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Cost-coloring : a generalization of vertex coloring in graph theory

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Morriss, Patrick, M.S.

San Jose State University, 1994

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COST-COLORING
A Generalization of Vertex Coloring
in Graph Theory

A Thesis
Presented to
The Faculty of the Department of Mathematics
and Computer Science
San Jose State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

by
Patrick Morriss
August, 1994

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ABSTRACT

COST-COLORING

by Patrick Morriss

In this thesis, we generalize the problem of vertex-coloring graphs by associating a cost with each color, and attempting to minimize the cost of vertex-coloring a graph. We specify families of equivalent cost sets that preserve any minimum-cost vertex-coloring.

For a given graph and a given palette with associated costs, we define the cost-chromatic number, establish upper and lower bounds on it, and compute it for some common graphs. We introduce the Maximum Independent Set method of cost-coloring, and explore its strengths and weaknesses.

We establish that there exist graphs with small chromatic number, in particular, trees, outerplanar blocks, planar blocks, and maximal planar blocks, that have arbitrarily large cost-chromatic number. We also establish that the cost-chromatic number is not monotonic with respect to subgraph inclusion, and that it is possible to reduce the cost-chromatic number solely by adding edges to a graph.

For Marisa

also for Nikki

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Chapter One

Graph Coloring

1.1 Introduction

Graph colorings are some of the most popular, and heavily investigated, problems in graph theory. The Four Color Map Problem, a famously open question until Appel and Haken [1] solved it in 1976, is arguably the best known graph theory problem. Its appeal is no doubt enhanced by its deceptive simplicity: a child with a box of crayons can investigate it, yet it held some of the keenest mathematical minds at bay for over a century. Graph theory is filled with problems that are simple to state yet very difficult to solve, almost the definition of a good puzzle. The graph coloring problems we investigate here are no different, good puzzles all, some with unexpected solutions.

Last summer, a group of Italian computer chip designers put a query to the Internet: what is known about weighted vertex colorings? A weighted coloring (or cost-coloring, as we call it) is a generalization of a vertex coloring problem, of which the Four Color Map Problem is one. The generalization is this: assign a cost to each color, and try not only to color the map or whatever, but to do so at lowest cost. Typical of a graph theory problem, it is simple to state, but on closer investigation, holds several sometimes surprising subtleties.

We contacted the chip designers [5], and to the best of our knowledge, little to nothing is known about cost coloring, though the topic appears to be first

mentioned by Supowit [8] in 1987. We take that as our cue to define concepts, invent notation, and solve problems.

1.2 Terms and Notation

We assume that the reader is familiar with the language of basic graph theory. In particular, we assume that the reader knows the meanings of the terms *graph*, *vertex*, *edge*, *region*; $V(G)$, $E(G)$; the *order* and *size* of a graph; *adjacent* vertices and edges; the *degree* of a vertex, *end vertex*; *path*, *cycle*; the *length* of a path or cycle; the *distance* between two vertices; *subgraph*; *connected*, *n-connected*, and *disconnected* graphs, and *components*. We will define several more specialized terms here, and others as needed in the body of the text.

A *chord* is an edge joining two vertices on a cycle that is not itself on the cycle. A *cut-vertex* is a vertex whose removal increases the number of components in the graph. A *block* is a graph with no cut-vertices. A *tree* is a connected graph with no cycles. A graph is *planar* if it can be embedded in the plane in such a way that its edges intersect only at vertices. A planar graph of order at least three is *maximal planar* if every region is a triangle.

Two vertices are *independent* if they are not adjacent. An *independent set* is a set of mutually independent vertices. A *clique* is a set of mutually adjacent vertices. The *neighborhood* of a vertex v , $N(v)$, is the set of all vertices adjacent to v .

The maximum degree of a graph is denoted by $\Delta(G)$. A graph is *regular* if every vertex has the same degree. If the degree of every vertex is r , the graph

is *r-regular*. A *spanning subgraph* of G is a subgraph of G with the same order as G . An *r-factor* is an r -regular spanning subgraph.

A *complete* graph has an edge joining every pair of vertices. The complete graph of order n is denoted by K_n . A graph is *n-partite* if it is possible to partition its vertex set into n independent sets, called *partite* sets. If $n = 2$, the graph is *bipartite*. If each vertex in each partite set is adjacent to every vertex in the graph except those in its partite set, the graph is a *complete n-partite* graph, denoted by $K(s_1, s_2, \dots, s_n)$, where $s_1, s_2, \dots, s_n$ are the orders of the respective partite sets.

Since we will be dealing with coloring problems, we start with a set of colors we call our *palette*, P . A *vertex coloring* k of a graph G is a mapping $k : V(G) \rightarrow P$. A *proper vertex coloring* of a graph is a mapping of the vertex set into the palette in such a way that no two adjacent vertices map to the same color. Note that the mapping is not necessarily onto; the palette is the codomain, which may or may not equal the range. Thus, there is no requirement that any particular color actually have a vertex mapped to it; it is permissible for the coloring to utilize only a subset of the palette. We frequently refer to this mapping as an assignment of colors to vertices.

The size of the range of the coloring is important. A proper coloring that assigns t colors to the vertices of a graph is a *t-coloring*. A graph is *t-colorable* if there exists an s -coloring for some $s \leq t$. There is much literature on the problem of properly coloring the vertices of a given graph with the minimum number of colors. [3, Ch.10] [4, Ch.2] The *chromatic number* of a graph G , $\chi(G)$, is the minimum t such that G is t -colorable. That is, if k is a proper vertex coloring of G

such that the range of k is no bigger than the range of any other proper coloring of G , then $\chi(G)$ is the size of the range of k . Put another way, k assigns $\chi(G)$ colors to the vertices of G . Any proper coloring partitions the vertex set into disjoint *color classes*, that is, into independent sets of vertices in which every element of a given set receives the same color. We denote the family of proper vertex colorings of a graph G with colors from the palette P as $\mathcal{X}_P(G)$.

Cost coloring generalizes the vertex coloring problem. Suppose our palette is $P = \{p_1, p_2, \dots, p_n\}$. We introduce a set of associated costs $c_P = \{c_1, c_2, \dots, c_n\}$ such that c_i is the cost associated with color p_i for $i = 1, \dots, n$. This is the key step in our generalization of vertex coloring. We can see that $|P| = |c_P|$, and we assume that P is arranged so that c_P is in nondecreasing order.

In addition to our coloring function $k : V(G) \rightarrow P$, we now require a *cost function* $C_k : V(G) \rightarrow c_P$, defined for each vertex $v \in V(G)$ by $C_k(v) = c_i$ if and only if $k(v) = p_i$. Thus, our cost function maps a given vertex to the cost of the color that a given coloring assigns to that vertex. The total cost of coloring the vertices of a graph G with colors from the palette P and associated costs c_P , according to the coloring k , is then the sum over all vertices in $V(G)$ of the cost of the color that k assigns to each vertex. We denote this total cost of coloring G with P and c_P by

$$C_k(G) = \sum_{v \in V(G)} C_k(v).$$

When there is no ambiguity, we will omit reference to the palette and simply specify the cost set. The objective of cost-coloring, then, is to find a particular vertex coloring that minimizes the cost of properly coloring a given graph with a

given palette and cost set. That is, find $m \in \mathcal{X}_P(G)$ such that $C_m(G) \leq C_k(G)$, for all $k \in \mathcal{X}_P(G)$. We call m a *minimum-cost coloring* of G with c_P .

Finally, we define here a property of graphs and cost sets unique to cost-coloring, and analogous to vertex-coloring, and introduce notation to refer to it.

Definition The *cost-chromatic number* of a graph G , $\chi_{c_P}(G)$, with respect to a palette P and a cost set c_P is the minimum number of colors necessary to produce a minimum-cost coloring of G with P and c_P .

We can immediately put a lower bound on this new measure of a graph and a cost set. Since any cost coloring of a graph G is also a proper vertex coloring of G , the cost-chromatic number of a graph is greater than or equal to its chromatic number, or $\chi_{c_P}(G) \geq \chi(G)$.

1.3 Preview

This thesis was motivated in part by a Very Large Scale Integrated (VLSI) circuit design problem [5]. VLSI design frequently encounters problems that can be formulated in graph theory in terms of planar blocks. So that our work will be the most useful to VLSI circuit design, we will establish some cost-chromatic properties of planar blocks.

In chapter two, we work exclusively with the new cost sets. We explore the question of which, if any, cost sets preserve any minimum-cost coloring. That is, if we have a minimum-cost coloring, and we change cost sets, is our coloring still minimum-cost? We establish a theorem that establishes which cost sets are, in fact, interchangeable from this standpoint. We then use the theorem

to specify families of equivalent cost sets, and establish two corollaries, one of which we will use extensively.

In chapter three, we begin to explore the cost-chromatic number of graphs with small chromatic number, beginning with trees. We construct a tree, and introduce the intuitively appealing Maximum Independent Set (MIS) method of cost-coloring to help establish the somewhat unexpected result that there exist trees with arbitrarily large cost-chromatic number. We also show by construction that the MIS method can, in certain circumstances, produce an arbitrarily poor estimate of the cost-chromatic number of a given graph with respect to a given cost set. In section 3.11, we use our tree to construct a planar block with chromatic number three that has arbitrarily large cost-chromatic number.

Chapter four begins with some properties of minimum-cost colorings, then examines the analogy between the chromatic number and the cost-chromatic number. We establish two Brooks-type upper bounds on the cost-chromatic number, then proceed to show that unlike the chromatic number, the cost-chromatic number is *not* monotonic with subgraph inclusion. Also unlike the chromatic number, it is possible to reduce the cost-chromatic number of a graph by adding edges to the graph. The latter was an unexpected result, and it gave us mixed feelings. We were happy to establish this seemingly curious property, yet it seemed to limit the usefulness of our previous result on planar blocks.

In chapter five, we turn our attention to another class of graphs, the outerplanar blocks, graphs with chromatic number no more than three. Analogous to our result for trees, we show that there exist outerplanar blocks with arbitrarily large cost-chromatic number. The chapter begins with the

construction of an outerplanar block, describes a coloring of it, and establishes its cost-chromatic number with a detailed inductive proof. We then demonstrate that an outerplanar block with large cost-chromatic number can be contained in an outerplanar graph with cost-chromatic number three. That is, in the new context of outerplanar graphs, the cost-chromatic number is not monotonic with subgraph inclusion. In Section 5.8, we at last can construct, as we hoped, a maximal planar block with arbitrarily large cost-chromatic number.

Chapter six contains an amalgam of results and conjectures, the fruits of discovery. Here the reader will find miscellaneous results, calculations of the cost-chromatic number of some common graphs, and a conjecture. There is some discussion of a possible upper bound on the cost-chromatic number that is at least as tight as the bounds established in chapter four. Our final section points out some lines of inquiry not yet pursued.

Chapter Two

Cost Sets

2.1 Preserving a Minimum-Cost Coloring

It seems clear that the cost of coloring a given graph depends on the costs of the colors in the palette. Suppose we found a minimum-cost coloring of a graph with a particular cost set. It would be helpful to know whether the coloring would still be minimum-cost if we switched cost sets. It does not seem that we should be able to switch cost sets arbitrarily, but there may be conditions under which we can. This is the first question we explore.

Consider an arbitrary graph G , a palette P , and a cost set c_p . Suppose that we have found $m \in \mathcal{X}_p(G)$, a minimum-cost coloring of G with c_p . Does m color G with c_p' at minimum cost where $c_p' = \{c_1', \dots, c_n'\}$? As we shall establish in the following development, the answer is yes for $c_i' = \alpha c_i + \beta$, for $i = 1, \dots, n$, and where α and β are real, and α is positive.

To begin, construct two new cost sets. Let $d_p = \{d_1, \dots, d_n\}$, and $e_p = \{e_1, \dots, e_n\}$, be defined by $d_i = \gamma c_i$, and $e_i = c_i + \delta$, for $i = 1, \dots, n$, γ and δ real, and γ positive.

Lemma 1a The vertex coloring m colors G with c_p at minimum cost if and only if m colors G with d_p at minimum cost.

Proof Assume that m colors G with c_p at minimum cost r . Then m colors G with d_p at cost γr , since $d_i = \gamma c_i$ for $i = 1, \dots, n$. By way of contradiction,

assume that m does not color G with d_p at minimum cost. Then there exists a coloring $k \in \mathcal{X}_p(G)$ such that k colors G with d_p at cost $s < \gamma \cdot r$. Since $d_i = \gamma \cdot c_i$, and $\gamma \neq 0$, $c_i = d_i / \gamma$ for $i = 1, \dots, n$. Thus, k colors G with c_p at cost $s + \gamma < \gamma \cdot r + \gamma = r$, where r is the minimum-cost coloring of G with c_p , by our hypothesis. We have derived a contradiction, so we can conclude that m does color G with d_p at minimum cost.

Conversely, assume that m colors G with d_p at minimum cost. By the first part of this proof, we can multiply the costs in a cost set by any positive real number and still have a minimum-cost coloring. Since $\gamma > 0$, we can multiply the costs in d_p by $1/\gamma$. Since d_p is given by $c_i = (1/\gamma) \cdot d_i$ for $i = 1, \dots, n$, we conclude that m colors G with d_p at minimum cost. **QED Lemma**

Lemma 1b The vertex coloring m colors G with c_p at minimum cost if and only if m colors G with e_p at minimum cost.

Proof Assume that m colors G with c_p at minimum cost r . Then m colors G with e_p at cost $r + \delta \cdot |V(G)|$, since $e_i = c_i + \delta$, for $i = 1, \dots, n$. By way of contradiction, assume that m does not color G with e_p at minimum cost. Then there exists a $k \in \mathcal{X}_p(G)$ such that k colors G with e_p at cost $s < r + \delta \cdot |V(G)|$. Since $e_i = c_i + \delta$, $c_i = e_i - \delta$ for $i = 1, \dots, n$. Hence, we have that k colors G with c_p at cost $s - \delta \cdot |V(G)| < r + \delta \cdot |V(G)| - \delta \cdot |V(G)| = r$, contradicting our assumption that m is a minimum cost coloring of G with c_p . Hence, m colors G with e_p at minimum cost.

Conversely, assume that m colors G with e_p at minimum cost. By the first part of this proof, we can add any real number to each cost in a cost set and still have a minimum-cost coloring. Thus, we can add $-\delta$ to each cost in e_p . Since c_p

is given by $c_i = e_i + (-\delta)$ for $i = 1, \dots, n$, we conclude that m colors G with c_p at minimum cost. **QED Lemma**

We can now establish our result concerning coloring with an alternative cost set. Let G be any graph, let P be a palette with associated costs c_p , and let $m \in \mathcal{X}_P(G)$ be a proper vertex coloring of G with P .

Theorem 1 The vertex coloring m colors G with c_p at minimum cost if and only if m colors G with c_p' at minimum cost, where c_p' is given by $c_i' = \alpha c_i + \beta$, for $i = 1, \dots, n$, α and β real, and α positive.

Proof By Lemma 1a, m colors G with c_p at minimum cost if and only if m colors G with c_p'' at minimum cost, where c_p'' is given by $c_i'' = \alpha c_i$ for $i = 1, \dots, n$. By Lemma 1b, m colors G with c_p'' at minimum cost if and only if m colors G with c_p' , where c_p' is given by $c_i' = c_i'' + \beta = \alpha c_i + \beta$, for $i = 1, \dots, n$. Thus, m colors G with c_p at minimum cost if and only if m colors G with c_p' at minimum cost. **QED**

2.2 Equivalent Cost Sets

We now use the ideas leading up to Theorem 1 to define equivalent cost sets.

Definition Two cost sets of the same size are said to be *equivalent* if and only if each cost in one set can be obtained by multiplying the corresponding cost in the other set by a positive constant and adding a constant.

