10-26-2020

On the necessity of Chvátal’s Hamiltonian degree condition

Douglas Bauer
Stevens Institute of Technology

Linda Lesniak
Western Michigan University

Aori Nevo
Stevens Institute of Technology

Edward Schmeichel
San Jose State University, edward.schmeichel@sjsu.edu

AKCE International Journal of Graphs and Combinatorics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

On the necessity of Chvátal's Hamiltonian degree condition

Douglas Bauer , Linda Lesniak , Aori Nevo & Edward Schmeichel

To cite this article: Douglas Bauer , Linda Lesniak , Aori Nevo & Edward Schmeichel (2020) On the necessity of Chvátal's Hamiltonian degree condition, AKCE International Journal of Graphs and Combinatorics, 17:2, 665-669, DOI: 10.1080/09728600.2020.1834337

To link to this article: https://doi.org/10.1080/09728600.2020.1834337

Published online: 27 Oct 2020.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2020, VOL. 17, NO. 2, 665–669
https://doi.org/10.1080/09728600.2020.1834337

On the necessity of Chvátal's Hamiltonian degree condition

Douglas Bauer[a], Linda Lesniak[b], Aori Nevo[a], and Edward Schmeichel[c]

[a]Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, New Jersey, USA; [b]Department of Mathematics, Western Michigan University, Kalamazoo, Michigan, USA; [c]Department of Mathematics, San José State University, San José, California, USA

ABSTRACT
In 1972 Chvátal gave a well-known sufficient condition for a graphical sequence to be forcibly Hamiltonian, and showed that in some sense his condition is best possible. In this paper, we conjecture that with probability 1 as $n \to \infty$, Chvátal's sufficient condition is also necessary. In contrast, we essentially prove that for every $k \geq 1$, the sufficient condition of Bondy and Boesch for forcible k-connectedness is not necessary in the same way.

KEYWORDS
Degree condition; forcibly Hamiltonian degree sequence; forcibly k-connected degree sequence

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Our terminology and notation are standard except as indicated. A good reference for undefined terms is Chartrand et al. [5].

The connectivity of a graph is one measure of how strongly held together a graph is. The toughness [7] is another.

If G is a noncomplete graph then the *connectivity* $\kappa(G)$ of G is defined to be the minimum number of vertices whose removal disconnects G. If $\kappa(G) \geq k$ for some positive integer k, then G is called *k-connected*.

On the other hand, suppose G is a noncomplete graph and t is a nonnegative real number. Let $\omega(G-S)$ denote the number of components of $G - S$ for a vertex cut S of G. If $\omega(G-S) \leq |S|/t$ for every vertex cut S of G, then G is called *t-tough*. The *toughness* *t(G)* of G is the maximum t for which G is t-tough. It is straightforward to see that $t(G) \leq \kappa(G)/2$ for every graph G.

In Section 2, we begin with a conjecture based on compelling data regarding forcibly 1-tough graphical sequences, and as a corollary, we obtain a surprising conjecture regarding forcibly Hamiltonian graphical sequences. Section 3 contains data supporting these conjectures. In Section 4, we investigate forcibly k-connected graphical sequences, and prove—assuming Royle's Conjecture—another unexpected result. Finally, in Section 5, we suggest a direction for future work.

2. Forcibly 1-tough graphical sequences

A graphical n-sequence $\pi = (d_1 \leq \ldots \leq d_n)$ is said to be *forcibly 1-tough* if every graph with degree sequence π is 1-tough.

In 1972, Chvátal [6] established the following condition for a graphical sequence π to be forcibly 1-tough.

Theorem 2.1. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 3$. If*

$$d_i \leq i \Rightarrow d_{n-i} \geq n-i, \quad \text{for } 1 \leq i \leq \frac{n-1}{2}, \tag{1}$$

then π is forcibly 1-tough.

Chvátal also showed that in one sense his condition for a graphical sequence to be forcibly 1-tough is best possible. A sequence of real numbers $(p_1, p_2, \ldots, p_n)$ is said to be *majorized* by another sequence $(q_1, q_2, \ldots, q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$. A graph G is *degree-majorized* by a graph H if G and H have the same number of vertices and the nondecreasing degree sequence of G is majorized by that of H. The family of degree-maximal non-1-tough graphs (those that are degree-majorized by no others) are described next.

