For regular $\kappa > \omega$, a κ -Kurepa tree is a κ -tree with at least κ^+ many paths.

1. A model of $\text{type}(\kappa^*, \kappa)$ is a model with universe of cardinality κ^+ and the interpretation of the distinguished predicate has cardinality κ .

DEFINITION 1.28: $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is a κ -Kurepa family if $|\mathcal{F}| \geq \kappa^+$ and $\forall \alpha < \kappa$, $|\{A \cap \alpha : A \in \mathcal{F}\}| < \kappa$. (Although $\mathcal{F}$ is very large, below any fixed $\alpha < \kappa$, the number of different $A \cap \alpha$'s is small because there is much duplication.)

In Chapter 6, we will prove that for κ regular, there exists a κ -Kurepa tree iff there exists a κ -Kurepa family; and we will construct an ω_1 -Kurepa tree using the combinatorial principle $\Diamond^+$. Solovay showed that if no ω_1 -Kurepa trees exist, then ω_1 must be inaccessible in L . Silver showed that

$$\text{Con}(\text{"there exists an inaccessible cardinal"}) \Rightarrow \\ \text{Con}(\text{GCH} + \text{"there exists no } \omega_1\text{-Kurepa trees"}).$$

As for larger cardinals, Jensen proved that if κ is ineffable, then no κ -Kurepa trees exist. However, if we assume $V = L$ and that κ is an uncountable regular cardinal but not ineffable, then there exists a κ -Kurepa tree; and thus, if $V = L$, then there exists a κ^+ -Kurepa tree for any infinite cardinal κ .

Chapter 2

SOUSLIN LINES AND SOUSLIN TREES

As we mentioned in Chapter 1, Souslin was concerned with finding minimal characterizations of the reals, other than Cantor's. He questioned whether or not we could substitute the countable chain condition for separability in Cantor's characterization and still characterize the reals, up to isomorphism. In this chapter, we will show that ccc and separability are not equivalent under the conditions that there exists an ω_1 -Souslin tree. We will begin this discussion with a few necessary definitions.

DEFINITION 2.1: An **open interval** in a linear ordering $\langle X, < \rangle$, for $x, y \in X$, is a subset of X of the form $(x, y) = \{z \in X : x < z < y\}$.

DEFINITION 2.2: Let S be a nonempty set and let $\mathcal{T}$ be a class of subsets of S . Then $\mathcal{T}$ is a **topology on S** if $\mathcal{T}$ satisfies the following:

- a. $S, \emptyset \in \mathcal{T}$.
- b. The union of any subclass of $\mathcal{T}$ is also in $\mathcal{T}$.
- c. If $X, Y \in \mathcal{T}$, then $X \cap Y \in \mathcal{T}$ (finite intersections).

The pair $(X, \mathcal{T})$ is called a **topological space** and each $X \in \mathcal{T}$ is called an **open set**.

DEFINITION 2.3: For linear order $\langle X, < \rangle$, the **order topology** on X is the topology with open sets formed by the arbitrary unions of open intervals.

DEFINITION 2.4: Let X be a set. Then subset A is **dense in X** iff $\text{cl}(A) = X$ where the $\text{cl}(A)$ is the intersection of all closed subsets of X containing A .

(In Appendix D, we will show how the definition for a dense partial order and a dense topological space are related.)

DEFINITION 2.5: X is **dense in itself** (or **densely ordered**) iff $\forall x, y \in X$
 $\exists z \in X (x < z < y)$.

DEFINITION 2.6: A topological space is **connected** iff it cannot be represented as the union of two disjoint open sets.

DEFINITION 2.7: A topological space is **separable** iff it contains a countable dense subset.

EXAMPLE: $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

DEFINITION 2.8: A topological space X (and in particular, a linear ordering) has the **countable chain condition (ccc)** iff every family of pairwise disjoint non-empty open subsets (intervals) of X is countable.

EXAMPLES:

1. If X is countable, then every family of pairwise disjoint nonempty subsets of X is countable.
2. If X is uncountable with the discrete topology, then the family of all singletons is uncountable; so X does not have ccc.

LEMMA 2.9: If X is separable, then X has ccc.

PROOF: Let X be separable with a countably dense subset D . Suppose X does not have ccc; i.e., suppose X has uncountably many disjoint open sets. Since D is dense, every one of these sets must contain some point of D . But that's impossible (pigeon hole principle). Therefore, X must have ccc. $\square$

NOTE: The converse of this lemma is false.

DEFINITION 2.10: A **Souslin line** is a linear ordering $\langle X, < \rangle$ which is ccc but not separable.

SOUSLIN'S HYPOTHESIS (SH) 2.11: There are no Souslin lines.

NOTE: Clearly if SH is true, then separability $\Leftrightarrow$ ccc for connected, dense linear orderings.

So another way to pose Souslin's question is: Does SH hold true? We begin this discussion by proving the following theorem:

THEOREM 2.12: If there exists a Souslin line, then there exists a Souslin line $\langle X, < \rangle$ such that:

- a. $\langle X, < \rangle$ is dense in itself.
- b. No nonempty interval in $\langle X, < \rangle$ is separable.

PROOF: Let $\langle Y, < \rangle$ be an arbitrary Souslin line; then Y is not separable. Define $\sim$ on Y by $x \sim y$ if $x = y$ or if the open interval between them is separable.

Claim: $\sim$ is an equivalence relation.

Proof: $\sim$ is reflexive since $x = x$.

$\sim$ is symmetric because (x, y) is separable regardless of which direction one travels.

$\sim$ is transitive because if $S_1 \subset (x, y)$ and $S_2 \subset (y, z)$ are countable dense subsets, then $S_1 \cup \{y\} \cup S_2$ is a countable dense subset in (x, z) . Let $X = \{\sim \text{ equivalence classes of } Y\}$ and order X by declaring that if $I, J \in X$, then $I < J$ if for some (all) $x \in I$ and some (all) $y \in J$, $x < y$.

a. To show that X is dense in itself, we must show that

$$\forall I, J \in X \exists K \in X (I < K < J).$$

Claim i: Each $I \in X$ is separable.

Proof: Let $\mathcal{M}$ be a maximal collection of disjoint open intervals $(x, y) \subset I$.

Since $\langle Y, < \rangle$ is ccc (all Souslin lines are), $\mathcal{M}$ is countable (disjoint intervals form an antichain), so let $\mathcal{M} = \{(x_n, y_n) : n \in \omega\}$. Furthermore, since $x_n, y_n \in I$, (x_n, y_n) is separable (by definition of $\sim$). Hence, (x_n, y_n) contains a countable dense subset; call it D_n . Let

$$D = \bigcup_{n \in \omega} D_n \cup \{x_n : n \in \omega\} \cup \{y_n : n \in \omega\},$$

which is countable and dense; and let $\hat{D} = D \cup \{\text{endpoints of } I\}$. Then, since $\hat{D}$ is a countable dense subset of I , I is separable.

Claim ii: X is dense in itself.

Proof: Fix $I, J \in X$ arbitrarily such that $I < J$. We must find $K \in X$ such that $I < K < J$. So pick $x \in I$ and $y \in J$ such that $x < y$. Assume that no such K exists. Then $(I, J) = \emptyset$, which implies that $(x, y) \subseteq I \cup J$. But I and J are separable by Claim (i), and $x, y \in I \cup J$. Therefore, $x \sim y$. This implies that, $y \in I$ and $y \in J$, contradicting that $I \cap J = \emptyset$. Hence, $\exists K (I < K < J)$.

Therefore, X is dense in itself.

b. To prove that no interval in $\langle X, < \rangle$ is separable, suppose $(I, J) \subset X$ is separable. Let $\{K_n : 2 \leq n < \omega\} \subseteq (I, J)$ be dense. Set $K_0 = I$ and $K_1 = J$. For each n , choose $D_n \subset K_n$ such that D_n is countable and dense in K_n (which is possible since each K_n is separable). Then let $D = \bigcup_{n \in \omega} D_n$ which is countable and dense in $\bigcup_{n \in \omega} K_n$. Now pick $x \in I$ and $y \in J$ where $I \cap J = \emptyset$. Since each K_n is separable, $\bigcup_{n \in \omega} K_n$ is separable, and hence, (x, y) is separable. Therefore, $x \sim y$; this implies that $x, y \in I$ and $x, y \in J$ and thus contradicting that $I \cap J = \emptyset$. Therefore, (I, J) is not separable. Since (I, J) was an arbitrary, non nonempty interval, $\langle X, < \rangle$ is separable.

c. It remains to prove that $\langle X, < \rangle$ is a S-line.

Claim i: $\langle X, < \rangle$ is ccc.

Proof: Suppose not; then there are uncountably many disjoint intervals of equivalence classes. But since the equivalence classes are disjoint subsets of Y , a set composed of one element from each equivalence class forms an uncountable antichain in Y , contradicting that Y is ccc. Therefore, there are only countably many equivalence classes. Hence, $\langle X, < \rangle$ is ccc.

Claim ii: $\langle X, < \rangle$ is a linear ordering.

Proof: This ordering is inherited from $\langle Y, < \rangle$.

Claim iii: $\langle X, < \rangle$ is not separable.

Proof: Suppose there existed a countable dense subset in $\langle X, < \rangle$. Then each equivalence class in that dense subset is separable. Pick a D_n from each I_n and take the $\bigcup D_n$; this union is dense in Y , contradicting that Y is not separable. $\square$

DEFINITION 2.13: A well-pruned tree is a tree T with the following properties:

- a. $|\text{Lev}_0(T)| = 1$.
- b. If $x \in T$ and $\text{ht}(x) < \alpha < \text{ht}(T)$, then $\exists y \in T (h(y) = \alpha \wedge x < y)$.

LEMMA 2.14: If T is a κ -tree, then T has a well-pruned subtree.

PROOF: Let $T^* = \{x \in T : |\{y \in T : y > x\}| = \kappa\}$, the set of all nodes in T with κ many extensions.

Claim i: T^* is a subtree.

Proof: $T^* \subset T$ and $\forall x \in T$, x has κ many extensions. Let y be a predecessor of x ; then $y < x$ and y has κ many extensions. Therefore, $y \in T^*$.

Claim ii: If $x \in T^*$ and $\text{ht}(x) < \alpha < \text{ht}(T)$, then $\exists y \in T^* (h(y) = \alpha \wedge x < y)$.

Proof: Fix $x \in T^*$ and $\alpha \in \text{Ord}$ such that $\text{ht}(x) < \alpha < \text{ht}(T)$.

Set $Y = \{y \in T^* : y < x \wedge \text{ht}(y) = \alpha\}$. Then $Y \subset \text{Lev}_\alpha(T^*)$ and $|\text{Lev}_\alpha(T^*)| < \kappa$.

To show that $\exists y \in Y$ with κ many extensions, let $Z = \{z \in T^* : \text{ht}(z) > \alpha \wedge z < x\}$.

Now x has κ many extensions. Each level of extensions has cardinality $< \kappa$, and there are $\leq \alpha$ many levels; hence, there are $< \kappa$ many. Since κ is regular, the union of fewer than κ many subsets of κ of size $< \kappa$ cannot have cardinality κ ; this implies that the number of nodes below Z has cardinality $< \kappa$. Therefore, there are κ many nodes in Z . Furthermore, $\forall z \in Z \exists y \in Y (z > y)$. Hence, $|\{z \in Z : z > y\}| = \kappa$. Now Y is not empty and $|Y| < \kappa$.

To show that $\exists y \in Y$ with κ many extensions, assume that every $y \in Y$ has $< \kappa$ many extensions, and $|Y| < \kappa$. Then $\bigcup_{y \in Y} \{\text{the extensions of } y\} < \kappa$, contradicting that $\bigcup_{y \in Y} \{\text{the extensions of } y\} = Z$ with $|Z| = \kappa$. Therefore,

$\exists y \in Y$ with κ many extensions. So this y is in T^* , and $\text{ht}(y) = \alpha$.

Hence, T^* is well-pruned. $\square$

LEMMA 2.15: If κ is regular, T is a well-pruned κ -Aronszajn tree, and $x \in T$, then $\forall n < \omega \exists \alpha > \text{ht}(x) (|\{y \in \text{Lev}_\alpha T : y > x\}| \geq n)$.

(i.e. For any $x \in T$, there is a level in T for which the set of all extensions of x on that level, is of size $\geq n$.)

PROOF (by induction on n): Let $n = 2$. Since κ is regular, it is a limit cardinal implying that any $x \in T$ has κ many extensions. Since T is a well-pruned Aronszajn tree, x has an extension on every level above it. Furthermore, all of these extensions cannot be linearly ordered because then there would be a chain of length κ . Therefore, there exists a level with at least 2 extensions of x .

Now suppose that this lemma holds for $n = k$. We must show that it holds for $n = k + 1$. So fix $\alpha > \text{ht}(x)$ with $y_1, \dots, y_k \in \text{Lev}_\alpha(T)$ such that $y_i > x, i = 1, \dots, k$. Let $\beta > \alpha$ with $z_k, z_{k+1} \in \text{Lev}_\beta(T)$ (which is possible by the $n = 2$ case). Then $\forall i < k \exists z_i \in \text{Lev}_\beta(T)$ with $z_i > y_i$ (by induction hypothesis). Therefore, $\{z_1, \dots, z_k, z_{k+1}\}$ is a set of $k+1$ many extensions of x in $\text{Lev}_\beta(T)$ where $\beta > \text{ht}(x)$. $\square$

DEFINITION 2.16: For $S \subset L$, $S^* \subset L$ is a **superset** of S iff $S \subset S^* \subset L$.

DEFINITION 2.17: If $\mathcal{J}$ is a family of sets, then $\mathcal{I}$ is a **maximal disjoint subfamily** of $\mathcal{J}$ iff $\mathcal{I}$ is pairwise disjoint and $\bigcup \mathcal{I} = \mathcal{J}$ (i.e., if you add one more set from $\mathcal{J}$ to $\mathcal{I}$, $\mathcal{I}$ will no longer be pairwise disjoint).

REMARK 2.18: Any superset of a dense subset is also dense.

THEOREM 2.19: There is an ω_1 -Souslin tree iff there is a Souslin line.

PROOF:

($\Rightarrow$) Let T be an ω_1 -Souslin tree. Since ω_1 is regular, T is an ω_1 -tree (Lemma 1.20) and, hence, an ω_1 -Aronszajn tree. Furthermore, T can assumed to be well-pruned (Lemma 2.14). Let

$$L = \{C \subset T : C \text{ is a maximal chain in } T\}$$

(by maximal we mean that there is no chain $C^* \subset T$ such that $C \subset C^*$). Then for each $C \in L$, there is an ordinal $h(C) = \text{ht}(C)$ with the following properties:

- a. C contains exactly one element from each $\text{Lev}_\alpha(T)$ for $\alpha < h(C)$.
- b. C contains no elements from $\text{Lev}_\alpha(T)$ for $\alpha \geq h(C)$.
- c. $h(C) < \omega_1$ since T is Aronszajn.
- d. $h(C)$ is a limit ordinal.

For each $\alpha < h(C)$, let $C(\alpha)$ be the element of C on $\text{Lev}_\alpha(T)$; and for $C, D \in L$ such that $C \neq D$, let $d(C, D)$ be the least α such that $C(\alpha) \neq D(\alpha)$.

Claim i: $d(C, D) < \min(h(C), h(D))$.

Proof: Since $C \neq D$, C and D must split at some point; otherwise $C \subset D$ (w.l.o.g.) in which case C is not maximal. Therefore, there exists $\gamma \in C$ and $\delta \in D$ such that $d(C, D) < \gamma \leq h(C)$ and $d(C, D) < \delta \leq h(D)$.

Next we must define the order $\triangleleft$ on L : let $\angle$ be an arbitrary linear ordering of T ; then $C \triangleleft D$ if $C(d(C, D)) \angle D(d(C, D))$ ($C \triangleleft D$ is a lexicographic order on L where the lowest element in $C \setminus D$ is less than the lowest element $D \setminus C$).

Claim ii: $\langle L, \triangleleft \rangle$ is a Souslin line.

Proof: First we must show that $\langle L, \triangleleft \rangle$ is a linear ordering: $\triangleleft$ is transitive and irreflexive, and trichotomy holds in L . Let $C, D, E \in L$ such that $C \triangleleft D$ and $D \triangleleft E$. If $d(C, D) = d(D, E) = d(C, E)$, transitivity is inherited from $\angle$. If $d(D, E) < d(D, C)$, then $C = D$ at $d(D, E)$, so $d(D, E) = d(C, E)$. Therefore, $D \triangleleft E$ which implies $C \triangleleft E$. Similarly, if $d(C, D) < d(D, E)$, then $D = E$ and $C \triangleleft D$ which implies that $C \triangleleft E$. Therefore, $\triangleleft$ is transitive. $\neg C \triangleleft C$ follows from irreflexive property of $\angle$. To show trichotomy, assume that $C \neq D$ and $\neg(C \triangleleft D)$ (w.l.o.g.). Then $\neg(C(d(C, D)) \angle D(d(C, D)))$. Therefore, $D(d(C, D)) \angle C(d(C, D))$ which implies that $D \triangleleft C$. Hence, $\triangleleft$ linearly orders L .

To show that L has ccc, we must show that it does not have any uncountable family of disjoint intervals. Suppose that $\{(C_\xi, D_\xi) : \xi < \omega_1\}$ is a family of disjoint intervals of L . Pick $E_\xi \in (C_\xi, D_\xi)$ and pick α_ξ such that $\max\{(C_\xi, E_\xi), (E_\xi, D_\xi)\} < \alpha < h(E_\xi)$.

Claim: $\{E_\xi(\alpha_\xi) : \xi < \omega_1\}$ forms an uncountable antichain in T .

Proof: Assume $\{E_\xi(\alpha_\xi) : \xi < \omega_1\}$ is not an antichain. Then there exists a ξ and a ζ such that $E_\xi(\alpha_\xi) \sim E_\zeta(\alpha_\zeta)$. Without loss of generality (w.l.o.g.), assume $E_\xi(\alpha_\xi) \leq E_\zeta(\alpha_\zeta)$ which implies that $\alpha_\xi \leq \alpha_\zeta$. Therefore, E_ξ and E_ζ agree below α_ξ and $\alpha_\xi > d(E_\xi, D_\xi)$. Hence, $d(E_\xi, D_\xi) = d(E_\zeta, D_\xi)$ implying that $E_\xi = E_\zeta$. Furthermore, $E_\xi \triangleleft D_\xi$, so $E_\zeta \triangleleft D_\xi$. By a similar argument $C_\zeta \triangleleft E_\zeta$. So $E_\zeta \in (C_\xi, D_\xi)$, and, by construction, $E_\zeta \in (C_\zeta, D_\zeta)$ contradicting that (C_ξ, D_ξ) and (C_ζ, D_ζ) are disjoint. Therefore, $\{E_\xi(\alpha_\xi) : \xi < \omega_1\}$ forms an uncountable antichain in T . But T is a Souslin tree which can have no

ω_1 - antichains. Therefore, there does not exist a family of ω_1 many disjoint intervals in L .

Hence, L is ccc.

Now $L = \bigcup_{\delta < \omega_1} C_\delta$; therefore, any countable dense subset of L will be contained in some C_δ . So, to show that L is not separable, it suffices to show that $\forall \delta < \omega_1$, $C_\delta = \{C : h(C) < \delta\}$ is not dense in L . So fix $\delta < \omega_1$ and $x \in \text{Lev}_\delta(T)$. Then $\exists \alpha > \delta$ such that $\text{Lev}_\alpha(T)$ contains three distinct extensions of x : $y \in D$, $w \in E$, and $w \in F$ where $D, E, F \in L$. Suppose, w.l.o.g., that $D \triangleleft E \triangleleft F$. Then (D, F) is not empty. Furthermore, $x \in D \cap F$ since D and F agree on $\text{Lev}_\delta(T)$ and everywhere below x . Now since D and F split at x , everything between them must also split at x or above. Therefore, for every chain $E \in (D, F)$ ($h(E) > \delta$), thus implying that $E \notin C_\delta$. Therefore, C_δ is not dense in L . Since δ was arbitrary, C_δ is not dense in L . Furthermore, since every countable dense set is contained in some C_δ there is no dense subset of L (by contrapositive of Remark 2.18).

($\Leftarrow$) Suppose $\langle L, \triangleleft \rangle$ is a Souslin line such that $\langle L, \triangleleft \rangle$ is dense in itself and no nonempty interval in $\langle L, \triangleleft \rangle$ is separable (by Theorem 2.12). Let $\mathcal{J} = \{(C, D) : C \triangleleft D \wedge C, D \in L\}$. (Note that $\mathcal{J}$ is the set of all nonempty open intervals of L ordered by reverse inclusion.) To define $T \subset \mathcal{J}$, $\forall \beta < \omega_1$, we'll construct $\mathcal{I}_\beta \subset \mathcal{J}$ such that:

- Intervals in $\mathcal{I}_\beta$ are pairwise disjoint.
- $\bigcup_{\beta < \omega_1} \mathcal{I}_\beta$ is dense in L .
- If $\alpha < \beta$, $I \in \mathcal{I}_\beta$, and $J \in \mathcal{I}_\alpha$, then either $I \cap J = \emptyset$, or $J \subset I$ and

$I \setminus \text{cl}(J) \neq \emptyset$. (This third property guarantees that J is a proper subset.)

(The actual construction will appear at the end of the proof)

Define $T = \bigcup_{\beta < \omega_1} \mathcal{T}_\beta$.

Claim i: T is a tree (i.e. T is well ordered).

Proof: First we will show that $\mathcal{T}_\beta = \text{Lev}_\beta(T)$. Clearly, $\text{Lev}_\beta(T) \subset \mathcal{T}_\beta$. So we must show that $\mathcal{T}_\beta \subset \text{Lev}_\beta(T)$. (Note that $\text{Lev}_\beta(T) = \{J : \text{order type } \{I : I < J\} = \beta\}$.)

To show that $\forall J \in \mathcal{T}_\beta \forall \alpha < \beta \exists I \in \mathcal{T}_\alpha$ such that $J \subset I$, let $J \in \mathcal{T}_\beta$ be arbitrary and fix $\alpha < \beta$. Then $\exists x \in J$ such that $x \in \bigcup \mathcal{T}_\alpha$, and $\bigcup \mathcal{T}_\alpha$ is dense in L .

Therefore, $\exists I \in \mathcal{T}_\alpha$ such that $x \in I$, so $I \cap J \neq \emptyset$. Therefore, $J \subset I$ (by property (c) in the definition of T).

To show that for $J \in \mathcal{T}_\beta$, $\text{type}(\{I : I < J\}) \geq \beta$, note that since $\forall \alpha < \beta \exists I (I < J)$, we can define an order preserving function $f: \beta \rightarrow \{I : I < J\}$ by $f(\alpha) = I_\alpha$ where $I_\alpha \in \mathcal{T}_\alpha$ and $I_\alpha < J$. Furthermore, each such I_α is unique because the predecessors of J form a chain. Therefore, there are β many such I_α 's.

To show that $\mathcal{T}_\beta = \text{Lev}_\beta(T)$, note that since the predecessors of J form a chain, f is an onto function; hence, each I_α is unique. Therefore, $\text{type}(\{I : I < J\}) = \beta$.

Since J was arbitrary, $\forall J \in \mathcal{T}_\beta$, $\text{type}(\{I : I < J\}) = \beta$. Hence, $\mathcal{T}_\beta = \text{Lev}_\beta(T)$.

Now let I_β be any node on $\text{Lev}_\beta(T)$. Then $\{I_\alpha : I_\alpha < I_\beta\}$ is well ordered by the ordinals. Therefore, there is a lowest level of the tree; hence T is well ordered.

Claim ii: If $A \subset T$ is an antichain, then $|A| \leq \omega$.

Proof: Since L is a Souslin-line, it has ccc. So, all antichains in L are countable. Hence, all antichains in T are countable.

Claim iii: Every chain of T is countable.

Proof: Suppose $\{I_\xi : \xi < \omega_1 \wedge I_\xi \in \mathcal{T}_\xi\}$ is an uncountable chain in T . Let $\xi < \eta$

such that $I_\xi < I_\eta$ (i.e. $I_\xi \sim I_\eta$) which implies that $(I_\eta \subset I_\xi) \wedge (I_\xi \setminus \text{cl}(I_\eta) \neq \emptyset)$.

So $\{I_\xi \setminus \text{cl}(I_{\xi+1}) : \xi < \omega_1\}$ is a set of pairwise disjoint intervals and, hence, is an uncountable antichain, contradicting that T is ccc. Therefore, every chain is countable.

Claim iv: $|T| = \omega_1$

Proof: $\bigcup_{\beta < \omega_1} \mathcal{I}_\beta$ is dense in L , which implies that $\forall \beta (\bigcup_{\alpha < \beta} \mathcal{I}_\alpha \neq \emptyset)$; hence, $\forall \beta < \omega_1 (\mathcal{I}_\beta \neq \emptyset)$. Since there are ω_1 many such β 's, $|T| = \omega_1$.

So T is a tree of size ω_1 with no chains nor antichains of size ω_1 . Therefore, T is an ω_1 -Souslin tree.

Construction of $\mathcal{I}_\beta$ by induction on β :

Base case: Let $\mathcal{I}_0$ be any maximal disjoint subfamily of $\mathcal{J}$. Then (a) holds by construction and (c) holds vacuously. For (b), assume that $\bigcup \mathcal{I}_0$ is not dense. Then there exists an open interval in its complement, $\mathcal{I}$. Then $\bigcup \mathcal{I}_0 \cup \mathcal{I}^c$ is pairwise disjoint contradicting that $\mathcal{I}_0$ is maximal. Therefore, $\bigcup \mathcal{I}_0$ is dense.

Successor case: Given any $\mathcal{I}_\alpha$, we'll define $\mathcal{I}_{\alpha+1}$ as follows: for each

$I \in \mathcal{I}_\alpha$ let $\mathcal{K}_I$ be a maximal disjoint subfamily of $\mathcal{J}$ defined by

$\mathcal{K}_I = \{K \in \mathcal{J} : K \subset I \wedge I \setminus \text{cl}(K) \neq \emptyset\}$. Now $\mathcal{K}_I$ is nonempty since L is dense in itself. Let $\mathcal{I}_{\alpha+1} = \bigcup \{\mathcal{K}_I : I \in \mathcal{I}_\alpha\}$. Then (a) and (c) hold by construction and $\bigcup \mathcal{K}_I$ is dense in L because $\mathcal{K}_I$ is a maximal disjoint subfamily of I .

Limit case: Let γ be a limit ordinal. Assume that $\mathcal{I}_\alpha$ is defined $\forall \alpha < \gamma$. Let $\mathcal{K} = \{K \in \mathcal{J} : \forall \alpha < \gamma \forall I \in \mathcal{I}_\alpha ((I \cap K = \emptyset) \vee (K \subset I \wedge I \setminus \text{cl}(K) \neq \emptyset))\}$ (i.e. the set of all intervals which are proper subsets of some I in some $\mathcal{I}_\alpha$). Now let $\mathcal{I}_\gamma$ be a disjoint subfamily of $\mathcal{K}$ (note that $\mathcal{I}_\gamma \neq \mathcal{K}$; otherwise, if $K \in \mathcal{K}$ and $K^* \subset K$, then $K^* \in \mathcal{K}$ contradicting that $K \cap K^* \neq \emptyset$).

Now conditions (a) and (b) hold for $\mathcal{I}_\gamma$ by construction. It remains to show that $\bigcup \mathcal{I}_\gamma$ is dense in L . It suffices to show that $\forall J \in \mathcal{J} \exists K \in \mathcal{K}$ such that $K \subset J$. So let E be the set of all endpoints of intervals in $\bigcup_{\alpha < \gamma} \mathcal{I}_\alpha$. Then E is countable because $\gamma < \omega_1$. Let $J \in \mathcal{I}_\alpha$ for some $\alpha < \gamma$. By hypothesis, J is not separable. So fix $K_1 \in \mathcal{J}$ with $K_1 \subset J$ and $K_1 \cap E = \emptyset$ (note that $K_1 \in \mathcal{K}$; otherwise $K_1 \cap E \neq \emptyset$). If $I \in \bigcup_{\alpha < \gamma} \mathcal{I}_\alpha$, then K_1 does not contain endpoints of I . Therefore, $I \cap K_1 = \emptyset$ or $K_1 \subset I$. To insure that $K_1 \neq I$, we let $K \in \mathcal{J}$ such that $K \subset K_1$ and $K_1 \setminus \text{cl}(K) \neq \emptyset$. Then $K \in \mathcal{K}$. So for $J \in \mathcal{J}$, we have found $K \in \mathcal{K}$ such that $K \subset J$. Therefore, $\mathcal{K}$ is dense in L ; and hence $\mathcal{I}_\gamma$ is dense in L . $\square$

Chapter 3

AN ω_1 -SOUSLIN TREE

In this chapter we will construct an ω_1 -Souslin tree. However, we cannot construct this tree from ZFC alone, nor can we construct it with the addition of GCH (Devlin and Johnsbråten showed that neither CH nor GCH imply the existence of an ω_1 -Souslin tree). We need to use the combinatorial principle $\Diamond$, which requires the additional assumption of the Axiom of Constructibility, $V = L$. So the Souslin tree which we will build actually lies in the constructible universe, L . We will hold off until Chapter 13 to prove the validity of $\Diamond$ since it involves the development of L , which begins in Chapter 7.

DEFINITION 3.1: $\Diamond_\kappa$, for regular $\kappa > \omega$, is the statement: There exists a sequence of subsets of κ , $\langle A_\alpha : \alpha < \kappa \rangle$, such that $\forall A \subseteq \kappa$, the set $S_A = \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary in κ (what it means to be stationary is defined and discussed in Appendix C). $\Diamond$ abbreviates $\Diamond_{\omega_1}$, and the sequence $\langle A_\alpha : \alpha < \kappa \rangle$ is called a $\Diamond$ -sequence.

NOTE: $\Diamond_\kappa$ asserts the existence of some sequence of A_α 's, such that for any $A \subseteq \kappa$, $\langle A_\alpha \rangle$ predicts A correctly at a stationary collection of points.

REMARKABLE FACT 3.2: There are 2^κ many subsets of κ , and $\langle A_\alpha \rangle$ has only κ many guesses; yet it manages to anticipate all 2^κ many subsets in at most κ many guesses.

THEOREM 3.3: $\Diamond_\kappa \Rightarrow \text{GCH}$.

THEOREM 3.3: $\Diamond_\kappa \Rightarrow \text{GCH}$.

PROOF: First note that $\{\{\alpha\} : \alpha < \kappa\} \subseteq \{A \subseteq \kappa : |A| < \kappa\}$, that $|\{\{\alpha\} : \alpha < \kappa\}| = \kappa$ and that $|\{A_\alpha : \alpha < \kappa\}| = \kappa$. Also note that $|\{A \subseteq \kappa : |A| < \kappa\}| = 2^{<\kappa}$.

Claim i: $\{A \subseteq \kappa : |A| < \kappa\} \subseteq \{A_\alpha : \alpha < \kappa\}$

Proof: Fix $A \subseteq \kappa$ such that $|A| < \kappa$. We want to find an α such that $A = A_\alpha$.

Since $\langle A_\alpha \rangle$ is a $\Diamond_\kappa$ -sequence, $\forall A \subseteq \kappa$, $\{\alpha < \kappa : A \cap \alpha = A_\alpha\}$ is stationary;

hence, it must intersect every c.u.b. subset of κ . So look at $\{\alpha : \alpha > \sup(A)\}$

(since κ is regular, A is bounded and hence, has a supremum); it is c.u.b.

Therefore, $\{\alpha : \alpha > \sup(A)\} \cap \{\alpha < \kappa : A \cap \alpha = A_\alpha\} \neq \emptyset$. So pick an α in this intersection. For this α , $A \cap \alpha = A$ since $\alpha > \sup(A)$, and $A \cap \alpha = A_\alpha$ since $\alpha \in \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$. Hence, we have found an α such that $A = A_\alpha$. $\square$

COROLLARY 3.4: $\Diamond \Rightarrow \text{CH}$.

PROOF: $\Diamond \Rightarrow 2^{<\omega_1} = \omega_1$ (by Theorem 3.3). Furthermore, Cantor showed that $2^\omega \geq \omega_1$. Now if $\langle A_\alpha : \alpha < \omega_1 \rangle$ is a $\Diamond$ -sequence, then $\forall A \subset \omega \exists \alpha > \omega (A = A_\alpha)$ (as proved in Theorem 3.3). Therefore, $\{A_\alpha : \alpha < \omega_1\} = \mathcal{P}(\omega)$. Since $|\mathcal{P}(\omega)| = \omega_1$, we have that $2^\omega = \omega_1$, which is CH. $\square$

DEFINITION 3.5: A tree is **everbranching** iff $\forall t \in T$, the set $\{t^* \in T : t^* > t\}$ is not linearly ordered by $<_T$.

Our goal is to build a Souslin tree. The next three lemmas allow us to use $\Diamond_\kappa$ to guess at antichains needed in the construction of such trees.

LEMMA 3.6: If T is an everbranching κ -tree and T has no κ -antichains, then T is a κ -Souslin tree.

PROOF: Suppose there exists a κ -chain, call it C . Since T is everbranching, $\forall x_i \in C \exists x_i^* \in T (x_i^* > x_i \wedge x_i^* \notin C)$. Then $\forall x_i \in C$, pick exactly one $x_i^* > x_i$. The collection of these x_i^* 's forms a κ -antichain, contradicting that T has no κ -antichains. Hence, T is a κ -Souslin tree. $\square$

NOTATION 3.7: For any tree T , $T_\alpha = \bigcup_{\beta < \alpha} \text{Lev}_\beta(T)$.

LEMMA 3.8: If $T = \langle T, \triangleleft \rangle$ is an ω_1 -tree, then

- a. $\{\alpha < \omega_1 : T_\alpha = \alpha\}$ is c.u.b. in ω_1 .
- b. If $A \subseteq \omega_1$ is a maximal antichain in T , then the set

$$C = \{\alpha < \omega_1 : (T_\alpha = \alpha) \wedge (A \cap T_\alpha \text{ is a maximal antichain in } T_\alpha)\}$$

is c.u.b. in ω_1 (i.e. the part of A below α is a maximal antichain in T_α where if $\beta \in T_\alpha$, then $\exists \gamma \in T_\alpha \cap A$ such that $\gamma \sim \beta$).

PROOF:

- a. Let $f(\xi) = \text{ht}(\xi)$ and let $g(\xi) = \sup(\{\eta : \eta \in \text{Lev}_\xi(T)\})$. Now $\{\alpha < \omega_1 : T_\alpha = \alpha\}$ is a set of ordinals closed under both f and g : closure under f guarantees that every ordinal $< \alpha$ occurs in T_α , and closure under g sends any ordinal ξ to the supremum of all ordinals up to that level (under ordinal ordering), thus guaranteeing that $\forall \xi < \alpha$, everything in T_ξ is below α . (Note: These are only two ways in which $T_\alpha \neq \alpha$.) By Lemma C.31, the collection of ordinals closed under $< \omega_1$ many functions $f: \omega_1 \rightarrow \omega_1$ is c.u.b. Hence this set is c.u.b.
- b. Define functions f and g as in part (a). Let $h(\xi) =$ the least $\eta \in A$ such

that $\eta \sim \xi$. Then $\xi \in A$ implies that $h(\xi) = \xi$. If α is closed under f , g and h , then $T_\alpha = \alpha$ and $A \cap T_\alpha$ is a maximal antichain in T_α . So C is c.u.b. by Lemma C.30. $\square$

LEMMA 3.9: Let $T = \langle T, \triangleleft \rangle$ be an everbranching ω_1 -tree, and $\langle A_\alpha : \alpha < \omega_1 \rangle$ a $\Diamond$ -sequence. Suppose that for every limit $\alpha < \omega_1$,

$$\begin{aligned} ((T_\alpha = \alpha) \wedge (A_\alpha \text{ is a maximal antichain in } T_\alpha)) \rightarrow \\ \forall x \in \text{Lev}_\alpha(T) \exists y \in A_\alpha (y \triangleleft x). \end{aligned} \quad (*)$$

Then T is an ω_1 -Souslin tree.

PROOF: First note that $\forall \alpha < \omega_1$, if $A_\alpha = A \cap \alpha$, then $|A_\alpha| \leq \omega$.

Claim: Every maximal antichain in T is countable.