By Theorem 1, equivalent cost sets have the property that a vertex coloring of a graph that gives a minimum cost with one cost set will also give a minimum cost with any equivalent cost set. To justify choosing the term *equivalent*, we

demonstrate here that the definition is in fact an equivalence relation. Let f , g , and h be cost sets of the same size. Then $f_i = 1 \cdot f_i + 0$, so f is equivalent to f and the relation is reflexive. If f is equivalent to g , then $f_i = \alpha \cdot g_i + \beta$, and $g_i = (1/\alpha) \cdot f_i + (-1/\alpha) \cdot \beta$. Thus, g is equivalent to f and the relation is symmetric. If f is equivalent to g and g is equivalent to h , then $f_i = \alpha \cdot g_i + \beta$ and $g_i = \gamma \cdot h_i + \delta$, so that $f_i = \alpha \cdot (\gamma \cdot h_i + \delta) + \beta = \alpha \cdot \gamma \cdot h_i + (\alpha\delta + \beta)$. Thus, f is equivalent to h and the relation is transitive. Hence, we have an equivalence relation.

We can now present a pair of corollaries that demonstrate the equivalence of certain cost sets, in the new sense that any vertex coloring that produces a minimum cost with one cost set also produces a minimum cost with any equivalent cost set. We continue as before with the graph G , the palette P , and $m : V(G) \rightarrow P$, the vertex coloring of G with P .

Our first corollary states that any cost set that contains only positive rationals is equivalent to a cost set that contains only positive integers, with the lowest cost 1.

Corollary 1a Let c_i be a positive rational number for $i = 1, \dots, n$, and assume that

$c_i \leq c_j$ for $i \leq j$. Then there exists an equivalent cost set

$c_p' = \{c_1', \dots, c_n'\}$ such that c_i' is a positive integer for $i = 1, \dots, n$,

that $c_i' \leq c_j'$ for $i \leq j$, and that $c_1' = 1$.

Proof Since c_i is a positive rational number for $i = 1, \dots, n$, we can write $c_i = p_i / q_i$ for positive integers p_i and q_i . Let α be the least common multiple of $\{q_1, \dots, q_n\}$. Then α is a positive integer, and αc_i is a positive integer for $i = 1, \dots, n$. Let $\beta = 1 - \alpha c_1$. Then β is an integer. Define $c_p' = \{c_1', \dots, c_n'\}$ by $c_i' = \alpha c_i + \beta$ for $i = 1, \dots, n$. Then by the definition of equivalent cost sets, c_p is equivalent to

c_p' . Further, $c_1' = \alpha c_1 + \beta = \alpha c_1 + (1 - \alpha c_1) = 1$, and $c_i' = \alpha c_i + \beta$ is the sum of two integers, hence an integer. Since c_p is arranged in nondecreasing order, it follows that c_p' is arranged in nondecreasing order, and that c_i' is a positive integer for $i = 1, \dots, n$. **QED**

We will make extensive use of this corollary in the coming chapters.

Our second corollary shows that any cost set that contains only real numbers in an increasing arithmetic sequence is equivalent to the cost set $\{1, 2, \dots, n\}$.

Corollary 1b Let $c_p = \{c_1, \dots, c_n\}$ be arranged as an increasing arithmetic sequence of real numbers. Then c_p is equivalent to c_p' , where $c_i' = i$ for $i = 1, \dots, n$.

Proof Since $\{c_i\}$ forms an increasing arithmetic sequence, $c_i = c_1 + (i - 1)d$, where $d \neq 0$ is the common difference between consecutive elements of c_p . Let $\beta = -c_1$, and define c_p'' by $c_i'' = c_i + \beta = c_i - c_1$ for $i = 1, \dots, n$. By definition, c_p is equivalent to c_p'' . Now, let $\alpha = 1/d$ and $\beta = 1$, and define c_p' by $c_i' = \alpha c_i'' + \beta$. Then, by definition, c_p'' is equivalent to c_p' , and for $i = 1, \dots, n$, we have $c_i' = \alpha c_i'' + \beta = (1/d) \cdot (c_i - c_1) + 1 = (1/d) \cdot [c_1 + (i - 1)d - c_1] + 1 = (1/d) \cdot [(i - 1)d] + 1 = i - 1 + 1 = i$. Thus, c_p is equivalent to $c_p' = \{1, \dots, n\}$. **QED**

Chapter Three

Cost-Coloring Trees and Planar Blocks

3.1 An Example

We now turn to an interesting question concerning the cost-coloring of trees. The chromatic number of any tree is two, that is, any tree can be properly vertex-colored with only two colors. But when costs are assigned to those colors, coloring the tree with just the two lowest-cost colors may not produce the minimum-cost coloring of the tree. Suppose that our palette is $P = \{\text{white, light gray, dark gray}\}$, and the associated costs are $c_p = \{1, 2, 3\}$. Consider this tree, discovered by Dr. Edward Schmeichel [6].

Since it is a tree, it can be vertex-colored with two colors. Further, the coloring, with reflections, is unique.

Since the costs of white and light gray are one and two, respectively, this two-coloring of the tree gives a cost of twelve. Would coloring a vertex dark gray, at a cost of three, enable us to reduce the total cost of coloring the tree? Yes, it would. The cost can be reduced to eleven.

In effect, using a little of the most expensive color makes it possible to use a lot more of the least expensive color and still properly color the tree.

A more general question presents itself. Given a certain palette and set of associated costs, does there exist a tree such that the total-cost coloring is minimized by a coloring that utilizes the entire palette? We begin exploring this question by considering some properties of the cost set c_p .

3.2 Distinct Costs

We assume that P is arranged so that c_p is in increasing order, that is, $c_i < c_j$ for $i < j$. Further, we now assume that all the costs are positive and rational. Corollary 1a then gives us license to restrict our attention to cost sets comprising positive integers, with the smallest cost being 1.

We digress slightly here to pursue a thought. Note that the assumption that c_i is strictly less than c_j for $i < j$ is necessary. Consider what happens without it. For now, we will be coloring trees, graphs with chromatic number two. If

there are two costs c_i and c_{i+1} that are equal, then there is never any advantage to using a color more expensive than c_{i+1} .

This idea can be generalized to any graph for which the chromatic number is known, as shown in the following theorem.

Theorem 2 Let G be a graph with chromatic number $\chi(G)$, and let

$c_p = \{c_1, \dots, c_n\}$ be a cost set with a particular cost repeated $\chi(G)$ times. That is, for some $m = 0, \dots, n - \chi(G)$, $c_{m+1} = \dots = c_{m+\chi(G)}$.

Then no minimum-cost coloring of G with c_p utilizes any color with cost greater than $c_{m+\chi(G)}$.

Proof If $m + \chi(G) = n$, the theorem is trivially true, since there are no colors with cost greater than $c_{m+\chi(G)}$. Assume then that $m + \chi(G) < n$. By way of contradiction, suppose that $k \in \mathcal{X}_p(G)$ such that k colors G with c_p at minimum cost, and that there exists a vertex $v \in V(G)$ such that $C_k(v) = c_r$ for some $r > m + \chi(G)$. Now, construct the coloring k' by setting $k'(v) = k(v)$ for all $v \in V(G)$ such that $k(v) \in \{p_1, \dots, p_m\}$. That is, whenever k colors a vertex with a color that has cost less than c_m , k' colors the same vertex with the same color. Now, suppose that there are t vertices remaining to be colored. Since the chromatic number of G is $\chi(G)$, and the remaining t vertices along with all induced edges constitute a subgraph of G , k' can color the remaining t vertices with the colors with costs $c_{m+1}, \dots, c_{m+\chi(G)}$, at total cost $t \cdot c_{m+1}$. By our assumption, k colors at least one vertex with some color that has cost $c_r > c_{m+\chi(G)} = c_{m+1}$, so that k colors the remaining t vertices at cost at least $(t - 1) \cdot c_{m+1} + c_r > (t - 1 + 1) \cdot c_{m+1} = t \cdot c_{m+1}$, the cost that k' generates coloring the remaining t vertices. Since k and k' generate the same cost when coloring the other vertices of G , $C_{k'}(G) < C_k(G)$,

contradicting that k is a minimum-cost coloring. We can conclude that no minimum-cost coloring of G colors any vertex with a color having cost greater than $c_{m+\chi(G)}$. **QED**

Applying Theorem 2 to our discussion of cost-coloring trees produces this corollary.

Corollary 2a Let T be a tree, and let $c_p = \{c_1, \dots, c_n\}$ be a cost set such that for some i , $0 \leq i \leq n - 2$, $c_{i+1} = c_{i+2}$. Then no minimum-cost coloring of G with c_p utilizes any color with cost greater than c_{i+2} .

Proof Since T is a tree, $\chi(T) = 2$, and the result follows directly from Theorem 2. **QED**

Thus, for our purposes, we can restrict discussion to cost sets that contain no repeated costs. In other words, when we cost-color trees, we lose no generality by considering only cost sets of *distinct* positive integers with the smallest cost 1. We leave unexplored the question of the limiting effects of assuming distinct costs when cost-coloring graphs with cost-chromatic number greater than two.

We turn now to constructing our tree, whose minimum-cost coloring requires the entire palette P .

3.3 Growing the Tree

The tree we will grow depends on n , the number of colors in the palette, and c_n , the highest cost in the cost set. To simplify notation, we will let $x = c_n$, and call our tree $T_x^{(n)}$. Construct $T_x^{(n)}$ as follows. Begin with a graph, call it $T_x^{(1)}$ that consists of a single vertex. Construct $T_x^{(2)}$ by attaching c_n end vertices to the

vertex in $T_x^{(1)}$. Construct $T_x^{(3)}$ by attaching c_n end vertices to each vertex in $T_x^{(2)}$. In general, construct $T_x^{(i)}$ by attaching c_n end vertices to each vertex in $T_x^{(i-1)}$. Here is a drawing of $T_3^{(3)}$, for $c_i = i$, for $i = 1, 2, 3$.

We will show that for the given palette and cost set, the minimum-cost coloring of the tree $T_x^{(n)}$ utilizes the entire palette. That is, for every color in the palette, there is a vertex in the tree that the minimum-cost coloring colors with that color. The proof proceeds by

- 1) describing the maximum independent set (MIS) method of cost-coloring,
- 2) establishing some relevant properties of the tree $T_x^{(n)}$ in four lemmas,
- 3) establishing a lemma about the actual color that the MIS method assigns to a given vertex,
- 4) demonstrating that the MIS method produces the minimum-cost coloring of $T_x^{(n)}$ with c_p , (We will show by counterexample that the efficacy of the method depends on the cost set), and finally,
- 5) showing that the minimum-cost coloring of $T_x^{(n)}$ with c_p utilizes the entire palette.

3.4 The Maximum Independent Set Method of Cost-Coloring

Begin with a graph G , and find $M \subseteq V(G)$, a maximum independent set of vertices. That is, find the set of mutually non-adjacent vertices such that no other such set contains more vertices. In the general case, this problem is NP-complete, however, we will attempt to solve it only for the special case of the tree $T_x^{(n)}$. Furthermore, our interest is in the method's correctness, not necessarily in its computational efficiency. Color the vertices in M with the least-expensive color. Now, find a maximum independent set in $G - M$, and color its vertices with the next-least-expensive color. Continue until all vertices of G are colored.

To see that the efficacy of this method is cost-set-dependent, consider the $T_3^{(3)}$ given above for $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$.

Now, apply the method to the tree with $c_1 = 1$, $c_2 = 2$, and $c_3 = 5$. The twelve end vertices receive color p_1 at cost 1, three vertices receive color p_2 at cost 2, and one vertex receives color p_3 at cost 5, giving a total cost of $12(1) + 3(2) + 5 = 23$. However, if we change the coloring to assign color p_1 to the vertex with degree six, and color p_2 to the three end vertices adjacent to that vertex, we have a two-coloring of $T_3^{(3)}$ with ten vertices colored at cost 1 and six vertices colored at cost 2. The cost of this coloring is thus $10(1) + 6(2) = 22 <$

23, the cost produced by the MIS method. It follows that the MIS method is not necessarily effective cost-coloring an arbitrary tree with an arbitrary cost set. In Section 3.10, we will establish that the MIS method can give arbitrarily poor estimates of the cost-chromatic number of a graph.

3.5 Some Properties of $T_x^{(n)}$

To begin our formal development, we establish some properties of $T_x^{(n)}$ in four lemmas.

Lemma 3a For $n \geq 1$ and $i = 1, \dots, n$, $T_x^{(i)}$ has order $(c_n + 1)^{i-1}$.

Proof Let $n \geq 1$, and proceed by induction on i . Beginning with $i = 1$, $T_x^{(1)}$ is a single vertex, hence has order $1 = (c_n + 1)^0 = (c_n + 1)^{i-1}$, so the lemma holds in the base case. Assume now that $T_x^{(i-1)}$ has order $(c_n + 1)^{i-2}$. Now, $T_x^{(i)}$ was constructed by attaching c_n end vertices to each vertex in $T_x^{(i-1)}$, of which there are $(c_n + 1)^{i-2}$ by the induction hypothesis. Hence $T_x^{(i)}$ has order

$$\begin{aligned} & c_n \cdot (c_n + 1)^{i-2} + (c_n + 1)^{i-2} \\ &= (c_n + 1) \cdot (c_n + 1)^{i-2} \\ &= (c_n + 1)^{i-1}. \quad \text{QED Lemma} \end{aligned}$$

Lemma 3b For $n \geq 2$, and $i = 2, \dots, n$, $T_x^{(i)}$ has $c_n \cdot (c_n + 1)^{i-2}$ end vertices.

Proof For $i = 2, \dots, n$, step i in the construction of $T_x^{(n)}$ attaches c_n end vertices to every vertex in $T_x^{(i-1)}$. Thus, no vertex that is an end vertex in $T_x^{(i-1)}$ is an end vertex in $T_x^{(i)}$. Further, every vertex added to construct $T_x^{(i)}$ from $T_x^{(i-1)}$ is an end vertex, and there are c_n times the order of $T_x^{(i-1)}$ of them. By Lemma 3a, there are thus $c_n \cdot (c_n + 1)^{i-2}$ end vertices in $T_x^{(i)}$. **QED Lemma**

Lemma 3c For $n \geq 2$ and $i = 2, \dots, n$, the end vertices of $T_x^{(i)}$ form a unique maximum independent set.

Proof Let $n \geq 2$. Since constructing $T_x^{(i)}$ from $T_x^{(i-1)}$ attaches each new end vertex to a vertex in $T_x^{(i-1)}$, no new end vertices are adjacent, hence they form an independent set of size $c_n \cdot (c_n + 1)^{i-2}$, as given by Lemma 3b. To see that the set is maximum, assume it is not. Then there exists another set S of independent vertices in $T_x^{(i)}$ of size greater than $c_n \cdot (c_n + 1)^{i-2}$. Let A be the set of end vertices in $T_x^{(i)}$, and let $S = S_1 \cup S_2$, where $S_1 = S \cap A$ and $S_2 = S \setminus S_1$, that is, S_2 is the set of vertices in S which are not end vertices in $T_x^{(i)}$. Since $|S| > |A|$ and $S \neq A$, $S_2 \neq \emptyset$. Let $|S_2| = r$, for some $r \geq 1$, and let $v \in S_2$. Since v is not an end vertex in $T_x^{(i)}$, it is adjacent to c_n end vertices by the construction. Hence, $|S_1| \leq |A| - r \cdot c_n$, so that $|S| = |S_1| + |S_2| \leq |A| - r \cdot c_n + r = |A| - r \cdot (c_n - 1)$. Since $c_n > 1$, $c_n - 1 > 0$, so $|S| < |A|$, a contradiction with our assumption that $|S| > |A|$. We conclude that the set of end vertices in $T_x^{(i)}$ forms a unique maximum independent set for $i = 2, \dots, n$. **QED Lemma**

Lemma 3d The tree $T_x^{(n)}$ has one vertex of degree $(n - 1) \cdot c_n$, and all other vertices have degree $1 + (n - i) \cdot c_n$, for some $i = 2, \dots, n$.

