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Orbital stability of distant satellites of jovian planets

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ORBITAL STABILITY OF DISTANT SATELLITES OF JOVIAN PLANETS

A Thesis

Presented to the Department of Physics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Jose Luis Alberto Alvarelllos

August 1996

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ORBITAL STABILITY OF DISTANT SATELLITES OF JOVIAN PLANETS

This thesis consists in studying the stability of distant satellites of jovian planets as perturbed by the Sun. We use the Restricted Three-Body Problem, with Jupiter as the secondary mass. Surfaces of Section are presented for the planar case, and zones of periodic, quasi-periodic and chaotic motion are found. Critical distances beyond which orbits become unstable are found for initially circular, planar orbits; in the case of arbitrary inclinations, most near-polar orbits undergo collisions. In the case of arbitrary eccentricities and inclinations if the initial semi-major axis a_o is small, most orbits are stable, the exception being a few collision orbits, while if a_o is large, most orbits escape. No stable satellite orbits are found if $a_o > 0.97$ Hill radius. We find that we can apply analytical results if the satellite's initial semi-major axis is within the range $0.07 < a_o < 0.27$ Hill radii. These findings are tested against results obtained with an N-body integrator and the results are very similar.

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Nature free at once and rid of her haughty lords is seen to do all things spontaneously of herself without the meddling of the gods.

Lucretius Carus
De Rerum Natura

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Chapter 1

Introduction

1.1 Satellites

A small body revolving around a larger one due to the gravitational influence of the latter is denominated a "satellite." For instance, the Earth may be thought of as a satellite of the Sun, since the Earth revolves around the Sun. If we consider the two bodies to be point masses and consider them to be isolated from the rest of the universe, then this is an example of what is termed the two-body problem. Important empirical facts were discovered by Kepler in the 17th century (Kepler's three laws of planetary motion), but a complete and formal solution to the two-body problem was first obtained by Isaac Newton between the years 1665-1667 as a byproduct of two of his most important contributions to physics: the calculus and his theory of universal gravitation [1]. Specifically, his theory of gravitation states that every material body attracts any other body with a force which is directly proportional to the product of their masses and varies inversely as the square of the distance between them. According to Newton's Law of Gravitation, for two bodies of masses m and M , separated by a distance r , the force which one body exerts on the other is given by

$$F = - \frac{k^2 Mm}{r^2} \quad (1.1)$$

where k is a constant (the *Gaussian gravitational constant*); sometimes G is used instead of k^2 . Note that the force is negative, since the force is attractive. Newton completely solved the two-body problem and explained Kepler's Three Laws as a consequence of his theory of universal gravitation.

The problem of computing the positions of real satellites, of course, is usually more complicated than this. Real satellites may, in addition to the attractive force of their primary, experience a variety of other forces or *perturbations* such as the gravitational attractions of other bodies (other satellites and/or planets), and/or the fact that the primary may not be perfectly spherical (in which case it is not strictly valid to treat it as a point mass), etc.; these other forces make the solution of the motion of such satellites very difficult. The two-body problem represents a good approximation to physical reality only if all perturbations can be shown to be negligible compared to the direct gravitational attraction of the primary.

1.2 Irregular Satellites

Consider the case that the satellite and its parent planet are perfectly spherical, so that they may be treated as point masses. What if the satellite *is* subjected to an appreciable gravitational perturbation by a third body? Is there a solution to such a problem? It is well known that the gravitational N-body problem can not be solved in

closed form (that is, we can not find an analytical solution to the equations of motion) if $N > 2$. The next simplest case after the two-body problem is the three-body problem. The list of physicists, astronomers and mathematicians who have tackled this problem include Newton himself, Lagrange, Laplace, Euler, Jacobi, Poincaré and others. A simplification of the three-body problem occurs if we (i) let the mass of our satellite tend to zero (so that it does not influence the motion of the other two massive bodies) and (ii) assume that the two massive bodies (the primary and the perturbing body) move around each other in circular orbits. This is the famous *Circular Restricted Three-Body Problem*.

The problem we wish to study in this work is the motion of small, distant satellites, sometimes called *irregular satellites* (see [2], p. 17), of a planet which moves in an (approximately) circular orbit around the Sun. It is clear that if the satellite is close enough to its parent planet, then the solar gravitational perturbations on the satellite are negligible and the problem may be adequately treated as a two-body problem. On the other hand, if the satellite is very far away from its parent planet, then the solar perturbations should be very strong, and if the satellite is sufficiently far away the solar gravitational pull may be so strong that it may "steal" the satellite from the planet to orbit the Sun instead. The theory of the Restricted Three-Body Problem provides an excellent approximation for the study of such (real or hypothetical) irregular, distant satellites and we will discuss it in detail in the next chapter. It is true that there is no known analytical

solution to this case, but the problem is mathematically simple and numerical solutions for the equations of motion of the satellite can be obtained to any order of accuracy; furthermore there are important analytical results as we will see later. Let us specifically state the three goals we have set for this work:

1. *To obtain a qualitative ("global") understanding of the effect of solar gravitational perturbations on very distant and small (irregular) satellites and to find general regions of stability and instability using the restricted three body problem as a model.*
2. *To understand the extent to which we can apply analytical theories derived from the Restricted Three-Body Problem and when we must resort to purely numerical methods.*
3. *To gain an understanding of the applicability of this simple model (the Restricted Three-Body Problem) to the problem we wish to study. We will compare results obtained with this model to those obtained with a very complete solar system integrator which takes into account the perturbations by other planets, the eccentricity of the parent planet's orbit around the Sun, etc.*

Point number three above is also important because in explaining any phenomena we wish to adopt the simplest model possible; a very complex model may be cumbersome to

use and/or may obscure, rather than illuminate the basic physics observed. Throughout this work we will assume the following:

1. The larger mass is the Sun.
2. The second massive body is the planet Jupiter.
3. The third body is a "test particle" with negligible mass which could represent a real or hypothetical satellite of Jupiter and whose motion we wish to study.

It is also worth pointing out what this work does *not* address:

1. It does not pretend to serve as a model for nearby satellites where oblateness effects or even atmospheric drag must be taken into account.
2. It does not pretend to serve as a model for satellites where tidal effects must be taken into account due to the large mass of the satellite with respect to its parent planet (i.e., the Earth-Moon and/or the Pluto-Charon system).
3. The third case that this work does not address is the orbital motion of satellites in complicated gravitational configurations (i.e., resonances) with other satellites, such as is the case with the galilean satellites of Jupiter.

Jupiter possesses eight irregular satellites, with masses which are negligible compared to that of the planet itself. Some results derived in this work may be applicable to those

satellites. It also is hoped that the results presented here will be sufficiently general that they may be applied with a minimum of accommodations to distant satellites of any other jovian ("Jupiter-like") planet in an approximately circular orbit about the Sun.

1.3 Previous Work

Previous applications of the Restricted Three-Body Problem are numerous. One of the most relevant to the present work is that of Kozai [3], who wished to study the motion of asteroids with arbitrary inclinations and eccentricities in orbit around the Sun but undergoing gravitational perturbations due to the planet Jupiter. His work differs from ours in the sense that Kozai's approach was purely analytical and that it was applied to asteroids, not planetary satellites (although he does make it clear that his work could be applied to planetary satellites as well).

Michel Hénon was also interested in the nature of orbits in the Restricted Three-Body Problem [4,5]. He used a slightly different model however, namely Hill's problem which can be derived from the Restricted Three-Body Problem in the limit that the secondary mass tends to zero. Hénon used the method of the Surface of Section to investigate the nature of orbits and obtained important results concerning the stability of distant satellites. He also discovered the presence of chaos in some orbits, lately a phenomenon of increasing interest in all areas of science. Hénon's work differs from ours

in several aspects: (i) his approach was basically numerical, (ii) the model he used was slightly different (Hill's model), and (iii) he only considered satellite orbits in the plane of the planet's orbit around the Sun. We will discuss the Surface of Section and some of Hénon's results in chapter 3.

The most recent relevant study is that of Hamilton and Burns [6], who wished to investigate the stability of particle (debris) orbits around the asteroid Gaspra in preparation for the 1991 encounter the spacecraft Galileo with that body. Their work was purely numerical although it did include three-dimensional orbits.

This work also attempts to include unconventional satellite orbits. Comet Shoemaker-Levy 9 had undergone gravitational capture by as early as 1930-1940 [7] so that technically it had become a temporary satellite of Jupiter. We will also apply our results to this remarkable satellite-comet.

1.4 Outline of Research

Chapter 2 explains in some detail the mathematical theory of the model which we will be using, the Restricted Three-Body Problem. We will also examine in some detail the applicability of this model to a distant satellite of Jupiter.

In Chapter 3 we will define the Surface of Section, explain why it is a useful concept and apply it to the (planar) Restricted Problem of Three Bodies. We also show how to extract information from it. We will also as previously pointed out revisit some of Hénon's results. Finally we will discuss the concept of chaos, what it means in general and for satellite orbits in particular.

Chapter 4 is a detailed chapter where we outline our results regarding the stability of initially circular satellite orbits at all inclinations. Particular emphasis is placed on the concept of "limiting or critical" distances for planar orbits beyond which stability is impossible or highly unlikely. Simple and useful analytical and numerical formulae are derived. We also compare the numerical results with purely analytical results and find when we can and can not apply these analytical formulae.

Chapter 5 generalizes the work of Chapter 4 to include elliptical satellite orbits. Again, numerical results are derived and compared with analytical results. The numerical results are presented as plots which represent regions of stability and instability, where instability may occur by collisions with Jupiter or escapes from the gravitational influence of that planet. Finally we apply the analytical results to some of Jupiter's distant satellites.

In Chapter 6 we present answers to the question we posed in section 1.2: how accurately does the simple Restricted Three Body Problem represent the real problem we are studying? We compare the results obtained using the Restricted Three Body Problem to the results obtained with an N-body integrator.

Finally in Chapter 7 we summarize our findings and pose suggestions for future work.

Chapter 2

The Circular Restricted Three-Body Problem

The purpose of this chapter is to describe the Restricted Three-Body Problem and its applicability to the real case of the Sun-Jupiter satellite system. The satellite is assumed massless and far from Jupiter.

2.1 The Circular Restricted Three-Body Problem

The Circular Restricted Three-Body Problem (R3BP from now on) was chosen because of its simplicity and the availability of analytical results. What follows is a brief description of the model; see Szebehely [8] for the full details.

In the R3BP we consider two bodies with finite masses m_1 and m_2 that revolve about their common center of mass in circular orbits due to their mutual gravitational attraction. A test particle with infinitesimal mass m_3 is gravitationally influenced by (but does not itself influence) m_1 and m_2 . *We wish to find the motion of this third body.* Let us choose the unit of mass so that $m_1 + m_2 = 1$. Specifically let $m_1 = 1 - \mu$ and $m_2 = \mu$. Note that $\mu = m_2/(m_1 + m_2) \approx 9.54 \times 10^{-4}$ in the case of Jupiter and the Sun; see Appendix A for a more accurate value. Furthermore let us choose the unit of time such that the gravitational constant $k^2 = 1$; the unit of length is chosen to be the distance between the

two finite masses, i.e., $a = 1$. Consider an xyz coordinate system rotating about the origin (which is the center of mass of m_1 and m_2) with a constant angular velocity $\vec{\omega} = \omega \hat{z}$ as shown in Fig. 2.1, with the z axis pointing out of the page. Note that m_1 and m_2 are fixed in the rotating frame: their positions are $(\mu, 0, 0)$ and $(\mu-1, 0, 0)$ respectively. Let the coordinates of m_3 in the inertial, non-rotating frame be (x_o, y_o, z_o) , and (x, y, z) in the non-inertial rotating frame. The conversion between the inertial and non-inertial coordinates of m_3 is

$$x_o = x \cos(\omega t) - y \sin(\omega t) \quad (2.1a)$$

$$y_o = x \sin(\omega t) + y \cos(\omega t) \quad (2.1b)$$

$$z_o = z \quad (2.1c)$$

Fig. 2.1. The restricted three-body problem

By Kepler's Third Law, the period of m_1 and m_2 is

$$P = \frac{2\pi k^2 a^{1.5}}{\sqrt{(1-\mu) + \mu}} = 2\pi \quad (2.2)$$

since we have taken the gravitational constant k^2 and the distance between the two finite masses, a , to be unity. Since the angular speed or mean motion is $\omega = 2\pi/P = 1$ eqns.

(2.1) simplify to

$$x_o = x \cos t - y \sin t \quad (2.3a)$$

$$y_o = x \sin t + y \cos t \quad (2.3b)$$

$$z_o = z \quad (2.3c)$$

2.2 Equations of motion

We will use the Lagrangian method to obtain the equations of motion for m_3 . The kinetic energy is

$$\begin{aligned} T &= \frac{m_3}{2} (\dot{x}_o^2 + \dot{y}_o^2 + \dot{z}_o^2) \\ &= \frac{m_3}{2} [(\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{z}^2] \end{aligned} \quad (2.4)$$

The potential energy is

$$U = -m_3 \left(\frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right) \quad (2.5)$$

where $r_1 = [(x-\mu)^2 + y^2 + z^2]^{1/2}$ and $r_2 = [(x-\mu+1)^2 + y^2 + z^2]^{1/2}$.

The Lagrangian function is $L=T-U$ and Lagrange's equations of motion for the generalized coordinate q_i are given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) = 0$$

For the x-coordinate we have,

$$\frac{\partial L}{\partial \dot{x}} = m_3(\dot{x} - y) \quad (2.6)$$

and

$$\frac{\partial L}{\partial x} = m_3(\dot{y} + x) - \frac{m_3(1-\mu)(x-\mu)}{r_1^3} - \frac{m_3\mu(x-\mu+1)}{r_2^3} \quad (2.7)$$

We then obtain the x equation of motion,

$$\ddot{x} - 2\dot{y} = x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x-\mu+1)}{r_2^3} \quad (2.8a)$$

Similarly, for the y and z coordinates we obtain

$$\ddot{y} + 2\dot{x} = y \left[1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \right] \quad (2.8b)$$

$$\ddot{z} = -z \left[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} \right] \quad (2.8c)$$

We can write the equations of motion in a compact form by defining the function

$\Omega = \Omega(x,y,z)$ as (Szebehely [8], p. 558)

$$\Omega = \frac{1}{2} (x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2} \mu (1-\mu) \quad (2.9)$$

Then we can write (2.8a,b,c) as

$$\ddot{x} - 2\dot{y} = \Omega_x \quad (2.10a)$$

$$\ddot{y} + 2\dot{x} = \Omega_y \quad (2.10b)$$

$$\dot{z} = \Omega_z \quad (2.10c)$$

where Ω_x denotes the partial derivative of Ω with respect to x , etc.

2.3 The Jacobi Integral

Equations (2.10a,b,c) posses an integral of motion first derived by Jacobi.*

Multiply (2.10a) by $2\dot{x}$, (2.10b) by $2\dot{y}$ and (2.10c) by $2\dot{z}$ and add. We obtain

$$2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z} = 2\dot{x}\Omega_x + 2\dot{y}\Omega_y + 2\dot{z}\Omega_z \quad (2.11)$$

* *Comptes Rendus de l'Académie des Sciences de Paris*, vol. III, p. 59

Next we take the derivative of Ω with respect to time (recalling that $\Omega = \Omega(x,y,z)$ only):

$$\frac{d\Omega}{dt} = \Omega_x \dot{x} + \Omega_y \dot{y} + \Omega_z \dot{z} \quad (2.12)$$

This is one half of the right hand side of (2.11). Therefore we can write

$$2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z} = 2 \frac{d\Omega}{dt} \quad (2.13)$$

which can be written as

$$\frac{d}{dt} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 2 \frac{d\Omega}{dt} \quad (2.14)$$

This is easily integrated, yielding

$$\boxed{\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega - C} \quad (2.15)$$

where C is a constant of integration and it is called *Jacobi's constant*. It is the only known constant of motion in the Restricted Three-Body Problem. The three dimensional R3BP has three degrees of freedom, and since we have only one constant of motion, we see that the R3BP is non-integrable, that is we can find no closed form solution to the equations of motion, i.e., eqns. (2.10a,b,c) (For a problem with n -degrees of freedom we

need n constants of motion for the problem to be integrable; see Lichtenberg and Lieberman [9]).

2.4 The Equilibrium or Lagrangian Points

The equations of motion (2.10a,b,c) contain five equilibrium or fixed points. These points all lie on the xy -plane and are called the *Lagrangian Points*, and they are labeled $L_1, L_2, \dots, L_5$. Points L_1, L_2 and L_3 are the collinear Lagrangian points since they lie along the (rotating) x -axis. L_1 is located near m_2 (to the left of m_2 in the diagram below); L_2 is located between m_1 and m_2 , and L_3 is located near m_1 opposite to m_2 . The two remaining points (L_4 and L_5) are the triangular Lagrangian points since they form equilateral triangles with m_1 and m_2 ; see Figure 2.2.

For a fixed point, $\dot{q}_i = \ddot{q}_i = 0$, where the q_i represent x, y, z . Then, from the equations of motion, we have that $\Omega_x = \Omega_y = \Omega_z = 0$. It follows that in order to find the location of the collinear Lagrangian points we must solve for the roots of the polynomial

$$x - \frac{(1 - \mu)(x - \mu)}{|x - \mu|} - \frac{\mu(x + 1 - \mu)}{|x - \mu + 1|} = 0 \quad (2.16)$$

In the case of the equilateral Lagrangian points we have $r_1 = r_2 = 1$, so that the location of the L_4 and the L_5 points are $(\mu - 1/2, \pm \sqrt{3}/2, 0)$ respectively.

If we place our test particle at one of the Lagrangian points it will stay there forever. Now suppose we put a test particle at each of the Lagrangian points and give each a slight displacement away from the fixed points: will the particle oscillate back and forth or will it recede indefinitely from its equilibrium position, i.e., are the equilibrium

Fig. 2.2. Schematic diagram showing the positions of the Lagrangian points in the rotating frame.

points linearly stable? In general the collinear points are unstable, but the triangular points are stable provided that $\mu < \mu_o = (1 - \sqrt{23/27}) / 2 \approx 0.03852 \dots$ (Routh's

value; see Szebehely [8], p. 250). In the case of Jupiter and the Sun $\mu \approx 9.54 \times 10^{-4}$ so the triangular equilibrium points are stable; as of mid-1993, there were 173 known Trojan asteroids librating about Jupiter's L_4 and L_5 points.

2.5 Surfaces of Zero Velocity

It can be seen from equation (2.15) that the constant C depends on the initial position and velocity of the test particle. Furthermore, the speed $v = [\dot{x}^2 + \dot{y}^2 + \dot{z}^2]^{1/2}$ is a function of position in the rotating frame. If we set $v = 0$ in equation (2.15) we obtain *surfaces of zero velocity* given by $2\Omega = C$. It is clear that motion is possible only if $2\Omega - C > 0$. For a given value of the Jacobi constant C , the surfaces of zero velocity mark the limits of the regions within which motion is permissible.

The general behavior of these surfaces as C varies is shown on the xy -plane in Figures 2.3(a-d) where we assume that $\mu = 0.33$. The star and cross represent the positions of m_1 and m_2 respectively. At a high enough value of the Jacobi constant there are two closed ovals formed around each of the finite masses and a bigger oval exterior to these two (see Fig. 2.3(a)); motion is possible outside the larger oval and inside the two inner ovals only. As the Jacobi constant is decreased to the critical value $C = C_2$ the surfaces of zero velocity meet at the L_2 point (see Fig. 2.3(b)). As we lower the value of C even further the two ovals around each of the finite masses open, and it is now possible

for the particle to orbit m_1 and m_2 alternatively. Lowering C even more we reach a second critical value, $C = C_1$ and the surfaces of zero velocity meet at the L_1 point (see Fig.

Figures 2.3(a-d). Curves of zero-velocity for $\mu = 0.33$. (a) $C > C_2$; (b) $C = C_2$ (curves of zero velocity meet at L_2); (c) $C = C_1$ (curves of zero velocity meet at L_1); (d) $C = C_3$.

2.3(c)). Continuing with this process of lowering C we reach a third critical value $C = C_3$, in which case the surfaces of zero velocity meet at the L_3 point (see Fig. 2.3(d)). The process continues until a low enough value of C is reached such that the surfaces of zero velocity shrink down to progressively smaller and smaller ovals around the L_4 and L_5 points (not shown). Notice that the lower the value of the Jacobi constant, the greater the

space available for the test particle to explore. Eventually the surfaces of zero velocity disappear and there are no constraints on the particle's motion.

Consider the case of a high Jacobi constant (Fig. 2.3(a)). If our test particle starts with initial conditions such that it is inside the smaller oval around m_2 (left oval), then because the particle cannot cross the surfaces of zero velocity, it is forever confined to move within this oval. This is sometimes called Hill's stability criterion, since he used this analysis to demonstrate the stability of the Moon's orbit about the Earth.*

2.6 The Planar Restricted Three Body Problem

A simplified but important version of the circular R3BP occurs if we confine the motion of our test particle to a plane. If the initial conditions are such that $z = 0$ and $\dot{z} = 0$ then the particle experiences no forces to make it leave the xy -plane. We then have the particular case of the Planar, Circular Restricted Three-Body Problem. The equations of motion are (Szebehely [8], p. 22)

$$\ddot{x} - 2\dot{y} = \Omega^{(2)}_x \quad (2.17a)$$

$$\ddot{y} + 2\dot{x} = \Omega^{(2)}_y \quad (2.17b)$$

with

* *Lunar Theory*, *Am. Jour. Math.*, vol. I, p. 23

$$\Omega^{(2)} = \frac{1}{2} [(1 - \mu)r_1^2 + \mu r_2^2] + \frac{(1 - \mu)}{r_1} + \frac{\mu}{r_2} \quad (2.18)$$

and $r_1 = [(x - \mu)^2 + y^2]^{1/2}$ and $r_2 = [(x - \mu + 1)^2 + y^2]^{1/2}$. The Jacobi constant C is determined from

$$\dot{x}^2 + \dot{y}^2 = 2\Omega^{(2)} - C \quad (2.19)$$

It is interesting to note that although $\Omega^{(2)} \neq \Omega$, equations (2.17) are identical to equations (2.10) if we set $z = 0$ in equation (2.9). The positions of the Lagrange equilibrium points in the planar case are identical to the three dimensional case. The shape of the curves of zero-velocity are also completely analogous to the three-dimensional case (although we now speak of *curves*, not surfaces of course; see Figs. 2.3(a) to 2.3(d)).

Notice that we have reduced the number of degrees of freedom of the problem from three to two by confining the motion to a plane. However the problem is still non-integrable since we still have only one constant of motion, the Jacobi constant.

2.7 Applicability to the Sun-Jupiter-Test Particle Problem

The problem we wish to study is the stability of distant (small) satellites, subject only to solar perturbations. Having described the R3BP in detail we ask, how valid is this simplified model in the case of the Sun, Jupiter and a massless distant satellite? The first problem with the model is the non-zero eccentricity of Jupiter's orbit; however it is quite small (approximately 0.048) and therefore as a first order approximation we are quite justified in neglecting it.

A second effect we are ignoring is the fact that Jupiter is not a perfect sphere. We are not, however, interested in the regions very close to Jupiter where we have to take into account oblateness effects. All of our test particles have initial conditions such that $r_2 \geq 50 R_J$ (where r_2 is the distance between the satellite and Jupiter and $R_J = 71398 \text{ km} = 1$ Jupiter radius). Let us estimate the order of magnitude of the effect of neglecting the oblateness of Jupiter. If we confine ourselves to the equatorial plane, the gravitational potential due to an oblate planet is (Danby [10], p. 115)

$$V = -\frac{k^2 M_J}{r_2} - \frac{k^2 M_J J_2}{2r_2^3} + \dots \quad (2.20)$$

where k^2 is the gravitational constant, M_J is Jupiter's mass, r_2 is the distance from Jupiter (in R_J) and J_2 is a constant. Notice that the first term in (2.20) is just the gravitational potential of a perfectly spherical planet; for convenience set $r = r_2$ and $M_J = 1$; furthermore $k^2=1$, and for Jupiter $J_2 = 14.75 \times 10^{-3}$ (Danby [10], p. 432). Then we can write the second term in (2.20) as

$$V_{obl} \cong - \frac{J_2}{2r^3} \quad (2.21)$$

The $\cong$ sign is needed to indicate that we have only included terms up to J_2 (there are higher order J terms which we are neglecting). We wish to compare the perturbations due to an oblate Jupiter with the gravitational perturbations due to the Sun. Confining ourselves to the equatorial plane, the averaged solar perturbation on a planetary satellite can be written as (Dobrovolskis [11], 1993)

$$V_{sol} = - \frac{k^2 M_s r^2}{4A^3} = - \frac{M_s r^2}{4A^3} \quad (2.22)$$

where $A \approx 1.0917 \times 10^4$ is the semi-major axis of Jupiter's orbit in Jupiter radii and $M_s \approx 1052.3$ is the mass of the Sun in units of Jupiter's mass. If we equate equations (2.21) and (2.22) and solve for r (in R_J) we obtain

$$r \cong \left(\frac{2A^3 J_2}{M_s} \right)^{1/5} \cong 33 R_J \quad (2.23)$$

That is, the oblateness perturbations approximately equal the solar perturbations at $33 R_J$.

However, notice that the solar perturbations grow as r^2 , while the oblateness perturbations go as r^{-3} . The ratio of the perturbation strength is

$$\frac{V_{obl}}{V_{sol}} = \left(\frac{2A^3 J_2}{M_s} \right) r^{-5} \quad (2.24)$$

at $r = 50 R_J$ we have $V_{obl}/V_{sol} \cong 0.1$, while at $r = 100 R_J$ we have $V_{obl}/V_{sol} \cong 4 \times 10^{-3}$.

All of our test particles are initially started at $r \geq 50 R_J$, and the great majority are started such that $r > 100 R_J$. Therefore to first order we can neglect the oblateness effects.

Other effects we are ignoring are the gravitational perturbations due to other inner, massive satellites; in the case of Jupiter, this amounts to ignoring the perturbations due to the galilean satellites. Dobrovolskis [11] has shown that, to first order, the perturbation due to an inner satellite in the equatorial plane is equivalent to an oblateness term

$$J_2 = \frac{m_s a_s^2}{2 M_J R_J} \quad (2.25)$$

where m_s is the mass of the inner satellite, a_s is the semi-major-axis of its orbit, M_j is Jupiter's mass and R_j is Jupiter's equatorial radius (71398 km). Table 2.1 lists the various galilean satellites and their equivalent "oblateness" terms. Data for the galilean satellites are from Danby [10]. Comparing the galilean satellites' equivalent oblateness terms with Jupiter's real oblateness term we see that the galilean satellites' effects on a small and distant satellite should be negligible, except maybe for the case of Ganymede or Callisto. However, it turns out that even Callisto's "oblateness" term may be ignored: plugging in Callisto's J_2 term into equation (2.23), we find that the solar perturbations approximately

Table 2.1 : Equivalent Oblateness of Jupiter due to the Galilean Satellites

Satellite	m_s/M_j	a_s/R_j	J_2
Io	4.68×10^{-5}	5.91	0.82×10^{-3}
Europa	2.52×10^{-5}	9.40	1.11×10^{-3}
Ganymede	7.80×10^{-5}	14.99	8.76×10^{-3}
Callisto	5.66×10^{-5}	26.37	19.70×10^{-3}

equal Callisto's perturbations at roughly $34 R_j$. We have thus shown that, for sufficiently distant satellites, the perturbation due to the galilean satellites may be considered negligible.

The other effects we are ignoring are the gravitational perturbations due to the other planets (mainly Saturn). Nonetheless, we are mostly interested in the stability of test particles orbiting relatively close to Jupiter, so that the perturbations due to the other planets are orders of magnitude smaller than the effects due to the Sun. For simplicity consider the perturbations due to Saturn only (keeping in mind that the perturbations due to the other planets will be even smaller). If we think of the Sun and Saturn as a distant point masses perturbing the two-body motion of a particle orbiting Jupiter, then we can use (2.18) for both the Sun and Saturn. Comparing the ratios of the averaged Saturn-perturbations and the averaged solar perturbations using (2.22) we find

$$\frac{V_{sat}}{V_{sol}} = \left(\frac{M_{sat}}{M_s} \right) \left(\frac{A_s}{A_{sat}} \right)^3 \quad (2.26)$$

where M_s and M_{sat} are the masses of the Sun and Saturn respectively. In order to find the maximum value for V_{sat}/V_{sol} we take A_s to be the *maximum* distance from Jupiter to the Sun, and A_{sat} to be the *minimum* distance between Saturn and Jupiter. We then obtain $A_s = a_j(1 + e_j) \approx 5.46 AU$ and $A_{sat} = a_{sat}(1 - e_{sat}) - A_s \approx 3.55 AU$, where a_j , e_j and a_{sat} , e_{sat} are Jupiter and Saturn's semi-major axis and eccentricity respectively. We find that

$$\left(\frac{V_{sat}}{V_{sol}} \right)_{\max} = 2.94 \times 10^{-7} \quad (2.27)$$

where we have assumed $M_{sat} = 0.299 M_J$ and $M_s = 1052.3 M_J$. We can see that the effect of including Saturn in the calculations is even smaller than the effect due to the oblateness perturbations. Therefore we can safely ignore Saturn and the other planets in our discussions.

We have seen that to first order accuracy we can neglect the eccentricity of Jupiter's orbit, the oblateness effects of Jupiter (if we are far away enough, $r_2 \geq 100 R_J$), the perturbing effects due to the galilean satellites and the effects of the other planets on our test particle. Later we will compare our results from the relatively simple R3BP to those results found using a more realistic N-body code to test the generality of the results.

Chapter 3

Surfaces of Section for the Planar, Circular Restricted Three-Body Problem

This chapter describes the concept of the Poincaré Surface of Section and applies the method to the Planar, Circular Restricted Three-Body Problem . We then examine the Surfaces of Sections and show how information may be extracted from them. Finally we proceed to consider the relative stability of the various orbits found (escape, chaotic, quasi-periodic and in particular periodic orbits). The stability of initially circular orbits will be examined in the next chapter.

