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Chapter 27: Bloch Sphere and Single-Qubit Arbitrary Unitary Gate

Hiu Yung Wong
San Jose State University, hiuyung.wong@sjsu.edu

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Chapter 27
Bloch Sphere and Single-Qubit Arbitrary Unitary Gate

27.1 Learning Outcomes

Able to describe how to map a qubit state to the surface of the Bloch sphere; able to perform rotation on the Bloch sphere for a given set of Euler angles; be aware of the correct and incorrect relationship between the qubit space and the real 3D space; able to construct arbitrary unitary rotation using the $\boldsymbol{U_{\theta,\phi,\lambda}}$ gate.

27.2 Bloch Sphere

Very often the **Bloch Sphere** is taught at the beginning of a quantum computing class. However, it is not necessary to understand the Bloch Sphere first to do quantum computing. But Bloch Sphere is a very useful tool (but can be confusing) for us to understand the operation of quantum gates. That is why we introduce until almost the end. If we use it correctly, we can use it to help us construct quantum gates and understand the underlying physics.

We are all familiar with the concept that a single qubit resides in the $\mathbb{C}^2$ space. This is a two-dimensional complex space. It means that it has two basis vectors, $|0\rangle$ and $|1\rangle$, which are *orthonormal* to each other. Any qubit state is a vector in this space formed by a linear combination of $|0\rangle$ and $|1\rangle$ with *complex coefficients*. We have been using a real 2D space to illustrate the properties of a qubit state vector, but we emphasize that this is just an illustration and **the $\mathbb{C}^2$ space is NOT the real 2D space that we can feel** (e.g. Fig. 5.1).

How many **degrees of freedom (DOF)** does a qubit state have? It means how many real numbers do we need to specify in order to fix a qubit. We know that any single-qubit state $|\Psi\rangle$ can be expressed as $|\Psi\rangle = \alpha |0\rangle + \beta |1\rangle$, where α and β are complex numbers. Each of the α and β is determined by two real numbers.

H. Y. Wong, *Introduction to Quantum Computing*,
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For example, $\alpha = |\alpha|e^{i\delta_\alpha}$ and $\beta = |\beta|e^{i\delta_\beta}$, where $|\alpha|$, δ_α, $|\beta|$, and δ_β are the real numbers. Therefore, it *looks like* it has 4 DOFs.

However, for any physical qubit, it has to be normalized. Therefore, $|\alpha|^2+|\beta|^2 = 1$. This reduces the DOF by 1 because if I specify $|\alpha|$, $|\beta|$ is specified at the same time due to this normalization equation. This also means that I can determine $|\alpha|$ and $|\beta|$ using 1 single real number parameter as long as I make the vector normalized. For example, I can set $|\alpha| = \cos\frac{\theta}{2}$ and $|\beta| = \sin\frac{\theta}{2}$ as $\cos\frac{\theta}{2}^2 + \sin\frac{\theta}{2}^2 = 1$ will help satisfy the normalization criteria. So, now, the amplitudes of α and β can be described by a single real parameter θ, and the DOF of the qubit state is 3.

There is also another thing that can help us further reduce the DOF. The global phase of a qubit does not have any physical meaning (this was discussed after Eq. (6.12)). For example,

$$
\begin{aligned}
|\Psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\
&= |\alpha|e^{i\delta_\alpha}\,|0\rangle + |\beta|e^{i\delta_\beta}\,|1\rangle \\
&= \cos\frac{\theta}{2}e^{i\delta_\alpha}\,|0\rangle + \sin\frac{\theta}{2}e^{i\delta_\beta}\,|1\rangle \\
&= e^{i(\delta_\alpha+\delta_\beta)/2}\left(\cos\frac{\theta}{2}e^{i(\delta_\alpha-\delta_\beta)/2}\,|0\rangle + \sin\frac{\theta}{2}e^{i(-\delta_\alpha+\delta_\beta)/2}\,|1\rangle\right) \\
&= e^{i(\delta_\alpha+\delta_\beta)/2}\left(\cos\frac{\theta}{2}e^{-i\phi/2}\,|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}\,|1\rangle\right) \\
&= e^{i\gamma}\left(\cos\frac{\theta}{2}e^{-i\phi/2}\,|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}\,|1\rangle\right)
\end{aligned}
\tag{27.1}
$$

