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Exploring the Poincaré models of hyperbolic geometry using Geometer's Sketchpad

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EXPLORING THE POINCARÉ MODELS OF HYPERBOLIC GEOMETRY
USING *GEOMETER'S SKETCHPAD*

A Thesis

Presented to

The Faculty of the Department of Mathematics
and Computer Science
San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Marlene Chiaramonte Dwyer

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ABSTRACT

EXPLORING THE POINCARÉ MODELS OF HYPERBOLIC GEOMETRY USING *GEOMETER'S SKETCHPAD*

by Marlene Chiaramonte Dwyer

In this thesis we discuss the fundamentals of circles and inversions in the extended complex plane, define and illustrate the basic types of Möbius transformations, and study, in particular, the subgroup of Möbius transformations which form the basis of motions in two models of the hyperbolic plane. We develop factorization methods for Möbius transformations that will be used in the presentation of direct and opposite motions in our models of the hyperbolic plane. We conclude with a study of hyperbolic geometry including parallelism, distance function, hyperbolic motions, classification of isometries, trigonometry, and area.

We illustrate the concepts presented with the *Geometer's Sketchpad* in the inversive plane, the Poincaré disk model, and the Poincaré half-plane, as appropriate. We discuss methods for using the *Geometer's Sketchpad* and provide suggestions for further study and exploration. The Appendix consists of a list of the *Sketchpad* scripts and illustrations that are contained on the included disks.

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Chapter 1

Introduction

The computer software *Geometer's Sketchpad, 3.0* is one of several programs available that allow quick and dynamic geometric constructions. The program facilitates the demonstration of geometric theorems and the exploration of conjectures. In this thesis, we illustrate many of the theorems of hyperbolic geometry using the *Geometer's Sketchpad*, and provide suggestions for further exploration. In most cases, illustrations are provided in both the Poincaré Disk and the Poincaré Half-Plane models of hyperbolic geometry. Additional construction tools are required for the Poincaré Disk model. These are available on the Internet through The Geometry Forum, Swarthmore College. (The reader may contact <http://forum.swarthmore.edu/sketchpad/gsp.gallery/poincare/poincare.html>.)

Most students are introduced to hyperbolic geometry as one of the non-Euclidean geometries that arose historically when attempts to prove Euclid's Fifth Postulate failed. The Hyperbolic Axiom states "if P is a point off line ℓ , then there exist at least two lines through P that are parallel to ℓ ." The geometry based on Euclid's first four postulates and the Hyperbolic Axiom has been proven to be a relatively consistent system. There are several models that adequately display the relationships of hyperbolic geometry. We may consider these models to be isomorphic representations of the hyperbolic plane.

In order to understand the Poincaré models of hyperbolic geometry, we first study the operation of inversion in a Euclidean circle. This operation will be analogous to the reflection across a line in the hyperbolic geometry models. In Chapter 2 we provide both synthetic and analytic explanations of inversion. The standard development of theorems is adapted from Smart [17] and Greenberg [6]. A different approach using the analytic geometry of circles may be found in Schwerdtfeger [15].

In Chapter 3, we study the Möbius transformations, which are compositions of an even number of inversions. These transformations are essential to understanding the isometries of the hyperbolic plane models. We have adapted, for the most part, the presentation in Schwerdtfeger, which is quite thorough. It also contains some very beautiful mathematics that we have not been able to include. We have utilized some material on complex numbers from Greenleaf [7] and Hahn [8]. Greenleaf provides straightforward proofs and excellent diagrams. Both of these authors provide useful examples and problems.

We pay particular attention to the factorization of a general Möbius transformation. We introduce and illustrate, using *Sketchpad*, two useful factorizations that can be easily derived from the matrix of a Möbius transformation. We define the factorization which consists of inversion in a circle followed by reflection in a line to be the Standard Factorization. The Symmetric Factorization consists of inversions in two circles that are symmetric with respect to a particular line.

The *Geometer's Sketchpad* is used for all the illustrations in this thesis, but in Chapter 4 we finally put it to use to illustrate the Poincaré models. The descriptions of the structures within the Disk model and the theoretical development have been taken largely from Hahn [8] and Martin [13]. We have relied on Stahl [18] for the discussion of the Half-Plane model. We illustrate representative examples of motions and their factorizations. We also provide illustrations of various theorems, some of which have Euclidean analogues.