3.1 The Surface of Section

Consider the equations of motion for the planar case (eqns. 2.17): they are two second-order O.D.E.'s (ordinary differential equations). It is clear that they can be re-written as four first-order O.D.E.'s if we define $\frac{dx}{dt} = \dot{x}$ and $\frac{dy}{dt} = \dot{y}$. Then, equations (2.17) can be written as

$$\frac{dx}{dt} = \dot{x} \quad (3.1a)$$

$$\frac{dy}{dt} = \dot{y} \quad (3.1b)$$

$$\frac{d\dot{x}}{dt} = 2\dot{y} + \Omega_x^{(2)} \quad (3.1c)$$

$$\frac{d\dot{y}}{dt} = -2\dot{x} + \Omega_y^{(2)} \quad (3.1d)$$

A solution to equations (3.1) can be viewed as a trajectory in a four-dimensional phase space $(x, y, \dot{x}, \dot{y})$. However, because of the existence of the Jacobi constant, the trajectory is actually confined to a three-dimensional subspace. Hence, for a given value of C , only three coordinates are needed to specify a trajectory, for example $(x, y, \dot{x})$, while $\dot{y}$ can be obtained from eqn. (2.19). Consider now an intersection of a two-dimensional surface with this three-dimensional subspace: we choose $y = 0$, i.e. the $x\dot{x}$ – plane; furthermore we choose to investigate only the positive crossings of the trajectory with this plane. Simply stated, every time $y = 0$ and $\dot{y} > 0$ we record $(x, \dot{x})$. This is the concept of a *Poincaré Map* or *Surface of Section*.

A result due to Hénon [5] is that not all of the $x\dot{x}$ – plane is available to the test particle: for a given value of the Jacobi constant, eqn. (2.19) shows that for $y = 0$ we obtain

$$\dot{x}^2 = (1 - \mu)(x - \mu)^2 + \mu(x - \mu + 1)^2 + \frac{2(1 - \mu)}{|x - \mu|} + \frac{2\mu}{|x - \mu + 1|} - C \quad (3.2)$$

Equation (3.2) represents the projection of the surfaces of zero-velocity in the $x\dot{x}$ – plane.

The method of surface of section is an excellent tool for obtaining a qualitative picture of the overall dynamics of the model. Let us apply the method of surface of section to the planar R3BP for different values of the Jacobi constant. Each surface of section is obtained by numerically integrating fifty-six direct and fifty-six retrograde orbits around Jupiter. All the orbits used in a given calculation have the same value of the Jacobi constant C (Direct orbits are those such that the initial velocity vector $(\dot{x}, \dot{y})$ is in the same direction of rotation as the rotating frame, while the opposite is true for the retrograde orbits). Once an initial condition or orbit is chosen, we integrate it with a fifth-order Runge-Kutta (Cash-Karp) variable step-size integrator [12]. The Jacobi constant is monitored so as to maintain at least six digits of accuracy throughout the integration of any particular initial condition. During the integration of any initial condition, whenever $y = 0$ and $\dot{y} > 0$ we save $(x, \dot{x})$, i.e., the points for our surface of section. Given the discontinuous nature of the data points produced in any numerical integration, we use linear interpolation to assure the constraint $y = 0$. Since we are interested in orbits around the second massive body (Jupiter), our surface of sections will be centered on m_2 (in all our surface of sections Jupiter is at $x = \mu - 1 \cong -0.999$). The surfaces of section plots should be compared with the corresponding plots of the curves of zero velocity in the xy -plane for the Sun-Jupiter case ($\mu \approx 9.5388 \times 10^{-4}$); see Figs. 3.9(a) to 3.9(h).

The first surface of section (Fig. 3.1) is for a relatively high value of the Jacobi constant, $C=3.08$. Note that a relatively high value of C , as in this case, means that the particle is tightly bound to m_2 (see Fig. 3.9(a)). The dashed curves represent the boundaries given by setting eqn. (3.2) equal to zero. The first thing one notices is the appearance, on either side of m_2 , of a set of concentric curves. According to Hagihara ([13], p. 64), the angle between the i^{th} and $(i+1)^{\text{th}}$ piercing of the surface of section, α , is a constant for all i : this is the *rotation angle*. If $\alpha/2\pi$ is equal to an irrational number, then "an infinite sequence of points is obtained which tends to cover the curve completely and the orbit fills up a ring domain completely. Such an orbit is called *quasi-periodic*". The sets of concentric curves on either side of m_2 represent the quasi-periodic orbits. In general, those concentric curves to the left of m_2 represent the retrograde orbits while those to the right of m_2 represent the direct or prograde orbits. The existence of these concentric curves, also called *invariant curves*, is significant because it indicates the existence of another "constant" of motion besides the Jacobi constant. This second constant is approximately equal to the angular momentum (per unit mass) h of m_3 about m_2 (see [13], p. 65) where

$$h = r_2^2 + (x - \mu + 1)\dot{y} - y\dot{x} \approx \text{const.} \quad (3.3)$$

Figure 3.1. Surface of Section for $C=3.08$

Strictly speaking, h is not a constant of the motion in the same sense as the Jacobi constant. At high values of C and in the vicinity of m_2 , the R3BP resembles the two-body problem; therefore at high values of C , h is *approximately* constant (and so is the energy per unit mass), this approximation becoming better and better as $C \rightarrow \infty$.

If, on the other hand $\alpha/2\pi$ is a rational number p/q (p, q integers), then we get a *periodic orbit* with q revolutions around m_2 before returning to its starting value. Periodic orbits are represented on a Poincaré Map by isolated points; a *simply periodic orbit* is represented by a single, isolated point, i.e. $q = 1$. We will be mostly concerned with simply periodic orbits, so henceforth we will take the term "periodic orbit" to mean "simply periodic orbit" unless indicated otherwise. As the curves representing the quasi-periodic orbits shrink down to a point, that point represents a periodic orbit. In our first surface of section we can deduce the existence of two periodic orbits, represented by two isolated points on either side of m_2 . The point to the left of m_2 represents a periodic, retrograde orbit, or using Strömberg's notation, a class f orbit* (or f -family orbit). Similarly the point to the right of m_2 represents a periodic direct orbit or a class g orbit (or g -family orbit).

As we go to higher values of the Jacobi constant the picture remains qualitatively similar to Fig. 3.1: let us then lower the Jacobi constant and investigate what happens.

* *Ergebnisse d. exakt. Naturw.*, IV, 233, Springer, Berlin (1925)

Figure 3.2. Surface of Section for $C=3.041$

For our next Surface of Section, Fig. 3.2 we choose $C = 3.041$. We notice two things: (i) the envelope (dashed line) due to eqn. (3.2) is wider, and (ii) we notice a qualitative change in the right side of the plot (representing the direct orbits), while the retrograde orbits remain qualitatively similar to those in Fig. 3.1. On both sides we still find mostly quasi-periodic orbits, and as those shrink to a point we find the f (periodic, retrograde) orbit to the left, and the g (periodic, direct) orbit to the right.

In Fig. 3.3 we lowered the Jacobi constant by just 10^{-3} to $C = 3.04$. Not much has changed with the retrograde orbits; we still have quasi-periodic orbits and an f periodic orbit. However, profound changes have occurred with the direct orbits (right side of the plot). We first notice that most concentric curves representing the direct, quasi-periodic orbits have disappeared, giving way to many dispersed points. These dispersed points cannot be connected by any simple curve and represent several *chaotic (or ergodic) orbits* intersecting the $xx -$ plane. This onset of chaotic motion in the R3BP was studied in detail by R. H. Smith and V. Szebehely [14] who found that it occurs very close to the critical value C_2 (in our case, we find it occurs within 0.04 % of C_2). Furthermore, we notice that the periodic g -family has split into two direct, periodic orbits. These two branches constitute the g' -family, in Hénon's notation [4]. There are some direct quasi-periodic orbits surrounding the left and right branches of the g' orbits. Note also the appearance of five tiny "islands" on the right side of the plot; they belong to the same

Figure 3.3. Surface of Section for $C=3.04$

quasi-periodic orbit, and as the five islands shrink to points, these five points represent a single periodic orbit which closes after five revolutions around m_2 (i.e. $q=5$).

The next surface of section (Fig. 3.4) is for the critical value $C_2 = 3.03971...$

Recall that at this value of the Jacobi constant the curves of zero velocity meet at the L_2 point (see Fig. 3.9(d)) which is located at $x = -0.93236...$. For any $C < C_2$ the particle is, in principle, free to escape the general vicinity of Jupiter. The general picture is similar to Fig. 3.3: we find quasi-periodic orbits (mostly on the retrograde side), chaotic orbits (mostly on the direct side), as well as three periodic orbits, namely the f , and the left and right branches of the g' periodic orbits.

In Fig. 3.5 we have the surface of section for $C = 3.0395 < C_2$. Notice how the dashed curve representing the curve of zero velocity is now open (see also Fig. 3.9(e)). Orbits, and in particular the chaotic orbits, are now free to escape the general vicinity of Jupiter towards the Sun (which is at the right of the plot, at $x = \mu = 9.5388... \times 10^{-4}$). We still have the usual quasi-periodic and periodic orbits.

As we lower the Jacobi constant even more we reach a second critical value, $C_1 = 3.03844....$; see Fig. 3.6. The curves of zero velocity meet at the L_1 point, which is located at $x = -1.06883...$. Notice how most of the "order" has disappeared from the right-hand side of the plot (direct orbits) while the retrograde orbits have hardly changed.

Figure 3.4. Surface of Section for $C_a = 3.0397139\dots$

Figure 3.5. Surface of Section for $C=3.0395$

Figure 3.6. Surface of Section for $C=C_1=3.0384418\dots$

Figure 3.7. Surface of Section for $C=3.038$

Figure 3.8. Surface of Section for $C=3.02$

Figs. 3.9(a-d). Surfaces of zero-velocity for different values of the Jacobi constant C in the vicinity of the smaller primary m_2 (which itself is represented by a plus "+" sign). The mass ratio is that for the Sun-Jupiter system, i.e., $\mu = 9.5388 \times 10^{-4}$. Each value of the Jacobi constant is also represented in the Surface of Section plots (Figs. 3.1 to 3.4).
(a) $C = 3.08$; (b) $C = 3.041$; (c) $C = 3.04$; (d) $C = C_2$.

Figs. 3.9(e-f). Surfaces of zero-velocity for different values of the Jacobi constant C in the vicinity of the smaller primary m_2 (which itself is represented by a plus '+' sign). The mass ratio is that for the Sun-Jupiter system, i.e., $\mu = 9.5388... \times 10^{-4}$. Each value of the Jacobi constant is also represented in the Surface of Section plots (Figs. 3.5 to 3.8).
(e) $C = 3.0395$; (f) $C = C_I$; (g) $C = 3.038$; (h) $C = 3.02$.

In the next plot, Fig. 3.7 the picture is qualitatively similar. Here $C = 3.038 < C_1$; notice how the dashed curve to the left of the plot is now open: orbits can, in principle, escape the Jupiter *and* Sun vicinity and recede to infinity (see also Fig. 3.9(g)). Such orbits are called *escape orbits*. The next and last surface of section plot is for the value $C = 3.02$ (Fig. 3.8). Notice how open the curves of zero velocity have become; see also Fig. 3.9(h). At such low values of C we find that most of the direct orbits have disappeared; i.e., there are no periodic or quasi-periodic orbits to be found; we do find, however a few (temporary) points here and there representing chaotic orbits. In contrast, the retrograde orbits are still present, and notice the (theoretical) presence of a periodic f orbit as the retrograde quasi-periodic orbits shrink to a point.

3.2 Orbits for $C = C_2$

The surfaces of section technique is an excellent tool for obtaining an enormous amount of information in a compact form about a dynamical system. However, it is a somewhat abstract tool; to make things easier to understand we have plotted (in the rotating frame) different types of orbits for a given value of the Jacobi constant. Figures 3.10(a) to 3.10(d) show four types of orbits found for $C = C_2$; information about each orbit's initial conditions can be found in Table 3.1. The plus sign represents the position of Jupiter in the rotating frame, which is located at $(\mu-1, 0)$. Recall that for these four plots, $C = C_2$; hence, the curve of zero velocity just encloses these four orbits (see Fig.

3.9(d)). The initial conditions were obtained from the corresponding surface of section (Fig. 3.4); they are such that $x(0)$ is given in Table 3.1; $\dot{x}(0) = 0$; $y(0) = 0$ (from the

Table 3.1. Initial conditions for the orbits in Figs. 3.10 ($C=C_2$)

Fig.	$x(0)$	Comments
3.10a	-0.940000	Chaotic orbit
3.10b	-0.949000	Quasi-periodic, direct orbit
3.10c	-0.979773...	g' , left-branch direct, periodic orbit
3.10d	-1.017901...	f -retrograde, periodic orbit

definition of the surface of section) and $\dot{y}(0)$ is found using the given value of C and equation (2.19), with $\dot{y}(0) = +\sqrt{(2\Omega^{(2)} - C)}$. The first plot shows a typical chaotic orbit (Fig. 3.10(a)); $x(0)$ was obtained from the chaotic "sea." Note that this orbit is prograde. There is a lack of organization or periodicity of any sort. The successive crossings of this orbit with the $x\dot{x}$ - plane form the dispersed set of points seen in the right side of Fig. 3.4. Figure 3.10(b) shows a direct, quasi-periodic orbit; this orbit is represented in the surface of section by the oval which surrounds the right branch of the g' periodic orbit. Figure 3.10(c) represents a g' -family orbit (left branch); the direction of motion is counterclockwise. The orbit's initial conditions are repeated exactly after a time $P \approx 1.38$, where P is the period of the "satellite" around Jupiter (recall that the orbital

Figs 3.10. (a) A chaotic orbit; (b) A quasi-periodic orbit; (c) A direct, periodic orbit (direction of motion is counter-clockwise); (d) A retrograde, periodic orbit (direction of motion is clockwise). The Jacobi constant for these four orbits is the critical value $C = C_2$. All orbits to the same scale.

period of the Jupiter-Sun system is 2π , see eqn. (2.2)). Finally Fig. 3.10(d) shows an f -family orbit; the direction of motion is clockwise and the orbital period is $P \approx 0.49$.

A word or two must be said about how the initial conditions for the periodic orbits were obtained. Recall that, in a surface of section, quasi-periodic orbits are represented by concentric curves and as these concentric curves shrink to a point it represents a periodic orbit. We will illustrate the method with an example: in Fig. 3.4 ($C = C_2$) consider the smallest concentric curve at roughly $x \approx -1.018$, $\dot{x} \approx 0$. If we take the average value of x for this quasi-periodic orbit, we obtain $x = -1.017901... = x(0)$ which is the last value in Table 3.1. We have thus obtained our initial conditions for the f periodic, retrograde orbit. We shall now consider the stability of some of these orbits.

3.3 Stability of the Orbits Found with the Surface of Section Technique

We have found at least four different types of orbits from our studies of the surface of sections. We have found *periodic*, *quasi-periodic*, *chaotic* and *escape orbits*; it should also be pointed out that escape orbits can be chaotic and vice versa. Let us examine the relative stability of each of these orbits in some detail, paying particular attention to the periodic orbits.

The periodic orbits form a very important subset of the total number of orbits in the R3BP and have been extensively studied ; see for example Szebehely [8] or Hagihara [13]. Many groups or families of periodic orbits have been found, but here we are concerned with only three: the f - family, representing retrograde satellites around m_2 (Jupiter) and the g and g' families, representing direct or prograde satellites around m_2 . Hénon [4] studied in detail the stability of periodic orbits in the R3BP: he considered the linear stability of the fixed points in the Poincaré Map (recall that the fixed points represent periodic orbits) and obtained a *stability index* a ; periodic orbits are stable provided that $|a| < 1$. If $|a| = 1$, then he classified the orbit as a *critical orbit*.

3.3.1 The f-Family of Retrograde Satellites

The f -family starts with an infinitesimal, retrograde, circular orbit around m_2 ; the orbits then grow in size, changing in shape from circular to bean-shaped. A remarkable result is that if $0 \leq \mu < 0.0477\dots$ "essentially all retrograde periodic orbits around the lighter body m_2 are stable" (a condition satisfied by the Sun and Jupiter; see Hénon and Guyot [15]). Recall that in all the surface of section plots (Figs. 3.1-3.8) we could *always* find retrograde, quasi-periodic orbits and as these retrograde quasi-periodic orbits shrink to a point, it represents an f -family periodic orbit. In Fig. 3.11 we have plotted in the rotating frame five periodic retrograde orbits; the direction of orbital motion is clockwise. The plus sign represents the position of m_2 while the crosses represent the positions of the L_1 and L_2 Lagrangian points.

Fig. 3.11. f -family orbits in the rotating frame. The plus sign (+) corresponds to the position of Jupiter, while the (x) signs represent the positions of the L_1 (left) and L_2 (right) points. The numbers correspond to initial conditions on Table 3.2. The direction of motion is clockwise.

Initial conditions for these five f -family orbits can be found in Table 3.2. The first column refers to the orbit number in Fig. 3.11; the second column is the value of the Jacobi constant for that particular orbit. The third column gives the approximate initial x -position of the orbit, found by the same method outlined in section 3.2, while the fourth column gives the approximate initial distance from m_2 in terms of the *Hill radius* R_H , where $R_H = (\mu/3)^{1/3}$; the distance from m_2 to the Lagrange points L_1 and L_2 is

Table 3.2 Family f Orbits for Fig. 3.11

Orbit #	C	$x(0)$	$D (R_H)$	P
1	3.020000	-1.027849...	0.422	0.88...
2	2.990000	-1.091258...	1.351	3.69...
3	2.975000	-1.149651...	2.207	5.27...
4	2.960613...	-1.191422...	2.819	5.76...
5	2.939409...	-1.240148...	3.532	6.01...

approximately equal to one R_H . The fifth column gives the period of the particular orbit. Note the size of some of the larger orbits compared with the positions of the Lagrangian points; note also that the larger the value of the Jacobi constant, the smaller the orbit. Again, $\dot{y}(0)$ is given by eqn. (2.19). From the surface of section plots, it can also be seen that the retrograde, quasi-periodic orbits are remarkably stable; according to Hagihara (p. 68.[13]), "there's apparently no upper bound on the dimension of retrograde quasi-periodic orbits."

3.3.2 The g and g' families of Prograde Satellites

The g -family starts with an infinitesimal circular, prograde orbit around m_2 . As we lower the Jacobi constant the orbits grow in size. In the surface of section plots we have found that the g -family "splits" into two new orbits representing the g' -family (see

Figs. 3.2 and 3.3). The exact orbit where this occurs is the critical orbit g_1 [15]; we have found empirically that this occurs at $3.04015 < C_{g_1} < 3.04020$, where C_{g_1} is the value of the Jacobi constant where the splitting occurs. The g -orbits are all stable down to C_{g_1} , while the g' -orbits are stable from C_{g_1} (at which point they form) until $C \approx C_1$ (see Fig. 3.6, which is the last surface of section plot for which we find quasi-periodic, prograde orbits). Two g -family orbits (solid lines) and two g' -family orbits (dashed lines) are plotted in the rotating frame in Figure 3.12; the direction of orbital motion is counterclockwise for all four orbits. Data for each orbit is tabulated in Table 3.3 below.

Table 3.3. Families g and g' Orbits for Fig. 3.12

Orbit #	C	$x(0)$	$D(R_H)$	P	Comments
1	3.08	-0.986529...	0.183	0.30...	g -family; See Fig. 3.1
2	3.04020	-0.95605	0.630	1.40...	g -Family; close to g_1 crit. orbit
3	3.0395	-0.947573...	0.754	1.73...	g' -family right branch; See Fig. 3.5
4	3.038537...	-0.986671...	0.181	1.79...	g' -family left branch; $C \approx C_1$

The largest g -family orbit we have found is approximately equal to orbit #2 in Fig. 3.12. Note that the value of C for this orbit is approximately equal to C_{g_1} . The largest g' orbit we have obtained is orbit # 4 in Fig. 3.12, which is also the largest prograde orbit we have found. Comparing Figs. 3.11 and 3.12 and Tables 3.2 and 3.3 we can see how much

Fig. 3.12. g (solid) and g' (dashed) family orbits in the rotating frame. The plus (+) sign represents the position of Jupiter. The numbers correspond to initial conditions on Table 3.3. The direction of motion is counterclockwise.

smaller the largest prograde periodic orbits (g and g') are than the retrograde periodic orbits. The "stability" evolution of the prograde orbits (periodic, quasi-periodic and ergodic) can be deduced from the surface of section plots. At high values of the Jacobi constant the prograde orbits resemble perturbed two-body orbits (for example, orbit #1 in Fig. 3.12) and they are stable. As the Jacobi constant approaches the first critical value (C_2) from above, the prograde orbits start to "deform" into chaotic orbits (recall that the orbit in Fig. 3.10(a) is initially prograde). Chaotic orbits are ergodic, i.e., they tend to fill

up most if not all the available space; if one is trapped inside a closed curve of zero velocity, then it is guaranteed that it will never cross that curve of zero velocity, but sooner or later such an orbit may suffer a close approach or even a collision with the central planet. As the Jacobi constant is lowered, more and more prograde orbits become ergodic; at the same time, the curves of zero velocity open and the prograde orbits (chaotic and non-chaotic) tend to escape, becoming escape orbits, which are unstable by definition.

3.4 Osculating Orbital Elements of the Test Particles

In a two-body problem, the orbital elements are constant, except for the Mean Anomaly which is always increasing. In the R3BP, the orbital elements of the third particle are *not* constant and they are denominated *osculating* orbital elements. It is instructive to study how some of the orbital elements (with respect to m_2) of the orbits we have just found behave as a function of time. In particular we shall study the variations in semi-major axis a and eccentricity e . We will need expressions for both a and e in rotating coordinates. We can obtain the semi-major axis a from the conservation of energy equation of the two body problem [10], which can be written in the following way:

$$V^2 = \mu \left(\frac{2}{r_2} - \frac{1}{a} \right) \quad (3.4)$$

where r_2 is the distance to m_2 and V is the speed (in the non-rotating frame) of m_3 with respect to m_2 . The coordinates in the non-rotating frame of the test particle are (x_o, y_o) and are given by eqns. (2.3a) and (2.3b). Let the coordinates of m_2 in the non-rotating frame be (x', y') where

$$x' = (\mu - 1) \cos(t) \quad (3.5a)$$

$$y' = (\mu - 1) \sin(t) \quad (3.5b)$$

The relative coordinates of m_3 with respect to m_2 in the non-rotating frame are (x_{rel}, y_{rel}) where

$$x_{rel} = x_o - x' = (x - \mu + 1) \cos(t) - y \sin(t) \quad (3.6a)$$

$$y_{rel} = y_o - y' = (x - \mu + 1) \sin(t) + y \cos(t) \quad (3.6b)$$

Taking the derivative with respect to time of eqns. (3.6a,b) we find

$$\dot{x}_{rel} = (\dot{x} - y) \cos(t) - (x + \dot{y} - \mu + 1) \sin(t) \quad (3.7a)$$

$$\dot{y}_{rel} = (\dot{x} - y) \sin(t) + (x + \dot{y} - \mu + 1) \cos(t) \quad (3.7b)$$

so that

$$V^2 = \dot{x}_{rel}^2 + \dot{y}_{rel}^2 \quad (3.8)$$

After some algebra we can write

$$V^2 = (\dot{x} - y)^2 + (x + \dot{y} - \mu + 1)^2 \quad (3.9)$$

From eqn. (3.4) we can solve for the semi-major axis a :

$$a = \left(\frac{2}{r_2} - \frac{V^2}{\mu} \right)^{-1} \quad (3.10)$$

where in our case $r_2 = [(x-\mu+1)^2 + y^2]^{1/2}$, and V^2 is given by eqn. (3.9).

The eccentricity e is given by

$$e = \sqrt{\left(1 - \frac{h^2}{\mu a} \right)} \quad (3.11)$$

where h is the angular momentum per unit mass of the test particle and is given by eqn. (3.3). It is convenient when plotting the orbital elements to set the units of the semi-major axes in units of Jupiter's equatorial radius R_J , where $R_J = 71398$ km; to convert to units of Hill's radius, we multiply the semi-major axis in Jupiter radii by the conversion factor $H = 1.342 \times 10^{-3}$. Likewise, let us use as units of time Jupiter's orbital period $P_J = 2\pi = 11.86$ years.

We start with the retrograde orbits. Figure 3.13 shows the variations in semi-major axis and eccentricity for orbit #1 in Fig. 3.11 (a medium size , periodic, retrograde orbit). The variations in a are periodic, with an average value $\langle a \rangle = 316 R_J$ and a relatively small amplitude of $11 R_J$. The eccentricity variations are quasi-periodic, reaching a maximum value of 0.065. Recall that the orbital period of this orbit is 0.88 or 0.14 Jupiter years. Note that this is a rather large orbit, and yet the semi-major axis and the eccentricity are "well behaved"

Fig. 3.13. Semi-major axis and eccentricity of retrograde orbit #1 in Fig. 3.11

For the other larger retrograde orbits shown in Fig. 3.11, eqns. (3.4) through (3.11) as they stand are inadequate: if we calculate a and e for orbits numbered 3, 4 and 5 we obtain large negative values for a , and eccentricities which are never less than unity. The total energy per unit mass of the satellite with respect to Jupiter is $-\mu/2a$ so that a negative semi-major axis implies positive energy, i.e., an "unbound" orbit. These "satellites" are no more satellites than the Trojan asteroids are satellites of Jupiter. For example, we can plot the eccentricities with respect to both m_1 and m_2 ; to calculate the

Fig. 3.14. Osculating e with respect to m_1 (a) and m_2 (b) for retrograde orbit #5 in Fig. 3.11.

eccentricity with respect to m_1 (the Sun) we use eqns. (3.10) and (3.11), making the appropriate substitutions. Figure 3.14(a) shows the eccentricity of orbit # 5 in Fig. 3.11 with respect to m_1 ; the eccentricity varies semi-sinusoidally between 0.247 and 0.256. Figure 14(b) shows the eccentricity for the *same* orbit but with respect to m_2 (i.e., using eqns. (3.10) and (3.11)); the eccentricity varies in a rather complicated manner between 1.0 and 21.5 - clearly this is not the eccentricity of a "normal" satellite. In Figure 3.15

Fig. 3.15. Plot of Jupiter (solid) and orbit # 5 (dashed) in Fig. 3.11 in an inertial frame. The direction of motion is counterclockwise for both Jupiter and the test particle.

we plot the positions of the Sun, Jupiter and this test particle in an inertial frame centered on the Jupiter-Sun barycenter at intervals of $P_J/4$. The open circles represent the

positions of Jupiter (assuming a circular orbit), the plus signs represent the “satellite” and the central star represents the Sun; the direction of motion is counterclockwise. At $t = 0$ (Jupiter at $(-0.999, 0)$) the particle is in conjunction with m_2 . At $t = P_J/4$ (Jupiter at $(0, -0.999)$) the particle is lagging behind Jupiter, while at $t = P_J/2$ (Jupiter at $(0.999, 0)$) the particle is only slightly ahead of the planet. At $t = 3P_J/4$ the particle is still ahead, but by the time Jupiter has completed an orbital period (Jupiter at $(-0.999, 0)$), the particle has not yet reached its starting position; in fact, its orbit never closes: it is a precessing ellipse. The distance to Jupiter of this particle oscillates between 2.4 and 1.2 A.U.’s. This type of behavior is typical of a short-period Jupiter-Family comet rather than a jovian satellite (Jupiter-Family comets are comets with periods less than 20 years).

Fig. 3.16. Osculating a and e for prograde orbit #1 in Fig. 3.12

Next we investigate the prograde orbits. Figure 3.16 shows the osculating semi-major axis and eccentricity for orbit # 1 in Fig. 3.12, a small, periodic prograde orbit. Note that the Jacobi constant for this orbit is relatively high ($C=3.08$) so we expect this orbit to resemble a two-body problem. The orbital period of this orbit is 0.3 , or 0.05 Jupiter periods. The variations in a are sinusoidal, with an average value $\langle a \rangle = 137.3 R_J$ and a small amplitude of $0.5 R_J$ while the variations in e are quasi-periodic, with an average value $\langle e \rangle = 0.006$ and a maximum value of ~ 0.01 .

Fig. 3.17. Osculating a and e for prograde orbit # 4 in Fig. 3.12

At lower values of the Jacobi constant the orbital elements of the prograde orbits suffer large perturbations. In Fig. 3.17 we plot the osculating a and e for orbit # 4 in Fig.3.12; recall that this is a g' family orbit, the largest prograde orbit we have found. The variations are largely sinusoidal, but note that the amplitudes are very large: the average value of a is $344 R_J$, with a relatively large amplitude of $51 R_J$. Likewise the eccentricity ranges from a maximum value of 0.58 to a minimum of 0.35, with an average value $\langle e \rangle = 0.51$.

Fig. 3.18. (a) An escape orbit in the rotating frame; (b) e vs. t

The a and e variations for the quasi-periodic orbits are very similar to those of the periodic orbits. Finally in Figure 3.18(a) we plot in the rotating frame an example of an escape orbit; the eccentricity vs. t plot is shown on Fig. 3.18(b). The Jacobi constant for this orbit is $C = 3.038$ and $x_o = -0.975$; note that $C < C_1 < C_2$. The satellite completes two and a half orbits around m_2 before escaping the Sun-Jupiter vicinity.

3.5 Chaotic Orbits

3.5.1 Orbital Elements for a Chaotic Orbit

Chaotic orbits are sufficiently interesting to merit separate consideration. We shall investigate the chaotic orbit shown in Fig. 3.10(a); recall that $C = C_2$ and hence the curve of zero velocity just encloses the orbit. We have plotted, in addition to a and e , the *periapse distance* $q = a(1 - e)$. Note how different this plot is from the ones we have seen thus far. The semi-major axis seems to stay fairly constant, albeit oscillating in a complex fashion about the value $320 R_J$. The eccentricity varies by more than order of magnitude from $e_{min} = 0.05$ to $e_{max} = 0.84$. The periapse distance tells us how close this chaotic satellite can approach Jupiter; during the course of this integration $q_{min} = 45 R_J$. Note that since this orbit does not show any periodicity of any sort, it is impossible to predict upper or lower bounds valid for all time for a , e and q ; since the orbit is chaotic,

Fig. 3.19. Variations in a , e and q for the chaotic orbit in Fig. 3.10(a)

its eccentricity may suddenly increase to a value close to unity and therefore the particle may suffer a close approach to the planet or even a collision. In the real case of the Jupiter system, it would be very unlikely to find a small, distant satellite in a chaotic orbit of this sort: even if the particle does not collide with Jupiter, its eccentricity may become high enough that the particle's orbit eventually crosses the orbits of the galilean satellites. Eventually, such a body may collide with Jupiter or with one of the galilean satellites and thus be removed from the Jupiter system. Another possible outcome for a chaotic satellite may be gravitational scattering so that it escapes the Jupiter system. Note that

this particular test particle is Hill stable (because the curve of zero velocity is closed) but its chaotic nature means that in practical terms the orbit would probably be unstable.