Here, we factorize a **global phase** $(\delta_\alpha + \delta_\beta)/2$ and call it γ. And we also define a new parameter $\phi = \delta_\beta - \delta_\alpha$. So we still have 3 DOFs characterized by θ, ϕ, and γ. The global phase means the phase shared by both $|0\rangle$ and $|1\rangle$. It has no physical significant because the physical properties for this single qubit are all computed by involving the *bra* and *ket* versions of the vector. For example, the expectation value of a matrix, $\boldsymbol{M}$, is $\langle\Psi|\,\boldsymbol{M}\,|\Psi\rangle$. We can take $e^{i\gamma}$ out from $|\Psi\rangle$ to be $e^{i\gamma}$ and take $e^{i\gamma}$ out from $\langle\Psi|$ to be $e^{-i\gamma}$ (note that we need to take the complex conjugate of the coefficient when it is a *bra* version, see Eq. (5.7)). $e^{i\gamma}$ and $e^{-i\gamma}$ multiply together to be 1. Therefore, the global phase factor does not affect the physical results. We can ignore γ when we describe a single qubit. As a result, we only need two real parameters, θ and ϕ, to describe a qubit. *Without losing its physics, a single qubit can then be completely described as*

$$
|\Psi\rangle = \cos\frac{\theta}{2}e^{-i\phi/2}\,|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}\,|1\rangle \tag{27.2}
$$

Since it has only two parameters, we can describe it on a real 2D plane. But the 2D plane is too much for it (it can describe also vectors with non-unit lengths) and

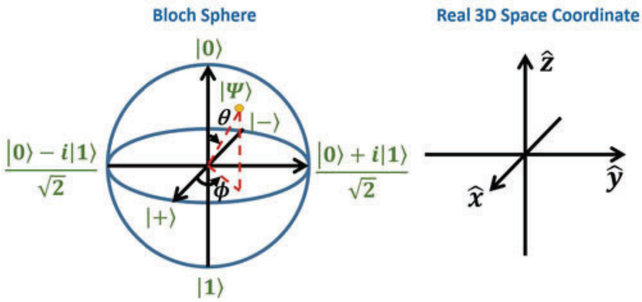

Fig. 27.1 The Bloch sphere (left) and its relationship to the 3-D real space (right)

also does not give a lot of insights. We can also describe it on the *surface* of a unit sphere (sphere with unit length), and this is called the **Bloch Sphere**. Figure 27.1 shows the Bloch sphere and its relationship to the real 3D space we live in. The *surface* of the Bloch sphere can be mapped to the space of a qubit in Eq. (27.2). We may say that we **embed/map** the qubit space in our real 3D space. It is very important to understand that *we only embed the qubit space to our real 3D space, and it does not mean that the qubit state is in our real 3D space*. This is just like we draw a map on 2D paper. First, San Francisco is not on your paper although you see it on the map. Second, the Earth's surface is the surface of a sphere, but you map it to the 2D plane on the paper. However, there are many advantages in describing the qubit using the Bloch sphere. First, it allows us to "visualize" how a qubit state evolves. Second, some of the physical properties of the qubit are linked to the Bloch sphere orientation in the real 3D space.

In Fig. 27.1, it shows the location of state $|\Psi\rangle$ on the Bloch sphere when it has the parameters θ and ϕ. We see that if we put the Bloch sphere in the 3D orientation shown in Fig. 27.1, the θ and ϕ are just the **polar angle** and **azimuthal angle** of a spherical coordinate, respectively. Note again, we can *embed* the surface in our 3D space in any way, but embedding in this way gives us the most intuition.

Let us try to understand how some of the important qubit states are mapped to the Bloch sphere.

Example 27.1 To which qubit states do the extrema on the Bloch sphere correspond?