Given positive integers i, n for which $1 \leq i \leq n/2$, let $C_{n,i}$ denote the graph $K_i \vee (\overline{K_i} \cup K_{n-2i})$. Clearly $C_{n,i}$ is not 1-tough since if S denotes the i vertices of degree $n-1$ in $C_{n,i}$, we have $\omega(C_{n,i} - S) = i + 1 > |S|$.

Theorem 2.2. *If G is a non-tough graph of order $n \geq 3$ then G is degree-majorized by some $C_{n,i}$.*

Not every forcibly 1-tough graphical sequence satisfies Chvátal's condition in Theorem 2.1. But in the following section, we present compelling data suggesting the following.

Conjecture 2.3. *For some $\epsilon < 1$ and for n sufficiently large, the fraction of forcibly 1-tough graphical n-sequences that satisfy Chvátal's condition in Theorem 2.1 is greater than $1 - \epsilon^n$; in other words, this fraction approaches 1 exponentially fast.*

CONTACT Aori Nevo aorinevo@gmail.com

Table 1. c_n, S_n, and S_n/c_n for $5 \le n \le 13$.

n	c_n	S_n	S_n/c_n
5	6	2	0.3333
6	24	3	0.1250
7	67	14	0.2090
8	263	31	0.1170
9	823	117	0.1422
10	3,276	278	0.0849
11	10,839	956	0.0882
12	43,287	2,578	0.0596
13	147,943	8,106	0.0548

Chvátal's condition in Theorem 2.1 does, in fact, give us more than forcibly 1-tough. A graphical $n-$sequence $\pi = (d_1 \le \dots \le d_n)$ is said to be *forcibly Hamiltonian* if every graph with degree sequence π is hamiltonan.

Theorem 2.4. *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical sequence with $n \ge 3$. If*

$$d_i \le i \Rightarrow d_{n-i} \ge n-i, \quad \text{for } 1 \le i \le \frac{n-1}{2}, \qquad (2)$$

then π is forcibly Hamiltonian.

Of course, not every forcibly Hamiltonian graphical sequence satisfies Chvátal's condition. St and Nash-Williams [8] showed that for each odd $n \ge 5$ with $(n-1)/2$ even, the graphical $n-$ sequence with $d_i = \frac{n-1}{2}$ for $1 \le i \le n$ is forcibly Hamiltonian. However, our main conjecture for this section is an interesting weakening of Conjecture 2.3.

Conjecture 2.5. *For some $\epsilon < 1$ and for n sufficiently large, the fraction of forcibly Hamiltonian graphical n-sequences that satisfy Chvátal's condition in Theorem 2.1 is greater than $1-\epsilon^n$; in other words, this fraction approaches 1 exponentially fast.*

Informally, Conjecture 2.5 says that with probability approaching 1 as $n \to \infty$, a graphical n-sequence π is forcibly Hamiltonian if and only if π satisfies Chvátal's condition.

3. Data supporting Conjecture 2.3

We noted in Section 2 that our main conjecture (Conjecture 2.5) regarding forcibly Hamiltonian sequences would follow immediately from the analogous conjecture (Conjecture 2.3) for forcibly 1-tough sequences.

Before giving the data supporting Conjecture 2.3, we need a definition and some notation.

A graphical sequence π is *spurious for 1-tough* (or just *spurious*) if and only if

(1) π fails Chvátal's degree condition, and
(2) π is forcibly 1-tough.

So by Theorem 2.1 (Chvátal's theorem), the set of forcibly 1-tough graphical n-sequences partitions into those which satisfy Chvátal's condition and those which are spurious. We will denote the number of Chvátal-satisfying (respectively, spurious) graphical n-sequences by c_n (respectively, s_n). With this notation, Conjecture 2.3 may be expressed: For some $\epsilon < 1$, $\frac{c_n}{c_n+s_n} > 1-\epsilon^n$, for n sufficiently large.

It is known [2] that $c_n = \Omega\left(\frac{4^n}{n^3}\right)$, so that c_n grows exponentially faster than b^n, for any $b < 4$.

On the other hand, an upper bound S_n for s_n appears as shown in Table 1, for $5 \le n \le 13$, and we'll explain how S_n is obtained algorithmically, and why it is an upper bound for s_n, later in this section.

The ratio S_{n+1}/S_n, for $5 \le n \le 12$, is shown in Table 2.

The values of S_{n+1}/S_n, considered separately for n odd and for n even, suggest that S_{n+1}/S_n is approaching some number near 3 in an oscillatory manner. Afortiori, these values suggest that for some $b' < 4, S_{n+1}/S_n \le b'$ for n sufficiently large. This in turn implies that for any b such that $b' < b < 4$, we have

(1) $S_n < b^n$, for n sufficently large.