3.5.2 Lyapunov Exponents

The plots of the orbital elements in Fig 3.18 and the orbit itself in Fig. 3.10(a) indeed do look complicated, but that does not necessarily mean that the orbit is chaotic. A qualitative test of chaos is the surface of section, as we have already seen: if we obtain a random distribution of points on the $x\dot{x}$ – plane then the orbit is chaotic. A more precise and quantitative way to measure chaos takes advantage of the fact that *chaotic orbits show sensitive dependence on initial conditions*; this is equivalent to saying that initially nearby phase-space trajectories diverge exponentially with time. The rate of divergence is called the *Lyapunov Characteristic Number* λ or LCN for short (Lichtenberg and Lieberman [9]).

Consider again the four, first order ODEs for the Planar, R3BP, i.e., equations (3.1). They could be written in vector form as

$$\frac{d\vec{s}}{dt} = \vec{F} \quad (3.12)$$

where $\vec{s} = (s_1, s_2, s_3, s_4) = (x, y, \dot{x}, \dot{y})$ is the "state" vector and $\vec{F}$ represents the right-hand side of eqns. 3.1. Now consider an initial condition $\vec{s}_0$ and an orthonormal set of infinitesimal vectors $\vec{\eta}_i$ ($i = 1, \dots, 4$) centered on $\vec{s}_0$. The evolution in time of

$\bar{s}_o = \bar{s}_o(t)$ is governed by eqn. (3.12), but we also wish to monitor the time-evolution of the infinitesimal vectors of slightly displaced initial conditions. We can perform a Taylor expansion of eqn. (3.12) about $\bar{s}_o(t)$, keeping terms of order two or less:

$$\frac{d}{dt} [\bar{s}_o(t) + \bar{\eta}_i(t)] = \bar{F}[\bar{s}_o(t) + \bar{\eta}_i(t)] \quad (3.13)$$

$$\frac{d\bar{s}_o(t)}{dt} + \frac{d\bar{\eta}_i(t)}{dt} = \bar{F}[\bar{s}_o(t)] + J[\bar{s}_o(t)] \cdot \bar{\eta}_i(t) + O(\eta_i^2) \quad (3.14)$$

where J is the Jacobian matrix of $\bar{F}$, $J \equiv \frac{\partial \bar{F}}{\partial \bar{s}}$, evaluated at $\bar{s}_o(t)$. For the planar

R3BP the Jacobian matrix takes the form

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_{XX}^{(2)} & \Omega_{XY}^{(2)} & 0 & 2 \\ \Omega_{XY}^{(2)} & \Omega_{YY}^{(2)} & -2 & 0 \end{pmatrix} \quad (3.15)$$

where the function $\Omega^{(2)}$ was defined by eqn. (2.18). By eqn. (3.12), the first term in the left hand side of eqn. (3.14) is equal to the first term in the r.h.s. of eqn. (3.14).

Therefore we can write the *linearized equations* as

$$\frac{d\bar{\eta}_i}{dt} = J[\bar{s}_o(t)] \cdot \bar{\eta}_i + O(\eta_i^2) \quad (3.16)$$

The i^{th} Lyapunov Exponent (or LE) is defined as (see Wolf et. al. [16] or [9])

$$\lambda_i \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{|\tilde{\eta}_i(t)|}{|\tilde{\eta}_i(0)|} \quad (3.17)$$

For the Planar, R3BP we obtain four LE's ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$); this set is known as the *Lyapunov Spectrum*. There are 2 important facts regarding the Lyapunov Spectrum of conservative dynamical systems, as in the case of the R3BP: (1) the LE's occur in associated pairs of equal magnitude but opposite sign, and (2) the existence of a constant of motion, such as the Jacobi constant, implies a pair of zero LE's. What this means is that the Lyapunov Spectrum of the planar, circular R3BP looks like $\lambda_1 = \lambda$, $\lambda_2 = \lambda_3 = 0$ and $\lambda_4 = -\lambda$ or $(\lambda, 0, 0, -\lambda)$; see Lichtenberg and Lieberman [9]. For our purposes then, we are only concerned with the largest Lyapunov exponent, λ the Lyapunov Characteristic Number. For a chaotic trajectory $\lambda > 0$ while for a non-chaotic trajectory (a periodic or quasi-periodic orbit) $\lambda \leq 0$. Note that in order to find the LCN we must integrate, in addition to the four equations of the Planar R3BP, 16 linearized, first order ODEs of the form of eqn. (3.16), or a total of 20 first order ODE's. The total number of first order ODE's to integrate to find the Lyapunov Spectrum is $n + n^2$, where n is the order of the system. For the fully three-dimensional, R3BP we would have to integrate a total of 42 first order ODE's to find the spectrum. Since nearby orbits diverge

exponentially, we must periodically re-normalize the infinitesimal $\bar{\eta}_i$ vectors, otherwise they would grow without bound. To re-normalize, we use the Gram-Schmidt re-normalization procedure; typically this is done once per time-step. The details of the method to compute a Lyapunov spectrum can be found in Wolf et al. [16]. In Figure 3.20 we have plotted the natural logarithm of the LCN versus the natural logarithm of the time for the chaotic orbit in Fig. 3.10(a) and for the quasi-periodic orbit in

Fig. 3.20. LCNs for a chaotic and a quasi-periodic orbit

Fig. 3.10(b). Note how the plot quickly converges to a definite value for the chaotic orbit, while it never converges for the non-chaotic orbit. The LCN has units of $(\text{time})^{-1}$, so we can define a *Lyapunov Time* $T_L = (\text{LCN})^{-1}$ as the time it takes for a nearby initial

condition to increase its phase-space displacement by a factor of e . In the case of this chaotic orbit the LCN converges to a value close to 0.15 or 1.06 Jupiter periods, so that the chaotic nature of the orbit quickly becomes apparent.

3.6 Discussion

One of the most important facts we have found is the qualitative difference between the prograde orbits and the retrograde orbits, namely the retrograde orbits are much more stable than the direct orbits. The prograde orbits are stable from very high values of the Jacobi constant until we reach the critical value C_2 , at which point the curves of zero velocity open and the prograde orbits are able to escape the general Jupiter vicinity; most if not all in fact do so. Furthermore, as we approach C_2 from above, some of the prograde orbits become chaotic. We found this onset of chaos to occur within 0.04% of C_2 . At lower values of C we find few or no stable, prograde orbits. A cutoff point for the stability of direct orbits then is the critical value C_2 . Note that this is the same as the Hill stability criterion: an orbit is stable if the curves of zero velocity are closed. If a prograde orbit's computed Jacobi constant indicates Hill-type instability, then the orbit is most likely unstable. The same is not true for the retrograde orbits, however; we have seen that these orbits may actually be stable even though the curves of zero velocity are open. In fact, apparently there is no limit to the size of quasi-periodic retrograde orbits. Strictly speaking, this is true only in the R3BP as a mathematical

model. However, the same general pattern is seen in the distant satellites of the Solar System. In the case of Jupiter system, the distant satellites are divided into two distinct groups, with the retrograde ones being more distant: (1) the prograde satellites Himalia (J VI) Elara (J VII), Lysithea (J X) and Leda (J XIII), at an average distance of $160 R_J$ and (2) the retrograde satellites Pasiphae (J VIII), Sinope (J IX), Carme (J XI), and Ananke (J XII), at an average distance of $320 R_J$. In the Saturn system the most distant satellite is Phoebe, at 216 Saturn radii: it is a retrograde satellite.

One of the aims of this chapter has been to put the planar R3BP in perspective. We have seen what a useful tool the Surface of Section is for mapping different regions of phase space at different values of the Jacobi constant. It should be readily apparent from inspection of the Surfaces of Section shown in Figs. 3.1-3.8 that the satellite problem we wish to study, even if we constrain ourselves to a plane, is very complex. A comparison with purely two-body motion may help to put the R3BP in perspective. We have seen that at high values of the Jacobi constant and in the vicinity of m_2 (the satellite problem), motion resembles the two-body problem. In the two-body problem we have only two types of orbits: periodic (circular and elliptical) orbits and escape (parabolic and hyperbolic) orbits. The introduction of a perturbing mass makes the problem non-integrable and the phase space more complex with two new types of orbits: quasi-periodic and chaotic. Quasi-periodic orbits are present in some integrable problems, as in the two-

dimensional oscillator such that the two main frequencies are incommensurable, but chaotic orbits only manifest themselves in non-integrable problems.

A more realistic model for the satellite problem is the full three dimensional R3BP. Can we produce Surfaces of Section so as to classify the different regions of phase space in the three dimensional case? Recall the Surface of Section for the planar case: we have seen that it was very useful because we have an enormous amount of quantitative information about phase-space available in a compact form, but more importantly we can extract all the information we need from it (as when we extracted the initial conditions for the orbits in section 3.2). The equations of motion for the three-dimensional case are eqns. (2.10a-c); this is a problem with three degrees of freedom so that the phase space is *six* dimensional. Now we still have only one constant of motion, namely the Jacobi constant so that the trajectories are confined to a five-dimensional subspace. We could do as before and consider the intersection of a two-dimensional surface, say the $y = 0$ plane with this five-dimensional subspace. This means that every time $y = 0$ and $\dot{y} > 0$ we must not only plot $(x, \dot{x})$ but also $(z, \dot{z})$ if we want to have all of the information we need: the Surfaces of Section for the three-dimensional R3BP are *four dimensional plots*. Attempts have been made to produce such four dimensional Surfaces of Section (see for example, Froeschlé [17]) but in such cases the results are very hard to interpret.

Chapter 4

Stability of Initially Circular Orbits

In this chapter we examine the behavior of test particles with initially circular orbits ($e_o = 0$) around the smaller mass in the R3BP. The justification for this is that most moons in the solar system are in approximately circular orbits. First we examine the behavior of test particles in the planar case and establish empirical limiting distances for the prograde and retrograde cases. Then we examine the empirical behavior of test particles with initial inclinations between 0 and 180 degrees. Finally we test analytical models derived from the averaged R3BP with the empirical results.

4.1 Orbital Elements

In studying the problem of the stability of possible distant Jovian satellites, it is convenient to study how the orbital elements of the satellite behave as a function of time. We have already studied the behavior of the semi-major axis, a , and eccentricity e in the planar R3BP. For an orbit in the plane of Jupiter and the Sun we only need two additional orbital elements, namely the *argument of pericenter* (or *perijove* in the case of Jupiter) ω and the *mean anomaly* M . In the general case that the satellite does *not* remain in the orbital plane of the Sun and Jupiter, we need two additional quantities: the *longitude of the ascending node* Ω (not to be confused with the function defined in eqn.

(2.9)) and the orbit's *inclination* i . These six independent coordinates a , e , i , Ω , ω , and M are called the *orbital elements*. Here we will only give a very brief and general description of the orbital elements. The reader is referred to any standard text of Celestial Mechanics (such as Danby [10] or Roy [18]). Briefly, a refers to the size, and hence the energy of the orbit; the satellite's energy per unit mass is equal to $-\mu/2a$. The eccentricity e is related to the ellipticity of the orbit. If $e = 0$ the orbit is circular; if $0 < e < 1$ the orbit is elliptical; if $e = 1$ the orbit is parabolic and finally if $e > 1$ the orbit is hyperbolic. The inclination i is measured with respect to some reference plane, which in our case we take to be the plane of the Sun-Jupiter orbit, and ranges from $0^\circ \leq i \leq 180^\circ$. If $0^\circ \leq i < 90^\circ$ the satellite is direct or prograde while if $90^\circ < i \leq 180^\circ$ the satellite is retrograde. The mean anomaly M is the angle which the satellite would have swept up since periape passage if it moved at the uniform rate given by the mean motion n , where

$$n = \sqrt{\frac{\mu}{a^3}} \quad (4.1)$$

The mean anomaly is related to the *eccentric anomaly* E (see Danby [10], p. 131) by Kepler's equation

$$M = E - e \sin E \quad (4.2)$$

and hence to the *true anomaly* υ by

$$\tan\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}} \tan\left(\frac{E}{2}\right) \quad (4.3)$$

The two remaining elements Ω and ω denote the orientation of the orbital plane in space.

The orbital elements are related to the *inertial* Cartesian coordinates of the satellite with respect to m_2 (Jupiter) by (see Roy [18], p. 101, or Fitzpatrick [19])

$$x_{rel} = r_2 [\cos \Omega \cos(\omega + \nu) - \sin \Omega \sin(\omega + \nu) \cos i] \quad (4.4a)$$

$$y_{rel} = r_2 [\sin \Omega \cos(\omega + \nu) + \cos \Omega \sin(\omega + \nu) \cos i] \quad (4.4b)$$

$$z_{rel} = r_2 \sin(\omega + \nu) \sin i \quad (4.4c)$$

where r_2 is the distance to m_2 (given in terms of the rotating coordinates in Chapter 2). In terms of the orbital elements it is given by

$$r_2 = \frac{a(1 - e^2)}{1 + e \cos(\nu - \omega)} \quad (4.5)$$

The position of the satellite in its own orbital plane is given by the polar coordinates r_2 and υ , where υ is the true anomaly which can be obtained from eqn. (4.3). Next we define the following auxiliary quantities,

$$G_1 = \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i \quad (4.6a)$$

$$H_1 = \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i \quad (4.6b)$$

$$I_1 = \sin \omega \sin i \quad (4.6c)$$

$$G_2 = -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i \quad (4.6d)$$

$$H_2 = -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i \quad (4.6e)$$

$$I_2 = \cos \omega \sin i \quad (4.6f)$$

plus the semi-minor axis $b = a(1 - e^2)^{1/2}$. We define the components of the (inertial, relative) velocity as follows:

$$\dot{x}_{rel} = \frac{na}{r_2} (bG_2 \cos E - aG_1 \sin E) \quad (4.7a)$$

$$\dot{y}_{rel} = \frac{na}{r_2} (bH_2 \cos E - aH_1 \sin E) \quad (4.7b)$$

$$\dot{z}_{rel} = \frac{na}{r_2} (bI_2 \cos E - aI_1 \sin E) \quad (4.7c)$$

where E is the eccentric anomaly and n is the satellite's mean motion.

Finally we note an important relationship between the velocity and the semi-major axis and distance (a and r_2):

$$V^2 = \dot{x}_{rel}^2 + \dot{y}_{rel}^2 + \dot{z}_{rel}^2 = \mu \left(\frac{2}{r_2} - \frac{1}{a} \right) \quad (4.8)$$

4.2 Rotating and Inertial (Relative) Coordinates

We also need the relationship between the inertial coordinates of the satellite and the rotating R3BP units at any given time t . This was already worked out in eqns. (3.6) and (3.7) of last chapter; in the general three-dimensional case we add the equations $z_{rel} = z$ and $\dot{z}_{rel} = \dot{z}$. Let us define the state vectors $\bar{s} = (x, y, z, \dot{x}, \dot{y}, \dot{z})$, which contains the rotating coordinates and $\bar{s}_{rel} = (x_{rel}, y_{rel}, z_{rel}, \dot{x}_{rel}, \dot{y}_{rel}, \dot{z}_{rel})$, which contains the inertial relative coordinates of m_3 with respect to m_2 ; then the relationship between the two vectors is

$$\bar{s}_{rel} = M \cdot \bar{s} + (\mu - 1) \cdot \bar{e} \quad (4.9)$$

where the matrix M is

$$M = \begin{bmatrix} \cos t & -\sin t & 0 & 0 & 0 & 0 \\ \sin t & \cos t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin t & -\cos t & 0 & \cos t & -\sin t & 0 \\ \cos t & -\sin t & 0 & \sin t & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.10)$$

and the vector $\bar{e}$ is

$$\bar{e} = \begin{bmatrix} -\cos t & -\sin t & 0 & \sin t & -\cos t & 0 \end{bmatrix}^T \quad (4.11)$$

where the superscript T denotes transpose. To go from the rotating coordinates to the inertial coordinates we solve for $\bar{s}$ in eqn. (4.9) above, i.e.,

$$\bar{s} = M^{-1} \cdot (\bar{s}_{rel} + (1 - \mu) \cdot \bar{e}) \quad (4.12)$$

Notice that in the transformation from the rotating to non-rotating coordinates, the x and y components involve cosine and sine terms while the z components remain unchanged. This is due to the fact that the axis of rotation of the non-inertial reference frame is the z -axis.

4.3 Orbital Integrations

4.3.1 Integration Scheme and Initial Conditions

We have written a program that integrates equations (2.10a,b,c) i.e., the three-dimensional R3BP, to model the effects of solar perturbations on possible distant satellites of Jupiter. The inputs to the code are the satellite's initial conditions in the form of orbital elements plus an extra set of parameters specifying the time-step to be used, the

total number of time steps the integration will last, etc., as well as the error tolerance to be used by the integrator (which we set to 10^{-13}). To integrate the equations of motion we use a Bulirsch-Stoer interpolation method [12]. We decided to use a Bulirsch-Stoer integrator with a variable time step because it is slightly faster than the classic Runge-Kutta method. During the course of the integrations we monitor the Jacobi constant to ensure it is kept accurate to at least eight or nine digits.

The code accepts any value of the initial eccentricity, but in this chapter we only consider the case $e_o = 0$; this implies that the *initial* value of the argument of periapse ω_o , is arbitrary, since it is only defined for elliptical orbits. *Therefore in all our integrations*

Fig. 4.1. Schematic diagram showing the initial position of a planar satellite (small dot) as a function of the initial mean anomaly M in the rotating frame. A retrograde satellite moves clockwise while a prograde satellite moves counterclockwise. The large filled circle represents Jupiter; its position on the rotating frame is $x = \mu - l$, $y = 0$.

for this chapter we set $\omega_o = 0$. As the orbital integration progresses, however, the osculating eccentricity may no longer be zero and therefore the osculating argument of periaapse ω is no longer arbitrary. If $i_o = 0^\circ$ or $i_o = 180^\circ$ then the orbit is confined to the xy -plane, the orbit being planar-prograde if $i_o = 0^\circ$ and planar-retrograde if $i_o = 180^\circ$. Furthermore if the orbit is planar, the longitude of the ascending node Ω is completely arbitrary since it is only defined for inclined orbits; therefore if the initial conditions are such that the orbit is in the xy -plane, we set $\Omega_o = 0^\circ$. Finally, in the planar case, if $M_o = 0^\circ$ the particle starts on the (rotating) x axis between m_1 (the Sun) and m_2 (Jupiter). If $M_o = 90^\circ$, then the particle starts on the $+y$ axis. If $M_o = 180^\circ$ the particle starts on the x -axis opposite the Sun and finally if $M_o = 270^\circ$ then the particle starts on the $-y$ axis; see Fig. 4.1.

4.3.2 Regularized Equations of Motion

All equations that deal with gravitational attraction contain singularities at the positions of the centers of attraction. The equations of motion of the R3BP contain two such singularities at the positions of m_1 and m_2 . We are not studying the case of initially close satellites of m_2 , but in some instances the eccentricity grows to a value close to unity and at those times the periaapse distance q (and hence r_2) might get very small. When r_2 is very small the integrator tries to use a very small time step to satisfy the tolerance imposed by the user and the integration takes a large amount of CPU time.

The key to avoiding such problems is to change variables to eliminate the singularity in the original equations of motion. Stiefel and Schiefle [20] have shown that a problem of the type

$$\ddot{\bar{X}} + \frac{k^2 M_p}{R^3} \bar{X} = \bar{P} \quad (4.13)$$

where $\bar{X} = (x, y, z, 0)$, M_p is the mass of the "primary" (in our case $M_p = m_2$, i.e., Jupiter), and $\bar{P}$ is some "perturbing force," can be transformed to the regularized equation

$$\bar{U}'' + \frac{1}{2} h_k \bar{U} = R L^T \bar{P} \quad (4.14)$$

where the primes denote a derivative with respect to a new *regularized time* τ such that $d\tau = dt/R$, $\bar{U} = (u_1, u_2, u_3, u_4)$, $h_k = (k^2 M_p - 2u'^2) / R$, u'^2 will be defined later and L is the matrix

$$L = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix} \quad (4.15)$$

The two coordinate systems are related by

$$\bar{X} = L \bar{U} \quad (4.16)$$

and

$$R = (x^2 + y^2 + z^2)^{1/2} = U^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2 \quad (4.17)$$

(regularization can not be done with only three variables for a three dimensional system).

This is known as a Kustaanheimo-Stiefel or K.S. transformation. In the R3BP equations of motion (eqns. 2.10a,b,c) the origin is set as the center of mass of m_1 and m_2 ; in eqn.

(4.13) the origin is assumed to be at m_2 . To shift the origin to m_2 we make the transformations

$$x \rightarrow x - (\mu - 1) \quad (4.18a)$$

$$y \rightarrow y \quad (4.18b)$$

$$z \rightarrow z \quad (4.18c)$$

where (x, y, z) are the usual rotating coordinates of the R3BP. We plug these transformations in equations (2.8a,b,c) to obtain

$$\ddot{\bar{X}} + \mu \frac{\bar{X}}{R^3} = \bar{P} \quad (4.19)$$

or

$$\ddot{\bar{x}} + \frac{\mu \bar{x}}{R^3} = -2\dot{y} + x + \mu - 1 - \frac{(1 - \mu)(x - 1)}{r_1^3} \quad (4.20a)$$

$$\ddot{\bar{y}} + \frac{\mu \bar{y}}{R^3} = -2\dot{x} + y - \frac{y(1 - \mu)}{r_1^3} \quad (4.20b)$$

$$\ddot{\bar{z}} + \frac{\mu \bar{z}}{R^3} = -\frac{z(1 - \mu)}{r_1^3} \quad (4.20c)$$

where the right hand side of eqns. (4.20) represent the components of $\bar{P}$ and contains the direct terms such as the direct attraction of the Sun and the coriolis terms. Observe that these are simply the equations of the R3BP with the origin at the position of m_2 . The quantity r_1 is the distance to m_1 ; using (4.18a,b,c), we obtain $r_1 = [(x-1)^2 + y^2 + z^2]^{1/2}$. In order to write the regularized equations of motion for the R3BP we must write the perturbing force in terms of the regularized coordinates. The time derivative of x is

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{R d\tau} = \frac{x'}{R} \quad (4.21)$$

and similarly for $\dot{y}$, so that the components of $\bar{P}$ are

$$P_1 = 2 \frac{y'}{R} + x + \mu - 1 - \frac{(1-\mu)(x-1)}{r_1^3} \quad (4.22a)$$

$$P_2 = -2 \frac{x'}{R} + y - \frac{y(1-\mu)}{r_1^3} \quad (4.22b)$$

$$P_3 = -\frac{z(1-\mu)}{r_1^3} \quad (4.22c)$$

$$P_4 = 0 \quad (4.22c)$$

From eqns. (4.15) and (4.16) we can write

$$x = u_1^2 - u_2^2 - u_3^2 + u_4^2 \quad (4.23a)$$

$$y = 2(u_1 u_2 - u_3 u_4) \quad (4.23b)$$

$$z = 2(u_1 u_3 + u_2 u_4) \quad (4.23c)$$

Next we need the time derivatives *with respect to the regularized time* τ of x , and y :

$$x' = \frac{dx}{d\tau} = 2(u_1 u_1' - u_2 u_2' - u_3 u_3' + u_4 u_4') \quad (4.24a)$$

$$y' = \frac{dy}{d\tau} = 2(u_1' u_2 + u_1 u_2' - u_3' u_4 - u_3 u_4') \quad (4.24b)$$

Plugging in the expressions in eqns. (4.23-4.24) into eqns. (4.22) we obtain the components of the "perturbing force" in terms of the regularized coordinates.

$$P_1 = 4(u_1' u_2 + u_1 u_2' - u_3' u_4 - u_3 u_4') / R + u_1^2 - u_2^2 - u_3^2 + u_4^2 - (1 - \mu)(u_1^2 - u_2^2 - u_3^2 + u_4^2 - 1) / r_1^3 + \mu - 1 \quad (4.25a)$$

$$P_2 = -4(u_1 u_1' - u_2 u_2' - u_3 u_3' + u_4 u_4') / R + 2(u_1 u_2 - u_3 u_4) - 2(u_1 u_2 - u_3 u_4)(1 - \mu) / r_1^3 \quad (4.25b)$$

$$P_3 = -2(u_1 u_3 + u_2 u_4)(1 - \mu) / r_1^3 \quad (4.25c)$$

The distance to m_I , r_I , was defined in terms of the non-regularized coordinates x, y, z earlier, but it can be defined in terms of the u_i 's using eqns. (4.23). The regularized equations of motion for the R3BP then are (from eqn. 4.14)

$$u_1'' + \frac{h_k}{2} u_1 = \frac{R}{2} (u_1 P_1 + u_2 P_2 + u_3 P_3) \quad (4.26a)$$

$$u_2'' + \frac{h_k}{2} u_2 = \frac{R}{2} (-u_2 P_1 + u_1 P_2 + u_4 P_3) \quad (4.26b)$$

$$u_3'' + \frac{h_k}{2} u_3 = \frac{R}{2} (-u_3 P_1 - u_4 P_2 + u_1 P_3) \quad (4.26c)$$

$$u_4'' + \frac{h_k}{2} u_4 = \frac{R}{2} (u_4 P_1 - u_3 P_2 + u_2 P_3) \quad (4.26d)$$

where

$$h_k = (\mu - u'^2) / R \quad (4.27)$$

$$u'^2 = u_1'^2 + u_2'^2 + u_3'^2 + u_4'^2 \quad (4.28)$$

and the P_i 's were defined in equations (4.25). To this we add the differential equation relating the regularized time τ to the non-regularized time t , i.e.,

$$\frac{dt}{d\tau} = t' = R \quad (4.29)$$

The initial conditions for the regularized problem are determined as follows: if $x \geq 0$ then

$$u_1 = \left[\frac{1}{2} (R + x) \right]^{1/2}, \quad u_2 = \frac{y}{2u_1}, \quad u_3 = \frac{z}{2u_1}, \quad u_4 = 0 \quad (4.30)$$

while if $x < 0$.

$$u_2 = \left[\frac{1}{2} (R - x) \right]^{1/2}, \quad u_1 = \frac{y}{2u_2}, \quad u_3 = 0, \quad u_4 = \frac{z}{2u_2} \quad (4.31)$$

For the regularized velocities we write

$$u_1' = (u_1 \dot{x} + u_2 \dot{y} + u_3 \dot{z}) / 2 \quad (4.32a)$$

$$u_2' = (-u_2 \dot{x} + u_1 \dot{y} + u_4 \dot{z}) / 2 \quad (4.32b)$$

$$u_3' = (-u_3 \dot{x} - u_4 \dot{y} + u_1 \dot{z}) / 2 \quad (4.32c)$$

$$u_4' = (u_4 \dot{x} - u_3 \dot{y} + u_2 \dot{z}) / 2 \quad (4.32d)$$

The regularized R3BP equations represent much extra work; however, for close approaches to Jupiter integrating the regularized equations is not only faster but also more accurate, since we do not have to deal with an approach to a singularity as the distance to m_2 gets smaller. In practice we integrate the non-regularized equations of motion (eqns. 2.10) while the eccentricity $e < 0.8$ (because it is more accurate to integrate the non-regularized equations in such a case), while if $e \geq 0.8$ (meaning close encounters can occur) we switch to and integrate the regularized equations of motion (eqns. (4.26) plus eqn. (4.29)).

4.3.3 Definition of Stability

So far we have encountered several definitions of stability. Hill type stability occurs when the initial conditions are such that the surface or curve of zero velocity completely encloses the orbit; such a criterion is useful for prograde orbits, although not very useful for retrograde ones. The fact that the curves of zero velocity are open does *not* guarantee that the satellite will in fact escape the vicinity of the secondary mass; this

is especially true in the case of the retrograde orbits. Another definition of stability is Hénon's numerically-computed stability index a for periodic orbits; however, this stability index is not useful in the case of non-periodic orbits (quasi-periodic and chaotic orbits). We will use as our criterion of stability the following somewhat arbitrary but practical definition: *a satellite of m_2 will be considered stable if it lasts 100 m_1 - m_2 periods (or 200π units of time; see section 2.1).* In conventional units, the satellite is considered stable if it lasts 100 Jupiter periods or 1186 years orbiting Jupiter. Possible fates of various test particles are (a) collision with the planet, (b) escape and (c) stability, i.e., the particle neither collides with nor escapes the general vicinity of m_2 . We now need definitions of collision and escape orbits. If the distance to the secondary r_2 is less than $100 R_j$ (9.16×10^{-3} in R3BP units, where the Sun-Jupiter distance is unity) *and* if the periape distance $q = a(1 - e)$ is less than $1 R_j$ (9.16×10^{-5} in R3BP units) then we consider this a *collision orbit* and stop the integration. Likewise, if r_2 exceeds some maximum value, which in our case we take to be 2 AU's ($4191 R_j$ or 0.384 in R3BP units) we consider this an *escape orbit* and we stop the integration. The large distance for escape orbits is needed to assure that a particle has indeed escaped.

4.4 Stability of Initially Circular, Planar Orbits

In this section we consider the stability of initially circular ($e_o = 0$) and planar ($i_o = 0^\circ$ or 180°) orbits. We consider the direct orbits first.

4.4.1 Planar-Prograde Orbits

We have integrated many prograde test particles on initially circular orbits at a variety of distances from m_2 . At first we integrated cases such that the particle was initially in conjunction with the Sun and Jupiter (i.e., $M_o = 0^\circ$ or $M_o = 180^\circ$, see Fig. 4.1). Each of the two cases gave a slightly different stability result, with the transition from stable to unstable cases occurring fairly sharply; all the unstable cases were observed to escape the vicinity of Jupiter. The initial conditions we used were $e_o = 0$ (initially circular orbit), $i_o = 0^\circ$ (prograde orbit), $\omega_o = 0^\circ$ (arbitrary since $e_o = 0$), $\Omega_o = 0^\circ$ (also arbitrary since $i_o = 0^\circ$) and $M_o = 0^\circ$ or 180° , while $a_o = r_2$ is the variable we change to find regions of stability. If the test particle's distance from Jupiter exceeds 2 AU we stop the integration and consider it to have escaped. Our findings are as follows: (1) if we set $M_o = 0^\circ$, we find that $r_c \cong 357 R_J$ (see Figure 4.2) and (2) if we set $M_o = 180^\circ$, we find that $r_c \cong 369 R_J$ where r_c in this case is the "critical" distance to Jupiter. Taking the average of these two values we find a critical distance of about $363 R_J$ or 0.49 Hill radii, where a Hill radius was defined in section 3.3.1; this value agrees with the critical distance found by Hamilton and Burns [6], who also studied the stability of initially circular orbits by starting their test particles along the $m_1 - m_2$ line; they obtained a critical distance of $\sim 0.49 R_H$.