We will only discuss 3 of them here. You will try the rest in the problems. Let us consider the "North Pole." It has $\theta = 0$ and $\phi = 0$. Therefore,

$$\begin{aligned} \left|\Psi_{\theta=0,\phi=0}\right\rangle &= \cos\frac{\theta}{2}e^{-i\phi/2}\,|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}\,|1\rangle \\ &= \cos\frac{0}{2}e^{-i0/2}\,|0\rangle + \sin\frac{0}{2}e^{i0/2}\,|1\rangle \\ &= |0\rangle \end{aligned} \tag{27.3}$$

What if we choose ϕ to be non-zero? It is still the "North Pole" as long as $\theta = 0$. Then we will get $e^{-i\phi/2}|0\rangle$. However, since $e^{-i\phi/2}$ is a global phase (as the coefficient for $|1\rangle$ is 0), this does not matter to the physics. Remember that we ignore the global phase of the qubit state when we construct the Bloch sphere; therefore, the Bloch sphere has lost the global phase information. *For the same point on the Bloch sphere, we can add any global phase to it without affecting its physical properties*. And a point on the Bloch sphere is not unique, but it corresponds to qubits that differ in only a global phase (e.g. $e^{-i\phi/2}|0\rangle$ and $|0\rangle$ in this case).

Let us consider the point corresponds to $x = -1$ and $y = z = 0$ in the 3D real space. It has $\theta = \pi/2$ and $\phi = \pi$. Therefore,

$$\begin{aligned}\left|\Psi_{\theta=\pi/2,\phi=\pi}\right\rangle &= \cos\frac{\theta}{2}e^{-i\phi/2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|1\rangle \\ &= \cos\frac{\pi}{4}e^{-i\pi/2}|0\rangle + \sin\frac{\pi}{4}e^{i\pi/2}|1\rangle \\ &= \frac{1}{\sqrt{2}}(-i)|0\rangle + \frac{1}{\sqrt{2}}i|1\rangle \\ &= -i\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= -i|-\rangle \end{aligned} \tag{27.4}$$

Again, the global phase $-i = e^{-i\pi/2}$ can be discarded. Therefore, this point corresponds to $|-\rangle$.

Let us now consider the point corresponds to $y = 1$ and $x = z = 0$ in the 3D real space. It has $\theta = \pi/2$ and $\phi = \pi/2$. Therefore,

$$\begin{aligned}\left|\Psi_{\theta=\pi/2,\phi=\pi/2}\right\rangle &= \cos\frac{\theta}{2}e^{-i\phi/2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|1\rangle \\ &= \cos\frac{\pi}{4}e^{-i\pi/4}|0\rangle + \sin\frac{\pi}{4}e^{i\pi/4}|1\rangle \\ &= \frac{1}{\sqrt{2}}e^{-i\pi/4}(|0\rangle + e^{i\pi/2}|1\rangle) \\ &= e^{-i\pi/4}\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \end{aligned} \tag{27.5}$$

Again, the global phase $e^{-i\pi/4}$ can be discarded. Therefore, this point corresponds to $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, which is one of the eigenvalues of $\boldsymbol{\sigma_y}$.

From the examples, we see that the extrema of the Bloch sphere in the x, y, and z directions correspond to the eigenvalues of the $\boldsymbol{\sigma_x}$, $\boldsymbol{\sigma_y}$, and $\boldsymbol{\sigma_z}$ matrices, respectively (e.g. Eq. (6.11)). This is the first place where we see it is related to our real 3D space. This is because, for example, if it is a spin qubit under an external magnetic field (we will not discuss here), the **Pauli matrices** are related to the directions of the magnetic field. So now we see the link between the extrema in a direction (e.g. z direction) on the Bloch sphere to the external magnetic direction (e.g. z direction). But again, *having* $|0\rangle$ *at the "North pole" and* $|1\rangle$ *at the "South pole" does not mean that the states are on the opposite direction in the real 3D space!* We are just embedding the space in our real 3D space.

27.3 Expectation Values of Pauli Matrices

Although we will not cover it in detail, I already told you that the Pauli matrices are related to the directions of the external magnetic field in the spin qubit case. Moreover, the expectation values of the Pauli matrices of any state are related to the energy it gains under the external magnetic field. Therefore, the calculation of the expectation value of the Pauli matrices is very important.

Example 27.2 Find the expectation value of $\boldsymbol{\sigma_z}$ of $|\Psi\rangle$.

