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Braids and the braid groups

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BRAIDS AND THE BRAID GROUPS

A Thesis

Presented to The Faculty of the
Department of Mathematics and Computer Science
San Jose State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

By

Jeffrey P. O'Connell

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ABSTRACT

BRAIDS AND THE BRAID GROUPS

by Jeffrey P. O'Connell

This thesis is a study of Artin's Braid Groups, B_n , from an algebraic point of view, as well as of the tools necessary for this study. The ultimate goal is to give an algebraic based proof of the fact that B_n is torsion free. This thesis is primarily based on the paper "The Algebraic Braid Groups are Torsion-Free: an Algebraic Proof" by Joan L. Dyer.

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INTRODUCTION

The braid groups were first defined by Artin in his paper "Theorie der Zöpfe." The fact that the braid groups are torsion-free was first discovered by Fadell and Neuwirth. Their proof was done using topological methods. The proof we produce here is done using primarily algebraic methods.

We begin by defining the braid group on n strings. We will then show the relationship between the braid group B_n and automorphisms of the free group F_n . After a description of the Reidemeister-Schreier rewriting procedure, which is used to find a presentation for a subgroup of a group, we will use it to find a presentation of certain subgroups of B_n . Following some results about automorphisms of free groups, we will prove that B_n is torsion-free. Then we will give an algebraic solution of the conjugate-power problem, which was first solved by Birman.

CHAPTER 1

BRAIDS AND THE BRAID GROUP

In this chapter we will define the braid group on n strings and give a presentation for this group. Then we will show a relationship between the braid group on n strings and the group of automorphisms of the free group on n generators.

1.1 THE BRAID GROUP

We begin by defining a braid on n strings and then defining the braid group B_n .

1.1.1 Definition. A *polygonal path* is a path that is a finite union of straight line segments in $\mathbf{R}^3$.

Let $P_1, P_2, \dots, P_n$ be the n points in $\mathbf{R}^3$ given by $P_i = (i, 0, 1)$ and let $Q_1, Q_2, \dots, Q_n$ be the n points in $\mathbf{R}^3$ given by $Q_i = (i, 0, 0)$.

1.1.2 Definition. An n -braid or a braid on n strings is a collection of n polygonal paths from P_i to $Q_{\mu(i)}$ where μ is a permutation of $\{1, 2, \dots, n\}$ and such that the z -coordinate of each path strictly decreases from P_i to $Q_{\mu(i)}$ and no two paths intersect.

1.1.3 Definition. The polygonal path from P_i to $Q_{\mu(i)}$ is called the i -th string and μ is called the *permutation* of the braid.

We can view a braid by looking at its projection onto the plane $y = 0$ from the negative y -axis while preserving information about the relative size of the y coordinates of the strings at crossing points, i.e., indicating overcrossings and undercrossings. For example, Figure 1.1.1 shows a braid on three strings.

Figure 1.1.1

1.1.4 Definition. An *isotopic deformation* of $\mathbf{R}^3$ is a family of homeomorphisms h_t , $0 \leq t \leq 1$, of $\mathbf{R}^3$ onto itself such that h_0 is the identity function and the function denoted by $H(p, t) = h_t(p)$ is continuous in both t and p , for all $p \in \mathbf{R}^3$.

1.1.5 Definition. We say that two braids β_1, β_2 are *equivalent* or *string isotopic* if and only if there exists an isotopic deformation h_t of $\mathbf{R}^3$ taking β_1 to β_2 such that the points P_i and Q_i for $1 \leq i \leq n$ are fixed by h_t for all t , and the z -coordinate of all paths strictly decreases from P_i to $Q_{\mu(i)}$ throughout the deformation. This is an equivalence relation. Therefore, two braids will be in the same equivalence class if they are string isotopic. We will say that $[\beta]$ is the equivalence class that contains the n -braid β .

For example, the two braids in Figure 1.1.2 are string isotopic by the isotopic deformation suggested in Figure 1.1.3.

Figure 1.1.2

Figure 1.1.3

1.1.6 Definition. For $1 \leq i \leq n-1$, we define the i -th elementary n braid, denoted by σ_i , to be the braid where the i -th string crosses over the $(i+1)$ -th string and all other strings go directly from P_j to Q_j for $j \neq i, i+1$. (Figure 1.1.4)

Figure 1.1.4

1.1.7 Definition. Let β_1 and β_2 be two n -braids. Define the product $\beta_1\beta_2$ to be the braid obtained by the concatenation of β_2 to the bottom of β_1 . More specifically, let $\theta_{i,\beta_1}(t)$ be the function that describes the polygonal path of the i -th string of β_1 , where $0 \leq t \leq 1$. Let $\theta_{i,\beta_1}(t) = (x_{i_1}(t), y_{i_1}(t), z_{i_1}(t))$ where $x_{i_1}(t)$, $y_{i_1}(t)$, and $z_{i_1}(t)$ describe the x , y , and z coordinates of the polygonal paths of β_1 respectively. Then if μ is the permutation of β_1 , the polygonal path of the i -th string of the product $\beta_1\beta_2$ is described by:

$$\theta_{i,\beta_1\beta_2} = \begin{cases} (x_{i_1}(2t), y_{i_1}(2t), \frac{1}{2}z_{i_1}(2t) + \frac{1}{2}), & 0 \leq t \leq \frac{1}{2} \\ (x_{\mu(i)_2}(2t-1), y_{\mu(i)_2}(2t-1), \frac{1}{2}z_{\mu(i)_2}(2t-1)), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We have defined all braids to be between the planes $z = 0$ and $z = 1$ in $\mathbb{R}^3$ and our definition of the product preserves this. For convenience, when drawing braids we will not draw them all with the same height. This will not cause any contradictions.

Given an n -braid, we can find an n -braid in the same equivalence class such that each string crossing happens on a different level with respect to the z -axis as shown in Figure 1.1.1 as well as in the next example.

1.1.8 Example. Figure 1.1.5 shows $\beta_1 = \sigma_2\sigma_3$, $\beta_2 = \sigma_1\sigma_4$, and the product $\beta_1\beta_2 = \sigma_2\sigma_3\sigma_1\sigma_4$, where these are all 5-string braids.

Figure 1.1.5

1.1.9 Definition. The *trivial braid*, e , is the braid where all of the strings go directly from P_i to Q_i without any crossings, i.e. where the polygonal paths are just the vertical line segments connecting P_i to Q_i . Note that if β is an n -braid then $e\beta$ and βe are both in the same equivalence class as β . That is, $[e\beta] = [\beta e] = [\beta]$. So the trivial braid acts as a multiplicative identity up to string isotopy.

1.1.10 Definition. Let β be an n -braid. Define the *inverse braid* of β , denoted β^{-1} , to be the braid obtained by reflecting β about the plane $z = 1/2$. We will see that our use of the word "inverse" is justified.

1.1.11 Example. Figure 1.1.6 shows σ_i^{-1} , the inverse of the i -th elementary n -braid. This is the braid where the $(i+1)$ -th string crosses over the i -th string and all other strings go directly from P_j to Q_j for $j \neq i, i+1$.

Figure 1.1.6

1.1.12 Example. Figure 1.1.7 shows the 5-braid $\beta = \sigma_1 \sigma_2 \sigma_3^{-1}$, its inverse $\beta^{-1} = \sigma_3 \sigma_2^{-1} \sigma_1^{-1}$, and the product $\beta \beta^{-1}$, which is string isotopic to the trivial braid, $[\beta \beta^{-1}] = [e]$.

Figure 1.1.7

1.1.13 Example. Probably the most well known braid is the common hair braid on three strings. It is shown in Figure 1.1.8 and is the braid $\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$.

Figure 1.1.8

1.1.14 Definition. Let B_n denote the set of equivalence classes of all n -braids.

As previously stated, if β is an n -braid, let $[\beta]$ denote the equivalence class that contains β . For $[\beta_1], [\beta_2] \in B_n$, define the product $[\beta_1][\beta_2] = [\beta_1 \beta_2]$, where $[\beta_1 \beta_2]$ is the equivalence class that contains the product $\beta_1 \beta_2$ as defined in Definition 1.1.7. We now show that this is a well-defined product which makes B_n into a group.

Pick $[\beta_1], [\beta_2] \in B_n$. The product $[\beta_1 \beta_2]$ is an equivalence class of n -braids and is an element of B_n . Say that β_1 is string isotopic to α_1 and β_2 is string isotopic to α_2 . Then there is an isotopic deformation of $\mathbf{R}^3$ from β_1 to α_1 and another one from β_2 to α_2 , where both of these deformations satisfy the constraints of Definition 1.1.5. We can use a combination of these maps to get an isotopic deformation from

$\beta_1\beta_2$ to $\alpha_1\alpha_2$. Therefore $[\beta_1\beta_2] = [\alpha_1\alpha_2]$, and multiplication in B_n is well defined. If β_1, β_2 , and β_3 are n -string braids then consider the two braids $(\beta_1\beta_2)\beta_3$ and $\beta_1(\beta_2\beta_3)$. These two braids will be different since they are different polygonal paths. To see that the polygonal paths are different look at what happens between $1 \leq t \leq 1/2$. The first braid will go through $\beta_1\beta_2$ and the second braid will go through β_1 . But these two braids will be string isotopic. Therefore

$$([\beta_1][\beta_2])[\beta_3] = [(\beta_1\beta_2)\beta_3] = [\beta_1(\beta_2\beta_3)] = [\beta_1]([\beta_2][\beta_3])$$

and multiplication in B_n is associative. Similarly $e\beta$ is string isotopic to βe as well as β . So

$$[e][\beta] = [\beta][e] = [\beta].$$

Hence, $[e]$ is the multiplicative identity of B_n . If β^{-1} is the inverse braid of β then $\beta\beta^{-1}$ is string isotopic to $\beta^{-1}\beta$ as well as e . So

$$[\beta][\beta^{-1}] = [\beta^{-1}][\beta] = [e].$$

Therefore the inverse element of $[\beta]$ is $[\beta^{-1}] = [\beta]^{-1}$, and B_n is indeed a group.

Although the elements of B_n are equivalence classes, we will refer to the elements of B_n by representatives of these equivalence classes. For example, when we refer to σ_1 in B_3 we mean the equivalence class that contains σ_1 in B_3 .

When referring to elements of braid groups it is important to specify which group you are working with. For example, Figure 1.1.9 shows the element $\sigma_1\sigma_2\sigma_1$ in B_3 and in B_4 .

Figure 1.1.9

In Artin's paper [1] he shows that the group B_n is generated by the set $\{\sigma_1, \sigma_2, \dots, \sigma_{n-1}\}$ and has the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

The first relation simply states that if the two crossings involve strings that are far enough apart then it doesn't matter in which order the crossings are done. The second relation is illustrated in Figure 1.1.10, where close examination shows that the two braids are string isotopic, in fact Figure 1.1.3 shows the isotopic deformation.