Figure 4.2. Survival time (in Jupiter periods) vs. initial starting distance (in R_J) for prograde orbits such that $M_o = 0^\circ$. The circles at the top-left represent orbits which had not escaped or collided with Jupiter after 100 Jupiter periods; all others are escape orbits.

Note the slight but significant difference between the values found using $M_o = 0^\circ$ or $M_o = 180^\circ$; the region of stability seems to depend on the value of the initial mean anomaly. Setting $M_o = 90^\circ$ the region of stability seems to be somewhat smaller; empirically we find that in this third case $313 R_J < r_c < 325 R_J$. This is significant because it indicates that the stability "radius" obtained previously (approximately $363 R_J$ or $0.49 R_H$) is not general and is only valid for certain initial conditions. The results obtained by setting $M_o = 270^\circ$ are identical to the $M_o = 90^\circ$ case. What we seek is a result

which is valid for *all* initial conditions, provided that the orbit is initially circular and prograde.

Can we find an analytical solution to this problem? From the previous chapter we recall that planar-prograde orbits tend to escape as soon as the curves of zero velocity open. The value of the Jacobi constant depends on the initial mean anomaly; for example, starting four different test particles at $a_o = 325 R_J$ and with the initial mean anomalies given in table 4.1 we find :

Table 4.1: The Jacobi constant C as a function of initial mean anomaly M_o for $a_o = 325 R_J$.

$M_o(\text{deg.})$	C	Stable?
0	3.04166247...	$C > C_2$, yes
90	3.03895238...	$C < C_2$, no
180	3.04155693...	$C > C_2$, yes
270	3.03895238...	$C < C_2$, no

Note that the value of the Jacobi constant is identical in the cases $M_o = 90^\circ$ and 270° ; in these two cases the Jacobi constant is lower than the critical value C_2 and hence the test particle escapes. Note also the slight difference in the values of C depending on whether $M_o = 0$ or 180 degrees, presumably accounting for the empirical findings above. We see that as a function of M_o , the minimum value is achieved at 90° (or 270°). Can we find a

maximum distance where we can place a prograde test particle with $M_o = 90^\circ$ and in an initially circular orbit, such that the curves of zero velocity are closed? The situation is shown schematically below in Fig. 4.3:

Fig. 4.3. Schematic diagram showing the relative positions in the xy -plane of m_1 , m_2 and m_3 on the rotating frame at $t = 0$ for a prograde, initially circular orbit such that $M_o = 90^\circ$.

We seek to find the largest $r_2 = y$ value such that the curves of zero velocity are closed.

From eqn. (2.19) we obtain

$$C = \left[(1 - \mu)r_1^2 + \mu r_2^2 \right] + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} - \dot{x}^2 - \dot{y}^2 \quad (4.33)$$

Now, from geometrical consideration in Fig. 4.3 we have (recall that the m_1 - m_2 distance is 1):

$$r_1^2 = r_2^2 + 1 = y^2 + 1 \quad (4.34)$$

$$\dot{y} = 0 \quad (4.35)$$

or

$$C = [y^2 + 1 - \mu] + \frac{2(1 - \mu)}{\sqrt{y^2 + 1}} + \frac{2\mu}{y} - \dot{x}^2 \quad (4.36)$$

Next, we need to express $\dot{x}$ in terms of y . For an orbit in the xy -plane we have

$$V_o^2 = \dot{x}_{rel}^2 + \dot{y}_{rel}^2 = \frac{\mu}{r_2} = \frac{\mu}{y} \quad (4.37)$$

From equations (3.7) and setting $t = 0$, we obtain

$$\dot{x}_{rel} = \dot{x} - y \quad (4.38a)$$

$$\dot{y}_{rel} = x + \dot{y} - \mu + 1 \quad (4.38b)$$

But $\dot{y} = 0$ and $x = \mu - 1$ so that $\dot{y}_{rel} = 0$. Therefore using eqns. (4.37) and (4.38a) we obtain $\dot{x} = y \pm V_o$. We are interested in the negative V_o case, which corresponds to the prograde orbits, so that

$$\dot{x} = y - \sqrt{\frac{\mu}{y}} \quad (4.39)$$

Finally we put this expression in eqn. (4.36):

$$C(y) = [y^2 + 1 - \mu] + \frac{2(1 - \mu)}{\sqrt{y^2 + 1}} + \frac{2\mu}{y} - \left(y - \sqrt{\frac{\mu}{y}}\right)^2 \quad (4.40)$$

Next we form the function

$$f(y) = C(y) - C_2 \quad (4.41)$$

where $C_2 = 3.0397139...$ is the value of the Jacobi constant at which the curves of zero-velocity open. We wish to find the root of this function; the easiest way is to proceed numerically via a Newton or secant method (see Burden and Fairies [21], or Garcia [22]). It turns out that the root of this function is $y = r_c = 2.899626 ... \times 10^{-2}$, or $316.5 R_J$. This is the lower limit of stability for planar, prograde, initially circular orbits. Recall we had obtained numerically a critical distance such that $313 R_J < r_c < 325 R_J$, so our semi-analytical critical distance is reasonable.

Similarly, let us do the same type of analysis for the case that $M_o = 0^o$; this would give us the maximum distance at which an initially circular, planar and prograde orbit can be stable. In this case, we have $r_1 + r_2 = 1$ and C is a function of r_2 . Omitting the details, we find

$$C(r_2) = 1 - r_2 + r_2^2 - \mu + 2\mu r_2 + \frac{2(1 - \mu)}{1 - r_2} + \frac{2\mu}{r_2} - \left(r_2 - \sqrt{\frac{\mu}{r_2}}\right)^2 \quad (4.42)$$

from which we form the function

$$g(r_2) = C(r_2) - C_2 \quad (4.43)$$

(note that the independent variable is now r_2 itself). In this case we find that the zero of this last function occurs at $r_2 = r_c = 3.2655203 \dots \times 10^{-2}$ or $356.5 R_J$. Recall that for this case we found empirically that the stability limit is approximately $357 R_J$, so the agreement is quite good.

For *planar, prograde and initially circular orbits* we then have the following analytical results:

- For $r_2 < 316.6 R_J$ ($0.43 R_H$), all orbits are stable
- For $r_2 > 356.5 R_J$ ($0.48 R_H$), all orbits are unstable

where r_2 is the satellite's initial distance. For values in between these two extremes, some orbits are stable and others are unstable, depending on the exact initial conditions. Our numerical experiments support these findings; however, for the case $M_o = 180^\circ$, we find that prograde orbits are stable out to slightly greater distances than what one would conclude from the analytical findings above; recall that in this case we found that $r_c \cong 369 R_J$, a 3.4% difference. We find that prograde orbits escape very quickly if the initial semi-major axis (or, equivalently, the initial distance) $a_o \geq 369 R_J$.

Figure 4.4. A Chaotic, Prograde Orbit

An interesting finding is that prograde orbits close to the lower stability limit ($316.6 R_J$ and $M_o \sim 90^\circ$) tend to be chaotic. In Fig. 4.4 we plot such a case; the initial conditions are such that $a_o = 316.54 R_J$, $e_o = 0.0$, $i_o = \Omega_o = \omega_o = 0^\circ$ and $M_o = 90^\circ$; the Jacobi constant in this case is $C = 3.0397145... > C_2$. We plot the semi-major axis and the eccentricity as a function of time, plus the orbit in the rotating frame and a log-log plot of the LCN vs. time. The orbit is clearly chaotic; nevertheless this orbit is stable for at least 100 Jupiter periods (furthermore this orbit is also stable in the sense of Hill). Since orbits so close to this lower stability limit tend to be chaotic, as soon as the particle is energetically able to escape it will quickly do so because chaotic orbits have a tendency to fill up the available space. When the particle's Jacobi constant is $C = C_2 - \varepsilon$, where ε is a small amount (barely below the critical value C_2), the "bubble" formed by the zero-velocity surface around Jupiter will have a small hole on its side facing the Sun and the chaotic orbit will bounce around inside this bubble until it finds the hole and escapes the enclosure. On the other hand, we find that for cases such that $M_o = 180^\circ$, orbits do not have a tendency to chaos and therefore will orbit Jupiter for a long time before escaping, even though the Jacobi constant is such that the curve of zero-velocity is barely open. In Fig. 4.5 we illustrate this point: the orbit at $M_o = 180^\circ$ (Fig. 4.5(a)) is initially further away than for the case $M_o = 90^\circ$, and yet the latter is less stable (escapes rather quickly) because of its chaotic nature. In fact, we have integrated the orbit in Fig. 4.5(a) for over 1000 Jupiter periods and even after that interval of time the particle had not escaped.

Fig. 4.5. Behavior of orbits close to the stability limit for different values of the initial mean anomaly. The open circle represents the position of Jupiter in the rotating frame, while the plus signs represent the initial position of the test particle; the enclosures represent the curves of zero-velocity. In Fig. 4.5(a), the initial starting distance is $a_o = 360 R_J$, $M_o = 180^\circ$; in Fig. 4.5(b), $a_o = 320 R_J$, $M_o = 90^\circ$.

4.4.2 Planar-Retrograde Orbits

We have already seen that retrograde orbits are stable out to much greater distances than direct orbits. As in the case for prograde orbits, we find that the critical distance seems to depend on the initial mean anomaly, the cases $M_o = 0^\circ$ or 180° being

stable out to greater distances than the case $M_o = 90^\circ$ (or 270°). The empirical findings are as follows: (1) $M_o = 0^\circ$, $r_c \cong 715 R_j$; (2) $M_o = 180^\circ$, $r_c \cong 726 R_j$; (3) $M_o = 90^\circ$, $538 < r_c < 544 R_j$ (the findings for $M_o = 270^\circ$ are identical to the case $M_o = 90^\circ$). Note the large difference in stability distances depending on whether we start the particle in conjunction or at right angles to the Sun-Jupiter line; the difference is even larger than for the direct

Fig. 4.6. A large, stable, retrograde orbit with $a_o = 726 R_j$ and $M_o = 180^\circ$. The open circle represents the position of Jupiter and the plus sign represents the initial position of the test particle. The patterns at the top and bottom of the figure are the curves of zero-velocity.

orbits. All of these large retrograde orbits have such low Jacobi constants that the curves of zero-velocity are completely open. Clearly then the opening of the curves of zero-velocity has no effect on the limiting distances for retrograde orbits in sharp contrast to the prograde case. In Figure 4.6 we plot in the rotating frame the most distant retrograde orbit we have found at $a_o = 726 R_J$, $M_o = 180^\circ$. The Jacobi constant for this orbit is $C = 3.003838\dots$, much lower than either C_1 or C_2 ; notice the curves of zero-velocity at the top and bottom of the picture. Also note how large the orbit is (recall that the Sun-Jupiter distance in dimensionless units is 1). In conclusion we can state that for *initially circular, retrograde orbits*, empirically we find that

- For $r_2 < 538 R_J$ ($0.72 R_H$), all orbits are stable
- For $r_2 > 726 R_J$ ($0.97 R_H$), all orbits are unstable

The larger limit is consistent with that found by Hamilton & Burns [6], who obtained also empirically $r_c \sim 1 R_H$. For distances between these two extremes some orbits are stable and others are unstable depending on the exact initial conditions, again the more stable cases tending to lie initially along the x-axis ($M_o = 0^\circ$ or 180°). Unfortunately we have found no analytical explanation for the critical distances of the retrograde orbits.

4.5 Stability of Initially Circular Orbits at all Inclinations

4.5.1 Numerical Results

In this section we relax the condition that the orbit must be in the xy -plane and consider empirically and analytically the case of initially circular orbits, but at all initial inclinations. We have integrated initially circular orbits for 100-Jupiter periods for inclinations such that $0^\circ \leq i_o \leq 180^\circ$ and for different distances from Jupiter. The particle's initial conditions are as follows: a_o and i_o have been previously explained, $e_o = 0$, and ω_o is arbitrary so we set it equal to 0° and M_o we set it equal to 90 degrees because as we have seen in the planar case it makes the test particles go unstable sooner than the cases 0 or 180 degrees. The last orbital element we need is Ω_o ; arbitrarily we set it to 90 degrees (we have checked several integrations with different values of Ω_o and they do not significantly affect the quantitative results). Typical results are seen in Table 4.2; in the left table we present the results such that the initial semi-major axis $a_o = 150 R_J$ ($0.20 R_H$) and in the table on the right we present the results for $a_o = 300 R_J$ ($0.40 R_H$). The first column shows the orbit's initial inclination, while the second column shows the orbit's maximum eccentricity achieved during the length of the integration. The most obvious trend is a definite differentiation into two distinct regimes. We can summarize the numerical findings as follows: (1) For $i_o \lesssim 40^\circ$ or $i_o \gtrsim 140^\circ$, the eccentricity varies quasi-periodically, reaching the maximum values ~ 0.014 for $150 R_J$ and ~ 0.12 for $300 R_J$, i.e., the maximum eccentricity is independent of the initial inclination, but depends on the

Table 4.2: Maximum eccentricity reached as a function of the initial inclination; $e_o = 0$,
 $\omega_o = 0$, $\Omega_o = M_o = 90$. Left table $a_o = 150 R_J$; right table $a_o = 300 R_J$.

$i_o(\text{deg.})$	e_{max}	Comments	$i_o(\text{deg.})$	e_{max}	Comments
0	0.018	Stable for 100 P_J	0	0.116	Stable for 100 P_J
10	0.0144	Stable for 100 P_J	10	0.1372	Stable for 100 P_J
20	0.0140	Stable for 100 P_J	20	0.1235	Stable for 100 P_J
30	0.0135	Stable for 100 P_J	30	0.1076	Stable for 100 P_J
40	0.0130	Stable for 100 P_J	40	0.1080	Stable for 100 P_J
45	0.3478	Stable for 100 P_J	45	0.0987	Stable for 100 P_J
50	0.5580	Stable for 100 P_J	50	0.3644	Stable for 100 P_J
60	0.7980	Stable for 100 P_J	60	0.8400	Stable for 100 P_J
70	0.9299	Stable for 100 P_J	70	0.9742	Stable for 100 P_J
80	0.9916	Stable for 100 P_J	80	0.9996	Coll. at $t = 13.0 P_J$
90	0.9963	Coll. at $t = 11.4 P_J$	90	0.9998	Coll. at $t = 54.2 P_J$
100	0.9848	Stable for 100 P_J	100	0.9966	Coll. at $t = 30.9 P_J$
110	0.9202	Stable for 100 P_J	110	0.9504	Stable for 100 P_J
120	0.8003	Stable for 100 P_J	120	0.8679	Stable for 100 P_J
130	0.6170	Stable for 100 P_J	130	0.7187	Stable for 100 P_J
140	0.2958	Stable for 100 P_J	140	0.4648	Stable for 100 P_J
150	0.0130	Stable for 100 P_J	150	0.0985	Stable for 100 P_J
160	0.0135	Stable for 100 P_J	160	0.0953	Stable for 100 P_J
170	0.0139	Stable for 100 P_J	170	0.1050	Stable for 100 P_J
180	0.0140	Stable for 100 P_J	180	0.106	Stable for 100 P_J

distance to the planet. A typical case is shown in Figure 4.7 where $a_o = 150 R_j$, and $i_o = 30^\circ$: the short-term variations in the eccentricity occur on a time scale comparable to the satellite's period around Jupiter. The inclination shows periodic variations, confined to the narrow range $30^\circ \lesssim i \lesssim 31^\circ$, while the eccentricity always stays fairly low; the maximum eccentricity in this case is 0.014. These orbits are very stable. (2) If the satellite's initial inclination i_o is within a broad range near 90 degrees (roughly $40^\circ \lesssim i_o \lesssim 140^\circ$), its eccentricity e grows exponentially with time until it approaches a maximum value e_{max} , i.e., the orbit's maximum eccentricity strongly depends on the initial inclination; the overall behavior of e is quasi-periodic as e varies from 0 to e_{max} . A typical case is shown in Figure 4.8 where $a_o = 150 R_j$, and $i_o = 60^\circ$. Note how, if the eccentricity is low, the inclination oscillates sinusoidally about 61° with a small amplitude of about 1° . However, when the eccentricity starts to grow, there's a sudden and sharp drop in inclination, in this case down to the value $i_{min} \approx 41^\circ$. At the same time that the inclination is at a minimum the eccentricity is reaching its maximum value, in this case ≈ 0.798 . (3) Most near-polar orbits ($i_o \sim 90^\circ$) result in collisions of the satellite with Jupiter. Note also that in the table on the right ($300 R_j$) the "collision zone" has a spread of about 10 degrees on either side of $i_o = 90^\circ$. At this distance none of the orbits escape, since the Jacobi constant for all of these orbits is such that $C < C_2$. In these examples, instability of an orbit has to do with initial conditions that lead to collisions rather than escapes from the vicinity of Jupiter (as was the case for the planar orbits). For

Fig. 4.7. $a_a = 150 R_j(0.201 R_H)$ [$e_a = 0$, $i_a = 30$ deg.]

Fig. 4.8. $a_a = 150 R_J$, $(0.201 R_H)$ [$e_a = 0$, $i_a = 60$ deg.]

Figure 4.9. Survival time vs. initial inclination for initially circular orbits at initial distance of $450 R_J$ ($0.6 R_H$). Other initial conditions are $e_o = 0$, $\omega_o = 0$ and $\Omega_o = M_o = 90^\circ$. The crosses represent *escape orbits* while the open circles represent *stable orbits*.

distances greater than $300 R_J$, the general pattern changes and we find very few if any collisions, with many prograde orbits becoming escape orbits. A typical example of this is shown in Figure 4.9 where we plot the survival time (in Jupiter periods) vs. the initial inclination for the case $a_o = 450 R_J$; recall that we consider an orbit to have escaped if its distance from Jupiter exceeds 2 A.U.'s. If the initial inclination is greater than about 120 degrees, the orbit is stable for at least 100 Jupiter periods. Note that all the stable orbits are retrograde orbits. Regarding the choice of Ω_o and M_o , we have made a variety of

different runs with different initial values for these two variables and they do not seem to affect the overall behavior of the satellite; therefore our choice of setting $\Omega_o = M_o = 90^\circ$, although arbitrary, seems appropriate.

4.5.2 Analytical Results

Is there an analytical explanation for the type of behavior we have encountered numerically? The usual analytical approach makes use of perturbation methods to obtain an approximate solution to the satellite's motion; usually these approximations assume that the orbit's eccentricity and inclination are so small that they can be ignored. In our case, however, we can not ignore the orbit's inclination since we are considering the behavior of initially circular orbits at all inclinations. Kozai [3] was among the first to treat a three-body problem where the third particle could have any arbitrary values of the eccentricity and the inclination; he considered a Sun-Jupiter-asteroid three-body problem. Kovalevsky and also Froeschlé [17] considered the solar perturbations on a satellite at arbitrary eccentricities and inclinations; we will follow the latter's approach to obtain a disturbing function R .

Consider the R3BP in a *non-rotating system* and with m_2 (Jupiter) at the origin; for convenience we set the coordinates as $m_1 (x', y', z'=0)$ (since it is confined to the xy -plane), $m_2 (0,0,0)$ (since it is located at the origin) and $m_3 (x, y, z)$. Recall that the masses are $m_1 = 1-\mu$ (the Sun), $m_2 = \mu$ (Jupiter) and $m_3 \sim 0$ (the satellite). The situation is

shown schematically in Fig. 4.10. Let the distance between Jupiter and the satellite be r , and the distance between the satellite and the Sun be ρ , the distance between Jupiter and the Sun is unity, $r' = 1$. Furthermore, let the angle between the satellite radius vector and the Sun's radius vector be γ . The coordinates of the Sun as a function of time are

$$x' = \cos t \quad (4.44a)$$

$$y' = \sin t \quad (4.44b)$$

$$z' = 0 \quad (4.44c)$$

Fig. 4.10. Relative positions of m_1 (the Sun), m_2 (Jupiter) and m_3 (the satellite).

It is clear that if the mass of m_1 were equal to zero, the satellite problem would be reduced to a two-body problem; the satellite's orbit suffers perturbations due to the non-zero mass of m_1 . The perturbing (or disturbing) function for a three-body problem is given by (see Danby [10], p. 339, or Kovalevsky [23], p. 56)

$$R = (1 - \mu) \left\{ \frac{1}{\rho} - \frac{xx' + yy' + zz'}{r'^3} \right\} \quad (4.45)$$

or, using equations (4.44) and the fact that $r' = 1$ we get

$$R = (1 - \mu) \left\{ \frac{1}{\rho} - x \cos t - y \sin t \right\} \quad (4.46)$$

where the quantity ρ , the distance from the satellite to the Sun, is given by

$$\rho = \sqrt{r^2 + 1 - 2(x \cos t + y \sin t)} \quad (4.47)$$

Using the law of cosines, we can also write ρ as

$$\rho = \sqrt{r^2 + 1 - 2r \cos \gamma} \quad (4.48)$$

so that we can write

$$\frac{1}{\rho} = \frac{1}{\sqrt{1 + r^2 - 2r \cos \gamma}} \quad (4.49)$$

Equation (4.49) is a form of the generating function for the Legendre Polynomials (see Danby [10], p. 112). Therefore we can write (4.49) as follows (keeping in mind that for our purposes $r \ll 1$):

$$\frac{1}{\rho} = \sum_{n=0}^{\infty} r^n P_n(\alpha) \quad (4.50)$$

where, using equations (4.47) and (4.48) we have

$$\alpha = \cos \gamma = \frac{x}{r} \cos t + \frac{y}{r} \sin t \quad (4.51)$$

The first few Legendre polynomials are

$$\begin{aligned} P_0(\alpha) &= 1 \\ P_1(\alpha) &= \alpha \\ P_2(\alpha) &= \frac{1}{2}(3\alpha^2 - 1) \\ &\text{etc.} \end{aligned}$$

We expand eqn. (4.50), keeping terms of order 2 and less:

$$\frac{1}{\rho} = 1 + x \cos t + y \sin t + \frac{3}{2} (x^2 \cos^2 t + y^2 \sin^2 t + 2xy \sin t \cos t) - \frac{r^2}{2} + O(r^3) \quad (4.52)$$

We now put equation (4.52) in eqn. (4.46) to obtain

$$R = (1 - \mu) \left\{ 1 + \frac{3}{2} (x^2 \cos^2 t + y^2 \sin^2 t + 2xy \sin t \cos t) - \frac{r^2}{2} \right\} + O(r^3) \quad (4.53)$$

We can simplify this disturbing function considerably using eqn. (4.51) and ignoring terms that are constant; then we obtain:

$$R = (1 - \mu) \frac{r^2}{2} \{ 3 \cos^2 \gamma - 1 \} + O(r^3) \quad (4.54)$$

Next we "average" this disturbing function twice: first over one "revolution" of the perturbing body (the Sun) m_1 , and second over one revolution of the satellite m_3 itself. The reason for this is to eliminate short period terms in the disturbing function. We can represent these operations schematically as

$$\mathfrak{R} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R \, dm_1 \, dm_3 \quad (4.55)$$

After performing these averaging operations and writing the resulting averaged disturbing function in terms of the satellite's orbital elements we obtain [17]:

$$\mathfrak{R} = \frac{1-\mu}{8} a^2 \left\{ 2 + 3e^2 - 3 \sin^2 i \left[1 + e^2 (4 \sin^2 \omega - \cos^2 \omega) \right] \right\} + O(r^3) \quad (4.56)$$

Next, we use this perturbing function in the *Lagrange Planetary Equations*. The Lagrange Planetary equations are differential equations which when solved give the time evolution of the osculating elements of the perturbed orbit; they are (see Kovalevsky [23], p. 41)

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial \mathfrak{R}}{\partial M} \quad (4.57a)$$

$$\frac{de}{dt} = \frac{-\sqrt{1-e^2}}{na^2 e} \frac{\partial \mathfrak{R}}{\partial \omega} + \frac{1-e^2}{na^2 e} \frac{\partial \mathfrak{R}}{\partial M} \quad (4.57b)$$

$$\frac{di}{dt} = \frac{-1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial \mathfrak{R}}{\partial \Omega} + \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial \mathfrak{R}}{\partial \omega} \quad (4.57c)$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial \mathfrak{R}}{\partial i} \quad (4.57d)$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial \mathfrak{R}}{\partial e} - \frac{\cos i}{na^2 \sqrt{1-e^2} \sin i} \frac{\partial \mathfrak{R}}{\partial i} \quad (4.57e)$$

$$\frac{dM}{dt} = n - \frac{2}{na} \frac{\partial \mathfrak{R}}{\partial a} - \frac{1-e^2}{na^2 e} \frac{\partial \mathfrak{R}}{\partial e} \quad (4.57f)$$

so that we get

$$\frac{da}{dt} = 0 \quad (4.58a)$$

$$\frac{de}{dt} = \frac{15(1-\mu)}{8n} e\sqrt{1-e^2} \sin^2 i \sin(2\omega) \quad (4.58b)$$

$$\frac{di}{dt} = \frac{-15(1-\mu)e^2}{4n\sqrt{1-e^2}} \sin i \cos i \sin \omega \cos \omega \quad (4.58c)$$

$$\frac{d\Omega}{dt} = \frac{-3(1-\mu)}{4n\sqrt{1-e^2}} \cos i [1 + e^2(4 \sin^2 \omega - \cos^2 \omega)] \quad (4.58d)$$

$$\begin{aligned} \frac{d\omega}{dt} = & \frac{3(1-\mu)\sqrt{1-e^2}}{4n} \{1 - \sin^2 i(4 \sin^2 \omega - \cos^2 \omega) + \\ & \frac{3(1-\mu)}{4n\sqrt{1-e^2}} \cos^2 i[1 + e^2(4 \sin^2 \omega - \cos^2 \omega)]\} \end{aligned} \quad (4.58e)$$

$$\begin{aligned} \frac{dM}{dt} = & n - \frac{1-\mu}{2n} \{2 + 3e^2 - 3 \sin^2 i[1 + e^2(4 \sin^2 \omega - \cos^2 \omega)]\} - \\ & \frac{3(1-\mu)(1-e^2)}{4n} \{1 - \sin^2 i(4 \sin^2 \omega - \cos^2 \omega)\} \end{aligned} \quad (4.58f)$$

where n is the mean motion of the satellite about Jupiter and is given (in radians per unit time) by eqn. (4.1).

This set of differential equations looks daunting and difficult to solve. However, we need not solve them; what is important to note is that there are some quantities which are conserved. Besides the obvious constancy of the semi-major axis (from eqn. (4.58a)), we also get (see Froeschlé [17])

$$(1 - e^2) \cos^2 i = \Theta \quad (4.59a)$$

$$e^2(5 \sin^2 i \sin^2 \omega - 2) = K \quad (4.59b)$$

How well are these quantities conserved in practice? In Figure 4.11 we can see that they are rather well conserved; the initial conditions for this particular orbit are $a_o = 150 R_j$, $e_o = 0$, $i_o = 30^\circ$, $\Omega_o = 90^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$ (this is the fourth orbit from the top of the left table 4.2). The semi-major axis' average value is $149.5 R_j$, with a standard deviation of $0.4 R_j$; $\langle \Theta \rangle = 0.74$ with a s.d. equal to 0.006 . The last conserved quantity K we expect to stay close to 0 , since $e_o = 0$; indeed this is the case: its average value is -6.9×10^{-5} , with a s.d. of 4.8×10^{-5} .

From the averaged R3BP we obtain three quasi-integrals of motion: what can they tell us? Consider an initially circular orbit; at $t = 0$, $e_o = 0$ and eqn. (4.59a) gives us

$$\cos^2 i_o = \Theta \quad (4.60)$$

where i_o is the orbit's initial inclination. At a later $t \neq 0$, we obtain

$$(1 - e^2) \cos^2 i = \cos^2 i_o = \Theta \quad (4.61)$$

and solving for e we obtain

Figs. 4.11. Three "conserved" quantities in the averaged R3BP. (a) the semi-major axis (in R_j); (b) the quantity Θ , defined in eqn. (4.59a); (c) the quantity K (defined in eqn. (4.59b)).

$$e = \sqrt{1 - \left(\frac{\cos i_o}{\cos i} \right)^2} \quad (4.62)$$

Can we solve for the *maximum* eccentricity attained by the orbit? In order to do that we need to eliminate $\cos^2 i$; let us make use of eqn.(4.59b). For initially circular orbits, $K=0$, so that we can write eqn. (4.59b) as

$$5 \sin^2 i \sin^2 \omega = 2 \quad (4.63)$$

or

$$\sin^2 i = \frac{2}{5 \sin^2 \omega} = 1 - \cos^2 i \quad (4.64)$$

We now need to eliminate $\sin^2 \omega$. We note that in equation (4.58b), an *extremum* for e occurs if $de/dt = 0$; for which values of ω do we get an extremum? This occurs if $\omega = 0^\circ, 90^\circ, 180^\circ$ or 270° . The cases $\omega = 0^\circ$ or 180° are not useful, although if we set $\omega = 90^\circ$ or 270° , $\sin^2 \omega = 1$; using this result we rewrite eqn. (4.64) as

$$\cos^2 i = \frac{3}{5} \quad (4.65)$$

which is the result we needed. Putting this expression in equation (4.62) we finally obtain an expression for the maximum eccentricity:

$$e_{\max} = \sqrt{1 - \frac{5}{3} \cos^2 i_o} \quad (4.66)$$

Equation (4.66) is very important for it *predicts the maximum eccentricity for an initially circular orbit given only the initial inclination i_o* . Observe also that eqn. (4.65) predicts two *critical inclinations*: if we set $e_{\max} = 0$ we obtain

$$\cos i_c = \pm \sqrt{\frac{3}{5}} \quad (4.67)$$

so that the critical inclinations are 39.2° and 140.8° , and this is also the range of validity of eqn. (4.66). In Fig. 4.12 we plot the maximum eccentricity vs. initial inclination (values from Table 4.2) as well as the *predicted* values of the maximum eccentricity as given by equation (4.66) for two values of the initial distance from Jupiter. The agreement is very good at $150 R_J$ ($0.201 R_H$) and not as good for the more distant case at $300 R_J$ ($0.403 R_H$). We must remember that eqn. (4.66) is obtained from a disturbing function good to order $O(r^3)$ so that it is expected to become less accurate the further out we place our test particle. The other behavior we can explain analytically is the extreme shifts in inclination that the orbit experiences while the eccentricity achieves its maximum value (see Fig. 4.8). This was already worked out in eqn. (4.67), because

Figs. 4.12. Numerical vs. analytical results for initially circular orbits at two different initial distances ($a_o = r$). The open circles represent the results of numerical experiments while the solid line is given by eqn. (4.66) and the dotted line is given by eqn. (4.68). (a) initial distance is $a_o = 300 R_J$ (0.403 Hill radii); (b) initial distance is $a_o = 150 R_J$ (0.201 Hill radii).

while the orbit's eccentricity is at its maximum value ($e = e_{max}$), the orbit's inclination is equal to the critical inclination $i_c = 39.2^\circ$ or 140.8° . This can also be seen by first solving eqn. (4.61) for $\cos i$, and then substituting eqn. (4.66) for e ; we obtain equation (4.67).