Proof: Let A be a maximal antichain in T . Now the set

$C = \{\text{limit } \alpha < \omega_1 : (T_\alpha = \alpha) \wedge (A \cap T_\alpha \text{ is a maximal antichain in } T_\alpha)\}$ is c.u.b. in ω_1 (by Lemma 3.8.b). Also, by $\Diamond$, $\mathcal{A} = \{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary; so $\mathcal{A} \cap C \neq \emptyset$. Fix $\alpha \in \mathcal{A} \cap C$. For this α , $A \cap \alpha = A_\alpha$ (since $\alpha \in \mathcal{A}$) and $T_\alpha = \alpha$ (since $\alpha \in C$). Hence, $A \cap T_\alpha = A_\alpha$. So, A_α is a maximal antichain in T_α (by definition of C). Therefore, this α fulfills the hypotheses of $(*)$; so $\forall x \in \text{Lev}_\alpha(T) \exists y \in A (y \triangleleft x)$. Now if $z \in T$ such that $\text{ht}(z) \geq \alpha$, then z extends some $a \in A_\alpha$. But $A_\alpha = A \cap \alpha$; therefore, $z \notin A$ (otherwise A would not be maximal). Thus, $A = A_\alpha$, so A must be countable.

Then by Lemma 3.6, T is an ω_1 -Souslin tree. $\square$

THEOREM 3.10: $\Diamond \Rightarrow$ "There exists an ω_1 -Souslin tree."

PROOF: Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond$ -sequence where $A \subseteq \omega_1$ and $A \cap \alpha = A_\alpha$.

Let $I_\beta = \{\omega \cdot \beta + n : n \in \omega\}$, a collection of ordinals less than ω_1 and let $\triangleleft$ be a tree ordering on ω_1 . We'll inductively construct a tree $T = \langle \omega_1, \triangleleft \rangle$ with the following properties:

- a. $\text{Lev}_\beta(T) = I_\beta$. (T has ω many roots, and each level of T is an interval from one limit ordinal to the next in ω_1).
- b. $\forall \beta < \omega_1$ and $n < \omega$, each ordinal $\omega \cdot \beta + n$ has exactly two extensions (to insure that T is everbranching):

$$(\omega \cdot \beta + n) \triangleleft (\omega \cdot (\beta + 1) + 2n) \text{ and}$$

$$(\omega \cdot \beta + n) \triangleleft (\omega \cdot (\beta + 1) + 2n + 1).$$

- c. If $\beta < \alpha < \omega_1$, and $x \in I_\beta$, then $\exists y \in I_\alpha$ such that $x \triangleleft y$. (This property insures that every branch goes to the top.)
- d. If α is a limit ordinal and $A_\alpha \subseteq \bigcup_{\beta < \alpha} I_\beta$ is a maximal antichain in $\langle \bigcup_{\beta < \alpha} I_\beta, \triangleleft \rangle$, then $\forall x \in I_\alpha$, $\exists y \in A_\alpha$ such that $y \triangleleft x$. (This insures that T is Souslin.) Note that for limit ordinals α , $T_\alpha = \alpha$ since each level is an interval from one limit ordinal to the next.

To construct T, we'll assume that $\triangleleft$ has been defined $\forall x \in \omega \cdot \alpha$ so that all four properties hold below α . We will extend $\triangleleft$ to include all $x \in \omega \cdot \alpha \cup \text{Lev}_\alpha(T)$.

Successor stage: Let $\alpha = \beta + 1$. Set $(\omega \cdot \beta + n) \triangleleft (\omega \cdot (\beta + 1) + 2n)$ and $(\omega \cdot \beta + n) \triangleleft (\omega \cdot (\beta + 1) + 2n + 1)$. Then extend $\triangleleft$ to be transitive (i.e. all ordinals below $(\omega \cdot \beta + n)$ under $\triangleleft$ are also below both $(\omega \cdot (\beta + 1) + 2n)$ and $(\omega \cdot (\beta + 1) + 2n + 1)$). Clearly, this order is still a tree ordering since all predecessors of new elements are still linearly well ordered. Properties (b) and (c) hold by construction.

Limit stage: Let α be a limit ordinal. Now $\forall x \in \bigcup_{\beta < \alpha} I_\beta$, choose a maximal path $B(x) \subseteq \langle \bigcup_{\beta < \alpha} I_\beta, \triangle \rangle$ with $x \in B(x)$. Using property (c) along with the fact that $\text{cf}(\alpha) = \omega$, we get that $B(x)$ intersects each $\text{Lev}_\beta(T) \forall \beta < \alpha$.² Now α is a limit ordinal $< \omega_1$, so $\text{cf}(\alpha) = \omega$. Using this fact along with property (c), we get that $B(x)$ intersects each I_β .

Now $I_\alpha = [\omega \cdot \alpha, \omega \cdot \alpha + \omega)$. To build $\text{Lev}_\alpha(T)$, we must figure out what points in $\bigcup_{\beta < \alpha} I_\beta$ each point of I_α extends.

Case 1: Suppose that A_α is not a maximal antichain in $\langle \bigcup_{\beta < \alpha} I_\beta, \triangle \rangle$.

Let $\langle x_i : i < \omega \rangle$ enumerate all nodes in $\langle \bigcup_{\beta < \alpha} I_\beta, \triangle \rangle$. Then $\forall x \in B(x_n)$, set $x \triangle \omega \cdot \alpha + n$. We need only to check that properties (a) and (c) hold (A_α is not an antichain so (d) is not applicable). $\triangle$ is still well-ordered by construction: we added something to the top of each branch to insure property (c).

Case 2: Suppose that A_α is a maximal antichain in $\langle \bigcup_{\beta < \alpha} I_\beta, \triangle \rangle$.

Let $\langle x_i : i < \omega \rangle$ enumerate all nodes in A_α . Then $\forall y \in B(x_n)$, set $y \triangle \omega \cdot \alpha + n$. In this case we need to check Properties (a), (c), and (d). To prove that (c) and (d) hold, pick any $x \in \bigcup_{\beta < \alpha} I_\beta = \alpha$. If for some $y \in A_\alpha$, $y \triangle x$, then $x = x_n$; so, set $x \triangle \omega \cdot \alpha + n$. Otherwise, choose $y \in A_\alpha$ such that $x \triangle y$; then $y = x_n$, so set $y = x_n \triangle \omega \cdot \alpha + n$. Thus $x \triangle \omega \cdot \alpha + n$. $\square$

2. To build $B(x)$, let $\langle \alpha_i \rangle_{i < \omega}$ be an increasing cofinal sequence in α such that $\alpha_0 > h(x)$ (we can find such a sequence because α is a countable limit ordinal; thus, $\text{cf}(\alpha) = \omega$). Then construct $\langle x_i \rangle_{i < \omega}$ by letting $x_0 = x$ and x_{i+1} = the least $x \in I_{\alpha_i}$ such that $x \triangle x_i$. Set $B(x) = \{y \in \bigcup_{\beta < \alpha} I_\beta : \exists i < \omega (y \triangle x_i)\}$.

Chapter 4

A MODEL WITH NO ω_1 -SOUSLIN TREES

In this chapter we will develop Martin's Axiom and show that when this axiom holds, there cannot exist an ω_1 -Souslin tree. We begin by giving some definitions which hold for any partial order $\mathbb{P}$.

DEFINITION 4.1: A subset D of $\mathbb{P}$ is **dense** in $\mathbb{P}$ if $\forall p \in \mathbb{P} \exists q \in D$ such that $q \geq p$. $\mathbb{P}$ is **dense in itself** if $\forall p, q \in \mathbb{P}$ such that $p < q$, $\exists r \in \mathbb{P}$ such that $p < r < q$.

DEFINITION 4.2: A subset $\mathcal{F} \subseteq \mathbb{P}$, is a **filter** if the following hold:

- a. All elements in $\mathcal{F}$ are compatible.
- b. If $p \in \mathcal{F}$ and $q \leq p$, then $q \in \mathcal{F}$.

DEFINITION 4.3: If $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$, then a filter $\mathcal{F}$ is **$\mathcal{D}$ -generic** if $\forall D \in \mathcal{D} (D \cap \mathcal{F} \neq \emptyset)$.

MARTIN'S AXIOM (MA) 4.4: $\forall \kappa < 2^\omega$, MA_κ holds where MA_κ is the following property: If $\mathbb{P}$ is a ccc partial order and $\mathcal{D}$ is a collection of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$, then there exists a $\mathcal{D}$ -generic filter on $\mathbb{P}$.

NOTE: Martin's Axiom is a tool used to prove that there exists a path through a tree which has some desired properties. If we can find a collection of dense

subsets of $\mathbb{P}$ which has the desired properties and MA_κ holds, then the path has these properties.

LEMMA 4.5 (ZFC):

- a. If $\kappa^* \leq \kappa$, then $MA_\kappa \rightarrow M_{\kappa^*}$.
- b. M_ω holds.
- c. MA_{2^ω} does not hold.

PROOF:

a. Assume MA_κ . If there exists a $\mathcal{D}$ -generic filter on $\mathbb{P}$ with $|\mathcal{D}| < \kappa$, then this filter must intersect every dense set in $\mathcal{D}$. If we throw enough elements from $\mathcal{D}$ to form $\mathcal{D}^*$ such that $|\mathcal{D}^*| < \kappa^*$, then this filter will still intersect every element of $\mathcal{D}^*$.

b. Let $\mathbb{P}$ be a ccc partial order with $\mathcal{D} = \langle D_n : n \in \omega \rangle$ a collection of countably many dense subsets of $\mathbb{P}$. Define $\langle p_n : n \in \omega \rangle$ recursively by:

- i. p_0 is an arbitrary element of $\mathbb{P}$
- ii. $p_n \leq p_{n+1}$ such that $p_{n+1} \in D_n$

Let $\mathcal{F} = \{q \in \mathbb{P} : \exists i (p_i \geq q)\}$.

Claim i: $\mathcal{F}$ is a filter on $\mathbb{P}$.

Proof: $\mathcal{F} \subseteq \mathbb{P}$. Let $q, q^* \in \mathcal{F}$ be arbitrary. Then $\exists i, j$ such that $q \leq p_i$ and $q^* \leq p_j$ (by definition of $\mathcal{F}$). Also, $p_i < p_j$ whenever $i < j$ by construction.

Therefore, q and q^* are below $p_{\max\{i, j\}} = \min\{p_i, p_j\}$. So $q \sim q^*$.

Claim ii: $\mathcal{F}$ is $\mathcal{D}$ -generic.

Proof: $\forall n, p_{n+1} \in D_n$ and $p_{n+1} \in \mathcal{F}$. Therefore, $D_n \cap \mathcal{F} \neq \emptyset$.

So we have found a $\mathcal{D}$ -generic filter on $\mathbb{P}$ when $|\mathcal{D}| \leq \omega$; therefore MA_ω holds.

c. To show MA_{2^ω} does not hold, we will construct $\mathcal{D}$, a collection of dense subsets of ${}^\omega 2$ with $|\mathcal{D}| \leq 2^\omega$ and show that there is no $\mathcal{D}$ -generic filter. Let $\mathbb{P} = \langle {}^\omega 2, \leq \rangle$ where $p^* < p$ iff $p^* \subseteq p$. Then $\mathbb{P}$ is a ccc partial order where each $p \in \mathbb{P}$ is a sequence of 0's and 1's. Let $f \in {}^\omega 2$ and define the following sets:

$D_f = \{p \in \mathbb{P} : p \not\subset f\}$, the set of all sequences not on the path of f .

$D_n = \{p \in \mathbb{P} : \text{dom}(p) \supseteq n\}$, the set of all sequences of length n or longer.

Let $\mathcal{D} = \{D_f : f \in {}^\omega 2\} \cup \{D_n : n \in \omega\}$.

Claim i: D_f is dense.

Proof: Let $p^* \in \mathbb{P}$. Suppose $\text{dom}(p^*) = n$. Set $p = p^* \cup \{(n, 1 - f(n))\}$. If $p^* \not\subset f$, then $p \in D_f$. If $p^* \subseteq f$, by turning away from f at the next node, $p \in D_f$. Therefore D_f is a dense.

Claim ii: D_n is dense.

Proof: Let $p^* \in \mathbb{P}$. Set $p(i) = \begin{cases} p^*(i) & \text{if } i \in \text{dom}(p) \\ 0 & \text{if } i \in n \setminus \text{dom}(p) \end{cases}$

(p agrees with p^* , and then adds 0's until its length is n .) Then $p^* \leq p$ and $p \in D_n$. Therefore, D_n is dense.

Claim iii: $\mathbb{P}$ does not have a $\mathcal{D}$ -generic filter.

Proof: Suppose $\mathcal{F}$ is a $\mathcal{D}$ -generic filter of $\mathbb{P}$. Then $\bigcup \mathcal{F} : \omega \rightarrow 2$ where $\bigcup \mathcal{F}$ is the path through $\mathbb{P}$ connecting all points of $\mathcal{F}$. Since $\mathcal{F}$ is a filter, all points of $\mathcal{F}$ must be compatible if they are on the same path. Therefore, if $p, q \in \mathcal{F}$,

$p \cup q \in \mathcal{F}$. Thus $\bigcup \mathcal{F} : \alpha \rightarrow 2$ for some $\alpha \leq \omega$. Since $\mathcal{F} \cap D_n \neq \emptyset \ \forall n$, $n \subseteq \text{dom}(\bigcup \mathcal{F})$. So $\bigcup \mathcal{F} : \omega \rightarrow 2$ and $\mathcal{F} \cap D_{\bigcup \mathcal{F}} = \emptyset$, which contradicts the hypothesis that $\mathcal{F}$ is a $\mathcal{D}$ -generic. $\square$

In Chapter 2 we showed that if a topological space is separable, then it has ccc. The question arises: does the product of any two ccc spaces also have ccc? It turns out that it is a theorem of ZFC that if ccc is preserved by the product of two factors, then it is preserved by arbitrary products. It is easy to prove by induction that if ccc is preserved by the product of two factors, then it is preserved by n products. To prove this is also true for arbitrary products, we will need MA_{ω_1} and the Δ -System Lemma.

DEFINITION 4.6: A Δ -system is a family of sets $\mathcal{A}$ such that the pairwise intersection of any two sets in $\mathcal{A}$ is a fixed set called the **root** of $\mathcal{A}$.

Δ -SYSTEM LEMMA 4.7 (a special case): Let $\mathcal{A}$ be an uncountable family of finite sets. Then there is an uncountable sub-family $\mathcal{B}$ of $\mathcal{A}$ which forms a Δ -system. (i.e., We can reduce any uncountable collection of finite sets to an uncountable Δ -system.)

PROOF: It suffices to consider the case of $|\mathcal{A}| = \omega_1$ where each $X \in \mathcal{A}$ is a set of ordinals since the type of sets in $\mathcal{A}$ is irrelevant. Then $|\bigcup \mathcal{A}| = \omega_1$ since X is finite for each $X \in \mathcal{A}$.

Claim i: We may assume that there is a fixed $n \in \omega$ such that $|X| = n$ for all $X \in \mathcal{A}$.

Proof: Let $\mathcal{A}_n = \{X \in \mathcal{A} : |X| = n\}$. Since ω_1 is regular and since there are only countably many $n \in \omega$ but uncountably many X 's, there must be some n for which $\mathcal{A}_n$ is uncountable; for this n , let $A = \mathcal{A}_n$.

Claim ii: We can reduce A to a Δ -system, B , and still have $|B| = \omega_1$.

Proof: We have $\forall X \in A, |X| = n$. Let $X = \{X_0 < X_1 < \dots < X_{n-1}\}$. Let

$A_i = \{X_i : X \in A\}$, the set of i^{th} elements of all $X \in A$. Then for some $k < n$,

A_k is unbounded in ω_1 (i.e., $\forall \alpha < \omega_1 \exists X \in A$ for which $X_k > \alpha$) because if

$\bigcup_{k < n} A_k$ is uncountable, then for some $k < n$, A_k is uncountable. Choose the

least such k so that A_k goes cofinally into ω_1 . We must show that, w.l.o.g.,

there exists a fixed $r = \{X_0, \dots, X_{k-1}\}$ which is contained in each X . Let

$\beta = \sup(\{\sup_{i < k} X_i : X \in \mathcal{A}\}) < \omega_1$ (note that $\sup_{i < k} X_i = X_{k-1}$ for $k > 0$ and 0

otherwise). Then $\forall X \in \mathcal{A}, \{X_0, \dots, X_{k-1}\} \subseteq \beta$. Since $\beta < \omega_1$ and k is finite,

$|\beta| = \omega$. Now there are uncountably many $X \in \mathcal{A}$ and only countably many

possibilities for $\{X_0, \dots, X_{k-1}\}$ since β is countable; therefore, one of these

possibilities must be used uncountably many times. Therefore, we may assume

that r is the first k elements of each of the X 's. To insure that the pairwise

intersection of any two X 's is no larger than r , define $\langle X^\xi : \xi < \omega_1 \rangle$ recursively:

choose X^ξ such that $X^\xi_k > \sup_{\eta < \xi} (\{X^\eta_{n-1}\})$. Observe that the recursion doesn't

break down for any $\xi < \omega_1$ (a breakdown can only occur if $\sup_{\eta < \xi} (\{X^\eta_{n-1}\}) = \omega_1$

which can only occur if $\xi = \omega_1$ since ω_1 is regular). So $\mathcal{B}$ is uncountable and

forms a Δ -system. $\square$

DEFINITION 4.8: Let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be a collection of topological spaces.

and $X = \prod_{i \in I} X_i$. Then the **product topology $\mathcal{T}$ on X** is the topology

generated by the set of points $X = \prod_{i \in I} X_i$. The basic open sets are defined as follows: fix finite $r \subseteq I$ and for each $i \in r$, let U_i be an open set in X_i ; then $U = \{f \in \prod_{i \in I} X_i : \text{for each } i \in r (f(i) \in U_i)\}$ is a basic open set. Call r the **support of U** written $\text{sup}(U)$.

THEOREM 4.9: Suppose X_i for $i \in I$ are spaces such that for every finite $r \subseteq I$, $\prod_{i \in r} X_i$ is ccc. Then $\prod_{i \in I} X_i$ is ccc.

PROOF: We'll prove the contrapositive.

Assume $\prod_{i \in I} X_i$ is not ccc. Let $\{U_\alpha : \alpha < \omega_1\}$ be a family of pairwise disjoint nonempty open subsets of $\prod_{i \in I} X_i$. By shrinking U_α if necessary, we may assume that U_α is a basic open set of $\prod_{i \in I} X_i$. Let $a_\alpha = \text{sup}(U_\alpha)$; so a_α is a finite subset of I . Then by the Δ -system Lemma, there is an uncountable $B \subseteq \omega_1$ such that $\{a_\alpha : \alpha \in B\}$ which forms a Δ -system with root r .

Claim: $a_\alpha \cap a_\beta = r$ is not empty.

Proof: Assume $a_\alpha \cap a_\beta = \emptyset$. We want to construct a sequence which is in both U_α and U_β . Let $\langle x_i : i \in I \rangle$ be our sequence such that $x_i \in \text{proj}_i(U_\alpha)$ for $i \in a_\alpha$ and $x_i \in \text{proj}_i(U_\beta)$ for $i \in a_\beta$. So $\langle x_i \rangle \in U_\alpha$ and U_β . Therefore, $U_\alpha \cap U_\beta \neq \emptyset$ contradicting that the U_α 's are disjoint. Therefore, $r \neq \emptyset$.

Let $\pi(U_\alpha)$ be the projection of U_α onto $\prod_{i \in r} X_i$ where r is the root of $\{a_\alpha : \alpha \in B\}$. Then $\{\pi(U_\alpha) : \alpha \in B\}$ forms an uncountable disjoint family in $\prod_{i \in r} X_i$ because there are uncountably many $\alpha \in B$ and $\pi(U_\alpha)$'s are all disjoint since U_α 's are all disjoint. Therefore, $\prod_{i \in I} X_i$ is not ccc. $\square$

DEFINITION 4.10: A family $\mathcal{A}$ of sets has the **finite intersection property** if the intersection of every finitely many elements of $\mathcal{A}$ is nonempty.

LEMMA 4.11: Assume MA_{ω_1} . Let X be ccc, and let $\{U_\alpha : \alpha < \omega_1\}$ be an uncountable family of nonempty open subsets of X . Then there exists an uncountable $A \subset \omega_1$ such that $\{U_\alpha : \alpha \in A\}$ has the finite intersection property.

PROOF: Let $V_\alpha = \bigcup_{\gamma > \alpha} U_\gamma$

Claim i: $\alpha < \beta \rightarrow V_\beta \subseteq V_\alpha$.

Proof: $V_\alpha = \bigcup_{\gamma > \alpha} U_\gamma = \left(\bigcup_{\gamma > \alpha} U_\gamma \right) \cup \left(\bigcup_{\gamma > \beta} U_\gamma \right) = \left(\bigcup_{\gamma > \alpha} U_\gamma \right) \cup V_\beta$. Hence, $V_\beta \subseteq V_\alpha$.

Claim ii: $\exists \alpha (\forall \beta > \alpha (\bar{V}_\beta = \bar{V}_\alpha))$.

Proof: Assume not. Then there exists a sequence of ordinals $\langle \alpha_\xi : \xi < \omega_1 \rangle$ such that $\forall \xi (\bar{V}_{\alpha_{\xi+1}} \neq \bar{V}_{\alpha_\xi})$, in which case $V_{\alpha_\xi} \setminus \bar{V}_{\alpha_{\xi+1}} \neq \emptyset$ (if $V_{\alpha_\xi} = \bar{V}_{\alpha_{\xi+1}}$, then $\bar{V}_{\alpha_\xi} = \bar{V}_{\alpha_{\xi+1}} = \bar{V}_{\alpha_{\xi+1}}$ which contradicts our assumption). Now these $V_{\alpha_\xi} \setminus \bar{V}_{\alpha_{\xi+1}}$ are open, disjoint, and uncountable contradicting that X is ccc (all antichains are countable). Therefore, $\exists \alpha \forall \beta > \alpha (\bar{V}_\beta = \bar{V}_\alpha)$. Now fix α satisfying: $\forall \beta > \alpha (\bar{V}_\beta = \bar{V}_\alpha)$. Let $P = \{p \subset V_\alpha : p \text{ is open and } p \neq \emptyset\}$; which P inherits ccc from X . Now if $\mathcal{G}$ is a filter in P , then $\mathcal{G}$ has the finite intersection property. Let $A = \{\gamma < \omega_1 : \exists p \in \mathcal{G} (p \subset U_\gamma)\}$; then $\{U_\gamma : \gamma \in A\}$ has the finite intersection property inherited from $\mathcal{G}$.

Now for each β , let $D_\beta = \{p \in P : \exists \gamma > \beta (p \subset U_\gamma)\}$

Claim iii: D_β is dense in P .

Proof: $\forall \beta > \alpha (V_\beta \subseteq V_\alpha)$. Therefore, if $p \in P$, then $p \cap V_\beta \neq \emptyset$. Hence, $p \cap U_\gamma \neq \emptyset$ for some $\gamma > \beta$; so $p \cap U_\gamma$ is an extension of p in D_β . Therefore,

D_β is dense in P .

Furthermore, since $\mathcal{G}$ has the finite intersection property and MA_{ω_1} holds, $\mathcal{G} \cap D_\beta \neq \emptyset$. Therefore, A contains some $\gamma > \beta$. Thus A is unbounded in ω_1 and, hence, uncountable. $\square$

THEOREM 4.12: If MA_{ω_1} holds, then any product of ccc spaces is ccc.

PROOF: Assume MA_{ω_1} , that is for any nonempty ccc partial ordering $\langle \mathbb{P}, \leq \rangle$ and any family $\mathcal{D}$ of $\leq \omega_1$ many dense subsets of $\mathbb{P}$, there exists a $\mathcal{D}$ -generic filter $\mathcal{G}$ such that $\forall D \in \mathcal{D} (D \cap \mathcal{G} \neq \emptyset)$.

Claim: The product of any two ccc spaces is ccc.

Proof: Let X and Y be ccc and suppose that $X \times Y$ is not ccc. Let $\mathcal{D} = \{W_\alpha : \alpha < \omega_1\}$ be an uncountable family of pairwise disjoint nonempty open subsets of $X \times Y$. Let $U_\alpha \times V_\alpha \subset W_\alpha$ with $\{U_\alpha : \alpha < \omega_1\}$ a nonempty subset of X . By Lemma 4.16, $\exists A \subset \omega_1$ such that $\{U_\alpha : \alpha \in A\}$ has that finite intersection property. If $\alpha, \beta \in A$ with $\alpha \neq \beta$, then $U_\alpha \cap U_\beta \neq \emptyset$. However, $U_\alpha \times V_\alpha \cap U_\beta \times V_\beta = \emptyset$. Therefore, $\{V_\alpha : \alpha \in A\}$ is uncountable, contradicting that Y has ccc. Therefore, $X \times Y$ must be ccc.

Then by induction on n , any product of countably many ccc spaces is ccc.

Therefore, by Theorem 4.14, any arbitrary product of ccc spaces is ccc. $\square$

LEMMA 4.18: If X is a Souslin line, then X^2 is not ccc.

PROOF: We will let our open sets be the open intervals (a, b) such that $a, b \in X$ with $a < b$. By induction on $\alpha < \omega_1$, we will find $a_\alpha, b_\alpha, c_\alpha \in X$ with the following properties:

- i. $a_\alpha < b_\alpha < c_\alpha$
- ii. $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$
- iii. $(a_\alpha, c_\alpha) \cap \{b_\xi : \xi < \alpha\} = \emptyset$ (i.e. b_ξ is not an isolated point).

Let $U_\alpha = (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha)$. If the above three properties hold, then $U_\alpha \neq \emptyset$ (by ii).

Claim: If $\xi < \alpha$, then $U_\xi \cap U_\alpha = \emptyset$.

Proof: Since $\xi < \alpha$, then either $b_\xi \leq a_\alpha$ or $b_\xi \geq c_\alpha$ because $b_\xi \notin (a_\alpha, c_\alpha)$ (by iii). If $b_\xi \leq a_\alpha$, then $(a_\xi, b_\xi) \cap (a_\alpha, b_\alpha) = \emptyset$. If $b_\xi \geq c_\alpha$, then $(b_\xi, c_\xi) \cap (b_\alpha, c_\alpha) = \emptyset$. Now for $U_\alpha = (a_\alpha, b_\alpha) \times (b_\alpha, c_\alpha)$ and $U_\xi = (a_\xi, b_\xi) \times (b_\xi, c_\xi)$, either the first or the second coordinates are disjoint; therefore, all the U_α 's are disjoint.

Since there exists ω_1 many such α 's that are nonempty and disjoint, there exists an uncountable antichain. Therefore, X^2 is not ccc.

It remains to prove that there exist a_α, b_α , and, c_α with the properties (i), (ii), and (iii). To find a_α, b_α , and, c_α , let W be the set of all isolated points of X . Since an isolated point is an open set and since X has ccc, $|W| \leq \omega$. Now let $\xi < \alpha$. Our induction hypothesis says that we can find a_ξ, b_ξ , and c_ξ such that (i), (ii), and (iii) hold for ξ . Now $X \setminus \text{cl}(W \cup \{b_\xi : \xi < \alpha\})$ is an open set because $\text{cl}(W \cup \{b_\xi : \xi < \alpha\})$ is closed. So we've thrown out only countably many points which implies that $X \setminus \text{cl}(W \cup \{b_\xi : \xi < \alpha\})$ contains a nonempty open interval, say (a_α, c_α) . Since (a_α, c_α) contains no isolated points (we threw them out), it is infinite. Therefore, for any $b \in (a_\alpha, c_\alpha)$, $(a_\alpha, b_\alpha) \neq \emptyset$ and $(b_\alpha, c_\alpha) \neq \emptyset$. $\square$

THEOREM 4.13: $MA_{\omega_1} \Rightarrow SH$.

PROOF: Assume MA_{ω_1} holds but that SH is false, that there exists a S-line $\langle X, < \rangle$, which is ccc but not separable. Now MA_{ω_1} implies that the product of ccc spaces is ccc (Theorem 4.11). Therefore, X^2 is ccc contradicting Lemma 4.12 which says that X^2 is not ccc. Therefore, SH must hold. $\square$

Chapter 5

ARONSZAJN TREES

In this chapter we will present two different proofs of the existence of an ω_1 -Aronszajn tree, one by Kunen and the second by Specker. Although both are constructive proofs, their approach to the problem is quite different. Kunen constructed a tree with no ω_1 -chains, but with very wide levels and then extracted a subtree in which all the levels are of size smaller than ω_1 . Specker's approach, on the other hand, was to construct the ω_1 -Aronszajn from the bottom up limiting level size as he went along. This method of constructing an ω_1 -Aronszajn tree also allowed him to generalize under GCH and construct a κ^+ -Aronszajn tree for κ regular. We conclude this chapter with that construction.

THEOREM 5.1: There exists an ω_1 -Aronszajn tree.

PROOF A (Kunen): Set $T = \{s \in {}^{<\omega_1}\omega : s \text{ is a 1-1 function}\} \subseteq {}^{<\omega_1}\omega$.

Claim i: $\text{ht}(T) = \omega_1$.

Proof: $\forall \alpha < \omega_1$ there is a 1-1 function $s: \alpha \rightarrow \omega$. Now $\text{ht}(T)$ is the smallest α such that $\text{Lev}_\alpha(T) = \emptyset$. Since there are ω_1 many $\alpha < \omega_1$, ω_1 is the smallest such α .

Claim ii: Every chain in T is countable.

Proof: Assume not; let $C \subseteq T$ be an uncountable chain. Then $\bigcup C: \omega_1 \rightarrow \omega$ would be 1-1 which is impossible (by the Pigeon Hole Principle). Therefore, C must be countable.

Claim iii: There are uncountable levels in T .

Proof: Since $\omega < \omega_1$, $\text{Lev}_\omega(T) = \{s \in {}^\omega\omega : s \text{ is 1-1}\}$ exists. Now $|{}^\omega\omega| \geq \omega_1$.

In general, $\text{Lev}_\alpha(T)$ is uncountable for all α between ω and ω_1 .

Therefore, T is not an ω_1 -tree, and hence, not an ω_1 -Aronszajn tree.

Since T is not Aronszajn, we will define a subtree $T^* \subset T$ which is Aronszajn (i.e. which has no uncountable antichains):

a. For $s, t \in {}^\alpha\omega$ and $\alpha < \omega_1$, define a relation $\sim$ on ${}^\alpha\omega$ by:

$s \sim t$ if $|\{\beta < \alpha : s(\beta) \neq t(\beta)\}| < \omega$ (i.e. s and t differ on only finitely many domain elements).

Note that $\sim$ is an equivalence relation.

b. Construct a sequence $S = \langle s_\alpha \in {}^\alpha\omega : \alpha < \omega_1 \rangle$ as follows:

i. s_α is 1-1 for $\alpha > 0$ and $s_\alpha \in \text{Lev}_\alpha(T)$.

ii. If $\alpha < \beta$, then $s_\alpha \sim s_\beta|_\alpha$.

iii. $\omega \setminus \text{rng}(s_\alpha)$ is infinite.

The proof of the existence of such a sequence will appear at the end of this proof.

c. Define $T^* = \{t \in T : \exists \alpha < \omega_1 (t \sim s_\alpha)\}$

$$= \bigcup_{\alpha < \omega_1} \{t \in \text{Lev}_\alpha(T) : t \sim s_\alpha\}.$$

Claim iv: T^* is a subtree of T .

Proof: Let $t \in T^*$ and $\alpha < \text{ht}(t)$. Now $\text{ht}(t) < \omega_1$ so the $t \sim s_{\text{ht}(t)}$. Furthermore, $\alpha < \text{ht}(t)$ implies $t|_\alpha \sim s_{\text{ht}(t)}|_\alpha$. But $s_{\text{ht}(t)}|_\alpha \sim s_\alpha$ (by (ii) in definition of S).

Therefore, $t|_\alpha \sim s_\alpha$ (by transitivity of $\sim$). So T^* is a subtree of T .

Now $\text{ht}(T^*) = \omega_1$ since $\forall \alpha < \omega_1 (s_\alpha \in T^*)$, and $\text{ht}(s_\alpha) = \alpha$. Also, T^* has no

ω_1 -chains since $T^* \subset T$. Furthermore, $\text{Lev}_\alpha(T^*) = \{t \in T : t \sim s_\alpha\}$ is countable since there are only finitely many $t \sim s_\alpha$ and there are only countably many finite subsets of a countable ordinal. Since T^* has no uncountable levels, T^* is an ω_1 -Aronszajn.

The existence of such a sequence $\langle s_\alpha \rangle$ is proved by induction on α :

Base case: $s_0: \emptyset \rightarrow \omega$ is the empty function.

Successor case: Suppose s_α is defined. Define $s_{\alpha+1} = s_\alpha \cup \{(\alpha, n)\}$ where $n \in \omega \setminus \text{rng}(s_\alpha)$. Then $\text{dom}(s_{\alpha+1}) = \text{dom}(s_\alpha) \cup \{\alpha\}$ and $s_{\alpha+1}(\alpha) \notin \text{rng}(s_\alpha)$ (by construction). Furthermore, since s_α is 1-1, $s_{\alpha+1}$ is 1-1.

Limit case: Suppose α is a limit ordinal $< \omega_1$ and s_β is defined $\forall \beta < \alpha$. Choose an increasing cofinal sequence $\langle \alpha_i : i \in \omega \rangle$ in α and construct $\langle s_\alpha \rangle$ as follows:

First construct sequence $\langle t_i : i \in \omega \rangle$: let $t_0 = s_{\alpha_0}$ and $t_{i+1} > t_i$ with $t_i \sim s_{\alpha_i}$. Then $\text{ht}(t_i) = \alpha_i$ and t_{i+1} extends t_i . To construct t_{i+1} , copy $s_{\alpha_{i+1}} \setminus \alpha_i$ onto the end of t_i . Then $t_{i+1} \sim s_{\alpha_{i+1}}$.

Claim i: t_{i+1} is 1-1.

Proof: Since $s_{\alpha_{i+1}}$ is 1-1 and t_{i+1} differs in only finitely many places, there can be only finitely many glitches. However, we can easily redefine these trouble spots since $\omega \setminus \text{rng}(t_i)$ is infinite. Therefore, t_{i+1} is 1-1.

Now set $s_\alpha = \bigcup_{i=1}^\omega t_i = \sup(\{t_i\})$.

Claim ii: s_α does indeed have the three defining properties of S .

Proof: i. $s_\alpha \in {}^\alpha \omega$ and is 1-1 by construction.

ii. $\forall \beta < \alpha \ s_\alpha \upharpoonright_{\alpha_1} = t_i$ and $s_\alpha \upharpoonright_\beta$ differs from s_β in only finitely many places.

iii. We know that $\omega \setminus \text{rng}(t_i)$ is infinite for all i . To show that $\omega \setminus \text{rng}(s_\alpha)$

is infinite, define $s_\alpha(\xi) = \begin{cases} s_\alpha(\xi) & \text{for } \xi \neq \alpha_i \text{ for all } i, \text{ and} \\ s_\alpha(\alpha_{2i}) & \text{for } \xi = \alpha_i \text{ for any } i. \end{cases}$

Now $|\{\alpha_i\}| = \omega$, and $\text{rng}(s_\alpha)$ uses only elements in α which are α_i for only even i 's or which are not α_i 's at all; therefore, $\omega \setminus \text{rng}(s_\alpha) = \{2n+1 : n \in \omega\}$ which is infinite.

Now s_α is still 1-1 and $s_\alpha \in {}^\alpha\omega$ and $s_\alpha \sim s_\beta|_\alpha$ because we've changed only finitely many places. Therefore, $\langle s_\alpha \rangle$ exists. $\square$

PROOF B (Specker): Let T be a set of increasing sequences of rational numbers ordered by extension: if $x = \langle q_\xi : \xi < \alpha \rangle$ and $y = \langle r_\xi : \xi < \beta \rangle$ then $x \leq y$ means that $\alpha \leq \beta$ and $\forall \xi < \alpha (q_\xi = r_\xi)$.

We will use the following properties of $\mathbb{Q}$:

- a. $\mathbb{Q}$ is countable; this insures that any tree constructed from an increasing sequence of rational numbers does not contain an ω_1 -branch.
- b. Every $\alpha < \omega_1$ can be embedded in an interval of $\mathbb{Q}$.

Note that T cannot consist of all increasing sequences of rationals; otherwise there would exist an uncountable ω^{th} level since $|\omega^\omega| = \omega_1$. Also note that since each $x \in T$ is an increasing sequence of rationals, it is bounded above by ω : therefore, each such sequence has a supremum.

We will construct T by recursion by assuming the construction of $T|_\alpha$. Let $U_\alpha = \text{Lev}_\alpha(T)$ consist of increasing α -sequences of rationals. Since we cannot use all increasing sequences, we will restrict T by imposing the following condition:

$$\forall x \in T|_\alpha \quad \forall q > \sup(x) \quad \exists y \in U_\alpha \quad (y \supseteq x \wedge \sup(y) \leq q) \quad (*)$$

Base case : $U_0 =$ the empty sequence.