Figure 1.1.10

There is a natural homomorphism π from the braid group on n strings onto the symmetric group on n letters, S_n , defined by sending σ_i to the transposition $\tau_i = (i, i+1)$. Recall that S_n is generated by the transpositions τ_i ($i = 1, 2, \dots, n-1$) and that $\tau_i \tau_j = \tau_j \tau_i$ for $|i - j| \geq 2$. It can be shown that $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$. In addition, S_n has the relations $(\tau_i)^2 = e$.

This homomorphism leads to a set of subgroups of B_n that will be of significant importance to us later in this paper. These subgroups are $T_j = \{\beta \in B_n \mid \pi(\beta)(j) = j\}$ for $1 \leq j \leq n$. T_j is the set of braids where the j -th string ends in the j -th position or, using the terminology of Definition 1.1.2, P_j and Q_j are joined by a polygonal path. Note that this does not imply that the j -th string goes uncrossed. Each of these sets of braids contain the identity since the identity fixes the j -th string. The inverse of a braid that fixes the j -th string will fix the j -th string so this set contains inverses. If two braids each fix the j -th string then the product will also fix the j -th

string and this set is closed under concatenation, so these sets are subgroups of B_n .

1.2 THE BRAID GROUP AND AUTOMORPHISMS OF THE FREE GROUP

The following theorem, which is due to Artin and Magnus, will allow us to view the braid group, B_n , as a subgroup of $\text{Aut}(F_n)$, the group of automorphisms of the free group with n generators.

1.2.1 Theorem. (Artin – Magnus). Let $\{x_1, x_2, \dots, x_n\}$ be a free basis of the free group F_n . The set of automorphisms of F_n that satisfy the two conditions

- (1) each x_i is mapped to a conjugate of some x_j , and
- (2) The product $x_1 x_2 \cdots x_n$ is fixed

form a group isomorphic to B_n .

This isomorphism may be defined by sending $\sigma_i \in B_n$ to the automorphism α_i of F_n defined by

$$\alpha_i = \begin{cases} x_i \rightarrow x_{i+1}, \\ x_{i+1} \rightarrow x_{i+1}^{-1} x_i x_{i+1}, \\ x_j \rightarrow x_j, & j \neq i, i+1. \end{cases}$$

Let α be an automorphism of a group G . We denote by $G(\alpha)$ the maximal quotient group of G upon which α induces the trivial automorphism. This group can be viewed as G/H , where H is the

normal subgroup of G generated by the elements $\alpha(y)y^{-1}$ for each $y \in G$. The automorphism α will induce an automorphism $\hat{\alpha}$ on $\text{Ab } G$ defined by $\hat{\alpha}(gG') = \alpha(g)G'$. Now $(\text{Ab } G)(\hat{\alpha})$ can be viewed as $(\text{Ab } G)/K$ where K is the normal subgroup of $\text{Ab } G$ generated by

$$\hat{\alpha}(gG')(gG')^{-1} = (\alpha(g)g^{-1})G'$$

for all $g \in G$.

1.2.2 Lemma. Let G be a group and α an automorphism of G . Then

$$\text{Ab } (G(\alpha)) = (\text{Ab } G)(\hat{\alpha}),$$

where $\hat{\alpha}$ is the automorphism of $\text{Ab } G$ induced by α .

Proof: As previously stated, H is the normal subgroup of G generated by the elements $\alpha(y)y^{-1}$ for each $y \in G$, and K is the normal subgroup of $\text{Ab } G$ generated by $(\alpha(g)g^{-1})G'$ for all $g \in G$. Note that $K = (HG')/G'$.

Consider the map

$$G \rightarrow G(\alpha) \rightarrow \text{Ab } (G(\alpha))$$

defined by the canonical homomorphisms

$$G \rightarrow G/H \rightarrow (G/H)/(G/H)'.$$

In G/H , $(G/H)' = (G'H)/H$. Therefore $(G/H)/(G/H)' \cong G/(G'H)$.

Next consider the map

$$G \rightarrow \text{Ab } G \rightarrow (\text{Ab } G)(\hat{\alpha})$$

defined by the canonical homomorphisms

$$G \rightarrow G/G' \rightarrow (G/G')/K = (G/G')/(HG'/G') \cong G/(G'H).$$

Now we have the commuting diagram

$$\begin{array}{ccc} G & \longrightarrow & G/G' \\ \downarrow & & \downarrow \\ G/H & \longrightarrow & G/(G'H) \end{array}$$

and $\text{Ab}(G(\alpha)) = (\text{Ab } G)(\hat{\alpha})$.

q.e.d.

1.2.3 Corollary. Let $\beta \in B_n < \text{Aut}(F_n)$. Then

- (1) $\text{Ab } F_n(\beta)$ is torsion free, and
- (2) The fixed point set of β in F_n is nontrivial.

Proof: To prove (1) we need to notice that the automorphism β on F_n induces an automorphism $\hat{\beta}$ on $\text{Ab } F_n$ as follows (recall from Theorem 1.2.1 that β sends x_i to a conjugate of some x_j):

$$\hat{\beta}([x_i]) = [\beta(x_i)] = [g_i x_{\mu(i)} g_i^{-1}] = [x_{\mu(i)}]$$

where μ is a permutation of $\{1, 2, \dots, n\}$. By Lemma 1.2.2 we have

$$\text{Ab}(F_n(\beta)) = (\text{Ab } F_n)(\hat{\beta}).$$

But $\text{Ab } F_n = \mathbb{Z}^n$ so forming the maximal quotient group of $\text{Ab } F_n$ upon which $\hat{\beta}$ induces the trivial automorphism merely assigns one generator to each orbit of μ . For example, say that $\overline{x_1}$, $\overline{x_2}$, and $\overline{x_3}$ are the images of x_1 , x_2 , and x_3 in $\text{Ab } F_n$, and $\hat{\beta}(\overline{x_1}) = \overline{x_2}$, $\hat{\beta}(\overline{x_2}) = \overline{x_3}$, and $\hat{\beta}(\overline{x_3}) = \overline{x_1}$. The set $\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}$ is an orbit of μ . In $(\text{Ab } F_n)(\hat{\beta})$ we have the relation $\hat{\beta}(\overline{x_1}) = \overline{x_1}$, which implies that in $(\text{Ab } F_n)(\hat{\beta})$, $\overline{x_1} = \overline{x_2} = \overline{x_3}$. Therefore the generators x_1 , x_2 , and x_3 reduce to just one generator in $(\text{Ab } F_n)(\hat{\beta})$. Now $\text{Ab } F_n(\beta)$ is a free abelian group with generators corresponding to the orbits of μ . Therefore $\text{Ab } F_n(\beta)$ is torsion-free.

The fact that the fixed point set of β in F_n is nontrivial is a direct consequence of the fact that β fixes the product $x_1 x_2 \cdots x_n$, so (2) is also true. q.e.d

CHAPTER 2

THE REIDEMEISTER–SCHREIER REWRITING PROCEDURE

In this chapter we will discuss the Reidemeister–Schreier rewriting procedure. This procedure will allow us to find a presentation for a subgroup of a group whose presentation is known. This work is primarily based on the corresponding material in [12].

2.1 SCHREIER TRANSVERSALS

2.1.1 Definition. Let H be a subgroup of a group G . A set X is called a *left transversal* of $G \bmod H$ if it contains exactly one representative of each left coset gH .

In what follows, let us assume that a transversal X contains e as the coset representative of $eH = H$. We will also assume that G is a group with a known presentation.

2.1.2 Definition. A left transversal X is called a left *Schreier Transversal* if every terminal segment of an element in X is also in X .

For example a_3 and a_2a_3 are terminal segments of $a_1a_2a_3$. Note that some books define a Schreier Transversal as a transversal in which every *initial* segment of an element in the transversal X is also in X . We define it the way we do because of the way that we choose to look at the braid group. This will not cause any contradictions.

2.1.3 Theorem. Suppose G is a group and H is a subgroup of G . Then there exists a left Schreier Transversal for H in G .

Proof: For this proof I will use the same argument found in [12]. Define the *length of a coset* gH to be the length of the shortest reduced word in gH , where the identity, e , has length 0. We proceed by induction on the length of a coset. If a coset gH has length 0 then $e \in gH$ and $gH = H$. Pick e as the representative of H . Next say that the coset fH has length n and all cosets with length less than n already have representatives that satisfy the definition. Pick an element of length n , $a_1a_2 \cdots a_n \in fH$. Clearly $a_2a_3 \cdots a_n \notin fH$, since it has length $n-1$, so this element must be in another coset that already has a representative, say $a_2a_3 \cdots a_n \in (b_1b_2 \cdots b_m)H$. So $a_1(a_2a_3 \cdots a_n) \in a_1(b_1b_2 \cdots b_m)H$. This last coset must be fH since left

cosets partition G/H . Therefore pick $a_1 b_1 b_2 \cdots b_m$ as the coset representative of fH and the definition is satisfied. q.e.d.

It should be noted that in the proof of Theorem 2.1.3 we constructed a *minimal Schreier system*, that is, a Schreier Transversal where each element of the set has length not exceeding the length of any word in the coset that it represents. It is not the case that every Schreier Transversal is minimal.

Consider the subgroup $T_n < B_n$ where $T_n = \left\{ \beta \in B_n \mid \pi(\beta)(n) = n \right\}$.

Recall that this subgroup is the set of braids where the permutation of the braid leaves n fixed. The elements e and $\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}$ for $1 \leq i \leq n-1$ form a left Schreier Transversal for $T_n < B_n$. The cosets are indexed by looking at the permutation of the braid and seeing which string goes to the n -th position. For example, the coset that consists of the braids where the permutation sends the first string to the n -position will have $\sigma_1 \cdots \sigma_{n-2} \sigma_{n-1}$ as its Schreier coset representative.

2.2 THE REIDEMEISTER-SCHREIER REWRITING PROCEDURE

In this section we begin by defining rewriting procedures in general and then we will look at the Reidemeister-Schreier rewriting procedure. We will use this procedure several times in this paper.

Let G be a group with the presentation

$$(1) \quad G = \left\langle a_1, a_2, \dots, a_n; R_\mu(a_v), \dots \right\rangle,$$

where the a_i are generators and the $R_\mu(a_v)$ are relators.

2.2.1 Definition. Let G be presented as in (1), and let H be the subgroup of G generated by the words $J_i(a_v), \dots$. Then a *rewriting procedure for H* (with respect to the generators $J_i(a_v)$) is a mapping

$$\tau: U(a_v) \mapsto V(s_i)$$

of words $U(a_v)$ which define elements of H into words in the symbols s_i , such that the words

$$U(a_v) \quad \text{and} \quad V(J_i(a_v))$$

define the same element of H . (See [12], p. 86.)