We also recall that the maximum eccentricity does not seem to depend on the initial inclination if i_o is outside of the range $39.2^\circ < i_o < 140.8^\circ$, so that for a given initial semi-major axis, e_{max} is constant. However, we do find empirically that the maximum eccentricity for $i_o < 39.2^\circ$ or $i_o > 140.8^\circ$ does depend on the initial semi-major axis a_o . After recording the maximum eccentricity for various initially circular orbits outside of the range $39.2^\circ < i_o < 140.8^\circ$ and at different distances from Jupiter we find using a least squares method [22] the following dependence of e_{max} on a_o :

$$e_{max} \cong 2a_o^3 \quad (4.68)$$

where a_o is the orbit's initial semi-major axis in *Hill Radii*. We show this dependence in Fig. 4.12 as a dotted line; the agreement is quite good in both Figs. 4.12(a) and 4.12(b). In Fig. 4.13 we compare the fit given by eqn. (4.68) to the actual data points used in the fit.

Figure 4.13. Least square fit to the maximum eccentricities (open circles) as a function of semi-major axis attained by initially circular orbits with inclinations *outside* of the range $39.2^\circ - 140.8^\circ$.

4.5.3 Collision zones for Near-Polar Orbits

(The author would like to acknowledge Dr. Luke Dones for bringing this point to his attention). Using equation (4.66) we can easily obtain an equation which predicts the width of a zone around 90° within which initially circular orbits will collide with the planet. We note that if the perijove distance $q = a(1-e)$ is less than a Jupiter radii, then the test particle will suffer a collision with the planet. We set

$$a(1 - e_{\max}) = 1 \quad (4.69)$$

so that solving for $e_{\max}$, and setting this equal to the expression in eqn. (4.66) we obtain

$$e_{\max} = \sqrt{1 - \frac{5}{3} \cos^2 i_o} = 1 - \frac{1}{a} \quad (4.70)$$

Using the fact that for large a ,

$$\left(1 - \frac{1}{a}\right)^2 \cong 1 - \frac{2}{a} \quad (4.71)$$

we then obtain a simple expression for the dependence of the "collision inclination" i_{coll}

$$\cos i_{coll} = \pm \sqrt{\frac{6}{5a}} \quad (4.72)$$

where a is the semi-major axis (in Jupiter radii) of the (initially) circular orbits. In Fig. 4.14(a) we plot the width (in degrees) of this collision region for different values of the semi-major axis. The graph is centered on 90° ; in theory all exactly polar orbits suffer a collision with the planet. For example, this graph tells us that at 150 planetary radii the collision zone for initially circular orbits extends from roughly 85° to 95° , a range of 10

degrees; any initially circular orbit with inclinations within this range will probably suffer a collision. Furthermore, it can be seen that the width of the collision zone gets wider the

Fig. 4.14. (a) Width of the "collision zone" for initially circular orbits at different distances from Jupiter as a function of initial inclination as given by eqn. (4.72). (b) Detail of graph 4.14a, showing a comparison between numerical experiments and eqn. (4.72) at initial semi-major axis $a_o = 150 R_J$; crosses represent collisions while open circles are orbits which did not collide (data from Table 4.3)

closer one places the test particle. Equation (4.72) breaks down at close distances to the planet (roughly closer than 4 or 5 planetary radii) because of the approximation in eqn. (4.71); it also breaks down at large distances (roughly $a > 200 R_J$) because it is derived

from a second-order perturbing function; this latter point can be clearly seen in Table 4.2, where the collision zone is wider at $300 R_J$ than $150 R_J$, contrary to what is predicted. In Table 4.3 we test equation (4.72) for initially circular orbits at $a_o = 150 R_J$ and different values of the initial inclination by integrating a test particle at the given distance and inclinations for 100 Jupiter periods and recording whether a collision took place or not. It can be seen that the agreement, although not excellent, is good; see also Fig. 4.14(b). Equation (4.72) can be used as a rough guide to predict which initially circular orbits will collide with the planet.

Table 4.3: Comparison between eqn. (4.72) and numerical experiments. A star in parenthesis and to the right of the initial inclination indicates disagreement between theory (eqn. 4.72) and experiment. See also Fig. 4.14(b). In all cases $a_o = 150 R_J$.

i_o (deg.)	Predicted to collide?	Collision after 100 P_J ?
80	No	No
82.5 (*)	No	Yes, at $t = 11.6 P_J$
85	Yes	Yes, at $t = 11.3 P_J$
87.5	Yes	Yes, at $t = 11.3 P_J$
90	Yes	Yes, at $t = 11.4 P_J$
92.5	Yes	Yes, at $t = 11.4 P_J$
95 (*)	Yes	No
97.5	No	No
100	No	No

4.6 Discussion

We have covered much ground in this chapter concerning the stability of initially circular orbits as perturbed by the Sun. We considered planar orbits first, and saw that the critical distance at which the satellite becomes unstable depends to a large extent on the test particle's initial mean anomaly; those test particles with Mean Anomalies close to 0° or 180° are stable out to greater distances than test particles with Mean Anomalies close to 90° or 270° . There was a significant difference in the critical distances found for the retrograde and direct orbits, retrograde orbits being stable out to much greater distances than direct ones. We find that no initially circular, planar-prograde orbits are stable if the initial distance is greater than $0.48 R_H$; in the case of retrograde orbits the critical distance extends roughly twice as far, as no initially circular retrograde orbit is stable if the initial distance is greater than $0.97 R_H$. These values also closely agree with those found by Hamilton and Burns [6]; what is also significant is that in their work they assume $\mu = 5 \times 10^{-12}$, a value much smaller than ours, and yet the results are quite similar. We also found an analytical explanation for the instability of direct orbits, i.e., the distance at which the curves of zero velocity open is the critical distance at which those orbits become unstable. No such explanation was found for the retrograde orbits. Lastly, we noted that initially circular, prograde direct orbits close to the critical distance and with initial Mean Anomalies close to 90° tend to be chaotic. None of the planar, initially circular orbits we tested suffered collisions with Jupiter.

For inclined orbits, the situation is quite different. We found first numerically and then analytically that the orbit's initial inclination greatly affects the survival probability of the test particle. Most importantly, we found that if the satellite's initial inclination is within the values 39° and 141° , its eccentricity tends to grow exponentially up to a value e_{max} , that value increasing as $i_o \rightarrow 90^\circ$; at 90° that value is unity and the satellite suffers a collision with the planet. It must be pointed out that this range $39^\circ < i < 141^\circ$ does not represent an instability zone *per se*; it only represents instability if the maximum eccentricity grows large enough so that the particle can experience gravitational scattering by massive, inner satellites, or, in an extreme case, collision with the planet. This type of behavior can be modeled quite accurately using an averaged perturbing function (eqn. (4.56)), from which we obtain the simple expression given by eqn. (4.66) for the value of e_{max} . It should be obvious that eqn. (4.66) also works well when $e_o \cong 0$; in practice we find it gives satisfactory results if $e_o < 0.1$. Furthermore, we have derived a simple result (eqn. (4.72)) to predict which initially circular orbits at given initial inclinations centered on 90° will actually collide with the planet. We find that the numerical results agree reasonably well with the analytical predictions (obtained from the perturbing function given by (4.56)) out to distances of roughly $200 R_J$ or $0.27 R_H$. We have also found that if the orbit's initial inclination is outside of the range 39° to 141° , its eccentricity varies quasi-periodically, the maximum value depending on the test particle's distance (see eqn. (4.68)). Beyond $\sim 0.27 R_H$ we find that many orbits start to escape the vicinity of Jupiter and that those orbits with inclination i approaching 180° are the most stable ones. At

greater and greater distances, few test particles collide with Jupiter and more and more of the test particles tend to escape, until finally at about $r_c \sim 1 R_H$ only the planar, retrograde orbits are stable ($i_o = 180^\circ$), such as the orbit depicted in Fig. 4.6. No stable test particles were found at any inclination beyond $r_c \sim 1 R_H$.

The 39° to 141° range at which the eccentricity grows exponentially is interesting, because no small distant satellite in the solar system has an inclination in this range with respect to the planet's equator. In the case of Jupiter, if there were a small satellite with an inclination within this range, the orbit would probably not be stable if it crossed the orbit of one of the galilean satellites (which have appreciable masses themselves). For example, the most distant of the galilean satellites is Callisto at roughly $26.4 R_J$. Proceeding in the same manner as in section 4.5.3 but setting the right side of eqn. (4.69) equal to 26.4 we find that in the real case of the Jupiter system no small distant satellite could be stable within the inclinations given by

$$\cos i \cong \pm \sqrt{\frac{31.7}{a_o}} \quad (4.73)$$

where a_o is the satellite's semi-major axis (in Jupiter radii); otherwise these distant satellites would suffer close passages to the galilean satellites. For example at $\sim 160 R_J$, which is the average distance of the prograde satellites of Jupiter, this range of

"forbidden" inclinations is 63° to 117° . For these satellites of Jupiter the inclinations range from 24.8° (Elara) to 29.0° (Lysithea), well below the forbidden range.

Chapter 5

General Stability of Orbits

In this chapter we relax the constraint that the test particle (satellite) to be studied follows an initially circular orbit, i.e., we generalize the work of the last chapter to include cases such that $e_o \neq 0$. In particular we are interested in the "topology of the e - i space" as we vary the semi-major axis. First we examine the numerical results and then we compare those results with the theory of the averaged R3BP developed in last chapter.

5.1 Orbital Integrations

We use a modified version of the code we used for the last chapter. The code takes in a set of parameters which include controls for the program, such as the tolerance to be used by the Bulirsch-Stoer integrator (the tolerated truncation error has been set, as in the previous chapter, to 10^{-13}), the number of time steps, etc., as well as part of the initial conditions in the form of the semi-major axis (in R_J), plus the orbit's longitude of the ascending node (Ω_o), argument of perijove (ω_o) and Mean Anomaly (M_o). The code then produces a set of initial eccentricities and initial inclinations in steps determined by the user: in effect the program is producing a set of many initial conditions, each representing a test particle orbiting Jupiter. Once the initial orbital elements for a given test particle are computed, the code transforms this set to $(x_o, y_o, z_o, \dot{x}_o, \dot{y}_o, \dot{z}_o)$, i.e., the

rotating R3BP coordinates, and given this set, integrates the equations of motion (eqns. 2.10). As before, whenever the eccentricity of any test particle is greater than 0.8, we switch to regularized variables (see section 4.3.2). Again, if the test particle survives orbiting Jupiter for 100 Jupiter periods (1186 years) without escaping or colliding with Jupiter, we consider the test particle to be stable.

The code itself produces the test particles' initial eccentricities and inclinations, while the user must supply the rest of the orbital elements. We decided to do a variety of different runs with increasing semi-major axis, since one of the principal objectives is to get a qualitative understanding of the e - i space at different distances from the planet.

The rest of the orbital elements we decided to arbitrarily set as follows: $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$. We did not know in advance how the first two choices would affect the results; it turned out that these arbitrary values do not change the qualitative results.

However, we know from last chapter that orbits such that $M_o = 90^\circ$ tend to be *less* stable than otherwise, which is desirable since we wish to know the *minimum* as well as the maximum extent of any regions of stability. Later we will repeat these runs with a variety of combinations of these three parameters to see how they affect the generality of the results.

5.2 Numerical Results

We start with the case such that the initial semi-major axis is $160 R_J$ ($0.215 R_H$); this is roughly the average distance of the prograde distant satellites of Jupiter. In Fig. 5.1 we show the results of running many test particles for 100 Jupiter periods. In this e_o vs. i_o diagram a plus '+' symbol represent a stable orbit (i.e., an orbit which during the course of the integration neither collided with nor escaped from Jupiter) and a star '*' represents a collision orbit. Therefore, a stability "zone" could be defined roughly as a region in the e - i plane where we find mostly '+' signs. The open circles represent the four prograde satellites of Jupiter, which lie at an average distance of $160 R_J$; note that they are in a stable region, as it should be. The "resolution" of this plot is 0° to 180° in steps of 10° in inclination (a total of 19 different initial inclinations) and 0 to 0.9 in steps of 0.1 in eccentricity (a total of 10 different initial eccentricities) so that the total number of test particles is 190.

At this initial semi-major axis most orbits are stable, the exception being a region somewhat symmetric about $i_o \sim 90^\circ$ where the solar perturbations drive the test particles to collide with the planet (a phenomenon we already encountered with the initially circular orbits); this region gets wider as the initial eccentricity increases: at $e_o = 0.9$ the collision zone extends from $80^\circ \lesssim i_o \lesssim 130^\circ$. Of 190 orbits integrated 165 (87%) were

Fig. 5.1. Stability at $a_o = 160 R_J$; a plus ('+') sign represents a stable orbit while a star ('*') sign represents a collision orbit. The open circles represent the positions of Jupiter's four prograde satellites as given by Danby. There are a total of 190 different initial conditions.

stable, while 25 (13%) underwent collisions. The average time scale for a collision is about ~ 5 Jupiter periods, with the most eccentric orbits colliding in a shorter time than the less eccentric ones. Note that no test particles escaped the Jupiter vicinity: in fact, the range of Jacobi constants for this set of test particles is $3.0545... \leq C \leq ... 3.0695$, well above the critical value C_2 so that the curves of zero-velocity are completely closed around Jupiter (meaning no escapes can occur). This means that Jupiter's four distant prograde satellites, Leda, Himalia, Lysithea and Elara, are all *Hill stable* in the context of the R3BP, implying long-term stability for them (see Appendix B). Hagihara was the first to show this in 1961.* We had already seen from analytical considerations in the last chapter how the semi-major axis is approximately constant if the distance from Jupiter is not too great, i.e., if $r_2 < 200 R_J$; in fact, at $a_o = 160 R_J$ we find the average semi-major axis of all the particles which did not suffer a collision is $160.4 R_J$, almost the same as the initial semi-major axis a_o .

At $a_o = 200 R_J$ the picture is qualitatively very similar to $160 R_J$; the next interesting value of the semi-major axis is $300 R_J$, at which distance lie (roughly) the two closest of Jupiter's retrograde satellites: Carme and Ananke. In Fig. 5.2 we present the stability results for $a_o = 300 R_J$. Since at this semi-major axis there seems to be much more structure than in the previous case we choose a higher resolution than in Fig. 5.1:

* Hagihara, Y. 1961. The stability of the solar system. In *Planets and Satellites* (G. Kuiper and B.M. Middlehurst, eds.), pp. 95-158. Chicago: University of Chicago Press.

Fig. 5.2. Stability at $a_o = 300 R_J$; a plus ('+') sign represents a stable orbit, a star ('*') sign represents a collision and a dot ('.') an escape orbit. The two open circles represent the two closest of Jupiter's retrograde satellites, Carme and Ananke (as given by Danby). There are a total of 740 different initial conditions.

the eccentricities are from 0 to 0.95 in steps of 0.05 and the inclinations go from 0° to 180° in steps of 5° for a total of 740 total initial conditions.

Roughly speaking Fig. 5.2 is similar to Fig. 5.1; for example, most orbits close to $i_o \sim 90^\circ$ result in collisions with Jupiter. However, there are several important differences. First, as the initial eccentricity increases, most orbits tend to collide with the planet; at $e_o \geq 0.85$, all retrograde and several prograde orbits undergo collisions with the planet. Second, there are now a few escape orbits, all of them being high eccentricity, prograde orbits: this is now possible because the range of Jacobi constant for this set is $3.020... \leq C \leq 3.041...$, so that some orbits have C 's less than the critical value C_2 . In fact, the Jacobi constants for Carme and Ananke are approximately 3.020 and 3.026 respectively (see Appendix B). Third, not all near-polar orbits ($i_o = 90^\circ$) undergo collisions: there are several near-polar orbits that are stable at least for 100 Jupiter periods; in particular, there is a stable case at $i_o = 90^\circ$ and $e_o = 0.3$. Roughly speaking, most prograde orbits such that $0^\circ \leq i_o \leq 70^\circ$ and such that $e_o \leq 0.75$ are stable, as are most retrograde orbits such that $120^\circ \leq i_o \leq 180^\circ$ and $e_o \leq 0.5$; at higher eccentricities the region of stability gets smaller, although there are several scattered stable orbits. All retrograde orbits such that $e_o \geq 0.85$ result in collisions. Note that Carme and Ananke are safely in a stable region. About 72% (530) of the 740 test particles were stable, 26% (194) underwent collisions and 2% (16) underwent escapes from the Jupiter vicinity.

The next figure (Fig. 5.3) shows the stability results for $a_o = 330 R_J$, which is roughly the semi-major axis of the two most distant (retrograde) satellites of Jupiter (Pasiphae and Sinope, open circles). This plot shows a marked contrast to the previous ones and shows much intricate structure. The resolution is the same as in the previous figure. The percentage of orbits which underwent escape has increased to 54% (396 out of 740), the percentage for collision orbits is 11% (82) and the corresponding number for orbits which survive is 35% (262). Not surprisingly most of the stable orbits are retrograde, with a relatively small region of stability for prograde orbits in a region centered around $i_o \sim 45^\circ$ and $e_o \leq 0.4$. There are no stable orbits of any kind for $e_o \geq 0.8$. The region between 60° and 120° and $e_o \leq 0.7$ is very complicated with mostly escape and collision orbits present, although there are a few stable orbits. In fact, at least some orbits in this particular sub-region are very sensitive to small changes in the initial conditions, suggesting the possibility of chaos. Note that the collision zone is no longer confined to near polar orbits; it is a wide and complicated "band" which curves to the right (towards the retrograde orbits) as the initial eccentricity increases. The range of Jacobi constants for the set of initial conditions shown in Fig. 5.3 is $3.017... \leq C \leq 3.039...$ so that *none* of the orbits in Fig. 5.3 are "stable" in the sense of Hill. Note that in spite of this a large proportion of retrograde orbits are actually stable, including of course Pasiphae and Sinope. We have already encountered this fact when studying planar orbits: in general, if the curves of zero-velocity are open, this represents actual instability for prograde orbits, while for retrograde orbits the same is not true.

Fig. 5.3. Stability at $a_o = 330 R_J$; a plus ('+') sign represents a stable orbit, a star ('*') sign represents a collision orbit and a dot ('.') an escape orbit. The two open circles represent the retrograde satellites Pasiphae and Sinope (as given by Danby). There are a total of 740 different initial conditions.

Next we show our results for $a_o = 400 R_j$. In this case the plot has lost the intricate structure we have seen in Figs. 5.2 and 5.3. The resolution of this plot is

Figure 5.4. Stability at $a_o = 400 R_j$ (0.537 Hill radii); a plus ('+') sign represents a stable orbit and a dot ('.') sign represents an escape orbit.

the same as in Fig. 5.1 (since apparently there is no complicated structure we decided to keep the resolution on the low side, which saves computing time).

Compared to previous plots, Fig. 5.4 is fairly simple. Most orbits (166 out of 190 orbits, or 87%) escape the vicinity of Jupiter and there are no collision orbits, so that 12.6% (24) of the test particles are on stable orbits. The escape orbits leave the vicinity

of Jupiter rather quickly, on average taking ~ 1.5 Jupiter periods to do so (although a few ones do take much longer). The most stable orbits are those which have a low eccentricity, and inclinations close to 180° . At $a_o = 500 R_J$, of 190 test particles only three (2 %) are stable while the rest escape; the stable objects are on initially circular orbits with inclinations of 160, 170 and 180 degrees. Further out, at $a_o = 600 R_J$, we found no stable orbits of any kind. We conclude then that at large distances, *the most stable orbits are those which have inclinations close to 180° and are most nearly circular orbits*. In fact, the most distant orbit we have found is the initially circular orbit at $a_o = 726 R_J$ and $i_o = 180^\circ$ (see section 4.4.2 and Fig. 4.6). This brings us back to the choices we have made about Ω_o , ω_o , M_o in our numerical experiments for this chapter so far.

Recall that we arbitrarily set $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$. To test the generality of these choices we have repeated our runs for $a_o = 200 R_J$ but setting different values for Ω_o , ω_o , M_o and recording any changes. Setting $\Omega_o = 90^\circ$ does not appreciably change our results from $\Omega_o = 0^\circ$; the only differences are in the timing of collision for a few orbits. Setting $\omega_o = 90^\circ$ makes even less of a difference: all orbits are practically the same as the case $\omega_o = 0^\circ$. Finally, setting $M_o = 0^\circ$ does not seem to make any appreciable difference either in our results for $a_o = 200 R_J$. However, we do know from the last chapter (see sections 4.4.1 and 4.4.2) that setting $M_o = 0^\circ$ does make a difference in stability *if* the initial semi-major axis is large enough (say $a_o \sim 500$ or $600 R_J$) and the orbits are nearly planar retrograde. We have already seen that if $M_o = 90^\circ$ and $a_o = 600 R_J$ there are no stable orbits; we expect that if $M_o = 0^\circ$, we should find a few stable orbits at inclinations close to

180° and eccentricities close to zero. Indeed that is the case: at $a_o = 600 R_J$ and $M_o = 0^\circ$, of 190 test particles (same resolution as in Figs. 5.1 and 5.4), only 7 (4%) are stable; the rest escape the vicinity of Jupiter. All these stable orbits had very low eccentricities (0.0 or 0.1) and inclinations close to 180°.

In Table 5.1 we summarize our findings: first column is the initial semi-major axis, given in terms of Jupiter radii; the second column shows N , the total number of

Table 5.1: Fraction of stable, collision and escape orbits at a given semi-major axis

$a_o (R_J)$	N	fraction of stable orbits	fraction of collision orbits	fraction of escape orbits
100	190	168/190 = 0.884	22/190 = 0.116	0/190 = 0.0
160	190	165/190 = 0.868	25/190 = 0.132	0/190 = 0.0
200	190	160/190 = 0.842	30/190 = 0.158	0/190 = 0.0
250	190	153/190 = 0.805	37/190 = 0.195	0/190 = 0.0
300	740	530/740 = 0.716	194/740 = 0.262	16/740 = 0.022
330	740	262/190 = 0.354	82/740 = 0.111	396/740 = 0.535
400	190	24/190 = 0.126	0/190 = 0.0	166/190 = 0.874
500	190	3/190 = 0.016	0/190 = 0.0	187/190 = 0.984
600	190	0/190 = 0.0	0/190 = 0.0	190/190 = 1.000

initial conditions integrated; the third, fourth and fifth columns indicate the number and fraction of stable, collision and escape orbits for the case $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$.

In Fig. 5.5 we plot the percentage of stable orbits, collision orbits and escape orbits as a function of initial semi-major axis as found on Table 5.1. The percentage of stable orbits is above 70% at $a_o \leq 300 R_j$, then it drops very fast until at $a_o \geq 600 R_j$ there are no stable orbits. The percentage of collision orbits is never above 26%, and at $a_o \geq 400 R_j$, there are no more collision orbits. Escape orbits are non-existent at small semi-major axes ($a_o \leq 200 R_j$) since the Jacobi constants are too high to allow escapes. At $a_o > 200 R_j$ the percentage of escape orbits rises quickly and at $a_o \geq 600 R_j$ all orbits are escape orbits. In general then, the question of why a given satellite orbit is unstable depends on the initial semi-major axis. If a_o is relatively "small" ($a_o \leq 200\text{-}300 R_j$) "instability" comes about because at certain inclinations ($i_o \sim 90^\circ$) solar perturbations tend to drive the satellite's orbital eccentricity close to unity so that a collision with the central planet is likely. If a_o is rather large ($a_o > 200\text{-}300 R_j$), instability comes about mainly because the Jacobi constant is low enough and the strong solar attraction tears the satellite away from the gravitational influence of Jupiter; retrograde orbits are more stable than prograde ones, but at large enough distances all satellites are lost due to the very strong solar perturbations.

Figure 5.5. Percentage of stable ('+'), collision ('*') and escape ('o') orbits as a function of initial semi-major axis. In all cases $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$. See text for a detailed discussion of the cases $a_o = 160, 300, 330, 400, 500$ and $600 R_j$.

5.3 Analytical Results

From the last chapter we recall that we could get useful analytical results from a simple perturbing function (eqn. (4.56)) if the distance from Jupiter is not too great (say r_2 less than about $200 R_j$ or 0.268 Hill radii). We saw that, besides the semi-major axis, two quantities are conserved in the averaged R3BP; these quantities are useful because they

give us insight into the long term behavior of test particles, since we have "filtered" out short period terms. We reproduce here the two quantities that are conserved:

$$(1 - e^2) \cos^2 i = \Theta \quad (5.1a)$$

$$e^2(5 \sin^2 i \sin^2 \omega - 2) = K \quad (5.1b)$$

Figure 5.6. Contour plot of Θ as a function of initial eccentricity and initial inclination. The contour levels are drawn at intervals of 0.1 from 0.1 to 0.9. The allowed range of Θ is $0 \leq \Theta \leq 1$. The maximum values of Θ occur at $e_o = 0$ and $i_o = 0^\circ$ or 180° while the minimum values occur at $i_o = 90^\circ$ and/or $e_o = 1$.

From inspection of eqns. (5.1a,b) it is easily seen that the ranges of these two quasi-conserved quantities (for a satellite with $e \leq 1$) are $0 \leq \Theta \leq 1$ and $-2 \leq K \leq 3$. In Figure 5.6 we show a contour plot of Θ as a function of initial inclination and initial eccentricity. Because Θ is a quantity which is (approximately) conserved, a satellite will move along the lines of constant Θ as its osculating e and i change. This is sometimes referred to as the "Kozai coupling mechanism." For example, an orbit with low Θ will alternate between periods of high inclination and low eccentricity and vice versa. This can be clearly be seen in Fig. 4.8, where during the time that the orbit is at its highest inclination ($i \sim 61^\circ$) its eccentricity is at its lowest ($e < 0.2$), while when the orbit achieves its lowest inclination ($i \sim 40^\circ$), the eccentricity is at its highest value ($e \sim 0.8$); for this particular orbit $\Theta = 0.25$. It should be pointed out that for any given initial condition (i_o, e_o) the satellite does not travel along the whole length one of the curves of constant Θ ; it only travels along a small arc of the curve and this is due to the constraint given by eqn. (5.1b).

5.3.1 A Criterion for Collisions

It can be seen from Fig. 5.6 (and also from eqn. (5.1a)) that Θ attains its maximum values if the initial eccentricity is low and the initial inclination far from 90° (i.e. $i_o \sim 0^\circ$ or $i_o \sim 180^\circ$); conversely, Θ attains its *minimum* values if the initial eccentricity is high and/or the initial inclination is close to 90° . But from Figures 5.1 and 5.2 we recall that orbits close to $i_o \sim 90^\circ$ and/or $e_o \sim 1$ undergo collisions with Jupiter. Therefore there seems to be a correlation between the value of Θ and the likelihood of the orbit to collide.

i.e., the lower the value of Θ , the more likely is the orbit to collide. In Fig. 5.7 below we again show the "stability" diagram from Fig. 5.1 (at $a_o = 160 R_j$) with superimposed contour lines for three values of Θ (0.01, 0.05 and 0.10). It can be seen in Fig. 5.7 that *all* of the orbits we integrated with $\Theta < 0.01$ undergo collisions; for $0.01 < \Theta < 0.05$, 9 out of 22 orbits undergo collisions and for $0.05 < \Theta < 0.10$ only 2 orbits out of 10 undergo collisions (One may also notice a curious lack of symmetry in this plot: the high inclination orbits with low values of Θ are more likely to collide than the corresponding

Fig. 5.7. Stability of orbits at $a_o = 160 R_j$ ($0.215 R_H$) as a function of i_o and e_o (a star '*' represents a collision orbit while a plus sign '+' represents a stable orbit). Also superimposed is a contour plot for three values of Θ : 0.01, 0.05 and 0.10. Note the correlation between a low value of Θ and the likelihood of collision. See also Fig. 5.6 and Table 5.2.

prograde orbits). We can also construct a table which shows the percentage of orbits within a given range of values of Θ which undergo a collision. For example, in Fig. 5.7 there are 14 orbits with $0 < \Theta < 0.01$ and all 14 undergo collisions. In Table 5.2 below we show the corresponding percentages for the ranges $0.00 < \Theta < 0.01$, $0.01 < \Theta < 0.05$ and $0.05 < \Theta < 0.10$ for three different values of the semi-major axis:

Table 5.2: Percentage of orbits within a given range of Θ at a given a_o that collide.

Θ_i	Percentage ($a_o = 100 R_j$)	Percentage ($a_o = 160 R_j$)	Percentage ($a_o = 200 R_j$)
$0.00 < \Theta < 0.01$	100% (14/14)	100 % (14/14)	100% (14/14)
$0.01 < \Theta < 0.05$	36% (8/22)	41% (9/22)	54% (12/22)
$0.05 < \Theta < 0.10$	0% (0/10)	20% (2/10)	30% (3/10)

The quantity Θ may be then used to roughly predict which orbits will suffer a collision. As a general rule of thumb one may take a "critical value" $\Theta_c \cong 0.1$; any satellite or test particle with a value of Θ close to or below this value is likely to end in a collision. To take a recent and very relevant case, comet Shoemaker-Levy 9 had, immediately previous to its fiery demise, an orbit characterized by a high jovicentric eccentricity ($e \approx 1$) and inclinations close to 90° [7], which would make its value of $\Theta \ll \Theta_c$. It should also be

pointed out that Θ is proportional to h_z^2 , where h_z is the z -component of the satellite's angular momentum, and orbits with low values of h_z are likely to collide.