Successor case : Assume U_α is constructed. Let

$U_{\alpha+1} = \{x \hat{\ } q : x \in U_\alpha \wedge q > \sup(x)\}$ where $x \hat{\ } q$ = the sequence x with the rational q added as the $(\alpha + 1)^{\text{th}}$ element. Now $U_{\alpha+1}$ is countable since it has countably many levels with $\omega \times \omega$ many elements on each level. Furthermore, $(*)$ holds by construction.

Limit case : Let λ be a limit ordinal and assume $T|_\lambda$ is constructed such that $(*)$ holds $\forall \alpha < \lambda$. Observe that $T|_\lambda$ has λ -branches since there are λ many levels below $\text{Lev}_\lambda(T)$.

Claim: $\forall x \in T|_\lambda$ and $\forall q > \sup(x)$, there is a λ -branch passing through x such that $\sup(y) \leq q$ (if $q = \sup(x) + 1$, then $\sup(y) = q$; if $q > \sup(x) + 1$, then $\sup(y) < q$; and $\sup(y) > \sup(x)$ since both sequences are strictly increasing).

Proof: Let $\langle \alpha_n \rangle$ be a chain of sequences such that each α_n is a sequence of rationals on the n^{th} level, and let $\langle q_n \rangle$ be the sequence of rationals such that q_m is added to α_{m-1} on the m^{th} level. Let $\lambda = \lim_{n \rightarrow \omega} \alpha_n$ and $q \geq \lim_{n \rightarrow \omega} q_n$. Then both $\langle \alpha_n \rangle$ and $\langle q_n \rangle$ are strictly increasing. By $(*)$, there exists a sequence $\langle y_n \rangle$

such that $\sup(y_n) \leq q_n$ where $x \subset y_0 \subset y_1 \subset \dots$. Let $y = \bigcup_{n=0}^{\infty} y_n = \lim_{n \rightarrow \omega} y_n$.

(Note that since λ is a limit ordinal and there are ω many levels between any x and $\text{Lev}_\alpha(T)$, such a sequence of y_n 's exists.)

We construct U_λ as follows: $\forall x \in T|_\lambda$ and each $q > \sup(x)$, choose only one y to put into U_λ thus restricting the number of branches which get make it up to U_λ . Then U_λ is countable because $|T|_\lambda| = \omega$, and there are ω many q 's

such that $q > \sup(x)$; and $\omega \times \omega = \omega$. Furthermore, (*) holds by construction.

Hence, $T = \bigcup_{\alpha < \omega_1} T|_\alpha$ is an ω_1 -Aronszajn tree. $\square$

Next the question arises, can these results be generalized to higher cardinals κ to form κ -Aronszajn trees? (i.e., *Does there exist a κ -tree whose levels are of size $2^{<\kappa}$ which has no κ -branches?*)

DEFINITION 5.2: The **tree property (TP)** for the cardinal κ states that there is no κ -Aronszajn tree (i.e. if κ has the TP then every κ -tree has a κ -branch).

NOTE: According to Jech, singular cardinals do not have the TP (there are κ -Aronszajn trees for singular cardinals). Also, under GCH, for κ regular, κ^+ -Aronszajn trees exist. However, little is known about κ^+ -Aronszajn trees for κ singular.

THEOREM 5.3: Assuming GCH, there exists a κ^+ -Aronszajn tree for κ regular.

PROOF: Let Q equal the set of all finite sequences of ordinals less than κ (i.e., $^{<\omega}\kappa$) with no sequence ending in zero. We'll order Q lexicographically by ordinal ordering.

Claim i: $|Q| = \kappa$

Proof: $|^{<\omega}\kappa| = |^n\kappa| = \kappa$ since the domain is finite.

Claim ii: Q is dense in itself.

Proof: Let $s, t \in Q$ such that $s < t$. If s is greater than or equal to t in length,

then choose any β such that $0 < \beta < \kappa$. Then $s < s \cap \beta < t$. If, on the other hand, s is shorter than t , add on 0's to s until s and t are equal in length and then add on β . Then $s < s \cap \langle 0, \dots, 0, \beta \rangle < t$.

Claim iii: Every ordinal $\alpha < \kappa^+$ can be order embedded in any interval of Q .

Proof: It suffices to prove that $\forall \alpha < \kappa^+$, given any interval, $[s, t) \subset Q$, there exists an increasing sequence of length α which can be order embedded in $[s, t)$ (i.e., $\exists \langle s_\xi \rangle_{\xi < \alpha}$ such that $s \leq s_1 < s_2 < \dots < s_i < \dots < t$ where each $s_i \in Q$).

Base case: For $\alpha = 0$ this claim is vacuously true.

Successor case: Assume that an α -sequence can be order embedded in any interval in Q . Now fix $[s, t) \subset Q$. Since Q is dense in itself, there exists a $t_0 \in Q$ such that $s < t_0 < t$. Also there exists a sequence $\langle s_\xi \rangle_{\xi < \alpha}$ contained in $[s, t_0)$ (by induction hypothesis). Set $s_\alpha = t_0$. Then $\langle s_\xi \rangle_{\xi < (\alpha+1)}$ is embedded in $[s, t)$.

Limit case: Let $\alpha = \lambda$, a limit ordinal such that $\text{cf}(\lambda) \leq \kappa$ because $\lambda < \kappa^+$ (but it may be that $\kappa < \lambda$: e.g. $\lambda = \kappa + \kappa$, $\kappa \times \kappa$, or $\kappa + n$). Then there exists an increasing sequence $\langle \gamma_i \in \lambda : i < \text{cf}(\lambda) \rangle$ which is cofinal in λ and which divides λ into intervals $[\gamma_i, \gamma_{i+1})$. Now $\delta_i = \text{type}([\gamma_i, \gamma_{i+1}))$ which is an ordinal which can be order embedded as required by our hypothesis. Now we must divide $[s, t)$ into $\text{cf}(\lambda)$ many intervals. We can assume, w.o.l.g., that length of s equals length of t and then partition $[s, t)$ by $\{s, s \cap 1, s \cap 2, \dots, s \cap \xi_{< \text{cf}(\lambda)}, t\}$ where $s_i = s_{i-1} \cap i$. Then $\delta_{i < \text{cf}(\lambda)}$ can be embedded in $[s \cap i, s \cap (i+1))$ since δ_i can be mapped cofinally into λ and $\text{cf}(\lambda) < \kappa = |Q| = |\lambda|$.

Therefore, any ordinal $< \kappa^+$ can be embedded in any interval of Q .

The construction of a κ^+ -Aronszajn tree T goes as follows:

The elements of T are increasing α -sequences of elements from Q where

$\alpha < \kappa^+$ (i.e. if $x \in T$, the $x \in {}^{<\kappa^+}Q$).

This tree will have the same restriction found in the previous construction:

$$(\forall x \in T|_\alpha)(\forall q > \sup(x))(\exists y \in U_\alpha)(y \supseteq x \wedge \sup(y) \leq q) \quad (*)$$

Base case: Again, the restriction holds vacuously.

Successor case: Assume that U_α is constructed. Then extend each sequence in U_α by one more element $q \in Q$: $U_{\alpha+1} = \{x \cap q : x \in U_\alpha \wedge q > \sup(x)\}$. (Note that since x is made up of increasing finite sequences, $\sup(x)$ is also a finite sequence; we can always add one more ordinal to the sequence $\sup(x)$ to find a $q \in Q$ such that $q > \sup(x)$.) Then $|U_{\alpha+1}| < \kappa^+$ because there exists κ many choices for each sequence in U_α and there are at most κ many nodes in U_α . Therefore, $|U_{\alpha+1}| < |\kappa \times \kappa| = \kappa < \kappa^+$. Hence $(*)$ holds by construction.

Limit case: Let λ be a limit ordinal such that $\text{cf}(\lambda) \leq \kappa$.

Case 1: Assume $\text{cf}(\lambda) < \kappa$ (e.g. let $\lambda = \kappa + \kappa$). Here put every branch of $T|_\lambda$ into U_λ . To insure that $|U_\lambda| < \kappa^+$, we must require that GCH holds (under GCH $\kappa^{<\kappa} = \kappa$): branches in $T|_\lambda$ are sequences of length λ picked from a set of size κ ; therefore $\kappa^{\text{cf}(\lambda)} = \kappa$.

Case 2: $\text{cf}(\lambda) = \kappa$ which implies $\kappa^{\text{cf}(\lambda)} = \kappa^\kappa > \kappa$ with or without GCH. Therefore, we cannot put all the branches that would reach U_λ into the tree (if we did there would be a κ^+ -antichain). Hence, for each $x \in T|_\lambda$ and each $q > \sup(x)$, a λ -branch $y \in T|_\lambda$ exists which goes through x such that $\sup(y) \leq q$. Now to construct U_λ , let $\lambda = \lim_{\xi \rightarrow \kappa} \alpha_\xi = \lim_{\xi \rightarrow \text{cf}(\lambda)} \alpha_\xi$ (Note that $\{\alpha_\xi\} \subseteq T|_\lambda$ contains $\text{cf}(\lambda)$ many levels) and the sequence $\langle \alpha_\xi : \xi < \text{cf}(\lambda) \rangle$ is cofinal in λ . Next let $\langle q_\xi : \xi < \text{cf}(\lambda) \rangle$ be an increasing sequence whose supremum is $\leq q$. Then by $(*)$, there exists a sequence $\langle y_\xi \rangle$ such that $\forall \xi < \kappa$ ($\sup(y_\xi) < q_\xi$). So we have

$x \subset y_0 \subset y_1 \subset \dots$. Then $y = \bigcup_{\xi < \kappa} y_\xi \in U_\lambda$ (this path goes all the way through the tree up to U_λ picking up λ many nodes from each level).

It remains to check that $\forall \lambda (|U_\lambda| < \kappa^+)$. The number of all λ -branches in $T|_\lambda$ is $\kappa^{cf(\lambda)}$ because $|T|_\lambda| = \kappa$ and branches which reach U_λ can be identified with sequences of length $cf(\lambda)$. Furthermore, there exists $\kappa^{cf(\lambda)}$ many such sequences since ${}^{cf(\lambda)}\kappa$ is the set of all sequences of length $cf(\lambda)$ with elements from κ and $|{}^{cf(\lambda)}\kappa| = \kappa^{cf(\lambda)}$. $\square$

Chapter 6

KUREPA TREES

KUREPA'S HYPOTHESIS (KH) 6.1: There is an ω_1 -Kurepa tree.

More generally, κ -KH is the statement: There is a κ -Kurepa tree.

REMARK 6.2: SH + Theorem 2.19 implies there are no Souslin trees. This is in contrast to KH which states the existence of a Kurepa tree.

Now it turns out that $GCH + \Diamond + \neg KH$ is consistent. To prove the existence of a Kurepa tree we need something stronger than $\Diamond$, namely $\Diamond^+$.

DEFINITION 6.3: $\Diamond^+$ is the statement: There exists a sequence, $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$, called a $\Diamond^+$ -sequence, where for each $\alpha < \omega_1$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ and $|\mathcal{A}_\alpha| < \omega_1$ such that $\forall A \subseteq \omega_1$, there exists a c.u.b. $C \subseteq \omega_1$ with the following properties:

- a. $\forall \alpha \in C, A \cap \alpha \in \mathcal{A}_\alpha$. (This property states that $\langle \mathcal{A}_\alpha \rangle$ guesses correctly at A on a c.u.b. set; furthermore, since for each α , $|\mathcal{A}_\alpha| < \omega_1$, there are only countably many guesses.)
- b. $\forall \alpha \in C, C \cap \alpha \in \mathcal{A}_\alpha$. (This property states that $\langle \mathcal{A}_\alpha \rangle$ guesses correctly at A where it guesses correctly.)

NOTE: $\Diamond$ and $\Diamond^+$ are combinatorial principles which are used to construct subsets of ω_1 which have some specific desired properties. Both $\Diamond$ and $\Diamond^+$

each give a sequence of ω_1 many sets to build the desired subset, say T :
 $\Diamond$ gives a sequence of subsets of ω_1 and $\Diamond^+$ gives a sequence of subsets of $\mathcal{P}(\omega_1)$. Now there are 2^{ω_1} subsets, $A \subseteq \omega_1$. It turns out that even though we have only ω_1 many stages in which to build T , both $\Diamond$ and $\Diamond^+$ guess correctly at all 2^{ω_1} many A 's: $\Diamond$ gives a single guess for A at each stage $\alpha < \omega_1$ and guesses correctly on a stationary many α 's. $\Diamond^+$, on the other hand, gives countably many guesses for A at each α , but it guesses correctly more often: it guesses correctly on a c.u.b. set.

To show that $\Diamond^+ \Rightarrow \text{KH}$, we need to define the following notion of a Kurepa family. We will use the construction of such a family to extract a κ -Kurepa tree.

DEFINITION 6.4: $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is a κ -Kurepa family if $|\mathcal{F}| \geq \kappa^+$ and $\forall \alpha < \kappa$, $|\{A \cap \alpha : A \in \mathcal{F}\}| < \kappa$.

NOTE: Although $\mathcal{F}$ is very large, below any fixed $\alpha < \kappa$, the number of different $A \cap \alpha$'s is small since there is much duplication.

NOTATION 6.5: If $\mathcal{F} \subseteq \mathcal{P}(I)$ and $X \subseteq I$, then $\mathcal{F}_X = \{A \cap X : A \in \mathcal{F}\}$.

LEMMA 6.6: If $X \subseteq Y \subseteq I$, $|\mathcal{F}_X| \leq |\mathcal{F}_Y|$.

PROOF: We can map $\mathcal{F}_Y$ onto $\mathcal{F}_X$ by sending $A \cap Y$ to $A \cap X$. $\square$

LEMMA 6.7: For regular κ , $\forall \alpha < \kappa$ ($|\mathcal{F}_\alpha| < \kappa$) iff $\forall X \subseteq \kappa$ ($|X| < \kappa \rightarrow |\mathcal{F}_X| < \kappa$).

PROOF:

($\Rightarrow$) Assume κ is regular and $\forall \alpha < \kappa (|\mathcal{F}_\alpha| < \kappa)$. Let $X \subseteq \kappa$ with $|X| < \kappa$. Then, since κ is regular, $\exists \alpha < \kappa (X \subseteq \alpha)$. Therefore, $|\mathcal{F}_X| \leq |\mathcal{F}_\alpha| < \kappa$.

($\Leftarrow$) Assume $\forall X \subseteq \kappa (|X| < \kappa \rightarrow |\mathcal{F}_X| < \kappa)$. Fix $\alpha < \kappa$. Then α is such an X , so $|\mathcal{F}_\alpha| < \kappa$. $\square$

LEMMA 6.8: For κ regular, there exists a κ -Kurepa family iff there exists a set I with $|I| = \kappa$ and an $\mathcal{F} \subseteq \mathcal{P}(I)$ with $|\mathcal{F}| \geq \kappa^+$ such that

$$\forall X \subseteq I (|X| < \kappa \rightarrow |\mathcal{F}_X| < \kappa).$$

PROOF:

($\Leftarrow$) Assume $\exists I$ and an $\mathcal{F} \subseteq \mathcal{P}(I)$ as in hypothesis. Then $\forall \alpha < \kappa (|\mathcal{F}_\alpha| < \kappa)$ (Lemma 6.7). But $\mathcal{F}_\alpha = \{A \cap \alpha : A \in \mathcal{F}\}$. Therefore, $\mathcal{F}$ is a κ -Kurepa family.

($\Rightarrow$) Just let $I = \kappa$. $\square$

THEOREM 6.9: For κ regular, there exists a κ -Kurepa tree iff there exists a κ -Kurepa family.

PROOF:

($\Rightarrow$) Let T be a κ -Kurepa tree. Then $|T| = \kappa$. Let $\mathcal{F}$ be the set of all paths through T . Then $|\mathcal{F}| > \kappa$ since T is Kurepa. Therefore, it suffices to show that $\forall X \subseteq T$ with $|X| < \kappa$, $|\mathcal{F}_X| < \kappa$. So fix such an X . Now since κ is regular, there exists an $\alpha < \kappa$ such that $\forall x \in X$, $\text{ht}(x) < \alpha$. Furthermore, $\forall C \in \mathcal{F}$, C is a path; hence, C intersects $\text{Lev}_\alpha(T)$. Therefore, each element of $\mathcal{F}_X$ is of the form $\{x \in X : x < z\}$ for some $z \in \text{Lev}_\alpha(T)$. Hence, $|\mathcal{F}_X| \leq |\text{Lev}_\alpha(T)| < \kappa$.

($\Leftarrow$) Suppose $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is a κ -Kurepa family. Then for $B \in \mathcal{F}_\alpha$, let

$\mathfrak{X}_B \in {}^\alpha 2$ be its characteristic function³.

Claim: $\bigcup_{\alpha < \kappa} \{\mathfrak{X}_B : B \in \mathcal{F}_\alpha\} \subseteq {}^{<\kappa} 2$ is a κ -Kurepa tree.

Proof: Fix $\alpha < \kappa$. Then $\text{Lev}_\alpha(T) = \{\mathfrak{X}_B : B \in \mathcal{F}_\alpha\}$. Now $|\text{Lev}_\alpha(T)| < \kappa$ because $|\mathcal{F}_\alpha| < \kappa$. Therefore, all levels are of size $< \kappa$. Furthermore, $\forall B \in \mathcal{F}$, $\mathfrak{X}_B$ is a path. Since there are more than κ many such $\mathfrak{X}_B$'s (because $\mathcal{F}$ is a κ -Kurepa family), $|\mathcal{F}| > \kappa$. Hence, $\bigcup_{\alpha < \kappa} \{\mathfrak{X}_B : B \in \mathcal{F}_\alpha\}$ is a κ -Kurepa tree. $\square$

We shall next construct an ω_1 -Kurepa family and from that extract an ω_1 -Kurepa tree.

THEOREM 6.10: $\Diamond^+$ implies there exists an ω_1 -Kurepa tree.

PROOF: We shall construct a κ -Kurepa family and then invoke Theorem 6.9.

Let $C \subseteq \omega_1$ and $\xi < \omega_1$. Then define $s(C, \xi) = \sup((C \cup \{0\}) \cap (\xi + 1))$. (Note that if C is closed, then $s(C, \xi)$ is the largest element of $C \cup \{0\}$, which is $\leq \xi$.)

Next let $X(A, C) = \{\xi \in A : \neg \exists \eta \in A (s(C, \xi) \leq \eta < \xi)\}$ (i.e. $\forall \beta \in C$, $X(A, C)$ contains the least $\xi \in A$ such that $s(C, \xi) = \beta$, if such a ξ exists).

So $X(A, C) \subseteq A$.

Claim: If $|A| = \omega_1$ and C is c.u.b., then $|X(A, C)| = \omega_1$.

Proof: Since A is cofinal in ω_1 and C is unbounded in ω_1 , $\forall \gamma \in C$, $\exists \xi \in A$ such that $\xi > \gamma$.

3. $B \in \mathcal{F}_\alpha$ implies that $\exists A \in \mathcal{F}$ such that $B = A \cap \alpha$, and hence, $B \subseteq \alpha$; so the characteristic

function of B is defined by $x_B(x) = \begin{cases} 1 & \forall x \in B \\ 0 & \forall x \in \alpha \setminus B. \end{cases}$

Now let $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$, be a $\Diamond^+$ -sequence, and let F be the set of all $X(A,C)$'s with the following properties:

- a. $\forall \alpha \in C (A \cap \alpha \in \mathcal{A}_\alpha)$.
- b. $\forall \alpha \in C (C \cap \alpha \in \mathcal{A}_\alpha)$.
- c. $A, C \subseteq \omega_1$; $|A| = \omega_1$; and C is c.u.b.

So $\forall A \subseteq \omega_1 (|A| = \omega_1 \rightarrow \exists X(A,C) \in F$ such that $|X(A,C)| = \omega_1$ and $X(A,C) \subseteq A$.

Now there are more than ω_1 many subsets of ω_1 of size ω_1 . Therefore, $|F| > \omega_1$.

It remains to prove that $\forall \beta < \omega_1 (|\{X(A,C) \cap \beta : X(A,C) \in F\}| < \omega_1)$.

Claim i: $\forall A, C \subseteq \omega_1$ satisfying (a), (b), and (c) and $\forall \beta < \omega_1$

$$|X(A,C) \cap \beta| \leq 1 \quad \text{or}$$

$$\exists \alpha \leq \beta \exists x \subseteq \beta \exists B, D \in \mathcal{A}_\alpha (X(A,C) \cap \beta = X(B,D) \cup x \text{ and } |x| \leq 1). \quad (*)$$

Proof: Let $\alpha = s(C, \beta) =$ the largest element in $C \cap \{0\} \leq \beta$.

Case 1: $\alpha = 0$. If $A \cap \beta = \emptyset$, then $X(A,C) \cap \beta = \emptyset$; otherwise, $X(A,C) \cap \beta$ is a singleton, namely, the least element in $A \geq 0$. So $|X(A,C) \cap \beta| \leq 1$.

Case 2: If $\alpha > 0$, then $\alpha \in C$. Let $B = A \cap \alpha$ and $D = C \cap \alpha$. Then $B, D \in \mathcal{A}_\alpha$ by properties (a) and (b) above. Let $\eta =$ the least element of $A \setminus \alpha$ and $x = \{\eta\}$. If $\eta \geq \beta$, then $X(B,D) \subseteq X(A,C) \cap \beta$ by definitions of B and D , and $X(A,C) \cap \beta \subseteq X(B,D)$ because $\eta \geq \beta$ (i.e., *the least element in* $A \setminus \alpha \geq \sup(C \cap \beta)$ *since* C *is closed*); so, $X(A,C) \cap \beta = X(B,D)$. On the other hand, if $\eta < \beta$, then $X(A,C) \cap \beta \subseteq X(B,D) \cup x$ as argued above and $X(B,D) \cup x \subseteq X(A,C) \cap \beta$ by (a) and (b). Therefore, $X(A,C) \cap \beta = X(B,D) \cup x$. Hence, $\exists \alpha \leq \beta \exists x \subseteq \beta \exists B, D \in \mathcal{A}_\alpha (X(A,C) \cap \beta = X(B,D) \cup x \wedge |x| \leq 1)$.

Thus, $(*)$ holds.

Claim ii: Claim i $\Rightarrow \forall \beta < \omega_1$ ($|\{X(A,C) \cap \beta : X(A,C) \in F\}| < \omega_1$).

Proof: There are countably many B's and D's in $\mathcal{A}_\alpha$ and the number of possibilities for η is $|\beta| < \omega_1$.

Hence, F is a ω_1 -Kurepa family.

Then by Theorem 6.10, there exists an ω_1 -Kurepa tree. $\square$

Chapter 7

WELL FOUNDED SETS

In this chapter we will develop some of the theory of well-founded sets. This theory, will then be used in the development of the constructible universe L since L is well-founded.

DEFINITION 7.1: A **class** is a collection of things which can be defined by a formula: anything of the form $\{x : \phi(x)\}$. A **proper class** is a class which is "too large" to be a set.

EXAMPLES:

1. $V = \{x : x = x\}$. (This is the universe of ZFC.)
2. $\text{Ord} = \{x : x \text{ is an ordinal}\}$.

DEFINITION 7.2: $\forall \alpha \in \text{Ord}$, we define $R(\alpha)$ recursively as follows:

- i. $R(0) = 0$.
- ii. $R(\alpha+1) = \mathcal{P}(R(\alpha))$.
- iii. For α a limit ordinal, $R(\alpha) = \bigcup_{\xi < \alpha} R(\xi)$.

DEFINITION 7.3: The class of **well founded sets** is defined by:

$$WF = \bigcup \{R(\alpha) : \alpha \in \text{Ord}\}.$$

(So WF is the class of sets constructed from 0 by iterating the set-theoretic operations.)

REMARK 7.4: Under the Axiom of Foundation, $V = WF$.

DEFINITION 7.5: A set A is **transitive** iff $\forall x \in A (x \subset A)$.

EXAMPLE: Set $S = \{\emptyset, \{\emptyset\}\}$ is transitive because $\emptyset, \{\emptyset\} \in S$ and $\emptyset, \{\emptyset\} \subseteq S$.

LEMMA 7.6: For every ordinal α :

- a. $R(\alpha)$ is transitive.
- b. $\forall \xi \leq \alpha (R(\xi) \subseteq R(\alpha))$.

PROOF (by induction on α): Assume that the lemma holds $\forall \beta < \alpha$.

If $\alpha = 0$, the lemma holds vacuously.

a. Let $\alpha = \beta + 1$. Then $R(\alpha) = R(\beta + 1) = \mathcal{P}(R(\beta))$. Since $R(\beta)$ is transitive (induction hypothesis), $\mathcal{P}(R(\beta))$ is transitive. Hence, $R(\alpha)$ is transitive.

Let α be a limit ordinal. Then $R(\alpha) = \bigcup_{\beta < \alpha} R(\beta)$ is transitive because $R(\beta)$ is transitive and the union of transitive sets is transitive.

b. Let $\alpha = \beta + 1$.

Case i: Let $\xi = \beta$. Then $R(\xi) = R(\beta) \subseteq \mathcal{P}(R(\beta)) = R(\beta + 1) = R(\alpha)$ by part (a).

Case ii: Let $\xi < \beta$. Then $R(\xi) \subseteq R(\beta) \subseteq \mathcal{P}(R(\beta)) = R(\beta + 1) = R(\alpha)$ by induction hypothesis, part (a), and the definition of $R(\beta + 1)$.

If α is a limit ordinal, then part (b) follows immediately from the induction hypothesis and the definition of $R(\alpha)$. $\square$

LEMMA 7.7: For $x \in WF$, the least α for which $x \in R(\alpha)$ is a successor ordinal.

PROOF: Assume that the least such α is a limit ordinal. Then for this α , $R(\alpha) = \bigcup_{\xi < \alpha} R(\xi)$. Therefore, $\exists \xi < \alpha$ such that $x \in R(\xi)$ which contradicts the "leastness" of α . $\square$

DEFINITION 7.8: For $x \in WF$, the **rank(x)** is defined to be the least β such that $x \in R(\beta+1)$.

NOTE: Since $x \in R(\beta+1)$, $x \subseteq R(\beta)$.

LEMMA 7.9: For all α , $R(\alpha) = \{x \in WF : \text{rank}(x) < \alpha\}$.

PROOF: Let $x \in R(\alpha)$.

If $\alpha = \beta+1$, then $x \in R(\alpha)$ iff $x \in R(\beta+1)$ iff $\text{rank}(x) \leq \beta < \alpha$ by definition of $\text{rank}(x)$.

If α is a limit ordinal, then $R(\alpha) = \bigcup_{\beta < \alpha} R(\beta)$. So $x \in R(\alpha)$ iff $\exists \beta < \alpha$ such that $x \in R(\beta)$. Since α is a limit ordinal, $\beta+1 < \alpha$. Now $x \in R(\beta)$ implies that $x \in R(\beta+1)$ (Lemma 7.6b). Hence, $x \in R(\beta)$ iff $\text{rank}(x) \leq \beta < \alpha$. $\square$

LEMMA 7.10: If $y \in WF$, then

- a. $\forall x \in y (x \in WF \wedge \text{rank}(x) < \text{rank}(y))$, and
- b. $\text{rank}(y) = \sup(\{\text{rank}(x)+1 : x \in y\})$.

PROOF:

a. Let $\alpha = \text{rank}(y)$. Then $y \in R(\alpha+1) = \mathcal{P}(R(\alpha))$. If $x \in y$, then $x \in R(\alpha)$ (by def. of rank); and hence, $\text{rank}(x) < \alpha$ (by Lemma 7.9) and $\alpha = \text{rank}(y)$.

b. Let $\alpha = \sup(\{\text{rank}(x)+1 : x \in y\})$. Then $\alpha \leq \text{rank}(y)$ (by part (a)).

Furthermore, $\forall x \in y (\text{rank}(x) < \alpha)$; so $y \subseteq R(\alpha)$ (by Lemma 7.9). But $R(\alpha)$ is

transitive, so $y \in R(\alpha) \subseteq R(\alpha+1)$; hence, $y \in R(\alpha+1)$. Therefore,
 $\text{rank}(y) \leq \alpha = \sup(\{\text{rank}(x)+1 : x \in y\})$.

Hence, $\text{rank}(y) = \sup(\{\text{rank}(x)+1 : x \in y\})$. $\square$

NOTE: Part (a) of this lemma says that WF is transitive and its elements are constructed from smaller ranked well-founded elements. Thus, the possibility that $x \in x$ is eliminated; otherwise, we could have the situation of $\text{rank}(x) < \text{rank}(x)$.

LEMMA 7.11:

- a. $\forall \alpha \in \text{Ord} (\alpha \in WF \wedge \text{rank}(\alpha) = \alpha)$.
- b. $\forall \alpha \in \text{Ord} (R(\alpha) \cap \text{Ord} = \alpha)$.

PROOF:

a. (by transfinite induction on α) Assume that $\forall \beta \in \alpha (\beta \in WF \wedge \text{rank}(\beta) = \beta)$. Then $\forall \beta < \alpha (\beta \in R(\beta+1) \subseteq R(\alpha))$ (by Lemma 7.10b), which implies that $\alpha \subseteq R(\alpha)$. Hence, $\alpha \in R(\alpha+1)$; thus, $\alpha \in WF$. Furthermore,

$$\begin{aligned} \text{rank}(\alpha) &= \sup(\{\text{rank}(\beta)+1 : \beta \in \alpha\}) \quad (\text{by Lemma 7.10b}) \\ &= \sup(\{\beta+1 : \beta \in \alpha\}) \quad (\text{by induction hypothesis}) \\ &= \alpha. \end{aligned}$$

b. We have that $R(\alpha) = \{x \in WF : \text{rank}(x) < \alpha\}$ (Lemma 7.9). Then $R(\alpha) \cap \text{Ord} = \{\beta \in \text{Ord} : \text{rank}(\beta) < \alpha\}$. But $\forall \beta \in \text{Ord} (\text{rank}(\beta) = \beta)$ (by part (a)). Hence, $R(\alpha) \cap \text{Ord} = \{\beta \in \text{Ord} : \beta < \alpha\} = \alpha$. $\square$

LEMMA 7.12: If $x, y \in WF$, then the following are also in WF :

- a. $\bigcup x$,

- b. $\mathcal{P}(x)$,
- c. $\{x\}$,
- d. $x \times y$,
- e. $x \cup y$,
- f. $x \cap y$,
- g. $\{x, y\}$,
- h. $\langle x, y \rangle$, and
- i. $\forall x$, and

Furthermore, the ranks of sets (a) - (c) are less than $\text{rank}(x) + \omega$, and the ranks of sets (d) - (i) are all less than $\max(\{\text{rank}(x), \text{rank}(y)\}) + \omega$.

PROOF: a. Let $\alpha = \text{rank}(x)$. Then $x \subseteq R(\alpha)$.

a. $x \subseteq R(\alpha) \Rightarrow \forall x_i \in x (x_i \in R(\alpha))$. But $R(\alpha)$ is transitive, so $x_i \subseteq R(\alpha)$, which implies that $\bigcup x \subseteq R(\alpha)$, which implies $\bigcup x \in R(\alpha+1)$. Hence, $\bigcup x \in \text{WF}$.

b. $x \subseteq R(\alpha) \Rightarrow \mathcal{P}(x) \subseteq \mathcal{P}(R(\alpha)) = R(\alpha+1)$.

Then $\mathcal{P}(x) \subseteq R(\alpha+1) \Rightarrow \mathcal{P}(x) \in \mathcal{P}(R(\alpha+1)) = R(\alpha+2) \subseteq \text{WF}$. Hence, $\mathcal{P}(x) \in \text{WF}$.

c. $x \subseteq R(\alpha) \Rightarrow x \in \mathcal{P}(R(\alpha)) = R(\alpha+1) \Rightarrow x \subseteq R(\alpha+1) \Rightarrow \{x\} \in \mathcal{P}(R(\alpha+1)) = R(\alpha+2)$.

e. - f. Let $\alpha = \max \{\text{rank}(x), \text{rank}(y)\}$. Then the proofs are similar to those above. $\square$

LEMMA 7.13: $\forall x (x \in \text{WF} \leftrightarrow x \subseteq \text{WF})$.

PROOF: $x \in \text{WF} \Rightarrow x \subseteq \text{WF}$ since WF is transitive (Lemma 7.10a). To prove the other direction, let $x \in \text{WF}$ and let $\alpha = \sup \{\text{rank}(y) + 1 : y \in x\}$. Then $x \subseteq R(\alpha)$ which implies that $x \in R(\alpha+1)$. Hence, $x \in \text{WF}$. $\square$

Our goal to keep in mind is that we want to prove that $V = L \Rightarrow \Diamond$ and $\Diamond^+$.

The next step in that direction is to develop the concept of well-founded relations, a concept which generalizes the notion of well-order.

DEFINITION 7.14: A relation R on set S is **well-founded** iff

$$\forall X \subseteq S (X \neq \emptyset \rightarrow \exists y \in X (\neg \exists z \in X (z R y))). \quad (*)$$

The y in $(*)$ is called an **R -minimal** element of X .

So a well-founded relation R on S is a binary relation which applies to non-linear orderings and which has an R -minimal element in every nonempty subset of S .

NOTES:

1. If R is a linear order, then well-founded becomes well-ordered.
2. If R is not a linear order, then the X in $(*)$ may have several minimal elements.

DEFINITION 7.15: x is an **$\in$ -minimal** element of a class C if $x \cap C = \emptyset$.

LEMMA 7.16 (ZF⁻): If $A \in \text{WF}$, then $\in$ is well-founded on A .

PROOF: Let $X \neq \emptyset$ be a subset of A , and let $\alpha = \min \{\text{rank}(y) : y \in X\}$ (this is possible since ranks are ordinals and ordinals are well ordered). Then y is an $\in$ -minimal element of X (Lemma 7.10a). $\square$

LEMMA 7.17 (ZF⁻): If A is transitive and $\in$ is well-founded on A , then $A \in \text{WF}$.

PROOF: To show that $A \in WF$, it suffices to show that $A \subseteq WF$ (Lemma 7.13). To show that $A \subseteq WF$, assume not and let $X = A \setminus WF$, which is not empty. Let y be an ϵ -minimal element of X . Suppose $z \in y$; then $z \notin X$. But $z \in A$ (since A is transitive), so $z \in WF$. Therefore, $y \subseteq WF$ (Lemma 7.13) contradicting that $y \in A \setminus WF$. $\square$

NOTE: Lemma 7.17 is a partial converse to Lemma 7.16. The actual converse is not always true. For example, look at $y = \{x\}$ with $x = \{y\}$ with $y \neq x$. In this case, ϵ is well-founded on y (i.e., empty on y), but $y \notin WF$.

Now the Axiom of Foundation,

$$\forall x (\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))),$$

states that every non-empty set has an ϵ -minimal element. The following lemma expands this idea to classes.

LEMMA 7.18: If C is a nonempty class, then C has an ϵ -minimal element.

PROOF: Since $C \neq \emptyset$, let $x \in C$ be an arbitrary set in C . Then $C \cap x \subseteq x$, so $C \cap x$ is a set. Therefore, by Foundation, $C \cap x$ has an ϵ -minimal element. But $C \cap x \subseteq C$ and an ϵ -minimal element of $C \cap x$ is also an ϵ -minimal element of C . $\square$

To show the equivalence of the Axiom of Foundation and $V = WF$, we will recursively define the transitive closure of A .

DEFINITION 7.19 (ZF-P): Let A be any set. Then define the following:

- i. $\bigcup^0 A = A$
- ii. $\bigcup^{n+1} A = \bigcup(\bigcup^n A)$

Then the **transitive closure** of A is defined by:

$$\text{tr cl}(A) = \bigcup\{\bigcup^n A : n \in \omega\}.$$

LEMMA 7.20 (ZF-P):

- a. $A \subseteq \text{tr cl}(A)$.
- b. $\text{tr cl}(A)$ is transitive.

PROOF:

a. Follows immediately from the definitions.

b. $y \in \text{tr cl}(A) \Rightarrow \exists n (y \in \bigcup^n A) \Rightarrow y \subseteq \bigcup^{n+1} A \Rightarrow y \subseteq \text{tr cl}(A)$. $\square$

THEOREM 7.21 (ZF): For any set A , the following are equivalent:

- a. $A \in \text{WF}$.
- b. $\text{tr cl}(A) \in \text{WF}$.
- c. $\in$ is well-founded on $\text{tr cl}(A)$.