2.2.2 Definition. Let G be presented as in (1), and let H be a subgroup of G and X a left transversal for $G \bmod H$. Then a *left coset representative function for $G \bmod H$* is a mapping which assigns to each word in the a_v its unique left coset representative from X ,

$$W(a_v) \mapsto \overline{W}(a_v),$$

where $\overline{W}(a_v) \in X$ is the unique left coset representative of the word $W(a_v)$.

Let $W \mapsto \overline{W}$ be a left coset representative function for G/H which sends an element of G to its unique left coset representative. If $K \in X$ and a_v is a generator of G then $a_v K \in (\overline{a_v K})H$, so there is some unique $S_{a_v, K} \in H$ for which

$$a_v K = \overline{a_v K} \cdot S_{a_v, K}.$$

We will now show that H is generated by the elements $S_{a_v, K}$.

The following properties of a left coset representative function will be needed. They are direct consequences of the definitions:

- (i) $\overline{W} = e$ if and only if $W \in H$;
- (ii) If W is freely equal to V then $\overline{W} = \overline{V}$;
- (iii) $\overline{\overline{W}} = \overline{W}$;
- (iv) $\overline{WV} = \overline{W}\overline{V}$.

2.2.3 Theorem. If $W \mapsto \overline{W}$ is a left coset representative function for $G \bmod H$, then H is generated by the words

$$(2) \quad S_{a_v, K} = \overline{a_v^{-1} K}^{-1} a_v K$$

where a_v is a generator for G and K is an arbitrary left coset representative.

Proof: As previously stated $S_{a_v, K} \in H$. It must now be shown that each element in H can be written as a product of words of the form (2) and their inverses. Observe that words of the form

$$(3) \quad \overline{a_v^{-1} K}^{-1} a_v^{-1} K$$

are the inverses of words of the form (2). To see this let $M = \overline{a_v^{-1} K}$, then

$$\overline{a_v M} = \overline{a_v \overline{a_v^{-1} K}} = \overline{a_v a_v^{-1} K} = \overline{K} = K.$$

Consider

$$\begin{aligned}
 S_{a_v^{-1}, K}^{-1} &= \overline{(a_v^{-1} K \ a_v^{-1} K)^{-1}} \\
 &= K^{-1} a_v a_v^{-1} K \\
 &= \overline{a_v M \ a_v M}^{-1} \\
 &= S_{a_v, M}.
 \end{aligned}$$

Therefore, $S_{a_v^{-1}, K}^{-1}$ is of the form (2).

Suppose that

$$(4) \quad U = a_{v_1}^{\varepsilon_1} a_{v_2}^{\varepsilon_2} \cdots a_{v_r}^{\varepsilon_r}, \quad (\varepsilon_i = \pm 1)$$

is an element of H . We need to write U in terms of words of the form (2) and (3). To do this we will first insert the words $\overline{W_j}^{-1}$ and

$\overline{a_{v_j}^{-\varepsilon_j} W_j}$ before and after each $a_{v_j}^{\varepsilon_j}$ in U respectively. Then we try to choose the W_j so that the product,

$$(5) \quad \overline{W_1}^{-1} a_{v_1}^{\varepsilon_1} \overline{a_{v_1}^{-\varepsilon_1} W_1} \cdot \overline{W_2}^{-1} a_{v_2}^{\varepsilon_2} \overline{a_{v_2}^{-\varepsilon_2} W_2} \cdots \overline{W_r}^{-1} a_{v_r}^{\varepsilon_r} \overline{a_{v_r}^{-\varepsilon_r} W_r}$$

defines the same element of H as the product in (4). We choose the W_j as follows:

$$W_1 = U, \quad W_2 = a_{v_1}^{-\varepsilon_1} W_1, \quad W_3 = a_{v_2}^{-\varepsilon_2} W_2, \dots, \quad W_r = a_{v_{r-1}}^{-\varepsilon_{r-1}} W_{r-1}.$$

W_j will be the j -th terminal segment of U , that is,

$$W_j = a_{v_j}^{\varepsilon_j} a_{v_{j+1}}^{\varepsilon_{j+1}} \cdots a_{v_r}^{\varepsilon_r}.$$

If the W_j are chosen this way then (5) will define the same element of H as (4) since

$$\overline{W_1}^{-1} U a_{v_r}^{-\varepsilon_r} W_r = \overline{U}^{-1} U e = e^{-1} U e = U$$

(recall that $U \in H$ so $\overline{U} = e$). Also note that

$$\overline{W_j}^{-1} a_{v_j} \overline{a_{v_j}^{-1} W_j}$$

is a word of the form (2) since it can, by (iv) above, be written as

$$\overline{a_{v_j} (a_{v_j}^{-1} W_j)}^{-1} a_{v_j} \overline{a_{v_j}^{-1} W_j}.$$

Similarly,

$$\overline{W_j}^{-1} a_{v_j}^{-1} \overline{a_{v_j} W_j}$$

is the inverse of the element

$$\overline{a_{v_j} W_j}^{-1} a_{v_j} \overline{W_j}$$

which is of the form of (2).

Therefore, every word of the form $U(a_v)$ in H can be written as a product of words of the form (2) and their inverses. Hence, the $S_{a_v, K} = \overline{a_v K}^{-1} a_v K$ generate H . q.e.d.

2.2.4 Corollary. Let $W \mapsto \overline{W}$ be a left coset representative function for $G \bmod H$. Define the mapping τ of the word U in (4), which defines an element of H , by

$$(6) \quad \tau(U) = S_{a_{v_1}, K_1}^{\varepsilon_1} S_{a_{v_2}, K_2}^{\varepsilon_2} \cdots S_{a_{v_r}, K_r}^{\varepsilon_r},$$

where K_j is the coset representative of the terminal segment of U after a_{v_j} if $\varepsilon_j = 1$ and K_j is the representative of the terminal segment of U after and including $a_{v_j}^{-1}$ if $\varepsilon_j = -1$. Then τ is a rewriting procedure for H .

As an example of this rewriting procedure let $U = a_1 a_2^{-1} a_3^2$, then $\tau(U)$ will be

$$\tau(a_1 a_2^{-1} a_3^2) = S_{a_1, \overline{a_2^{-1} a_3^2}} \cdot S_{a_2, \overline{a_2^{-1} a_3^2}}^{-1} \cdot S_{a_3, \overline{a_3}} \cdot S_{a_3, \overline{1}}.$$

Proof of Corollary 2.2.4: In order to show that τ satisfies the definition of a rewriting procedure, it must be shown that if $S_{a_{v_j}, K_j}$ is replaced by $\overline{a_{v_j} K_j}^{-1} a_{v_j} K_j$ in (6) then the result is a word that defines the same element as U in H . That is, U and $\tau(U)$ must define the same element of H .

As before, let W_j be the j -th terminal segment of U . Then

$$K_j = \overline{W_{j+1}}, \quad \text{if } \varepsilon_j = 1,$$

and

$$K_j = \overline{W_j}, \quad \text{if } \varepsilon_j = -1.$$

Therefore, if $\varepsilon_j = 1$ then

$$\begin{aligned} S_{a_{v_j}, K_j} &= \overline{a_{v_j} K_j}^{-1} a_{v_j} K_j \\ &= \overline{W_j}^{-1} a_{v_j} \overline{a_{v_j}^{-1} W_j}, \end{aligned}$$

and if $\varepsilon_j = -1$ then

$$\begin{aligned} S_{a_{v_j}, K_j}^{-1} &= (\overline{a_{v_j} K_j}^{-1} a_{v_j} K_j)^{-1} \\ &= (\overline{a_{v_j} W_j}^{-1} a_{v_j} \overline{W_j})^{-1} \\ &= \overline{W_j}^{-1} a_{v_j}^{-1} \overline{a_{v_j} W_j}. \end{aligned}$$

But now $\tau(U) = S_{a_{v_1}, K_1}^{\epsilon_1} S_{a_{v_2}, K_2}^{\epsilon_2} \dots S_{a_{v_r}, K_r}^{\epsilon_r}$ just reduces to $U = a_{v_1}^{\epsilon_1} a_{v_2}^{\epsilon_2} \dots a_{v_r}^{\epsilon_r}$.

Therefore $\tau(U)$ defines the same element in H as U . Hence, τ is a rewriting procedure. q.e.d.

2.2.5 Definition. The rewriting procedure given in Corollary 2.2.4 is called a *Reidemeister rewriting procedure*.

2.2.6 Definition. A Reidemeister rewriting procedure that uses a Schreier Transversal is called a *Reidemeister-Schreier rewriting procedure*.

2.2.7 Theorem. Let G be presented as in (1), and let H be the subgroup of G . If τ is a Reidemeister-Schreier rewriting procedure, then H will have the following presentation:

$$H = \left\langle S_{a_v, K}, \dots ; S_{a_\lambda, M}, \dots, \tau(K^{-1} R_\mu K), \dots \right\rangle$$

where K is an arbitrary Schreier representative, a_v is an arbitrary generator of G , R_μ is an arbitrary defining relator of G , M is a Schreier representative and a_λ is a generator such that $a_\lambda M$ is freely equal to $\overline{a_\lambda M}$.

The proof of this result may be found in [12]. As we have already shown, the elements $S_{a_v, K}$ will generate H , but now we are able to find the defining relators of H . The term $S_{a_\lambda, M}$ will be a relator since

$$\begin{aligned} S_{a_\lambda, M} &= \overline{a_\lambda M}^{-1} a_\lambda M \\ &= 1. \end{aligned}$$

When rewriting the relators there is some simplification that occurs. For example consider

$$\tau((a_i a_{i+1} \cdots a_n)^{-1} R_\mu (a_i a_{i+1} \cdots a_n))$$

where R_μ is a relator and $a_i a_{i+1} \cdots a_n$ is an element of the Schreier transversal. Working from right to left, the first $n-i+1$ terms will be of the form

$$S_{a_j, a_{j+1} a_{j+2} \cdots a_n} = 1.$$

The last $n-i+1$ terms will be of the form

$$\begin{aligned} S^{-1} \frac{a_j, a_j^{-1} a_{j+1}^{-1} \cdots a_i^{-1} R_\mu a_i a_{i+1} \cdots a_n}{a_j, a_{j+1} a_{j+2} \cdots a_n} \\ = S_{a_j, a_{j+1} a_{j+2} \cdots a_n}^{-1} = 1, \end{aligned}$$

since R_μ is a relator in G and $\overline{a_j a_{j+1} \cdots a_n} = a_j a_{j+1} \cdots a_n$. So we only need to look at the terms remaining in the middle.