5.3.2 Range of eccentricities

Equations (5.1a,b) can be used to solve for the maximum and minimum eccentricities attained by an orbit. From eqn. (5.1a) we solve for e^2 :

$$e^2 = 1 - \frac{\Theta}{\cos^2 i} \quad (5.2)$$

Proceeding in a similar manner from last chapter, we seek to eliminate $\cos^2 i$; we do that by solving for $\sin^2 i$ in eqn. (5.1b) and using $\sin^2 i + \cos^2 i = 1$ so that we obtain

$$\cos^2 i = 1 - \frac{K + 2e^2}{5e^2 \sin^2 \omega} \quad (5.3)$$

Putting eqn. (5.3) into eqn. (5.2) we finally obtain a polynomial of order four involving e :

$$(5\alpha - 2)e^4 + (5\Theta\alpha + 2 - 5\alpha - K)e^2 + K = 0 \quad (5.4)$$

where $\alpha \equiv \sin^2 \omega$. It is clear that $0 \leq \alpha \leq 1$; we consider the two extreme cases:

Case 1: $\alpha = 0$; eqn. (5.4) becomes

$$-2x^2 + (2 - K)x + K = 0 \quad (5.5)$$

where $x \equiv e^2$.

Case 2: $\alpha = 1$; in this case eqn. (5.4) becomes

$$3x^2 + (5\Theta - 3 - K)x + K = 0 \quad (5.6)$$

Recall that Θ and K are constants (given by eqns. (5.1)) fixed by the orbit's initial conditions. Eqns. (5.5-6) are two quadratic equations which are to be solved for the maximum and minimum eccentricities. But these two quadratic equations give us *four* roots, x_1, x_2, x_3, x_4 ; clearly only *two* of the roots are useful. Let us examine these equations in detail to determine when, if ever, these equations might be useful.

We start examining the simplest of the two, eqn. (5.5). It can be seen that the two roots of this equation are *always* real. Furthermore, it can be easily seen that one of the roots of this equation is 1, i.e., $x_1 = 1$; this root is of no interest and we ignore it. The second root is $x_2 = -K/2$; we recall that $-2 \leq K \leq 3$ so that in order to get a meaningful eccentricity (a real quantity or a non-zero answer) eqn. (5.5) is useful *only if* $-2 \leq K < 0$.

The situation in the second equation is a bit more tricky, although it can be shown that in most cases of interest (see Appendix C) we can extract a minimum and a maximum eccentricity from eqns. (5.5) and (5.6) as follows:

1. *Obtain four (real) roots from eqns. (5.5) and (5.6)*
2. *Discard the first root (since it is always $= 1$) and any negative root.*
3. *Take the square root of the two remaining real, positive roots to obtain e_{min} , e_{max} .*

Obviously the procedure will fail if: (i) the roots of eqn. (5.6) are complex or (ii) of the four possible (real) roots we do not obtain *exactly one* negative root (which we discard) for the case $K \neq 0$. This approach, which might seem haphazard and empirical, nevertheless does seem to give good results when applied to the four prograde distant satellites of Jupiter, as we will see later.

As a consistency check, consider an initially circular orbit again. In this case, $K = 0$ so that eqn. (5.5) is no longer useful, but eqn. (5.6) gives us

$$3x^2 + (5\Theta - 3)x = 0 \quad (5.7)$$

so that the two roots are

$$x = 0 \quad (5.8a)$$

$$x = 1 - \frac{5}{3}\Theta \quad (5.8b)$$

Recall that $x \equiv e^2$, so that taking the square roots we obtain $e_{min} = 0$, $e_{max} = \sqrt{1 - 5\Theta/3}$, i.e, eqn. (4.66) for the maximum eccentricity of an initially circular orbit.

5.3.3 Range of Inclination

Assuming we can obtain e_{min} and e_{max} from eqns (5.5-6) we can also obtain the extreme values (or equivalently, the range) for the inclination. From eqn. (5.1a) we obtain

$$\cos^2 i_{min} = \frac{\Theta}{1 - e_{max}^2} \quad (5.9a)$$

$$\cos^2 i_{max} = \frac{\Theta}{1 - e_{min}^2} \quad (5.9b)$$

since we already know that the orbit achieves its maximum inclination when its eccentricity is at its minimum and vice versa. The quantity Θ is fixed by the initial conditions and eqn. (5.1a).

5.4 Applications to Natural Satellites

As we have seen, the analytical results we have developed give fairly accurate results if $a_o \leq 200 R_J$ or 0.27 Hill radii. This suggests that analytical results may be applied to Jupiter's four distant, prograde satellites. We have already pointed out that

these prograde satellites are Hill stable. Their (approximate) semi-major axes, eccentricities and inclinations are (see Danby [10], p. 430) given below in Table 5.3:

Table 5.3: Approximate semi-major axes, eccentricities and inclinations for Jupiter's prograde satellites; data from Danby.

Satellite	$a (R_J)$	e	i (deg.)
VI Himalia	160.7	0.16	27.6°
VII Elara	164.4	0.21	24.8°
X Lysithea	164.2	0.11	29.02°
XIII Leda	155.4	0.15	26.1°

Given the satellites' initial conditions in the forms of orbital elements we should be able to predict the ranges of the eccentricities and inclinations. More accurate initial conditions for the outer satellites of Jupiter can be obtained from the Supplement to the Astronomical Almanac ([24], p. 353) in the form of positions x , (in AU's) and velocities $\dot{x}$, (in AU/day). These jovicentric positions and velocities are transformed to orbital elements, which in turn are converted to R3BP coordinates and integrated. A summary of our findings can be found in Tables 5.4, 5.5, 5.6 and 5.7 below.

Our findings for VII Elara can be found in Table 5.4; the computed values Θ and K are also given. The first column shows the quantities we are interested in (e_{min}, e_{max} ,

i_{min} , i_{max}), the second column shows the *predicted* values (from eqns. (5.5-6) and (5.9a,b)) while the third column shows the *measured* values from a 100 Jupiter-period integration; the fourth column shows the difference Δ between the predicted and measured values and finally the fourth column gives the percentage difference where we assume the measured value is the "accepted" value.

Table 5.4: Comparison between predicted and measured values of e_{min} , e_{max} , i_{min} , i_{max} for VII Elara. For this satellite $\Theta = 0.7225$ and $K = -6.0086 \times 10^{-2}$.

Quantity	Predicted	Measured	Δ	% Difference
e_{min}	0.173	0.178	-0.005	2.8%
e_{max}	0.262	0.318	-0.056	-17.6%
i_{min} (deg.)	28.27°	27.74°	0.53°	1.9%
i_{max} (deg.)	30.34°	31.30°	-0.96°	3.1%

As an example of obtaining e_{min} and e_{max} , the roots of eqns. (5.5-6) in this case are as follows: $x_1 = 1$, $x_2 \approx 0.06843$, $x_3 \approx -0.29268$ and $x_4 \approx 0.03004$; we discard x_1 and x_3 so that $e_{min} = +\sqrt{x_4} \approx 0.173$ and $e_{max} = +\sqrt{x_2} \approx 0.262$ (obviously we want the positive roots). The largest percentage difference (-17.6%) comes from the predicted and the actually measured maximum eccentricities. The percentage differences between the predicted and measured values of the inclination are surprisingly small, especially i_{min} , considering the relatively large percentage difference between the predicted and measured e_{max} .

In Table 5.5 we present our results for VI Himalia. The results are quite similar, only that in this case the largest percentage difference comes from the predicted and measured values for the *minimum* eccentricity. Again, the percentage differences for the minimum and maximum inclinations are surprisingly small.

Table 5.5: Comparison between predicted and measured values of e_{min} , e_{max} , i_{min} , i_{max} for VI Himalia. For this satellite $\Theta = 0.7303$ and $K = -2.5171 \times 10^{-2}$.

Quantity	Predicted	Measured	Δ	% Difference
e_{min}	0.112	0.093	0.019	20.4%
e_{max}	0.180	0.184	-0.004	-2.2%
i_{min} (deg.)	29.68°	29.59°	0.09°	0.3%
i_{max} (deg.)	30.68°	31.66°	-0.98°	-3.1%

In Table 5.6 below we present our results for XIII Leda:

Table 5.6: Comparison between predicted and measured values of e_{min} , e_{max} , i_{min} , i_{max} for XIII Leda. For this satellite $\Theta = 0.7666$ and $K = -4.0121 \times 10^{-2}$.

Quantity	Predicted	Measured	Δ	% Difference
e_{min}	0.142	0.134	0.008	6%
e_{max}	0.201	0.233	-0.032	-13.7%
i_{min} (deg.)	26.65°	26.15°	0.5°	1.9%
i_{max} (deg.)	27.81°	28.45°	-0.64°	-2.2%

Finally, in Table 5.7 we present our findings for X Lysithea:

Table 5.7: Comparison between predicted and measured values of e_{min} , e_{max} , i_{min} , i_{max} for X Lysithea. For this satellite $\Theta = 0.78065$ and $K = -1.4784 \times 10^{-2}$.

Quantity	Predicted	Measured	Δ	% Difference
e_{min}	0.086	0.070	0.016	22.9%
e_{max}	0.124	0.133	-0.009	-6.8%
i_{min} (deg.)	27.08°	26.19°	0.89°	3.4%
i_{max} (deg.)	27.52°	27.73°	-0.21°	-0.8%

We note two things about the prograde satellites of Jupiter: (i) $0.722... < \Theta < 0.780...$ so that for all of these satellites, $\Theta \gg \Theta_c$; (ii) $K < 0$ in all four cases so that we can use eqn. (5.5) to compute the maximum and/or minimum eccentricities of these satellites.

In order to put things in perspective in Figure 5.8 we plot the orbit of XIII Leda up to ~ 1.85 Jupiter periods (21.9 years), so that since the approximate period of Leda's orbit about Jupiter is 239 days (see Danby [10], p. 430) this represents approximately 33 revolutions of Leda. The orbit is plotted in a *non-rotating* coordinate system with Jupiter at the origin, i.e., we are plotting the quantities x_{rel} , y_{rel} , and z_{rel} defined in eqn. (4.9); the units of distance are Jupiter radii. In the terminology of Chapter 3, the orbit is *quasi-periodic*. It can be seen that this orbit suffers large perturbations due to the Sun, as evidenced by the fact that the orbit never remains a fixed Keplerian orbit. The time history of the semi-major axis, eccentricity and inclination of Leda are shown Fig. 5.11.

In addition we show in Figs. 5.9, 5.10 and 5.12 the time histories of the semi-major axis, eccentricity and inclination for the rest of the prograde satellites of Jupiter. We integrated these orbits with the integrator discussed in Chapter 4, and the error tolerance (for the Bulirsch-Stoer integrator) was set to 10^{-13} . Compare Figs. 5.9 to 5.12 with the values found in tables 5.4 to 5.7.

From our previous discussions we conclude that the analytical methods we have discussed in this and the previous chapter give reasonably accurate results when applied to the distant, direct satellites of Jupiter. In context of the model we are using (R3BP), it gives answers which show at worst a $\pm 23\%$ difference (see Table 5.7) between the predicted and the values measured from a numeric integration of the equations of motion (R3BP) for the maximum or minimum eccentricity although the actual percentage differences are usually quite smaller than this. Alternatively, we can say that the largest absolute difference between the predicted and measured eccentricities may be no larger than $|\Delta_{\max}(e)| \approx 0.06$ (see Table 5.4). The percentage differences for the inclinations are (surprisingly) much better than this, and they are at worst $\pm 3.4\%$, with an absolute difference between predicted and measured inclinations of $|\Delta_{\max}(i)| \approx 1^\circ$ (see Table 5.5). The average value of the absolute difference for the eccentricities is $\langle \Delta(e) \rangle \approx 0.02$ and for the inclinations is $\langle \Delta(i) \rangle \approx 0.6^\circ$ (to compute these quantities we added the absolute value of all the $\Delta(e)$'s -or $\Delta(i)$'s- and divided by 8). The purpose of the equations

Fig. 5.8. The orbit of XIII Leda in a non-rotating coordinate system with Jupiter (open circle) at the origin. The units of distance are Jupiter radii.

Fig. 5.9. VII Elara

Fig. 5.10. VI Himalia

Fig. 5.11. XIII Leda

Fig. 5.12. X Lysithea

developed in this chapter (eqns. (5.5) and (5.6) and eqns. (5.9a,b)) is not precise predictions, but rather to enable one to get a good idea of the long term behavior of these distant satellites. It should be noted that we also expect (although we have not tested this) these analytical methods to work on some other irregular satellites in the solar system, such as Neptune's Nereid ($a_o = 5511 \times 10^3 \text{ km} \sim 0.05 R_H$) and Saturn's Phoebe ($a_o = 12952 \times 10^3 \text{ km} \sim 0.2 R_H$).

Unfortunately, these analytical methods fail when applied to the more distant jovian retrograde satellites. The solar perturbations are so strong that their orbits are very deformed: in Figure 5.13 we illustrate the orbit of Jupiter's satellite Pasiphae in a non-rotating, inertial frame centered on Jupiter; the length of the integration is ~ 3.2 Jupiter periods. For such orbits the percentage difference between the predicted and measured e_{min} and e_{max} can be quite large. Developing a more accurate perturbing function is not very difficult; all we would have to do is keep more terms in the expansion of eqn. (4.46). The difficult part is obtaining quasi-conserved quantities such as those in eqns. (5.1a,b) from where we could develop useful analytical expressions. As an illustration of the failure of these equations for the retrograde satellites of Jupiter, we compare in Table 5.7 the predicted vs. the measured values of $e_{min}, e_{max}, i_{min}, i_{max}$ for VIII Pasiphae ($a_o \sim 330 R_J$):

Table 5.8: Comparison between predicted and measured values of e_{min} , e_{max} , i_{min} , i_{max} for VIII Pasiphae. For this satellite $\Theta = 0.6029$ and $K = -0.2111$. * : Retrograde degrees.

Quantity	Predicted	Measured	Δ	% Difference
e_{min}	0.325	0.219	0.106	48%
e_{max}	0.480	0.721	-0.241	-33%
i_{min} (deg.)*	27.74° (R)	24.39° (R)	3.35°	14%
i_{max} (deg.)*	34.81° (R)	42.26° (R)	-7.45°	-18%

For completeness we also plot the time-histories of the semi-major axis, eccentricity and inclination of the retrograde satellites of Jupiter: VIII Pasiphae (Fig. 5.14), IX Sinope (Fig. 5.15), XI Carme (Fig. 5.16) and XII Ananke (Fig. 5.17). We can see that the orbital elements change appreciably even over short time scales. This raises the problem of how accurate is it to plot the position of the satellites (prograde and retrograde) in the $e-i$ plane given at a given time (a point) as seen in Figs. 5.1 to 5.3 and state that they are in fact in a zone of stability. The changing of the osculating eccentricities and inclinations imply that a given satellite actually occupies a certain area in the $e-i$ plane, rather than a single point as we have previously plotted. Taking into account these changes, however, it is still possible to state the stability of the satellites, i.e., the areas that these satellites cover as their e and i change does not overlap any unstable zone (be it a collision zone or an escape zone). For example, in Fig. 5.18(a) we show the area covered by Elara in the $e-i$ plane; Elara covers the greatest area of any prograde satellite and even so the area covered is quite small and never overlaps the collision zone seen in Fig. 5.1. Pasiphae is the

Fig. 5.13. The orbit of VIII Pasiphae in a non-rotating coordinate system with Jupiter (open circle) at the origin. The units of distance are Jupiter radii.

Fig. 5.14. VIII Pasiphae

Fig. 5.15. IX Sinope

Fig. 5.16. X1 Carne

Fig. 5.17. XII Ananke

retrograde satellite with the most extreme variations in eccentricity and inclination. In Fig. 5.18(b), we see the area covered by Pasiphae (the largest area covered by any prograde *or* retrograde satellite) never overlaps the unstable zone seen in Fig. 5.3. Pasiphae does come close to a "collision zone" at the points it achieves its highest eccentricities ($e \sim 0.7$); however, it is safe from collisions because at such points its inclination tends towards $i \sim 155^\circ$ (to the right in Fig. 5.18(b)), away from the collision zone.

Fig. 5.18. Extent of e - i variations for two satellites of Jupiter. (a) Elara (prograde satellite); (b) Pasiphae (retrograde satellite). We find that Pasiphae has the largest e - i variations of any other Jovian satellite.

It should be pointed out that, at least for Pasiphae and Sinope, numerical experiments suggest that secular resonances may play a very important role in the long term behavior of these satellites and may even be a source of "mild chaos" [25, 26], so that therefore the simple model of the R3BP may not be adequate here. Precisely, the next thing we need to do is to assess how accurately does the R3BP represent the actual behavior of very distant satellites (real or hypothetical); in our simple model the only perturbation the satellite experiences is the solar perturbation. In Chapter 2 we made order of magnitude estimates of the relative strength of other perturbations (oblateness, the effects of Jupiter's eccentricity, etc.) with respect to the solar one and concluded that these other perturbations are much smaller than the solar perturbations. In the next chapter we will test these assumptions with a very complete solar system integrator.

Chapter 6

Generality of the Restricted Three-Body Problem

6.1 Introduction

In the course of the previous chapters (especially chapters 4 and 5), we have gained an understanding of the qualitative effects of solar gravitational perturbations on very distant and small satellites of Jupiter and have found general regions of stability and instability using the R3BP as a model. Furthermore, we have found that if the satellite distance is smaller than about $\sim 200 R_J$ or $0.27 R_H$ we may use analytical methods to study their behavior; otherwise, we must resort to numerical integrations (or develop a more accurate perturbing function). This fulfills the first two of the three objectives we set at the beginning of this study (see section 1.2). The third objective of this study is to try to understand the generality of the results obtained thus far. As previously mentioned, the R3BP is a very simple model. Its simplicity is a double sword, however, in that it may make analytical results relatively easy to derive and numerical solutions to the equations of motion inexpensive to (numerically) integrate; however there may be important details we are ignoring without which an accurate picture of the stability of distant satellites is impossible. Previously we have made order of magnitude estimates of other perturbing effects (at the end of Chapter 2); here we will actually test those assumptions. In this

chapter we compare the numerical and analytical results obtained with the R3BP in Chapters 4 and 5 to numerical results obtained with a fairly complete N-body integrator.

6.2 SWIFT

The integrator we will use to test the validity of the R3BP is called SWIFT and was recently developed by M. Duncan and H. Levison [27]. The SWIFT package was developed to numerically integrate a number of mutually gravitationally interacting particles along with a number of test particles; it even allows for the oblateness parameters of the central planet to be used. The code is capable of using several integration techniques, including a Mixed Variable Symplectic (MVS) method; a Regularized Mixed Variable Symplectic Method (RMVS2), which allows for close approaches between the test particles and massive bodies (planets), and finally a Bulirsch-Stoer (BU) routine. The advantage of the Symplectic integration methods (MVS and RMVS2) is the large time step which can be used resulting in much faster integrations than otherwise; everything being equal, the RMVS2 method can be about ten times faster than the BU method. To check the validity of our R3BP simulations, we will use the RMVS2 method (since it is regularized and allows close approaches) and occasionally the BU method as a check on the symplectic method itself.

6.3 General Description of the Integrations

There are two important findings that we wish to test: (1) The stability limits of planar, initially circular orbits (see section 4.4) and (2) the "topology" of e - i space (see chapter 5). Does the presence of an exterior massive body (i.e., Saturn for example) *and* the non-zero eccentricity of Jupiter's orbit affect the results previously obtained? If so, how important are these effects? The presence of Saturn appreciably affects the motion of Jupiter and vice versa; in fact, the two planets are near a 2:5 mean motion commensurability with a 900 year period known as the "Great Inequality" [18]. The corresponding orbital elements of the two planets are therefore not fixed. In Fig. 6.1 we plot the eccentricity of Jupiter as a function of time (up to 100 Jupiter periods) as obtained with SWIFT using the RMVS2 integration technique; the only massive bodies present in the integration were Jupiter, the Sun and Saturn. Initial conditions for the massive bodies were provided by Paul Chodas of the Jet Propulsion Laboratory at epoch JD 2448896.5 (October 1.0, 1992). Since the reference frame for these data is the Earth mean equator and equinox of J2000, a rotation into the frame of the Jovian mean equator of date was performed by Luke Dones. Figure 6.1 indicates the behavior of Jupiter's eccentricity in *all* our numerical integrations for this chapter, since the initial conditions of the massive bodies are exactly the same for all subsequent integrations. The same dynamical system was integrated using SWIFT with a Bulirsch-Stoer integration technique and the two Jupiter eccentricities differed by at most 0.08% after 100 Jupiter

periods. For the length of these integrations Jupiter's eccentricity spans the range $0.0469 < e_j < 0.0499$, with an average value around $\langle e_j \rangle \approx 0.048$. Unless specified

Figure 6.1. The eccentricity of Jupiter as a function of time, as obtained with the SWIFT N-body code and the RMVS2 integration technique.

otherwise, the only massive bodies present in the following integrations will be Saturn, Jupiter and the Sun, since we are assuming the gravitational perturbations of the other giant planets (Uranus and Neptune) to be much smaller than those of Saturn. We determined the (fixed) time step using the following rule of thumb: it is to be

approximately 1/50 of the orbital period of the closest body (massive or otherwise) to Jupiter, i.e., 1/50 of the orbital period of Jupiter's closest satellite.

As before, if the test particle survives 100 Jupiter periods orbiting Jupiter, the particle is considered stable. If the distance from Jupiter exceeds 2 AU's we discard that test particle; likewise if the test particle approaches too closely to the planet ($r_2 < 100 R_J$ and $q < 1 R_J$) we consider this a collision orbit and also discard the test particle.

6.4 Stability of Initially Circular, Planar Orbits with SWIFT

We will start with the planar, initially circular orbits. We produced a set of initial conditions such that the test particles were in the xy -plane ($i_o = 0^\circ$ if prograde or $i_o = 180^\circ$ if retrograde), with $e_o = 0$, $\Omega_o = \omega_o = 0^\circ$, a given value of the initial mean anomaly M_o and with a range of initial semi-major axes $a_{o,i}$, $i = 1, 2, 3, \dots$, that included the "critical distances" r_c obtained previously. As expected, the critical distances are dependent on the initial mean anomaly. In all the unstable cases the particles escaped the vicinity of Jupiter. We present our findings for a given value of the initial mean anomaly followed by the constraints found for the critical distances. For prograde orbits ($i_o = 0^\circ$) we find: (i) $M_o = 0^\circ$; $320 R_J \leq r_c \leq 330 R_J$; (ii) $M_o = 90^\circ$; $290 R_J \leq r_c \leq 300 R_J$; (iii) $M_o = 180^\circ$; $330 R_J \leq r_c \leq 340 R_J$; (iv) $M_o = 270^\circ$; $300 R_J \leq r_c \leq 310 R_J$. Note that the more stable cases are those such that $M_o = 0^\circ$ or 180° , the latter being the most stable case. The

critical distances for $M_o = 90^\circ$ and 270° are no longer identical, presumably because of the lack of symmetry that the presence of another perturbing body (Saturn) implies. If we take the smallest critical distance $r_{c,min}$ to be the average of the maximum and minimum constraint values when $M_o = 90^\circ$ we obtain $r_{c,min} \approx 295 R_j$; likewise, the maximum critical distance is obtained when $M_o = 180^\circ$: $r_{c,max} \approx 335 R_j$. That is, for prograde, planar and initially circular orbits and taking into account the perturbations due to Saturn and the non-zero eccentricity of Jupiter's orbit we can write:

- For $r_2 < 295 R_j$ ($0.40 R_H$), all orbits are stable
- For $r_2 > 335 R_j$ ($0.45 R_H$), all orbits are unstable

where r_2 is the distance of the test particle to Jupiter. These results are highly encouraging since they are quite similar to those obtained with our simpler model. If we assume these values to represent the "true" critical distances, then they are approximately only a 6-7% smaller than our old results.

Next we investigate the retrograde orbits ($i_o = 180^\circ$). Our findings are as follows:

(i) $M_o = 0^\circ$; $680 R_j \leq r_c \leq 690 R_j$; (ii) $M_o = 90^\circ$; $540 R_j \leq r_c \leq 550 R_j$; (iii) $M_o = 180^\circ$; $700 R_j \leq r_c \leq 710 R_j$; (iv) $M_o = 270^\circ$; $530 R_j \leq r_c \leq 540 R_j$. Again these results are encouraging. Taking the minimum critical distance to be the average of the constraint values for the case $M_o = 270^\circ$, we find $r_{c,min} \approx 535 R_j$; similarly, we find that $r_{c,max} \approx 705$

R_J (for the case $M_o = 180^\circ$). We can summarize our findings for initially circular, planar retrograde orbits (taking into account the perturbations due to Saturn and the non-zero eccentricity of Jupiter's orbit) thus:

- For $r_2 < 535 R_J$ ($0.72 R_H$) all orbits are stable
- For $r_2 > 705 R_J$ ($0.95 R_H$) all orbits are unstable

Surprisingly, the result for the minimum critical distance is virtually identical to what we had found previously ($r_{c,min} \approx 538 R_J$), while the maximum critical distance is smaller by about $\sim 3\%$.

The presence of Saturn and the non-zero eccentricity of Jupiter's orbit reduce the critical distances we had obtained previously somewhat. This is to be expected: in the more complex dynamical system comprised of the Sun-Jupiter-Saturn-test particle, it is reasonable that the particles which were located right at the stability boundaries in the R3BP (see section 4.4) should be lost from Jupiter due to the extra gravitational perturbations of another distant body and the changing Sun-Jupiter distance, reducing the critical distances. However, as we pointed out, the effect is small. Therefore we conclude that, for planar orbits at least, the search for stability regions of distant satellites of Jupiter is quite accurately modeled using the R3BP.

6.5 General Stability Results with SWIFT

In this section we explore the "topology" of $e-i$ space, as we have previously done in section 5.2. We use SWIFT to integrate the orbits of Jupiter, Saturn and a set of test particles (which orbit Jupiter) in the same way that we did in the last chapter: given an initial jovicentric semi-major axis a_o we produce a set of initial conditions with the given a_o and a variety of eccentricities and inclinations; the rest of the orbital elements are set as before: $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$. Unless otherwise noted, the integration technique is RMVS2.

In Figure 6.2 we plot the results for $a_o = 160 R_J$. A star ('*') represents a collision orbit while a plus ('+') sign represents a stable orbit. We integrated a total of 190 different initial conditions ($i_o = 0^\circ$ to 180° in steps of 10° and 0 to 0.9 in steps of 0.1 in eccentricity). Here 163 out of 190 (86%) test particles are actually stable after 100 Jupiter orbits and the rest, 27 (14%) suffer collisions. Our results using the R3BP were that 165 out of 190 (87%) particles were stable and 25 (13%) underwent collisions. Comparing Figs. 5.1 and 6.2 it can be seen that the only differences in the two plots are two orbits with initial conditions $i_o = 100^\circ$ and $e_o = 0.0$ and 0.1 : these orbits did not suffer a collision in Fig. 5.1 but did collide in Fig. 6.2. The results are, considering the different integration techniques and the different models used, surprisingly similar. At this semi-major axis, the R3BP is an excellent approximation to reality.

Fig. 6.2. Stability at $a_o = 160 R_J$ obtained with SWIFT using the RMVS2 integration technique; a plus ('+') sign represents a stable orbit while a star (*) sign represents a collision orbit. The open circles represent the positions of Jupiter's four prograde satellites as given by Danby. This plot should be directly compared to Fig. 5.1. There are a total of 190 different initial conditions.

Next we plot the results for $a_o = 300 R_J$ (Fig. 6.3). Direct comparison with Fig. 5.2 shows that the total number and percentage of escape orbits (each escape orbit is denoted by a point ('.')) has increased with respect to the R3BP integration. Basically all orbits with initial conditions which led to collisions in Fig. 5.2 also underwent collisions in Fig. 6.3; in addition, many other orbits also underwent collisions. Out of 740 test particles 432 (58%) are stable, 171 (23%) underwent collisions and 137 (19%) achieved escape. The number of stable orbits decreased compared to the corresponding R3BP simulation, and most of this decrease is due to the increase in number of escape orbits. The two open circles represent the positions of Carme and Ananke, and as before, they are located in a stable region.

In Figure 6.4 we present the stability results at $a_o = 330 R_J$. A comparison with Fig. 5.3 shows that the curved "band" of collision orbits has mostly disappeared, to be replaced by a few scattered collision orbits. Notice that there still exists a region of stability for prograde orbits, although it is now much smaller than in the R3BP case (see Fig. 5.3); this region extends roughly from $e_o = 0$ to $e_o \sim 0.2$ and $i_o = 45^\circ$ to $i_o = 60^\circ$, with a few scattered stable orbits at higher eccentricities. We find that of 740 test particles 174 (24%) are stable, with most stable orbits being retrograde orbits, 62 (8%) suffered collisions and 504 (68%) escaped the vicinity of Jupiter. Sinope and Pasiphae (open circles) are in a stable region. Note that Pasiphae is now closer to an unstable region so that, according to Fig. 5.18(b) of last chapter, it may cross over into the unstable region.

Fig. 6.3. Stability at $a_o = 300 R_J$ obtained with SWIFT using the RMVS2 integration technique; a plus ('+') sign represents a stable orbit, a star ('*') sign represents a collision and a dot ('.') an escape orbit. The two open circles represent Carne and Ananke (as given by Danby). This plot should be directly compared with Fig. 5.2. There are a total of 740 different initial conditions.

Fig. 6.4. Stability at $a_o = 330 R_J$ obtained with SWIFT using the RMVS2 integration technique; a plus ('+') sign represents a stable orbit, a star ('*') sign represents a collision orbit and a dot ('.') an escape orbit. The two open circles represent Pasiphae and Sinope (as given by Danby). This plot should be directly compared with Fig. 5.3. There are a total of 740 different initial conditions.

However, it is not proper to compare Fig. 5.18(b) and Fig. 6.4 for the purposes of establishing Pasiphae's stability since they are derived from different dynamical systems. According to Whipple and Shelus [25], in the more complete dynamical system comprised of the Sun, the four giant planets, Pluto and Pasiphae, the presence of a secular resonance (in the form of a near equality between the rates of advance of Jupiter's longitude of perihelion and Pasiphae's longitude of perijove^{*}) limits the solar perturbations to between 45 and 85% of what they would be in the absence of this resonance. This in turn limits the maximum and minimum osculating eccentricities and inclinations of Pasiphae. Whipple and Shelus obtained $0.177 < e < 0.675$ and $138.2^\circ < i < 156.9^\circ$ for this satellite; compare these values with those on Table 5.8 of last chapter.