Since $c_n \ge C\frac{4^n}{n^3}$ for some constant C and n sufficiently large, it follows from (1) that for any ϵ such that $b/4 < \epsilon < 1$, we have

(2) $\frac{S_n}{c_n} < \frac{b^n}{C\frac{4^n}{n^3}} = \frac{n^3}{C}\left(\frac{b}{4}\right)^n < \epsilon^n$, for n sufficiently large.

Finally, (2) gives Conjecture 2.3 as follows:

$$\frac{c_n}{c_n+s_n} \ge \frac{c_n}{c_n+S_n} = 1 - \frac{S_n}{c_n+S_n} \ge 1 - \frac{S_n}{c_n} > 1-\epsilon^n,$$

for n sufficiently large.

It remains to explain how the values S_n in Table 1 are obtained algorithmically, and why they are upper bounds for s_n.

Let X_n be a set of edge-maximal, non-1-tough graphs on n vertices. We give below an algorithm, algo(X_n), which uses X_n to declare as spurious certain graphical n sequences π which fail Chvátal's condition.

algo(X_n)

For each graph $G \in X_n$, assign the degrees in π to the vertices of G in monotone order (smallest degree in π to the smallest degree vertex in G, etc.), and use Tutte's factor theorem [9] to decide: Does G contain a spanning subgraph realizing π with the degrees in π as assigned? If the answer is "no" for every $G \in X_n$, then declare π to be spurious (i.e., forcibly 1-tough). □ (algo(X_n))

If X_n were the set of all edge-maximal, non-1-tough graphs on n vertices, and if the monotone assignment restriction in algo(X_n) was replaced by "in all possible ways," then a declaration by algo(X_n) that π is spurious would, of course, be correct. But otherwise (in particular, if X_n is a proper subset of all the edge-maximal, non-1-tough graphs on n vertices) algo(X_n) might error by declaring a nonspurious π to be spurious. Thus for any set X_n, the number of sequences π declared spurious by algo(X_n) will be an upper bound for s_n.

Table 2. Ratio S_{n+1}/S_n for $5 \le n \le 12$.

n	5	6	7	8	9	10	11	12
S_{n+1}/S_n	1.5	4.66	2.21	3.77	2.37	3.43	2.63	3.21

The number S_n in Table 1 is the number of graphical n-sequences π declared spurious by algo(X_n), when X_n is taken to be the n-vertex Chvátal graphs $C_{n,i}$ for $1 \le i < n/2$. As noted above, S_n is therefore an upper bound for s_n.

If instead we let X_n be the set of all edge-maximal, non-1-tough graphs on n vertices, and let S'_n denote the number of π declared spurious by algo(X_n) with this choice of X_n, then of course $S'_n \le S_n$. But perhaps surprisingly, the data suggests that $S'_n/S_n \to 1$ as $n \to \infty$. Briefly, enlarging X_n from the Chvátal graphs to all edge-maximal, non-1-tough graphs doesn't appear to substantially enhance algo(X_n)'s effectiveness in correctly identifying spurious π.

4. Forcibly *k*-connected sequences

In Sections 2 and 3, we conjectured, and provided data, that for some $\epsilon < 1$ and n sufficiently large, the fraction of forcibly Hamiltonian n-sequences which satisfy Chvátal's condition $> 1 - \epsilon^n$. In this section we consider the analogue of this conjecture for forcibly k-connected n-sequences.

A graphical n-sequence $\pi = (d_1 \le \dots \le d_n)$ is called *forcibly k-connected, for* $k \le n-1$, if every graph with degree sequence π is k-connected. In 1972, Bondy [4] gave a sufficient condition for an n-sequence to be forcibly k-connected. The form given below is due to Boesch [3].

Theorem 4.1. *Let* $\pi = (d_1 \le \dots \le d_n)$ *be a graphical* n-*sequence with* $n \ge 2$ *and* $1 \le k \le n-1$. *If*