First, we can express $|\Psi\rangle$ in its column form. That is $|\Psi\rangle = \cos\frac{\theta}{2}e^{-i\phi/2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi/2}|1\rangle = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}$. Therefore,

$$\begin{aligned}
\langle\Psi|\,\boldsymbol{\sigma_z}\,|\Psi\rangle &= \left(\cos\tfrac{\theta}{2}e^{i\phi/2}\ \sin\tfrac{\theta}{2}e^{-i\phi/2}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\\
&= \left(\cos\tfrac{\theta}{2}e^{i\phi/2}\ \sin\tfrac{\theta}{2}e^{-i\phi/2}\right)\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ -\sin\frac{\theta}{2}e^{i\phi/2}\end{pmatrix}\\
&= \cos\frac{\theta}{2}e^{i\phi/2}\cos\frac{\theta}{2}e^{-i\phi/2} + \sin\frac{\theta}{2}e^{-i\phi/2}\left(-\sin\frac{\theta}{2}e^{i\phi/2}\right)\\
&= \cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2}\\
&= \cos\theta
\end{aligned} \tag{27.6}$$

Here we have taken the complex conjugates of the coefficients when we write the $\langle\Psi|$. At the end, we use the identity $\cos^2 x - \sin^2 x = \cos 2x$.

What is the geometric meaning of $\cos\theta$ on the Bloch sphere in Fig. 27.1? Since it is a unit sphere, this is just *the projection of the state* $|\Psi\rangle$ *on the* z*-axis*! This is the second place where we see how the Bloch sphere is related to our real 3D space. Although the states on the Bloch sphere have no direct relationship to the real 3D space, their Paul matrices' expectation values turn out to be the projections on the axes from the states on the Bloch sphere. Therefore, although $|0\rangle$ and $|1\rangle$ do not lie in our real 3D space on the opposite sides of the z-axis, their $\boldsymbol{\sigma_z}$ expectation values do have opposite values. These expectation values are related to the energy splitting and magnetic moments of the qubits, which we will not discuss in this book.

We can also calculate the expectation values of $\boldsymbol{\sigma_x}$ and $\boldsymbol{\sigma_y}$, and we will find that they are just the projections of the state on the x- and y-axis, respectively. If so, without any derivation, based on the geometry, we know that

$$\langle\Psi|\,\boldsymbol{\sigma_x}\,|\Psi\rangle = \sin\theta\cos\phi \tag{27.7}$$

$$\langle\Psi|\,\boldsymbol{\sigma_y}\,|\Psi\rangle = \sin\theta\sin\phi \tag{27.8}$$

We will prove this in the problems.

27.4 Single-Qubit Arbitrary Unitary Rotation

We have been emphasizing that quantum computing is nothing but just the rotation of the quantum states in the hyperdimensional space that we cannot see. For example, applying a NOT gate to the $|0\rangle$ state will rotate it to the $|1\rangle$ state. Now, since we have mapped the qubit space to the Bloch sphere embedded in the real 3D space, we will be able to "see" how a vector rotates in the 3D space. This is another great benefit of using the Bloch sphere. But I need to emphasize again, the rotation on the Bloch sphere is NOT the rotation of the state in the 3D space. The qubit space is not something we can feel. This is just a convenient visualization.

How do we describe this rotation? This rotation is a rotation in the hyperspace where the qubit state resides, and it must be a unitary rotation and it must be a 2×2 matrix just like any other 1-qubit quantum gate. In general, it is described by a **Single-Qubit Arbitrary Unitary Gate**, $\boldsymbol{U_{\theta,\phi,\lambda}}$.

$$\boldsymbol{U_{\theta,\phi,\lambda}} = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\lambda+\phi)}\cos\frac{\theta}{2} \end{pmatrix} \tag{27.9}$$

We will not derive this equation, but there is a lot to appreciate in this equation. First, the parameters, θ, ϕ, and λ, are angles, and they are the **Euler angles** in the **Euler rotation**. We will not discuss the Euler rotation in detail, but it is a 3D real space rotation of a rigid body. Euler rotation is a sequence of three rotations. It can be described in two ways. One is the *intrinsic rotation* in which it rotates about the axes embedded in/moving with the body. We will NOT use this one. Another equivalent way is the *extrinsic rotation* that it rotates about fixed 3D coordinate axes (Fig. 27.2). In the extrinsic rotation, the body will first rotate about the z-axis by λ (also called γ in some literature) and then rotate about the y-axis by θ (also called β in some literature) and finally rotate about the z-axis by ϕ (also called α in some literature). If you are studying Euler rotation for comparison, make sure that we are using the so-called $z-y-z$ basis. Also be careful that in the $\boldsymbol{U_{\theta,\phi,\lambda}}$ notation, we do not put the rotation angles in the order of rotation (i.e. θ, ϕ, λ instead of ϕ, θ, λ or λ, θ, ϕ). These are the confusions you might have if you try to compare to Euler rotation.