The final plot we show regarding the stability regions in " e - i space" is that for $a_o = 400 R_J$ (see Fig. 6.5). This plot is very similar to the analogous plot for the R3BP (Fig. 5.4), with the difference that there are fewer stable orbits. Only 16 out of 190 test particles are stable (8%) while the rest (174 or 92%) all achieved escape. As in Fig. 5.4, all of the stable orbits have low eccentricities (in this case $e_o < 0.3$) and inclinations approaching 180° .

^{*} The longitude of periapse is defined as $\varpi \equiv \Omega + \omega$.

Fig. 6.5. Stability at $a_o = 400 R_j$ for the Sun-Jupiter-Saturn-satellite system using SWIFT with the RMVS2 integration technique. A plus sign ('+') indicates a stable orbit while a dot ('.') indicates an escape orbit. This plot should be compared with Fig. 5.4.

We can summarize our results in tabular form as we did in last chapter. The first column of Table 6.1 gives the initial semi-major axis of the satellite in R_j ; the second column gives N , the total number of test particles for a given run; the third column gives the fraction of stable test particles while the fourth and fifth columns gives the fraction of collision and escape orbit respectively. Note that the corresponding N for a given initial semi-major axis is not necessarily the same as the N in the corresponding R3BP run (see Table 5.1); however, we believe this should not appreciably affect our findings.

Table 6.1: Fraction of stable, collision and escape orbits at a given semi-major axis

$a_o (R_J)$	N	fraction of stable orbits	fraction of collision orbits	fraction of escape orbits
100	740	632/740 = 0.854	108/740 = 0.146	0/740 = 0.0
160	190	163/190 = 0.858	27/190 = 0.142	0/190 = 0.0
200	740	598/740 = 0.808	142/740 = 0.192	0/740 = 0.0
250	190	152/190 = 0.800	38/190 = 0.200	0/190 = 0.0
300	740	432/740 = 0.584	171/740 = 0.231	137/740 = 0.185
330	740	174/740 = 0.235	62/740 = 0.084	504/740 = 0.681
400	190	16/190 = 0.084	0/190 = 0.0	174/190 = 0.912
500	740	11/740 = 0.015	0/740 = 0.0	729/740 = 0.985
600	740	0/740 = 0.0	0/740 = 0.0	740/740 = 1.000

The best way to represent these data is in the form of a plot. In Fig. 6.6 we plot the percentage of stable orbits for the RMVS2 runs compared to the R3BP runs, as well as the corresponding percentages of collision and escape orbits. The solid curves represent the R3BP runs (Table 5.1) while the dashed curves represent the RMVS2 results (Table 6.1). Perhaps these plots more than anything else are indicative of the applicability of the R3BP to the study of distant satellites of Jovian planets. In Fig. 6.1(a), it can be seen that beyond $\sim 250 R_J$, the percentage of stable orbits diminishes slightly faster in

Figure 6.6. A comparison of stability results using the R3BP (solid line) and the SWIFT-RMVS2 (dashed line) integrator (which includes the Sun, Jupiter, Saturn and the test particles). (a) Percentage of stable orbits; (b) Percentage of collision orbits; (c) Percentage of escape orbits.

the more realistic integrations done with SWIFT (dashed line) than the integrations done with the R3BP. A glance at Figs. 6.6(b) and 6.6(c) indicates that most of these additional unstable orbits are escape orbits, since the percentage of collision orbits remains about the same for both dynamical models. This indicates that the addition of an exterior planetary perturber (Saturn) and the non-zero eccentricity of Jupiter have little if any effect on the

orbits which are driven towards collisions with the central planet (i.e., Jupiter); therefore we conclude that solar gravitational perturbations are the sole cause of collision orbits.

The fact that the percentage of escape orbits is, in general, higher in the SWIFT runs than in the R3BP runs is consistent because in the former the test particles not only experience additional perturbations by Saturn: Jupiter's non-zero eccentricity means that at times the planet is closer to the Sun, meaning larger solar perturbations. The particles which are on the verge of stability in the R3BP runs (where we assume $e_J = 0.0$) escape in the more realistic dynamical model because *all* particles periodically experience stronger than normal solar gravitational perturbations when Jupiter is at perihelion: satellites which are well inside stability zones are hardly affected by these periodic variations in the solar perturbation strength, but particles on the edges of the stability zones are removed, i.e., they achieve escape. We can quantify this statement by computing an estimate of how much stronger is the solar gravitational force on a satellite of Jupiter when Jupiter is in on a circular orbit and when Jupiter is on an elliptical orbit and at perihelion. Let us assume that Jupiter's eccentricity has its average value $e_J = 0.048$. Let the *solar tidal force* on the particle be F_1 if Jupiter is on a circular orbit and F_2 if Jupiter on an eccentric orbit and at perihelion; then it can be shown that the ratio of these two forces is

$$\frac{F_2}{F_1} \approx (1 + 3e_J) = 1.14 \quad (6.1)$$

Thus, once every Jupiter period the test particles experience approximately this extra amount of gravitational pull from the Sun. Given these periodic greater than normal solar gravitational pulls combined with the extra gravitational perturbations of Saturn and it is easy to see why the percentage of escape orbits is greater in the SWIFT runs than in the R3BP runs.

We have also integrated the cases $a_o = 160 R_j$, $300 R_j$ and $400 R_j$ and $e_o = 0$ to 0.9 in steps of 0.1 and $i_o = 0^\circ$ to 180° in steps of 10° (190 different initial conditions) with $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$, using SWIFT with the RMVS2 method and the Bulirsch-Stoer (BU) method for 100 Jupiter periods (the case $a_o = 400 R_j$ has been done already with the RMVS2 method; see Fig. 6.5). The RMVS2 and BU methods give identical results at $a_o = 160 R_j$ while the results at $a_o = 400 R_j$ are almost identical: the only difference is that a test particle at $i_o = 140^\circ$ and $e_o = 0.2$ is stable in the BU run and escapes in the RMVS2 run. Finally, at $a_o = 300 R_j$, there is a total of 18 test particles with different fates: however, the overall topological pattern in the e - i plane for the two runs is very similar, only the fates of certain individual particles are different: see Figs. 6.7(a,b). This may also be indirect evidence of the chaotic nature of this region. Recall that chaotic orbits are very sensitive to small changes in the initial conditions; integrating a given initial condition using a different integrator is equivalent to changing the initial conditions after

Fig 6.7. e - i stability results for $a_o = 300 R_J$ and $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$ obtained using SWIFT with two different integration techniques: (a) SWIFT with RMVS2; (b) BU (Bulirsch-Stoer). There are a total of 18 test particles with different outcomes in the two runs.

the second time-step due to the different integration techniques, i.e., different truncation methods and finite computer arithmetic. Hence, a given initial condition which is chaotic may show different fates using different integrators. A definitive answer can only come from the numerical computation of Lyapunov characteristic numbers for each test particle suspected of chaotic behavior.

Finally, we have also performed integrations containing the galilean satellites with SWIFT using the RMVS2 method. The galilean satellites have considerable masses themselves (in fact, Ganymede is larger - although not more massive - than the planet Mercury) so that it is not realistic to treat them as zero-mass test particles. This implies using a very small time step, on the order of $\sim 1/50$ the orbital period of the closest body in orbit around Jupiter, in this case Io. Io's orbital period is approximately 1.8 days, or 4.2×10^{-4} Jupiter periods, so that the time step used must be $\Delta t \sim 8.3 \times 10^{-6} P_J$ (Jupiter periods). A total of 25 test particles at $a_o = 200 R_J$, and $\Omega_o = 0^\circ$, $\omega_o = 0^\circ$ and $M_o = 90^\circ$, with a range of inclinations 80° to 100° in steps of 5° and a range of eccentricities $e_o = 0.5$ to 0.9 in steps of 0.1 were integrated with SWIFT for 100 Jupiter periods using (i) the RMVS2 method with the galilean satellites ($\Delta t \sim 8.3 \times 10^{-6} P_J$) and (ii) the RMVS2 method without the galilean satellites ($\Delta t \sim 0.06 P_J$). Case (i) took approximately one week to run, while case (ii) took just a few minutes. Of the 25 test particles in both runs, only one had a different fate or outcome: it collided in case (i) and remained stable in case (ii); the rest had identical outcomes. From this experience it can be concluded that the gravitational effects of the galilean satellites on distant test particles are almost negligible. This is good news, for otherwise we would have to integrate along with the test particles an N-body problem with the galilean satellites in place, and as we have seen it is extremely time consuming to do so. Kary and Dones [28] investigated the outcome of a set of "synthetic comets" in the vicinity of Jupiter and also neglected the gravitational influence of the galilean satellites; their argument for doing so also applies in this case,

namely that "during a passage deep within Jupiter's potential well, a [test particle] has a small chance of undergoing a close encounter with one of the galilean satellites". This also gives validity to the R3BP for modeling the behavior of distant test satellites, in which the only massive bodies perturbing these satellites are Jupiter and the Sun.

6.6 Discussion

The purpose of this chapter was to try to compare stability results using the R3BP and an N-body code (SWIFT) to see if the former is in fact a realistic dynamical system to model the behavior of distant satellites of Jupiter. In most SWIFT runs we only included the Sun, Jupiter, Saturn and the test particles. The stability results are very similar for both dynamical models, suggesting that the R3BP is an adequate model to investigate the stability of distant satellites. In the case of the planar, initially circular orbits, the critical distances are at most 6-7% smaller in the SWIFT runs than in the R3BP runs. In the general case (arbitrary eccentricities and inclinations), a number of test particles were integrated; we have seen that the percentage of collision orbits is approximately the same for both models. The percentage of escape orbits, however, is slightly higher in the SWIFT runs; as we have seen, this is to be expected due to the extra gravitational perturbations by another massive body (Saturn) and Jupiter's non-zero eccentricity, which makes the Jupiter-Sun distance vary periodically. Finally, the gravitational attraction of the galilean satellites was also taken into account in a single SWIFT run and their effects

on the test particles were found to be negligible. All of these effects validate the order of magnitude estimates which were made at the end of chapter 2 regarding the applicability of the R3BP to the problem at hand; therefore, we conclude that the R3BP is an accurate dynamical system to qualitatively model the behavior of distant satellites of the planet Jupiter.

Chapter 7

Summary and Conclusions

The question of what regions of phase space are stable in the vicinity of Jupiter has been investigated in this work. In particular, we were interested in the relative stability of small, distant (real and/or hypothetical) satellites of the planet Jupiter. These satellites are sometimes termed irregular satellites. Since for these bodies the main cause of deviations from true Keplerian motion is the solar gravitational attraction, we have chosen to model the behavior of these satellites using the (circular) Restricted Three Body Problem as a model. This has the advantage that the equations of motion are very simple and inexpensive to integrate, but the model does ignore other effects. Therefore, the question of the applicability of the R3BP to the problem we wish to study is also of importance.

Despite its simplicity, the R3BP is a non-integrable problem, since there is only one constant of motion, the Jacobi constant C . The method of Surface of Sections has been applied to the planar case; the four-dimensional phase space has been reduced without any loss of information to a two dimensional phase "sub-space." For each of eight different fixed values of the Jacobi constant (in the range $3.08 < C < 3.02$) we have integrated 112 different initial conditions and recorded their intersections with the plane

$y = 0$ and plotted the results in the $\dot{x}x$ -plane. Regions of stability and instability have been found. In particular we have found periodic orbits, quasi-periodic orbits and chaotic orbits, and investigated the variations in time of these orbits' eccentricities and semi-major axes. We have found that the onset of chaotic motion occurs very close to the critical value C_2 , that is, the value of the Jacobi constant where the curves of zero-velocity meet at the L_2 Lagrangian point. Two tests were used to verify chaotic motion: the Surfaces of Section and the computation (via a set of linearized equations) of Lyapunov Exponents. More importantly, we have confirmed the generally held view that retrograde orbits are more stable than prograde orbits.

We have also investigated the stability of initially circular orbits around Jupiter. A set of test particles were integrated in the three-dimensional R3BP, and if they remained in orbit about Jupiter after 100 Jupiter revolutions around the Sun, we considered the particle stable. We also showed the necessity of using regularized equations whenever the distance between the test particle and Jupiter became very small. In particular, in the planar case we have found that all prograde orbits are stable if the (initial) distance r_o from Jupiter is less than $0.43 R_H$ (Hill radii), while there are no stable prograde orbits if the initial distance is greater than $0.48 R_H$. For retrograde orbits we find that all test particles are stable if r_o is less than $0.72 R_H$, while there are no stable orbits if r_o is greater than $0.97 R_H$. There is an analytical explanation for the stability of (planar)

prograde orbits, namely that these orbits tend to be unstable whenever the curves of zero velocity are open.

In investigating the instability of initially circular test particles around Jupiter we find that if the initial inclination is within the range 39° to 141° , the orbital eccentricity increases up to the maximum value e_{max} , then it decreases again to zero, the whole process itself being periodic. If the orbit's initial inclination is sufficiently close to 90° then e_{max} approaches unity and the particle may suffer a collision with Jupiter. Following the work of Kozai, Kovalevsky and Froeschlé we develop a perturbing function R from which simple analytic formulae can be derived. From this perturbing function we derive the simple equation $e_{max} = \sqrt{1 - 5 \cos^2 i_o / 3}$, where i_o is the satellite's initial inclination (assumed to be in the range 39° to 141°). If the test particle's initial inclination is outside of this range, then we find that the maximum eccentricity is independent of the initial inclination; instead, the maximum eccentricity is empirically seen to be proportional to the initial semi-major axis cubed, i.e., $e_{max} \propto a^3$. We compared the analytical versus the numerical results (in the R3BP) and observed that the agreement is excellent if the distance from Jupiter is such that $0.07 R_H < r_o < 0.27 R_H$; if the distance is smaller than $0.07 R_H$ ($50 R_J$), other effects become important, such as the oblateness of Jupiter, while if the distance is larger than $0.27 R_H$ ($200 R_J$) the perturbing function we have used (eqn. 4.56) is not accurate enough to represent the very strong solar gravitational perturbations.

We have also extended this work to orbits with arbitrary initial eccentricities. The numerical results have been presented as plots with discrete points (each point representing an initial condition) in the $e-i$ space. Regions of collision, escape and stability were found. The topology of such plots greatly depends on the test particles' initial semi-major axis a_o . If a_o is within the range $0.07 R_H (50 R_J) < a_o < 0.27 R_H (200 R_J)$, very few orbits escape; instability is seen to occur in the form of collisions with Jupiter for orbits such that their initial inclination are within a narrow range centered on 90° , this range widening with increasing eccentricity. Further away from Jupiter ($a_o > 0.27 R_H (200 R_J)$), less and less orbits collide with Jupiter and/or are stable and more and more orbits achieve escape, until finally if $a_o > 0.95 R_H$ we found no more stable orbits. Furthermore, for the range $0.07 R_H (50 R_J) < a_o < 0.27 R_H (200 R_J)$ relatively simple analytical results were obtained, i.e., formulae to predict the maximum and minimum eccentricities and inclinations. These (analytical) results were then successfully applied to the prograde distant satellites of Jupiter; it is believed that these same results may be applied also to Saturn's distant satellite Phoebe and to Neptune's Nereid. Also, a criterion for collision orbits has been established and this criterion has been applied to comet Shoemaker-Levy 9.

A finding that needs further work is the fact that certain orbits with $300 R_J < a_o < 330 R_J$ and inclinations close to 90° seem to be chaotic. Confirmation of this would require the computation of Lyapunov Characteristic Numbers for these orbits. Further,

we have not yet found an analytical explanation for the behavior of the eccentricity when the initial inclination is less than 39° or greater than 141° .

Finally we have made a comparison of numerical results obtained with the R3BP and a general N-body code which takes into account the gravitational attractions of the Sun, Jupiter and Saturn. In the latter case the test particles are, in general, slightly more likely to escape the vicinity of Jupiter due to the extra gravitational perturbations of Saturn and the (varying) non-zero eccentricity of Jupiter. However, detailed comparisons of the results obtained with both methods suggest that the R3BP is an excellent approximation to the actual motion of distant satellites. This is very encouraging, since the equations of motion for the R3BP are much less expensive to integrate than the equations of motion for the general N-body problem .

In closing, we would like to point out that since we find that test particles are stable at greater distances than actually existing satellites, it is not too far-fetched to think that previously unknown distant satellites of Jupiter lie waiting to be discovered. For example, Jupiter's most distant satellite is Sinope, which has a semi-major axis is equal to $\sim 330 R_J$, while the most distant stable test particle we have found had a semi-major axis of $726 R_J$. Perhaps the Galileo spacecraft, in orbit around Jupiter at the time of this writing, will reveal more members of the already large Jovian family.

Appendix A

Numbers Used Throughout this Project

A.1 R3BP Numerical Quantities

Ratio of Jupiter mass (m_2) to Solar mass (m_1):

$$m_2/m_1 = 9.547919384243220 \times 10^{-4}$$

Source: Luke Dones (personal communication), but see also Campbell and Synott [29]. These values refer to the total mass of the Jovian system, i.e., they include the galilean satellites.

Value $\mu = m_2/(m_1 + m_2)$ used in the R3BP:

$$\mu = 9.538811803630968 \times 10^{-4}$$

Note: obtained from mass ratio above using the additional fact that $m_1 + m_2 = 1$.

x-positions of Sun and Jupiter in the rotating coordinate system (y -position = 0):

$$\text{Sun: } x_1 = \mu = 9.538811803630968 \times 10^{-4}$$

$$\text{Jupiter: } x_2 = \mu - 1 = -0.99904611881964$$

A.2 Positions and distances to the Lagrangian Points; critical values of C

Finding the distances from the second mass (m_2) to the Lagrangian points involves the solving of quintic equations (see Szebehely [8]). These quintic equations are best solved

numerically using a Secant or Newton-type root finder (see Garcia, [22]). See Figure A.1 for the relative positions and an explanation of the distances.

Figure A.1. Schematic diagram showing the positions and distances (on the rotating frame) from m_2 (Jupiter) to the collinear Lagrangian points.

Distance d_1 from m_2 to the L_1 point:

In order to find the distance from m_2 to the L_1 point we must solve for the quintic equation (see [8], p. 135),

$$d_1^5 + (3 - \mu)d_1^4 + (3 - 2\mu)d_1^3 - \mu d_1^2 - 2\mu d_1 - \mu = 0 \quad (\text{A.1})$$

Using a Newton root finder with a first guess for $d_1 \approx (\mu/3)^{\frac{1}{3}}$ and a tolerance of 1×10^{-12} we obtain,

$$d_1 = 0.06978454158042$$

Hence position of L_1 point $x(L_1)$ is

$$x(L_1) = \mu - 1 - d_1 = -1.06883066040006$$

To obtain the critical value of $C = C_l$ (the value of C where the two zero-velocity curves meet at the L_1 pt.) is we use eqn. (2.9), i.e.,

$$C_l = 2\Omega[x(L_1)] = 3.03844186452533$$

Distance d_2 from m_2 to the L_2 point:

To obtain d_2 we solve the following quintic equation ([8], p. 136) with a Newton root finder, using as a first guess $d_2 \approx (\mu/3)^{\frac{1}{3}}$ and a tolerance of 1×10^{-12} :

$$d_2^5 - (3 - \mu)d_2^4 + (3 - 2\mu)d_2^3 - \mu d_2^2 + 2\mu d_2 = 0 \quad (A.2)$$

to obtain,

$$d_2 = 0.06668066976385$$

Hence the position of L_2 point $x(L_2)$ is

$$x(L_2) = \mu - 1 + d_2 = -0.93236544905579$$

The critical value of $C = C_2$ (the value of C where the two zero-velocity curves meet at the L_2 pt.) is

$$C_2 = 2\Omega[x(L_2)] = 3.03971395931538$$

Distance from m_2 to the L_3 point:

To obtain d_3 we solve for the quintic

$$d_3^5 + (2 + \mu)d_3^4 + (1 + 2\mu)d_3^3 - (1 - \mu)d_3^2 - 2(1 - \mu)d_3 - (1 - \mu) = 0 \quad (\text{A.3})$$

Again, using a Newton root finder with a first guess $d_3 \approx 1$ and a tolerance 1×10^{-12} we obtain.

$$d_3 = 0.99944356926426$$

Hence position of the L_3 point $x(L_3)$ is

$$x(L_3) = \mu + d_3 = 1.00039745044462$$

The critical value $C = C_3$ is

$$C_3 = 2\Omega[x(L_3)] = 3.00190683334300$$

A.3 Jupiter Data

Equatorial Radius of Jupiter

$$1 R_J = 71398 \text{ km}$$

Source: Danby [9], p. 432.

Semi-major axis of Jupiter's orbit:

$$a_J = 5.21021558 \text{ AU}$$

Source : Danby [9], p. 428.

Eccentricity of Jupiter's orbit:

$$e_J = 0.04849485$$

Source : Danby [9], p. 428.

Period of Jupiter's orbit:

$$P_J = 11.85652502$$

Source: Explanatory Supp. to the Astrn. Alm. [24], p. 704.

A.4 Miscellaneous Astronomical Data

Astronomical Unit

$$1 \text{ AU} = 1.4959787066 \times 10^8 \text{ km}$$

Source: Explanatory Supplement to the Astronomical Almanac [23].

Appendix B

Distant Satellites of Jupiter Data

Table B.1: Distant prograde satellites of Jupiter (Danby [8])

Name	$a(10^3 \text{ km})$	$a(R_J)$	$a(R_H)$	e	$i \text{ (deg.)}^*$
VI Himalia	11,480	160.7	0.22	0.16	27.6
VII Elara	11,737	164.4	0.22	0.21	24.8
X Lysithea	11,720	164.2	0.22	0.11	29.0
XIII Leda	11,094	155.4	0.21	0.15	26.1

Table B.2: Distant retrograde satellites of Jupiter (Danby [8])

Name	$a(10^3 \text{ km})$	$a(R_J)$	$a(R_H)$	e	$i \text{ (deg.)}^*$
VIII Pasiphae	23,500	329.1	0.44	0.38	145
IX Sinope	23,700	331.9	0.45	0.28	153
XI Carme	22,600	316.5	0.42	0.21	164
XII Ananke	21,200	296.9	0.40	0.17	147

* Inclinations are with respect to Jupiter's equatorial plane.

The following tables give the initial conditions in the rotating frame for the distant (prograde and retrograde) satellites of Jupiter. The initial position and velocity components can be found in the Explanatory Supplement to the Astronomical Almanac [24] in AU's and AU/year respectively. To get these quantities in a form useful to use as

starting conditions for the R3BP equations of motion we must perform several transformations: (1) A conversion from (Earth) equatorial to ecliptic coordinates; (2) a conversion of units from AU's to units such that the Sun-Jupiter distance is unity; (3) a conversion from (Earth) years to Jupiter years; (4) finally a conversion from inertial units to rotating units using equation (4.12). Note: we are ignoring Jupiter's inclination to the ecliptic, $i_J \sim 1.3^\circ$.

Table B.3: Initial conditions (in the rotating frame) for VI Himalia

Variable	Numerical Value
x	-1.006620234142865
y	-1.4029924352581335E-02
z	2.5094802065936794E-03
$\dot{x}$	0.1675334657701909
$\dot{y}$	-7.9233068928695638E-02
$\dot{z}$	-0.1137111756263601
C (Jacobi cont.)	3.068232389560198

Table B.4: Initial conditions (in the rotating frame) for VII Elara:

Variable	Numerical Value
x	-1.010940011654528
y	1.2495685632220872E-03
z	5.0111986216588482E-03
$\dot{x}$	5.9351071484530206E-02
$\dot{y}$	-0.2632477873827646
$\dot{z}$	6.1990112636973738E-02
C (Jacobi cont.)	3.067996830160242

Table B.5: Initial conditions (in the rotating frame) for X Lysithea:

Variable	Numerical Value
x	-0.9859236406090136
y	-4.3400171817985691E-03
z	-4.4797259083446507E-03
$\dot{x}$	7.6755587301689202E-02
$\dot{y}$	0.2145564344063786
$\dot{z}$	9.6050756571607249E-02
C (Jacobi cont.)	3.067789662342346

Table B.6: Initial conditions (in the rotating frame) for XIII Leda:

Variable	Numerical Value
x	-0.9922123433465114
y	-1.3876365435976142E-02
z	7.1163440891094214E-03
$\dot{x}$	0.1671871263936235
$\dot{y}$	0.1003159833925438
$\dot{z}$	3.7426770208858401E-02
C (Jacobi cont.)	3.069855688676445

Table B.7: Initial conditions (in the rotating frame) for VIII Pasiphae

Variable	Numerical Value
x	-1.008157491214865
y	-1.7697312130235857E-02
z	7.4742640206715187E-03
$\dot{x}$	-0.1523977323052820
$\dot{y}$	0.2021126452055033
$\dot{z}$	4.8566982224634798E-02
C (Jacobi cont.)	3.020632884108875

Table B.8: Initial conditions (in the rotating frame) for IX Sinope:

Variable	Numerical Value
x	-1.010163360393200
y	3.2540948550297731E-02
z	4.4109181770951812E-03
$\dot{x}$	0.1786972284379025
$\dot{y}$	2.7069792788883218E-02
$\dot{z}$	-5.9208041943108393E-02
C (Jacobi cont.)	3.016380037835019

Table B.9: Initial conditions (in the rotating frame) for XI Carme

Variable	Numerical Value
x	-1.022495966821416
y	-2.3692912819307105E-02
z	-4.0748238595627335E-03
$\dot{x}$	-0.1175737515555154
$\dot{y}$	0.1442500843013288
$\dot{z}$	-3.1131768272157329E-02
C (Jacobi cont.)	3.019990396916315

Table B.10: Initial conditions (in the rotating frame) for XII Ananke

Variable	Numerical Value
x	-0.9751223534828483
y	-6.0768230403831433E-03
z	-1.6298840288687224E-03
$\dot{x}$	-9.3154767546749784E-02
$\dot{y}$	-0.1729350434463859
$\dot{z}$	0.1071191617770327
C (Jacobi cont.)	3.025938962040981

Appendix C

Usefulness of the Quadratic Equations Obtained in Chapter 5

In Chapter 5 we derived two quadratic equations to predict the minimum and maximum eccentricities of a distant satellite. We saw that such equations are useful if the distance to Jupiter is $50 R_J < r < 200 R_J$. We reproduce the two equations here:

$$-2x^2 + (2 - K)x + K = 0 \quad (\text{C.1})$$

$$3x^2 + (5\Theta - 3 - K)x + K = 0 \quad (\text{C.2})$$

where $x = e^2$, and the other quantities are fixed by the initial conditions thus:

$$\Theta = (1 - e^2) \cos^2 i \quad (\text{C.3})$$

$$K = e^2(5 \sin^2 i \sin^2 \omega - 2) \quad (\text{C.4})$$

These two constants are limited to the ranges $0 \leq \Theta \leq 1$ and $-2 \leq K \leq 3$. Equations (C.1) and (C.2) give us four roots, x_1, x_2, x_3 , and x_4 . The two roots of eqn. (C.1) are $x_1 = 1$ (a root which we ignore) and $x_2 = -K/2$, so that in order to obtain meaningful results equation (C.1) is useful for the range $-2 \leq K < 0$. If K is within this "useful" range, then eqn. (C.1) gives us one of the two needed roots, i.e., $x_2 = -K/2$. We need only one more root, which we obtain from eqn. (C.2). If K is *not* within the useful range $-2 \leq K < 0$, then both roots must come from eqn. (C.2). So the equation is, when is (C.2) useful?

Unfortunately, for the ranges of Θ and K that we are interested in, one can show that the two roots of eqn. (C.2) are not even guaranteed to be real. From the quadratic equation, and eqn. (C.2) we form the quantity $f = b^2 - 4ac$, i.e.,

$$f(\Theta, K) = (5\Theta - 3 - K)^2 - 12K \quad (\text{C.5})$$

The range of this auxiliary function is $-35 \leq f \leq 40$ if $0 \leq \Theta \leq 1$ and $-2 \leq K \leq 3$.

However, the range changes to $0 \leq f \leq 40$ if we limit ourselves to $-2 \leq K \leq 0$, i.e., we are *guaranteed* two real roots if $-2 \leq K \leq 0$. Since eqn. (C.1) gives is only one possible useful root, one other useful root *must* come from eqn. (C.2) when $K \leq 0$. The situation is as follows: if $K \leq 0$ one "useful" root (a positive root) should come from eqn. (C.1) and another useful root should come from eqn. (C.2). This other useful root can be either x_3 or x_4 , depending on the sign (we are interested in the non-negative root).

We have seen that, in general, the usefulness of these equations depends on whether $K < 0$. One could also use simple "averaging" arguments to show that, in most cases of interest, K will be less than zero and equations (C.1) and (C.2) will be useful. Consider the definition of K , eqn. (C.4). Most (distant) satellites are within 30° of the parent planet's orbital plane, whether the satellite is prograde or retrograde. Therefore we can set

$$\sin^2 i = (\sin^2 i)_{\max} \approx \sin^2 30 = 0.25 \quad (\text{C.6})$$

The same argument may not be valid for the value of the argument or periapse ω ; however, even if we set $\sin^2 \omega$ to its maximum value 1 we obtain in eqn. (C.2),

$$\langle K \rangle = e^2(5 \times 0.25 \times 1 - 2) = -0.75e^2 \quad (\text{C.7})$$

Therefore, the question of the usefulness of eqn. (C.2) is (in an average sense at least), answered, since we are only interested in the sign of $\langle K \rangle$. This is also supported by the fact that signs of K for the distant satellites of Jupiter are also all negative (see Chapter 5).