Proof The construction of $T_x^{(n)}$ begins with a single vertex. At each subsequent step in the construction, c_n end vertices are attached to each vertex in the graph at the beginning of that step. Thus, the single vertex that is $T_x^{(1)}$ has c_n vertices attached to it $(n - 1)$ times, so its degree is $(n - 1) \cdot c_n$. Now, for $i = 2, \dots, n$, any vertex v that is added to the graph in step i has degree one after that step, since it was added as an end vertex, and has c_n vertices attached to it for

each of the remaining $(n - i)$ steps. Thus the degree of v is $1 + (n - i) \cdot c_n$. **QED**

Lemma

An immediate consequence of this result is that since $c_n > 1$, $(n - 1) \cdot c_n > 1 + (n - i) \cdot c_n$, for $i = 2, \dots, n$, so that there is no ambiguity in referring to the vertex with degree $(n - 1) \cdot c_n$ as the vertex in $T_x^{(n)}$ with highest degree.

3.6 A Property of the MIS Method of Cost-Coloring $T_x^{(n)}$

We now prove a lemma establishing that the color that the MIS method assigns to any vertex in $T_x^{(n)}$ depends only on the degree of the vertex.

Lemma 3e For $i = 1, \dots, n - 1$, the MIS method of cost-coloring $T_x^{(n)}$ with c_p assigns color p_i to all vertices of degree $1 + (i - 1) \cdot c_n$. Further, the method assigns color p_n to the vertex of degree $(n - 1) \cdot c_n$.

Proof By Lemma 3c, the end vertices of $T_x^{(n)}$ form the only maximum independent set, and the MIS method colors them with color p_1 , which verifies Lemma 3e for $i = 1$. After the end vertices of $T_x^{(n)}$ are colored, the remaining uncolored graph is $T_x^{(n-1)}$. Again, Lemma 3c gives us that the end vertices in $T_x^{(n-1)}$ form the only maximum independent set, so that the MIS method colors them with p_2 . When considered as part of $T_x^{(n)}$, however, the end vertices of $T_x^{(n-1)}$ have degree $1 + c_n = 1 + (2 - 1) \cdot c_n$, verifying Lemma 3e for $i = 2$, and leaving the graph $T_x^{(n-2)}$ uncolored. Continuing this argument gives us that for $i = 1, \dots, n - 1$, the end vertices in $T_x^{(n-i)}$ have degree $1 + (i - 1) \cdot c_n$ in $T_x^{(n)}$, and the MIS method colors them with color p_i , leaving the graph $T_x^{(n-i)}$ uncolored. Thus, after $n - 1$ colors have been assigned, the remaining uncolored graph is $T_x^{(n-(n-1))} = T_x^{(1)}$, which is a single vertex, to be colored with p_n . **QED Lemma**

3.7 An Alternative Characterization of $T_x^{(n)}$

It is helpful at this point to consider an alternative characterization of $T_x^{(i)}$, suggested by Dr. Maurice Stanley [7]. Instead of constructing $T_x^{(i)}$ by attaching c_n end vertices to each vertex in $T_x^{(i-1)}$, construct $T_x^{(i)}$ from $c_n + 1$ copies of $T_x^{(i-1)}$. Choose one copy to be the center, and join the highest-degree vertex of the chosen copy to the highest-degree vertex of each other copy, producing $T_x^{(i)}$. Here are two diagrams illustrating the construction of $T_4^{(3)}$ from five copies of $T_4^{(2)}$, for the cost set $\{1, 2, 4\}$ or $\{1, 3, 4\}$.

We establish the equivalence of these two characterizations in the following inductive argument.

Lemma 3f Let $T_x^{(1)}$ be the graph consisting of a single vertex, and let $T_x^{(n)}$ be the tree constructed by joining the highest-degree vertex of each of c_n copies of $T_x^{(n-1)}$ with the highest-degree vertex of another copy of $T_x^{(n-1)}$. Then $T_x^{(n)} = T_x^{(n)}$.

Proof By induction on n . First note that $T_x^{(1)} = T_x^{(1)}$, since both graphs consist of a single vertex. As the base case, let $n = 2$. By our original

construction, $T_x^{(2)}$ is produced by attaching c_n end vertices to the single vertex that is $T_x^{(1)}$, so that $T_x^{(2)}$ is the complete bipartite graph $K(1, c_n)$. By the alternative characterization, $T_x^{(2)}$ is obtained from $T_x^{(1)}$ by arranging an additional c_n copies of $T_x^{(1)}$ around the first copy, and joining the highest-degree vertex in each of the c_n copies to the highest-degree vertex in the first copy. Since each copy contains only one vertex, the resulting graph is $K(1, c_n)$, so that $T_x^{(2)} = T_x^{(2)}$.

Now, assume that $T_x^{(j)} = T_x^{(j)}$ for $j = 1, 2, \dots, i$, where $i < n$. Then $T_x^{(i)}$ can be considered to be made up of $c_n + 1$ copies of $T_x^{(i-1)}$, with the highest degree vertex in each of c_n copies joined to the highest-degree vertex in the remaining copy. Now, construct $T_x^{(i+1)}$ by the original construction, that is, by attaching c_n end vertices to each vertex in $T_x^{(i)}$. By the induction hypothesis, $T_x^{(i)} = T_x^{(i)}$, so $T_x^{(i)}$ is made up of $c_n + 1$ copies of $T_x^{(i-1)}$ joined in a particular fashion. Thus, constructing $T_x^{(i+1)}$ from $T_x^{(i)}$ attaches c_n vertices to each copy of $T_x^{(i-1)}$, creating $c_n + 1$ copies of $T_x^{(i)}$. As a result, since $T_x^{(i)} = T_x^{(i)}$, $T_x^{(i+1)}$ is made up of $c_n + 1$ copies of $T_x^{(i)}$, joined in the particular fashion. But this is simply the definition of $T_x^{(i+1)}$. Thus, $T_x^{(i+1)} = T_x^{(i+1)}$, so we conclude by induction that $T_x^{(n)} = T_x^{(n)}$. **QED Lemma**

3.8 Why the MIS Method?

We now move to the *raison d'être* for the MIS method. The proof is by induction, with both the base case and the inductive case established by contradiction. Keep in mind that c_p is a set of distinct positive integers in increasing order, and that $c_1 = 1$.

Lemma 3g For $n \geq 2$ and $i = 2, \dots, n$, the MIS method colors $T_x^{(i)}$ with c_p at smaller cost than any other coloring.

Proof Let $n \geq 2$, and proceed by induction on i . Let the coloring of $T_x^{(i)}$ with c_p by the MIS method be called M . As the base case, let $i = 2$. Now, $T_x^{(2)}$ has order $(c_n + 1)^{2-1} = c_n + 1$ by Lemma 3a. By Lemma 3b, $T_x^{(2)}$ has c_n end vertices. M colors the end vertices with p_1 , at cost one each, and colors the remaining vertex, call it v , with p_2 , at cost c_2 . Thus, $C_M(T_x^{(2)}) = c_n \cdot 1 + 1 \cdot c_2 = c_n + c_2$. Now, let K be any other coloring of $T_x^{(2)}$. Since $K \neq M$, there is a vertex u in $T_x^{(2)}$ such that $C_K(u) \neq C_M(u)$. If u is an end vertex, $C_K(u) = c_i$, for $i \neq 1$. Since $c_i > c_1$ for $i \neq 1$, $C_K(u) > C_M(u)$. Coloring an end vertex differently than M simply adds to the cost of coloring $T_x^{(2)}$. Further, there is no advantage to be gained unless all the end vertices are assigned a color other than p_1 , freeing up p_1 to assign to v . But doing this produces $C_K(T_x^{(2)}) \geq c_2 \cdot c_n + 1 \geq 2c_n + 1 = c_n + c_n + 1 \geq c_n + c_2 + 1 > c_n + c_2 = C_M(T_x^{(2)})$. Thus, no advantage is to be gained from assigning any color other than p_1 to any end vertex of $T_x^{(2)}$. But assigning a color other than p_2 to v also increases the total cost of the coloring. Thus, the lemma holds in the base case.

As our induction hypothesis, assume that $C_M(T_x^{(j)}) < C_K(T_x^{(j)})$ for $j = 2, \dots, i - 1$, where K is any other coloring of $T_x^{(j)}$, and consider $C_M(T_x^{(i)})$. Lemma 3e gives $M(w)$ for any $w \in V(T_x^{(i)})$, in particular, $M(w) = p_j$ whenever the degree of w is $1 + (j - 1) \cdot c_n$. Using the alternative characterization of $T_x^{(i)}$, arrange c_n copies of $T_x^{(i-1)}$ around another copy of $T_x^{(i-1)}$, to be known as the central copy. By the induction hypothesis, $C_M(T_x^{(i-1)}) < C_K(T_x^{(i-1)})$, where K is any other coloring of $T_x^{(i-1)}$. Lemma 3d gives us that the highest-degree vertex in $T_x^{(i-1)}$ has degree $(i - 2) \cdot c_n$ and Lemma 3e gives that M colors that highest-degree vertex with p_{i-1} . Now, if we simply join the highest-degree vertex in each of the c_n copies with the highest-degree vertex in the central copy, we obtain $T_x^{(i)}$, but the graph is not

properly colored because we made c_n vertices colored with p_{i-1} adjacent to another vertex that was also assigned p_{i-1} . However, if we assign the color p_i to the central vertex in $T_x^{(i)}$, we obtain a proper vertex coloring. In fact, it is M . It remains to show $C_M(T_x^{(i)}) < C_K(T_x^{(i)})$, where K is any other coloring of $T_x^{(i)}$ with c_p .

By way of contradiction, assume that $C_K(T_x^{(i)}) \leq C_M(T_x^{(i)})$, for some coloring K of $T_x^{(i)}$ with c_p , $K \neq M$. Thus, there exists at least one vertex, u_1 , such that $C_K(u_1) < C_M(u_1)$. Lemma 3d gives us that the degree of u_1 is either $(i-1) \cdot c_n$, or $1 + (j-1) \cdot c_n$, for some $j = 1, \dots, i-1$. Assume for the moment that the degree of $u_1 \neq (i-1) \cdot c_n$, that is, $u_1 \neq v$, the highest-degree vertex in $T_x^{(i)}$. Then the degree of $u_1 = 1 + (j-1) \cdot c_n$, for some $j = 1, \dots, i-1$. Now, identify the unique path in $T_x^{(i)}$ joining u_1 with v . Remove from $T_x^{(i)}$ the edge in that path that is incident with v . The component containing u_1 is thus a copy of $T_x^{(i-1)}$. By the induction hypothesis, $C_M(T_x^{(i-1)}) < C_K(T_x^{(i-1)})$, so u_1 is in a subgraph of $T_x^{(i)}$ that M colors at lower cost than K . Since all of our costs are integers, the advantage that K had by coloring u_1 at lower cost than M is turned to a net disadvantage of at least one cost unit over some subgraph of $T_x^{(i)}$ that contains u_1 . Thus, K cannot gain any cost advantage over M by coloring any vertex in $T_x^{(i)}$ at lower cost than M with the possible exception of the single vertex, v , of degree $(i-1) \cdot c_n$.

Now, Lemma 3e gives us that $M(v) = p_i$, so that $C_M(v) = c_i$. Since we assumed that $C_K(T_x^{(i)}) \leq C_M(T_x^{(i)})$, we have that $C_K(v) = c_j$ for some j , $1 \leq j \leq i-1$. Thus, the maximum advantage that K could have over M is to produce a cost that is $(c_i - 1)$ lower. That is, $C_M(T_x^{(i)}) - C_K(T_x^{(i)}) \leq c_i - 1$. It remains to show that if $C_K(v) = c_j$ for some j , $1 \leq j \leq i-1$, then $C_K(T_x^{(i)}) > C_M(T_x^{(i)})$.

Now, $K(v) = p_j$ for some j , $1 \leq j \leq i - 1$, so $C_K(v) = c_j$, producing a savings of $(c_i - c_j)$ over $C_M(v)$. But the construction of $T_x^{(i)}$ gives us that v is adjacent to the c_n vertices in $W = \{w_1, w_2, \dots, w_{c_n}\}$ of degree $1 + (j - 1) \cdot c_n$. By Lemma 3e, $M(w) = p_j$ for $w \in W$. Since $K(v) = p_j$, $K(w) \neq p_j$, since v is adjacent to each vertex in W . So K must color each w differently than M , some perhaps at lower cost, some perhaps at higher cost. Let $w_a \in W$ be such that $C_K(w_a) < C_M(w_a)$. Similarly to our previous argument, delete the edge that joins w_a to a higher degree vertex. Then the component that contains w_a is a copy of $T_x^{(i)}$ for some j , $1 \leq j \leq i - 1$. By the induction hypothesis, M colors this component, call it g_a , at lower cost than K . Thus, we have found a set g_a of vertices in $T_x^{(i)}$ such that $w_a \in g_a$, and $C_K(g_a) \geq C_M(g_a) + 1$. So for each w that K colors at lower cost than M , a net advantage of at least one cost unit goes to M . On the other hand, let $w_b \in W$ such that $C_K(w_b) > C_M(w_b)$. Since all our costs are integers, we have $C_K(w_b) \geq C_M(w_b) + 1$. Now, K cannot recoup this deficit anywhere else, since coloring any vertex other than v at lower cost than M produces a net cost advantage to M over some group of vertices that includes the one colored at lower cost. Thus, for each vertex in W , K must generate a cost of at least one more than M . Since $|W| = c_n$, coloring v at lower cost than M implies that K must incur costs at least c_n higher than M elsewhere in $T_x^{(i)}$. Thus, we have

$$\begin{aligned}
 C_K(T_x^{(i)}) &\geq C_M(T_x^{(i)}) - (c_i - 1) + c_n \\
 &\geq C_M(T_x^{(i)}) - c_i + 1 + c_i \\
 &= C_M(T_x^{(i)}) + 1 \\
 &> C_M(T_x^{(i)}).
 \end{aligned}$$

This contradicts our assumption that $C_K(T_x^{(i)}) \leq C_M(T_x^{(i)})$. We conclude that for $n \geq 2$ and $i = 2, \dots, n$, the maximum independent set method colors $T_x^{(i)}$ with c_p at smaller cost than any other coloring. **QED Lemma**

3.9 The Cost-Chromatic Number of a Tree

With all the pieces now in place, we can proceed to our main result on the cost-coloring of trees.

Theorem 3 For any given palette with associated costs that are distinct, rational, and positive, there exists a tree such that any minimum-cost coloring of the tree utilizes the entire palette.

Proof Let P be any palette with associated costs c_p that are distinct, positive, and rational. By Corollary 1a, c_p is equivalent to a set of costs that are distinct, positive, and integral, with $c_1 = 1$. Thus, for purposes of this argument, we can assume that each c_i is an integer, and that $c_1 = 1$. Let $x = c_n$, and construct the tree $T_x^{(n)}$. It is sufficient to show that if M is a minimum-cost coloring of $T_x^{(n)}$ that uses the smallest number of colors of all minimum-cost colorings, then M assigns each color in P to at least one vertex in $T_x^{(n)}$.