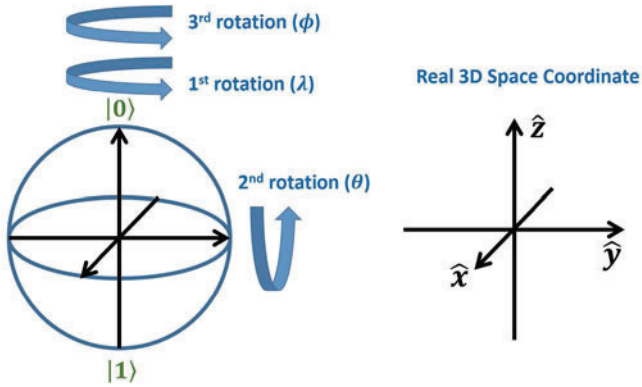

Fig. 27.2 Relationship between the Bloch sphere and Euler rotation

However, regardless of the notations, you see another benefit of using the Bloch sphere, i.e. the angles in Eq. (27.9) have the meanings corresponding to the real 3D space rotations when you embed the Bloch sphere in the 3D space. This can help us understand the transformation/rotation of the state vectors.

Example 27.3 Construct a NOT gate using $\boldsymbol{U_{\theta,\phi,\lambda}}$ by matching the matrices.

We can match the elements in Eq. (27.9) to the NOT gate matrix elements in Eq. (15.6).

$$\boldsymbol{U_{\theta,\phi,\lambda}} = \boldsymbol{U_{NOT}}$$
$$\begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\lambda+\phi)}\cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{27.10}$$

I can set up four equations to match each of the elements.

$$\cos\frac{\theta}{2} = 0$$
$$-e^{i\lambda}\sin\frac{\theta}{2} = 1$$
$$e^{i\phi}\sin\frac{\theta}{2} = 1$$
$$e^{i(\lambda+\phi)}\cos\frac{\theta}{2} = 0 \tag{27.11}$$

But since $\boldsymbol{U_{NOT}}$ is simple, I can see $\theta = \pi$ is required to make the diagonal element zero as $\cos\pi/2 = 0$. Then $\sin\theta/2 = \sin\pi/2 = 1$. And I can put $\lambda = \pi$ so that $-e^{i\lambda} = 1$ and $\phi = 0$ so that $e^{i\phi} = 1$. So the solution is $(\theta, \phi, \lambda) = (\pi, 0, \pi)$. Figure 27.3 shows the path of the rotation of the $|0\rangle$ state under this quantum gate. When it is at the “North” or “South” poles, the first and third rotations have no