2.2.8 Example. Let us consider $T_4 = \{\beta \in B_4 \mid \pi(\beta)(4) = 4\}$, the subgroup of B_4 that consists of all braids where the fourth string finishes in the fourth position. We choose T_4 in B_4 since B_4 is the smallest example of a B_n where both types of relations are nontrivial. B_4 is generated by $\{\sigma_1, \sigma_2, \sigma_3\}$ and the set $\{1, \sigma_3, \sigma_2 \sigma_3, \sigma_1 \sigma_2 \sigma_3\}$ is a Schreier Transversal for T_4 in B_4 . The cosets of this subgroup are indexed by looking at which string goes to

the fourth position. For instance, the coset with the representative $\sigma_2\sigma_3$ is the set of braids where the second string ends in the fourth position. So by Theorem 2.2.7, T_4 is generated by:

$$\begin{array}{ll} S_{\sigma_1,1}, & S_{\sigma_2,1}, \\ S_{\sigma_3,1}, & S_{\sigma_1,\sigma_3}, \\ S_{\sigma_2,\sigma_3}, & S_{\sigma_3,\sigma_3}, \\ S_{\sigma_1,\sigma_2\sigma_3}, & S_{\sigma_2,\sigma_2\sigma_3}, \\ S_{\sigma_3,\sigma_2\sigma_3}, & S_{\sigma_1,\sigma_1\sigma_2\sigma_3}, \\ S_{\sigma_2,\sigma_1\sigma_2\sigma_3}, & S_{\sigma_3,\sigma_1\sigma_2\sigma_3}. \end{array}$$

First of all, $S_{\sigma_3,1}$, S_{σ_2,σ_3} , and $S_{\sigma_1,\sigma_2\sigma_3}$ are all equal to the identity in G ,

so they become relators in our new presentation. Now we must rewrite the relations $\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1} = 1$ and $\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$ for $i = 1, 2$. The way to find the coset representative of $a_{v_i}K_i$ is to look at the braid $a_{v_i}K$ and determine which string goes to the fourth

position. Let $M_1 = \sigma_1\sigma_2\sigma_3$, $M_2 = \sigma_2\sigma_3$, $M_3 = \sigma_3$, $M_4 = 1$.

$$(7) \quad \tau(M_4^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}M_4) = S_{\sigma_1,1} S_{\sigma_3,\sigma_3} S_{\sigma_1,\sigma_3}^{-1} S_{\sigma_3,\sigma_3}^{-1}$$

$$(8) \quad \tau(M_3^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}M_3) = S_{\sigma_1,\sigma_3} S_{\sigma_1,1}^{-1}$$

$$(9) \quad \tau(M_2^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}M_2) = S_{\sigma_1,\sigma_1\sigma_2\sigma_3} S_{\sigma_3,\sigma_1\sigma_2\sigma_3} S_{\sigma_1,\sigma_1\sigma_2\sigma_3}^{-1} S_{\sigma_3,\sigma_2\sigma_3}^{-1}$$

$$(10) \quad \tau(M_1^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}M_1) = S_{\sigma_3,\sigma_2\sigma_3} S_{\sigma_3,\sigma_1\sigma_2\sigma_3}^{-1}$$

$$(11) \quad \tau(M_4^{-1}\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}M_4) = S_{\sigma_1,1} S_{\sigma_2,1} S_{\sigma_1,1}^{-1} S_{\sigma_2,1}^{-1} S_{\sigma_1,1}^{-1} S_{\sigma_2,1}^{-1}$$

$$\begin{aligned}
(12) \quad \tau(M_3^{-1} \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} M_3) &= S_{\sigma_1, \sigma_3} S_{\sigma_2, \sigma_2 \sigma_3} S_{\sigma_1, \sigma_1 \sigma_2 \sigma_3} \cdot \\
&\quad S_{\sigma_2, \sigma_1 \sigma_2 \sigma_3}^{-1} S_{\sigma_1, \sigma_1 \sigma_2 \sigma_3}^{-1} S_{\sigma_2, \sigma_2 \sigma_3}^{-1} \\
(13) \quad \tau(M_2^{-1} \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} M_2) &= S_{\sigma_1, \sigma_1 \sigma_2 \sigma_3} S_{\sigma_2, \sigma_1 \sigma_2 \sigma_3} S_{\sigma_2, \sigma_2 \sigma_3}^{-1} S_{\sigma_1, \sigma_3}^{-1} \\
(14) \quad \tau(M_1^{-1} \sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} M_1) &= S_{\sigma_1, \sigma_3} S_{\sigma_2, \sigma_1 \sigma_2 \sigma_3}^{-1} \\
(15) \quad \tau(M_4^{-1} \sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} M_4) &= S_{\sigma_2, 1} S_{\sigma_3, \sigma_3} S_{\sigma_2, \sigma_2 \sigma_3} \cdot \\
&\quad S_{\sigma_3, \sigma_2 \sigma_3}^{-1} S_{\sigma_2, \sigma_2 \sigma_3}^{-1} S_{\sigma_3, \sigma_3}^{-1} \\
(16) \quad \tau(M_3^{-1} \sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} M_3) &= S_{\sigma_2, \sigma_2 \sigma_3} S_{\sigma_3, \sigma_2 \sigma_3} S_{\sigma_3, \sigma_3}^{-1} S_{\sigma_2, 1}^{-1} \\
(17) \quad \tau(M_2^{-1} \sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} M_2) &= S_{\sigma_2, 1} S_{\sigma_3, \sigma_2 \sigma_3}^{-1} \\
(18) \quad \tau(M_1^{-1} \sigma_2 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} M_1) &= S_{\sigma_2, \sigma_1 \sigma_2 \sigma_3} S_{\sigma_3, \sigma_1 \sigma_2 \sigma_3} S_{\sigma_2, \sigma_1 \sigma_2 \sigma_3} \cdot \\
&\quad S_{\sigma_3, \sigma_1 \sigma_2 \sigma_3}^{-1} S_{\sigma_2, \sigma_1 \sigma_2 \sigma_3}^{-1} S_{\sigma_3, \sigma_1 \sigma_2 \sigma_3}^{-1}
\end{aligned}$$

By (8), (10), (14), and (17) we can write all of the generators and relators in terms of $S_{\sigma_1, 1}$, $S_{\sigma_2, 1}$, S_{σ_3, σ_3} , $S_{\sigma_2, \sigma_2 \sigma_3}$, and $S_{\sigma_1, \sigma_1 \sigma_2 \sigma_3}$. Now

T_4 is generated by the following:

$$\begin{aligned}
S_{\sigma_1, 1} &= \sigma_1, \\
S_{\sigma_2, 1} &= \sigma_2, \\
S_{\sigma_3, \sigma_3} &= \sigma_3^2 \\
&= M_4^{-1} \sigma_3^2 M_4, \\
S_{\sigma_2, \sigma_2 \sigma_3} &= \sigma_3^{-1} \sigma_2^2 \sigma_3 \\
&= M_3^{-1} \sigma_2^2 M_3, \\
S_{\sigma_1, \sigma_1 \sigma_2 \sigma_3} &= (\sigma_2 \sigma_3)^{-1} \sigma_1^2 (\sigma_2 \sigma_3) \\
&= M_2^{-1} \sigma_1^2 M_2.
\end{aligned}$$

Let $a_i = M_{i+1}^{-1} \sigma_i^2 M_{i+1}$. We note that T_4 is generated by $\{\sigma_1, \sigma_2, a_1, a_2, a_3\}$ and has the following relations:

$$\begin{aligned}
 a_3 &= \sigma_1 a_3 \sigma_1^{-1} \\
 a_3 &= \sigma_2^{-1} a_2 \sigma_2 \\
 \sigma_1 a_2 a_1 &= a_2 a_1 \sigma_1 \\
 a_2 &= \sigma_1^{-1} a_1 \sigma_1 \\
 \sigma_2 a_3 a_2 &= a_3 a_2 \sigma_2 \\
 \sigma_1 \sigma_2 \sigma_1 &= \sigma_2 \sigma_1 \sigma_2 \\
 a_1 &= \sigma_2 a_1 \sigma_2^{-1}.
 \end{aligned}$$

Later we will show that the subgroup of T_4 generated by the a_i is free, so we choose not to use the relations to reduce our generating set to $\{\sigma_1, \sigma_2, a_1\}$.

CHAPTER 3

AUTOMORPHISMS OF THE FREE GROUP F_n

As shown in Chapter 1, the braid group B_n may be viewed as a subgroup of the group of automorphisms of the free group F_n . In this chapter we will look at some results that deal with automorphisms of the free group F_n . These results will be needed in the proof that B_n is torsion free.

3.1 A DECOMPOSITION OF THE FREE GROUP F_n

The following theorem which may be found in [10] shows a way to factor the free group F_n using an automorphism of prime order. The main result that we will need is a corollary to the theorem.

3.1.1 Theorem. Let F be free, and α an automorphism of F of prime order p . Then $F = F^{\langle \alpha \rangle} * (\ast_{i \in I} F_i) * (\ast_{\lambda \in \Lambda} F_\lambda)$, where $F^{\langle \alpha \rangle}$ is the fixed point set of $\langle \alpha \rangle$ and each factor is α -invariant. Moreover,

(i) For each $i \in I$, F_i has a free basis $\{x_{i,0}, x_{i,1}, \dots, x_{i,p-1}\}$ such that

$$\alpha(x_{i,r}) = x_{i,r-1} \pmod{p}$$

(ii) For each $\lambda \in \Lambda$, F_λ has a free basis,

$$\{x_{\lambda,1}, x_{\lambda,2}, \dots, x_{\lambda,p-1}\} \cup \{y_j \mid j \in J_\lambda\}$$

such that

$$\alpha(x_{\lambda,r}) = x_{\lambda,r-1}, \quad r = 2, \dots, p-1$$

$$\alpha(x_{\lambda,1}) = (x_{\lambda,p-1} x_{\lambda,p-2} \cdots x_{\lambda,1})^{-1}$$

$$\alpha(y_j) = x_{\lambda,p-1}^{-1} y_j x_{\lambda,p-1}.$$

Proof: The automorphism α has prime order p . We may therefore construct the semi-direct product $G = F \rtimes \mathbb{Z}_p$. An element in G will have the form (f, ζ^r) , where ζ generates $\mathbb{Z}_p$ and $f \in F$; multiplication in G will be defined by:

$$(f, \zeta^r)(g, \zeta^s) = (f \cdot \alpha^r(g), \zeta^{r+s}).$$

By Theorem 1 of [10], $G \cong (\bigast_{\lambda \in \Lambda} G_\lambda) * H$ where $G_\lambda \cong \mathbb{Z}_p \times H_\lambda$ with both H and H_λ free. The proof of this result involves graphs of groups and is beyond the scope of this paper. G now has a presentation:

$$(1) \quad \begin{array}{ll} \text{generators:} & \{c_\lambda \mid \lambda \in \Lambda\}; \{t_j \mid j \in J_\lambda\}; \{t_i \mid i \in I\} \\ \text{relations:} & c_\lambda^p = 1; t_j c_\lambda t_j^{-1} = c_\lambda, j \in J_\lambda, \end{array}$$

where the c_λ generate the Z_p 's, the $\{t_j \mid j \in J_\lambda\}$ generate the H_λ , and the $\{t_i \mid i \in I\}$ generate H . Pick $c \in G$ such that $c = (e, \zeta)$. If $x \in F \leq G$ then x will have the form (f, e) . So for $x \in F$, then

$$cxc^{-1} = (e, \zeta)(f, e)(e, \zeta^{-1}) = (\alpha(f), e) = \alpha(x),$$

or,

$$(2) \quad cxc^{-1} = \alpha(x).$$

I claim that c is conjugate to some element in some G_{λ_0} . Since $c^p = 1$ and H is free, $c \notin H$ and either $c \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$ or c is a reduced word in G with a length greater than one. If $c \in G_{\lambda_0}$ then done, if not

then both the first and last syllable of the word must be from H or both must be from the same G_{λ_i} for some $\lambda_i \in \Lambda$. Also the first and

last syllables must be inverses of each other since $c^p = 1$. So c is conjugate to some element in G with length less than the length of c , say $c = fgf^{-1}$. Now $g^p = 1$, so either $g \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$ or f is a

reduced word in G with length greater than one. By the same argument as above, g must be conjugate to some element in G with length less than the length of g . If we repeat this process, eventually we will get an element of length one that must be in some G_{λ_0} , and we will have c conjugate to some element in some G_{λ_0} . Say that $c = fg_0f^{-1}$, then $c \in f(G_{\lambda_0})f^{-1}$ for some $\lambda_0 \in \Lambda$. Since conjugation by f is an automorphism of G , we can conjugate everything in G by f to get a new free product representation of G and a new presentation of G via conjugates of the generators in (1). But this new free product

representation will have $\widehat{G}_{\lambda_0} = fG_{\lambda_0}f^{-1}$ as a factor, with $c \in \widehat{G}_{\lambda_0}$. We may therefore assume that $c \in G_{\lambda_0}$ for some $\lambda_0 \in \Lambda$.

Our goal now is to find a presentation for F . Since $G = F \rtimes \mathbf{Z}_p$, F is a subgroup of G and we can choose $c^r = (e, \zeta^r)$, $0 \leq r \leq p-1$, as left coset representatives for F in G . (But these are also right coset representatives, since F is normal in G .) We will now replace the generators of G with new generators that are amenable to the Reidemeister–Schreier rewriting procedure.

We would like to have $c_\lambda c^{-1} \in F$. (Recall that c_λ generates the $\mathbf{Z}_p$ factor of G_λ .) We do this by replacing c_λ with c_λ^m for some $m \in \{1, 2, \dots, p-1\}$ as follows. Since $c_\lambda \in G$, $c_\lambda = (f, e)(e, \zeta^n)$ for some $f \in F$ and integer n . Pick m such that $nm \equiv 1 \pmod{p}$. Then,

$$\begin{aligned} c_\lambda^m &= [(f, e)(e, \zeta^n)]^m \\ &= (g, \zeta^{nm}) \\ &= (g, \zeta) \\ &= (g, e)(e, \zeta) \\ &= (g, e)c \end{aligned}$$

for some $g \in F$. Note that c_λ^m will function just as well as c_λ in generating $\mathbf{Z}_p$ since p is prime. We will refer to c_λ^m as c_λ from now on, and $c_\lambda c^{-1} \in F$. Note that this implies that each c_λ is in the right coset Fc , and hence in the left coset cF by normality of F in G .

Now we would like $t_j \in F$. (Recall that the t_j generate the H_λ .)

We replace $t_j, j \in J_\lambda$, by $c_\lambda^n t_j = q_j$ for some $n \in \{1, 2, \dots, p-1\}$. Since $t_j = (f, \zeta^r)$ for some r , pick n such that $n+r \equiv 0 \pmod{p}$ then,

$$\begin{aligned} q_j &= c_\lambda^n t_j = (f_1, \zeta^n)(f, \zeta^r) \\ &= (f_2, \zeta^{n+r}) \\ &= (f_2, e). \end{aligned}$$

Note that the n may be different for each t_j , and $q_j \in F$. We may make this substitution since the t_j freely generate H_λ and $t_j c_\lambda = c_\lambda t_j$, so a word in $G_\lambda = \mathbb{Z}_p \times H_\lambda$ will have the form:

$$c_\lambda^g t_{j_1}^{\varepsilon_1} t_{j_2}^{\varepsilon_2} \dots t_{j_s}^{\varepsilon_s}$$

which can be written as

$$\begin{aligned} c_\lambda^g (c_\lambda^{-n_1 \varepsilon_1} q_{j_1}^{\varepsilon_1}) (c_\lambda^{-n_2 \varepsilon_2} q_{j_2}^{\varepsilon_2}) \dots (c_\lambda^{-n_s \varepsilon_s} q_{j_s}^{\varepsilon_s}) \\ = c_\lambda^{g_1} q_{j_1}^{\varepsilon_1} q_{j_2}^{\varepsilon_2} \dots q_{j_s}^{\varepsilon_s} \end{aligned}$$

for some new g_1 . The q_j generate a possibly new $\hat{H}_\lambda$, but we still have $G_\lambda = \langle c \rangle \times \hat{H}_\lambda$, and $q_j c_\lambda q_j^{-1} = c_\lambda$. From now on we will refer to q_j as t_j .

We also would like $t_i \in F$ so we similarly replace t_i by $c_\lambda^n t_i$ for some $n \in \{0, 1, \dots, p-1\}$ and refer to it as t_i .

Now we have another presentation for G that satisfies (1) and also:

$$\begin{aligned}
 (3) \quad & c = c_{\lambda_0} \text{ for some } \lambda_0 \in \Lambda, \\
 & cc_{\lambda}^{-1} \in F \text{ for all } \lambda \in \Lambda, \\
 & t_j, t_i \in F \text{ for all } j \in J_{\lambda}, i \in I.
 \end{aligned}$$

We can choose $1, c, \dots, c^{p-1}$ as left coset representatives of F in G and this is a Schreier Transversal. The coset representative is chosen by looking at the power of ζ in the element. For example,

$$\overline{c_{\lambda} c^r} = \overline{(g, \zeta)(f, \zeta^r)} = \overline{(h, \zeta^{r+1})} = c^{r+1}.$$

Recall that our new t_j and t_i are both of the form (f, e) for $f \in F$, so

$$\overline{t_j c^r} = \overline{(f, c \chi e, \zeta^r)} = \overline{(f, \zeta^r)} = c^r.$$

Similarly, $\overline{t_i c^r} = c^r$.

Now the Reidemeister-Schreier rewriting procedure yields a presentation of F as follows: $F = F_{\lambda_0} *_{i \in I} (*_{\lambda \in \Lambda} F_{\lambda})$. F_{λ_0} has:

$$\text{generators: } y_{j,r} = c^{-r} t_j c^r, \quad j \in J_{\lambda_0}, \quad r \in \{0, \dots, p-1\}$$

$$\text{relations: } y_{j,r} = y_{j,r-1}, \quad r \in \{1, \dots, p-1\}$$

For each $i \in I$, F_i is freely generated by:

$$x_{i,r} = c^{-r} t_i c^r, \quad r \in \{0, \dots, p-1\}.$$

For each $\lambda \in \Lambda, \lambda \neq \lambda_0$, F_{λ} has :

$$\text{generators: } x_{\lambda,r} = c^{-1-r} c_{\lambda} c^r,$$

$$y_{j,r} = c^{-r} t_j c^r, \quad r \in \{0, \dots, p-1\}$$

$$\text{relations: } x_{\lambda,p-1} x_{\lambda,p-2} \cdots x_{\lambda,0} = 1$$

$$x_{\lambda,r-1}^{-1} y_{j,r-1} x_{\lambda,r-1} = y_{j,r}, \quad r \in \{1, \dots, p-1\}$$

Recall that $G \cong G_{\lambda_0} * (\bigstar_{\substack{\lambda \in \Lambda \\ \lambda \neq \lambda_0}} G_{\lambda}) * H$. The F_{λ_0} term came from rewriting the generators and relators of G_{λ_0} , the F_i , for $i \in I$, terms came from rewriting the generators of H , and the $F_{\lambda}, \lambda \neq \lambda_0$, terms came from rewriting the generators and relators of the $G_{\lambda}, \lambda \neq \lambda_0$. For an example of how the rewriting procedure works, let us look at the generators and relators for $F_{\lambda}, \lambda \neq \lambda_0$. Recall that $G_{\lambda} \cong \mathbb{Z}_p \times H_{\lambda}, \lambda \neq \lambda_0$, where $\mathbb{Z}_p$ is generated by c_{λ} , and $\{t_j \mid j \in J_{\lambda}\}$ generate H_{λ} .

Generators :

$$\begin{aligned} x_{\lambda,r} &= S_{c_{\lambda}, c^r} \\ &= \overline{(c_{\lambda} c^r)^{-1}} (c_{\lambda} c^r) \\ &= c^{-1-r} c_{\lambda} c^r, \end{aligned}$$

$$\begin{aligned} y_{j,r} &= S_{t_j, c^r} \\ &= \overline{(t_j c^r)^{-1}} (t_j c^r) \\ &= c^{-r} t_j c^r. \end{aligned}$$

Relations : The relation $c_{\lambda}^p = 1$ gives us the following:

$$\begin{aligned} \tau(c^{-r} c_{\lambda}^p c^r) &= S_{c_{\lambda}, c^{r+p-1}} \cdot S_{c_{\lambda}, c^{r+p-2}} \cdots S_{c_{\lambda}, c^r} \\ &= x_{\lambda, r+p-1} \cdot x_{\lambda, r+p-2} \cdots x_{\lambda, r} \end{aligned}$$

If we let $r = 0$ then we get the relation

$$1 = x_{\lambda, p-1} \cdot x_{\lambda, p-2} \cdots x_{\lambda, 0}.$$

The relation $t_j c_{\lambda} t_j^{-1} = c_{\lambda}$ gives us the following:

$$\begin{aligned}\tau(c^{-r} t_j c_\lambda t_j^{-1} c_\lambda^{-1} c^r) &= S_{c_\lambda, c^{r+1}}^{-1} \cdot S_{t_j, c^{r+1}}^{-1} \cdot S_{c_\lambda, c^{r+1}} \cdot S_{t_j, c^r} \\ &= y_{j,r} \cdot x_{\lambda, r-1} \cdot y_{j,r-1}^{-1} \cdot x_{\lambda, r-1}^{-1}\end{aligned}$$

Therefore we get the relation

$$y_{j,r-1} = x_{\lambda, r-1}^{-1} \cdot y_{j,r} \cdot x_{\lambda, r-1}.$$

We need to find the centralizer of c for reasons that will become apparent in a moment. We know by (3) that $c \in G_{\lambda_0}$, and since $G_{\lambda_0} \cong Z_p \times H_{\lambda_0}$, c will have the form $c = (\xi, e)$ where $\xi \in Z_p$. Recall that $G \cong (\bigast_{\lambda \in \Lambda} G_\lambda) * H$ and $G = F \rtimes Z_p$. Up until this point we have said that $c = (e, \zeta)$; this is viewing $c \in G = F \rtimes Z_p$. Now we are looking at $c \in G \cong (\bigast_{\lambda \in \Lambda} G_\lambda) * H$, and this is why it will also have the form $c = (\xi, e)$.