PROOF:

(a) $\Rightarrow$ (b): Let $A \in \text{WF}$. Then $\bigcup^0 A \in \text{WF}$. Now assume that $\bigcup^n A \in \text{WF}$.

Then $\bigcup^{n+1} A = \bigcup(\bigcup^n A) \in \text{WF}$ (Lemma 7.12a). Therefore,

$\forall n (\bigcup^n A \in \text{WF})$. Hence, $\text{tr cl}(A) \in \text{WF}$.

(b) $\Rightarrow$ (c): is an application of Lemma 7.16.

(c) $\Rightarrow$ (a): Our hypothesis that $\in$ is well-founded on $\text{tr cl}(A)$, along with the fact that $\text{tr cl}(A)$ is transitive (Lemma 7.20b), implies that $\text{tr cl}(A) \in \text{WF}$

(Lemma 7.17). Therefore, $\text{tr cl}(A) \subseteq \text{WF}$. Furthermore, since $A \subseteq \text{tr cl}(A)$, $A \in \text{tr cl}(A)$. Hence, $A \in \text{WF}$. $\square$

THEOREM 7.22 (ZF⁻): The following are equivalent:

- a. the Axiom of Foundation
- b. $\forall A (\in \text{ is well-founded on } A)$
- c. $V = \text{WF}$.

PROOF:

(a) $\Leftrightarrow$ (b) follows immediately from the definition of well-founded.

(b) $\Rightarrow$ (c): Clearly, $\forall A \in \text{WF}, A \in V$. For the other direction, fix $A \in V$ and let $\in$ be well-founded on A (by (b)). Then $A \in \text{WF}$ (by Theorem 7.21). Hence $V = \text{WF}$.

(c) $\Rightarrow$ (b) is just an application of Lemma 7.16. $\square$

THEOREM 7.23 (ZF-P): $A \in \text{Ord}$ iff A is transitive and linearly ordered by $\in$.

PROOF:

($\Rightarrow$) This is just the definition of what it means to be an ordinal.

($\Leftarrow$) Let X be an arbitrary non-empty subset of A . Then apply Foundation to X to establish the existence of an $\in$ -minimal element in X . Then $\in$ well-orders A . Hence A is an ordinal. $\square$

PRINCIPAL OF TRANSFINITE INDUCTION ON WELL-FOUNDED

RELATIONS 7.24: If R is well-founded on A and ϕ is a formula, then

$$\forall x \left(\forall y \in A (y R x \rightarrow \phi(y)) \rightarrow \phi(x) \right) \rightarrow \forall x \in A \phi(x).$$

PROOF: We shall prove the contrapositive. So assume that $\neg (\forall x \in A \phi(x))$, that is $\exists x \in A (\neg \phi(x))$. We shall show that $\neg (\forall y \in A (y R x \rightarrow \phi(y)) \rightarrow \phi(x))$; that is, $\exists y \in A \neg ((y R x \rightarrow \phi(y)) \rightarrow \phi(x))$. So assume that

$$\forall y \in A (y R x \rightarrow \phi(y)) \rightarrow \phi(x). \quad (*)$$

Consider set $C = \{x \in A : \neg \phi(x)\}$, the set of all counterexamples. Since C is not empty (by our initial assumption), let $x \in C$ be R -minimal in C . Then $\phi(x)$ is false. But this contradicts $(*)$ which says that for every $y \in A$ such that $y R x$, $\phi(y)$ is true. Hence, $(*)$ must be false. $\square$

DEFINITION 7.25: R is **set-like** on A iff $\forall x \in A$, $\{y \in A : y R x\}$ is a set (i.e. the collection of all points R -below x forms a set, $\text{pred}(A, x, R)$, rather than a proper class).

THEOREM (TRANSFINITE RECURSION) 7.26: Suppose that R is well-founded and set-like and that $F: A \times V \rightarrow V$ where F is a definable function and V is the universe. Then there is a unique function $g: A \rightarrow V$ such that $\forall x \in A (g(x) = F(x, g|_{\text{pred}(A, x, R)}))$

PROOF: See Kunen, p. 103, Theorem 5.6 and then p.25-26, Theorem 9.3. $\square$

We conclude this section with the Mostowski Collapsing Theorem which will be used to prove several important results in later sections.

DEFINITION 7.27 ($\text{ZF}^- - P$): Let R be a well-founded, set-like relation on A . Then we define the **Mostowski collapsing function**, $G: A \rightarrow M$, by

$$G(x) = \{G(y) : y \in A \wedge y R x\}.$$

$M = \text{rng}(G)$ is called the **Mostowski collapse** of A, R .

DEFINITION 7.28 ($\text{ZF}^- - P$): R is an **extensional** relation on A iff

$\forall x, y, x \in A$, whenever $x R y \leftrightarrow x R z$, then $y = z$.

MOSTOWSKI COLLAPSING THEOREM 7.29: If E is an extensional, well-founded, set-like relation on class M , then there is a unique isomorphism, $\pi : \langle M, E \rangle \cong \langle \bar{M}, \epsilon \rangle$, for some transitive class $\bar{M}$.

PROOF: Define π by $\pi(x) = \{\pi(y) : y E x\}$ for $x, y \in M$ (this is possible by transfinite recursion on E). Note that if x is minimal, then $\pi(x) = \emptyset$ since there is nothing E -below x . Also note that π is a definable class function. Next set $\bar{M} = \text{rng}(\pi)$.

Claim i: $\bar{M}$ is a transitive class.

Proof: Since π is a definable function, $\bar{M}$ is a class. Now suppose that $z \in \pi(x) \in \bar{M}$. Then $\exists y \in M$ such that $y E x$ where $z = \pi(y)$. Therefore, $z \in \text{rng}(\pi)$ which implies that $z \in \bar{M}$. Hence, $\bar{M}$ is transitive.

Claim ii: π is one-to-one.

Proof: We shall prove by induction on x that $\forall x \forall y ((\pi(x) = \pi(y)) \rightarrow x = y)$.

So assume that $\pi(x) = \pi(y)$. By the definition of π , this says that

$$\{\pi(z) : z E x\} = \{\pi(z^*) : z^* E y\} \quad (*)$$

We want to show $x = y$. Since E is extensional, it suffices to show that $z E x \leftrightarrow z E y$. Our induction hypothesis says that $\forall z, z^* E$ -below x ,

$\pi(z) = \pi(z^*) \rightarrow z = z^*$. So suppose that $z E x$. Then $\pi(z) \in \text{LHS of } (*)$; therefore, $\pi(z) \in \text{RHS of } (*)$. Then $\exists z^*$ with $z^* E y$ such that $\pi(z) = \pi(z^*)$. But z is E -beneath x ; so by induction hypothesis, $z = z^*$. Therefore, $z E y \leftrightarrow z^* E y$. Hence, $z E x \rightarrow z^* E y \leftrightarrow z E y$.

The argument for the other direction is not quite symmetrical here because our induction hypothesis is on x and not on y . So suppose that $z^* E y$. Then $\exists z$ such that $z E x$ where $\pi(z) = \pi(z^*)$. Now z is E -beneath x ; hence by induction, $z = z^*$. So $z^* E x \rightarrow z E x \leftrightarrow z E y$.

Therefore, π is one-to-one.

Claim iii: π is a homomorphism.

Proof: If $x E y$, then $\pi(x) \in \pi(y)$ (by definition of π). If $\pi(x) \in \pi(y)$, then $\exists x^*$ such that $x^* E y$ and $\pi(x) = \pi(x^*)$. But π is one-to-one, so $x = x^*$. Therefore, $x E y$. Hence, order is preserved.

Claim iv: π is unique.

Proof: Assume not. Then there exists a function $\mu : \langle M, E \rangle \cong \langle \bar{M}, \in \rangle$. It suffices to show that $\mu(x) = \{\mu(y) : y \in x\}$ (since two functions with the same definition are the same function). So suppose $x, y \in M$ such that $y E x$. Then $\mu(y) \in \mu(x)$ since isomorphisms preserve order. Therefore, $\{\mu(y) : y \in x\} \subseteq \mu(x)$. Next let $z \in \mu(x)$; then $\exists \hat{y} \in M$ ($\mu(\hat{y}) = z$). Hence, $\mu(\hat{y}) \in \mu(x)$; so $\hat{y} E x$ (μ is an isomorphism). Therefore, $z = \mu(\hat{y}) \in \{\mu(y) : y \in x\}$. Thus, $\mu(x) \subseteq \{\mu(y) : y \in x\}$. Hence, $\mu(x) = \{\mu(y) : y \in x\}$. $\square$

So this theorem says that if you have a nonstandard model and the relation is extensional, well-founded, and set-like, then this model is isomorphic to a standard model (see Definition 10.2).

Chapter 8

ABSOLUTENESS

The next goal will be to look at develop the notion of truth in a class M .

DEFINITION 8.1: The **bounded formulas** are formulas of the forms:

$\forall x \in y \phi$ which abbreviates $\forall x (x \in y \rightarrow \phi)$, or

$\exists x \in y \phi$ “ $\exists x (x \in y \wedge \phi)$.

DEFINITION 8.2: We define a Δ_0 formula, ϕ , recursively on the length of ϕ :

- i. If ϕ is $x = y$, then ϕ is Δ_0 .
- ii. If ϕ is $x \in y$, then ϕ is Δ_0 .

For Δ_0 formulas σ and ψ :

- iii. $\sigma \wedge \psi$ is Δ_0 .
- iv. $\neg \psi$ is Δ_0 .
- v. $\exists x \in y (\psi(x))$ is Δ_0 .

Δ_0 is the set of bounded formulas (i.e. the closure of the atomic formulas, $x = y$ and $x \in y$ under $\rightarrow$, $\neg$, and bounded quantification.)

DEFINITION 8.3: Let M be any set and ϕ be any formula. Then ϕ^M , the **relativization of ϕ to M** is defined recursively as follows:

- i. $(x = y)^M$ is $x = y$.
- ii. $(x \in y)^M$ is $x \in y$.
- iii. $(\phi \wedge \psi)^M$ is $\phi^M \wedge \psi^M$.

iv. $(\neg\phi)^M$ is $\neg(\phi)$.

v. $(\exists x \phi)^M$ is $\exists x (x \in M \wedge \phi^M)$.

vi. $(\forall x \phi)^M$ is $\forall x (x \in M \rightarrow \phi^M)$.

NOTE: ϕ^M is the formula obtained from ϕ by replacing $\exists x$ with $\exists x \in M$. So if $\phi(x_1, \dots, x_n)$ is a formula with $x_1, \dots, x_n$ free variables, then for $x_1, \dots, x_n \in M$, $\phi^M(x_1, \dots, x_n)$ is the statement that ϕ^M is true of $x_1, \dots, x_n$ if bound variables range over M .

DEFINITION 8.4: Let M be any class and ϕ be any formula. Then ϕ^M is defined as above except for parts (v) and (vi):

v. $(\exists x \phi)^M$ is $\exists x (\psi(x) \wedge \phi^M)$

vi. $(\forall x \phi)^M$ is $\forall x (\psi(x) \rightarrow \phi^M)$

where $(\psi \rightarrow \phi)$ abbreviates $\neg(\psi \wedge \neg\phi)$ and ψ defines M .

DEFINITION 8.5: For any class M :

a. For any sentence ϕ , ϕ is true in M means ϕ^M .

b. For any set of sentences S , M is a **model for S** means $\forall \phi \in S, \phi$ is true in M .

NOTATION 8.6: For a set of well-founded formulas S , $S \models \phi$ means S tautologically implies ϕ . $S \vdash \phi$ means ϕ is a theorem of S . (A complete discussion of these concepts can be found in Enderton.)

The following metatheorem gives a notion of what it means for a formula "to be true" in a set structure.

METATHEOREM 8.7: Let M be any set. If $x_1, \dots, x_n \in M$. then

$$\langle M, \in \rangle \models \phi(x_1, \dots, x_n) \leftrightarrow \phi^M(x_1, \dots, x_n).$$

PROOF: This can be proved easily by induction using the definition of relativization. $\square$

Now this metatheorem only makes sense for sets, not for proper classes because $\langle M, \in \rangle, \phi(x_1, \dots, x_n)$ is an element in the relation $\models$ which only holds for pairs of sets. So the expression $\phi^M(x_1, \dots, x_n)$ will become our notion of what it means for a formula "to be true" in a class structure. One way to check if a particular formula holds in a class structure, M , is to show that any formula of the same form which is true in the universe, also holds in M .

DEFINITION 8.8: Let M, N be any classes. Then a formula ϕ with free variables a subset of $\{x_1, \dots, x_n\}$ is said to be **absolute** as follows:

a. If $M \subseteq N$, then ϕ is **absolute for M, N** iff

$$\forall x_1, \dots, x_n \in M (\phi^M(x_1, \dots, x_n) \leftrightarrow \phi^N(x_1, \dots, x_n)).$$

b. ϕ is **absolute for M** iff ϕ is absolute for M, V . Formally,

$$\forall x_1, \dots, x_n \in M (\phi^M(x_1, \dots, x_n) \leftrightarrow \phi(x_1, \dots, x_n)).$$

So ϕ is absolute for any two classes M, N iff ϕ is true for $M \Leftrightarrow \phi$ is true for N .

DEFINITION 8.9: If $M \subseteq N$ and the function $F(x_1, \dots, x_n)$ is defined to be the

unique y such that $\phi(x_1, \dots, x_n, y)$, assuming that $\forall x_1, \dots, x_n \exists! y \phi(x_1, \dots, x_n, y)$ is true in both M and N . Then F is **absolute for M, N** iff $\forall x_1, \dots, x_n \in M (F^M(x_1, \dots, x_n) = F^N(x_1, \dots, x_n))$ (i.e. F is absolute for M, N if $F(x_1, \dots, x_n) = y$).

LEMMA 8.10: If $M \subseteq N$ and ϕ and ψ are absolute for M, N , then $\neg\phi$ and $\phi \wedge \psi$ are also absolute for M, N .

PROOF: $(\neg\phi)^M = \neg\phi^M \Leftrightarrow \neg\phi^N = (\neg\phi)^N$.

$(\phi \wedge \psi)^M = \phi^M \wedge \psi^M \Leftrightarrow \phi^N \wedge \psi^N = (\phi \wedge \psi)^N$. $\square$

LEMMA 8.11: Any quantifier free formula ϕ is absolute for any class.

PROOF (by induction on ϕ): All atomic formulas are absolute and by Lemma 8.10, all other quantifier free formulas are absolute. $\square$

LEMMA 8.12: If M and N are both transitive classes with $M \subseteq N$, then any bounded formula ϕ is absolute for M, N .

PROOF: (By induction on ϕ)

i. Any quantifier free formula is absolute.

Let ψ be absolute.

ii. Let $\phi = (\exists x \in y \psi)^M$ with $y \in M$. Then,

$$\begin{aligned} (\exists x \in y \psi)^M &\Leftrightarrow \exists x (x \in y \wedge \psi)^M \\ &\Leftrightarrow \exists x \in M (x \in y \wedge \psi^M) \\ &\Leftrightarrow \exists x \in y \psi^M \quad (\text{by transitivity of } M) \\ &\Leftrightarrow \exists x \in y \psi^N \quad (\text{by absoluteness of } \psi) \\ &\Leftrightarrow \exists x \in N (x \in y \wedge \psi^N) \quad (\text{by transitivity of } N) \end{aligned}$$

$$\leftrightarrow \exists x (x \in y \wedge \psi)^N.$$

iii. The proof for $\phi = (\forall x \in y \psi)^M$ is similar. $\square$

LEMMA 8.13: If M is a transitive class and ϕ is Δ_0 , then ϕ is absolute for M .

PROOF: The proof of this lemma follows by induction directly from the definitions of relativization and absoluteness and from Lemmas 8.10, 8.11, and 8.12. $\square$

Rarely do we actually encounter Δ_0 formulas, but rather find formulas logically equivalent to Δ_0 formulas. So often the following lemma is needed to in conjunction with Lemma 8.13 to show absoluteness.

LEMMA 8.14: Suppose $M \subseteq N$ with M, N transitive class models for a set of sentences S such that

$$S \vdash \forall x_1, \dots, x_n (\phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)). \quad (*)$$

Then ϕ is absolute for M, N iff ψ is absolute for M, N .

PROOF: Let $x_1, \dots, x_n \in M$.

($\Rightarrow$) Assume ϕ is absolute for M, N . Then

$$\begin{aligned} S \vdash \phi(x_1, \dots, x_n)^M &\Leftrightarrow \phi^M(x_1, \dots, x_n) \quad (\text{by } (*)) \\ &\Leftrightarrow \phi^N(x_1, \dots, x_n) \quad (\text{by absoluteness of } \phi) \\ &\Leftrightarrow \psi^N(x_1, \dots, x_n) \quad (\text{by } (*)). \end{aligned}$$

($\Leftarrow$) Assume ψ is absolute. The proof is similar. $\square$

LEMMA 8.15: Absolute notions are closed under composition: Suppose $M \subseteq N$ and $\phi(x_1, \dots, x_n)$, $F(x_1, \dots, x_n)$, $G_i(y_1, \dots, y_m)$ with $i = 1, \dots, n$ are all

absolute for M, N ; then $\phi(G_1(y_1, \dots, y_m), \dots, G_n(y_1, \dots, y_m))$ and $F(G_1(y_1, \dots, y_m), \dots, G_n(y_1, \dots, y_m))$ are absolute for M, N .

PROOF: For simplicity of notation, we shall prove the case for $m = n = 1$.

However, the proof is similar $\forall m, n \in \omega$. So let $m = n = 1$. Then

$$\begin{aligned} \phi(G(y))^M &\Leftrightarrow \phi^M(G^M(y)) \Leftrightarrow \phi^N(G^N(y)) \Leftrightarrow \phi(G(y))^N \quad \text{and} \\ F(G(y))^M &\Leftrightarrow F^M(G^M(y)) \Leftrightarrow F^N(G^N(y)) \Leftrightarrow F(G(y))^N. \quad \square \end{aligned}$$

Next we shall prove several absolute results by letting M be a transitive class with $N = V$ and $S =$ empty set of sentences. Then we shall apply Lemmas 8.9 and 8.10.

THEOREM 8.16: The following functions and relations are definable by Δ_0 formulas in $ZF^- - P - \text{Inf.}$ and hence are absolute:

a. $x = y$	l. $z = x \cup \{x\}$ (the successor function, $S(x)$)
b. $x \in y$	m. $z = \bigcup x$
c. $x \subseteq y$	n. $z = \bigcap x$ where we define $\bigcap \emptyset = \emptyset$
d. $z = \{x, y\}$	o. z is an ordered pair
e. $z = \{x\}$	p. $z = x \times y$ (cross product)
f. $z = \langle x, y \rangle$	q. R is a relation
g. $z = \emptyset$	r. $z = \text{dom}(R)$ where r is a binary relation
h. $z = x \cup y$	s. $z = \text{rng}(R)$ where R is a binary relation
i. $z = x \cap y$	t. F is a function
j. $z = x \setminus y$	u. $z = R(x)$
k. x is transitive	v. F is a 1-1 function

PROOF:

- a. and b. are atomic formulas, so are absolute.
- c. $x \subseteq y \Leftrightarrow \forall z \in x (z \in y)$.
- d. $z = \{x, y\} \Leftrightarrow x \in z \wedge y \in z \wedge \forall w \in z (w = x \vee w = y)$.
- e. $z = \{x\} \Leftrightarrow x \in z \wedge \forall y \in z (y = x)$.
- f. $z = \langle x, y \rangle \Leftrightarrow z = \{\{x\}, \{x, y\}\}$.
- g. $z = \emptyset \Leftrightarrow \forall x \in z \neg(x = x)$.
- h. $z = x \cup y \Leftrightarrow x \subseteq z \wedge y \subseteq z \wedge \forall w \in z (w \in x \vee w \in y)$.
- i. $z = x \cap y \Leftrightarrow \forall w \in z (w \in x \wedge w \in y) \wedge \forall w ((w \in x \wedge w \in y) \rightarrow w \in z)$.
- j. $z = x \setminus y \Leftrightarrow \forall w \in z (w \in x \wedge \neg(w \in y)) \wedge \forall w \in x (\neg(w \in y) \rightarrow w \in z)$.
- k. x is transitive $\Leftrightarrow \forall y \in x \forall z \in y (z \in x)$.
- l. The proof of $z = x \cup \{x\}$ uses parts (e) and (i).
- m. $z = \bigcup x \Leftrightarrow \forall y \in x (y \subseteq z) \wedge \forall w \in z \exists y \in x (w \in y)$.
- n. $z = \bigcap x \Leftrightarrow \forall y \in x (z \subseteq y) \wedge \forall w \in z (\forall y \in x (w \in y))$.
- o. z is an order pair $\Leftrightarrow \exists y \exists x (z = \langle x, y \rangle)$.
- p. $z = A \times B$ (cross product) $\Leftrightarrow \forall x \in A \forall y \in B (\langle x, y \rangle \in z)$
 $\wedge \forall w \in z \exists x \in A \exists y \in B (w = \langle x, y \rangle)$.
- q. R is a relation $\Leftrightarrow \forall z \in R (z \text{ is an ordered pair})$.
- r. $A = \text{dom}(R) \Leftrightarrow \forall x \in A \exists y \in \bigcup \bigcup R (\langle x, y \rangle \in R)$
 $\wedge \forall x \in \bigcup \bigcup R \forall y \in \bigcup \bigcup R (\langle x, y \rangle \in R \rightarrow x \in A)$.
- s. $B = \text{rng}(R) \Leftrightarrow \forall y \in B \exists x \in \text{dom}(R) (\langle x, y \rangle \in R)$
- t. F is a function $\Leftrightarrow F$ is a relation
 $\wedge \forall x \in \bigcup \bigcup F \forall y \in \bigcup \bigcup F \forall z \in \bigcup \bigcup F$
 $(\langle x, y \rangle \in F \wedge \langle x, z \rangle \in F \rightarrow y = z)$.

u. $z = F(x) \Leftrightarrow (\phi(x) \wedge \langle x, y \rangle \in F) \vee (\neg\phi(x) \wedge y = 0)$, where $\phi(x)$ is

$$\begin{aligned} & \exists v \in \bigcup \bigcup F (\langle x, v \rangle \in F \\ & \wedge \forall w \in \bigcup \bigcup F (\langle x, y \rangle \in F \rightarrow v = w)). \end{aligned}$$

v. F is a 1-1 function $\Leftrightarrow F$ is a function

$$\wedge \forall x, y \in \text{dom}(F) (F(x) = F(y) \rightarrow x = y). \quad \square$$

LEMMA 8.17: Let M be a transitive class model for $ZF^- - P - \text{Inf}$. If $\omega \in M$, then Infinity holds in M ; and if M is a model of Infinity, then $\omega \in M$.

PROOF: Infinity relativized to M is:

$$\exists x \in M (0 \in x \wedge \forall y \in x (y \cup \{y\} \in x)) \quad (*)$$

If $x \in M$, then ω is such an x ; so M is a model for Infinity.

On the other hand, if M is a model of Infinity, then $\exists x \in M$ such that $\omega \subseteq x$ (take the x which satisfies $(*)$). But then

$$\omega = \{z \in x : z \text{ is transitive} \wedge \forall y \in z \neg(y \text{ is a limit ordinal})\}.$$

Hence, $\omega \in M$. $\square$

LEMMA 8.18: Let M be a transitive class model of $ZF^- - P - \text{Inf}$. and let $R \in M$ be a well-founded relation. Then $(R \text{ is well-founded})^M$.

PROOF: Assume that R is well-founded in V but that $(R \text{ is well-founded})^M$ fails. Then $\exists x \in M (x \text{ has no } R\text{-least element})^M$. Now this last statement is a Δ_0 formula, so it is absolute. Thus, x has no R -least element in V contradicting that R is well-founded in the universe. Therefore, $(R \text{ is well-founded})^M$. $\square$

Next we want to prove the absoluteness of several ordinal notions and of well-ordering.

THEOREM 8.19: The following notions are absolute for transitive models of ZF - P:

a. x is an ordinal.	f. $x = 0$.
b. x is a limit ordinal.	g. $x = 1$.
c. x is a successor ordinal.	h. $x = 2$.
d. x is a finite ordinal.	...
e. $x = \omega$.	z. $x = 20$.

PROOF:

a. x is an ordinal iff x is transitive and linearly ordered by $\in$. Now " x is transitive" is absolute by Theorem 8.16k. By Foundation, which is absolute for transitive models, $\in$ is well-founded and, thus, absolute. Finally, " x is linearly ordered" can be expressed by the Δ_0 formula:

$$\forall y \in x \forall z \in x (y < z \vee y > x \vee y = z).$$

b. To be a limit ordinal, $x \neq \emptyset$. The notion of being an ordinal is absolute by part (a), and the notion of being a limit can be expressed by the Δ_0 formula:

$$\forall y \in x \exists z \in x (y \in z).$$

c. " x is a successor ordinal" is absolute by part (a) and Theorem 8.16.l.

d. x is a finite ordinal iff x is a successor ordinal and x does not contain a limit ordinal. All of this can be expressed by a Δ_0 formula using parts (a) - (c) and parts of Theorem 8.16.

e. $x = \omega$ is absolute because ω can be expressed as a limit ordinal which contains no limit ordinals.

f. See Theorem 8.16g.

g. $x = 1 \Leftrightarrow \exists y \in x (y = 0 \wedge x = S(y))$.

h. $x = 2 \Leftrightarrow \exists y \in x (y = 1 \wedge x = S(y))$.

.....

z. $x = 20 \Leftrightarrow \exists y \in x (y = 19 \wedge x = S(y))$. $\square$

LEMMA 8.20: If M is a transitive model of ZF-P, then ${}^{<\omega}M \subseteq M$ (so M is closed under finite sequences from M).

PROOF: We'll show that ${}^n M \subseteq M$ by induction on n . If $n = 0$, then $\emptyset \subseteq M$.

Now assume ${}^n M \subseteq M$ and consider ${}^{n+1}M$. Let $y \subseteq M$ such that $|y| = n+1$. Pick $x \in y$. Then $y \setminus \{x\} \in M$ by induction hypothesis.

Then by Union^M and Pairing^M, $\exists z (z = (y \setminus \{x\} \cup \{x\}))^M$ which is equivalent to $\exists z \in M (z = (y \setminus \{x\} \cup \{x\}))$. But $y = (y \setminus \{x\} \cup \{x\})$ so $z = y \in M$. $\square$

LEMMA 8.21: The following are absolute for transitive models, M , of ZF-P:

a. x is finite.

b. $B = A^n$ for $n \in \omega$.

c. $B = A^{<\omega}$ where $A^{<\omega} = \bigcup_{n < \omega} A^n$.

PROOF:

a. x is finite iff $\exists f \in M \phi(x, f)$ where $\phi(x, f)$ is a formula which stands for

" f is a function" $\wedge x = \text{dom}(f) \wedge \text{rng}(f) \in \omega \wedge f$ is 1-1.

$\phi(x, f)$ is absolute by Theorems 8.16 and 8.19. Therefore, it suffices to show

$$\forall x \in M (\exists f \in M \phi(x, f) \leftrightarrow \exists f \phi(x, f)).$$

Clearly, $f \in M$ implies $f \in V$: so the direction from left to right holds. To show $(\exists f \phi(x, f)) \rightarrow (\exists f \in M \phi(x, f))$, note that $\phi(x, f)$ implies there is a finite set of ordered pairs (since $\text{rng}(f) \in \omega$) of elements from M (since $x = \text{dom}(f)$ and $x \in M$, a transitive model). Now M is closed under pairing (pairing is absolute) so $f \in M$.

- b. $B = A^n$ iff $\forall f \in B (f \text{ is a function} \wedge \text{dom}(f) = n \wedge \forall x \in \text{dom}(f) (f(x) \in A))$.
- c. This proof is similar to that in (b). $\square$

THEOREM 8.22: The following are absolute for transitive models of ZF-P:

- a. R well-orders A .
- b. $\text{type}(A, R)$.

PROOF:

- a. R well-orders A iff R linearly orders A and R is well-founded on A .

Now " R linearly orders A " can be represented by the Δ_0 formula

$\forall x, y \in A (x < y \wedge x > y \wedge x = y)$. This together with Lemma 8.18 shows that if R well-orders A in the universe, then $(R \text{ well-orders } A)^M$.

Now if $(R \text{ well-orders } A)^M$, then $(\exists f \exists \alpha (f : \langle A, R \rangle \cong \langle \alpha, \in \rangle))^M$ because there's a theorem in ZF that every well-ordering is isomorphic to an ordinal (Kunen, p. 17, Theorem 7.6). Since the ordinals are absolute and f is an isomorphism, R must well-order A in the universe in the way that $f(A)$ is ordered in α .

Hence, " R well-orders A " is absolute.

- b. " $\text{type}(A, R)$ is absolute" follows directly from the argument in part (a). $\square$

Though a class A is just a formula $A(x)$, we tend to think of A as $A = \{x : A(x)\}$.

In these terms, A^M means $\{x \in M : A(x)^M\}$; so A is absolute for M iff

$A^M = A \cap M$. Some examples are:

- i. $V(x)$ is the formula $x = x$.
- ii. $v^M = M$.
- iii. $\text{Ord}^M = \text{Ord} \cap M$ if M is a transitive model of ZF-P.

Classes of more than one variable work similarly. For example, look at

$R \subseteq V \times V$. $R(x, y)$ is the formula which we think of as $R = \{\langle x, y \rangle : R(x, y)\}$.

Then $R^M = \{\langle x, y \rangle \in M : R(x, y)^M\}$, and R is absolute for M iff $R^M = R \cap M \times M$.

We think of class functions, say $G : V \rightarrow V$, as a class of ordered pairs

$\{\langle x, y \rangle : G(x, y)\}$ where $G(x, y)$ abbreviates the formula $\forall x \exists! y G(x, y)$. Thus

$$G^M \Leftrightarrow (\forall x \exists! y G(x, y))^M \Leftrightarrow \forall x \in M \exists! y \in M G(x, y)^M.$$

So $G^M : M \rightarrow M$, and G is absolute for M iff $G^M = G|_M$.

THEOREM 8.23: Let R be a well-founded, set-like relation on A , and let $F : A \times V \rightarrow V$. Then define $G : A \rightarrow V$ by $G(x) = F(x, G|_{\text{pred}(A, x, R)})$. Let M be a transitive model of ZF-P such that:

- a. F is absolute for M ,
- b. R and A are absolute for M ,
- c. $(R \text{ is Set-like on } A)^M$, and
- d. $\forall x \in M (\text{pred}(A, x, R) \subseteq M)$.

Then G is absolute for M .

PROOF:

Claim i: $(R \text{ is well-founded on } A)^M$.

Proof: $A^M = A \cap M \subseteq A$; so any non-empty subset of A^M has an R^M -minimal element since R is well-founded. Furthermore, $R^M = R \cap (M \times M)$ and $A^M \subset M$ so an R -minimal element of A^M is an R^M -minimal element as well. Now, by claim (i), we can recursively define $G^M : A^M \rightarrow M$ by

$$G^M(x) = F^M(x, G^M|_{\text{pred}(A^M, x, R^M)}). \quad (*)$$

Claim ii: G is absolute for M .

Proof: Assume not. Suppose $G^M \neq G|_{A^M}$. Then there exists an R -minimal element at which they disagree; call it x . Now look at $\text{pred}(A^M, x, R^M)$: G^M and $G|_{A^M}$ agree on this set. But $G^M(x)$ is defined by $(*)$, and $G|_{A^M}$ is just G in the outer model restricted to A^M . By the absoluteness of F , $G|_{A^M} = G^M$, which is a contradiction. Therefore, G is absolute for M . $\square$

THEOREM 8.24: The following are absolute for any transitive model of ZF-P:

- a. α^β for $\alpha, \beta \in \text{Ord}$.
- b. $\text{rank}(x)$
- c. $\text{tr cl}(x)$.

PROOF:

a. α^β , ordinal exponentiation, is defined recursively by

- i. $\alpha^0 = 1$.
- ii. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$.
- iii. For limit ordinal β , $\alpha^\beta = \sup\{\alpha^\xi : \xi < \beta\}$.

Hence, by Theorem 8.23, α^β absolute.

b. $\text{rank}(x) = \text{least } \beta \text{ such that } x \in R(\beta+1)$. Since $R(\alpha)$ is defined recursively, $\text{rank}(x)$ is also defined recursively. Hence, $\text{rank}(x)$ is absolute (Theorem 8.23).

NOTE: It is important to think of $\text{rank}(x)$ as officially defined recursively, rather than defined solely in terms of $R(\alpha)$ because, if M does not satisfy the Power Axiom, then $R(\alpha)^M$ is not defined. Furthermore, if M does satisfy Power, $R(\alpha)^M$ is defined, but is not necessarily absolute even though rank is.

c. Define $\bigcup^n x = \begin{cases} x & \text{if } n = 0 \\ \bigcup(\bigcup^{n-1} x) & \text{if } 0 < n \in \omega. \end{cases}$

Then by Theorem 8.23, $\bigcup^n x$ is an absolute function of variables n and x . Hence, $\text{tr cl}(x) = \{\bigcup(\bigcup^n x : n \in \omega)\}$ is absolute. $\square$

LEMMA 8.25: Let M be a transitive model of ZF. Then

- a. $\forall x \in M (\mathcal{P}(x)^M = \mathcal{P}(x) \cap M)$.
- b. $\forall \alpha \in M (R(\alpha)^M = R(\alpha) \cap M)$.

PROOF:

a. $\mathcal{P}(x)$ can be represented by the formula $\exists y (\forall z \in y (z \subseteq x))$. Now

$$\begin{aligned} (\exists y (\forall z \in y (z \subseteq x)))^M &\Leftrightarrow \exists y \in M (\forall z \in y (z \subseteq x))^M \\ &\Leftrightarrow \exists y \in M (\forall z \in y (z \subseteq x)^M) \\ &\Leftrightarrow \exists y \in M (\forall z \in y (z \subseteq x)) \text{ since } \subseteq \text{ is absolute.} \end{aligned}$$

Hence, $\mathcal{P}(x)^M = \mathcal{P}(x) \cap M$.

b. Since $R(\alpha) = \{x : \text{rank}(x) < \alpha\}$ (Lemma 7.9),

$$\begin{aligned} R(\alpha)^M &\Leftrightarrow (\exists y (\forall x \in y (\text{rank}(x) < \alpha)))^M \\ &\Leftrightarrow \exists y \in M (\forall x \in y (\text{rank}(x) < \alpha))^M \\ &\Leftrightarrow \exists y \in M (\forall x \in y (\text{rank}(x) < \alpha)^M) \\ &\Leftrightarrow \exists y \in M (\forall x \in y (\text{rank}(x) < \alpha)). \end{aligned}$$

Since $\text{rank}(x)$ is absolute and $\text{rank}(x) \in \text{Ord}$, $<$ represents $\in$, which is absolute.

Hence, $R(\alpha)^M = R(\alpha) \cap M$. $\square$

LEMMA 8.26: Let $G: A \rightarrow M$ be an isomorphism for the relation $\in$. Then for each formula $\phi(x_1, \dots, x_n)$,

$$\forall x_1, \dots, x_n \in A \left(\phi(x_1, \dots, x_n)^A \leftrightarrow \phi(G(x_1), \dots, G(x_n))^M \right).$$

PROOF: The proof is by induction on ϕ . $\square$

Chapter 9

L: THE CONSTRUCTIBLE UNIVERSE

(For this chapter, we will assume ZF.)

In Chapter 7, we defined a class to be a collection of things which could be defined by a formula. Then in Chapter 8, we defined the “relativization of ϕ to M ” and what it means for ϕ to be true in M where M is any class. We shall now define what it means to be “definable” and develop the constructible universe in which everything is “definable.”

DEFINITION 9.1: A set, A , is **definable** iff it is a collection of elements which satisfy some particular formula.

The problem with this definition, as proved by Tarsky, is that the notion of what it means to be “true” in a class structure is not definable. It turns out that “the class of all formulas” is definable by a formula; however, the “the class of all true formulas” is not because the notion of “truth” for classes is not definable.

We want to define a function $Df(A, n)$ to be the set of n -place relations on A which are definable by a formula with n free variables relativized to A . It would be nice if we could define $Df(A, n)$ by

$$Df(A, n) = \{X \subseteq A^n : X = \{\langle x_1, \dots, x_n \rangle : \phi^A(x_1, \dots, x_n)\}\}$$

where ϕ is a formula with n free variables. To do this, we would have to formalize both ZF and the notion of “truth” for sets in first-order logic. Because

"truth" is so complex for set structure, this would be a very long and tedious job. Kunen has developed a shorter, though less intuitive, method which allows definability to be independent of the syntactical details of first-order logic. We shall adopt Kunen's approach.