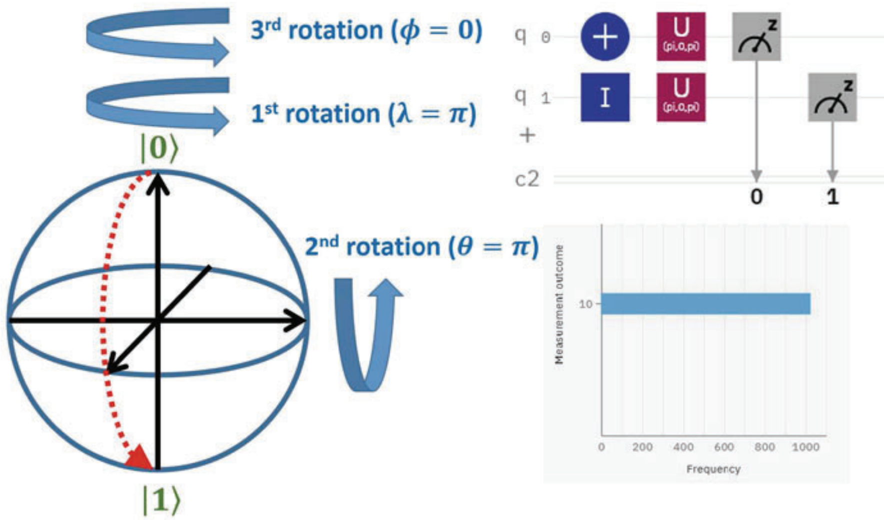

Fig. 27.3 Implementation of $\boldsymbol{U_{NOT}}$ using $\boldsymbol{U_{\theta,\phi,\lambda}}$. Left shows how $|0\rangle$ is rotated on the Bloch sphere under a NOT operation. Right shows the implementation on IBM-Q with $|01\rangle$ as the input. MSB is at the bottom of the circuit

effects. We can see the second rotation (about y-axis by $\theta = \pi$) brings it to $|1\rangle$. The figure also shows the implementation on IBM-Q. Two unentangled qubits are shown. The MSB (bottom) has an input of $|0\rangle$, and the LSB (top) has an input of $|1\rangle$ (after a NOT gate) to the $\boldsymbol{U_{\theta,\phi,\lambda}} = \boldsymbol{U_{\pi,0,\pi}}$. Therefore, the input is $|01\rangle$, and we can see the output is $|10\rangle$ as expected and, thus, $\boldsymbol{U_{\pi,0,\pi}}$ behaves as a $\boldsymbol{U_{NOT}}$.

Now we have the Bloch sphere to help the visualization. Can we construct matrices by identifying the initial and final states and the path only?

Example 27.4 Construct the matrix corresponding to the rotation of $|0\rangle$ to $|1\rangle$ on the Bloch sphere.

Figure 27.3 clearly shows that the initial state $|0\rangle$ is at the "North" pole and the final state is at the "South" pole. To go from the "North" pole to the "South" pole, the most straightforward way is to rotate about the y-axis by π in the second rotation and do nothing in the first and third rotations. This corresponds to $\boldsymbol{U_{\theta,\phi,\lambda}} = \boldsymbol{U_{\pi,0,0}}$. Then we obtain

$$\begin{aligned}\boldsymbol{U_{\pi,0,0}} &= \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & e^{i(\lambda+\phi)}\cos\frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\frac{\pi}{2} & -e^{i0}\sin\frac{\pi}{2} \\ e^{i0}\sin\frac{\pi}{2} & e^{i(0+0)}\cos\frac{\pi}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{aligned} \tag{27.12}$$

Unfortunately, this is *not* a NOT gate. But this matrix correctly describes the transformation from $|0\rangle$ to $|1\rangle$ because

$$\boldsymbol{U_{\pi,0,0}}\,|0\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle \tag{27.13}$$

The reason is that many gates can bring $|0\rangle$ to $|1\rangle$, but they are not necessarily the NOT gate. We need to use the rotation of a general vector on the Block sphere to derive the NOT gate if we want to construct a quantum gate using $\boldsymbol{U_{\theta,\phi,\lambda}}$. We need to match the elements carefully like in Eqs. (27.10) and (27.11). Be aware not to just pick one or two rotation examples of a special state (like $|0\rangle$) and construct the gate directly using the θ, ϕ, and λ seen in the Bloch sphere (like in Eqs. (27.12) and (27.13)). On the other hand, once we have the correct $\boldsymbol{U_{\theta,\phi,\lambda}}$ for a certain quantum gate, we can see the "path" of how a qubit is transformed on a Bloch sphere.

Example 27.5 Describe how $|+\rangle$ rotates on the Bloch sphere when the NOT gate is applied.

First, we know that $\boldsymbol{U_{NOT}}\,|+\rangle = \boldsymbol{U_{NOT}}\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = |+\rangle$. We used the definition of $\boldsymbol{U_{NOT}}$ that it changes $|0\rangle$ and $|1\rangle$ to each other. So, $|+\rangle$ is transformed/rotated to itself.

The NOT gate is $\boldsymbol{U_{\pi,0,\pi}}$ as proved in Example 27.3. So, on the Bloch sphere, it will first rotate about the z-axis by π, and then about the y-axis by π, and do nothing in the third rotation as $\phi = 0$. Figure 27.4 shows that it is brought to $|-\rangle$ in the first rotation and brought back to $|+\rangle$ in the second rotation.