The centralizer of c in G consists of all of the elements b such that $cb = bc$ or $c = bcb^{-1}$. G is a free product, so in order for this last equation to be defined, b must be in G_{λ_0} ; in fact for all $b \in G_{\lambda_0}$,

$b = (\xi^s, h)$ then

$$bcb^{-1} = (\xi^s, h)(\xi, e)(\xi^{-s}, h^{-1}) = c.$$

So G_{λ_0} is the centralizer of c in G . I claim that $F \cap G_{\lambda_0} = F_{\lambda_0} = F^{\langle \alpha \rangle}$, the fixed point set of α . Pick $x \in F \cap G_{\lambda_0}$. Now $x \in G_{\lambda_0}$ so $x = c^r w_t$, (where w_t is a word in the t_j) since $G_{\lambda_0} \cong Z_p \times H_{\lambda_0}$. But the c^j are the coset representatives for F in G , so $x \in F$ implies that $r = 0$. Therefore $x = w_t$ and $x \in F_{\lambda_0}$. Next pick $x \in F_{\lambda_0}$, then $x = (e, h)$ with $h \in H_{\lambda_0}$. Now by (2)

$$\alpha(x) = cxc^{-1} = (\zeta, e)(e, h)(\zeta, e)^{-1} = (e, h) = x,$$

and $x \in F^{\langle \alpha \rangle}$. Next pick $x \in F^{\langle \alpha \rangle}$, then $x \in F$ and $\alpha(x) = cxc^{-1} = x$ implies that x is in the centralizer of c which is G_{λ_0} , so $x \in F \cap G_{\lambda_0}$. If

we put all of this together we get,

$$F \cap G_{\lambda_0} \subseteq F_{\lambda_0} \subseteq F^{\langle \alpha \rangle} \subseteq F \cap G_{\lambda_0}.$$

Hence,

$$F \cap G_{\lambda_0} = F_{\lambda_0} = F^{\langle \alpha \rangle}.$$

Therefore $F_{\lambda_0} = F^{\langle \alpha \rangle}$, the fixed point set of $\langle \alpha \rangle$, which is freely generated by $\{y_{j,0} = t_j \mid j \in J_{\lambda_0}\}$.

Within $F_{\lambda}, \lambda \neq \lambda_0$, we may eliminate $y_{j,r}, r=1, 2, \dots, p-1$, and $x_{\lambda,0}$ since these can be written in terms of the other generators, hence $F_{\lambda}, \lambda \neq \lambda_0$, has free generators $\{x_{\lambda,1}, x_{\lambda,2}, \dots, x_{\lambda,p-1}\} \cup \{y_{j,0} = t_j \mid j \in J_{\lambda}, \lambda \neq \lambda_0\}$. We will refer to $y_{j,0}$ as y_j . We can now use (2) to verify that the action of α is as specified in the theorem.

$$\begin{aligned} \alpha(x_{i,r}) &= c x_{i,r} c^{-1} & r = 0, \dots, p-1 \\ &= c c^{-r} t_i c^r c^{-1} \\ &= c^{-(r-1)} t_i c^{r-1} \\ &= x_{i,r-1 \pmod{p}}. \end{aligned}$$

$$\begin{aligned} \alpha(x_{\lambda,r}) &= c x_{\lambda,r} c^{-1} & r = 2, \dots, p-1 \\ &= c c^{-1-r} c_{\lambda}^r c^{-1} \\ &= c^{-1-(r-1)} t_i c^{r-1} \\ &= x_{\lambda,r-1}, \end{aligned}$$

$$\begin{aligned}
\alpha(x_{\lambda,1}) &= c x_{\lambda,1} c^{-1} \\
&= c c^{-1-1} c_{\lambda} c^1 c^{-1} \\
&= c^{-1} c_{\lambda} c^0 \\
&= x_{\lambda,0} \\
&= (x_{\lambda,p-1} x_{\lambda,p-2} \cdots x_{\lambda,1})^{-1},
\end{aligned}$$

$$\begin{aligned}
\alpha(y_j) &= c y_j c^{-1} \\
&= c t_j c^{-1} \\
&= y_{j,p-1} \\
&= x_{\lambda,p-1} y_j x_{\lambda,p-1}^{-1},
\end{aligned}$$

and we are done.

q.e.d.

The next corollary shows that in some cases there will be fewer factors in this decomposition of F_n . Recall from Section 1.2 that if α is an automorphism of F_n then $F_n(\alpha)$ is the maximal quotient group of F_n upon which α induces the trivial automorphism.

3.1.2 Corollary. Let F_n, α be given as above. If $\text{Ab } F_n(\alpha)$ is torsion-free then no F_{λ} factors occur in this factorization of $F_n, \lambda \neq \lambda_0$.

Recall that in Corollary 1.2.2 it was shown that if $\beta \in B_n < \text{Aut}(F_n)$ then $\text{Ab } F_n(\beta)$ is torsion-free. We will use this to decompose F_n in the proof that B_n is torsion-free.

Proof of 3.1.2: Assume that $\text{Ab } F_n(\alpha)$ is torsion-free. $F_n(\alpha)$ is a free product of groups of the form $F^{(\alpha)}$, $F_i(\alpha)$, and $F_\lambda(\alpha)$. It follows that $\text{Ab } F_n(\alpha)$ is a direct product of groups of the form $\text{Ab } F^{(\alpha)}$, $\text{Ab } F_i(\alpha)$, and $\text{Ab } F_\lambda(\alpha)$. In F_λ we have,

$$\alpha(x_{\lambda,r}) = x_{\lambda,r-1}, \quad r = 1, 2, \dots, p-1.$$

So if $\overline{x_{\lambda,r}}$ and $\overline{x_{\lambda,r-1}}$ are the images of $x_{\lambda,r}$ and $x_{\lambda,r-1}$ in $F_\lambda(\alpha)$, we get $\overline{x_{\lambda,r}} = \overline{x_{\lambda,r-1}}$, $r = 1, 2, \dots, p-1$. All of the generators in F_λ will reduce to one generator in $F_\lambda(\alpha)$. We will refer to this generator in $F_\lambda(\alpha)$ as $x_{\lambda,p-1}$. Recall that $x_{\lambda,r} = c^{-1-r} c_\lambda^r$ and consider

$$\begin{aligned} (x_{\lambda,p-1})^p &= x_{\lambda,p-1} x_{\lambda,p-2} \cdots x_{\lambda,0} \\ &= (c^{-p} c_\lambda^{p-1}) (c^{-(p-1)} c_\lambda^{p-2}) \cdots (c^{-1} c_\lambda^0) \\ &= c_\lambda^p \\ &= e. \end{aligned}$$

So this generator has order p in $F_\lambda(\alpha)$ as well as $\text{Ab } F_\lambda(\alpha)$. More

importantly, this generator is a torsion element. Therefore, if

$\text{Ab } F_n(\alpha)$ is torsion-free no F_λ factors occur in this factorization of

F_n .

q.e.d.

It should also be noted that $\text{Ab } F^{(\alpha)}$ is a free abelian group with rank equal to the rank of $F^{(\alpha)}$. Also the $\text{Ab } F_i(\alpha)$ will be infinite cyclic since in F_i , $\alpha(x_{i,r}) = x_{i,r-1} \pmod{p}$. As a result, these generators

in F_i will reduce to only one in $\text{Ab } F_i(\alpha)$, and it will have infinite order.

3.2 AUTOMORPHISMS OF THE FREE GROUP AND MATRICES

We will need to look at the function $\text{Tr}: \text{Aut}(F_n) \rightarrow \mathbb{Z}$ defined by the composite

$$\text{Aut}(F_n) \rightarrow \text{Aut}(\text{Ab } F_n) \cong \text{GL}(n, \mathbb{Z}) \xrightarrow{\text{trace}} \mathbb{Z},$$

where $\text{Aut}(F_n) \rightarrow \text{Aut}(\text{Ab } F_n)$ is the homomorphism induced by the canonical projection $F_n \rightarrow \text{Ab } F_n$. We will show that

$\text{Aut}(\text{Ab } F_n) \cong \text{GL}(n, \mathbb{Z})$ and then show that even though this isomorphism depends on the basis that we choose for $\text{Ab } F_n$, its composition with the trace function does not.

3.2.1 Theorem. $\text{Aut}(\text{Ab } F_n) \cong \text{GL}(n, \mathbb{Z})$.

Proof: Let $\{x_1, x_2, \dots, x_n\}$ be a basis for F_n . Pick $f \in \text{Aut}(\text{Ab } F_n)$. Then there exist $r_{ij} \in \mathbb{Z}$ such that

$$\begin{aligned} f(x_1) &= r_{11}x_1 + r_{21}x_2 + \cdots + r_{n1}x_n \\ f(x_2) &= r_{12}x_1 + r_{22}x_2 + \cdots + r_{n2}x_n \\ &\vdots \\ f(x_n) &= r_{1n}x_1 + r_{2n}x_2 + \cdots + r_{nn}x_n. \end{aligned}$$

The r_{ij} are unique since the x_i form a basis. Define

$\theta: \text{Aut}(\text{Ab } F_n) \rightarrow \text{GL}(n, \mathbb{Z})$ by $\theta(f) = A$ where A is the $n \times n$ matrix (r_{ij}) .

θ is a multiplicative homomorphism. If $\theta(f) = I$ the identity matrix then $f(x_i) = x_i$ for all x_i which implies that f is the identity

automorphism. Therefore the kernel is trivial and θ is one-to-one.

Next pick $(r_{ij}) = A \in \text{GL}(n, \mathbb{Z})$ define f by

$f(x_i) = r_{1i}x_1 + r_{2i}x_2 + \cdots + r_{ni}x_n$ for $1 \leq i \leq n$. This is an automorphism of $\text{Ab } F_n$ since $\det A \neq 0$. Hence θ is onto and

$\text{Aut}(\text{Ab } F_n) \cong \text{GL}(n, \mathbb{Z})$.

q.e.d.