DEFINITION 9.2: If $n \in \omega$ and $i, j < n$, then we define $\mathbf{Df}(A, n)$ as follows:

- a. $\mathbf{Proj}(A, R, n) = \{s \in A^n : \exists t \in R (t|_n = s)\}.$
- b. $\mathbf{Diag}_\in(A, n, i, j) = \{s \in A^n : s(i) \in s(j)\}.$
- c. $\mathbf{Diag}_=(A, n, i, j) = \{s \in A^n : s(i) = s(j)\}.$
- d. $\mathbf{Df}'(k, A, n)$ is defined recursively on $k \in \omega$:
 - i. $\mathbf{Df}'(0, A, n) = \{\mathbf{Diag}_\in(A, n, i, j) : i, j, < n\}$
 $\cup \{\mathbf{Diag}_=(A, n, i, j) : i, j, < n\}.$
 - ii. $\mathbf{Df}'(k+1, A, n) = \mathbf{Df}'(k, A, n) \cup \{A^n \setminus R : R \in \mathbf{Df}'(k, A, n)\}$
 $\cup \{R \cap S : R, S \in \mathbf{Df}'(k, A, n)\}$
 $\cup \{\mathbf{Proj}(A, R, n) : R \in \mathbf{Df}'(k, A, n+1)\}.$

Then we define: $\mathbf{Df}(A, n) = \bigcup_{k \in \omega} \{\mathbf{Df}'(k, A, n)\}.$

NOTE: In (a) we have defined the relation of existence, in (b) the relation of "is an element of", and in (c) the relation of equality. So we define $\mathbf{Df}(A, n)$ to be the least set of relations on A containing these basic relations which, are closed under the operations of complementation, intersection, and projection, by (d).

LEMMA 9.3:

- a. If $R, S \in \mathbf{Df}(A, n)$, then $A^n \setminus R \in \mathbf{Df}(A, n)$ and $R \cap S \in \mathbf{Df}(A, n).$

- b. If $R, S \in \text{Df}(A, n)$ then $R \cup S \in \text{Df}(A, n)$.
- c. If $R \in \text{Df}(A, n+1)$, then $\text{Proj}(A, R, n) \in \text{Df}(A, n)$.

PROOF:

- a. Let $R, S \in \text{Df}(A, n)$; then there exists k_R and k_S such that $R \in \text{Df}'(k_R, A, n)$ and $S \in \text{Df}'(k_S, A, n)$. Let $k = \max \{k_R, k_S\}$. Then $R, S \in \text{Df}'(k, A, n)$.
Therefore, $A^n \setminus R, R \cap S \in \text{Df}'(k+1, A, n) \subset \text{Df}(A, n)$ (by Definition 9.2.d.ii.).
Hence, $A^n \setminus R, R \cap S \in \text{Df}(A, n)$.
- b. Let $R, S \in \text{Df}(A, n)$; then $A^n \setminus R, A^n \setminus S \in \text{Df}(A, n)$. Hence,
 $(A^n \setminus R) \cap (A^n \setminus S) \in \text{Df}(A, n)$; and then $A^n \setminus ((A^n \setminus R) \cap (A^n \setminus S)) \in \text{Df}(A, n)$.
But $R \cup S = A^n \setminus ((A^n \setminus R) \cap (A^n \setminus S))$. Therefore, $R \cup S \in \text{Df}(A, n)$.
- c. This part is immediate from the definition of $\text{Df}(A, n)$.

LEMMA 9.4: Let $\phi(x_0, \dots, x_{n-1})$ be a formula whose free variables form a subset of $\{x_0, \dots, x_{n-1}\}$. Then, for every A ,

$$\{s \in A^n : \phi^A(s(0), \dots, s(n-1))\} \in \text{Df}(A, n). \quad (*)$$

PROOF (by induction on the length of ϕ):

Let ϕ be $x_i \in x_j$. Then $(*)$ holds because $\text{Diag}_\in(A, n, i, j) \in \text{Df}(A, n)$.

Let ϕ be $x_i = x_j$. Then $(*)$ holds because $\text{Diag}_=(A, n, i, j) \in \text{Df}(A, n)$.

In the remaining cases, let ψ and σ be formulas for which $(*)$ holds.

Let ϕ be $\psi \wedge \sigma$. Then $(*)$ holds because $\text{Df}(A, n)$ is closed under finite intersections.

Let ϕ be $\neg\psi$. Then $(*)$ holds because $\text{Df}(A, n)$ is closed under complementations.

Let ϕ be $\exists y \phi$.

Case i: If $y \notin \{x_0, \dots, x_{n-1}\}$, then we may eliminate the quantifier in front of the y by extending each $s \in A^n$ to a $t \in A^{n+1}$ and then applying $(*)$ to the smaller formula ψ using the induction hypothesis:

$$\{s \in A^n : \phi^A(s(0), \dots, s(n-1))\} = \text{Proj}(A, T, n) \in \text{Df}(A, n)$$

where $T = \{t \in A^{n+1} : \psi^A(t(0), \dots, t(n))\}$.

Case ii: If $y = x_j \in \{x_0, \dots, x_{n-1}\}$, then $\phi(x_0, \dots, x_{n-1}) = \exists x_j \psi(x_0, \dots, x_{n-1})$, so x_j is not free in ϕ . Let z be a variable which does not occur anywhere in ϕ , let $\psi^*(x_0, \dots, x_{n-1}, z)$ be $\psi(x_0, \dots, x_{j-1}, z, x_{j+1}, \dots, x_{n-1})$, and let ϕ^* be $\exists z \psi^*$. Since the argument in case (i) holds for ϕ^* , it will also hold for ϕ since ϕ is logically equivalent to ϕ^* .

Hence, $(*)$ holds for all ϕ . $\square$

NOTE: This lemma states that the formula $(*)$ is provable in ZF for every ϕ .

Intuitively, it makes sense that $\text{Df}(A, n)$ should be countable since there are only countably many formulas. Formally, this can be proved in ZF.

However, we adopt Kunen's approach instead, as it presents a tool which will become useful with later material: we shall define a countable collection of functions which equals $\text{Df}(A, n)$.

DEFINITION 9.5: We shall define $\text{En}(m, A, n)$ by recursion on $m < \omega$.

- i. $\text{En}(m, A, n) = \text{Diag}_\in(A, n, i, j)$ if $m = 2^i \cdot 3^j$ and $i, j < n$.
- ii. $\text{En}(m, A, n) = \text{Diag}_=(A, n, i, j)$ if $m = 2^i \cdot 3^j \cdot 5$ and $i, j < n$.
- iii. $\text{En}(m, A, n) = A^n \setminus \text{En}(i, A, n)$ if $m = 2^i \cdot 3^j \cdot 5^2$ and $i, j < n$.
- iv. $\text{En}(m, A, n) = \text{En}(i, A, n) \cap \text{En}(j, A, n)$ if $m = 2^i \cdot 3^j \cdot 5^3$ and $i, j < n$.
- v. $\text{En}(m, A, n) = \text{Proj}(A, \text{En}(i, A, n+1), n)$ if $m = 2^i \cdot 3^j \cdot 5^4$ and $i, j < n$.

vi. $En(m, A, n) = 0$ otherwise.

So En is a function which enumerates definable n -place relations.

LEMMA 9.6: $Df(A, n) = \{En(m, A, n) : m < \omega\}$ for any n and A .

PROOF:

($\Rightarrow$) First, it can be shown by induction on k that for all n ,

$$Df'(k, A, n) \subseteq \{En(m, A, n) : m < \omega\}.$$

Hence, $Df(A, n) = \bigcup_{n < \omega} Df'(k, A, n) \subseteq \{En(m, A, n) : m < \omega\}.$

($\Leftarrow$) Next, it is easy to show by induction on m that $En(m, A, n) \in Df(A, n)$ for all n . Therefore, $Df(A, n) \supseteq \{En(m, A, n) : m < \omega\}.$ $\square$

COROLLARY 9.7: $|Df(A, n)| \leq \omega$ for all n .

PROOF: Since $|\{En(m, A, n) : m < \omega\}| \leq \omega$, this corollary follows immediately from the lemma. $\square$

Next we want to define L , the class of constructible sets, which is a transitive proper class model of ZFC + GCH, $\Diamond$, and $\Diamond^*$. To do so we need several definitions and lemmas.

DEFINITION 9.8: Let s and t be functions with $\text{dom}(s) = \alpha$ and $\text{dom}(t) = \beta$ (i.e. s and t are sequences of length α and β , respectively). Then the **concatenation** of s and t is the function $s \wedge t$, with $\text{dom}(s \wedge t) = \alpha + \beta$ defined by: $(s \wedge t)|_\alpha = s$ and $\forall \xi < \beta ((s \wedge t)(\alpha + \xi) = t(\xi)).$

DEFINITION 9.9: The definable power set operation, $\mathcal{D}$, on any set A , is defined by:

$$\mathcal{D}(A) = \{X \subseteq A : \exists n \in \omega \exists s \in A^n \exists R \in \text{Df}(A, n+1)(X = \{x \in A : s \wedge \langle x \rangle \in R\})\}.$$

Intuitively, this says that $\mathcal{D}(A)$ is a collection of subsets of A which are definable from a finite number of elements of A by a formula relativized to A . So $X \subseteq A$ gets into $\mathcal{D}(A)$ if $\exists s \in A \forall x \in X (s \wedge \langle x \rangle \in R)$ where s is a finite sequence and R is an $n+1$ -definable relation.

LEMMA 9.10: Let $\phi(v_0, \dots, v_{n-1}, x)$ be a formula with $v_0, \dots, v_{n-1}, x$ free variables, let $s \in A^n$, and let $V = \{x \in A : \phi^A(s(0), \dots, s(n-1), x)\}$. Then $V \in \mathcal{D}(A)$. (i.e. Any subset of A which is definable with free variables ranging over A is in $\mathcal{D}(A)$).

PROOF: Let $s(i) = v_i$ for $i < n$. Then we may rewrite $V = \{a \in A : \phi^A(s \wedge \langle a \rangle)\}$ which makes $V \in \mathcal{D}(A)$ where $R = \{s \in A^n : \phi^A(s(0), \dots, s(n-1))\}$. $\square$

NOTE: So we may now express $\mathcal{D}(A)$ as follows:

$$\mathcal{D}(A) = \{X \subseteq A : \exists \phi \exists a_1, \dots, a_n \in A (X = \{x \in A : \langle A : \epsilon \rangle \models \phi(a_1, \dots, a_n)\})\}.$$

LEMMA 9.11: For any set A ,

- $\mathcal{D}(A) \subseteq \mathcal{P}(A)$.
- If A is transitive, then $A \subseteq \mathcal{D}(A)$.
- $\forall X \subseteq A (|X| < \omega \rightarrow X \in \mathcal{D}(A))$.
- (AC) $|A| \geq \omega \rightarrow |\mathcal{D}(A)| = |A|$.

PROOF:

- Immediate from definitions of $\mathcal{D}(A)$.

b. Assume A is transitive and $x \in A$. Then $x \subseteq A$. Let ϕ be the formula $x \in v$. Then by Lemma 9.10,

$$\forall v \in A (\{x \in A : x \in v\} \in \mathcal{D}(A)). \quad (*)$$

Claim: $\{x \in A : x \in v\} = v$.

Proof: $\{x \in A : x \in v\} \subseteq v$ is trivial. To show containment in the other direction, let $x \in v \in A$. Since A is transitive, $x \in A$.

So $(*)$ reduces to $\forall v \in A (v \in \mathcal{D}(A))$. Therefore, $A \subseteq \mathcal{D}(A)$.

c. Let $E_n^m = \{t \in A^{n+1} : \exists i < m (t(n) = t(i))\}$ for $m \leq n$.

Claim i: $E_n^m \in \text{Df}(A, n+1)$.

Proof (by induction on m): $E_n^0 = \emptyset \in \text{Df}(A, n+1)$. Next assume $E_n^m \in \text{Df}(A, n+1)$, and let $T = \{t \in A^{n+1} : t(n) = t(m)\}$. Then

$$E_n^{m+1} = \{t \in A^{n+1} : \exists i < m+1 (t(n) = t(i))\} = E_n^m \cup T.$$

Now $T \subseteq \text{Diag}_=(A, n+1, i, j)$; so $T \in \text{Df}(A, n+1)$. Furthermore, $E_n^m \in \text{Df}(A, n+1)$ by induction hypothesis. Hence, $E_n^{m+1} = E_n^m \cup T \in \text{Df}(A, n+1)$ by Lemma 9.3b. Therefore, $\forall m E_n^m \in \text{Df}(A, n+1)$.

Claim ii: $\forall s \in A^n, \text{rng}(s) = \{x \in A : s \wedge \langle x \rangle \in E_n^n\} \in \mathcal{D}(A)$.

Proof: First note that E_n^n is a subset of A^{n+1} such that the last element in each sequence appears somewhere earlier in the sequence. Now $\text{rng}(s) \subseteq A$. Let $R = E_n^n$ which is in $\text{Df}(A, n+1)$. Therefore, $\text{rng}(s) \in \mathcal{D}(A)$.

Now $|\text{rng}(s)| \leq n$ because the last element is a repeat. Therefore,

$$\forall n < \omega \forall X \subseteq A (|X| \leq n \rightarrow X \in \mathcal{D}(A)).$$

d. Assume AC and $|A| \geq \omega$. $|\mathcal{D}(A)| \geq |A|$ because $\forall x \in A (\{x\} \in \mathcal{D}(A))$.

To show that $|\mathcal{D}(A)| \leq |A|$, first note that $|\text{Df}(A, n+1)| \leq \omega$ (Corollary 9.7). Also note that a typical element of $\mathcal{D}(A)$ is of the form $X = \{a \in A : s \wedge a \in R\}$. Since

there are only countably many R 's and $|A^{<\omega}|$ many s 's, so there are $\omega \times A^{<\omega}$ many X 's. Therefore, $|\mathcal{D}(A)| = |\omega \times A^{<\omega}| = |A^{<\omega}| = |A|$. $\square$

LEMMA 9.12: $A \in \mathcal{D}(A)$.

PROOF: We know that $\{s \in A^1 : (s(0) = s(0))^A\} = \text{Diag}_=(A, 1, i, j) \in \text{Df}(A, 1)$ where $R = \{s \in A^1\} = \{\langle x \rangle : x \in A\}$ and $A = \{x \in A : \langle \rangle \wedge \langle x \rangle \in R\}$. Recall that $\mathcal{D}(A) = \{X \subseteq A : \exists n \in \omega \exists s \in A^n \exists R \in \text{Df}(A, n+1) (X = \{x \in A : s \wedge \langle x \rangle \in R\})\}$. So A is just such an X . $\square$

LEMMA 9.13: If A is transitive, then $\mathcal{D}(A)$ is transitive.

PROOF: Let A be transitive and $x \in y \in \mathcal{D}(A)$. We must show that $y \subseteq \mathcal{D}(A)$. By definition of $\mathcal{D}(A)$, $y \subseteq A$; hence, $x \in A$. Also, since A is transitive, $A \subseteq \mathcal{D}(A)$ (Lemma 9.11b) and so $x \in \mathcal{D}(A)$. Now x was arbitrary in y , so we have $\forall x \in y (x \in \mathcal{D}(A))$. Therefore, $y \subseteq \mathcal{D}(A)$. $\square$

DEFINITION 9.14: For any ordinal α , we define $L(\alpha)$ recursively as follows:

- i. $L(0) = 0$.
- ii. $L(\alpha+1) = \mathcal{D}(L(\alpha))$.
- iii. $L(\alpha) = \bigcup_{\xi < \alpha} L(\xi)$ for α a limit ordinal.

DEFINITION 9.15: $L = \bigcup_{\alpha \in \text{Ord.}} \{L(\alpha)\}$ is the **constructible universe**.

LEMMA 9.16: For every ordinal α :

- a. $L(\alpha)$ is transitive.
- b. $\forall \xi \leq \alpha (L(\xi) \subseteq L(\alpha))$.

PROOF (by induction on α): Assume that the lemma holds $\forall \beta < \alpha$.

If $\alpha = 0$, the lemma holds vacuously.

a. Let $\alpha = \beta + 1$. Then $L(\alpha) = L(\beta + 1) = \mathcal{D}(L(\beta))$. Since $L(\beta)$ is transitive (induction hypothesis), $\mathcal{D}(L(\beta))$ is transitive (Lemma 9.13). Hence, $L(\alpha)$ is transitive.

Let α be a limit ordinal. Then $L(\alpha) = \bigcup_{\beta < \alpha} L(\beta)$ is transitive because $L(\beta)$ is transitive and the union of transitive sets is transitive.

b. Let $\alpha = \beta + 1$.

Case i: Let $\xi = \beta$. Then $L(\xi) = L(\beta) \subseteq \mathcal{D}(L(\beta)) = L(\beta + 1) = L(\alpha)$ by part (a) and Lemma 9.11b.

Case ii: Let $\xi < \beta$. Then $L(\xi) \subseteq L(\beta) \subseteq \mathcal{D}(L(\beta)) = L(\beta + 1) = L(\alpha)$ by induction hypothesis, part (a), and Lemma 9.11b.

If α is a limit ordinal, then part (b) follows immediately from the induction hypothesis and the definition of $L(\alpha)$. $\square$

LEMMA 9.17: For $x \in L$, the least α for which $x \in L(\alpha)$ is a successor ordinal.

PROOF: Assume that the least such α is a limit ordinal. Then for this α ,

$L(\alpha) = \bigcup_{\xi < \alpha} L(\xi)$. Therefore, $\exists \xi < \alpha$ such that $x \in L(\xi)$ which contradicts the "leastness" of α . $\square$

DEFINITION 9.18: For $x \in L$, the **L-rank** of x , $\rho(x)$, is the least β such that $x \in L(\beta+1)$.

LEMMA 9.19: For all α , $L(\alpha) = \{x \in L : \rho(x) < \alpha\}$.

PROOF: Let $x \in L(\alpha)$.

If $\alpha = \beta+1$, then $x \in L(\alpha)$ iff $x \in L(\beta+1)$ iff $\rho(x) \leq \beta < \alpha$ by definition of $\rho(x)$.

If α is a limit ordinal, then $L(\alpha) = \bigcup_{\beta < \alpha} L(\beta)$. So $x \in L(\alpha)$ iff $\exists \beta < \alpha (x \in L(\beta))$.

Since α is a limit ordinal, $\beta + 1 < \alpha$. Now $x \in L(\beta)$ implies that

$x \in L(\beta+1)$ (by Lemma 9.16b). Hence, $x \in L(\beta)$ iff $\rho(x) \leq \beta < \alpha$. $\square$

LEMMA 9.20:

- a. $\forall \alpha \in \text{Ord} (L(\alpha) \cap \text{Ord} = \alpha)$.
- b. $\forall \alpha \in \text{Ord} (\alpha \in L \wedge \rho(\alpha) = \alpha)$.

PROOF:

a. (by induction α) Assume that $\forall \beta < \alpha (L(\beta) \cap \text{Ord} = \beta)$.

Let $\alpha = 0$. Then $L(\alpha) \cap \text{Ord} = 0 \cap \text{Ord} = 0 = \alpha$.

Let $\alpha = \beta+1$. To show $L(\beta+1) \cap \text{Ord} \subseteq \beta+1$, let $\gamma \in L(\beta+1) = \mathcal{D}(L(\beta))$. Now $\mathcal{D}(L(\beta))$ is a collection of subsets of $L(\beta)$; hence, $\gamma \subseteq L(\beta) \cap \text{Ord} = \beta$ (by induction hypothesis). Since both $\gamma, \beta \in \text{Ord}$, either $\gamma \in \beta$ or $\gamma = \beta$. But $\beta \subseteq \beta+1$. Hence, $\gamma \in \beta+1$.

To show $\beta+1 \subseteq L(\beta+1) \cap \text{Ord}$, let $\gamma \in \beta+1$. Then $\gamma \leq \beta$. But both $\gamma, \beta \in \text{Ord}$, so either $\gamma \in \beta$ or $\gamma = \beta$. Furthermore, $\beta = L(\beta) \cap \text{Ord}$ (by induction hypothesis) which equals $\{x \in L(\beta) : \phi^{L(\beta)}(x)\}$ where $\phi(x)$ is the formula for "x is an ordinal"

Lemmas 8.19). Therefore, $\beta \in \mathcal{D}(L(\beta)) = L(\beta+1)$ (by Lemma 9.11). So if $\gamma = \beta$, $\gamma \in L(\beta+1)$. But we also have $\gamma \in \beta \subseteq L(\beta+1)$ since $L(\beta+1)$ is transitive. Let α be a limit ordinal. Then $L(\alpha) = \bigcup_{\beta < \alpha} L(\beta)$. But $L(\beta) \cap \text{Ord} = \beta$ for all $\beta < \alpha$. Hence, $L(\alpha) \cap \text{Ord} = \alpha$.

b. $\alpha = \{\beta \in L(\alpha) : \beta \in \text{Ord}\}$ by part (a). Now $\alpha \in L(\alpha+1)$ by transitivity of L . Furthermore, $\alpha \notin L(\alpha)$ (because if it were, then we would have $\alpha \in \alpha$ which contradicts the definition of an ordinal). So α is the least β such that $\alpha \in L(\beta+1)$. Hence, $\alpha = \rho(\alpha)$. $\square$

LEMMA 9.21: $L(\alpha) \in L(\alpha+1)$.

PROOF: $L(\alpha) = \{x \in L(\alpha) : (x = x)^{L(\alpha)}\} \in \mathcal{D}(L(\alpha)) = L(\alpha+1)$ (by Lemma 9.10). $\square$

LEMMA 9.22: $\forall \alpha (L(\alpha) \subseteq R(\alpha))$.

PROOF: If $\alpha = 0$, then $L(0) = 0 = R(0)$.

If $\alpha = \beta+1$, $L(\beta+1) = \mathcal{D}(L(\beta)) \subseteq \mathcal{P}(L(\beta)) \subseteq \mathcal{P}(R(\beta)) = R(\beta+1)$.

If α is a limit ordinal, then $L(\alpha) = \bigcup_{\xi < \alpha} L(\xi) = \bigcup_{\xi < \alpha} R(\xi) = R(\alpha)$. $\square$

LEMMA 9.23: Every finite subset N of $L(\alpha)$ is in $L(\alpha+1)$.

PROOF: Fix $N \subseteq L(\alpha)$ such that $|N| < \omega$. Then $N \in \mathcal{D}(L(\alpha)) = L(\alpha+1)$ (by Lemma 9.11c). Since $L(\alpha+1)$ is transitive, $N \subseteq L(\alpha+1)$. $\square$

LEMMA 9.24:

a. $\forall n \in \omega (L(n) = R(n))$.

b. $L(\omega) = R(\omega)$.

PROOF:

a. (by induction on n) $L(0) = 0 = R(0)$. Now assume $L(n) = R(n)$.

Then $L(n+1) = \mathfrak{D}(L(n)) = \mathfrak{D}(R(n)) = \mathfrak{P}(R(n)) = R(n+1)$ since all finite sets are definable and the power set of a finite set is finite.

$$b. L(\omega) = \bigcup_{\xi < \omega} L(\xi) = \bigcup_{\xi < \omega} R(\xi) = R(\omega) \quad \square$$

LEMMA 9.25: For all $\alpha \geq \omega$, $|L(\alpha)| = |\alpha|$.

PROOF: Clearly, $|L(\alpha)| \geq |\alpha|$ since $\alpha \subseteq L(\alpha)$. We shall prove that $|L(\alpha)| = |\alpha|$ by induction. We'll assume $\alpha \geq \omega$ and $\forall \beta < \alpha (\beta \geq \omega \rightarrow |L(\beta)| = |\beta|)$; hence, $\forall \beta < \alpha (|L(\beta)| \leq |\alpha|)$.

Let $\alpha = \beta + 1$ with $\beta \geq \omega$. Now $L(\alpha) = \mathfrak{D}(L(\beta))$ (by def. of $L(\alpha)$), and $|\mathfrak{D}(L(\beta))| = |L(\beta)| = |\beta| = |\alpha|$ (by Lemma 9.11d, by induction hypothesis, and by definition of an infinite cardinal, respectively).

Therefore, $|L(\alpha)| = |\alpha|$.

Next let α be a limit ordinal. Then $L(\alpha) = \bigcup_{\beta < \alpha} L(\beta)$. Now there are α many β 's less than α , but each β is of size less than α . Hence, $|L(\alpha)| = |\alpha|$. $\square$

REMARK 9.26: $L(\alpha) \neq R(\alpha)$ for most $\alpha > \omega$. For example, if $\omega < \alpha < \omega_1$, $L(\alpha)$ is countable but $R(\alpha)$ is uncountable. Furthermore, $\mathfrak{P}(\omega) \subseteq R(\omega+1)$, but not every subset of ω is in $L(\alpha+1)$. However, if $V = L$, then $L(\alpha) = R(\alpha)$.

Chapter 10

L: A MODEL FOR ZF

In this chapter our goal is to show that L is a model for ZF (i.e., *every axiom of ZF holds in L*). Along the way, we shall also prove a version of the Reflection Theorem, which will be an important tool for proving later results.

DEFINITION 10.1: For any set M , $\langle M, \in \rangle$ is a **standard structure** for the language of set theory. $\langle M, R \rangle$, where R is a relation other than $\in$, is considered to be a **nonstandard structure**.

DEFINITION 10.2: If T is a collection of axioms and $\langle M, \in \rangle \models T$, then M is a **standard model** of T . If $\langle M, R \rangle \models T$ for some other relation R , then M is a **nonstandard model** of T .

LEMMA 10.3: Let S and T be sets of sentences in the language of set theory. If M is a nonempty class which can be proved from T to be a model for S , then $\text{Con}(T) \Rightarrow \text{Con}(S)$.

PROOF: Assume $\text{Con}(T)$ and $\neg \text{Con}(S)$. Then there exists a formula ϕ such that S proves both ϕ and $\neg \phi$. Now we can argue from T that M is a model for S , so $\forall \sigma \in S$, T proves σ^M ; hence, T proves both ϕ^M and $\neg \phi^M$, which contradicts the consistency of T . $\square$

REMARK 10.4: $\exists x (x = x)^M$ is equivalent to $M \neq \emptyset$.

LEMMA 10.5: Extensionality holds in L .

PROOF: We have Extensionality in V :

$$(\forall x, y (x = y \leftrightarrow \forall z \in L (z \in x \leftrightarrow z \in y)))^V \quad (*)$$

We want to prove:

$$(\forall x, y (x = y \leftrightarrow \forall z \in L (z \in x \leftrightarrow z \in y)))^L,$$

that is to say,

$$\forall x, y \in L (x = y \leftrightarrow \forall z \in L (z \in x \leftrightarrow z \in y)). \quad (**)$$

So assume $x, y \in L$ with $x = y$, and let $z \in x$ in V . Then $z \in y$ in V by $(*)$.

But $x, y \in L$ and L is transitive; so $z \in L$.

Similarly, if we assume $x, y \in L$ and $z \in x \leftrightarrow z \in y$ with $z \in x$ in V , we get that $x = y$ in L .

Hence, $(**)$ holds. $\square$

Next we want to prove Comprehension in L . If ϕ is a formula of Set Theory, we want to show $\{y \in x : \phi(y)^L\} \in L$. The problem here lies in knowing whether $\phi(y)$ holds in all of L , not just in some specific $L(\alpha)$; what is true for all of L may not be true for all of the $L(\alpha)$'s and visa versa. There exist $L(\alpha+1)$'s, for example, in which there is a largest ordinal, whereas in L there is no largest ordinal. To show that $\{y \in x : \phi(y)^L\} \in L$, we must show that there exists an α such that $x \in L(\alpha) \wedge \forall y (\phi(y)^L \leftrightarrow \phi(y)^{L(\alpha)})$. We shall use a generalized version of the Reflection Theorem to prove this.

DEFINITION 10.6: Any set of formulas $F = \{\phi_1, \dots, \phi_n\}$ is **subformula closed** iff every subformula of some $\phi_i \in F$ is also in F .

LEMMA 10.7: Let M and N be classes such that $M \subseteq N$. Let $\{\phi_1, \dots, \phi_n\}$ be a subformula closed set of formulas. Then the following are equivalent:

- a. $\phi_1, \dots, \phi_n$ are absolute for M, N .
- b. Whenever $\phi_i \in \{\phi_1, \dots, \phi_n\}$ is of the form $\exists x \phi_j(x, y_1, \dots, y_r)$ with free variables a subset of $\{x, y_1, \dots, y_r\}$, then

$$\forall y_1, \dots, y_r \in M (\exists x \in N \phi_j^N(x, y_1, \dots, y_r) \rightarrow \exists x \in M \phi_j^M(x, y_1, \dots, y_r)).$$

PROOF:

(a) $\Rightarrow$ (b) Fix $y_1, \dots, y_r \in M$ and assume $\exists x \in N \phi_j^N(x, y_1, \dots, y_r)$. But $\exists x \in N \phi_j^N(x, y_1, \dots, y_r) \Leftrightarrow \phi_i^N(y_1, \dots, y_r) \Leftrightarrow \phi_i^M(y_1, \dots, y_r) \Leftrightarrow \exists x \in M \phi_j^M(x, y_1, \dots, y_r)$.

Now $\{\phi_1, \dots, \phi_n\}$ subformula closed and absolute, so

$$\phi_j^N(y_1, \dots, y_r) \Leftrightarrow \phi_j^M(y_1, \dots, y_r). \text{ Hence, } \exists x \in M \phi_j^N(x, y_1, \dots, y_r).$$

(b) $\Rightarrow$ (a) (Proof by induction on the length of $\phi_i \in \{\phi_1, \dots, \phi_n\}$). By hypothesis, $\{\phi_1, \dots, \phi_n\}$ is subformula closed. If ϕ_i is atomic, then ϕ_i is absolute since our only predicate symbol is $\in$ and both $x = y$ and $x \in y$ are absolute.

If $\phi_i = \phi_j \wedge \phi_k$, then ϕ_j and ϕ_k are absolute by the induction hypothesis.

$$\text{Hence, } (\phi_j \wedge \phi_k)^M \Leftrightarrow \phi_j^M \wedge \phi_k^M \Leftrightarrow \phi_j^N \wedge \phi_k^N \Leftrightarrow (\phi_j \wedge \phi_k)^N.$$

If $\phi_i = \neg \phi_j$, the proof is similar.

If $\phi_i = \exists x \phi_j(x, y_1, \dots, y_r)$, fix $y_1, \dots, y_r \in M$. Then $\phi_i^M(y_1, \dots, y_r) \Leftrightarrow \exists x \in M \phi_j^M(x, y_1, \dots, y_r) \Leftrightarrow \exists x \in M \phi_j^N(x, y_1, \dots, y_r) \Leftrightarrow \exists x \in N \phi_j^N(x, y_1, \dots, y_r) \Leftrightarrow \phi_i^N(y_1, \dots, y_r)$. $\square$

DEFINITION 10.8: $\forall \alpha \in \text{Ord}$, $Z(\alpha)$ is a set such that:

- a. $\alpha < \beta \rightarrow Z(\alpha) \subseteq Z(\beta)$.
- b. For any limit ordinal γ , $Z(\gamma) = \bigcup_{\alpha \in \gamma} Z(\alpha)$.

Then class $Z = \bigcup_{\alpha \in \text{Ord}} Z(\alpha)$.

NOTE: Z is an arbitrary class with properties (a) and (b).

THEOREM 10.9 (Reflection): For any formulas $\phi_1, \dots, \phi_n$, $\forall \alpha \in \text{Ord} \exists \beta > \alpha$ such that $\phi_1, \dots, \phi_n$ are absolute for $Z(\beta)$, Z .

PROOF: We shall use Lemma 10.7 with $N = Z$. Assume $\{\phi_1, \dots, \phi_n\}$ is a subformula closed set (if not, just extend it until it is). We seek a β such that $M = Z(\beta)$. Now $\forall i \in \{1, \dots, n\}$ we shall define an increasing function

$F_i: \text{Ord} \rightarrow \text{Ord}$ as follows:

If ϕ_i is of the form $\exists x \phi_j(x, y_1, \dots, y_r)$, we then define a function G_i by:

$$G_i(y_1, \dots, y_r) = \begin{cases} \text{least } \eta \text{ such that } \exists x \in Z(\eta) \phi_j^Z(x, y_1, \dots, y_r) \\ \text{if } \exists x \in Z \phi_j^Z(x, y_1, \dots, y_r) \\ 0 \text{ if } \neg \exists x \in Z \phi_j^Z(x, y_1, \dots, y_r) \end{cases}$$

(If $\exists x \in Z$, then G_i picks out the smallest η such that $x \in Z(\eta)$.)

Set $F_i(\xi) = \sup(\{G_i(y_1, \dots, y_r) : y_1, \dots, y_r \in Z(\xi)\})$, which is guaranteed to exist by Replacement. (So first we pick a level ξ and look at all the $x \in Z$ on that level which make $\phi_j^Z(x, y_1, \dots, y_r)$ true. Then we use G_i to pick the lowest level in which each of these x 's occurs. F_i picks out the sup of all of these levels.)

If ϕ_i is not of the form $\exists x \phi_j(x, y_1, \dots, y_r)$, we set $F_i(\xi) = 0$.

Then by Lemma 10.7. (b) $\Rightarrow$ (a), for a limit ordinal β , if $\forall i \forall \xi < \beta (F_i(\xi) < \beta)$, then $\phi_1, \dots, \phi_n$ will be absolute $Z(\beta)$, Z .

So fix $\alpha \in \text{Ord}$. To find such a β , for $p \in \omega$, we'll define β_p recursively by:

$$\beta_0 = \alpha$$

$$\beta_{p+1} = \max \{ \beta_p + 1, F_1(\beta_p), \dots, F_n(\beta_p) \}.$$

Then $\beta = \sup(\{\beta_p : p \in \omega\})$. Now $\alpha = \beta_0 < \beta_1 < \beta_2 < \dots$; so β is a limit ordinal greater than α . Therefore, if $\xi < \beta$, then there exists a p such that $\xi < \beta_p$.

Hence, $F_i(\xi) \leq F_i(\beta_p) \leq \beta_{p+1} < \beta$. So we have found a β , namely, $\sup(\{\beta_p : p \in \omega\})$, such that

$$\forall y_1, \dots, y_r \in Z(\beta) (\exists x \in Z \phi_j^Z(x, y_1, \dots, y_r) \rightarrow \exists x \in Z(\beta) \phi_j^Z(x, y_1, \dots, y_r)).$$

Therefore, $\phi_1, \dots, \phi_n$ are absolute for $Z, Z(\beta)$ (by Lemma 10.7, (b) $\Rightarrow$ (a)). $\square$

LEMMA 10.10: Suppose that $\forall \phi(x, z, w_1, \dots, w_n)$ with free variables a subset of $\{x, z, w_1, \dots, w_n\}$, $\forall z, w_1, \dots, w_n \in M (\{x \in z : \phi^M(x, z, w_1, \dots, w_n)\} \in M)$, where M is any class. Then Comprehension holds in M .

PROOF: Fix a ϕ as described in the lemma. We must check that

$$\forall z, w_1, \dots, w_n \in M \exists x \in y (x \in y \leftrightarrow (x \in z \wedge \phi^M(x, z, w_1, \dots, w_n))). \quad (*)$$

So fix $z, w_1, \dots, w_n$ and let $y = \{x \in z : \phi^M(x, z, w_1, \dots, w_n)\} \in M$. Then $\forall x$, and hence, $\forall x \in M (x \in y \leftrightarrow \phi^M(x, z, w_1, \dots, w_n))$ (by definition of relativization).

Therefore, $(*)$ holds. $\square$

LEMMA 10.11: Comprehension holds in L .