Let M be the cost-coloring of $T_x^{(n)}$ given by the MIS method. By Lemma 3g, M colors $T_x^{(n)}$ at smaller cost than any other coloring. By Lemma 3c, the end vertices in $T_x^{(i)}$ form a maximum independent set, for $i = 1, \dots, n$. By the construction of $T_x^{(n)}$, coloring only the end vertices in $T_x^{(i)}$ leaves the tree $T_x^{(i-1)}$ as the uncolored graph, so there are vertices that receive color p_i , for $i = 1, \dots, n - 1$. After vertices have been assigned colors $p_1, \dots, p_{n-1}$, the remaining uncolored graph is $T_x^{(n-(n-1))} = T_x^{(1)}$, which is a single vertex, which M assigns color p_n . We

conclude that there exists a tree such that any minimum-cost coloring of the tree utilizes the entire palette. **QED**

This result is curious because while the chromatic number of any tree is two, Theorem 3 shows that there exist trees for which the cost-chromatic number is arbitrarily large. You'll notice, however, that $T_x^{(n)}$ was carefully constructed, and branches heavily. The key attribute that we built into $T_x^{(n)}$ was that a large fraction of the vertices have degree one. By using small amounts of expensive colors, we freed up large numbers of end vertices to be colored with the least-expensive color.

3.10 The Denouement of the MIS Method

It has been suggested that the MIS method should, in general, produce a minimum-cost coloring of a graph. The reasoning is appealing: give the lowest-cost color to each vertex in a maximum independent set in the graph, the next lowest-cost color to each vertex in a maximum independent set of the remaining uncolored vertices, and so on until the graph is colored. The flaw in this reasoning is that it fails to consider the actual costs of the colors. As we argued informally in Section 3.4, the efficacy of the method depends on the particular graph and the particular costs. Here we formalize this result.

Theorem 4 For $n \geq 2$, there exists a tree T and a cost set $c_p = \{c_1, \dots, c_n\}$ such that $\chi_{c_p}(T) = 2$ and that the MIS method colors T with n colors.

Proof Let $c_p' = \{c_1', \dots, c_n'\}$ be a cost set. Construct $T_x^{(n)}$. Theorem 3 gives that $\chi_{c_p'}(T_x^{(n)}) = n$, and Lemmas 3d and 3e give that the MIS method colors $T_x^{(n)}$ with n colors. Now, construct $c_p = \{c_1, \dots, c_n\}$ where $c_1 = c_1'$, $c_2 = c_2'$, and $c_i =$

$c_i + c_2 \cdot |V(T_x^{(n)})|$ for $i = 3, \dots, n$. Since $T_x^{(n)}$ is a tree, $\chi(T_x^{(n)}) = 2$, so let m be the lower-cost two-coloring of $T_x^{(n)}$ with p_1 and p_2 .

To show that $\chi_{c_p}(T) = 2$, it is sufficient to show that m is a minimum-cost coloring of $T_x^{(n)}$ with c_p . Assume that it is not. Then there exists a coloring k of $T_x^{(n)}$ with c_p such that $C_k(T_x^{(n)}) < C_m(T_x^{(n)})$. Since no two-coloring of $T_x^{(n)}$ with c_p has lower cost than m , k must use at least three colors. Thus, there exists a vertex v in $T_x^{(n)}$ that k colors with cost at least $c_3 > c_2 \cdot |V(T_x^{(n)})|$. It follows that $C_k(T_x^{(n)}) > c_2 \cdot |V(T_x^{(n)})| = c_2 \cdot |V(T_x^{(n)})| > C_m(T_x^{(n)})$, contradicting that k colors $T_x^{(n)}$ with c_p at lower cost than m . **QED**

3.11 The Cost-Chromatic Number of a Planar Block

As we stated in the preview, we wish to discover some properties of the cost-chromatic number of planar blocks. While planar blocks are in general much more complex than trees, the structure of our tree $T_x^{(n)}$ gives us an opportunity to construct a particular planar block with arbitrarily large cost-chromatic number. Since our tree is fresh in mind, we include this section here.

As always, we begin with a palette P of n colors, $n \geq 3$, and let $c_p = \{c_1, \dots, c_n\}$ be the associated costs. To simplify notation, let $x = c_n$. Construct $T_x^{(n-1)}$ with respect to c_p . As before, let v be the highest-degree vertex in $T_x^{(n-1)}$. Now, construct a planar block from $T_x^{(n-1)}$ by adding one additional vertex u , and joining it to every vertex in $T_x^{(n-1)}$. Call the resulting graph G . First, it's important to note that G has the property we need.

Lemma 5a G is a planar block.

Proof Since $T_x^{(n-1)}$ is a tree, every vertex can be considered to be on the boundary of the external region. Thus, it is possible to create a plane drawing of G so that no edges cross, hence G is planar. To be a block, G must be connected and contain no cut-vertices. Since every vertex is adjacent to u , G is connected. Let r and s be any two vertices in G . If neither vertex is u , then there is an r - s path that lies in the tree, and an edge-disjoint r - u - s path. Since there are two edge-disjoint paths between any two vertices other than u , no vertex other than possibly u is a cut-vertex. However, deleting u leaves a tree, which is connected, so u is not a cut-vertex. Thus, G contains no cut-vertices. We can conclude that G is a planar block. **QED Lemma**

We include the following technical lemma to aid in establishing our anticipated proof by contradiction.

Lemma 5b If $P' = P \setminus \{p_i\}$ for some i , $1 \leq i \leq n - 1$, so that $c_{P'} = c_P \setminus \{c_i\}$,
then $\chi_{c_{P'}}(T_x^{(n-1)}) = n - 1$.

Proof We have that $|P'| = n - 1$, and the largest element in $c_{P'}$ is c_n . By the construction in Section 3.3, $T_x^{(n-1)}$ is the tree we would construct for any palette of $n - 1$ colors, the largest of which is c_n . By Lemmas 3d and 3e, substituting $n - 1$ for n , the MIS method assigns $n - 1$ colors to the vertices of $T_x^{(n-1)}$. By Lemma 3g with $i = n - 1$, the MIS method gives the unique minimum-cost coloring of $T_x^{(n-1)}$ with $c_{P'}$. The result follows from the definition of cost-chromatic number. **QED Lemma**

Since G is constructed by adding one new vertex which is adjacent to every vertex in a tree, $\chi(G) = 3$. Furthermore, G is not outerplanar, since it will always contain as a subgraph the one shown here.

We are now ready to prove our next theorem.

Theorem 5 For any given palette containing at least three colors, with associated costs that are distinct, rational, and positive, there exists a planar block such that any minimum-cost coloring of the block utilizes the entire palette.

Proof Let $P = \{p_1, \dots, p_n\}$ be a palette with associated costs $c_p = \{c_1, \dots, c_n\}$ that are distinct, positive, and rational, where $n \geq 3$. By Corollary 1a, c_p is equivalent to a set of costs that are distinct, positive, and integral. Thus, for purposes of this argument, we can assume that each cost is an integer.

Let $x = c_n$, and construct $T_x^{(n-1)}$. Construct the graph G from $T_x^{(n-1)}$ by adding an additional vertex, call it u , and adding edges joining u with each vertex in $T_x^{(n-1)}$. By Lemma 5a, G is a planar block.

To establish the theorem, it is sufficient to show that $\chi_{c_p}(G) = n$. By way of contradiction, assume that $\chi_{c_p}(G) \neq n$. Since $|P| = n$, $\chi_{c_p}(G) \leq n$, so we have that $\chi_{c_p}(G) = t < n$. Thus, there exists a t -coloring $k \in \mathcal{X}_p(G)$ such that $C_k(G) \leq C_m(G)$ for all colorings $m \in \mathcal{X}_p(G)$. Since k must assign some color to u , we can assume that $k(u) = p_j$, for some j , $1 \leq j \leq t$.

Define $m \in \chi_p(G)$ as follows. Let $m(u) = p_j$, and let m color $G - u = T_x^{(n-1)}$ by the MIS method. Since m cannot assign p_j to any other vertex in G , the remaining palette available to m to color $T_x^{(n-1)}$ is $P' = \{p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n\}$, with associated costs $c_{P'} = \{c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_n\}$. Lemma 5b gives us that $\chi_{c_{P'}}(T_x^{(n-1)}) = n - 1$. Further, Lemma 3g gives us that the MIS method, that is, m , colors $T_x^{(n-1)}$ at lower cost than any other coloring, in particular, at lower cost than k . Hence, we have $C_m(T_x^{(n-1)}) < C_k(T_x^{(n-1)})$. Since $T_x^{(n-1)} = G - u$, and $C_m(u) = C_k(u)$, we have

$$\begin{aligned} C_m(G) &= C_m(G - u) + C_m(u) \\ &< C_k(G - u) + C_k(u) \\ &= C_k(G). \end{aligned}$$

Since we assumed that k was a minimum-cost coloring, we have reached a contradiction. Thus, we can conclude that $\chi_{c_p}(G) = n$. **QED**

Even though Theorem 5 is stated for planar blocks, the block that we constructed has chromatic number three, and is not maximal planar. In Section 5.8 we construct maximal planar blocks with arbitrarily large cost-chromatic number.

Chapter Four

The Analogy with the Chromatic Number

Since we introduced the cost-chromatic number as being analogous to the chromatic number, it is tempting to think of that analogy as complete. That is, it is tempting to think that if the chromatic number has a certain property, then the cost-chromatic number has the property as well. As should be evident by now, the cost-chromatic number behaves in some very different ways than the chromatic number. For instance, we have already established that the cost-chromatic number is not a property of a graph, but of a graph and a cost set, so that changing the cost set can change the cost-chromatic number, and that a graph with small chromatic number can have large or small cost-chromatic number. In this chapter, we explore several other similarities and differences between the chromatic number and the cost-chromatic number.

4.1 Minimum-Cost Colorings

We begin with a pair of observations concerning properties of minimum-cost colorings. Our first observation, despite its seeming simplicity, will prove useful later on.

Theorem 6 Let P be any palette of n colors with associated costs c_p , and let G be a graph that is t -colorable for some $t < n$. Then any minimum-cost t -coloring of G uses the t least-expensive colors of P .

Proof Denote the t least-expensive colors of P by $P_t = \{p_1, \dots, p_t\}$. By way of contradiction, suppose K is a minimum-cost t -coloring of G with c_p that assigns to some vertex of G a color $p \in \{p_{t+1}, \dots, p_n\}$. Since K is a t -coloring, there is a color q in P_t that K did not assign to any vertex of G , and that has lower cost than p . Create the t -coloring K' to be the same coloring as K , except that any vertex K colors p , K' colors q . Thus, $C_{K'}(G) < C_K(G)$, contradicting the assumption that K is a minimum-cost t -coloring of G . We can conclude that any minimum-cost t -coloring of G with c_p uses the t least-expensive colors. **QED**

We observe next that the minimum cost of coloring any graph is monotonic with subgraph-inclusion.

Theorem 7 Let G be any graph, and let H be a subgraph of G . Let P be a palette of n colors, $n \geq \chi(G)$, with associated costs c_p . Then the minimum cost of coloring H with c_p does not exceed the minimum cost of coloring G with c_p .

Proof Let the minimum cost of coloring G with c_p be C , and let the minimum cost of coloring H with c_p be C' . By way of contradiction, assume $C' > C$. Since H is a subgraph of G , we can take G with its minimum-cost coloring, and remove all vertices and edges that are not in H . This produces a coloring of H at cost $C'' \leq C < C'$, where C is the minimum cost of coloring H , a contradiction. We conclude that the minimum cost of coloring H does not exceed the minimum cost of coloring G . **QED**

Thus, it is impossible to color a graph at lower cost than the minimum cost of coloring any of its subgraphs.

4.2 Brooks-Type Upper Bounds

Although we are beginning to see some pitfalls in carrying the analogy between chromatic number and cost-chromatic number too far (and more await), there is still something to be gained from pursuing the analogy. In 1941, Brooks [2] established that for any graph G that is not a complete graph nor an odd cycle, $\chi(G) \leq \Delta(G)$. An immediate corollary is that for any graph G , $\chi(G) \leq \Delta(G) + 1$. Theorem 6 enables us to use an analogous argument to establish an upper bound on the cost-chromatic number.

Theorem 8 Let G be a graph, and let $P = \{p_1, \dots, p_n\}$ be a palette with associated costs $c_p = \{c_1, \dots, c_n\}$ for some $n > \Delta(G) + 1$, then $\chi_{c_p}(G) \leq \Delta(G) + 1$.

Proof Let G be a graph, let $n > \Delta(G) + 1$, and let $P = \{p_1, \dots, p_n\}$ be a palette with associated costs $c_p = \{c_1, \dots, c_n\}$. By way of contradiction, assume that $\chi_{c_p}(G) > \Delta(G) + 1$. Then there exists a coloring $k \in \mathcal{X}_p(G)$ such that k is a minimum-cost coloring of G with c_p , and k uses t colors, for some $t \geq \Delta(G) + 2$. By Theorem 6, we know that $k : V(G) \rightarrow \{p_1, \dots, p_t\}$.

Let $S = \{v \in V(G) \mid C_k(v) \geq c_{\Delta(G)+2}\}$. Since k uses at least $\Delta(G) + 2$ colors, $S \neq \emptyset$, so let $S = \{v_1, \dots, v_r\}$. Construct a new coloring m of G that differs from k only on the vertices in S . Consider v_1 . For $v \in V(G)$, $\deg v \leq \Delta(G)$, so v_1 is adjacent to at most $\Delta(G)$ other vertices. Thus, of the colors $p_1, \dots, p_{\Delta(G)+1}$, there is at least one that k does not assign to any vertex adjacent to v_1 . Let i be the smallest index of all the colors that k does not assign to vertices adjacent to v_1 , and let $m(v_1) = p_i$. Thus, $C_m(v_1) = c_i < c_{\Delta(G)+1} \leq C_k(v_1)$.

Apply the same reasoning to each vertex v_j for $j = 2, \dots, r$. Thus, for $v \in S$, $C_m(v) < C_k(v)$, and for $v \in V(G) \setminus S$, $C_m(v) = C_k(v)$. Hence, $C_m(G) < C_k(G)$, contradicting our assumption that k is a minimum-cost coloring of G with c_p . We conclude that $\chi_{c_p}(G) \leq \Delta(G) + 1$. **QED**

A direct counterpart to Brooks's Theorem holds for the cost-chromatic number of a connected graph. The proof is intricate and rather lengthy.

Theorem 9 (Mitchem) Let P be a palette of n colors with associated costs c_p .

Let G be a connected graph that is neither an odd cycle nor a complete graph, and let $3 \leq \Delta(G) \leq n$. Then $\chi_{c_p}(G) \leq \Delta(G)$.

Proof Denote $\Delta(G)$ simply by Δ . Since $\Delta \geq 3$, G is not a cycle. By way of contradiction, assume $\chi_{c_p}(G) \geq \Delta + 1$. By Theorem 8, $\chi_{c_p}(G) \leq \Delta + 1$, so we have $\chi_{c_p}(G) = \Delta + 1$.

Let k color G with c_p at minimum cost. Then $k : V(G) \rightarrow \{p_1, \dots, p_{\Delta+1}\}$, and there is a nonempty set $V = \{v_1, \dots, v_t\}$ such that $k(v_i) = p_{\Delta+1}$ for $i = 1, \dots, t$. Of all the minimum-cost $\Delta+1$ -colorings of G with c_p , choose k first so that $|V|$ is minimized, and second so that $\min \{d(v_i, v_j) \mid 1 \leq i < j \leq t\}$ is also minimized.

Claim $t = 1$.

Proof To establish the claim, assume that it is false, so that $t \geq 2$. Let $v_1, v_m \in V$ such that $d(v_1, v_m)$ is minimum. We have that $\deg v_1 \leq \Delta$ and $\deg v_m \leq \Delta$. Now, for $i = 1, \dots, \Delta$, v_1 and v_m are each adjacent to one vertex u_i such that $k(u_i) = p_i$. Otherwise, it would be possible to recolor v_1 or v_m with the missing color, producing a coloring of G with lower cost than k , a contradiction. Further, we know v_1 and v_m are not adjacent, since the proper coloring k assigns them the

same color. Therefore, since G is connected, there exists a path Q of length at least two that joins v_1 and v_m .