$$d_i \le i + k - 2 \Rightarrow d_{n-k+1} \ge n - i$$

for every $1 \le i \le \frac{n-k+1}{2}$, *then* π *is forcibly k-connected.*

We saw that every graphical n-sequence that fails Chvátal's condition for hamiltonicity is majorized by the degree sequence of a non-Hamiltonian graph. Analogously, suppose a graphical n-sequence π fails the Bondy-Boesch condition for k-connectedness for some i satisfying $1 \le i \le \frac{n-k+1}{2}$. Let $BB_{n,k,i} \doteq K_{k-1} \vee (K_i \cup K_{n-k-i+1})$, with degree sequence π_k. Then $BB_{n,k,i}$ is not k-connected, and π is degree-majorized by π_k. Despite this similarity, we believe that a conjecture analogous to Conjecture 2.5 for forcibly k-connected n-sequences that satisfy the Bondy–Boesch condition is almost certainly false, for every $k \ge 1$. Our argument depends on a compelling conjecture of Gordon Royle (see [10, p. 889]) on the asymptotics of g_n, where g_n is the number of n-sequences $\pi = (d_1 \le \dots \le d_n)$ that are graphical.

Conjecture 4.2 (Royle's Conjecture).

$$\frac{g_n}{g_{n-1}} \to 4 \text{ as } n \to \infty.$$

Royle's Conjecture, if true, implies that for j independent of n,

$$g_{n-j} \approx \frac{1}{4^j} g_n \text{ for large } n. \tag{3}$$

Let bb_n^k denote the number of graphical n-sequences that satisfy the Bondy-Boesch condition for k-connectedness. We will call a graphical n-sequences π *spurious for k-connectedness* ($k \ge 1$) if

(i) π does not satisfy the Bondy-Boesch condition for k-connectedness, but
(ii) π is forcibly k-connected.

Let s_n^k denote the number of graphical n-sequences which are spurious for k-connectedness. The analogue for forcibly k-connected n-sequences of our work in Section 3 for forcibly Hamiltonian n-sequences would be:

For some $\epsilon < 1$ and for n sufficiently large, $s_n^k/(bb_n^k) < \epsilon^n$.

Of course, this would imply $\frac{s_n^k}{bb_n^k} \to 0$ exponentially fast. However, we show that almost certainly $\frac{s_n^k}{bb_n^k} \ge \frac{1}{2^{11}4^{k-1}}$ for every $k \ge 1$ and $n \ge k + 8$. Our argument will require Royle's Conjecture.

Theorem 4.3. *If Royle's Conjecture is true, then for any* $k \ge 1$ *and* $n \ge k + 8$,

$$\frac{s_n^k}{bb_n^k} \ge \frac{1}{2^{11}4^{k-1}}.$$

Proof. We first establish the result for $k = 1$. That is, we show that

$$\frac{s_n^1}{bb_n^1} \ge \frac{1}{2^{11}}$$

for $n \ge 9$, assuming Royle's Conjecture.

Let $n \ge 9$ and let $\pi' = (d'_1 \le \dots \le d'_{n-5})$ be a graphical $(n-5)$-sequence with $d'_1 \ge 1$ and $d'_{n-5} \le n-7$, i.e., $d'_1 \ne 0$ and $d'_{n-5} \ne n-6$.

Claim 1. There are $g_{n-5} - 2g_{n-6}$ such sequences π'.

Proof of Claim 1. There are g_{n-6} graphical $(n-5)$-sequences π' with $d'_1 = 0$; these are obtained by adding a new first term 0 to a graphical $(n-6)$-sequence. Similarly, there are g_{n-6} graphical $(n-5)$-sequences π' with $d'_{n-5} = n-6$, obtained by adding a new last term n – 6 to a graphical $(n-6)$-sequence, and increasing by 1 each term in the $(n-6)$-sequence. Moreover, these two types of graphical $(n-5)$-sequences do not overlap. Removing them from the set of all g_{n-5} graphical $(n-5)$-sequences gives the proof of Claim 1. □

Given a graphical sequence $\pi' = (d'_1 \le \dots \le d'_{n-5})$ as described in Claim 1, we form an n-sequence $\pi = (d_1 \le \dots \le d_n)$ as follows: Set $d_1 = 1$ and $d_2 = 2 = d_3$. For $4 \le i \le n-3$, set $d_i = d'_{i-3} + 2$ (noting that $d_4 = d'_1 + 2 \ge 3$ and $d_{n-2} = d'_{n-5} + 2 \le n-5$). Finally, set $d_{n-1} = n-5$ and $d_n = n-4$. By Claim 1, there are $g_{n-5} - 2g_{n-6}$ such n-sequences π, since the map π' to π is $1-1$.

Claim 2. Each such π is spurious for 1-connectedness.

Proof of Claim 2. We need to show that (1) π is graphical, (2) π does not satisfy the Bondy-Boesch condition for 1-connectedness, and (3) π is forcibly 1-connected.