Appendix D

FORTRAN Programs

Included: ORBITS3D.F, POINCARE_R3BP.F, RKWR3BP.F

```

      program orbits3d
c-----
c This program integrates test particles in the 3-D r3bp model. Inputs
c are in the file orbit.par (inputs are orbital elements). Uses
c M. Duncan's routines, a Bulirsch-Stoer integrator and a
c regularizing routine.
c To change mu change it in main & eq_of_3Dmotion().
c Compile as f77 -r8 -o orbits3d.x orbits3d.f -lswiftold
c-----
      parameter(N = 6)           !dimension of phase space
      parameter(M=9)             ! regularized order
      parameter(rj = 71398)       !radius of Jupiter in km
      parameter(AU = 1.4959787066d8) !Astrn. Unit in km
      parameter(aj = 5.21021558)  !Semi-major axis of Jup. in AU

      external bsstep,eq_of_3dmotion,reg_eqns,convertx6_to_v9
      external convertv9_to_x6

      real x(N),v(M),xrel,yrel,zrel,vxrel,vyrel,vzrel
      real mu,xj,C,C_o,errC,conv,check
      real tol,tau, t,tend,h1st,hmin
      real a_o,e_o,i_o,lan_o,man_o,omega_o,gm
      real a,e,inc,lan,man,omega,mpo,torb,dtwrite,twrite
      real delta_tr,tr,trend,dt,temp,t_total
      integer i,k,nstep,ialpha,NEQR,p ,io

      mu = 9.538811803630968e-4      !m2/(m1+m2)
      pi = 4.*atan(1.)
      errC = 1.e-6                   !tol desired for err. in C
      conv = rj/(AU*aj)              !1 jupiter radii
      r2_max=2./aj                   !2 AU from m2
      r2_c = 100.                    !100 jovian radii separates r3bp from
                                   !a perturbed 2-body problem

```

```
open(5,file='orbit.par')    !get initial data in elements and other parameters.
```

```

read(5,*) a_o      ! initial semi-major axis (in rj)
read(5,*) e_o      ! "    eccentricity
read(5,*) i_o      ! "    inclination
read(5,*) lan_o    ! "    long. of asc. node
read(5,*) omega_o  ! "    arg. of perijove
read(5,*) man_o    ! "    mean anomaly
read(5,*) h1st     !first guess for t-step in odeint
read(5,*) hmin     !min. t-step in odeint
read(5,*) tau      !by how much advances each step
read(5,*) tol      !acc parameter in odeint
read(5,*) nstep    !number of time steps
read(5,*) mpo      !number of times elem. written per orbit of m3
read(5,*) io       !how often to write x,y,z, etc.
```

c Need to convert from orb. elements around Jupiter to x,y,z, etc. in
c the rotating frame ...

c define a few more constants ...

```

gm = mu              !G*m2 = k^2*mu = mu
ialpha = -1          !assumes an initially circular orbit
a_o = conv*a_o       !convert from jup. radii to r3bp units
torb=(2.*pi*(sqrt(a_o))**3)/sqrt(mu) !2-body period of m3 around m2
dtwrite=torb/mpo      !how often to write elements
twrite = 0.0         !initialize twrite
```

```

call orbel_el2xv(gm,ialpha,a_o,e_o,i_o,lan_o,omega_o,man_o,
&               xrel,yrel,zrel,vxrel,vyrel,vzrel)
```

```
xj = mu - 1.         !x-pos of Jupiter ...
```

c Now transform to "good" coordinates in context of r3bp (t = 0)
c These are the initial cond. which are integrated ...

```

x(1) = xrel + xj      !x-pos
x(2) = yrel           !y-pos
x(3) = zrel           !z-pos
x(4) = vxrel + x(2)   !x-vel
x(5) = vyrel + xj - x(1) !y-vel
x(6) = vzrel          !z-vel
```

```

call jacobi3(x,mu,C)      !check initial value of C
C_o = C

open(14,file='ini_cond.dat')    !put initial cond. here
open(16,file='orbit.dat')       !orbit in rot. frame
open(17,file='elements1.dat')   !orbital elements
open(18,file='elements2.dat')   ! " "
open(19,file='elements3.dat')   ! " "
open(25,file='jacobi.dat')      !jacobi const.
c

write(14,*) x(1),x(2),x(3)      !write out the initial
write(14,*) x(4),x(5),x(6)      !conditions, the initial
write(14,*) a_o/conv,C_o        !a and jacobi const.

NEQ = N
t = 0.0

write(17,99) t,a_o/conv,e_o      !write the initial orbital el.
write(18,99) i_o,lan_o,omega_o   ! " " " " " "
write(19,*) man_o,ialpha         ! " " " " " "

c -----Main Loop-----

t_total = tau*nstep
i = 0

do while(t .lt. t_total) !total int. time = tau*nstep

  i = i + 1
  tend = tau*float(i)

  if ( t .ge. tend) then
    dt = t - tend
    temp = dt/tau - mod(dt,tau)/tau
    p = int(temp) + 1
    i = i + p      !increase i by p if t > tend
    tend = tau*float(i)
  end if

```

```

        call odeint(x,NEQ,t,tend,tol,h1st,hmin,nok,
&          nbad,eq_of_3Dmotion,bsstep)    !Num. Rec.

        t = tend                                !update the time

c      Compute the rel. pos w.r.t. m2 (in inertial coordinate)
c      of test particle to convert to osculating orbital elements

        call inertial(xj,x,t,xrel,yrel,zrel,vxrel,vyrel,vzrel)

c      Now compute the orbital elements using orbel_xv2el.f

        gmsum = gm    !gmsum=G*(m2+m3) but m3 = 0;
                    !gmsum=gm=G*m2=mu

        call orbel_xv2el(xrel,yrel,zrel,vxrel,vyrel,vzrel,gmsum,
&          ialpha,a,e,inc,lan,omega,man)

        r2 = sqrt((x(1)+1.-mu)**2 + x(2)**2 + x(3)**2) !dist. to m2

c      If e > 0.8 then Regularize the equations of motion!!

        if( e .gt. 0.8) then
          call convertx6_to_v9(t,r2,xj,x,v) !change to (initial) reg. coord.
          k=0

          delta_tr = tau/r2                ! !try this

          NEQR = M                        !order of the regularized problem
          tr = 0.0                        !initial regularized time

          do while ( e .gt. 0.8 )          !while ecc. > 0.8
            k=k+1
            trend=delta_tr*float(k)        !advance by a reg. time step!

            call odeint(v,NEQR,tr,trend,tol,h1st,hmin,nok,
&              nbad.reg_eqns,bsstep)

            tr = trend                    !update the (regularized) time

            call convertv9_to_x6(xj,v,x) !convert to r3bp coord.

```

```

if(mod(k,io).eq. 0.)then
    write(16,99) t,x(1),x(2),x(3)
end if

    call jacobi3(x,mu,C)          !Don't forget to check C

if(mod(k,800).eq.0.) write(25,*) t,C

check=abs(C_o-C)

if(check .gt. errC) goto 90

t = v(9)          ! the new (unregularized) time

call inertial(xj,x,t,xrel,yrel,zrel,vxrel,vyrel,vzrel)

call orbel_xv2el(xrel,yrel,zrel,vxrel,vyrel,vzrel,gmsum,
&               ialpha,a,e,inc,lan,omega,man) !get orbital elements

r2=sqrt((x(1)+1.-mu)**2 + x(2)**2 + x(3)**2)

if(r2 .gt. r2_max) goto 80

if(r2 .lt. 100*conv .and. a*(1.-e) .lt. conv) then !Collision
    write(40,*) '!!Collision!!'
    write(40,*) 'r2 =',r2
    write(40,*) 'q =' ,a*(1.-e)
    write(40,*) 'Finished ',i,' steps of ',nstep,' steps.'
    call orbel_xv2el(xrel,yrel,zrel,vxrel,vyrel,vzrel,
&               gmsum,ialpha,a,e,inc,lan,omega,man)
    write(40,*) 't =',t          !write the elem.
    write(40,*) 'ialpha =' ,ialpha
    write(40,*) 'a =' ,a/conv
    write(40,*) 'e =' ,e
    write(40,*) 'inc =' ,inc
    write(40,*) 'lan =' ,lan
    write(40,*) 'omega =' ,omega
    write(40,*) 'm. an. =' ,man
    goto 110 !finish this run
end if

if(t .gt. twrite) then
    write(17,99) t,a/conv,e

```

```

        write(18,99) inc,lan,omega
        write(19,*) man,ialpha
        twrite = twrite + dtwrite
    end if

end do    !finish the regularized integration

end if

80  if (r2 .gt. r2_max) then                !"Escape"
    write(50,*) 'r2 > r2_max =2AU'
    write(50,*) 'r2 =',r2,' > r2_max =',r2_max
    write(50,*) 'Finished ',i,' steps of ',nstep,' steps.'
    call orbel_xv2el(xrel,yrel,zrel,vxrel,vyrel,vzrel,gmsum,
&      ialpha,a,e,inc,lan,omega,man)
    write(50,*) 't =',t                    !write elements
    write(50,*) 'ialpha =',ialpha
    write(50,*) 'a =',a/conv
    write(50,*) 'e =',e
    write(50,*) 'inc =',inc
    write(50,*) 'lan =',lan
    write(50,*) 'omega =',omega
    write(50,*) 'm. an. =',man
    goto 110                                !finish this run
end if

if(mod(i,io).eq.0.) then
    write(16,99) t,x(1),x(2),x(3)          !coord. in rot. frame
end if

    if(t .gt. twrite) then
        write(17,99) t,a/conv,e            !write orb. elements
        write(18,99) inc,lan,omega        ! " " "
        write(19,*) man,ialpha            ! " " "
        twrite = twrite + dtwrite
    end if

c  check the jacobi constant..

    call jacobi3(x,mu,C)

    if(mod(i,500).eq.0.) write(25,*) t,C    !write C

```



```

    check = abs(C_o - C)

    if(check.gt.errC) then
90      write(60,*) '-----'
        write(60,*) 'Warning: errC exceeded'
        write(60,*) 'Computation halted'
        write(60,*) 'Only', i, 'steps of ', nstep, ' done'
        write(60,*) t, C_o, C
        write(60,*) 'Recommend redoing this run with smaller tol'
        write(60,*) '-----'
        goto 110 !finish this run & redo with smaller tol
    end if
99      format(4f17.10)
        end do
110 stop
    end

c*****
c Subroutines follow
c*****

c-----
    subroutine eq_of_3Dmotion(t,x,xprime)

c Equations of motion of the 3-D r3bp. See Szebehely (1967)
c Input : t,x(6)
c Output: xprime(6)

    parameter(N=6)

    real x(N),xprime(N),mu
    real r1,r2,Ox,Oy,Oz

    mu = 9.538811803630968e-4

c Define auxiliary equations here

    r1=sqrt((x(1)-mu)**2 + x(2)**2 + x(3)**2) !distance to m1
    r2=sqrt((x(1)+1.-mu)**2 + x(2)**2 + x(3)**2) !distance to m2

    Ox =x(1) - (1.-mu)*(x(1)-mu)/r1**3 - mu*(x(1)+1.-mu)/r2**3

    Oy =x(2) - (1.-mu)*x(2)/r1**3 - mu*x(2)/r2**3

```

$$Oz = -x(3)*((1.-mu)/r1**3 + mu/r2**3)$$

c equations of motion

c

```
xprime(1) = x(4)
xprime(2) = x(5)
xprime(3) = x(6)
xprime(4) = Ox + 2.*x(5)
xprime(5) = Oy - 2.*x(4)
xprime(6) = Oz
```

c

```
return
end
```

c-----

subroutine jacobi3(x,mu,C)

c Computes the Jacobi constant for the 3-D R3BP

c Input : x(6),nu

c Output: C

parameter(N=6)

real x(N),mu,r1,r2,O,C

c Define auxiliary equations here

```
r1=sqrt((x(1)-mu)**2 + x(2)**2 + x(3)**2)    !distance to m1
r2=sqrt((x(1)+1.-mu)**2 + x(2)**2 + x(3)**2) !distance to m2
```

```
O = (x(1)**2+x(2)**2)/2.+ (1.-mu)/r1 + mu/r2 +
& mu*(1.-mu)/2.
```

c Compute the Jacobi constant

```
C = 2.*O - x(4)**2 -x(5)**2 -x(6)**2
```

```
return
end
```

c-----

subroutine inertial(xj,x,t,xrel,yrel,zrel,vxrel,vyrel,vzrel)

c Computes the inertial positions and velocities w.r.t. m2 to
 c get the orbital elements
 c Input : xj,x(6),t
 c Output: xrel,yrel,zrel,vxrel,vyrel,vzrel

```

parameter(N=6)

real xj,x(N),t,xrel,yrel,zrel,vxrel,vyrel,vzrel

xrel = (x(1)-xj)*cos(t) - x(2)*sin(t)
yrel = (x(1)-xj)*sin(t) + x(2)*cos(t)
zrel = x(3)
vxrel = (x(4)-x(2))*cos(t) - (x(1)+x(5)-xj)*sin(t)
vyrel = (x(4)-x(2))*sin(t) + (x(1)+x(5)-xj)*cos(t)
vzrel = x(6)

return
end

```

c-----

```

subroutine reg_eqns(tr,v,vprime)

```

c The regularized, 3-D R3BP equations of motion
 c Input : tr(regularized time),v(9)
 c Output: vprime(9)

```

parameter(M=9)
parameter(N=6)

real v(M),mu,R,xj,vpsq,k,xo,y,z,xp,yp,rl
real P1,P2,P3,vprime(M),x(N)

mu=9.538811803630968e-4 !m2/(m1+m2)

xj = mu - 1. !position of Jupiter

R = v(1)**2 + v(2)**2 + v(3)**2 + v(4)**2

vpsq = v(5)**2 + v(6)**2 + v(7)**2 + v(8)**2

k = (mu - 2.*vpsq)/R

```

```

call convertv9_to_x6(xj,v,x)      !from v(9) get x(6)

xo = x(1) - xj                    !rot. x w.r.t. m2 (jupiter)

y = x(2)                          !rot. y

z = x(3)                          !rot. z

xp = x(4)                         !rot. x-vel

yp = x(5)                         !rot. y-vel.

zp = x(6)                         !rot. z-vel.

r1 = sqrt((xo-1.)**2 + y**2 + z**2) !dist. to m1 (sun)

P1 = 2.*yp + xo + mu-1. - (1-mu)*(xo-1.)/r1**3    !Pert. fn

P2 = -2.*xp + y - y*(1.-mu)/r1**3                ! "  "

P3 = -z*(1.-mu)/r1**3                            ! "  "

```

c Now define the equations of regularized motion

```

vprime(1) = v(5)
vprime(2) = v(6)
vprime(3) = v(7)
vprime(4) = v(8)
vprime(5) = (-k*v(1) + R*( v(1)*P1 + v(2)*P2 + v(3)*P3))/2.
vprime(6) = (-k*v(2) + R*(-v(2)*P1 + v(1)*P2 + v(4)*P3))/2.
vprime(7) = (-k*v(3) + R*(-v(3)*P1 - v(4)*P2 + v(1)*P3))/2.
vprime(8) = (-k*v(4) + R*( v(4)*P1 - v(3)*P2 + v(2)*P3))/2.
vprime(9) = R      !unregularized time

return
end

```

c-----
subroutine convertx6_to_v9(t,r2,xj,x,v)

c This routine converts from the unregularized, r3bp problem to
c the regularized r3bp problem. Notice we must change the x-coord.

c since we need the (rotating) x-coord. but w.r.t. m2

c Inputs : t,r2,xj,x(6)

c Output : v(9)

```
parameter(N=6)    !order of r3bp (3D) problem
parameter(M=9)    !order of regularized problem
```

```
real r2,xj,x1,x(N),v(M)
```

c change the x-pos. from w.r.t. barycenter to w.r.t. m2 (other
c coordinates are the same)

```
x1 = x(1) - xj
```

c get initial conditions

```
if(x1 .ge. 0.0 )then
  v(1) = sqrt((r2+x1)/2.)
  v(2) = x(2)/(2.*v(1))
  v(3) = x(3)/(2.*v(1))
  v(4) = 0.0
else
  v(2) = sqrt((r2 - x1)/2.)
  v(1) = x(2)/(2.*v(2))
  v(3) = 0.
  v(4) = x(3)/(2.*v(2))
end if
```

c

```
v(5) = ( v(1)*x(4) + v(2)*x(5) + v(3)*x(6))/2.
v(6) = (-v(2)*x(4) + v(1)*x(5) + v(4)*x(6))/2.
v(7) = (-v(3)*x(4) - v(4)*x(5) + v(1)*x(6))/2.
v(8) = ( v(4)*x(4) - v(3)*x(5) + v(2)*x(6))/2.
v(9) = t                                !initial unregularized time
```

```
return
end
```

```

c-----
      subroutine convertv9_to_x6(xj,v,x)
c Converts form regularized coordinates to unregularized coordinates
c appropriate for the r3bp
c Input :xj,v(9)
c Output:x(6)

      parameter(N=6)           !order of r3bp
      parameter(M=9)           !order of regularized problem

      real v(M),x(N),xj,R

      R = v(1)**2 + v(2)**2 + v(3)**2 + v(4)**2

c

      x(1) = v(1)**2 - v(2)**2 - v(3)**2 + v(4)**2 + xj    !good 4 r3bp
      x(2) = 2*(v(1)*v(2) - v(3)*v(4))
      x(3) = 2*(v(1)*v(3) + v(2)*v(4))
      x(4) = 2*(v(1)*v(5) - v(2)*v(6) - v(3)*v(7) +
&          v(4)*v(8))/R
      x(5) = 2*(v(2)*v(5) + v(1)*v(6) - v(4)*v(7) -
&          v(3)*v(8))/R
      x(6) = 2*(v(3)*v(5) + v(4)*v(6) + v(1)*v(7) +
&          v(2)*v(8))/R

      return
      end

c*****
c Numerical Recipes follow (odeint.f, bsstep.f, mmid.f, pzextr.f)
c*****

```

```

program poincare_r3bp
c-----
c  Produces a set of initial conditions to make a Poincare
c  surface of section for the (planar) R3BP. Inputs are in the file
c  poincare.par. Uses an RK integrator...
c  Compile as f77 -r8 -o poincare_r3bp.x poincare_r3bp.f
c-----

parameter(N=4)
external eq_of_motion, jacobi, rkqs, odeint, rkck

real pi, r1, r2, dx, dth, x0, y0, vsq, v, mu, vx, vy
real x2, y2, xj, th, errC, test, check, flag

real x(N), y_cur, x_cur, vx_cur, t_cur
real y_old, x_old, vx_old, t_old
real tol, tau, t, tend, h1st, hmin, C_o, C, m_y, b_y
real t_o, m_x, b_x, x_ok, m_xdot, b_xdot, xdot_ok

integer nstep, i, j, k, N1, N2, count

open(5, file='poincare.par')      !Input file

      read(5, *) h1st      !first guess for t-step in odeint
      read(5, *) hmin      !min. t-step in odeint
      read(5, *) tau       !by how much advances each step
      read(5, *) tol       !acc parameter in odeint
      read(5, *) nstep     !number of time steps per in. cond.
      read(5, *) C_o       !Jacobi constant desired!
      read(5, *) flag      !If flag=1, prograde; if flag=-1 retrogr.i

      if(abs(flag).ne. 1.) write(*,*) 'BAD VALUE FOR FLAG!!'

      open(10, file='poincare.dat')
      open(15, file='jacobi.dat')
      open(20, file='ini_cond.dat')
c
c  *****produce the initial conditions*****
c
c  Some constants

errC=1.e-6                      !We want |C_o-C| = check < errC

mu=9.538811803630968e-4        !m2/(m1+m2)

```

```

pi=3.141592653589793

dx=9.18e-3                !~100 Jupiter radii

xj=mu-1.                  !Position of m2 along x-axis

N1=7                      !Number of x-spacings

N2=8                      !if N2=4, dth=90; if N2=8, dth=45, etc. (even #)

count = 0                 !initialize the orbit counter

dth=2.*pi/float(N2)       !delta-theta

do i=1,N1
  r2=float(i)*(dx/2.)      !dist. from m2
  do 100 j=1,N2
    count=count+1          !increase orbit counter
    th=float(j)*dth        !theta
    x2=r2*cos(th)          !x-pos with respect to m2
    y2=r2*sin(th)          !y-pos " " " "
    x0=xj+x2               !x-pos. !****!
    y0=y2                  !y-pos. !****!
    r=sqrt(x0**2 + y0**2)   !dist from origin
    r1=sqrt((x0-mu)**2 + y0**2) !dist. from m1
    vsq=(1.-mu)*r1**2 + mu*r2**2 + 2.*(1.-mu)/r1 +
&    2.*mu/r2 - C_o
    v=sqrt(vsq)             !speed
    vx=flag*(-y2*v/r2)      !x-vel !*****!
    vy=flag*( x2*v/r2)      !y-vel !*****!
  100
c
c  write the initial conditions to file 20
c
  write(20,*) 'Orbit #',count
  write(20,*) x0,y0
  write(20,*) vx,vy
  write(20,*) '-----'

  x(1) = x0    !Initial conditions ...
  x(2) = y0
  x(3) = vx
  x(4) = vy

```



```

c
c  initialization for integrator...

NEQ=N
t=0.0

do k=1,nstep    !total int. time = tau*nstep

    x_old=x(1)    !remember the old values
    y_old=x(2)
    vx_old=x(3)
    t_old=t

    tend=tau*float(k)

c  call any ODE integrator ...

    call odeint(x,NEQ,t,tend,tol,h1st,hmin,nok,
&    nbad,eq_of_motion,rkqs)    !Num. Rec.

    t=tend !update the time

    x_cur=x(1)
    y_cur=x(2)
    vx_cur=x(3)
    t_cur=t

    test=y_old*y_cur !test if y changes sign ...

    if(test .lt. 0. .and. x(4) .gt. 0.) then    !(y=0 & vy>0)
c
c  Do a linear interpolation for the time y crosses the
c  y=0 axis

        m_y = (y_cur - y_old)/tau    !y-slope
        b_y = y_cur - m_y*t_cur    !y-intercept

        t_o = -b_y/m_y    !time between t_old & t_cur when y=0

        m_x=(x_cur - x_old)/tau    !x-slope
        b_x=x_cur - m_x*t_cur    !x-intercept
        x_ok=m_x*t_o + b_x    !**Point we want

```

```

    m_xdot = (vx_cur - vx_old)/tau      !xdot-slope
    b_xdot = vx_cur - m_xdot*t_cur      !xdot intercept
    xdot_ok=m_xdot*t_o + b_xdot        !***Point we want

    write(10,*) count,x_ok,xdot_ok      !surf. of section

end if

c  check the jacobi constant..

call jacobi(x,C)
check=abs(C_o - C)                    !Check error in C
if(k .eq. 1) then
    write(15,*) 'Orbit #'.count
    write(15,*) t,C                    !Write initial value of C
end if
if(check .gt. errC) then
    write(15,*) '.....'
    write(15,*) 'Warning: |C_o-C| > ',errC
    write(15,*) 'Finished ',k,' of ',nstep,' steps only.'
    write(15,*) t,C,C_o
    write(15,*) '.....'
    write(15,*) '-----'
    go to 100    !don't want this orbit any-more; next one
end if
if(k .eq. nstep) then
    write(15,*) t,C                    !Write final value of C
    write(15,*) '-----'
end if

c
    end do

100  continue

    end do

    stop
    end

c

```

```

c *****
c Equations of motion for R3BP
subroutine eq_of_motion(t,x,xprime)

parameter(N=4)

real x(N),xprime(N),mu
real r1,r2,Ox,Oy

mu = 9.538811803630968e-4

c Define auxiliary equations here

r1=sqrt((x(1)-mu)**2 + x(2)**2) !distance to m1
r2=sqrt((x(1)+1.-mu)**2 + x(2)**2) !distance to m2

Ox =x(1) - (1.-mu)*(x(1)-mu)/r1**3 - mu*(x(1)+1.-mu)/r2**3

Oy =x(2) - (1.-mu)*x(2)/r1**3 - mu*x(2)/r2**3

c equations of motion
c
xprime(1)= x(3)
xprime(2)= x(4)
xprime(3)= Ox + 2.*x(4)
xprime(4)= Oy - 2.*x(3)
c

return
end

c-----
subroutine jacobi(y,C)

parameter(N=4)

real y(N),mu,r1,r2,O,C

mu = 9.538811803630968e-4

c Define auxiliary equations here

r1=sqrt((y(1)-mu)**2 + y(2)**2) !distance to m1

```

```

r2=sqrt((y(1)+1.-mu)**2 + y(2)**2) !distance to m2

O = ((1.-mu)*r1**2+mu*r2**2)/2. + (1.-mu)/r1 + mu/r2

c  Compute the Jacobi constant

C = 2.*O - y(3)**2 -y(4)**2

return
end

c  *****
c  Numerical Recipes follow (odeint.f, rkqs.f, rkck.f, derivs.f)
c  *****

```

```

program rkwr3bp
c-----
c  Uses the Wolf et al. routine for calculating the spectra
c  of LCN's for the planar restricted 3-body problem ...
c  Uses a Runge-Kutta integrator
c  N=# of nonlinear eqns., NN=total number of eqns.
c  Compile as f77 -r8 -o rkwr3bp.x rkwr3bp.f
c  Input file is r3bp.par, BUT TO CHANGE MU change it
c  in jacobi() and fcn() (below)
c-----
parameter(N=4)
parameter(NN=20)
external fcn, jacobi, rkqs, odeint, rkck

real y(NN), znorm(N), gsc(N), cum(N)
real tol, tau, x, xend, h1st, hmin, C

integer io, nstep

open(5, file='r3bp.par')

      read(5,*) y(1)      !x-pos  [initial conditions]
      read(5,*) y(2)      !y-pos   "
      read(5,*) y(3)      !vx-vel  "
      read(5,*) y(4)      !vy-vel  "
      read(5,*) h1st      !first guess for t-step in odeint
      read(5,*) hmin      !min. t-step in odeint
      read(5,*) tau       !by how much advances each step
      read(5,*) tol       !acc parameter in odeint
      read(5,*) nstep     !number of time steps
      read(5,*) io

open(10, file='orbit.dat')
open(20, file='lyap1.dat')
open(21, file='lyap2.dat')
open(30, file='jacobi.dat')

c
c  initial conditions for linear system (orthonormal frame)
do 10 i=N+1, NN
      y(i)=0.0
10  continue
do 20 i=1, N

```

```

        y((N+1)*i)=1.0
        cum(i)=0.0
20    continue
c
c    initialization for integrator...

    NEQ=NN
    x=0.0

    do 100 i=1,nstep
        xend=tau*float(i)
c    call any ODE integrator ...

        call odeint(y,NEQ,x,xend,tol,h1st,hmin,nok,
& nbad,fcn,rkqs)    !Num. Rec.

        x=xend !update the time
        if(i.lt.10000) write(10,*) x,y(1),y(2) !t,x,y

c    jacobi constant...
        call jacobi(y,C)
        if(mod(i,io).eq.0) write(30,*) x,C
c
c    construct a new orthonormal basis by the Gram_Schmidt method
c
c    normalize the first vector

        znorm(1)=0.0
        do 30 j=1,N
            znorm(1)=znorm(1)+y(N*j+1)**2
30    continue
        znorm(1)=sqrt(znorm(1))

        do 40 j=1,N
            y(N*j+1)=y(N*j+1)/znorm(1)
40    continue

c    generate the new set of orthonormal vectors
        do 80 j=2,N
c    generate j-1 GSR coefficients
            do 50 k=1,(j-1)
                gsc(k)=0.0
                do 50 l=1,N

```

```

        gsc(k)=gsc(k)+y(N*l+j)*y(N*l+k)
50    continue
c
c    construct a new vector
c
    do 60 k=1,N
        do 60 l=1,(j-1)
            y(N*k+j)=y(N*k+j)-gsc(l)*y(N*k+l)
60    continue
c
c    calculate the vector's norm
c
    znorm(j)=0.0
    do 70 k=1,N
        znorm(j)=znorm(j)+y(N*k+j)**2
70    continue
    znorm(j)=sqrt(znorm(j))
c
c    normalize the new vector
c
    do 80 k=1,N
        y(N*k+j)=y(N*k+j)/znorm(j)
80    continue
c
c    update running vector magnitudes
c
    do 90 k=1,N
        cum(k)=cum(k)+alog(znorm(k)) !/alog(2.)
90    continue
c
c    normalize exponent and print every io iterations

    if(mod(i,io).eq.0) write(20,*) x,(cum(k)/x,k=1,N-2)
    if(mod(i,io).eq.0) write(21,*) (cum(k)/x,k=N-1,N)
    !if(mod(i,io).eq.0) write(*,*) x,i,nstep,cum(1)/x
100   continue
    stop
    end

c

```

c Equations of motion and linearized eqns. follow
subroutine fcn(x,y,yprime)

parameter(NN=20) !total number of equations=N + N^2

real y(NN),yprime(NN),mu
real r1,r2,Ox,Oy,Oxx,Oyy,Oxy

mu = 9.538811803630968e-4

c Define auxiliary equations here

r1=sqrt((y(1)-mu)**2 + y(2)**2) !distance to m1
r2=sqrt((y(1)+1.-mu)**2 + y(2)**2) !distance to m2

Ox =y(1) - (1.-mu)*(y(1)-mu)/r1**3 - mu*(y(1)+1.-mu)/r2**3

Oy =y(2) - (1.-mu)*y(2)/r1**3 - mu*y(2)/r2**3

Oxx=1. + (1.-mu)*((3.*(y(1)-mu)**2)/r1**5 - 1./r1**3)
& +mu*((3.*(y(1)+1.-mu)**2)/r2**5 - 1./r2**3)

Oxy=3.*(1.-mu)*(y(1)-mu)*y(2)/r1**5 +
& 3.*mu*(y(1)+1.-mu)*y(2)/r2**5

Oyy=1. + (1.-mu)*((3.*y(2)**2)/r1**5-1./r1**3) +
& mu*((3.*y(2)**2)/r2**5 - 1./r2**3)

c

c equations of motion

c

yprime(1)= y(3)
yprime(2)= y(4)
yprime(3)= Ox + 2.*y(4)
yprime(4)= Oy - 2.*y(3)

c

c 4 copies of the linearized eq.

c

do 10 i=0,3
yprime(5+i)= y(13+i)
yprime(9+i)= y(17+i)
yprime(13+i)=Oxx*y(5+i) + Oxy*y(9+i) +2.*y(17+i)


```
yprime(17+i)=Oxy*y(5+i) + Oyy*y(9+i) -2.*y(13+i)
```

```
10 continue
```

```
return
end
```

```
c-----
```

```
subroutine jacobi(y,C)
```

```
parameter(NN=20)
```

```
real y(NN),mu,r1,r2,O,C
```

```
mu = 9.538811803630968e-4
```

```
c Define auxiliary equations here
```

```
r1=sqrt((y(1)-mu)**2 + y(2)**2) !distance to m1
```

```
r2=sqrt((y(1)+1.-mu)**2 + y(2)**2) !distance to m2
```

```
O = ((1.-mu)*r1**2+mu*r2**2)/2. + (1.-mu)/r1 + mu/r2
```

```
c Compute the Jacobi constant
```

```
C = 2.*O - y(3)**2 -y(4)**2
```

```
return
end
```

```
c *****
```

```
c Numerical Recipes follow (odeint.f, rkqs.f, rkck.f, derivs.f)
```

```
c *****
```

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