PROOF: Suppose that $\phi(x, z, w_1, \dots, w_n)$ is a formula with free variables a subset of $\{x, z, w_1, \dots, w_n\}$. It suffices to show that

$$\forall z, w_1, \dots, w_n \in L (\{x \in z : \phi^L(x, z, w_1, \dots, w_n)\} \in L).$$

So suppose that $z, w_1, \dots, w_n \in L$. Then $\exists \alpha$ such that $z, w_1, \dots, w_n \in L(\alpha)$. We know from Lemma 9.11 that

$$\forall z, w_1, \dots, w_n \in L(\alpha) (\{x \in z : \phi^L(x, z, w_1, \dots, w_n)\} \in \mathcal{D}(L(\alpha))) = L(\alpha+1).$$

However, $\mathcal{D}(L(\alpha))$ involves relativization to $L(\alpha)$ rather than to L . We have $z \in L(\alpha)$, which implies that $z \subseteq L(\alpha)$ (by transitivity of $L(\alpha)$). We want to separate out a subset of z , and hence, of $L(\alpha)$, which is defined by ϕ . The problem is that the quantifiers range over L , not just over $L(\alpha)$. So we have $z, w_1, \dots, w_n \in L(\alpha)$. Now let $\beta > \alpha$ such that ϕ is absolute for $L(\beta)$, L (such a β exists by Theorem 10.9). Thus,

$$\forall x, z, w_1, \dots, w_n \in L(\beta) (\phi^{L(\beta)}(x, z, w_1, \dots, w_n) \leftrightarrow \phi^L(x, z, w_1, \dots, w_n)). \quad (*)$$

$$\begin{aligned} \text{Then } \{x \in z : \phi^L(x, z, w_1, \dots, w_n)\} &= \{x \in L(\beta) : x \in z \wedge \phi^L(x, z, w_1, \dots, w_n)\} \\ &= \{x \in L(\beta) : x \in z \wedge \phi^{L(\beta)}(x, z, w_1, \dots, w_n)\} \\ &= \{x \in z : \phi^{L(\beta)}(x, z, w_1, \dots, w_n)\} \\ &\in \mathcal{D}(L(\beta)) = L(\beta+1) \in L. \end{aligned}$$

The first equality holds because $x \in z \in L(\beta)$ and $L(\beta)$ is transitive, and the second equality holds by (*); the first $\in$ holds by Lemma 9.10, and finally $L(\beta+1) \in L$ holds (by definition of L). $\square$

LEMMA 10.12: Foundation holds in L .

PROOF: Foundation holds in any transitive $M \subseteq WF$ because any subclass of a well-founded class is well-founded; and $L \subseteq WF$ because $V \subseteq WF$ assuring the Axiom of Foundation in V . $\square$

LEMMA 10.13: Pairing holds in L .

PROOF: Let $x, y \in L$. Then $\exists \alpha_x, \alpha_y$ such that $x \in L(\alpha_x)$, $y \in L(\alpha_y)$. Let $\alpha = \max\{\alpha_x, \alpha_y\}$. Then $x, y \in L(\alpha)$. So if we let $z = L(\alpha)$, then $\forall x, y \in L \exists z \in L (x \in z \wedge y \in z)$. Hence, Pairing holds. $\square$

LEMMA 10.14: Union holds in L .

PROOF: Let $x \in L$. We want to show that $\bigcup x \in L$. So say $x \in L(\alpha)$. Then,

$$z \in \bigcup x \Leftrightarrow \exists y \in x (z \in y) \Leftrightarrow (\exists y \in x (z \in y))^{L(\alpha)},$$

since this formula is absolute for transitive models. Hence,

$$\bigcup x = \{z \in L(\alpha) : (\exists y \in x (z \in y))^{L(\alpha)}\} \in \mathcal{D}(L(\alpha)) = L(\alpha+1). \quad \square$$

LEMMA 10.15: Power Set holds in L .

PROOF: Let $x \in L$. We want to show that $(\mathcal{P}(x))^L$ is in L , where

$$(\mathcal{P}(x))^L = \mathcal{P}(x) \cap L \text{ (by Lemma 8.25). Now, in } V, \text{ we'll define}$$

$f: \mathcal{P}(x) \cap L \rightarrow \text{Ord}$ by $f(y) = p(y)$ where $p(y)$ is the least α such that

$y \in L(\alpha+1)$. Then by Replacement in V , $\exists \beta \in V$ such that

$\forall y \in \mathcal{P}(x) \cap L (f(y) < \beta)$. Then $\mathcal{P}(x) \cap L$ is definable over $L(\beta)$; hence,

$$\mathcal{P}(x) \cap L \in L(\beta+1). \quad \square$$

LEMMA 10.16: Infinity holds in L .

PROOF: Infinity holds in L since $\omega \in L(\omega+1) \subseteq L$. $\square$

LEMMA 10.17: Replacement holds in L .

PROOF: Assume

$$(\forall x \in z \exists! y \phi(x, y) \rightarrow \exists w \forall x \in z \exists y \in w \phi(x, y))^V.$$

To show

$$(\forall x \in z \exists! y \phi(x, y) \rightarrow \exists w \forall x \in z \exists y \in w \phi(x, y))^L,$$

assume $z \in L$ and $(\forall x \in z \exists! y \phi(x, y))^L$ or equivalently,

$$\forall x \in z \exists! y \in L \exists x \in L \phi^L(x, y).$$

We want to find an $\alpha \in \text{Ord}$ such that $\forall x \in z \exists y \in L(\alpha) \phi^L(x, y)$. So in V , $\forall x \in z$, choose $\alpha_x \in \text{Ord}$ such that $\exists y \in L(\alpha_x) \phi^L(x, y)$: in particular, let $\alpha_x = p(y) + 1$. Then using Replacement in V , let $\alpha = \sup(\{\alpha_x\})$; for this α , $\forall x \in z \exists y \in L(\alpha) \phi^L(x, y)$. $\exists x \in z \phi^L(x, y)$.

Hence, $(\text{Replacement})^V \rightarrow (\text{Replacement})^L$. $\square$

Chapter 11

THE AXIOM OF CONSTRUCTIBILITY

DEFINITION 11.1: The **Axiom of Constructibility** is the statement $V = L$.
(formally: $\forall x \exists \alpha (x \in L(\alpha))$).

It turns out that both $V = L$ and $V \neq L$ are consistent with ZF. We have already shown that L is a model for ZF. We shall prove that $ZF + V = L$ is consistent by showing that $L \models V = L$ (i.e. $V = L$ is true in L). Now this is not as trivial as it may seem. Clearly, $\forall x \in L \exists \alpha (x \in L(\alpha))$; however, $(V = L)^L$ means $\forall x \in L \exists \alpha \in L ((x \in L(\alpha))^L)$. To prove $(V = L)^L$ we must first show that $x \in L(\alpha)$ is absolute for L .

LEMMA 11.2: The function $L_\alpha: \alpha \rightarrow L(\alpha)$ is absolute for transitive models of ZF-P (i.e., M 's version of $L = V$'s version of L).

PROOF: $\text{Proj}_\in$, $\text{Diag}_\in$ and $\text{Diag}_\in$ are absolute notions, and since Df' is defined directly from these and other absolute notions, it is absolute as well. Df is absolute because it is recursively defined from Df' . Furthermore, $\mathcal{D}$ is also absolute since it is recursively defined from Df . And finally, $L(\alpha)$ is absolute since it is defined recursively from $\mathcal{D}$. $\square$

THEOREM 11.3 (ZF): L is a model of $ZF + V = L$.

PROOF: We have already shown that L is a model for ZF. Now

$$(V = L)^L \Leftrightarrow (\forall x \in L \exists \alpha (x \in L(\alpha))^L \Leftrightarrow \forall x \in L \exists \alpha \in L ((x \in L(\alpha))^L).$$

So fix $x \in L$. Then $\exists \alpha$ such that $x \in L(\alpha)$. Now $\text{Ord} \subseteq L$ (Lemma 9.20a).

Therefore, $\alpha \in L$. Hence, L is a model for $\text{ZF} + V = L$. $\square$

COROLLARY 11.4: $\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZF} + V = L)$.

PROOF: This corollary holds by ZFC in L and Theorem 11.3. $\square$

REMARK 11.5: The Axiom of Foundation, equivalent to $V = \text{WF}$, has been made a basic axiom of ZF because it can be argued that all mathematics takes place in WF . However, not all mathematics can be said to take place in L ; not even all subsets of ω lie in L . Hence, $V = L$ has not been made a basic axiom of ZF. Instead, $V = L$ is used as a tool to prove several relative consistency results. In particular, we shall use it to prove both AC and GCH as well as $\Diamond$ and $\Diamond^+$ since it is easier to prove these statements directly in L without having to worry about relativizing to L .

THEOREM 11.6: If M is any transitive proper class model of ZF-P, then $L = L^M \subseteq M$.

PROOF: Claim: $\text{Ord} \subseteq M$.

Proof: Fix $\alpha \in \text{Ord}$. Now $R(\alpha)$ is a set, so $M \not\subset R(\alpha)$. Therefore, $\exists x \in M$ such that $\text{rank}(x) \geq \alpha$ (recall that $\text{rank}(x) = \text{least } \beta \text{ such that } x \in R(\beta+1)$). But the rank function is absolute for M (Theorem 8.24b), so $\alpha \in \text{rank}(x) \in M$; and since M is transitive, $\alpha \in M$. Now, since $L^M = \{x \in M : (\exists \alpha (x \in L(\alpha))^M)\}$, $L^M \subseteq M$. But both Ord and $L(\alpha)$ are absolute for any transitive model of ZF-P

and α was fixed arbitrarily, so $L^M = \bigcup \{L(\alpha) : \alpha \in \text{Ord}\} = L$.

Hence, $L = L^M \subseteq M$. $\square$

NOTE: To avoid the problem that this theorem may not make sense for some proper classes, it may be valuable to restate this theorem as follows:

M is a transitive proper class $\wedge \phi^M \rightarrow L \subseteq M$,

where ϕ is a finite conjunction of axioms of ZF-P (since some of the axioms are really schema for many formulas).

Next we shall develop an analog of Theorem 11.6 for sets.

DEFINITION 11.7: $o(M) = M \cap \text{Ord}$.

LEMMA 11.8: If M is a transitive set, then $o(M) \in \text{Ord}$.

PROOF: To show $o(M) \in \text{Ord}$, we must show it is transitive and well-ordered by $\in$. Since $o(M) \subseteq \text{Ord}$, $o(M)$ is well-ordered by $\in$. Furthermore, since both M and Ord are transitive, $M \cap \text{Ord}$ is transitive. $\square$

NOTE: $o(M) = \sup(\{\beta \in \text{Ord} : \beta \in M\})$, which is the least ordinal α such that $\alpha \notin M$. So since M is transitive and " $x \in \text{Ord}$ " is absolute, anything which M "thinks" is an ordinal really is an ordinal.

THEOREM 11.9: There is a finite conjunction ψ of axioms of ZF-P such that for all transitive sets M , $(\psi^M \rightarrow L(o(M)) = L^M \subseteq M)$.

PROOF: Define ψ to be the conjunction of ϕ from the note on Theorem 11.6 with enough axioms thrown in to prove the following there is no largest ordinal.

Now assume M is transitive and ψ^M .

Claim: $o(M)$ is a limit ordinal.

Proof: From ψ^M we can prove that there is no greatest ordinal:

$$M \models \forall x ((x \in \text{Ord}) \rightarrow \exists y (y \in \text{Ord} \wedge x \in y)).$$

Since “ x is an ordinal” is absolute, M has no largest ordinal. Now using absoluteness of $L(\alpha)$,

$$L^M = \{x \in M : (\exists \alpha (x \in L(\alpha)))^M\} = \bigcup_{\alpha \in o(M)} L(\alpha)^M = \bigcup_{\alpha \in o(M)} L(\alpha) = L(o(M)).$$

Furthermore, $L^M \subseteq M$ by definition of L^M .

Hence, $L(o(M)) = L^M \subseteq M$. $\square$

THEOREM 11.10: There is a finite conjunction σ of axioms of $ZF - P + V = L$ such that:

- If M is a transitive proper class and σ^M , then $M = L$.
- For every set M (M transitive $\wedge \sigma^M \rightarrow M = L(o(M))$).

PROOF:

a. Let $\sigma = \psi$ from Theorem 11.9. Assume that $V = L$, that M is a transitive proper class, and that σ^M . Now $V = L$ can be expressed as $\forall x \exists \alpha (x \in L(\alpha))$.

So $\sigma^M \Rightarrow (\forall x (x \in L))^M \Rightarrow \forall x \in M (x \in L^M) \Rightarrow M = L^M$. However, L is absolute, so $L = L^M$. Hence, $M = L$.

b. Let σ be the ψ in Theorem 11.9. Then for all transitive sets M , if σ^M , then $M = L^M$. But by Theorem 11.9, $L(o(M)) = L^M$. Hence, $M = L(o(M))$. $\square$

COROLLARY 11.11: There is a finite conjunction of axioms χ of $ZF + V = L$ such that if M is a transitive set and χ^M , then $M = L(o(M))$.

PROOF: Let ψ be as in Theorem 11.9.

Claim: $\chi = \psi \wedge (V = M)$.

Proof: In any transitive set M in which χ is true, both ψ and $V = L$ are also true. Since ψ is true, $L^M = L(o(M))$. But M satisfies $V = L$, so M "thinks" all of its elements are in its version of L , which is $L(o(M))$. Hence,

$M = L(\alpha) = L(o(M))$. $\square$

Chapter 12

$V = L \Rightarrow AC + GCH$

The goal for this chapter is to prove, $V = L \Rightarrow AC + GCH$. In the process of accomplishing this, we will develop and use the Löwenheim-Skolem Method. This method will also be used in Chapter 13 to prove $\Diamond$, and $\Diamond^+$.

To begin, we'll define a sequence of well-orders $\triangleleft_\alpha$. The ordering $\triangleleft_\alpha$ well-orders $L(\alpha)$. This ordering will then induce a lexicographic order on $L(\alpha)^{<\omega}$ which will then allow us to well-order $L(\alpha+1) = \mathfrak{D}(L(\alpha))$ by enumerating the countable number of ways subsets of $L(\alpha)$ can be defined from elements of $L(\alpha)^{<\omega}$ (i.e. *the number of formulas without parameters*). To list out these formulas, we'll use the function En (Definition 9.6) which enumerates Df in order.

DEFINITION 12.1: We'll recursively define a set of well-orders of L , such that for each α , order $\triangleleft_\alpha$ orders $L(\alpha)$ as follows:

- i. If $\alpha = 0$, $\triangleleft_\alpha = 0$.
- ii. If $\alpha = \beta + 1$, we'll assume $\triangleleft_\beta$ and let $t, s \in L(\beta)^n$. Then we'll define a lexicographic ordering of $L(\beta)^n$ by

$$s \triangleleft_\beta^n t \text{ iff } \exists k < n (s|_k = t|_k \text{ and } s(k) \triangleleft_\beta t(k)).$$

Recall that

$$\mathfrak{D}(L(\beta)) = \{X \subseteq L(\beta) : \exists n \in \omega \exists s \in L(\beta)^n \exists R \in Df(L(\beta), n+1) \\ (X = \{x \in L(\beta) : s \wedge \langle x \rangle \in R\})\}.$$

So if $X \in L(\beta+1) = \mathcal{D}(L(\beta))$, then X is of the form

$$X = \{x \in L(\beta) : s \wedge \langle x \rangle \in R\}.$$

So define the following:

Let n_X be the least n such that

$$\exists s \in L(\beta)^{n_X} \exists R \in \text{Df}(L(\beta), n_X+1) (X = \{x \in L(\beta) : s \wedge \langle x \rangle \in R\}).$$

Let s_X be the $\triangleleft_\beta^{n_X}$ -least $s \in L(\beta)^{n_X}$ such that

$$\exists R \in \text{Df}(L(\beta), n_X+1) (X = \{x \in L(\beta) : s \wedge \langle x \rangle \in R\}).$$

Let m_X be the least $m \in \omega$ such that

$$X = \{x \in L(\beta) : s_X \wedge \langle x \rangle \in \text{En}(m, L(\beta), n_X)\}.$$

Then for $X, Y \in L(\beta+1) \setminus L(\beta)$, we define $X \triangleleft_{\beta+1} Y$ for the following three cases:

- a. $X, Y \in L(\beta) \wedge X \triangleleft_\beta Y$.
 - b. $X \in L(\beta) \wedge Y \notin L(\beta)$.
 - c. $X, Y \notin L(\beta) \wedge [(n_X < n_Y) \vee (n_X = n_Y \wedge s_X \triangleleft_\beta^{n_X} s_Y) \vee (n_X = n_Y \wedge s_X = s_Y \wedge m_X < m_Y)]$.
- iii. If α = a limit ordinal with $x, y \in L(\alpha)$, then $x \triangleleft_\alpha y$ iff
- $$(\rho(x) < \rho(y)) \vee (\rho(x) = \rho(y) \wedge x \triangleleft_{\rho(x)+1} y)$$

where $\rho(x)$ = rank(x), the first appearance of x in the ordering.

So $x \triangleleft_\alpha y$ when X occurs first on a lower level than Y first occurs, or they both occur initially on the same level and are ordered by the ordering of that level.

LEMMA 12.2: $\triangleleft_\alpha$ Well-orders $L(\alpha)$.

PROOF: (by induction on α):

Base case : The proof for $\alpha = 0$ is trivial.

Successor case : Let $\alpha = \beta + 1$ and suppose that $\triangleleft_\beta$ is a well-order. Now for cases (a) and (b) above,

$$\xi < \alpha \rightarrow \triangleleft_\xi = \triangleleft_\alpha|_\xi \text{ and } x \in L_\xi, y \in L_\alpha \setminus L_\xi \rightarrow x \triangleleft_\alpha y.$$

(This just says that x is least element of both $L(\beta)$ and $L(\beta+1)$. Hence, it suffices to show that $\triangleleft_{\beta+1}$ well-orders $L(\beta+1) \setminus L(\beta)$. Now $\triangleleft_{\beta+1}$ depends on $\triangleleft_\beta^n$, so we must first prove the following claim.

Claim: $\forall \alpha (\triangleleft_\alpha^n \text{ well-orders } L(\alpha)^n)$.

Proof (by induction on α): Fix $n \in \omega$. Assume $\forall \beta < \alpha$, that $\triangleleft_\beta^n$ is a well-order. Let $S \neq \emptyset$ be a set of n -tuples, $S \subseteq L(\alpha)^n$. Define s_i for $i = 1, \dots, n$, as follows:

Let $s_1 =$ the ϵ -least element of the first coordinates of all $s \in S$.

Let $s_2 =$ the ϵ -least element of the second coordinates of all $s \in S$ whose first coordinates are s_1 .

...

Let $s_n =$ the ϵ -least element of the n^{th} coordinates of all $s \in S$ whose first coordinates are s_1 , second coordinates are $s_2, \dots$, and $(n-1)^{\text{th}}$ coordinates are s_{n-1} .

Then $\hat{s} = (s_1, \dots, s_n)$ will be the $\triangleleft_\alpha^n$ -least element of S .

Hence, $\triangleleft_\alpha^n$ well-orders $L(\alpha)^n$.

Now to show that $\triangleleft_{\beta+1}$ well-orders $L(\beta+1) \setminus L(\beta)$, let $W \neq \emptyset$ be a subset of $L(\beta+1) \setminus L(\beta)$. Then define $\hat{n}$, $\hat{s}$, and $\hat{m}$ as follows:

$$\hat{n} = \min\{n_X : X \in W\}.$$

$$\hat{s} = \triangleleft_\beta^n\text{-least } \{s_X : X \in W \wedge \hat{n} = n_X\}.$$

$$\hat{m} = \text{least } \{m_X : X \in W \wedge \hat{n} = n_X \wedge \hat{s} = s_X\}.$$

Then there exists a unique $\hat{X}$ defined by:

$$\hat{X} = X \in W \ (\hat{n} = n_X \wedge \hat{s} = s_X \wedge \hat{m} = m_X).$$

Then $\hat{X}$ is the $\triangleleft_{\beta+1}$ -least element in W .

Limit case: Let α be a limit ordinal. The proof that $\triangleleft_\alpha = \bigcup_{\xi < \alpha} \triangleleft_\xi$ is trivial by induction from the definitions. $\square$

THEOREM 12.3: $V = L \Rightarrow AC$.

PROOF: So assume $V = L$. If $x \in L$, for any α such that $x \subseteq L(\alpha)$, $\triangleleft_\alpha$ well-orders x . $\square$

COROLLARY 12.4: $\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZFC})$.

PROOF: $\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZF} + V = L)$ and $V = L \rightarrow AC$. $\square$

NOTE: ZF does not prove AC, but if ZF is consistent, then so is ZFC.

Our next step is to show that $V = L \Rightarrow GCH$.

DEFINITION 12.5: We define an order $<_L$ of L as follows:

if $x, y \in L$, then $x <_L y$ iff $(\rho(x) < \rho(y)) \vee (\rho(x) = \rho(y) \wedge x \triangleleft_{\rho(x)+1} y)$.

LEMMA 12.6:

- Each $L(\alpha)$ is an initial segment of L under $<_L$.
- $<_L|_{L(\alpha)}$ is the $\triangleleft_\alpha$ given in Definition 12.1.
- $<_L$ well-orders L .

PROOF:

- a. Follow directly from the definition of $\rho(x)$.
- b. Follows inductively from the definition of $<_L$ and the fact that if $\alpha < \beta$, then $\triangleleft_\alpha = \triangleleft_\beta|_{L(\alpha)}$.
- c. Follows from (a) and (b) and the fact that $\triangleleft_\alpha$ is a well-order. $\square$

LEMMA 12.7:

- a. The function $\triangleleft_\alpha$ is absolute for transitive models of ZF-P.
- b. If M is a transitive proper class model of ZF-P, then $<_L$ is absolute for M .
- c. If M is a transitive set model for ZF-P, $x, y \in M$, and $x, y \in L^M$, then $x <_L y$ is absolute for M .

PROOF:

- a. $\triangleleft_\alpha$ can be represented by a Δ_0 formula and recursion. The fact that all Δ_0 formulas are absolute together with Theorem 8.23, implies that $\triangleleft_\alpha$ is absolute.
- b. For any transitive proper class model of ZF-P, $L \subseteq M$ (Theorem 11.6). Then $\forall \alpha (<_L|_{L(\alpha)} = \triangleleft_\alpha)$. Therefore, by part (a), $<_L$ is absolute.
- c. If $x, y \in L^M$, then $x, y \in L(\alpha)$ for some $\alpha < o(M)$. So $x <_L y$ iff $(x <_L y)^M$. $\square$

So in L , there is a fixed order of the universe that well-orders any set. The question often arises as to just how $\mathbb{R}$, the set of real numbers, can be well-ordered. One answer lies in $V = L$: the elements of $\mathbb{R}$ is ordered in the way L constructs them.

To prove $(\text{GCH})^L$, we will use a condensation argument: take a nontransitive elementary substructure of some $L(\alpha)$ and condense it down to some $L(\beta)$. This method requires the use of the Reflection Theorem, the Löwenheim-Skolem Theorem, and the Mostowski Collapsing Theorem. Both Reflection and Mostowski have already been proved. We shall next prove Löwenheim-Skolem.

Recall that in ZF we can prove that $\text{Df}(A, n)$, the set of n -place relations on A which are definable by a formula relativized to A , is countable since it was defined as the closure of a countable set under finitary operations (Lemma 12.9 below). Furthermore, Df is absolute for transitive models of $\text{ZF} - \text{P}$ because if $A \in M$ is transitive, then all elements of A are in M ; hence relativizing a formula to A means the same in M and V . So, Df shortens both the formal logical syntax and the notion of what it means to be true in ZF. The function $\text{En}(m, A, n)$ enumerates all the definable formulas and shortens Gödel's system of coding them.

DEFINITION 12.8: An **n -ary** function on A is a function $f : A^n \rightarrow A$ for $n > 0$ or an element of A if $n = 0$. A **finitary** function is an n -ary function for some fixed n . If $\mathcal{A}$ is a set of finitary functions, then the **closure** of $B \subseteq A$ under $\mathcal{A}$, written $\text{cl}(A)$, is the least $C \subseteq A$ which is closed under all functions in $\mathcal{A}$ such that $B \subseteq C$.

LEMMA 12.9 (AC): Let κ be an infinite cardinal. Suppose $B \subseteq A$ has cardinality $\leq \kappa$, and $\mathcal{A}$, a collection of finitary functions on A , has cardinality $\leq \kappa$. Then, if $C = \text{closure of } B \text{ under } \mathcal{A}$, then $|C| \leq \kappa$.

PROOF: The proof of this lemma is similar to that of Lemma C.30.

LEMMA 12.10: Let $\phi(x_0, \dots, x_{n-1})$ be a formula with free variables a subset of $\{x_0, \dots, x_{n-1}\}$. Then

$$\exists m \forall A [\{s \in A^n : \phi^A(s(0), \dots, s(n-1))\} = \text{En}(m, A, n)]. \quad (*)$$

PROOF: Then proof of this lemma is by induction on the length of ϕ and is similar to the proof of Lemma 9.4. $\square$

NOTE: This lemma is a metalemma which says that for each formula ϕ , we can always find an m such that $(*)$ is provable in ZF: ϕ is the m^{th} formula in n variables under the enumerating function En regardless of which A we choose.

DEFINITION 12.11: Set A is an **elementary substructure** of B , written $A \prec B$, iff $A \subseteq B$ and $\forall m, n (\text{En}(m, A, n) = \text{En}(m, B, n) \cap A^n)$.

LEMMA 12.12: For every formula $\phi(x_0, \dots, x_{n-1})$ and for any sets A and B , $A \prec B$ implies that ϕ is absolute for A, B .

PROOF: Follows from Lemma 11.2 and the definition of $A \prec B$. $\square$

DOWNWARD LÖWENHEIM-SKOLEM-TARSKI THEOREM 12.13 (AC):
 $\forall B \forall X \exists A (X \subseteq A \subseteq B \wedge A \prec B \wedge |A| \leq \max\{\omega, |X|\})$.

Given any sets X and B such that $X \subseteq B$, there exists a set A that is either countable or of the same cardinality as X (if $|X| > \omega$) for which all formulas are absolute for A, B .

PROOF: Fix B with well-order $\triangleleft$ (guaranteed by AC) and let $X \subseteq B$. Then $\forall m, n \in \omega$ we define the **Skolem function** for the m^{th} formula in n variables, $H_{mn} : B^n \rightarrow B$, as follows:

$$H_{mn}(s) = \begin{cases} \triangleleft\text{-first } x \in B \text{ such that } s \wedge \langle x \rangle \in \text{En}(i, B, n+1) & \text{if } m \text{ is of the form } 2^i 3^j 5^k \\ \triangleleft\text{-first } x \in B & \text{otherwise.} \end{cases}$$

(So if ϕ , the m^{th} formula in n variables, is an existential quantifier, then $H_{mn}(s)$ is the $\triangleleft$ -first $x \in B$ such that $s \wedge \langle x \rangle$ satisfies ϕ ; otherwise, $H_{mn}(s)$ is the $\triangleleft$ -first $x \in B$.)

Let $A = \text{cl}(X)$ under all H_{mn} . Then if $|X| \leq \omega$, $|A| = \omega$; if $|X| = \lambda > \omega$, then $|A| = \lambda$. (by Lemma 12.9).

It is a straightforward induction proof to show that

$$\forall n (\text{En}(m, A, n) = \text{En}(m, B, n) \cap A^n).$$

Thus, A is an elementary substructure of B ; i.e. $A \prec B$. $\square$

LÖWENHEIM-SKOLEM METHOD 12.14: (This method uses a condensation argument.)

Given a set B with subset X , we want to shrink B down to an elementary substructure A , which covers X and for which all formulas are absolute for A, B . A must be large enough such that if ϕ is of the form $\exists x \psi(x_0, \dots, x_{n-1})$,

then $\exists b \in A$ for which if $b \in B$, then $\exists x \psi(x_0, \dots, x_{n-1}, b)$ also holds. This method is used to prove GCH, $\Diamond$, and $\Diamond^+$. Their proofs all

- Taking a nontransitive elementary submodel of some $L(\alpha)$,
- Constructing $B \subseteq L(\alpha)$ such that all formulas are absolute for $B, L(\alpha)$.
- Applying Mostowski's Collapsing Theorem to obtain a transitive M such that $\langle M, \in \rangle \cong \langle B, \in \rangle$.

We use this approach to avoid having to talk about all formulas simultaneously; instead we find a W for which a finite collection of formulas are absolute.

LEMMA 12.15 (ZF): If $A \subseteq B$ and A is transitive and $\pi: B \rightarrow C$ is the Mostowski collapse, then $\pi|_A = \text{id}|_A$ (where id is the identity function).

PROOF (by induction on A): Suppose $x \in A$. Then

$$\begin{aligned}\pi(x) &= \{\pi(y) : y \in x \cap B\} \text{ (by definition of the collapse)} \\ &= \{\pi(y) : y \in x\} \text{ (since } x \in A \text{ and } A \text{ is transitive).}\end{aligned}$$

We want that $\pi(x) = x$; that is that $x = \{\pi(y) : y \in x\}$.

($\Rightarrow$) If $z \in x$, then $\pi(z) \in \{\pi(y) : y \in x\}$.

($\Leftarrow$) If $z \in \{\pi(y) : y \in x\}$, then $z = \pi(y)$ for some y . Since $y \in x$, by induction hypothesis, $\pi(y) = y$; so $z \in x$. $\square$

To prove that $V=L \Rightarrow \text{GCH}$, the following theorem will be needed.

THEOREM 12.16: If $V = L$, then for all infinite $\alpha \in \text{Ord}$, $\mathcal{P}(L(\alpha)) \subseteq L(\alpha^+)$.

PROOF: Assume $V = L$ and let χ be a finite conjunction of axioms of $\text{ZF} + V = L$ such that for all transitive M , $\chi^M \rightarrow M = L(o(M))$ (such a χ exists by

the Completeness Theorem). Fix $A \in \mathcal{P}(L(\alpha))$. Let $X = L(\alpha) \cup \{A\}$. Then $|X| = |L(\alpha) \cup \{A\}| = |L(\alpha)| = |\alpha|$. (The second equality holds because $|\{A\}| = 1$; the third holds by Theorem 9.25.) Now choose γ such that $L(\gamma) \models \chi$ (such a γ is guaranteed to exist by Reflection). Pick B such that B and an γ such that $X \subseteq B$, $B \prec L(\gamma)$, and $|B| \leq \max\{\omega, |X|\}$. Then by the Mostowski Collapsing Theorem, there exists a transitive M such that $\langle M, \in \rangle \cong \langle B, \in \rangle$ which implies that $|M| = |\alpha|$.

Furthermore, $\chi^M \leftrightarrow \chi^B \leftrightarrow \chi^{L(\gamma)} \leftrightarrow \chi^V$. (The first implication holds since $M \cong B$ and the second since $B \prec L(\gamma)$; the third holds choice of γ and Reflection). Also, $V = L \Rightarrow \chi^V$ holds $\Rightarrow \chi^M$ holds, so $M = L(o(M))$ by choice of χ .

Now $|M| = |\alpha|$; therefore $|o(M)| \leq |\alpha| < \alpha^+$.

Claim: $A \in M$.

Proof: $\pi(\delta) = \{\pi(\delta) : \delta \in A \cap B\}$
 $= \{\delta : \delta \in A \cap B\}$ (because $A \subseteq \alpha$ and π is the identity function
on $L(\alpha)$ by Lemma 12.15)
 $= \{\delta : \delta \in A\}$ (because $\alpha \subseteq B$)
 $= A$

So $A \in X \subseteq M = L(o(M)) \subseteq L(\alpha^+)$. $\square$

COROLLARY 12.17: $V = L \rightarrow \text{GCH}$.

PROOF: $\forall \kappa \geq \omega$, $|\mathcal{P}(\kappa)| = 2^\kappa$ and $\mathcal{P}(L(\kappa)) \subseteq L(\kappa^+)$. Furthermore, $\mathcal{P}(\kappa) \subseteq \mathcal{P}(L(\kappa)) \subseteq L(\kappa^+)$. Hence, $2^\kappa \leq |L(\kappa^+)| = \kappa^+$. And by Cantor, $\kappa^+ \leq 2^\kappa$. Therefore, $2^\kappa = \kappa^+$. $\square$

COROLLARY 12.18 (ZF): $(AC + GCH)^L$.

PROOF: Follows immediately from Theorem 12.3 and Corollary 12.15. $\square$

COROLLARY 12.19: $\text{Con (ZF)} \rightarrow \text{Con (ZFC + GCH)}$.

PROOF: Follows immediately from Corollary 12.18. $\square$

Chapter 13

$V=L \Rightarrow \Diamond^+ \Rightarrow \Diamond$

Recall that $\Diamond^+$ is the statement: There exists a sequence, $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$, called a $\Diamond^+$ -sequence, where for each $\alpha < \omega_1$, $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| < \omega_1$, and $\forall A \subseteq \omega_1$, there exists a c.u.b. $C \subseteq \omega_1$ with the following properties:

- a. $\forall \alpha \in C, A \cap \alpha \in \mathcal{A}_\alpha$.
- b. $\forall \alpha \in C, C \cap \alpha \in \mathcal{A}_\alpha$.

To prove $\Diamond^+$ from $V = L$ we will apply Löwenheim-Skolem to $\{\omega_1, A\}$ followed by an application of the Mostowski Collapsing Theorem.

LEMMA 13.1: $\{\rho < \omega_1 : L(\rho) \prec L(\omega_1)\}$ is unbounded in ω_1 .

PROOF: Let $H_{mn} : L(\omega_1)^n \rightarrow L(\omega_1)$ be the Skolem function for the m^{th} formula in n variables with $m, n \in \omega$ defined as follows:

$$H_{mn}(s) = \begin{cases} \Delta - \text{first } x \in L(\omega_1) \text{ such that } s \wedge \langle x \rangle \in \text{En}(i, L(\omega_1), n+1) & \text{if } m \text{ is of the form } 2^i 3^j 5^4 \\ & \text{and if } s \in \text{En}(m, L(\omega_1), n) \\ \Delta - \text{first } x \in L(\omega_1) & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} \text{En}(m, L(\omega_1), n) &= \text{Proj}(L(\omega_1), \text{En}(m, L(\omega_1), n+1), n) \\ &= \{s \in L(\omega_1)^n : \exists t \in \text{En}(i, L(\omega_1), n+1) (t \upharpoonright_n = s)\}. \end{aligned}$$

(This is saying that the m^{th} formula in n variables is an existential quantification and is true of s .)

Then $L(\rho) \prec L(\omega_1)$ whenever $L(\rho)$ is closed under all H_{mn} .

Claim: $\{p : L(p) \prec (\omega_1)\}$ is c.u.b in ω_1 .

Proof: $|p| \leq \omega$ and $\mathcal{H} = \{H_{mn} : m, n \in \omega\}$ has cardinality ω and is a set of finitary functions on ω_1 . Therefore, $\{p < \omega_1 : p \text{ is closed under } \mathcal{H}\}$ is c.u.b. in ω_1 (by Lemma C.30). $\square$

LEMMA 13.2: Let $G: A \rightarrow B$ be an isomorphism for the relation $\in$. Then

$$\forall n, m \forall x \in A^n (s \in \text{En}(m, A, n) \leftrightarrow (G \circ s \in \text{En}(m, B, n))).$$

PROOF: The proof is by induction on n . $\square$

Now to prove $V = L \Rightarrow \Diamond^+$, we shall work in $L(\omega_2)$ because by Theorem 12.16 every subset of $L(\omega_1)$ is an element of $L(\omega_2)$.

THEOREM 13.3: $V = L \Rightarrow \Diamond^+$.

PROOF: Assume $V = L$. Then for $\alpha < \omega_1$, define $q(\alpha)$ = the least $p > \alpha$ such that $L(p) \prec L(\omega_1)$.

Claim i: $q(\alpha) < \omega_1$.

Proof: $|L(\omega_1)| = \omega_1$ and $|\alpha| = \omega_1$. Now $\mathcal{R} = \{p > \alpha : L(p) \prec L(\omega_1)\}$ is unbounded and, hence, $\mathcal{R} \neq \emptyset$. Therefore, $\exists p > \alpha$ such that $p \in \mathcal{R}$. Since ordinals are well-ordered, there exists a least such $p = q(\alpha)$. Hence, $q(\alpha) \in \mathcal{R}$ implies $q(\alpha) < \omega_1$.

Next define $\mathcal{A}_\alpha = \mathcal{P}(\alpha) \cap L(q(\alpha))$. Since $|L(q(\alpha))| = |q(\alpha)| < \omega_1$, $\mathcal{A}_\alpha$ is countable. We shall now use the Löwenheim-Skolem method applied to $L(\omega_2)$ to show that $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ is a $\Diamond^+$ -sequence. Let $<_L$ be a well-order of $L(\omega_2)$.