(1) π is graphical.

Let $\pi' = (d'_1 \le \dots \le d'_{n-5})$ be the graphical $(n-5)$-sequence associated with π, and let G' be a graph with degree sequence π'. Then the vertices of G' can be labeled $v_1, \dots, v_{n-5}$ so that $deg_{G'} v_i = d_i'$ for $1 \le i \le n-5$.

Construct G from G' by adding 5 vertices a, b, c, d, e as follows. In G, vertices a and b are each adjacent to $v_1, \dots, v_{n-5}$. In addition, b is adjacent to c, c is adjacent to d, and d is adjacent to e. Then G has degree sequence π.

(2) π fails the Bondy-Boesch condition for 1-connectedness.

Consider $i = 3$. Then $i + k - 2 = 2$ and $d_3 \le 2$, but $d_n = n - 4 < n - 3 = n - i$.

(3) π is forcibly 1-connected.

Suppose, to the contrary, that π is the degree sequence of a disconnected graph $G = G_1 \cup G_2$. Since G has a vertex of degree $n - 4$, one of G_1 and G_2, say G_1, has at least $n - 3$ vertices. Therefore G_2 has no more than three vertices. If G_2 has exactly one vertex, that vertex has degree 0, a contradiction. If G_2 has exactly two vertices, then either $d_1 = d_2 = 0$ or $d_1 = d_2 = 1$, again a contradiction. Finally, if G_2 has exactly three vertices, then the degree of these vertices in G_2 must be 1, 2, 2 which is impossible.

Thus π is forcibly 1-connected and the proof of Claim 2 is complete. □

So, the number s_n^1 of spurious n-sequences for 1-connectedness satisfies

$$s_n^1 \ge g_{n-5} - 2g_{n-6}. \tag{4}$$

Assuming Royle's Conjecture, we then have from (4) and (3)

$$\begin{aligned} s_n^1 &\ge g_{n-5} - 2g_{n-6} \\ &\approx \frac{1}{4^5} g_n - 2\frac{1}{4^6} g_n \\ &= \frac{1}{2^{11}} g_n \\ &\ge \frac{1}{2^{11}} bb_n^1. \end{aligned} \tag{5}$$

So, from (5) we have $\frac{s_n^1}{bb_n^1} \ge \frac{1}{2^{11}}$, for $n \ge 9$. Thus the result holds for $k = 1$.

Let $n \ge 9$ and fix $k \ge 2$. Consider any spurious n-sequence π from Claim 2, i.e.,

$$\pi = (1, 2, 2, d_1' + 2, \dots, d_{n-5}' + 2, n - 5, n - 4).$$

We construct, in a similar fashion, a spurious N_k-sequence π_k for k-connectedness, where $N_k = n + k - 1$. Let

$$\pi_k \doteq (k, k+1, k+1, d_1' + k + 1, \dots, d_{n-5}' + k + 1, n + k - 6, n + k - 5, \underbrace{n + k - 2, \dots, n + k - 2}_{k-1}).$$

That is, create π_k from π by increasing each term of π by $k - 1$, and then adding $k - 1$ new terms with value $n + k - 2$.

We claim that π_k is a spurious $(n + k - 1)$-sequence for k-connectedness.

(1) π_k is graphical.

Let G be the graph associated with π. Define $G_k \doteq G \vee K_{k-1}$. Then G_k has degree sequence π_k.

(2) π_k fails the Bondy-Boesch condition for k-connectedness. Consider $i = 3$. Then $i + k - 2 = k + 1$ and $d_3 \le k + 1$. However

$d_{N_k - k + 1} = d_n = n + k - 5 = N_k - 4 < N_k - 3 = N_k - i$.

(3) π_k is forcibly k-connected.

Let G be any graph with degree sequence π_k and let S be a set of at most $k - 1$ vertices whose removal disconnects G. Furthermore, let B denote the set of vertices in G of degree $n + k - 2$. Then $|B| = k - 1$. If some vertex $v \in B$ is not in S, then v is in $G - S$. But v is adjacent to everything in $G - S$, which contradicts the fact that $G - S$ is disconnected. Thus, $B \subseteq S$, and so $B = S$ and $|B| = |S| = k - 1$. However, then the degree sequence of $G - S$ is π, which is forcibly 1-connected. This contradiction implies π_k is forcibly k-connected.

We conclude that π_k is a spurious N_k-sequence for k-connectedness, where $N_k = n + k - 1 \ge k + 8$. Furthermore, since the map $\pi \longrightarrow \pi_k$ is 1-1, there are at least $\frac{1}{2^{11}} g_n$ such sequences.