Assume for the moment that the length of Q is 2. Then there exists a vertex u_1 adjacent to both v_1 and v_m such that $k(u_1) = p_j$ for some j , $1 \leq j \leq \Delta$. Let $D = \{v \in V(G) \mid d(v, u_1) = 1 \text{ and } k(v) = p_{\Delta+1}\}$. Then $|D| \geq 2$, since v_1 and v_m are both in D . Define a new coloring k' of G by $k'(u_1) = p_{\Delta+1}$, $k'(v) = p_j$ for $v \in D$, and $k'(v) = k(v)$ otherwise. Thus we have

$$\begin{aligned} C_{k'}(G) &= C_k(G) + (c_{\Delta+1} - c_j) - |D| \cdot (c_{\Delta+1} - c_j) \\ &\leq C_k(G) + (c_{\Delta+1} - c_j) - 2 \cdot (c_{\Delta+1} - c_j) \\ &= C_k(G) - (c_{\Delta+1} - c_j) \\ &< C_k(G). \end{aligned}$$

This contradicts that k is a minimum-cost coloring of G . Thus, the length of Q is at least 3. Denote Q by $v_1, u_1, u_2, \dots, v_m$. We know $k(u_1) = p_j$ for some j , $1 \leq j \leq \Delta$, and that v_1 is adjacent to no other vertices that k colors p_j . Since the length of Q is at least 3, u_1 is adjacent to no other vertices that k colors $p_{\Delta+1}$. Define a new coloring k' of G by $k'(v_1) = p_j$, $k'(u_1) = p_{\Delta+1}$, and $k'(v) = k(v)$ otherwise. Then $C_{k'}(G) = C_k(G)$, and $d(u_1, v_m) < d(v_1, v_m)$, contradicting that we chose k from among all the minimum-cost colorings of G such that the minimum distance between any two vertices colored $p_{\Delta+1}$ is minimized. **QED Claim**

Thus, there exists a unique vertex $v \in V(G)$ such that $k(v) = p_{\Delta+1}$. Furthermore, for $i = 1, \dots, \Delta$, v is adjacent to exactly one vertex s such that $k(s) = p_i$.

Since G is not a complete graph on $\Delta + 1$ vertices, there exist two vertices adjacent to v that are not themselves adjacent. Since k colors them differently, it

is conceivable that we could reduce the cost of coloring G by coloring them both with the less-expensive of the two colors. However, since k is a minimum-cost coloring, assigning them both the same color would not produce a proper coloring of G . Thus, there must be at least one vertex adjacent to each of these two that k colors with the other's color. In particular, there exists a vertex in G that lies at distance at least two from v . Let $u \in V(G)$ such that $d(v, u) = 2$, and denote the vertex on the v - u path by w .

We summarize here what we know about the structure of G and the coloring k . There is a unique vertex v in G such that $k(v) = p_{\Delta+1}$; $\deg v = \Delta$; v is adjacent to exactly one vertex s such that $k(s) = p_i$ for $i = 1, \dots, \Delta$; there is a v - w - u path in G ; $k(w) = p_i$, and $k(u) = p_j$ for some $i \neq j$, $1 \leq i, j \leq \Delta$. Here is a diagram to illustrate.

Let $R = \{v \in V(G) \mid k(v) = p_i \text{ or } k(v) = p_j\}$, and let S be the subgraph of G induced by the vertex set R . Let H be that component of S that contains u . Thus, H is connected and contains at least two vertices, namely, w and u . Since k properly colors G , k is a proper coloring of H , so that every edge in H joins a vertex colored p_i with a vertex colored p_j . Thus, H is either an even cycle, a path, or contains a vertex with degree at least three. We consider each case separately.

Case i) H is an even cycle. Since k colors the vertices of H alternately with p_i and p_j , we have that every vertex colored p_i is adjacent to two vertices colored p_j . In particular, w is adjacent to two vertices colored p_j , u and x , and one other, call it y . Thus, the next diagram represents a subgraph of G with coloring k .

By a previous argument, no vertex in $N(v)$ is colored p_i except w . Further, no vertex other than v in $N(w)$ is colored $p_{\Delta+1}$, since v is the only vertex in G that k assigns $p_{\Delta+1}$. Thus, it is possible to interchange the colors of v and w without affecting the cost of the coloring, while preserving a proper coloring of G . We do so, giving the coloring in the next diagram.

Since $\deg w \leq \Delta$, and $k(w) = p_{\Delta+1}$, we have that $k(N(w)) \subseteq \{p_1, \dots, p_\Delta\}$. Since w is adjacent to u and x , two vertices that k assigns p_j , there is a color $p_k \in \{p_1, \dots, p_\Delta\}$ that k does not assign to any vertex adjacent to w . Thus, we can

recolor w with p_k . This reduces the cost of coloring G , contradicting that k is a minimum-cost coloring.

Case ii) H is a path. Let Q be the path $H - w$, denote the vertices in Q as $t_1 (= u)$, t_2 , ..., t_m , and consider the graph H' , as shown here.

If m is even, then $k(t_{m-1}) = p_j$, and $k(t_m) = p_i$. If m is odd, then $k(t_{m-1}) = p_i$, and $k(t_m) = p_j$.

By a previous argument, we know that $\deg v = \Delta$, $k(v) = p_{\Delta+1}$, and $k(N(v)) = \{p_1, \dots, p_\Delta\}$. Furthermore, we know that $\deg w = \Delta$, $k(w) = p_i$, and $k(N(w)) = \{p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{\Delta+1}\}$. Thus, it is possible to redefine k so that $k(v) = p_i$ and $k(w) = p_{\Delta+1}$, retaining a proper coloring of G without affecting the cost of the coloring. Applying the same argument to w and u under the new coloring gives us that $\deg w = \Delta$, $k(w) = p_{\Delta+1}$, $k(N(w)) = \{p_1, \dots, p_\Delta\}$, $\deg u = \Delta$, $k(u) = p_j$ and $k(N(u)) = \{p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_{\Delta+1}\}$. Thus, it is possible to redefine k so that $k(w) = p_j$ and $k(u) = p_{\Delta+1}$, again retaining a proper coloring of G without affecting the cost of the coloring.

We continue applying this argument successively to the vertices in Q , until we have $\deg t_{m-1} = \Delta$, $k(t_{m-1}) = p_{\Delta+1}$, $k(N(t_{m-1})) = \{p_1, \dots, p_\Delta\}$, and $\deg t_m = \Delta$. If m is even, $k(t_m) = p_i$, and t_m is adjacent to no vertex that k colors p_j . If m is odd, $k(t_m)$

$= p_i$ and t_m is adjacent to no vertex that k colors p_i . In either case, it is now possible to redefine k so that $k(t_{m-1})$ is the color that k originally assigned to t_m , and $k(t_m) = p_{\Delta+1}$, without affecting the cost of the coloring. At this point, there is a color in $\{p_1, \dots, p_\Delta\}$ that k does not assign to any vertex in the neighborhood of t_m , and that is less expensive than $p_{\Delta+1}$. Thus, it is possible to recolor t_m with this missing color, reducing the cost of coloring G . This contradicts that k is a minimum-cost coloring of G .

Case iii) H contains a vertex of degree at least three. Let t be the vertex in H such that $\deg t \geq 3$ and $d(u, t)$ in H is minimized. Then t is adjacent to two vertices, r and s , that are not on the shortest u - t path in H and that $k(r) = k(s)$. If $d(u, t)$ in H is even, then $k(r) = k(s) = p_i$, and if $d(u, t)$ in H is odd, then $k(r) = k(s) = p_j$. Consider the graph H' consisting of H together with the vertex v and the edge $\{v, w\}$. By the argument in case ii), we have $\deg t = \Delta$, and we can successively redefine k so that $k(t) = p_{\Delta+1}$. Since $r, s \in N(t)$ and $k(r) = k(s)$, there is a color $p_k \in \{p_1, \dots, p_\Delta\}$ that has lower cost than $p_{\Delta+1}$ and is not assigned to any vertex in $N(t)$. Thus, it is possible to recolor t with p_k , reducing the cost of coloring G . This contradicts that k is a minimum-cost coloring of G .

Since each case led to a contradiction, we know that our original assumption is false. Hence, $\chi_{c_p}(G) < \Delta + 1$, so that $\chi_{c_p}(G) \leq \Delta$. **QED**

4.3 Lack of Monotonicity (Order)

We establish in this section an unexpected result that illustrates a striking difference between the chromatic number and the cost-chromatic number: the size of the palette required to produce a minimum-cost coloring of a tree is *not* monotonic with subtree-inclusion. That is, if T is a tree that requires n colors to

produce a minimum-cost coloring, and $T \subseteq T'$, it does not necessarily follow that a minimum-cost coloring of T' requires at least n colors. In fact, a tree T' which contains T can be constructed that requires only two colors to produce a minimum-cost coloring. As an example, let $P = \{\text{white, light gray, dark gray}\}$, let $c_P = \{1, 2, 3\}$, and let T be the Schmeichel tree that required three colors to color at minimum cost 11.

Now, let T' be the following tree, which contains T as a subtree.

By coloring the three new vertices white, we can color T' with two colors at a cost of fifteen, as shown here.

Notice that if we were to color some vertex dark gray, at cost three, we would immediately incur an additional expense of at least one unit. Thus, to recoup the expense, we would be required to assign white to some vertices that are now colored light gray. But using white on any of those four vertices will require us to use light gray on at least one end vertex, canceling our savings from using white. Thus, using the higher-cost color will produce a coloring with higher cost. We formalize these ideas in the following theorem.

Theorem 10 Let P be a palette of n colors, $n \geq 2$, with associated costs c_p , and let T be any tree with $\chi_{c_p}(T) = n$. Then there exists a tree $T' \supseteq T$ such that $\chi_{c_p}(T') = 2$.

Proof Let $P = \{p_1, \dots, p_n\}$ be a palette of n colors, $n \geq 2$, with associated costs $c_p = \{c_1, \dots, c_n\}$, and let T be a tree with $\chi_{c_p}(T) = n$. Let K be a minimum-cost coloring of T with $\{p_1, p_2\}$. Thus, K partitions $V(T)$ into two disjoint color classes, $S_1 = \{v \in V(T) \mid k(v) = p_1\}$, and $S_2 = \{v \in V(T) \mid k(v) = p_2\}$.

Construct the tree T' from T by attaching c_n end vertices to each vertex in S_2 . Define the two-coloring K' of T' with $\{p_1, p_2\}$ by $K'(v) = K(v)$ for all $v \in V(T)$, and $K'(v) = p_1$ for all $v \in V(T') \setminus V(T)$. To show that $\chi_{c_p}(T') = 2$, it is sufficient to

show that K' colors T' at minimum cost, since K' uses the two least-expensive colors in P .

By way of contradiction, suppose there exists some other coloring K'' of T' with c_p such that $C_{K''}(T') \leq C_{K'}(T')$. Since $K'' \neq K'$, there exists a vertex $v \in V(T')$ such that $C_{K''}(v) < C_{K'}(v)$. Now, if $K'(v) = p_1$, K'' cannot color v at lower cost, so we have $K'(v) = p_2$. Since the only lower-cost color available is p_1 , we also have $K''(v) = p_1$, producing a savings for K'' over K' of $c_2 - c_1$. However, since $K'(v) = p_2$, $v \in V(T)$, so that $K'(v) = K(v)$, giving us that $v \in S_2$. Thus, the construction of T' attached c_n end vertices to v , each of which K' colored p_1 . Since K'' does not have color p_1 available, K' must color each of the c_n end vertices with a color at least as expensive as p_2 , producing an additional cost over K' of at least $c_n \cdot (c_2 - c_1) > c_2 - c_1$, which is the savings that K'' produced coloring v with p_1 . Hence, $C_{K''}(T') > C_{K'}(T')$, contradicting our assumption that $C_{K''}(T') \leq C_{K'}(T')$. Thus, K' colors T' at minimum cost, so we can conclude that $\chi_{c_p}(T') = 2$. **QED**

4.4 Lack of Monotonicity (Size)

We just observed that increasing the order of a graph can decrease the cost-chromatic number. Here we explore the effect of increasing the size of a graph on the cost-chromatic number.

The chromatic number of a graph has the intuitively appealing property that increasing the size of the graph without increasing the order, that is, adding edges but no vertices to the graph, can never decrease the chromatic number. We show here an example of adding edges to a graph to decrease the cost-chromatic number.

Suppose that our palette $P = \{\text{white, light gray, dark gray}\}$, and the respective costs are $c_p = \{1, 2, 3\}$. Again, we recall the Schmeichel tree, S .

We argued previously that this tree has cost-chromatic number three with respect to this cost set. Here is a three-coloring with cost eleven.

Observe that it is possible to reduce the cost-chromatic number to two by adding edges. For now, return to a two-coloring with cost twelve, as shown here.

Now, create the graph S' by adding two edges to this graph in such a way as to join the vertices in pairs, each vertex colored white to be paired in S' with a vertex colored light gray.

Note that the edges marked below with crosshatches form a 1-factor in such a way that each white vertex is paired with a light gray vertex, and vice versa.

Since each pair of adjacent vertices must receive two different colors, coloring each pair must cost at least three. Since there are four pairs, the total cost of coloring the graph cannot be less than twelve. Since $\chi(S') = 2$, and we have here a two-coloring that costs twelve, we have established that $\chi_{cp}(S') = 2$. Thus, to a graph with cost-chromatic number three, we have added edges to produce a graph with cost-chromatic number two.

We pursue this line of thought by digressing slightly. We next establish a theorem that under certain conditions will give us a lower bound on the cost of coloring a graph with a cost set. In cases where we are able to construct a coloring with this lower-bound cost, the theorem gives us the cost-chromatic number. The corollaries to this theorem will bring us back to our line of inquiry concerning reducing the cost-chromatic number of a graph by increasing its size.

Theorem 11 Let G be a graph, and let P be a palette of n colors with associated costs c_p . If $V(G)$ can be partitioned into subsets $V_1, \dots, V_t$ such that V_i induces a complete graph of order m_i for $i = 1, \dots, t$, then the minimum cost of coloring G with c_p is at least

$$\sum_{i=1}^t \sum_{j=1}^{m_i} c_j.$$

Proof Each subset V_i induces a complete graph of order m_i , which requires m_i colors to properly color. The lowest-cost way to color the clique is to assign the m_i lowest-cost colors, one to each vertex, giving a minimum cost of coloring V_i of $c_1 + c_2 + \dots + c_{m_i}$. The cost of coloring the graph is at least as large as the sum of the minimum-cost colorings of each of the t cliques, giving the double sum. **QED**

This result is useful because if it is possible to find a $\chi(G)$ -coloring of G that achieves the lower-bound cost, it follows immediately that $\chi_{c_p}(G) = \chi(G)$. It turns out that this lower bound on the cost of coloring a graph is easy to obtain for certain common graphs, which leads us to some corollaries.

Corollary 11a Let G be a complete r -partite graph, and let P be a palette with associated costs c_p . Then $\chi_{c_p}(G) = r$.

Proof Let the partite sets of G be $S_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$ for $i = 1, \dots, r$, and let the sets be arranged so that $|S_i| \geq |S_j|$ for $1 \leq i \leq j \leq r$. Thus, $n_1 \geq n_j$ for all j , $1 \leq j \leq r$. Now, for $i = 1, \dots, r$, let m be the coloring that assigns color p_i to all vertices in S_i . Partition $V(G)$ into subsets V_i for $i = 1, \dots, n_1$ as follows. For $i = 1, \dots, n_1$, let V_i be the set of all vertices in $V(G)$ with second subscript i .