In the same figure, it shows that if we use $\boldsymbol{U_{\pi,0,0}}$, we get the wrong result because $\boldsymbol{U_{\pi,0,0}}$ is not the NOT gate.

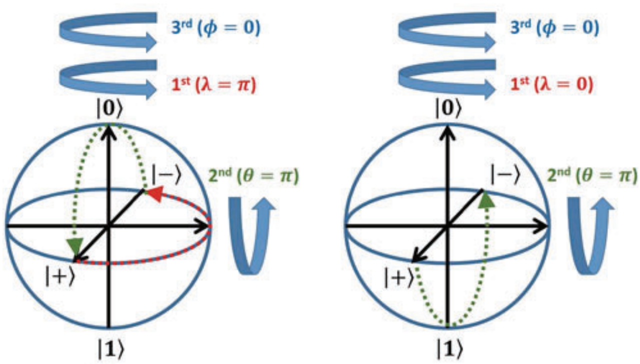

Fig. 27.4 Left: Rotation path of $|+\rangle$ under a NOT gate operation. Right: Rotation path of $|+\rangle$ under the $\boldsymbol{U_{\pi,0,0}}$ operation

27.5 Summary

In this chapter, we have introduced the Bloch sphere that is not necessary to understand quantum computing. However, it gives us intuition and is particularly useful if we want to link it to the physics construction and operation of qubits (which is not discussed in this book). But most importantly, we need to understand that the Bloch sphere is just a way to embed the qubit space into our real 3D space. If so, you will not ask why $|0\rangle + |1\rangle$ is not zero as it looks like the "vectors" at the "North" and "South" poles should get canceled. This shows again, they are not the states in our real 3D space. If you use it carefully, you will be able to understand the arbitrary unitary gate better. This can be used to construct other single-qubit gates. We appreciate that if we can represent the gate in the form of $\boldsymbol{U}_{\boldsymbol{\theta},\boldsymbol{\phi},\boldsymbol{\lambda}}$, we will know the evolution path of the qubit, and this will help us design quantum gate hardware (not covered in this book). For now, knowing the path is good for visualization.

Problems

27.1 Effect of Global Phase on a Single Qubit
Show that the global phase of a single qubit has no effect when finding its expectation value on a $\boldsymbol{\sigma}_z$ matrix.

27.2 Extrema on the Bloch Sphere
To which qubit states do the extrema on the Bloch sphere correspond? We have proved three of them in the text. Please prove the rest.

27.3 Expectation Values of Pauli Matrices of the States on Bloch Sphere
Prove Eqs. (27.7) and (27.8).

27.4 State on Bloch Sphere
Find θ and ϕ for $|\Psi\rangle = (\frac{3}{4} - i\frac{\sqrt{3}}{4})\,|0\rangle + (\frac{\sqrt{3}}{4} + i\frac{1}{4})\,|1\rangle$. Draw it on the Bloch sphere.

27.5 $\boldsymbol{U}_{\boldsymbol{\theta},\boldsymbol{\phi},\boldsymbol{\lambda}}$ as a Hadamard Gate
Construct the Hadamard gate using $\boldsymbol{U}_{\boldsymbol{\theta},\boldsymbol{\phi},\boldsymbol{\lambda}}$ like in Example 27.3. Hints: Answer is $\boldsymbol{U}_{\boldsymbol{\pi}/\mathbf{2},\mathbf{0},\boldsymbol{\pi}}$.

27.6 Rotation on Bloch Sphere by Hadamard Gate
Show on the Bloch sphere how $|+\rangle$ and $|1\rangle$ evolve when it is applied with the Hadamard gate.

27.7 Entanglement Circuit on IBM-Q
Construct an entanglement state using only $\boldsymbol{U}_{\boldsymbol{\theta},\boldsymbol{\phi},\boldsymbol{\lambda}}$ on IBM-Q. We know that we need a CNOT gate, and it can be done by constructing a controlled version of $\boldsymbol{U}_{\boldsymbol{\theta},\boldsymbol{\phi},\boldsymbol{\lambda}}$. Set $\gamma = 0$, which will be discussed in the next chapter.