As shown in the proof above, this isomorphism depends on the basis that we choose for $\text{Ab } F_n$. But changing the basis for $\text{Ab } F_n$ will change the matrices in $\text{GL}(n, \mathbb{Z})$ by a similarity. That is, if $A \in \text{GL}(n, \mathbb{Z})$ is the image of $f \in \text{Aut}(\text{Ab } F_n)$ relative to a basis B_1 and $D \in \text{GL}(n, \mathbb{Z})$ is the image of f relative to a different basis B_2 , then A and D are similar matrices, i.e., there exists an invertible matrix S such that $A = SDS^{-1}$. But

$$\text{Tr}(A) = \text{Tr}((SD)S^{-1}) = \text{Tr}(S^{-1}SD) = \text{Tr}(D),$$

since $\text{Tr}(XY) = \text{Tr}(YX)$ for any $n \times n$ matrices X and Y . Hence, this function does not depend on the basis of $\text{Ab } F_n$.

Recall from Theorem 1.2.1 that $\beta \in B_n$ can be viewed as an automorphism of F_n that sends each x_i to a conjugate of some x_j . The image of β in $\text{Aut}(\text{Ab } F_n)$ will just be a permutation of the generators, which will correspond to a permutation matrix in $\text{GL}(n, \mathbb{Z})$. Therefore $\text{Tr}(\beta)$ will be equal to the number of generators of F_n that are fixed

by the image of β in $\text{Aut}(\text{Ab } F_n)$ as well as the number of strings that β fixes in B_n . For example, consider $\sigma_1 \in B_3$. Recall that in Section 1.2 we showed that σ_1 can be viewed as the automorphism $\alpha_1 \in \text{Aut}(F_n)$ defined by

$$\alpha_1 = \begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_2^{-1} x_1 x_2, \\ x_3 \rightarrow x_3. \end{cases}$$

Under the projection $\text{Aut}(F_n) \rightarrow \text{Aut}(\text{Ab } F_n)$, α_1 will be sent to the automorphism

$$\hat{\alpha}_1 = \begin{cases} x_1 \rightarrow x_2, \\ x_2 \rightarrow x_1, \\ x_3 \rightarrow x_3, \end{cases}$$

of $\text{Aut}(\text{Ab } F_n)$. Under the map from $\text{Aut}(\text{Ab } F_n) \rightarrow \text{GL}(n, \mathbb{Z})$, defined in Theorem 3.2.1, the image of α_1 is the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, for $\sigma_1 \in B_3$, $\text{Tr}(\sigma_1) = 1$.

CHAPTER 4

SUBGROUPS OF B_n

The following theorem deals with the subgroups T_j of B_n that were described in Section 1.1. When we prove that B_n is torsion-free, one case will be to show that these subgroups do not contain torsion elements.

4.1 Theorem. For each j , $1 \leq j \leq n$ define $T_j < B_n$ by

$$T_j = \{\beta \in B_n \mid \pi(\beta)(j) = j\}$$

Then T_j fits into a short exact sequence

$$1 \rightarrow U_n \rightarrow T_j \rightarrow B_{n-1} \rightarrow 1$$

where U_j is free.

We shall prove the result for $T_n < B_n$. The proof for $T_j < B_n$ is similar.

Proof of Theorem 4.1: First we will use the Reidemeister-Schreier rewriting procedure to find a presentation for $T_n < B_n$. The elements $M_n = 1$ and $M_i = \sigma_i \cdots \sigma_{n-2} \sigma_{n-1}$ for $1 \leq i < n$ form a Schreier

Transversal for $T_n < B_n$. As in Example 2.2.8, the cosets are indexed by looking at which string goes to the n -th position.

The following relations will be of use in simplifying the generators and relations. They are consequences of the defining relations of B_n . Recall from Example 2.2.8 that $a_i = M_{i+1}^{-1} \sigma_i^2 M_{i+1}$.

$$\begin{aligned} M_j \sigma_k^\varepsilon &= \sigma_k^\varepsilon M_j & k < j-1, \varepsilon = \pm 1 \\ \sigma_{j-1} M_j &= M_{j-1} \\ \sigma_j M_j &= M_{j+1} a_j \\ \sigma_k^\varepsilon M_j &= M_j \sigma_{k-1}^\varepsilon & k \geq j+1, \varepsilon = \pm 1. \end{aligned}$$

Also, for $\varepsilon = \pm 1$,

$$\overline{\sigma_i^\varepsilon M_j} = \begin{cases} M_j, & i < j-1 \text{ or } i > j \\ M_i, & i = j-1 \\ M_{i+1}, & i = j \end{cases}$$

Now we use Reidemeister-Schreier rewriting procedure to find the generators of T_n and get the following:

$$(1) \quad S_{\sigma_i, M_j} = \begin{cases} \sigma_i, & i < j-1 \\ 1, & i = j-1 \\ M_{i+1}^{-1} \sigma_i^2 M_{i+1} = a_i, & i = j \\ \sigma_{i-1}, & i > j \end{cases}$$

Using this information, we get that T_n is generated by the elements

$$\begin{aligned} &\sigma_1, \sigma_2, \dots, \sigma_{n-2} \\ a_i &= M_{i+1}^{-1} \sigma_i^2 M_{i+1}, \quad i = 1, 2, \dots, n-1. \end{aligned}$$

We will show that the a_i are the generators of a free group U_n .

When we rewrite the relator $\sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$ we get

$$\tau(M_j^{-1} \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} M_j) = S \frac{\quad}{\sigma_i, \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} M_j} S \frac{\quad}{\sigma_{i+1}, \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} M_j} \\ S \frac{\quad}{\sigma_i, \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} M_j} S^{-1} \frac{\quad}{\sigma_{i+1}, \sigma_i^{-1} \sigma_{i+1}^{-1} M_j} S^{-1} \frac{\quad}{\sigma_i, \sigma_{i+1}^{-1} \sigma_i^{-1} M_j} S^{-1} \frac{\quad}{\sigma_{i+1}, \sigma_i^{-1} M_j}$$

This breaks down into five cases, the results of which are outlined below.

$$(2) \quad \tau(M_j^{-1} \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} M_j) = \begin{cases} \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}, & i < j-2 \\ \sigma_i a_{i+1} a_i \sigma_i^{-1} a_i^{-1} a_{i+1}^{-1}, & i = j-2 \\ a_i \sigma_i a_{i+1}^{-1} \sigma_i^{-1}, & i = j-1 \\ 1, & i = j \\ \sigma_{i-1} \sigma_i \sigma_{i-1} \sigma_i^{-1} \sigma_{i-1}^{-1} \sigma_i^{-1}, & i > j \end{cases}$$

Now we rewrite the relator $\sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}$ where $|i - k| \geq 2$.

$$\tau(M_j^{-1} \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1} M_j) = S \frac{\quad}{\sigma_i, \sigma_k \sigma_i^{-1} \sigma_k^{-1} M_j} S \frac{\quad}{\sigma_k, \sigma_i^{-1} \sigma_k^{-1} M_j} S^{-1} \frac{\quad}{\sigma_i, \sigma_i^{-1} \sigma_k^{-1} M_j} S^{-1} \frac{\quad}{\sigma_k, \sigma_k^{-1} M_j}$$

This breaks down into six cases:

$$(3) \quad \tau(M_j^{-1} \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1} M_j) = \begin{cases} \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}, & i, k < j-1 \\ \sigma_i a_k \sigma_i^{-1} a_k^{-1}, & k = j-1, i < j-2 \\ \sigma_{i-1} a_k \sigma_{i-1}^{-1} a_k^{-1}, & k = j-1, i > j \\ 1, & k = j \\ \sigma_{i-1} \sigma_{k-1} \sigma_{i-1}^{-1} \sigma_{k-1}^{-1}, & i, k > j \\ \sigma_i \sigma_{k-1} \sigma_i^{-1} \sigma_{k-1}^{-1}, & k > j, i < j-1 \end{cases}$$

So this subgroup has the following relations:

$$(4) \quad \sigma_i \sigma_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} = 1$$

$$(5) \quad \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1} = 1 \quad |i - k| \geq 2$$

$$(6) \quad \sigma_i a_k \sigma_i^{-1} = a_k \quad (k \neq i, i+1)$$

$$(7) \quad \sigma_i a_i \sigma_i^{-1} = a_{i+1}$$

$$(8) \quad a_{i+1} a_i \sigma_i = \sigma_i a_{i+1} a_i.$$

Let U_n denote the subgroup of T_n generated by $a_1, a_2, \dots, a_{n-1}$.

I claim that this subgroup is free. Consider the following automorphisms of F_{n-1} , the free group generated by $x_1, x_2, \dots, x_{n-1}$, which are associated with the $2n-3$ generators of T_n .

$$\hat{\sigma}_i(x_j) = \begin{cases} x_{j+1}, & j = i \\ x_j^{-1} x_{j-1} x_j, & j = i+1 \\ x_j, & j \neq i, i+1 \end{cases}$$

$$\hat{a}_k(x_i) = x_k^{-1} x_i x_k,$$

where $\hat{\sigma}_i$ is defined for $1 \leq i \leq n-2$ and $\hat{a}_k$ is defined for $1 \leq k \leq n-1$.

It can be verified that these automorphisms satisfy the relations (4) – (8). Therefore the subgroup of $\text{Aut}(F_{n-1})$ generated by these automorphisms is a homomorphic image of T_n since the

Reidemeister–Schreier rewriting procedure guarantees that (4) – (8) and their consequences are the only relations satisfied by the generators on T_n . It follows that the subgroup of $\text{Aut}(F_{n-1})$ generated by the $\hat{a}_i$ ($1 \leq i \leq n-1$) is a homomorphic image of U_n .

Consider the map

$$\begin{aligned}\Phi: F_{n-1} &\rightarrow \text{Aut}(F_{n-1}) \\ x_k &\rightarrow \hat{a}_k.\end{aligned}$$

This is a homomorphism whose kernel is the center of F_{n-1} and hence is trivial. This implies that the image of Φ , the inner automorphisms, is isomorphic to F_{n-1} and, therefore, is free. But we can define

$$\varphi: T_n \rightarrow \text{Aut}(F_{n-1})$$

by

$$\begin{aligned}\sigma_i &\rightarrow \hat{\sigma}_i, \\ a_i &\rightarrow \hat{a}_i.\end{aligned}$$

This is a well defined function since the $\hat{\sigma}_i$ and $\hat{a}_i$ satisfy (4) – (8).

The image of U_n under φ is $\langle \hat{a}_1, \hat{a}_2, \dots, \hat{a}_{n-1} \rangle$. If U_n were not free, any relation among the a_i would be sent to a relation among the $\hat{a}_i$, which would contradict the freeness of $\langle \hat{a}_1, \hat{a}_2, \dots, \hat{a}_{n-1} \rangle$.