Since G is a complete r -partite graph, and each set V_i contains at most one vertex from each partite set S_i , the vertex subset V_i induces a complete graph of order $|V_i|$ for $i = 1, \dots, n_1$. In addition, since each set V_i contains at most one vertex from each partite set S_i , m properly colors the complete graphs. Since the sets S are arranged in nonincreasing order of size, any complete graph of order s is colored with $p_1, p_2, \dots, p_s$, the s least-expensive colors in the palette. Thus,

$$C_m(G) = \sum_{i=1}^{n_1} \sum_{j=1}^{|V_i|} c_j.$$

By Theorem 11, this is the minimum cost of coloring G with c_p . Since we have an r -coloring of G with this cost, we have that $\chi_{c_p}(G) \leq r$. Since G is a complete r -partite graph, $\chi(G) = r$, so $\chi_{c_p}(G) \geq r$. Hence, the cost-chromatic number of any complete r -partite graph is r . **QED**

An immediate corollary of this result concerns the cost-chromatic number of a complete graph.

Corollary 11b Let G be a complete graph on r vertices, and let P be a palette with associated costs c_p . Then $\chi_{c_p}(G) = r$.

Proof Since any complete graph on r vertices can be considered to be a complete r -partite graph with partite sets of size 1, the result follows immediately from Corollary 11a. **QED**

A third corollary of Theorem 11 establishes the idea we illustrated above. That is, for any tree with cost-chromatic number $n \geq 2$, it is possible to add edges to the tree in such a way as to produce a graph with cost-chromatic number two.

Corollary 11c Given a palette P of n colors with associated costs c_p , and a tree T with $\chi_{c_p}(T) = n \geq 2$, there exists a graph G such that $V(G) = V(T)$, $E(G) \supseteq E(T)$, and $\chi_{c_p}(G) = 2$.

Proof Since T is a tree, $\chi(T) = 2$. Let m be the lower-cost two-coloring of T with p_1 and p_2 . Thus, m partitions $V(T)$ into two color classes, $S_1 = \{v \in V(T) \mid m(v) = p_1\}$ and $S_2 = \{v \in V(T) \mid m(v) = p_2\}$. Since m is the lower-cost two-coloring of T , $|S_1| \geq |S_2|$. Let $|S_1| = r_1$, and $|S_2| = r_2$. Further, we have that $C_m(T) = r_1 \cdot c_1 + r_2 \cdot c_2$. To construct G , it is necessary to identify a set of edges; we do so as follows.

Let $f : S_2 \rightarrow S_1$ be any one-to-one mapping. Let E be the set of pairs of the form $\{v, f(v)\}$ which are not edges of T . Let G be the graph with $V(G) = V(T)$, and $E(G) = E(T) \cup E$. If E is nonempty, E contains only edges that join vertices that m colored differently. Thus, m is a proper coloring of G , so that $C_m(G) = C_m(T) = r_1 \cdot c_1 + r_2 \cdot c_2$. To show that $\chi_{c_p}(G) = 2$, it is sufficient to show that m colors G with c_p at minimum cost.

Since f is a one-to-one mapping of S_2 into S_1 , the pairs $\{v, f(v)\}$ and the unmatched vertices in S_1 define a partition of $V(T)$ into $V_1, \dots, V_{r_1}$ such that $|V_i| = 2$ for $i = 1, \dots, r_2$, and $|V_i| = 1$ for $i = r_2 + 1, \dots, r_1$. Since each partition induces a complete subgraph of G , Theorem 11 gives us that the minimum cost of coloring G is $r_2 \cdot (c_1 + c_2) + (r_1 - r_2) \cdot c_1 = r_2 \cdot c_1 + r_2 \cdot c_2 + r_1 \cdot c_1 - r_2 \cdot c_1 = r_2 \cdot c_2 + r_1 \cdot c_1 = C_m(G)$. Since m is a 2-coloring of G , $\chi_{c_p}(G) = 2$. **QED**

Thus, while adding edges to any graph can never decrease the chromatic number of the graph, we have shown that adding edges to a tree can produce a

graph of smaller cost-chromatic number. As it turns out, this result holds as well for a general graph G with cost-chromatic number $n > \chi(G)$.

Theorem 12 Given a palette P of n colors with associated costs c_p , and a graph G with $\chi_{c_p}(G) = n > \chi(G)$, there exists a graph G' such that $V(G') = V(G)$, $E(G') \supseteq E(G)$, and $\chi_{c_p}(G') = \chi(G)$.

Proof Let $\chi(G) = t$. Let $m \in \mathcal{X}_p(G)$ be a minimum-cost t -coloring of G with $p_1, \dots, p_t$. Thus, m partitions $V(G)$ into t color classes, $S_i = \{v \in V(G) \mid m(v) = p_i\}$ for $i = 1, \dots, t$, and $|S_i| \geq |S_j|$ for $i \leq j \leq t$. Let $|S_i| = r_i$, so that we have $r_i \geq r_j$ for $i \leq j \leq t$, and we can write $S_i = \{v_{i,j} \mid 1 \leq j \leq r_i\}$. Partition $V(G)$ into subsets T_i , where T_i is the set of all vertices in G with second subscript i .

We construct the graph G' from G by adding edges as follows. For each set T_i , $i = 1, \dots, r_1$, add enough edges to G so that the subgraph of G induced on the vertices of T_i is complete. Since each T_i contains at most one vertex from any color class, the t -coloring m of G also properly colors G' . Furthermore, m colors each T_i clique at cost $c_1 + \dots + c_k$, where $k = |T_i|$. Since there are r_1 of these cliques, we can write

$$C_m(G) = \sum_{i=1}^{r_1} \sum_{j=1}^{|T_i|} c_j.$$

By Theorem 11, this is the minimum cost of coloring G' with c_p . Since m colors G' with $t = \chi(G)$ colors, we have $\chi_{c_p}(G') \leq \chi(G)$. Since $G \subseteq G'$, and m colors G' with t colors, it follows that $\chi(G') = \chi(G)$. Thus, we have $\chi_{c_p}(G') \geq \chi(G') = \chi(G)$, so we can conclude $\chi_{c_p}(G') = \chi(G)$. **QED**

The reader will notice that our Corollary 11c now follows directly from Theorem 12, with $t = 2$. However, we present the material in this sequence as this is the order of our discovery.

It should be evident now that the cost-chromatic number of a graph behaves in many ways differently than the chromatic number, and that it is easy to overextend the analogy between the two.

4.5 Dependence on the Cost Set

We close this chapter with an observation. The weakness in the analogy between the chromatic number and cost-chromatic number depends on the cost set. That is, with respect to certain cost sets, the cost-chromatic number behaves much like the chromatic number. In particular, if there is a cost in the set that is prohibitively expensive, no minimum-cost coloring will use it or any color more expensive, putting an upper bound on the cost-chromatic number that is smaller than the size of the palette. We establish this result formally here.

Theorem 13 Let P be a palette of n colors, and let G be a graph with $\chi(G) = s$,
for $s \leq n$. Then there exists a cost set $c_p = \{c_1, \dots, c_n\}$
such that $\chi_{c_p}(G) = s$.

Proof Let $t = |V(G)|$, and let $c_p = \{1, 2, \dots, s, t \cdot s + 1, t \cdot s + 2, \dots, t \cdot s + n - s\}$. We know that $\chi_{c_p}(G) \geq \chi(G) = s$. Let m be any s -coloring of G with c_p , and let k be any r -coloring for $r > s$. Then there exists $v \in V(G)$ such that $C_k(v) \geq t \cdot s + 1$. Thus, we have $C_m(G) < |V(G)| \cdot c_s = t \cdot s < t \cdot s + 1 = C_k(v) < C_k(G)$. Since any s -coloring colors G at lower cost than any r -coloring for $r > s$, we have $\chi_{c_p}(G) \leq s$. Thus, $\chi_{c_p}(G) = s$. **QED**

Chapter Five

Cost-Coloring Outerplanar Blocks and Maximal Planar Blocks

5.1 Definitions

We turn our attention now to another class of graph, the outerplanar blocks. We begin with two necessary definitions.

Definition A graph is *outerplanar* if it can be embedded in the plane so that every vertex lies on the boundary of the external region.

Definition An outerplanar graph of order at least three is *maximal outerplanar* if every interior region is a triangle.

Since a block is a graph with no cut-vertices, an outerplanar block is an outerplanar graph with no cut-vertices. For our purposes, we assume that the blocks we consider are embedded in the plane with all vertices on the border of the external region. We choose this class because the chromatic number of any outerplanar block is no more than three. (The chromatic number of some outerplanar blocks, for instance, the even cycles, is two.) We ask the same question that we did of trees. Given a palette of arbitrary size, do outerplanar blocks exist for which the minimum-cost coloring utilizes the entire palette?

As before, let the palette of n colors be $P = \{p_1, \dots, p_n\}$, and let the cost associated with color p_i be c_i for $i = 1, \dots, n$. Then denote the cost set by

$c_p = \{c_1, \dots, c_n\}$. Assume that the costs are distinct, positive, and rational, so that we can use Corollary 1a and restrict discussion to costs that are distinct positive integers. We do not require that the lowest cost be 1. Finally, assume that P is arranged so that c_p is in increasing order.

5.2 Building the Block

It is immediately helpful to have a way to identify particular pairs of vertices in an outerplanar block, so we make the following definition.

Definition Two vertices in an outerplanar block are *adjacent on the perimeter* if they are adjacent, and the edge that joins them separates an interior region of the block from the exterior region.

One property of outerplanar blocks is that inserting a path of arbitrary length between two vertices, u and v , that are adjacent on the perimeter produces another outerplanar block. Outerplanarity is also preserved by adding edges joining each vertex on the new path to u or joining each vertex on the new path to v . Here is an example of building a larger outerplanar block by inserting a path between two vertices adjacent on the perimeter of a given outerplanar block, and joining the vertices on the path with one of the two chosen vertices.

The outerplanar blocks B_i that we construct depend on the cost set in the same manner as did our tree $T_x^{(n)}$. While the construction procedure is similar, we take a different starting point for palettes of even and odd size.

We construct an outerplanar block B_i for $i = 2, \dots, n$, as follows. Let $B_2 = K_2$, and let $B_3 = K_3$. Construct B_{i+2} from B_i by choosing two vertices $v_1, v_2 \in V(B_i)$ that are adjacent on the perimeter of B_i . Insert a path with $m_i = 2 \cdot (c_n + 1) \cdot |V(B_i)|$ internal vertices between v_1 and v_2 , and construct a chord from each of the m_i new vertices in the path to v_1 . Now, find the vertex other than v_1 that is adjacent to v_2 on the perimeter of B_i , and call it v_3 . If there is no such vertex, use v_1 . Insert a path of length $m_i + 1$ between v_2 and v_3 , and construct a chord from each of the m_i new vertices in the path to v_2 . Continue inserting a path of length $m_i + 1$ and its chords between each pair of vertices adjacent on the perimeter of B_i until every vertex in $V(B_i)$ has m_i new vertices adjacent to it.

Notice that we have constructed a different type of block for palettes of even and odd size. Since for any odd size palette, we need to construct B_{2i+1} , the construction begins with $B_{2(1)+1} = B_3 = K_3$, and we sequentially build $B_{2(2)+1} = B_5$, $B_{2(3)+1} = B_7$, and so on until we have $B_{2i+1} = B_n$. If the palette has even size, we begin with $B_{2(1)} = B_2 = K_2$, and sequentially build $B_{2(2)} = B_4$, $B_{2(3)} = B_6$, and so on until we have $B_{2i} = B_n$. The result is that B_i is a very different block from B_{i+1} .

It would likely be helpful to see an example. The two drawings on the next page show versions of blocks B_5 and B_7 . The construction of both blocks began with $B_3 = K_3$. In the first, for convenience, we have assumed $m_3 = 5$. The construction of the second block also began with B_3 , used $m_3 = 5$ as with the first block, and for convenience used $m_5 = 7$. We emphasize that these values of m_3 and m_5 are deliberately chosen small for illustrative purposes.

Note that both K_2 and K_3 are maximal outerplanar blocks, and each step in the construction of B_n triangulated the interior of the block. Thus, B_n is a maximal outerplanar block.

5.3 Coloring the Block

We now describe a coloring M_n of B_n with c_p , where $n = 2t$ or $n = 2t+1$. Our coloring M_n will assign each color in P to at least one vertex in B_n , and we will eventually show that any other coloring of B_n that uses fewer colors has higher cost.

Our coloring M_n assigns the two least-expensive colors available, p_1 and p_2 , to the internal vertices in the paths that the construction added to B_{n-2} to create B_n , and to no other vertices in B_n . To the internal vertices in the paths that the construction added to B_{n-4} to create B_{n-2} , M_n alternately assigns the two least expensive colors not yet assigned to any vertex, namely, p_3 and p_4 . In general, for $i = 1, \dots, t-1$, M_n assigns colors p_{2i-1} and p_{2i} alternately to the internal vertices on the paths that the construction added to B_{n-2i} to create B_{n-2i+2} , and to no other vertices in B_n . Finally, if n is even, the construction of B_n began with $B_2 = K_2$, so M_n assigns colors p_{n-1} and p_n arbitrarily to the vertices of B_2 . If n is odd, the construction of B_n began with $B_3 = K_3$, so M_n assigns colors p_{n-2} , p_{n-1} , and p_n arbitrarily to the vertices of B_3 .

We have described a coloring M_n of B_n with c_p which uses every color in the palette. It remains to establish that any other coloring K of B_n with c_p which uses fewer colors has higher cost. We state our theorem here, with the proof to follow three lemmas.

Theorem 14 For any given palette with associated costs that are distinct, rational, and positive, there exists an outerplanar block such that any minimum-cost coloring of the block utilizes the entire palette.

We establish this result by induction on the number of steps in the construction of our block. Note that if n , the number of colors in the palette, is 1, 2, or 3, the appropriate outerplanar block will be K_n .

5.4 The Base Case Lemmas

We begin as before with a palette $P = \{p_1, \dots, p_n\}$ with associated costs $c_p = \{c_1, \dots, c_n\}$ that are distinct, rational, and positive, and use Corollary 1a to restrict discussion to costs that are distinct positive integers. We require that P is arranged so that the costs are in increasing order. Since any coloring of a graph G requires at least $\chi(G)$ colors, we choose as our base case two graphs which require more than $\chi(G)$ colors to obtain a minimum-cost coloring. Thus, our base case lemmas begin with the second step in our construction of B_n .

Lemma 14a Let $S \subseteq P$. For $j = 4$ or 5 , if $|S| \geq j$, $\chi_{c_S}(B_j) = j$.

Proof Let $S = \{p_1', p_2', \dots\} \subseteq P$ be such that $|S| \geq j$, for $j = 4$ or 5 , and be arranged so that the corresponding costs $c_S = \{c_1', c_2', \dots\} \subseteq c_p$ are in increasing order. Construct B_j . If $j = 4$, B_j consists of a copy of K_2 with two paths and some chords added. If $j = 5$, B_j consists of a copy of K_3 with three paths and some chords added. Our coloring M_j assigns p_1' and p_2' alternately to the internal vertices in the paths. For $j = 4$, M_j assigns p_3' and p_4' arbitrarily to the two adjacent vertices in the copy of K_2 . For $j = 5$, M_j assigns p_3' , p_4' , and p_5' arbitrarily to the three adjacent vertices in the copy of K_3 . Thus, for $j = 4$ or 5 , M_j

colors B_j with c_s using the j least-expensive colors in S . Now, for $j = 4$ or 5 , let $N_j \neq M_j$ be another coloring of B_j with c_s . To establish the lemma, it remains to show that

- i) if N_j uses more colors than M_j , then $C_{N_j}(B_j) > C_{M_j}(B_j)$, and
- ii) if N_j uses fewer colors than M_j , then $C_{N_j}(B_j) > C_{M_j}(B_j)$.