Define Skolem functions $K_{mn} : L(\omega_2)^n \rightarrow L(\omega_2)$ by

$$K_{mn}(s) = \begin{cases} \Delta - \text{first } x \in L(\omega_2) \text{ such that } s \wedge \langle x \rangle \in \text{En}(i, L(\omega_2), n+1) & \text{if } m \text{ is of the form } 2^i 3^j 5^4 \\ 0 & \text{otherwise.} \end{cases}$$

Now fix $X \subseteq L(\omega_2)$, and let $Y = \text{cl}(X)$ under all K_{mn} . Then $Y \prec L(\omega_2)$ (Y is the A in Theorem 10.33). Furthermore, all axioms of $\text{ZF} - P + V = L$ hold in $L(\omega_2)$ (the proof of this is similar to that for L using that ω_2^L is regular in L) so all axioms of $\text{ZF} - P + V = L$ hold relativized to Y . Since Extensionality is one such axiom, $\in$ is extensional in Y . Hence,

$$\forall x, y \in Y (\forall z \in Y (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

Now $\in$ is well-founded, set-like, and extensional on Y ; therefore, there exists a unique transitive class M and a unique isomorphism $G: Y \rightarrow M$ such that $\langle Y, \in \rangle \cong \langle M, \in \rangle$ by the Mostowski Collapsing Theorem.

Now let $\phi(x)$ represent " x is the first uncountable ordinal". Then $(\exists x \phi(x))^{L(\omega_2)}$ under $<_L$, and hence, $(\exists x \phi(x))^Y$ since $Y \prec L(\omega_2)$ implies that $\exists x \phi(x)$ is absolute. So if $\gamma \in Y$ satisfies $\phi(x)^Y$, then $\phi(x)^{L(\omega_2)}$. Hence, $\gamma = \omega_1$.

Furthermore, since $\gamma \in Y$, $\omega_1 \in Y$.

Similarly, there are formulas which represent " x is the empty set" and " x is the first infinite ordinal", so we can show $0, \omega \in Y$. And since we can show that Y is closed under the ordinal successor function S , $\omega+1, \omega+2, \dots \in Y$, implying that $\omega+\omega \in Y$ and $\omega+\omega \subseteq Y$. Furthermore, if $\xi < \omega_1$ and $\xi \in Y$, then $(\xi \text{ is countable})^Y$. Hence, $\exists f \in Y$ where $f''\omega = \xi$ (i.e. $f: \omega \xrightarrow{\text{onto}} \xi$) because $(f''\omega = \xi)^Y \Rightarrow (f''\omega = \xi)^{L(\omega_2)}$ (so what Y thinks is the ordinal ξ , actually is the ordinal ξ).

Claim ii: Either $\omega_1 \subseteq Y$ or $Y \cap \omega_1 \in \omega_1$.

Proof: Recall that $X \subseteq L(\omega_2)$. Now if $|X| \leq \omega$, then $|Y| = \omega$ since $Y = \text{cl}(X)$ under the Skolem functions. Furthermore, since Y contains all ordinals up to some ordinal, ξ , $Y \cap \omega_1 = \xi \in \omega_1$. On the other hand, if $|X| \geq \omega_1$, then $|Y| \geq \omega_1$; so $\omega_1 \subseteq Y$.

For the remainder of this proof, we'll assume that X is countable.

Let $\alpha = Y \cap \omega_1 \in \omega_1$. Then α is the first countable ordinal not in Y (such an ordinal exists since the ordinals are unbounded in ω_1). Let $G : Y \rightarrow M$ be the Mostowski Collapsing isomorphism with M transitive. Since all axioms of $ZF - P + V = L$ hold in M , there exists a $\delta \in \text{Ord}$ such that $M = L(\delta)$ (because $(V = L \wedge M \models \chi) \Rightarrow (M \subseteq L)$). Now $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$, so

$$Y \cap (\omega_1 + 1) = Y \cap (\omega_1 \cup \{\omega_1\}) = \alpha \cup \{\omega_1\}.$$

Since $G : Y \rightarrow L(\delta)$ is an isomorphism on the ordinals, $G(\xi) = \xi \ \forall \xi < \alpha$, and $\alpha = G(\omega_1)$. (Note that since $\alpha = Y \cap \omega_1 \in \omega_1$, $|\alpha| < \omega_1$; hence, this α is the ξ described in (*)). Furthermore,

$$Y \cap (\omega_1 + 1) = \alpha \cup \{\omega_1\} \Rightarrow G''\alpha \cup \{\omega_1\} = \alpha + 1.$$

Therefore, $\alpha < \delta < \omega_1$. So $Y = L(\alpha) \cup (\text{some subset of } L(\omega_2) \setminus L(\omega_1))$ and G collapses Y down to some $L(\delta)$ which has the same cardinality as Y but which is transitive. Now

$$\begin{aligned} & \phi(\omega_1)^V \text{ (since } \omega_1 \text{ is the first uncountable ordinal in } Y) \\ & \Rightarrow \phi(\alpha)^M \text{ (since } G : \omega_1 \rightarrow \alpha \text{ by Mostowski)} \\ & \Rightarrow (\alpha \text{ is uncountable})^M \text{ (by definition of } \phi) \\ & \Rightarrow (\alpha \text{ is uncountable})^{L(\delta)}. \end{aligned}$$

Also, $(\alpha \text{ is countable})^{L(\omega_1)} \Rightarrow (\alpha \text{ is countable})^{L(q(\alpha))}$ since $\alpha < \omega_1$ and $L(q(\alpha)) \prec L(\omega_1)$; so " α is countable" for $L(q(\alpha))$, $L(\omega_1)$. Thus $\delta < q(\alpha)$

because the function which witnesses that " α is countable" has not been constructed by $L(\delta)$; however, it has been constructed by $L(q(\alpha))$. Therefore, $\alpha < \delta < (q(\alpha)) < \omega_1$.

Next suppose $A \in Y$ and $A \subseteq \omega_1$. Then we can define Mostowski isomorphism restricted to A by $G(A) = \{G(x) : x \in A \cap Y\}$. Since Y is a set of ordinals, $G(A) = \{G(\xi) : \xi \in A \cap \alpha\} = A \cap \alpha$. Thus, we have

$$A \cap \alpha = G(A) \in L(\delta) \subseteq L(q(\alpha)) \text{ and}$$

$$A \cap \alpha \in \mathcal{P}(\alpha) \cap L(q(\alpha)) = \mathcal{A}_\alpha.$$

Our goal is to show that $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ is a $\Diamond^+$ -sequence. Fix $A \subseteq \omega_1$. Define

$$C = \{\alpha < \omega_1 : \alpha = \text{cl}(\alpha \cup \{A\}) \cap \omega_1\}.$$

Claim iii: C is c.u.b. in ω_1 .

Proof: By definition, $C \subseteq \omega_1$. In general, for any $f : \omega_1 \rightarrow \omega_1$, the set $D = \{\delta \in \omega_1 : f''\delta \subseteq \delta\}$ is c.u.b. in ω_1 . Then if $\sigma = \sup(D)$, $\forall \gamma \in \sigma$, $f''\gamma \subseteq \gamma < \sigma$. Now each Skolem function is such an f ; so $\forall \alpha \in C \forall K_{mn}$ ($K_{mn}(\alpha) \subseteq \alpha$). Therefore, α is closed. C is unbounded in ω_1 because ω_1 is regular. Thus C is c.u.b.

Hence, $\forall \beta < \omega_1 \exists \gamma > \beta (\gamma = \text{cl}(\alpha \cup \{A\}) \cap \omega_1)$.

Next, fix $\alpha \in C$.

Claim iv: $A \cap \alpha \in \mathcal{A}_\alpha$.

Proof: Let $Y = \text{cl}(\alpha \cup \{A\})$. Then $A \in Y$. Thus, by above argument $A \cap \alpha \in \mathcal{A}_\alpha$.

Claim v: $C \cap \alpha \in \mathcal{A}_\alpha$.

Proof: First note that $C \notin Y$, so we cannot use the above argument. We will begin by defining $L(\delta)$ analogs for K_{mn} . Let $K_{mn}^1 : L(\delta)^n \rightarrow L(\delta)$ be the function induced by the isomorphism G :

$$K_{mn}^1 (G(y_1), \dots, G(y_n)) = G(K_{mn}(y_1, \dots, y_n)).$$

(*) Let cl^1 be the closure under all K_{mn}^1 . Then by the definition of C and the fact that $G(A) = A \cap \alpha$,

$$C \cap \alpha = \{\beta < \alpha : \beta = cl^1(\beta \cup \{A \cap \alpha\}) \cap \alpha\}.$$

Next define $K_{mn}^2 : L(\delta)^n \rightarrow L(\delta)$ to be $K_{mn}|_\delta$ and let cl^2 be the closure under all K_{mn}^2 . Then

$$C^2 = \{\beta < \alpha : \beta = cl^2(\beta \cup \{A \cap \alpha\}) \cap \alpha\}.$$

Now $L(\delta) \in L(q(\alpha)) \prec L(\omega_1)$. Since $L(\omega_1)$ is a transitive model for $ZF - P + V = L$, so absoluteness of the necessary concepts implies that $C^2 \in L(q(\alpha))$. Hence, to show $C \cap \alpha \in \mathcal{A}_\alpha$, it suffices to show that $C \cap \alpha = C^2$ (for then $C \cap \alpha \in L(q(\alpha))$ and, therefore, is in $\mathcal{A}_\alpha$); thus, it suffices to show that $K_{mn}^1 = K_{mn}^2$.

Claim: For $y_1, \dots, y_n \in Y$ $(K_{mn}^2(G(y_1), \dots, G(y_n)) = G(K_{mn}(y_1, \dots, y_n)))$.

Proof: Assume $m = 2^i 3^j 5^4$ and $s = \langle y_1, \dots, y_n \rangle \in \text{En}(m, L(\omega_2), n)$. Then

$s \in \text{En}(M, Y, n)$ since $Y \prec L(\omega_2)$ and $K_{mn}(s)$ is the $<_L$ -first $x \in Y$ such that $s \wedge \langle x \rangle \in \text{En}(i, Y, n+1)$. Also, since $s \in \text{En}(m, L(\delta), n)$,

$\langle G(y_1), \dots, G(y_n) \rangle \in \text{En}(m, L(\delta), n)$ (by Lemma 13.2) and

$K_{mn}^2(G(y_1), \dots, G(y_n))$ is the $<_L$ -first element $G(x) \in L(\delta)$ such that

$s \wedge \langle x \rangle \in \text{En}(i, Y, n+1)$ (by applying Lemma 13.2 to $\text{En}(i, Y, n+1)$). Now for

$$x, z \in Y, \quad x <_L z \iff (x <_L z)^{L(\omega_2)} \quad (\text{by absoluteness})$$

$$\iff (x <_L z)^Y \quad (\text{since } Y \prec L(\omega_2))$$

$$\iff (G(x) <_L G(z))^{L(\delta)} \quad (\text{by Lemma 8.26})$$

$$\iff G(x) <_L G(z) \quad (\text{by absoluteness}).$$

Therefore, $K_{mn}^2(G(y_1), \dots, G(y_n)) = G(K_{mn}(y_1, \dots, y_n))$. But by (*),

$$G(K_{mn}(y_1, \dots, y_n)) = K_{mn}^1(G(y_1), \dots, G(y_n)).$$

So $K_{mn}^1 = K_{mn}^2$; and hence, $C \cap \alpha = C^2$.

Now $C \cap \alpha \in L(q(\alpha))$, and thus, $C \cap \alpha \in \mathcal{P}(\alpha) \cap L(q(\alpha)) = \mathcal{A}_\alpha$.

Hence, $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ is a $\Diamond^+$ -sequence; so $V = L \Rightarrow \Diamond^+$. $\square$

Our next goal is to show that $\Diamond^+ \Rightarrow \Diamond$.

DEFINITION 13.4: $\Diamond^-$ is the statement: there are sets $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$ for $\alpha < \omega_1$, such that $\forall \alpha (|\mathcal{A}_\alpha| \leq \omega)$ and $\forall A \subseteq \omega_1$, $\{\alpha < \omega_1 : A \cap \alpha \in \mathcal{A}_\alpha\}$ is stationary.

LEMMA 13.5: $\Diamond^+ \Rightarrow \Diamond^-$.

PROOF: $\Diamond^-$ is just a weaker version of $\Diamond^+$. The proof is immediate from the definitions. $\square$

Recall $\Diamond$ is the statement: There exists a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ where each $A_\alpha \subseteq \alpha$ and $\forall A \subseteq \omega_1$, $S_A = \{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary in ω_1 .

LEMMA 13.6: $\Diamond^- \Rightarrow \Diamond$.

PROOF: Let $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond^-$ -sequence which captures subsets of ω_1 . We begin by altering this sequence to $\langle B_\alpha : \alpha < \omega_1 \rangle$ which captures subsets of $\omega \times \omega_1$. Fix $f: \omega_1 \xrightarrow{1-1} \omega \times \omega_1$ arbitrarily. For $A \subseteq \omega_1$, let $A^\circ = f''A$, which abbreviates $\text{rng}(f|_A)$ and for $B \subseteq \omega \times \omega_1$, let $B^* = f^{-1}(B)$. Then $A^{*\circ} = A$ and $B^{*\circ} = B$. Next let

$$C = \{\alpha < \omega_1 : \alpha \geq \omega_1 \wedge f^{-1}(\omega \times \alpha) \subseteq \alpha \wedge f''(\alpha) \subseteq \omega \times \alpha\}.$$

Then C is c.u.b. (by Lemma C.30). So if $\alpha \in C$, $A \subseteq \alpha$, and $B \subseteq \omega \times \alpha$, then $A^\circ \subseteq \omega \times \alpha$ and $B^* \subseteq \alpha$. Then we define $\langle \mathcal{B}_\alpha : \alpha < \omega_1 \rangle$ by:

$$\mathcal{B}_\alpha = \begin{cases} \{A^\circ : A \in \mathcal{A}_\alpha\} & \text{for } \alpha \in C \\ 0 & \text{for } \alpha \notin C. \end{cases}$$

Claim i: $\forall B \subseteq \omega \times \omega_1$ ($\{\alpha < \omega_1 : B \cap (\omega \times \alpha) \in \mathcal{B}_\alpha\}$ is stationary).

Proof: Let $S = \{\alpha < \omega_1 : B^* \cap \alpha \in \mathcal{A}_\alpha\}$. Then by $\Diamond$, S is stationary and, hence, $S \cap C$ is stationary. Now fix $\alpha \in S \cap C$. Then $B^* \cap \alpha \in \mathcal{A}_\alpha$ since $\alpha \in S$, and $(B^* \cap \alpha)^\circ \in \mathcal{B}_\alpha$ since $\alpha \in C$ (by definition of $\mathcal{B}_\alpha$). Furthermore, $B \cap (\omega \times \alpha) = (B^* \cap \alpha)^\circ$ since $\alpha \in C$. Hence, $\{\alpha < \omega_1 : B \cap (\omega \times \alpha) \in \mathcal{B}_\alpha\}$ is stationary. Next, for each α , we can rewrite $\mathcal{B}_\alpha = \{B_\alpha^k : k \in \omega\}$ where each $B_\alpha^k \subseteq \omega \times \alpha$ since $|\mathcal{B}_\alpha| \leq \omega$. Then let $B_{\alpha,n}^k = \{\xi : \langle n, \xi \rangle \in B_\alpha^k\}$.

Claim ii: $\exists n$ ($\langle B_{\alpha,n}^n : \alpha < \omega_1 \rangle$ is a $\Diamond$ -sequence).

Proof: Assume not. Then for each n , we can find a $B_n \subset \omega_1$ such that $\{\alpha < \omega_1 : B_n \cap \alpha = B_{\alpha,n}^n\}$ is non-stationary. Now,

$$\{\alpha < \omega_1 : \exists n (B \cap (\omega \times \alpha) = B_{\alpha,n}^n)\} \subseteq \{\alpha < \omega_1 : B_n \cap \alpha = B_{\alpha,n}^n\}$$

because $B_{\alpha,n}^n$ is the n^{th} cross section of B_n . Therefore, since the right side of this inequality is non-stationary, the left side is as well. Thus,

$\bigcup_{n \in \omega} \{\alpha < \omega_1 : \exists n (B \cap (\omega \times \alpha) = B_{\alpha,n}^n)\}$ since the union of a countable collection of non-stationary sets is non-stationary. But

$$\bigcup_{n \in \omega} \{\alpha < \omega_1 : \exists n (B \cap (\omega \times \alpha) = B_{\alpha,n}^n)\} = \{\alpha < \omega_1 : \exists n (B \cap (\omega \times \alpha) = B_{\alpha,n}^n)\}$$

contradicting Claim i. $\square$

THEOREM 13.7: $V = L \Rightarrow \Diamond$.

PROOF: This follows from Theorem 13.3 and Lemmas 13.5 and 13.6. $\square$

Appendix A

THE AXIOMS OF ZFC + GCH

AXIOM OF SET EXISTENCE: $\exists x (x = x)$.

This axiom states the existence of a set.

AXIOM OF EXTENSIONALITY: $\forall x \forall y (x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y))$.

This axiom states that two sets are equal iff they have the same elements.

AXIOM OF COMPREHENSION SCHEME: For each formula ϕ with free variables a subset of $\{x, z, w_1, \dots, w_n\}$, the following is an axiom:

$$\exists y (\forall x (x \in y \leftrightarrow x \in z \wedge \phi)).$$

This axiom states that once you have a set z and a property named by ϕ , you can separate out from z a subset y of all objects which have property ϕ .

NOTE: Both Comprehension and Replacement (below) are schema for infinite collections of axioms, one for each formula ϕ .

AXIOM OF FOUNDATION: $\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))]$.

This axiom states that every non-empty set has an ϵ -minimal element (i.e. if $x \neq \emptyset (\exists y \in (x \cap y = \emptyset))$.

AXIOM OF PAIRING: $\forall x \forall y \exists z (x \in z \wedge y \in z)$.

This axiom states that there exists a set which contains any two given sets.

AXIOM OF UNION: $\forall x \exists y \forall z (\exists w (w \in x \wedge z \in w) \rightarrow z \in y)$.

This axiom states that for every sets of sets x , there exists a set y which contains the union of x .

AXIOM OF REPLACEMENT SCHEME: For each formula ϕ in which w is not free, the following is an axiom:

$$\forall x \in z \exists! y \phi(x, y) \rightarrow \exists w \forall x \in z \exists y \in w \phi(x, y).$$

NOTE: $\forall x \in z \psi$ abbreviates $\forall x (x \in z \rightarrow \psi)$,
 $\exists y \in w \psi$ " $\exists y (y \in w \wedge \psi)$, and
 $\exists! y \psi$ " $\forall y \psi \wedge \forall y_1 \forall y_2 (\psi(y_1) \wedge \psi(y_2) \rightarrow y_1 = y_2)$.

This axiom states that if for each x in the domain of ϕ , where ϕ is a formula which defines a function, there exists a unique y in the range of ϕ , then there exists

a set w which covers the range of ϕ .

From the above axioms we can define the following notions which appear in the remaining axioms:

nullset: $\emptyset$ abbreviates $\exists! y (\forall x (x \notin y))$.

subset: $x \subseteq y$ " $\forall z \in x (z \in y)$.

successor $S(x)$ " $\forall x \exists! y (y = x \cup \{x\})$

AXIOM OF INFINITY: $\exists x (\emptyset \in x \wedge \forall y \in x (S(y) \in x))$.

This axiom states the existence of an infinite set.

AXIOM OF POWER SET: $\forall x \exists y \forall z (z \subseteq x \rightarrow z \in y)$.

This axiom states that given any set, there exists a power set which is the set of all subsets of the given set.

AXIOM OF CHOICE (AC): Every set can be well-ordered.

(i.e. $\forall A \exists R (R \text{ well-orders } A)$.)

GENERALIZED CONTINUUM HYPOTHESIS (GCH) 2: $\forall \alpha (2^{\omega_\alpha} = \omega_{\alpha+1})$

where $\omega_{\alpha+1} = \omega^+$. The **Continuum Hypothesis (CH)** is the case of GCH

where $\alpha = 0$.

NOTE: Cantor showed that $2^{\omega_\alpha} \geq \omega_{\alpha+1}$.

Appendix B

COFINALITY

DEFINITION B.1: A function $f: \alpha \rightarrow \beta$ is a **cofinal** in β iff $\text{rng}(f)$ is unbounded in β : $\forall \gamma < \beta \exists \delta < \alpha (f(\delta) \geq \gamma)$.

NOTE: If a sequence is cofinal in λ , then λ is a limit point of the sequence.

EXAMPLES:

1. $f: \omega \rightarrow \omega$ defined by $f(n) = n \ \forall n \in \omega$, is a cofinal function because $\forall n_0 \in \omega$, $\exists n_1 \in \omega$ such that $f(n_1) = n_1 > n_0$.
2. $f: \omega \rightarrow \aleph_\omega$ defined by $\forall n \in \omega (f(n) = \aleph_n)$, is a cofinal function because $\forall n_0 \in \omega \exists n_1 \in \omega (f(n_1) > \aleph_{n_0})$.

DEFINITION B.2: The **cofinality** of β , written $\text{cf}(\beta)$, is the least α such that there exists a cofinal function $f: \alpha \rightarrow \beta$.

NOTE: $\text{cf}(\beta) \leq \beta$.

EXAMPLES:

1. $\text{cf}(\aleph_\omega) = \omega$ because there is no $n < \omega$ for which there exists a cofinal function with domain n because n is finite and, hence, bounded.
2. $\text{cf}(\beta) = 1$ if β is a successor ordinal. To show this, let $\beta = \gamma + 1$ and define $f(0) = \gamma$. This function is cofinal in β because $\forall \delta < \beta \exists \alpha \in \text{dom}(f) (f(\alpha) \geq \delta)$. Since 0 is the element only in $\text{dom}(f)$, everything in $\text{dom}(f)$ goes to γ and $\forall \delta \in \beta (\gamma \geq \delta)$.

LEMMA B.3: If β is a limit ordinal, then $\text{cf}(\beta)$ is a limit ordinal.

PROOF: Assume $\text{cf}(\beta)$ is a successor ordinal: $\text{cf}(\beta) = \gamma+1$. Then there exists a cofinal function $f: \gamma+1 \rightarrow \beta$.

Claim: $f: \gamma \rightarrow \beta$ is also unbounded (i.e. $\forall \delta < \beta \exists \alpha < \gamma (f(\alpha) \geq \delta)$).

Proof: If $\forall \alpha \in \gamma+1 (f(\gamma) \geq f(\alpha))$, then $f(\gamma) \geq \delta$ for all $\delta < \beta$, since f is cofinal in β . But then $f(\gamma) < \beta \leq f(\gamma)+1$; hence, $\beta = f(\gamma)+1$ contradicting that β is a limit ordinal. Therefore, there must exist $\alpha < \gamma$ such that $f(\gamma) < f(\alpha)$.

So fix $\alpha_0 < \gamma$ such that $f(\gamma) < f(\alpha_0)$ and suppose $\delta < \beta$. We must find an $\alpha < \gamma$ such that $f(\alpha) \geq \delta$. Choose any $\eta \in \gamma+1$ such that $f(\eta) \geq \delta$. If $\eta \neq \gamma$, set $\alpha = \eta$; otherwise, set $\alpha = \alpha_0$ noting that then $f(\alpha_0) > f(\gamma) = f(\eta) \geq \delta$, as required. So we have found $\alpha < \gamma$ such that $f(\alpha) \geq \delta$ which implies that $f: \gamma \rightarrow \beta$ is unbounded. Since f maps γ cofinally into β , $\gamma+1$ is not the smallest ordinal which can be mapped cofinally into β , contradicting that $\text{cf}(\beta) = \gamma+1$. So $\text{cf}(\beta)$ must be a limit ordinal. $\square$

LEMMA B.4: If β is a limit ordinal, then there exists a strictly increasing cofinal function $f: \text{cf}(\beta) \rightarrow \beta$.

PROOF: If $\text{cf}(\beta) = \beta$, then the identity map is a strictly increasing cofinal function. Otherwise let $g: \text{cf}(\beta) \rightarrow \beta$ cofinally. Set

$f(\xi) = \max \{g(\xi), \sup(\{f(\eta)+1 : \eta < \xi\})\}$. Now f is cofinal since g is cofinal.

Furthermore, $\forall \xi \in \text{cf}(\beta)$, if $\eta < \xi$, then $f(\xi) \geq \sup(\{f(\eta)+1 : \eta < \xi\}) > f(\eta)$, so

f is strictly increasing. (Note that by the definition of cofinality, the $\text{dom}(f)$

includes all of $\text{cf}(\beta)$.) $\square$

LEMMA B.5: If α is a limit ordinal and $f: \alpha \rightarrow \beta$ is a strictly increasing cofinal function, then $\text{cf}(\alpha) = \text{cf}(\beta)$.

PROOF:

Claim i: $\text{cf}(\beta) \leq \text{cf}(\alpha)$.

Proof: Let $g: \text{cf}(\alpha) \rightarrow \alpha$ be a cofinal function.

Claim: $f \circ g: \text{cf}(\alpha) \rightarrow \beta$ is cofinal.

Proof: Let $\xi \in \beta$ be arbitrary. Then $\exists \delta_0 \in \alpha$ such that $f(\delta_0) \geq \xi$ since f is cofinal in β . Furthermore, $\forall \delta^* > \delta_0$, $f(\delta^*) > f(\delta_0)$ since f is strictly increasing. Then there exists a $\gamma \in \text{cf}(\alpha)$ such that $g(\gamma) \geq \delta^*$ since g is cofinal. So $f \circ g(\gamma) > f(\delta_0) \geq \xi$. Therefore, $f \circ g: \text{cf}(\alpha) \rightarrow \beta$ is cofinal. Now $\text{cf}(\beta)$ is the smallest ordinal which can be mapped cofinally into β ; therefore, $\text{cf}(\beta) \leq \text{cf}(\alpha)$.

Claim ii: $\text{cf}(\alpha) \leq \text{cf}(\beta)$.

Proof: We're given that $f: \alpha \rightarrow \beta$ is a strictly increasing cofinal function. Let $g: \text{cf}(\beta) \rightarrow \beta$ be cofinal. We seek a cofinal function $h: \text{cf}(\beta) \rightarrow \alpha$. Let $\gamma \in \text{cf}(\beta)$. Define $h: \text{cf}(\beta) \rightarrow \alpha$ by $h(\gamma) = \tau_0$ where τ_0 is the $\min\{\tau \in \alpha : f(\tau) > g(\gamma)\}$. We know that this set is not empty since f cofinal implies the existence of such a τ .

Claim: h is cofinal in α (i.e. $\forall \eta < \alpha$, $\exists \gamma \in \text{cf}(\beta)$ such that $h(\gamma) > \eta$).

Proof: Choose an arbitrary $\eta \in \alpha$ and choose $\gamma \in \text{cf}(\beta)$ such that $g(\gamma) > f(\eta)$ which is possible since g is cofinal in β . So $h(\gamma)$ is the least $\tau \in \alpha$ for which $f(\tau) > g(\gamma)$, but $\tau_0 > \eta$ since f is strictly increasing. By definition, $\tau_0 \in \text{rng}(h)$. So $\forall \eta \in \alpha \exists \gamma \in \text{cf}(\beta)$ such that $h(\gamma) > \eta$. Therefore, h is cofinal.

Now $\text{cf}(\alpha)$ is the least ordinal which maps cofinally into α .

Hence, $\text{cf}(\alpha) \leq \text{cf}(\beta)$. $\square$

COROLLARY B.6: $\text{cf}(\text{cf}(\beta)) = \text{cf}(\beta)$.

PROOF: Immediate from Lemmas B.4 and B.5. $\square$

DEFINITION B.7: β is a **regular** ordinal iff β is a limit ordinal and $\text{cf}(\beta) = \beta$.

NOTES:

1. All regular ordinals are cardinals because if β is not a cardinal then there exists a 1-1 correspondence between β and some smaller ordinal. Therefore, $\text{cf}(\beta) = \beta \Rightarrow |\beta| = \beta$.
2. κ is regular if any cofinal sequence is of length κ .

EXAMPLES:

1. ω is regular.
2. $\forall \beta$, $\text{cf}(\beta)$ is regular.

DEFINITION B.8: β is a **singular** ordinal if β is a limit ordinal and if $\text{cf}(\beta) \neq \beta$.

THEOREM B.9: κ is regular iff $\bigcup_{<\kappa} S$ such that $|S| < \kappa$ has size $< \kappa$.

PROOF:

($\Rightarrow$) Assume $\text{cf}(\kappa) = \kappa$.

Claim: Then $\text{cf}(\kappa) = \kappa$ the union of less than κ many sets of size less than κ cannot have size κ .

Proof: Let $S = \langle S_\alpha : \alpha < \lambda \rangle$ for $\lambda < \kappa$ and $|S_\alpha| < \kappa$. Set $\mu_\alpha = \sup(S_\alpha)$ which is less than κ since $|S_\alpha| < \kappa$ and κ is regular. Now $\sup(\langle \mu_\alpha : \alpha < \lambda \rangle) < \kappa$

because $\lambda < \kappa$ and κ is regular. Also $S_\alpha \subset \sup(\langle \mu_\alpha \rangle) \forall \alpha$. Therefore,
 $\bigcup S \subset \sup(\langle \mu_\alpha \rangle)$. Since $\bigcup S$ is bounded below κ , $|\bigcup S| < \kappa$.
 $(\Leftarrow)$ (Contrapositive) Assume κ is singular. Then $\text{cf}(\kappa) < \kappa$. $\bigcup S$ such that
 $|S| < \kappa$ has size κ . Then there exists $\alpha < \kappa$ such that $\text{cf}(\kappa) = \alpha$. Therefore,
there exists a cofinal function $f: \alpha \rightarrow \kappa$. Therefore, κ is singular. $\square$

EXAMPLE: $\aleph_\omega$ is singular because $\text{cf}(\aleph_\omega) = \omega$.

LEMMA B.10: If α is a limit ordinal, then $\text{cf}(\omega_\alpha) = \text{cf}(\alpha)$.

PROOF: If α is a limit ordinal, there exists a strictly increasing cofinal function
 $f: \alpha \rightarrow \omega_\alpha$ defined by $f(\beta) = \omega_\beta \forall \beta < \alpha$. Then by Lemma 5, $\text{cf}(\alpha) = \text{cf}(\omega_\alpha)$. $\square$

EXAMPLES:

1. $\text{cf}(\omega_\omega) = \text{cf}(\omega) = \omega$ since ω is a limit ordinal.
2. $\text{cf}(\omega_{\omega_n}) = \text{cf}(\omega_n) = \omega_n$ for $n \in \omega$.
3. $\text{cf}(\omega_1 + \omega) = \omega$ where $f(n) = \omega_1 + n$. (This example uses ordinal (not) cardinal arithmetic.)

DEFINITION B.11: A cardinal κ is **weakly inaccessible** iff κ is an uncountable regular limit cardinal.

DEFINITION B.12: A cardinal κ is **strongly inaccessible** iff κ is weakly inaccessible with the additional property that $\forall \lambda < \kappa (2^\lambda < \kappa)$.

We will use **inaccessible** to abbreviate strongly inaccessible.

NOTES:

1. Under GCH, strongly inaccessible and weakly inaccessible are synonymous.
2. "If κ is inaccessible, then V_κ (and hence L_κ), is a model of ZFC. and hence, by Gödel's Second Incompleteness Theorem, the existence of inaccessible cardinals is not provable in ZFC." (Devlin, Constructibility, p. 170)

DEFINITION B.13: A cardinal κ is **strong limit cardinal** if κ is limit cardinal and $\forall \lambda < \kappa, 2^\lambda < \kappa$.

THEOREM B.14: There exist a proper class of strong limit cardinals.

PROOF: Let κ be any cardinal. To find a strong limit cardinal greater than κ , we'll define κ_n recursively by $\kappa_0 = \kappa$ and $\kappa_{n+1} = 2^{\kappa_n}$. Then let $\lambda = \sup_{n \in \omega} (\kappa_n)$. Since the sequence $\langle \kappa_n \rangle$ is increasing, λ is a limit cardinal. Furthermore, λ is a strong limit cardinal because if $\mu < \lambda$, then $\mu < \kappa_n$ for some n . Therefore, $2^\mu \leq 2^{\kappa_n} = \kappa_{n+1} < \lambda$. $\square$

KÖNIG'S LEMMA (AC) B.15: If κ is an infinite cardinal such that $\text{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda > \kappa$.

PROOF: Let $f: \lambda \rightarrow \kappa$ cofinally. To show that $\kappa^\lambda > \kappa$, we fix an arbitrary function $G: \kappa \rightarrow {}^\lambda \kappa$. We will show that G is not onto ${}^\lambda \kappa$. Now $f(\alpha) \in \kappa$ by definition of f , and $G(f(\mu))(\alpha) \in \kappa$ since $G(f(\mu)) \in {}^\lambda \kappa$. $\forall \mu \in \kappa \forall \alpha \in \lambda$. Define $h \in {}^\lambda \kappa$ as follows: for each $\alpha \in \lambda$ set $h(\alpha)$ to be the

least element of $\kappa \setminus \{G(\mu)(\alpha) : \mu < f(\alpha)\}$. We know that since this set has a least element because $f(\alpha) < \kappa$, there are fewer than κ many μ 's; therefore, there are fewer than κ many $G(\mu)$'s, each being evaluated at α . So there are fewer than κ many ordinals in $\{G(\mu)(\alpha) : \mu < f(\alpha)\}$ and ordinals are well ordered.

Claim: $h \notin \text{rng}(G)$.

Proof: Suppose $h \in \text{rng}(G)$. Choose $\alpha \in \lambda$ such that $f(\alpha) > \mu$ (this is possible since f is cofinal in κ). But $h(\alpha) =$ the least element in $\kappa \setminus \{G(\mu)(\alpha) : \mu < f(\alpha)\}$.

Hence, $h(\alpha) \neq G(\mu)(\alpha)$, which contradicts the hypothesis that $h = G(\mu)$.

So we have found a function $h \in {}^\lambda \kappa$ which has no pre-image in $\text{dom}(G)$.

Therefore, G is not onto ${}^\lambda \kappa$. Hence, $\kappa^\lambda > \kappa$. $\square$

COROLLARY B.16: If $\lambda \geq \omega$, then $\text{cf}(2^\lambda) > \lambda$.

PROOF: Follows directly from König's Lemma. $\square$

REMARK B.17: All infinite successor cardinals are regular but limit cardinals are not always regular.

THEOREM B.18: Successor cardinals are regular.

PROOF: Suppose not.

Claim : There exists a cofinal function $f: \kappa \rightarrow \kappa^+$.

Proof: $\text{cf}(\kappa^+) \leq \kappa$ so define $f(\alpha) = \begin{cases} \kappa^+ & \forall \alpha \in \text{cf}(\kappa^+) \\ 0 & \forall \alpha \in \kappa \setminus \text{cf}(\kappa^+) \end{cases}$

Now since κ is a limit cardinal, $\kappa^+ = \bigcup_{\beta < \kappa^+} \beta = \bigcup_{\alpha < \kappa} \text{rng}(f) = \bigcup_{\alpha < \kappa} f(\alpha)$. But $|f(\alpha)| < \kappa^+ \forall \alpha < \kappa$ which implies $\text{cf}(\kappa^+) \leq \kappa$. So we have a union of κ many

sets of size $\leq \kappa$ which has cardinality κ . But this implies that $\kappa^+ = \kappa$ which is impossible. Therefore, κ^+ must be regular. $\square$

LEMMA (AC + GCH) B.19: Assume $\kappa, \lambda \geq 2$ and at least κ is infinite.

Then the following hold:

- a. $\lambda \leq \kappa \Rightarrow \lambda^\kappa = \kappa^+$
- b. $\kappa > \lambda \geq \text{cf}(\kappa) \Rightarrow \kappa^\lambda = \kappa^+$
- c. $\lambda < \text{cf}(\kappa) \Rightarrow \kappa^\lambda = \kappa$

PROOF:

a. If $\kappa \geq \omega$ and $2 \leq \lambda \leq \kappa$, then

$$\kappa^+ = 2^\kappa = |\kappa^2| \leq |\kappa^\lambda| \leq |\kappa^\kappa| \text{ and } |\kappa^\kappa| \leq |\mathcal{P}(\kappa \times \kappa)| = |\mathcal{P}(\kappa)| = 2^\kappa = \kappa^+.$$

So by the Squeezing Lemma, $\lambda^\kappa = \kappa^+$.

b. κ is infinite ($\kappa > \lambda$ and at least one of κ and λ is infinite by hypothesis) together with $\text{cf}(\kappa) \leq \lambda$ implies that $\kappa^\lambda > \kappa$. Therefore, $\kappa^\lambda \geq \kappa^+$.

Claim: $\kappa^\lambda = |\lambda^\kappa| \leq |\kappa^\kappa|$.