Finally,

$$\begin{aligned} s_{N_k}^k \ge \frac{1}{2^{11}} g_n &\approx \frac{1}{2^{11} 4^{k-1}} g_{n+k-1} \\ &= \frac{1}{2^{11} 4^{k-1}} g_{N_k} \\ &\ge \frac{1}{2^{11} 4^{k-1}} bb_{N_k}^k \end{aligned}$$

and so

$$\frac{s_{N_k}^k}{bb_{N_k}^k} \ge \frac{1}{2^{11} 4^{k-1}},$$

for every $k \ge 2$.

This concludes the proof of Theorem 4.3. □

Another way of stating what we've shown in Theorem 4.3 is the following: Assuming Royle's Conjecture, for every $k \ge 1$ and $n \ge k + 8$, the fraction $bb_n^k / (bb_n^k + s_n^k)$ of forcibly k–connected graphical n-sequences which satisfy the Bondy-Boesch condition for k-connectedness is at most

$$\frac{2^{11} 4^{k-1}}{1 + 2^{11} 4^{k-1}} < 1,$$

and this fraction does not approach 1 as $n \longrightarrow \infty$.

5. Concluding remarks

The contrasting results conjectured in Sections 2 and 4 suggest an interesting general problem.

Let P be any montone[1] graph property. As shown in Bauer et al. [1], every such property has a best monotone degree condition (analogous to Chvátal's condition for Hamiltonian). For each n, consider the fraction $f_P(n)$ defined by

$$f_P(n) \doteq \frac{\text{Number of graphical } n\text{-sequencess at is fying the best monotone condition for } P}{\text{Number of graphical } n\text{-sequences which are forcibly } P}.$$

It seems interesting to consider what happens to $f_P(n)$ as $n \to \infty$. Indeed, when P is "Hamiltonian" ("k-connected"), this was the focus of Section 2 (Section 4). We conjecture those examples illustrate two important possibilities for $f_P(n)$ as $n \to \infty$, namely:

(i) $f_P(n) \to 1$, as $n \to \infty$; and
(ii) There exists $b < 1$ such that $f_P(n) \leq b$, for n sufficiently large.

Of course, there may be other possibilities as well. Possibility (*i*), in particular, suggests that the best monotone degree condition for P is essentially necessary and sufficient for a graphical n-sequence π to be forcibly P, for large n.

Best monotone degree conditions for many monotone graph properties are given in Bauer et al. [1]. It would be interesting to investigate $f_P(n)$ as $n \to \infty$ for other properties P in this survey.

Note

1. A graph property is *monotone increasing (decreasing)* if the addition (deletion) of edges preserves the property.

References

[1] Bauer, D., Broersma, H. J., van den Heuvel, J., Kahl, N., Nevo, A., Schmeichel, E., Woodall, D. R, Yatauro, M. (2015). Best monotone degree conditions for graph properties: A survey. *Graphs Combinatorics* 31(1): 1–22.
[2] Bauer, D., Lesniak, L., Nevo, A, Schmeichel, E. The number of graphical n-sequences which satisfy Chvátal's Hamiltonian degree condition, in preparation.
[3] Boesch, F. (1974). The strongest monotone degree condition for n-connectedness of a graph. *J. Combin. Theory Ser. B* 16(2): 162–165.
[4] Bondy, J. A. (1969). Properties of graphs with constraints on degrees. *Studia Sci. Math. Hungar* 4: 473–475.
[5] Chartrand, G., Lesniak, L., Zhang, P. (2016).*Graphs and Digraphs*. 6th ed. Boca Raton, FL: CRC Press.
[6] Chvátal, V. (1972). On Hamilton's ideals. *J. Combin. Theory Ser. B* 12(2): 163–168.
[7] Chvátal, V. (1973). Tough graphs and Hamiltonian circuits. *Discrete Math.* 5(3): 215–228.
[8] St, C, Nash-Williams, J. A. (1971). Hamiltonian arcs and circuits. In: *Recent Trends in Graph Theory, Lecture Notes in Mathematics*, Vol. 186. Berlin: Springer-Verlag, pp. 197–210.
[9] Tutte, W. T. (1954). A short proof of the factor theorem for finite graphs. *Can. J. Math.* 6: 347–352.
[10] Wang, K. (2019). Efficient counting of degree sequences. *Discrete Math.* 342(3): 888–897.