Part i) Assume that N_j uses more colors than M_j . It is necessary to assume for the moment that there are more colors available in S . Then N_j must use a color, call it p , at least as expensive as p_{j+1}' . Since M_j colors the paths alternately with the two least-expensive colors available, coloring a single vertex on one of the paths with p merely raises the cost of coloring the path, while creating no opportunity to reduce the cost elsewhere in B_j . The only way that an opportunity to reduce the cost would arise from using p on one of the paths is if p were assigned to half the internal vertices in the path, freeing up either p_1' or p_2' to be used in B_{j-2} . By the construction, however, half the number of internal vertices on the path is $(c_n + 1) \cdot |V(B_{j-2})|$, which exceeds the cost of coloring every vertex in B_{j-2} with the most expensive color in the palette, which in turn exceeds $C_{M_j}(B_{j-2})$. Coloring half of them with p instead of p_1' or p_2' generates an additional cost of at least one unit for each vertex colored p , raising the cost by more than $C_{M_j}(B_{j-2})$. Hence, N_j cannot make up the deficit. Thus, if N_j colors any vertex on any path with p , then $C_{N_j}(B_j) > C_{M_j}(B_j)$. In addition, coloring any vertex in B_{j-2} with p increases the cost over M_j , while offering no opportunity to recoup the deficit. We can thus conclude that if N_j uses more colors than M_j , then $C_{N_j}(B_j) > C_{M_j}(B_j)$.

Part ii) Assume that N_j uses fewer colors than M_j . Then N_j uses no more than $j - 1$ colors. If $j = 4$, $B_{j-2} = K_2$, so N_4 must assign two of its at most three different colors to B_2 . If $j = 5$, $B_{j-2} = K_3$, so N_5 must assign three of its at most four

different colors to B_3 . In either case, N_j must assign either p_1' or p_2' to one of the vertices in B_{j-2} , so that either p_1' or p_2' is unavailable to color any of the internal vertices on at least one of the paths. Thus, at least half of the internal vertices in one of the paths must be colored with a color at least as expensive as p_3' , producing an additional cost over M_j on that path of at least one cost unit for each vertex so colored. By the construction, half the number of internal vertices on any of the paths is $(c_n + 1) \cdot |V(B_{j-2})|$, which exceeds $C_{M_j}(B_{j-2})$, which in turn is greater than any savings N_j can generate coloring B_{j-2} at lower cost than M_j . We can thus conclude that if N_j uses fewer colors than M_j , then $C_{N_j}(B_j) > C_{M_j}(B_j)$.

We have established that for $j = 4$ or 5 , any minimum-cost coloring of B_j requires j colors, since any coloring that uses a different number of colors has a higher cost. Thus, we can conclude that $\chi_{c_p}(B_j) = j$ for $j = 4$ or 5 . **QED Lemma**

We continue by establishing that our coloring M_n colors B_n at minimum cost for $n = 4$ or 5 .

Lemma 14b Let $S \subseteq P$. For $j = 4$ or 5 , if $|S| \geq j$, then the coloring M_j colors B_j with c_S at minimum cost.

Proof By Lemma 14a, we have that for $j = 4$ or 5 , $\chi_{c_p}(B_j) = j$. Theorem 6 then gives us that any minimum-cost coloring of B_j with c_S uses the j least-expensive colors in S . Thus, to establish the lemma, it is sufficient to show that for any other j -coloring N_j of B_j with c_S , $C_{N_j}(B_j) \geq C_{M_j}(B_j)$.

Since M colors the paths added to B_{j-2} to construct B_j at minimum cost with the colors available, any savings that N_j might generate over M_j must occur on B_{j-2} . For $j = 4$ or 5 , the vertices in B_{j-2} are mutually adjacent, so N_j must assign different colors to each of them. If $j = 4$, $B_{j-2} = B_2 = K_2$, and M_j assigns

colors p_3' and p_4' to the two vertices. If $j = 5$, $B_{j-2} = B_3 = K_3$, and M_j assigns colors p_3' , p_4' , and p_5' to the three vertices. In either case, if N_j assigns the same colors as M_j to the vertices of B_{j-2} , and assigns p_1' and p_2' alternately to the internal vertices in the paths added to create B_j , then $C_{N_j}(B_j) = C_{M_j}(B_j)$. On the other hand, if N_j assigns either p_1' or p_2' to at least one vertex in B_{j-2} , then either p_1' or p_2' is unavailable to color any internal vertices on at least one of the paths. By the argument in part ii) of Lemma 14a, this implies that $C_{N_j}(B_j) > C_{M_j}(B_j)$. Thus, for $j = 4$ or 5 , any j -coloring N_j of B_j with c_S , $C_{N_j}(B_j) \geq C_{M_j}(B_j)$. Hence, M_j colors B_j with c_S at minimum cost. **QED Lemma**

We have now established that for $j = 4$ or 5 and any palette that contains at least j colors with distinct, rational, and positive costs, the cost-chromatic number of the outerplanar block B_j is j . Furthermore, we now know that for $j = 4$ or 5 , and for any subset S of our palette P such that $|S| \geq j$, our coloring M_j colors B_j with c_S at minimum cost.

5.5 The Induction Lemma

We continue with our inductive argument.

Lemma 14c For $n = 2t$ or $n = 2t + 1$, and $t \geq 2$, let $P = \{p_1, \dots, p_n\}$ be a palette with associated costs $c_P = \{c_1, \dots, c_n\}$, and let $S \subseteq P$. For $i = 2, \dots, t$, and for $j = 2i$ or $2i+1$, if $|S| \geq j$ then $\chi_{c_S}(B_j) = j$. Furthermore, M_j colors B_j with c_S at minimum cost.

Proof By induction on i . The lemma is true for $i = 2$ by Lemmas 14a and 14b. As our induction hypothesis, assume that the lemma is true for $i = 2, \dots, k$, where $k < t$. Now, let $i = k + 1$, and let $j = 2i$ or $2i + 1$, so that $j = 2k + 2$ or $2k + 3$.

Let $S = \{p_1', p_2', \dots\} \subseteq P$ be arranged so that the corresponding costs, $c_s = \{c_1', c_2', \dots\}$, are in increasing order, and be such that $|S| \geq j$. Let M_j be our coloring of B_j with c_s . Then M_j colors the internal vertices in each of the paths that the construction added to B_{j-2} to create B_j alternately with p_1' and p_2' , leaving a palette $\{p_3', p_4', \dots\}$ of size at least $j - 2$ to color B_{j-2} . Now, consider some other coloring N_j of B_j with c_s . To show that $C_{M_j}(B_j) \leq C_{N_j}(B_j)$, we assume the opposite, that is, we assume that $C_{N_j}(B_j) < C_{M_j}(B_j)$.

Since M_j colors the paths that the construction added to B_{j-2} to produce B_j at minimum cost with the colors available, N_j cannot gain any advantage over M_j on those paths by coloring them differently. Thus, N_j must color B_{j-2} at lower cost than M_j . By the induction hypothesis, M_j colors B_{j-2} at minimum cost with the palette $S \setminus \{p_1', p_2'\}$. Thus, N_j cannot produce any savings over M_j by coloring B_{j-2} with $S \setminus \{p_1', p_2'\}$. Therefore, N_j must assign either p_1' or p_2' to at least one vertex $v \in V(B_{j-2})$. By the construction of B_j , there exists a path Q in B_j such that every internal vertex on Q is adjacent to v . The coloring M_j colors half of the internal vertices in $V(Q)$ with the same color that N_j assigned to v , so N_j must assign to those vertices some other color, at least as expensive as p_3' , raising the cost of coloring the path by at least one cost unit for each of the vertices so colored. By the construction, half the number of internal vertices on the path Q is $(c_n + 1) \cdot |V(B_{j-2})|$, which exceeds the cost of coloring every vertex in B_{j-2} with the most-expensive color in P , which exceeds $C_{M_j}(B_{j-2})$, which is, in turn, larger than any possible savings N_j could generate coloring B_{j-2} at lower cost than M_j . Thus, $C_{M_j}(B_j) \leq C_{N_j}(B_j)$. Since N_j was an arbitrary coloring of B_j with c_s , we can conclude that M_j colors B_j with c_s at minimum cost.

To establish that $\chi_{c_s}(B_j) = j$, it remains to show that any coloring of B_j with c_s that uses fewer than j colors has higher cost. Let N_j be such a coloring of B_j with c_s .

Assume for the moment that N_j assigns a color $p \notin \{p_1', p_2'\}$ to an internal vertex on one of the paths that the construction added to B_{j-2} to produce B_j . Now, M_j colors the paths alternately with p_1' and p_2' , the two lowest-cost colors available. Thus, N_j increased the cost of coloring the path, and no opportunity to reduce the overall cost is created in B_j , unless the color p were assigned to half the internal vertices in the path, freeing up either p_1' or p_2' to be used in B_{j-2} . By the construction, however, half the number of internal vertices in the path is $(c_n + 1) \cdot |V(B_{j-2})|$. Coloring half the internal vertices on the path with p instead of p_1' or p_2' generates an additional cost of at least one unit for each vertex colored p , raising the cost by more than the cost of assigning the most-expensive color in P to every vertex in B_{j-2} , which exceeds the entire cost that M_j generates coloring B_{j-2} . Hence, N_j cannot make up the deficit by coloring B_{j-2} at lower cost than M_j . Thus, if N_j assigns a color $p \notin \{p_1', p_2'\}$ to a vertex on one of the paths, then $C_{N_j}(B_j) > C_{M_j}(B_j)$.

We can now assume that N_j colors all the internal vertices on the paths added to B_{j-2} to construct B_j alternately with p_1' and p_2' , leaving fewer than $j - 2$ colors available to color B_{j-2} . But by the induction hypothesis, $\chi_{c_s}(B_{j-2}) = j - 2$, so we know that if N_j uses fewer colors than M_j , then $C_{N_j}(B_j) > C_{M_j}(B_j)$.

We have shown that M_j colors B_j with c_s at minimum cost, and that M_j uses j colors do so. We have also shown that any coloring that uses fewer

colors to color B_j has higher cost. We can thus conclude by induction that the lemma holds for $i = 2, \dots, t$. **QED Lemma**

5.6 The Cost-Chromatic Number of an Outerplanar Block

We are now ready to restate and prove our main theorem about cost-coloring outerplanar blocks.

Theorem 14 For any given palette with associated costs that are distinct, rational, and positive, there exists an outerplanar block such that any minimum-cost coloring of the block utilizes the entire palette.

Proof Let $P = \{p_1, \dots, p_n\}$ be a palette with associated costs $c_p = \{c_1, \dots, c_n\}$ that are distinct, positive, and rational. By Corollary 1a, c_p is equivalent to a set of costs that are distinct, positive, and integral. Thus, for purposes of this argument, we can assume that each cost is an integer.

Let $|P| = |c_p| = n$, for $n = 2t$ or $2t + 1$. If $n = 1, 2$, or 3 , then the required outerplanar block is K_n . If $t \geq 2$, construct the outerplanar block B_n . Our proof proceeds by induction on t .

As the base case, let $t = 2$. Lemma 14a established that for $j = 4$ or 5 , that is, for $j = 2t$ or $2t + 1$ and for any $S \subseteq P$ such that $|S| \geq j$, $\chi_{c_S}(B_j) = j$. Thus, the theorem holds in the base case.

As the induction hypothesis, assume that $\chi_{c_S}(B_j) = j$ for $j = 2i$ or $2i + 1$ and $i = 1, \dots, k$, where $k < t$. Let $i = k + 1$. By Lemma 14c, for any $S \subseteq P$ such that $|S| \geq j$, we have $\chi_{c_S}(B_j) = j$. Hence, $\chi_{c_P}(B_n) = n$ by induction. We can thus conclude that for any palette with associated costs that are distinct, rational, and

positive, there exists an outerplanar block such that any minimum-cost coloring of the block utilizes the entire palette. **QED**

As we mentioned for our similar result concerning trees, this result is also curious because while the chromatic number of any outerplanar block is no more than three, there exist outerplanar blocks for which the cost-chromatic number is arbitrarily large. As with the tree $T_x^{(n)}$, our block B_n was carefully constructed, and is quite large. The objective of our construction was to produce a block with the property that by using small amounts of expensive colors, we are able to free up large numbers of vertices to be colored with the two least-expensive colors.

5.7 Expanding the Block

In this section, we establish another result about the lack of monotonicity of the cost-chromatic number with respect to subgraph inclusion. We will begin with an arbitrary outerplanar block B , and construct an outerplanar graph G that contains B and that has cost-chromatic number three. We note here that our construction preserves outerplanarity, but not the property of being a block.

As always, we begin with P , a palette of n colors, $n \geq 4$, with associated costs c_p . Let B be a maximal outerplanar block with $\chi_{c_p}(B) = n$. Since B is a maximal outerplanar block, we have that $\chi(B) = 3$, and that every region is a triangle. Thus, coloring any two adjacent vertices determines an entire three-coloring, so that any three-coloring of B is a partition of $V(B)$ into three independent sets, the color classes S_1 , S_2 , and S_3 . Without loss of generality, assume that the three classes are arranged in nonincreasing order of size. Let m be the three-coloring of B defined by $m(v) = p_i$ for all $v \in S_i$ for $i = 1, 2, 3$. Hence, m is a minimum-cost three-coloring of B .

We construct our outerplanar graph G from B in two stages. First, we use the coloring m to help us build an outerplanar block B' that contains B , in a manner similar to our construction of B_n from B_{n-2} . We then construct our outerplanar graph G by attaching end vertices to certain vertices in B' .

Choose a vertex $u \in S_3$, and a vertex v that is adjacent to u on the perimeter of B . Since $u \in S_3$, we have $m(u) = p_3$, and since u is adjacent to v , we have $m(v) \neq p_3$, so that $v \notin S_3$. If $v \in S_1$, let $t = 2 \cdot (c_n + 1) \cdot |V(B)|$. If $v \in S_2$, let $t = 2 \cdot (c_n + 1) \cdot |V(B)| + 1$. Now, extend a path Q with $t + 1$ internal vertices between u and v , so that $u-Q-v-u$ forms a cycle. Insert chords of the cycle so that u is adjacent to every vertex on Q . Perform this operation for each vertex $u \in S_3$, and call the resulting graph B' . Since our operation inserted paths between vertices adjacent on the perimeter of an outerplanar block, B' is itself an outerplanar block. Here is a diagram illustrating the construction of B' .

Now, extend the coloring m to the new paths of B' by starting at the vertex u colored p_3 , and coloring internal vertices on the path alternately with p_1 and p_2 , beginning with p_1 . We chose the number of vertices in the path carefully. If v , the last vertex in the cycle, the one adjacent to u on the perimeter of B , is colored p_1 , then Q has an even number of vertices, so that the vertex on Q that

is adjacent to v is colored with p_2 . If $m(v) = p_2$, the path has an odd number of internal vertices, so that the vertex on Q that is adjacent to v is colored with p_1 . Thus, the path is in fact colorable as we describe, and m is a proper coloring of B' . Let p_1 be white, p_2 light gray, and p_3 dark gray. This diagram illustrates the coloring of the path Q when $m(v) = p_1$.

This diagram illustrates the coloring of Q when $m(v) = p_2$.

To construct G from B' is less involved. To each vertex $v \in V(B')$ such that $m(v) = p_2$, attach $(c_n + 1) \cdot |V(B')|$ end vertices. Call the resulting graph G . Extend the coloring m to G by assigning p_1 to each of the new end vertices. The next diagram illustrates the construction of G . The coloring m assigns p_1 , or white, to all the new vertices.