Therefore U_n is free.

We must now show that U_n fits into the short exact sequence. If we consider T_n / U_n we will get a group isomorphic to B_{n-1} since, by

looking at (4) – (8) we see that the presentation of T_n / U_n is just the same as that of B_{n-1} . But now

$$1 \rightarrow U_n \rightarrow T_n \rightarrow T_n / U_n \cong B_{n-1} \rightarrow 1$$

gives us an exact sequence as desired with U_n free. q.e.d.

When we prove that B_n is torsion free, the proof will be done by induction. In the inductive step we will be assuming that B_{n-1} is torsion free and will need the following corollary to show that T_j does not contain any torsion elements.

4.2 Corollary. If B_{n-1} is torsion-free then T_j is also torsion-free.

Proof: As in the above theorem consider the short exact sequence

$$1 \rightarrow U_n \xrightarrow{\phi} T_j \xrightarrow{\theta} B_{n-1} \rightarrow 1.$$

Pick $x \in T_j$ with $x \neq e$ such that $x^r = e$ for some $r > 0$. Since ϕ is a one-to-one map and U_n is free, $x \notin \text{Im } \phi$, so by exactness $x \notin \text{Ker } \theta$. Now $\theta(x) = z$ not the identity of B_{n-1} . But $\theta(x^r) = z^r$ so $z^r = e$. Since we are assuming that B_{n-1} is torsion-free, $z = e$. This implies that $x = e$, a contradiction. Hence T_j is torsion-free. q.e.d.

CHAPTER 5

B_n IS TORSION-FREE

In this chapter we will prove that B_n is torsion-free. This result depends heavily on the previous work in this thesis. Then we will look at some results from group theory that will allow us to give an algebraic proof of the conjugate-power problem.

5.1 Theorem. B_n is torsion-free.

Proof: We prove this by induction on n , the number of strings. From the presentation of B_n it should be clear that B_1 is trivial and B_2 is infinite cyclic, generated by σ_1 . Let us assume that B_{n-1} is torsion-free and consider B_n .

Suppose there exists $\beta \in B_n$ with $\beta \neq e$ and $\beta^r = e$ for some $r > 1$.

We may also assume that r is prime since if it is not we can use some power of β to get an element of prime order. Because we are assuming that B_{n-1} is torsion-free, by Corollary 4.2 $\beta \notin T_j$ for $1 \leq j \leq n$. Since β does not fix any strings, $\text{Tr}(\beta) = 0$, where $\text{Tr}: \text{Aut}(F_n) \rightarrow \mathbb{Z}$ is the function defined in Section 3.2.

By Theorem 1.2.1, we may think of β as an element of $\text{Aut}(F_n)$.

By Theorem 3.1.1 and Corollary 3.1.2, the elements

$y_1, y_2, \dots, y_m, x_{i,0}, x_{i,1}, \dots, x_{i,p-1}$ ($1 \leq i \leq k$) are a basis for F_n . Recall that $F_n^{(\beta)}$ is freely generated by $\{y_j \mid j \in J_{\lambda_0}\}$ where $J_{\lambda_0} = \{1, 2, \dots, m\}$ for some m . The $x_{i,j}$ generate $(\ast_{i \in I} F_i)$ where $I = \{1, 2, \dots, k\}$ for some k . β

will act as follows on these generators:

$$\beta(y_i) = y_i \quad (1 \leq i \leq m)$$

$$\beta(x_{i,r}) = x_{i,r-1 \pmod{p}} \quad (1 \leq i \leq k, 0 \leq r \leq p-1).$$

Since each of the y_i is fixed by β , the matrix with respect the image of this basis in $\text{Ab } F_n$ will have trace m . Hence $\text{Tr}(\beta) = m = 0$. This implies that the fixed point set of β in F_n has rank 0, but this is a contradiction to Corollary 1.2.3 (2). Therefore B_n is torsion-free.

q.e.d.

5.2 Definition. Let Π be a type of group, e.g., finite, abelian, etc. A group is *residually* Π if G satisfies the following condition: For any element $g \neq e$ in G there exists a normal subgroup K of G such that $g \notin K$ and G/K is of the type Π .

For the next corollary we will look the group $P_n = \text{Ker } \{\pi: B_n \rightarrow S_n\}$.

This is called the set of *pure braids*. P_n is a subgroup of

$\text{Ker}\{\text{Aut}(F_n) \rightarrow \text{Aut}(\text{Ab } F_n)\}$, which is residually torsion-free

nilpotent (see [3]). By the following lemma this will imply that P_n is residually torsion-free nilpotent.

5.3 Lemma. A subgroup of a residually torsion-free nilpotent group is residually torsion-free nilpotent.

Proof: Let H be a subgroup of a group G where G is residually torsion-free nilpotent. So for any $g \neq e$ in H there exists a normal subgroup, K of G such that G/K is torsion-free nilpotent and $g \notin K$. I claim that $H/(H \cap K)$ is torsion-free nilpotent. First, $H \cap K$ is a normal subgroup of H and $g \notin H \cap K$. Next, by the Second Isomorphism Theorem, $H/(H \cap K) \cong HK/K$. HK/K is a subgroup of G/K and a subgroup of a torsion-free nilpotent group is torsion-free nilpotent. Hence $H/(H \cap K)$ is torsion-free nilpotent. Therefore H is residually torsion-free nilpotent. q.e.d.

5.4 Theorem. P_n is residually torsion-free nilpotent.

Proof: P_n is a subgroup of a residually torsion-free nilpotent group. Hence, by Lemma 5.3, P_n is residually torsion-free nilpotent. q.e.d.

5.5 Definition. If H and K are subgroups of group G , let $[H, K]$ be the subgroup of G generated by the elements $hkh^{-1}k^{-1}$, $h \in H$, $k \in K$. These elements are called *commutators*.

5.6 Definition. The *lower central series* of a group G is the chain of subgroups $G_{(i)}$ defined as follows:

$$G_{(1)} = G, G_{(2)} = [G, G], \dots, G_{(i)} = [G, G_{(i-1)}].$$

5.7 Theorem. A group is nilpotent if and only if $G_{(k)} = \{e\}$ for some k .

5.8 Theorem (Gruenberg) Let G be a finitely generated nilpotent group. If G is torsion-free then G is residually a finite p -group for every prime p .

The proof of Theorem 5.8 may be found in [16]. Note that B_n is a finitely generated group so P_n is a subgroup of a finitely generated group of index $n!$, so by the Reidemeister-Schreier rewriting procedure it is also finitely generated.

5.9 Lemma. Let $z \in P_n$. If z^r is conjugate to z^s in P_n then $r = s$.

Proof: If z^r is conjugate to z^s then $z^r = az^sa^{-1}$ for some $a \in P_n$. Note that $z^{r-s} = az^sa^{-1}z^{-s}$ is a commutator. We will show that every prime divides $r-s$, which will imply that $r-s = 0$ and $r = s$. P_n is residually torsion-free nilpotent so there exists a normal subgroup N of P_n such that $z \notin N$ and P_n/N is torsion-free nilpotent. Let $G = P_n/N$ and $\bar{z} = zN$. Now G is a finitely generated torsion-free nilpotent group, so by Gruenberg's Theorem G is residually a finite p -group for every prime p . Pick a prime p . There exists a normal subgroup K of G such that $\bar{z} \notin K$ and G/K is a finite p -group. Let $H = G/K$. H is a nilpotent group so $\bar{z}K \in H_{(t)}$ and $\bar{z}K \notin H_{(t+1)}$ for some t , but

$$(\bar{z}K)^{r-s} = (\bar{a}K)(\bar{z}K)^s(\bar{a}K)^{-1}(\bar{z}K)^{-s}$$

which implies that $(\bar{z}K)^{r-s} \in H_{(t+1)}$. So in $H_{(t)}/H_{(t+1)}$, the image of $\bar{z}K$ has order dividing $r-s$ and since $H_{(t)}/H_{(t+1)}$ is a p -group, p divides $r-s$. We may do this for every prime p , so $r-s = 0$. q.e.d.

5.10 Corollary (Birman). Let β be a non-trivial element of B_n . If β^r is conjugate to β^s in B_n (where r and s are integers), then $r = \pm s$. Moreover, $r = -s$ is possible when $n \geq 3$.

Proof: Consider the equation

$$\beta^r = \alpha^{-1} \beta^s \alpha$$

$(\alpha, \beta \in B_n, \beta \neq e, r, s \in \mathbb{Z})$. Now we will show that this implies

$$\beta^{r^k} = \alpha^{-k} \beta^{s^k} \alpha^k \quad (*)$$

by induction on k . Assume that $(*)$ is true and consider $\beta^{r^{k+1}}$.

$$\begin{aligned} (\beta^{r^k})^r &= (\alpha^{-k} \beta^{s^k} \alpha^k)^r \\ &= \alpha^{-k} \beta^{s^k r} \alpha^k \\ &= \alpha^{-k} (\beta^r)^{s^k} \alpha^k \\ &= \alpha^{-k} (\alpha^{-1} \beta^s \alpha)^{s^k} \alpha^k \\ &= \alpha^{-(k+1)} \beta^{s^{k+1}} \alpha^{k+1}. \end{aligned}$$

It is an immediate consequence that

$$\beta^{jr^k} = \alpha^{-k} \beta^{js^k} \alpha^k.$$

Since P_n has index $n!$ in B_n , if we let $j = k = n!$ we get

$$\beta^{(n!)r^{n!}} = (\alpha^{-n!}) (\beta^{(n!)s^{n!}}) (\alpha^{n!}),$$

where each of these terms is now in P_n . B_n is torsion free and $\beta \neq e$ so $\beta^{n!} \neq 1$. P_n is residually torsion-free nilpotent, hence by Lemma 5.9, $r^{n!} = s^{n!}$. Since r and s are integers, $r = \pm s$.

I claim that $\sigma_i \sigma_{i+1}^{-1}$, for $1 \leq i \leq n-2$, is conjugate to its inverse in B_n . If we conjugate $\sigma_i \sigma_{i+1}^{-1}$ by $\sigma_i \sigma_{i+1} \sigma_i$ we get

$$\begin{aligned} (\sigma_i \sigma_{i+1} \sigma_i) \sigma_i \sigma_{i+1}^{-1} (\sigma_i \sigma_{i+1} \sigma_i)^{-1} &= \sigma_{i+1} (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \\ &= \sigma_{i+1} \sigma_{i+1} \sigma_i \sigma_{i+1} \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \sigma_i^{-1} \\ &= \sigma_{i+1} \sigma_i^{-1}. \end{aligned}$$

Hence $r = -s$ is possible when $n \geq 3$.

q.e.d.

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