Proof: Define $F: \lambda^\kappa \rightarrow \kappa^\kappa$ by $F(f)(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha < \lambda \\ 0 & \text{if } \alpha \geq \lambda \end{cases}$

Furthermore, $|\kappa^\kappa| \leq |\mathcal{P}(\kappa \times \kappa)| = 2^{\kappa \times \kappa} = 2^\kappa = \kappa^+$. So $\kappa^\lambda \leq \kappa^+$. Hence, $\kappa^\lambda = \kappa^+$.

c. Claim i: $\lambda < \text{cf}(\kappa) \Rightarrow \kappa^\lambda = \bigcup_{\mu < \kappa} \lambda^\mu$.

Proof: $\bigcup_{\mu < \kappa} \lambda^\mu \leq \kappa^\lambda$ by containment. Now $\lambda < \text{cf}(\kappa)$ implies that λ^κ is bounded.

Therefore, no function from λ into κ is cofinal in κ . Hence, each function

$f: \lambda \rightarrow \kappa$ will be in λ^μ for some μ .

Claim ii: For $\mu < \kappa$, $|\lambda^\mu| \leq (\max\{\mu, \kappa\})^+$.

Proof: $|\lambda^\mu| \leq |\mu|^+$ by induction hypothesis on κ . Then by part (a), $|\lambda^\mu| = \lambda^+$.

Therefore, $(\max(\mu, \kappa))^+$ is an upper bound for λ^μ . So $\lambda < \text{cf}(\kappa)$ implies that

$$\kappa^\lambda = \sup_{\mu < \kappa} \mu^\kappa < \sup_{\mu < \kappa} (\max\{\mu, \lambda\})^+ \leq \kappa.$$

Now let $F: \alpha \rightarrow f_\alpha$ where $\alpha \in \kappa$, $f_\alpha \in {}^\lambda \kappa$, and $\forall \gamma \in \lambda (f_\alpha(\gamma) = \gamma)$. Since F is one-to-one, $|\lambda^\kappa| = \kappa$. $\square$

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Appendix C

THE C.U.B. FILTER

DEFINITION C.1: A filter $\mathcal{F}$ on a set S is a collection of subsets of S such that:

- i. $S \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
- ii. If $X, Y \subseteq S$ such that $X \in \mathcal{F}$ and $X \subseteq Y$, then $Y \in \mathcal{F}$.
- iii. If $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$.

DEFINITION C.2: A ideal $\mathcal{I}$ on set S is a collection of subsets of S such that:

- i. $S \notin \mathcal{I}$ and $\emptyset \in \mathcal{I}$
- ii. If $X, Y \subseteq S$ such that $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.
- iii. If $X, Y \in \mathcal{I}$ then $X \cup Y \in \mathcal{I}$.

EXAMPLE: If S is any infinite set, then $\{X \subset S : |X| < \omega\}$ is an ideal of S .

DEFINITION C.3: If $\mathcal{F}$ is a filter on set S , then the **dual ideal** is:

$$\mathcal{F}^* = \{X \subseteq S : X^c \in \mathcal{F}\}.$$

NOTE: For $X \subseteq S$, we will use the notation X^c to mean $S \setminus X$.

DEFINITION C.4: If $\mathcal{I}$ is an ideal on set S , then the **dual filter** is:

$$\mathcal{I}^* = \{X \subseteq S : X^c \in \mathcal{I}\}$$

LEMMA C.5: Let $\mathcal{F}$ be a filter and $\mathcal{I}$ be an ideal on set S . Then

- a. $\mathcal{F}^*$ is an ideal.
- b. $\mathcal{I}^*$ is a filter.
- c. $\mathcal{F}^{**} = \mathcal{F}$.
- d. $\mathcal{I}^* = \mathcal{I}$.

PROOF:

a. We must show that $\mathcal{F}^*$ has the three properties of ideals:

- i. $S \subseteq S$ and $S^c = \emptyset \notin \mathcal{F}$, so $S \notin \mathcal{F}^*$; $\emptyset \subseteq S$ and $\emptyset^c = S \in \mathcal{F}$, so $\emptyset \in \mathcal{F}^*$
- ii. If $X \in \mathcal{F}^*$, then $X^c \in \mathcal{F}$. If $Y \subseteq X$, then $X^c \subseteq Y^c$ so $Y^c \in \mathcal{F}$. Therefore, $Y \in \mathcal{F}^*$.
- iii. If $X, Y \in \mathcal{F}^*$, then $X^c, Y^c \in \mathcal{F}$. Then $X^c \cap Y^c \in \mathcal{F}$ and $X^c \cap Y^c = (X \cup Y)^c$; so $(X \cup Y)^c \in \mathcal{F}$. Therefore, $X \cup Y \in \mathcal{F}^*$.

- b. The proof that $\mathcal{I}^*$ is a filter is similar.
- c. $X \in \mathcal{F} \Leftrightarrow X^c \in \mathcal{F}^* \Leftrightarrow S \setminus X^c = X \in \mathcal{F}^{**}$
- d. The proof that $\mathcal{I}^* = \mathcal{I}$ is similar to part (c). $\square$

DEFINITION C.6: Let $\mathcal{F}$ be a filter of S . Then $\mathcal{F}$ is κ -complete iff for each

family $\mathcal{A} \subset \mathcal{F}$ such that $|\mathcal{A}| < \kappa$, $\bigcap_{X_i \in \mathcal{A}} X_i \in \mathcal{F}$.

NOTE: As κ increases, it becomes more difficult for a filter to be κ -complete because the intersection decreases as more sets are included.

DEFINITION C.7: Let $\mathcal{I}$ be an ideal of S . Then $\mathcal{I}$ is κ -complete if for each family $\mathcal{A} \subset \mathcal{I}$ such that $|\mathcal{A}| < \kappa$, $\bigcup_{X_i \in \mathcal{A}} X_i \in \mathcal{I}$.

NOTE: As κ increases, it becomes more difficult for an ideal to be κ -complete because the union increases as more sets are included.

LEMMA C.8: An ideal $\mathcal{I}$ is κ -complete iff its dual filter is κ -complete.

PROOF:

($\Rightarrow$) Let $\mathcal{I}$ be an ideal which is κ -complete with dual filter $\mathcal{F}$. Let $\mathcal{A} \subseteq \mathcal{I}^*$ such that $|\mathcal{A}| < \kappa$. Then $\mathcal{A}^* = \{X : X^c \in \mathcal{A}\} \subseteq \mathcal{I}^{**} = \mathcal{I}$. Then $\bigcup \mathcal{A}^* \in \mathcal{I}^{**} = \mathcal{I}$ by definition $\mathcal{I}$. Therefore, $(\bigcup \mathcal{A}^*)^c \in \mathcal{F}$ by definition of $\mathcal{F}$. But

$$(\bigcup \mathcal{A}^*)^c = \{X^c : X \in \mathcal{A}\} = \bigcap \{X : X \in \mathcal{A}\} = \bigcap \mathcal{A}.$$

Hence, $\bigcap \mathcal{A} \in \mathcal{F}$ which implies that $\mathcal{F}$ is κ -complete.

($\Leftarrow$) Next let $\mathcal{I}$ be an ideal with its dual filter $\mathcal{F}$ κ -complete. Let $\mathcal{A} \subset \mathcal{I}$ such that $|\mathcal{A}| < \kappa$. Then $\mathcal{A}^* = \{X : X^c \in \mathcal{A}\} \subset \mathcal{F}$. Therefore, $\bigcap \mathcal{A}^* \in \mathcal{F}$ by definition of κ -completeness. Therefore, $(\bigcap \mathcal{A}^*)^c \in \mathcal{I}$. But

$$\bigcup \{X : X \in \mathcal{A}\} = \bigcup \{X^c : X \in \mathcal{A}^*\} = (\bigcap \mathcal{A}^*)^c \in \mathcal{I}.$$

So $\mathcal{I}$ is κ -complete. $\square$

LEMMA C.9: Every filter and every ideal is ω -complete.

PROOF: Let $\mathcal{F}$ be a filter on S . Let $\mathcal{A} \subset \mathcal{F}$ such that $|\mathcal{A}| = n < \omega$.

Claim: $\bigcap \mathcal{A} \in \mathcal{F}$.

Proof by induction on n :

i. Let $n = 1$. Then $\mathcal{A} = \{X\} \subseteq \mathcal{F}$. So $X \in \mathcal{F}$ and $X = \bigcap \mathcal{A}$. Therefore,

$$\bigcap \mathcal{A} \in \mathcal{F}.$$

ii. Assume claim holds for m : $|\mathcal{A}| = m$ and $\bigcap_{i=1}^m X_i \in \mathcal{F}$ where $X_i \in \mathcal{A}$.

iii. Let $\mathcal{A} = \{X_1, \dots, X_{m+1}\}$. Then $\bigcap_{i=1}^m X_i \in \mathcal{F}$ (by ii). Now $X_{m+1} \in \mathcal{F}$. Then

$$\bigcap \mathcal{A} = \bigcap_{i=1}^{m+1} X_i = \left(\bigcap_{i=1}^m X_i \right) \cap X_{m+1} \in \mathcal{F}$$

by definition of filter. Therefore, $\mathcal{F}$ is ω -complete.

The proof that every ideal is ω -complete is similar. $\square$

REMARK C.10: If $\mathcal{I}$ is an ideal on S and $\{s\} \in \mathcal{I}$ for every $s \in S$, then $\mathcal{I}$ is not $|S|^+$ -complete because $S = \bigcup \{\{s\} : s \in S\} \notin \mathcal{I}$.

REMARK C.11: $\mathcal{I}$ could be $|S|$ -complete if $|S|$ is regular.

DEFINITION C.12: $\mathcal{U}$ is an **ultrafilter** iff $\mathcal{U}$ is a filter and $\forall X \subset S$ either $X \in \mathcal{U}$ or $X^c \in \mathcal{U}$.

DEFINITION C.13: A filter is **nonprincipal** iff $\forall s \in S (\{s\}^c \in \mathcal{F})$.

NOTE: If $\mathcal{F}$ is nonprincipal, then $\mathcal{F}$ is not $|S|^+$ -complete because $\bigcap_{s \in S} \{s\}^c = \emptyset$.

DEFINITION C.14: If $\text{cf}(\mu) > \omega$, then the **c.u.b. filter** on μ is defined to be:

$$\text{Cub}(\mu) = \{X \subseteq \mu : \exists C \subset X \text{ such that } C \text{ is c.u.b. in } \mu\}.$$

THEOREM C.15: The collection of c.u.b. sets does not form a filter.

PROOF: Let $C = \{\gamma : \gamma \text{ is a limit ordinal } \wedge \omega < \gamma < \omega_1\}$. Let $X = C \cup \omega$.

Claim i: $X \in \text{Cub}(\omega_1)$.

Proof: $C \subseteq X$ by definition. Now C is unbounded in ω_1 since the set of limit ordinals is unbounded in ω_1 . To show C is closed, we must show that $\alpha \in C \Rightarrow \forall \alpha \in \mu$ such that $C \cap \alpha$ is cofinal in α . But the limit of a set of limit ordinals is a limit ordinal, so $\alpha \in C$. Therefore, C is closed.

Claim ii: X is not c.u.b. in ω_1 .

Proof: $X = C \cup \omega \subset \omega_1$. X is unbounded in ω_1 but not closed in ω_1 because ω_1 is a limit point of X but $\omega_1 \notin X$.

Now C is c.u.b. but has a non-c.u.b. superset, X . Therefore, C is not a filter. (However, X is in $\text{Cub}(\omega_1)$, which is a filter.) $\square$

LEMMA C.16: If $\text{cf}(\mu) > \omega$ and $C, D \subseteq \mu$ are c.u.b., then $C \cap D$ is c.u.b.

PROOF: We must show that $C \cap D$ is closed and unbounded. First let p be a limit point of $C \cap D$. Then p is a limit point of both C and D . Since both C and D are c.u.b., $p \in C$ and $p \in D$. Therefore, $p \in C \cap D$. So $C \cap D$ is closed.

Next let $\delta < \mu$ be an arbitrary ordinal. We'll recursively define the following intertwining sequences :

$$\text{a. } \langle \alpha_i : i \in \omega \rangle \text{ by } \begin{cases} \alpha_0 = \min(C \setminus \delta) \\ \alpha_{i+1} = \min(C \setminus (\beta_i + 1)) \end{cases}$$

$$b. \langle \beta_i : i \in \omega \rangle \text{ by } \begin{cases} \beta_0 = \min(D \setminus \alpha_0) \\ \beta_{i+1} = \min(D \setminus (\alpha_{i+1} + 1)) \end{cases}$$

Then set $\gamma = \sup(\{\alpha_i\}) = \sup(\{\beta_i\})$. (Note that $\gamma < \mu$ because $\text{cf}(\mu) > \omega$ makes it impossible to get all the way to μ in countably many steps, and both $\langle \alpha_i \rangle$ and $\langle \beta_i \rangle$ are countable.) Since γ is a limit point of both c.u.b. sets C and D , $\gamma \in C \cap D$. Now by construction, $\gamma > \delta$. Therefore, $C \cap D$ is cofinal in μ (i.e. *unbounded*). $\square$

LEMMA C.17: $\text{Cub}(\mu)$ is a filter.

PROOF:

- i. μ is c.u.b. so $\mu \in \text{Cub}(\mu)$, and $\emptyset \notin \text{Cub}(\mu)$ because $\emptyset$ does not contain a c.u.b. subset of μ .
- ii. $\forall X \in \text{Cub}(\mu) \exists C \subseteq X$ such that C is c.u.b. in μ . Choose $X, Y \in \text{Cub}(\mu)$ such that $X \subseteq Y$. Then $\exists C \subseteq X$, which implies that $C \subseteq Y$. Therefore, $Y \in \text{Cub}(\mu)$.
- iii. Let $X, Y \in \text{Cub}(\mu)$. Then $\exists C_1 \subseteq X$ and $C_2 \subseteq Y$ with both C_1 and C_2 c.u.b. Then $C_1 \cap C_2 \subseteq X \cap Y$. But $C_1 \cap C_2$ is c.u.b. Therefore, $X \cap Y$ contains a c.u.b. subset. Hence, $X \cap Y \in \text{Cub}(\mu)$.

Therefore, $\text{Cub}(\mu)$ is a filter. $\square$

LEMMA C.18: If $\text{cf}(\mu) > \omega$ then

- a. The intersection of any family of $< \text{cf}(\mu)$ many c.u.b. subsets of μ is c.u.b.
- b. $\text{Cub}(\mu)$ is a $\text{cf}(\mu)$ -complete filter.

PROOF:

a. Let $\langle C_\alpha : \alpha < \lambda \rangle$ be a sequence of c.u.b. subsets of μ where $\lambda < \text{cf}(\mu)$. We must show that $\bigcap_{\alpha < \lambda} C_\alpha$ is closed and unbounded. First let ρ be a limit point of $\bigcap_{\alpha < \lambda} C_\alpha$. Then ρ is a limit point of each C_α . Since each C_α is c.u.b., $\rho \in C_\alpha$ for all α ; therefore, $\rho \in \bigcap_{\alpha < \lambda} C_\alpha$. Hence, $\bigcap_{\alpha < \lambda} C_\alpha$ is closed.

We shall prove that $\bigcap_{\alpha < \lambda} C_\alpha$ is unbounded in μ by induction on λ :

Let λ be a *successor ordinal*. Then $\bigcap_{\alpha < \lambda-1} C_\alpha$ is unbounded by the induction hypothesis. So $\bigcap_{\alpha < \lambda-1} C_\alpha \cap C_{\lambda-1}$ is c.u.b. by Lemma C.16. Therefore,

$\bigcap_{\alpha < \lambda} C_\alpha$ is c.u.b.

Let λ be a *limit ordinal*. To prove $\bigcap_{\alpha < \lambda} C_\alpha$ is c.u.b., let $\langle C_\alpha : \alpha < \lambda \rangle$ be a sequence of λ many c.u.b. subsets of μ and $\langle \gamma_i : i < \text{cf}(\lambda) \rangle$ an increasing sequence of ordinals defined by $\gamma_i \in \bigcap_{\alpha < \beta_i} C_\alpha$ where $\langle \beta_i : i < \text{cf}(\lambda) \rangle$ is cofinal in λ . Note that $\gamma_i \in C_\alpha$ for only those $\alpha < \beta_i$ so that, for each C_α we add to the intersection, we take out a γ_i from the beginning of the sequence $\langle \gamma_i \rangle$.

Since the tail of $\langle \gamma_i \rangle$ is contained in each C_α , it will always be in the intersection. Also note that sequence $\langle \beta_i \rangle$ tells us how many C_α 's to intersect each time. Now let $\gamma = \sup\{\gamma_i\} \in C_\alpha$. Since the tail of the sequence $\langle \gamma_i \rangle$ always gets into $\bigcap_{\alpha < \beta_i} C_\alpha$, and each C_α is c.u.b., $\gamma \in \bigcap_{\alpha < \lambda} C_\alpha \neq \emptyset$.

b. To show $\text{Cub}(\mu)$ is a $\text{cf}(\mu)$ -complete filter we must show that $\bigcap_{C \in \mathcal{A}} C$ contains a c.u.b. subset $\forall \mathcal{A} \subset \text{Cub}(\mu)$ such that $|\mathcal{A}| < \text{cf}(\mu)$. So let

$\mathcal{A} = \langle A_\alpha : \alpha < \lambda \rangle$. Now $\mathcal{A} \subset \text{Cub}(\mu)$ and $|\mathcal{A}| = \lambda < \text{cf}(\mu)$. Then $\forall A_\alpha \in \mathcal{A}$

$\exists$ c.u.b. $C_\alpha \subset A_\alpha$. Since $\lambda < \text{cf}(\mu)$, $\bigcap_{\alpha < \lambda} C_\alpha$ is still c.u.b. by part (a). But

$\bigcap_{\alpha < \lambda} C_\alpha \subset \bigcap_{\alpha < \lambda} A_\alpha = \bigcap \mathcal{A}$. Therefore, $\bigcap \mathcal{A} \in \text{Cub}(\mu)$, and hence, $\text{Cub}(\mu)$ is a $\text{cf}(\mu)$ -complete. $\square$

REMARKS C.19:

1. $\text{Cub}(\mu)$ is most frequently used for μ regular. By Lemma C.17, $\text{Cub}(\mu)$ is $\text{cf}(\mu)$ -complete. Therefore, for μ regular, $\text{Cub}(\mu)$ is μ -complete. (Note that $\text{Cub}(\mu) \subset \mathcal{P}(\mu)$ and $|\mathcal{P}(\mu)| > \mu$ so some families of subsets of μ may have more than μ many members.)
2. If μ is regular, then $\{X \subset \mu : |X| < \mu\} \subset \text{Cub}^*(\mu)$.

DEFINITION C.20: Suppose $\text{cf}(\mu) > \omega$ and $S \subseteq \mu$. Then S is **stationary** iff $S \cap C \neq \emptyset \ \forall \text{c.u.b. } C \subseteq \mu$.

THEOREM C.21: S is **stationary** iff $S \notin \text{Cub}^*(\mu)$.

PROOF:

($\Rightarrow$) (Contrapositive) Assume $S \in \text{Cub}^*(\mu)$. Then $S^c \in \text{Cub}(\mu)$. Therefore, $\exists \text{c.u.b. } C \subset S^c$. Hence, $S \cap C = \emptyset$. Therefore, S is not stationary.

($\Leftarrow$) Fix an arbitrary c.u.b. $C \subset \mu$. Assume $S \notin \text{Cub}^*(\mu)$. Then C is not contained in S^c . Therefore, $S^c \cap C = \emptyset$ which implies that $S \cap C \neq \emptyset$. Hence, S is stationary. $\square$

REMARKS C.22:

1. $S \text{ c.u.b.} \Rightarrow S \text{ stationary}$, but not conversely. (See Lemma C.26 below.)
2. Whereas filters consist of very large sets and ideals consist of very small sets, stationary sets are medium in size.

DEFINITION C.23: $S \subset \mu$ is **non-stationary** if $S \in \text{Cub}^*(\mu)$.

COROLLARY C.24: The union of any family of $< \text{cf}(\mu)$ many non-stationary subsets of μ is non-stationary.

PROOF: Let $\mathcal{A}$ a family of $< \text{cf}(\mu)$ many non-stationary subsets of μ . Then $\mathcal{A} \subset \text{Cub}^*(\mu)$. Therefore, A^c contains a c.u.b. subset of μ for every $A \in \mathcal{A}$. Hence, $\bigcap_{A \in \mathcal{A}} A^c$ contains a c.u.b. set, which implies that $(\bigcap_{A \in \mathcal{A}} A^c)^c \in \text{Cub}^*(\mu)$. But

$$\bigcup_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} A^c = (\bigcap_{A \in \mathcal{A}} A^c)^c \in \text{Cub}^*(\mu).$$

Therefore, $\bigcup \mathcal{A}$ is non-stationary. $\square$

COROLLARY C.25: If $\mu = \bigcup_{\alpha < \lambda} X_\alpha$ where $\lambda < \text{cf}(\mu)$, then X_α is stationary for some α .

PROOF: Assume that for every α , X_α is non-stationary. Then $\bigcup_{\alpha < \lambda} X_\alpha$ is non-stationary (by Corollary C.24) which, implies that $\bigcup_{\alpha < \lambda} X_\alpha \in \text{Cub}^*(\mu)$. But $\text{Cub}^*(\mu)$ is an ideal which implies $\mu \notin \text{Cub}^*(\mu)$. Therefore, there must be an α such that X_α is stationary. $\square$

LEMMA C.26: If $\lambda < \text{cf}(\mu)$ for λ regular, then $S = \{\alpha < \mu : \text{cf}(\alpha) = \lambda\}$ is stationary in μ .

PROOF: Let $C \subseteq \mu$ be c.u.b., which implies it is unbounded in μ . To obtain a sequence in C , define sequence $\langle \alpha_i : i \leq \lambda \rangle$ by $\alpha_i = \min(C \setminus (\alpha_j + 1)_{j < i})$ where if κ is a limit ordinal then $\alpha_\kappa = \sup_{\beta < \kappa} \alpha_\beta$. Then $\alpha_\lambda = \sup_{i < \lambda} \alpha_i \in C$ (since $\alpha_\lambda < \mu$), and it is a limit point of C . So we have an ordinal which contains a

cofinal λ -sequence. Therefore, α_λ has cofinality λ . Hence, $\alpha_\lambda \in S$. Thus, S is stationary. $\square$

NOTE: Though S is stationary, in general, it is not c.u.b. To show this, let

$\mu = \omega_2$, $\lambda = \omega_1$, and $S = \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega_1\}$. Let

$C = \{\gamma : \gamma \text{ is a limit ordinal and } \omega < \gamma < \omega_1\}$. C is unbounded in ω_1

since the set of limit ordinals is unbounded in ω_1 . To show C is closed, we must show that $\alpha \in C \forall \alpha \in \mu$ such that $C \cap \alpha$ is cofinal in α . But the limit of a set of limit ordinals is a limit ordinal, so $\alpha \in C$. Therefore, C is closed.

However, $S \cap C = \emptyset$. Hence, S is not c.u.b.

From this point on till the end of this section, we will only be concerned with regular cardinals.

THEOREM C.27 (Ulam): Let κ be a successor cardinal with κ -complete ideal $\mathcal{I}$ on κ such that each singleton is in $\mathcal{I}$. Then there exists a pairwise disjoint collection, $\{X_\alpha : \alpha < \kappa\}$, such that $X_\alpha \notin \mathcal{I}$ for all α .

PROOF: Let $\kappa = \lambda^+$.

Claim i: Every subset of κ of size $\leq \lambda$ is in $\mathcal{I}$.

Proof: $\emptyset \in \mathcal{I}$

Now for each ordinal $p < \kappa$, let $f_p : p \rightarrow \lambda$ be a 1-1 function; and $\forall \alpha < \kappa$ and each $\xi < \lambda$, let $X_\alpha^\xi = \{p > \alpha : f_p(\alpha) = \xi\}$. Then $\alpha \neq \beta$ implies $X_\alpha^\xi \cap X_\beta^\xi = \emptyset$ since f_p is 1-1. Furthermore, $\forall \alpha < \kappa \left(\bigcup_{\xi < \lambda} X_\alpha^\xi = \{p : \alpha < p < \kappa\} \in \mathcal{I} \right)$. Therefore, by κ -completeness of $\mathcal{I}$ $\exists \xi < \lambda (X_\alpha^\xi \notin \mathcal{I})$. Now let $h : \kappa \rightarrow \lambda$ such that $h(\alpha) < \lambda$ and

$X_\alpha^{h(\alpha)} \notin \mathcal{I}$. Then there exists a fixed $\xi < \lambda$ such that $|h^{-1}(\xi)| = \kappa$.

Therefore, $\{X_\alpha^\xi : h(\alpha) = \xi\}$ satisfies the theorem. $\square$

COROLLARY C.28 For regular $\kappa > \omega$, there exists a family of κ many disjoint stationary subsets of κ .

PROOF: If κ is a successor cardinal, we will apply Lemma C.26 to $\text{Cub}^*(\kappa) = \{X \subset \kappa : X^c \in \text{Cub}(\kappa)\}$. To show that $\text{Cub}^*(\kappa)$ is κ -complete, it suffices to show that $\text{Cub}(\kappa)$ is κ -complete. Since $\text{cf}(\kappa) = \kappa$, by Lemma C.17, $\text{Cub}(\kappa)$ is κ -complete. Then by Lemma C.9, $\text{Cub}^*(\kappa)$ is κ -complete. Now since each singleton of κ is in $\text{Cub}^*(\kappa)$, by Lemma C.26, there exists a disjoint $X_\alpha \subset \kappa$, $\forall \alpha < \kappa$ such that $X_\alpha \notin \text{Cub}^*(\kappa)$. Therefore, $\{X_\alpha\}_{\alpha < \kappa}$ is a family of stationary subsets of κ .

If κ is a limit cardinal, then there are κ many regular cardinals $\lambda < \kappa$, and for each such λ , the set $S_\lambda = \{\gamma < \kappa : \text{cf}(\gamma) = \lambda\}$ is stationary and the family of all the S_λ 's is disjoint. $\square$

THEOREM C.29: If κ is a regular cardinal, then any stationary $S \subset \kappa$ may be partitioned into κ many disjoint stationary subsets.

PROOF: Apply Theorem C.26 to the ideal

$$\mathcal{I} = \{X \subset \kappa : X \cap S \text{ is non-stationary}\}. \quad \square$$

NOTE: In general, stationary sets are "skinnier" than c.u.b. sets: a c.u.b. set cannot be split into two disjoint c.u.b. subsets.

LEMMA C.30: Suppose $\kappa > \omega$ is regular and $\mathcal{F}$ a set of finitary functions such that $|\mathcal{F}| < \kappa$. Then the set $C = \{\alpha < \kappa : \alpha \text{ is closed under each } f \in \mathcal{F}\}$ is c.u.b. in κ .

PROOF: Claim i: C is closed.

Proof: Suppose $\langle \alpha_i \rangle$ is an increasing sequence of ordinals in C , each of which is closed under all functions in $\mathcal{F}$. Let $\gamma = \sup(\{\alpha_i\})$. Now in C , pick any $\alpha < \gamma$ and any $\lambda > \alpha$ and apply any arbitrary $f \in \mathcal{F}$ to λ . Since $\lambda \in C$, λ is closed under f . So $f(\alpha) < \lambda$, and hence, γ is closed under f . Since f was arbitrary, γ is closed in $\mathcal{F}$.

Claim ii: C is unbounded.

Proof: Pick $\beta < \kappa$. We want to find a $\gamma \in C$ such that $\gamma > \beta$. Define sequence

$$\langle \alpha_i < \kappa : i < \omega \rangle \text{ by } \begin{cases} \alpha_0 = \sup(f(\delta)) \text{ for } f \in \mathcal{F}, \delta < \beta \\ \alpha_{i+1} = \sup(f(\delta)) \text{ for } f \in \mathcal{F}, \delta < \alpha_i. \end{cases}$$

So $\forall f \in \mathcal{F}$ and $\forall \delta < \beta$, $f(\delta) < \beta < \kappa$ since κ is regular and $\beta < \kappa$. Furthermore, by hypothesis, there are $< \kappa$ many f 's, so there are $< \kappa$ many such $f(\delta)$'s. At each step we repeat this process substituting α_i for β the first time and α_{i+1} for each successor. Set $\gamma = \sup(\{\alpha_i\}) < \kappa$. Since $\kappa > \omega$ and is regular, $\text{cf}(\kappa) > \omega$. Therefore, it takes $> \omega$ many steps to get to κ . But $\langle \alpha_i \rangle$ is an ω -sequence. Hence, $\gamma < \kappa$.

To show that $\gamma \in C$, we must show that γ is closed under $\mathcal{F}$. If $\delta < \alpha_i$ for some i , then $f(\delta) \leq \alpha_{i+1} \leq \gamma$. So $f(\gamma) \subseteq \gamma \forall f \in \mathcal{F}$. Hence, $\gamma \in C$. $\square$

DEFINITION C.31: Suppose $\{A_\alpha : \alpha < \kappa\}$ is a collection of subsets of κ . Then the **diagonal intersection** of $\{A_\alpha : \alpha < \kappa\}$ is

$$\Delta_{\alpha < \kappa} A_\alpha = \{\gamma < \kappa : \forall \alpha < \gamma (\gamma \in A_\alpha)\}.$$

(So $\gamma \in \Delta_{\alpha < \kappa} A_\alpha$ iff $\gamma \in \bigcap_{\alpha < \gamma} A_\alpha$).

NOTE: If $\kappa^{>\omega}$ is regular, then the intersection of $< \kappa$ many subsets of κ is c.u.b. However, in general, the intersection of κ many c.u.b. subsets of κ is empty.

PROPOSITION C.33: Let $\kappa > \omega$ be regular; and $\forall \alpha < \kappa$, let $C_\alpha \subseteq \kappa$ be c.u.b. Then $\Delta_{\alpha < \kappa} C_\alpha$ is c.u.b.

PROOF: Claim i: C_α is closed.

Proof: Suppose $\langle \alpha_i \rangle_{i < \lambda}$ is an increasing sequence of ordinals in $\Delta_{\alpha < \kappa} C_\alpha$. Let $\gamma = \sup_{i < \lambda} (\{\alpha_i\})$ which is less than κ for $\lambda < \kappa$. Fix $\delta < \gamma$. Then $\{\alpha_i > \delta : i < \lambda\} \subseteq C_\delta$ (this set exists since $\langle \alpha_i \rangle_{i < \lambda}$ is cofinal in γ ; it is contained in C_δ by definition of $\Delta_{\alpha < \kappa} C_\alpha$). But C_δ is c.u.b. so it contains all its limit points. Therefore, $\gamma \in C_\delta$, implying that $\gamma \in \Delta_{\alpha < \kappa} C_\alpha$. Hence, $\Delta_{\alpha < \kappa} C_\alpha$ is closed.

Claim ii: $\Delta_{\alpha < \kappa} C_\alpha$ is unbounded in κ .

Proof: Let $\beta < \kappa$. We must find an ordinal, say ξ , such that $\beta < \xi < \kappa$.

We define an ω -sequence $\langle \alpha_i \rangle_{i < \omega}$ as follows:

$$\alpha_0 = \min\left(\bigcap_{\delta < \beta} C_\delta\right) \setminus (\beta+1) \quad \text{(the least element in the intersection of the } C_\delta\text{'s above } \beta+1\text{).}$$

$$\alpha_{i+1} = \min\left(\bigcap_{\delta < \alpha_i} C_\delta\right) \setminus (\alpha_i+1) \quad \text{(the least element in the intersection of the } C_\delta\text{'s for } \delta < \alpha_i \text{ above } \alpha_i+1\text{).}$$

Let $\gamma = \sup_{i < \omega} (\{\alpha_i\})$. Then we have the following results:

- i. $\beta < \gamma < \kappa$ (since $\alpha_i > \beta+1 > \beta$),
- ii. $|\langle \alpha_i \rangle| = \omega$, and

iii. κ is regular.

Now choose any $\delta < \gamma$ and any i such that $\alpha_i > \delta$. Then $\forall j > i, \alpha_j \in \bigcap_{\eta < \alpha_{j+1}} C_\eta$ by definition of α_i . Hence, $\alpha_j \in C_\delta$ since $\delta < \alpha_i$. Furthermore, C_δ is closed so $\gamma \in C_\delta$ for all δ since δ was arbitrary. Therefore, $\gamma \in \Delta_{\alpha < \kappa} C_\alpha$ and $\gamma > \beta$. So γ is the ξ we seek. Therefore, $\Delta_{\alpha < \kappa} C_\alpha$ is unbounded. $\square$

The following definitions use the concepts just discussed and are needed in Chapter 1.

DEFINITION C.33: A **Mahlo** cardinal κ is inaccessible cardinal with the property that the set $\{\lambda \in \kappa : \lambda \text{ is inaccessible}\}$ is stationary .

DEFINITION C.34: κ is a **weakly compact** cardinal iff it is inaccessible and for every κ -complete field F of cardinality κ of subsets of κ , there exists a κ -complete nonprincipal ultrafilter on F . (κ is weakly compact cardinal iff κ is inaccessible and there exists no κ -Aronszajn tree.)

DEFINITION C.35: An uncountable regular cardinal κ to be **ineffable** iff for $\alpha < \kappa$ and for $A_\alpha \subseteq \alpha$, there exists a stationary set $S \subseteq \kappa$ such that

$$\forall \alpha, \beta \in S (\alpha < \beta \rightarrow A_\alpha = A_\beta \cap \alpha).$$

NOTE: Ineffability strengthens the weakness of weak compactness.

Appendix D

TWO DEFINITIONS OF DENSE

DEFINITION D.1: A subset D of $\mathbb{P}$ is **dense** in $\mathbb{P}$ if $\forall p \in \mathbb{P} \exists q \in D$ such that $q \geq p$.

DEFINITION D.2: A subset D of a topological space $\mathcal{T}$ is **dense** in $\mathcal{T}$ if $\text{cl}(D) = \mathcal{T}$.

To show that these two definitions for dense are related, we will use the topological space called the stone space.

DEFINITION D.3: We define the **stone space**, denoted $\text{st}(\mathbb{P})$, to be the topological space whose points are the maximal filters in $\mathbb{P}$. Let $p \in \mathbb{P}$ and $\mathcal{F} \in \text{st}(\mathbb{P})$. Then the basic open sets of $\text{st}(\mathbb{P})$ are of the form

$$U_p = \{\mathcal{F} \in \text{st}(\mathbb{P}) : p \in \mathcal{F}\}.$$

THEOREM D.4: Let $X \subseteq$ partial order $\mathbb{P}$. Define $\hat{X} \subseteq \text{st}(\mathbb{P})$ by $\mathcal{F} \in \hat{X}$ iff $\exists p \in X$ such that $p \in \mathcal{F}$. Then X is dense in $\mathbb{P}$ iff $\hat{X}$ is dense in $\text{st}(\mathbb{P})$.

PROOF:

($\Rightarrow$) Pick any $\mathcal{F} \in \text{st}(\mathbb{P})$ and any p such that $\mathcal{F} \in U_p$ (we can do this because if $\mathcal{F} \in \text{st}(\mathbb{P})$, it must be in some basic open set). Thus, $p \in \mathcal{F}$ (by definition of U_p). Now $X \subseteq \mathbb{P}$ is dense (by hypothesis) implies that there exists a $q \geq p$ such that $q \in X$. Let $\mathcal{F}' \in \text{st}(\mathbb{P})$ such that $q \in \mathcal{F}'$ since every node is in some maximal filter. Then $\mathcal{F}' \in \hat{X}$ (by definition of $\hat{X}$ and by how

we chose $\mathfrak{F}$. Since $q \geq p$, $\mathfrak{F} \in U_p$ as well (by definition of filter). So we have found something in $\hat{X}$ which is also in U_p . Therefore, $\mathfrak{F} \in \text{cl}(\hat{X})$, which implies that $\text{st}(\mathbb{P}) \subseteq \text{cl}(\hat{X})$.

Trivially, $\text{cl}(\hat{X}) \subseteq \text{st}(\mathbb{P})$.

Therefore, $\text{st}(\mathbb{P}) = \text{cl}(\hat{X})$, so $\hat{X}$ is dense in $\text{st}(\mathbb{P})$.

($\Leftarrow$) Assume $\text{st}(\mathbb{P}) = \text{cl}(\hat{X})$. To show that X is dense in $\mathbb{P}$, it suffices to show that $\forall p \in \mathbb{P}, \exists x \in X$ such that $x \leq p$. So let $p \in \mathbb{P}$. Then $p \in \mathfrak{F}$ for some $\mathfrak{F} \in \text{st}(\mathbb{P})$. Then by the definition of $\text{st}(\mathbb{P})$, $\mathfrak{F}$ is a maximal filter which implies that $\exists q \leq p$ such that $q \in \mathfrak{F}$. Therefore, X is dense in $\mathbb{P}$. $\square$

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