It is easy to see that G is no longer a block; every vertex in S_2 is a cut-vertex. Deleting any one of them leaves the $(c_n + 1) \cdot |V(B')|$ adjacent end vertices disconnected from the rest of G . G is, however, outerplanar, since every vertex is still on the boundary of the external region, and G clearly contains B as a subgraph.

We are ready to establish our next lemma.

Lemma 15a $\chi_{c_p}(G) = 3$.

Proof Since $\chi(G) = 3$, and m is a three-coloring of G , it is sufficient to show that m colors G at minimum cost. Recall that m partitions $V(G)$ into color classes, $S_i = \{v \in V(G) \mid m(v) = p_i\}$ for $i = 1, 2, 3$. By way of contradiction, assume that m does not color G with c_p at minimum cost. Then there exists a coloring $k \in \mathcal{X}_p(G)$ such that $C_k(G) < C_m(G)$. Thus, there exists a vertex $v \in V(G)$ such that $C_k(v) < C_m(v)$.

Since m partitions $V(G)$, we have that either $v \in S_1$, $v \in S_2$, or $v \in S_3$. We consider each case separately.

Case i) $v \in S_1$. Then $m(v) = p_1$, and $C_m(v) = c_1$. Then $C_k(v) < C_m(v)$ gives us that $C_k(v) < c_1$, a contradiction, since c_1 is the lowest cost in the cost set.

Case ii) $v \in S_2$. Then $m(v) = p_2$, so $C_k(v) < C_m(v)$ gives us $k(v) = p_1$, giving a savings of $c_2 - c_1$. By the construction of G , v is adjacent to $(c_n + 1) \cdot |V(B')|$ end vertices that m assigns color p_1 . Thus, k must assign a color at least as expensive as p_2 to those vertices, generating an additional cost of at least $(c_2 - c_1) \cdot (c_n + 1) \cdot |V(B')| = (c_2 - c_1) \cdot c_n \cdot |V(B')| + (c_2 - c_1) \cdot |V(B')| \geq c_n \cdot |V(B')| + (c_2 - c_1)$. The second term exactly offsets the savings to k from coloring v at lower cost than m , and the first term exceeds the total cost that m generates coloring B' . Thus, k cannot recoup this expense by coloring B' any more efficiently than m . Further, m assigns p_1 to every vertex in G that is not in B' , so there is no opportunity to recoup the expense in G . Thus we have $C_m(G) < C_k(G)$, a contradiction.

Case iii) $v \in S_3$. Then $m(v) = p_3$, so $C_k(v) < C_m(v)$ gives us $k(v) = p_1$ or $k(v) = p_2$, generating a savings for k of at most $c_3 - c_1$. By the construction of G , v is adjacent to at least $2 \cdot (c_n + 1) \cdot |V(B)|$ vertices on a path Q . Since m assigns color p_1 to at least one endpoint of Q , the path has at least as many vertices in S_1 as in S_2 . Since $k(v) = p_1$ or p_2 , k must color Q alternately with colors at least as expensive as p_1 and p_3 , generating an additional expense of at least $(c_3 - c_2) \cdot (c_n + 1) \cdot |V(B)| = (c_3 - c_2) \cdot c_n \cdot |V(B)| + (c_3 - c_2) \geq c_n \cdot |V(B)| + (c_3 - c_2)$. As before, the second term exactly offsets the savings to k from coloring v at lower cost than m , and the first term exceeds the total cost that m generates coloring B . Thus, k cannot recoup this expense by coloring B any more efficiently than m . Further, since m assigns p_1 to at least one endpoint of each path Q that was attached to B to construct B' , k cannot color any of the paths Q at lower cost than m . Finally, m assigns p_1 to every vertex in G that is not in B' , so there is no

opportunity to recoup the expense in G . Thus we have $C_m(G) < C_k(G)$, a contradiction.

Since every case led to a contradiction, we can conclude that m does in fact color G with c_p at minimum cost, and that $\chi_{c_p}(G) = 3$. **QED Lemma**

Since we have already proved it, we can state the result of this section as a theorem.

Theorem 15 Let P be a palette of n colors, $n \geq 3$, with associated costs c_p , and let B be any outerplanar block with $\chi_{c_p}(B) = n$. Then there exists an outerplanar graph $G \supseteq B$ such that $\chi_{c_p}(G) = 3$.

5.8 The Cost-Chromatic Number of a Maximal Planar Block

As we mentioned earlier, we are interested in addressing some issues in VLSI circuit design. Since planar blocks often occur in mathematical formulations of problems in this field, we wish to explore questions concerning cost-coloring these graphs. The motivation for this section comes from the fact that the particular planar blocks we studied in Section 3.11 have chromatic number three, while a planar block can have chromatic number four. Furthermore, we know by Theorem 10 that it is possible to reduce the cost-chromatic number of a graph by adding edges to the graph. In this section, we extend our construction of the maximal outerplanar block with large cost-chromatic number to build a maximal planar block with arbitrarily large cost-chromatic number. Here we construct a maximal planar block that will require every color in an arbitrary palette except one to produce a minimum-cost coloring.

Begin with a palette P of n colors, $n \geq 4$ and construct the block B_{n-2} . Since B_{n-2} is maximal outerplanar, it can be drawn so that every vertex is on a circle, every edge is a chord, and every interior region is a triangle. Now, construct the graph G by adding a vertex v , and edges joining v to every vertex in B_{n-2} , as in this diagram.

As before, it is necessary to establish that the graph we constructed has the property we need.

Lemma 16a G is a maximal planar block.

Proof Since we have a plane drawing of G above, it is sufficient to show that every region is a triangle. The external region is a triangle bounded by v and two vertices adjacent on the perimeter of B_{n-2} . Since B_{n-2} is a maximal outerplanar block, all its interior regions are triangles. Finally, any two vertices adjacent on the perimeter of B_{n-2} are mutually adjacent with v , completing the triangulation. **QED Lemma**

We can now establish our key lemma.

Lemma 16b $\chi_{c_p}(G) \geq n - 1$.

Proof Suppose the lemma is false, so that $\chi_{c_p}(G) < n - 1$. Then there exists a t -coloring $k \in \mathcal{X}_p(G)$ such that k colors G with c_p at minimum cost for some $t < n - 1$. Since k must assign some color to v , we have $k(v) = p_j$ for some j , $1 \leq j \leq t$, leaving colors in $S = \{p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n\} \subset P$ available to assign to vertices in B_{n-2} .

Since $|S| = n - 1 \geq n - 2$, Lemmas 14a and 14c give $\chi_{c_s}(B_{n-2}) = n - 2$. Construct a coloring m of G with c_p such that $m(v) = p_j$, and m colors B_{n-2} with c_s at minimum cost. Thus, m assigns $n - 2$ colors to B_{n-2} , and one additional color to v , so that m colors G with $n - 1$ colors. Since k assigns fewer than $n - 2$ colors from S to the vertices of B_{n-2} , we have that $C_k(B_{n-2}) > C_m(B_{n-2})$. Since $m(v) = k(v)$, we have $C_m(v) = C_k(v)$. Since $G - v = B_{n-2}$, it follows that

$$\begin{aligned} C_m(G) &= C_m(G - v) + C_m(v) \\ &= C_m(B_{n-2}) + C_m(v) \\ &< C_k(B_{n-2}) + C_k(v) \\ &= C_k(G - v) + C_k(v) \\ &= C_k(G). \end{aligned}$$

This contradicts that k is a minimum-cost coloring of G . We can thus conclude that the lemma holds, that is, $\chi_{c_p}(G) \geq n - 1$. **QED Lemma**

Since we began with an arbitrary palette of n colors, and constructed G , a maximal planar block with cost-chromatic number at least $n - 1$, we have established the key steps in the proof of the following theorem.

Theorem 16 For any given palette of n colors with associated costs that are distinct, rational, and positive, there exists a maximal planar block such that any minimum-cost coloring of the block utilizes at least $n - 1$ colors.

Proof Let P be any palette with associated costs c_p that are distinct, positive, and rational. By Corollary 1a, c_p is equivalent to a set of costs that are distinct, positive, and integral, with $c_1 = 1$. Thus, for purposes of this argument, we can assume that each c_i is an integer. Construct the maximal outerplanar block B_{n-2} , and construct graph G by adding one vertex v that is adjacent to every vertex in B_{n-2} . Then by Lemma 16a, G is a maximal planar block. By Lemma 16b, $\chi_{c_p}(G) \geq n - 1$. From the definition of cost-chromatic number, any minimum-cost coloring of the block utilizes at least $n - 1$ colors. **QED**

This maximal planar block with arbitrarily large cost-chromatic number thus fills a gap left by Theorem 5 and Theorem 10.

Chapter Six

Miscellany and Conclusion

As we explored this topic, we established some results that didn't appear immediately applicable to our goals, yet may be helpful for future investigators. We include some of them here.

6.1 The Maximum Independent Set Method

When we introduced this method in Chapter Three, we alluded to its intuitive appeal. Indeed, it seems a sensible way to approach a given cost-coloring problem. In our experience, exploring this method often gives insight into a problem, so the method is worthwhile as a starting point. In particular, the next theorem establishes that, for nontrivial graphs, if the MIS method colors the graph with two colors, then the cost-chromatic number is 2, regardless of the costs associated with any color in the palette.

Theorem 17 Let P be a palette of n colors, $n \geq 2$, with associated costs c_p , and let G be a nontrivial graph. If the MIS method colors G with p_1 and p_2 , then $\chi_{c_p}(G) = 2$.

Proof Let $m \in \mathcal{X}_p(G)$ be the MIS method coloring of G with c_p . Assume that m colors G with p_1 and p_2 . Since G is nontrivial and m colors G with two colors, $\chi(G) = 2$. Let $k \in \mathcal{X}_p(G)$ be any minimum-cost r -coloring of G with c_p , for $2 < r < n$. By Theorem 6, $k : V(G) \rightarrow \{p_1, \dots, p_r\}$, that is, k assigns the r least-expensive colors in P .

By way of contradiction, assume that $C_k(G) < C_m(G)$. Since $m : V(G) \rightarrow \{p_1, p_2\}$, m partitions $V(G)$ into two color classes, S_1 and S_2 , where $S_1 = \{v \in V(G) \mid m(v) = p_1\}$, $S_2 = \{v \in V(G) \mid m(v) = p_2\}$, S_1 is a maximum independent set in G , and S_2 is $G \setminus S_1$. Since k colors G with more than two colors, we can consider that k partitions $V(G)$ into three sets, T_1 , T_2 , and T_3 , where $T_1 = \{v \in V(G) \mid k(v) = p_1\}$, $T_2 = \{v \in V(G) \mid k(v) = p_2\}$, and $T_3 = \{v \in V(G) \mid k(v) = p_t \text{ for } t = 3, \dots, r\}$. Thus, there exists a vertex $v \in V(T_3)$ such that $C_k(v) > C_m(v)$. Since k cannot color any vertex in S_1 at lower cost than m , k must assign p_1 to more vertices than m does. Thus, $|T_1| > |S_1|$, contradicting that S_1 is a maximum independent set in G . Hence, $C_k(G) \geq C_m(G)$. Since k was an arbitrary r -coloring for $r > 2$, we can conclude that $\chi_{c_p}(G) = 2$. **QED**

In the course of our investigation, we have recognized that although using the MIS method to find the cost-chromatic number has serious shortcomings, the method still has its uses. For instance, it may be true that Theorem 17 holds for graphs more general than those with chromatic number two. That is, if the MIS method assigns $\chi(G)$ colors to an arbitrary graph G , then $\chi_{c_p}(G) = \chi(G)$.

Furthermore, it is also possible that the MIS method may provide an upper bound for $\chi_{c_p}(G)$. That is, if the MIS method assigns s colors to an arbitrary graph G , then $\chi_{c_p}(G) \leq s$.

Since $\chi(G)$ is a lower bound for $\chi_{c_p}(G)$, the first possibility follows from the second whenever $s = \chi(G)$. Theorem 17 established this for $\chi(G) = 2$.

Finally, there is one more field of inquiry that suggested itself to us as we studied this topic. The arguments in this thesis have often been made complex so that using a pathological cost set does not invalidate them. We expect that

there are many interesting and useful results to be discovered by investigating cost-coloring with a more well-behaved cost set such as $c_p = \{1, \dots, n\}$.

6.2 A Better Upper Bound?

Every graph of large cost-chromatic number that we constructed had vertices of extraordinarily large degree. Thus, the Brooks-type upper bounds on the cost-chromatic number, dependent on the maximum degree of the graph, do not in general give good estimates of the cost-chromatic number of our graphs. A central principle in our constructions was that to realize cost savings by using a more expensive color, we needed to create graphs with vertices of a degree that exceeded the cost of the more expensive color. We suspect that this is a necessary condition. If so, then we can place a more restrictive upper bound on the cost-chromatic number, as we conjecture here.

Let G be a graph with chromatic number $\chi(G)$ and maximum degree $\Delta(G)$. Let P be a palette of n colors with associated costs c_p that are distinct positive integers, with $c_1 = 1$.

Conjecture If $\Delta(G) < c_i$ for some i , $\chi(G) \leq i \leq n$, then $\chi_{c_p}(G) \leq i$.

6.3 The Cost-Chromatic Number of Common Graphs

In the corollaries to Theorem 11, we saw that the cost-chromatic number of any complete graph on t vertices is t , and that the cost-chromatic number of any complete r -partite graph is r . These results are independent of the cost set. Here we establish the corresponding results for cycles and paths.

Let P be a palette of n colors with associated costs c_p that are positive.

Theorem 18 If C is a cycle, then $\chi_{c_p}(C) = \chi(C)$.

Proof If C is an odd cycle, $\chi(C) = 3$, and $\Delta(C) = 2$. We have $\chi_{c_p}(C) \geq 3$, and Theorem 8 gives us that $\chi_{c_p}(C) \leq 2 + 1 = 3$. Hence, $\chi_{c_p}(C) = \chi(C) = 3$. If C is an even cycle, $\chi(C) = 2$. Furthermore, the maximum independent set method colors C with two colors. By Theorem 17, $\chi_{c_p}(C) = 2$. **QED**

We conclude our computations by establishing the cost-chromatic number of paths.

Theorem 19 If Q is a nontrivial path, then $\chi_{c_p}(Q) = 2$.

Proof Since Q is a nontrivial path, $\chi(Q) = 2$. Furthermore, the maximum independent set method colors Q with two colors. By Theorem 17, $\chi_{c_p}(Q) = 2$. **QED**

For each of the common graphs we examined, $\chi_{c_p}(G) = \chi(G)$ for every cost set of size at least three. These results suggest another question, that is, how to classify those graphs for which $\chi_{c_p}(G) = \chi(G)$, given a sufficiently large cost set.

6.4 In Closing

Cost-coloring is, as far as we have been able to determine, a new topic in graph theory. In the course of writing this thesis, many lines of inquiry have suggested themselves to us; we enjoyed the luxury of choosing the lines to pursue, and of changing course when another line looked more promising. We established these results illustrating how the cost-chromatic number behaves less predictably than the chromatic number: that the cost-chromatic number is a property of a graph with respect to a particular cost set; that the intuitively-

appealing Maximum Independent Set method of cost-coloring has serious shortcomings, but is still useful; that there exist graphs with small chromatic number and arbitrarily large cost-chromatic number; and that the cost-chromatic number is not monotonic with subgraph inclusion, not even with graphs of the same order.

This thesis raises many more questions than it answers. The seemingly simple extension of graph coloring to cost-coloring has opened a surprisingly rich topic, with subtle interplay between the structure of graphs and the sequence of costs. In particular, there is much to discover about the uses and limits of the Maximum Independent Set method, about cost-coloring with less-general cost sets such as consecutive positive integers, about classifying graphs according to their cost-chromatic number, and, of course, about applications to VLSI circuit design and other practical problems. We leave these questions for another day.

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