In Chapter 5, we present the basic theorems of hyperbolic trigonometry. We include, primarily, results from Martin and Greenberg. We emphasize visualization of the theorems with *Sketchpad* rather than the proofs, which are given in the references cited.

Chapter 6 is devoted to the definition of a suitable area function for the hyperbolic geometry model. We generally follow the presentation in Martin. Several theorems are

taken also from Moise [14], which provide some beautiful results and some challenging constructions on the *Sketchpad*.

We conclude with some remarks about using *Geometer's Sketchpad*, emphasizing the use of the Poincaré tools. Since there is a wealth of material that we are not able to include, we provide some suggestions for further topics to investigate. The general theory of hyperbolic geometry and a comparative study to the Euclidean model can be more thoroughly developed using *Sketchpad*. Additional topics related to the Möbius transformations include continued fractions, discrete groups (particularly the Modular group and Picard's group), frieze groups, and a new characterization of Möbius transformations by Haruki [9] that utilizes the concept of symmedians in geometry. In addition, the basic tools of *Sketchpad* include tracing and animation tools which have not been explored in this thesis. We also discuss other computer programs which may be useful in exploring models of hyperbolic geometry.

We have included (Macintosh compatible) computer disks containing the *Sketchpad* scripts and many of the illustrations developed for this project. Some of the illustrated theorems included on the disks were not included in the printed thesis. The Appendix lists an index to the contents of the disks.

Chapter 2

Circles and Inversion

§ 2.1 Analytic Geometry of Circles

We begin with the standard equation of a circle, $(x - a)^2 + (y - b)^2 = r^2$. This equation describes all the points (x, y) in the real plane that lie on the circle with center (a, b) and radius r . If we consider this in the complex plane, as in Schwerdtfeger [15], then the equation above represents the points $z = x + yi$ that lie on the circle with center $\gamma = a + bi$ having radius r . Using the properties of complex numbers, we can rewrite the left-hand side of the equation as $|z - \gamma|^2 = (z - \gamma)(\bar{z} - \bar{\gamma})$. Expanding the product and setting the equation equal to zero, we have $z\bar{z} - \bar{\gamma}z - \gamma\bar{z} + \gamma\bar{\gamma} - r^2 = 0$. Since $\gamma\bar{\gamma} = |\gamma|^2$, we have

$$z\bar{z} - \bar{\gamma}z - \gamma\bar{z} + |\gamma|^2 - r^2 = 0. \quad (2.1)$$

It will be convenient for our purposes to represent equation (2.1) in matrix form as

$$\begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0. \quad (2.2)$$

In this form, we consider the 2×2 matrix to be the matrix representation of the circle with center γ and radius r , and we use the notation

$$C_{(\gamma, r)} = \begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix}. \quad (2.3)$$

The matrix $\begin{pmatrix} z & 1 \end{pmatrix}$ for complex number z is called a homogeneous coordinate representation of z . Any non-zero multiple of the matrix $\begin{pmatrix} z & 1 \end{pmatrix}$ is another representation. Similar remarks hold for the matrix $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}$ as a homogeneous coordinate representation of $\bar{z}$.

The matrix representation of a circle has certain important properties to be noted. If we call the 2×2 matrix in (2.3) H , we notice that H is equal to the transpose of its conjugate:

$$\bar{H} = \overline{\begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix}} = \begin{pmatrix} 1 & -\gamma \\ -\bar{\gamma} & |\gamma|^2 - r^2 \end{pmatrix} \text{ and } \bar{H}^T = \begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix} = H.$$

Any matrix with this property is defined as a Hermitian matrix, named after Charles Hermite. We see that every circle can be represented by a 2×2 Hermitian matrix. We shall also see that every Hermitian matrix represents a “circle”, when suitably interpreted.

Let us consider the general form of a 2×2 complex matrix $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. If H is Hermitian, then $H = \bar{H}^T$, so $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix}$. Since $A = \bar{A}$ and $D = \bar{D}$, A and D must be real. Furthermore, B and C are complex conjugates. Finally, we note that the determinant $|H| = AD - BC = AD - |B|^2$ is a real number.

If $A \neq 0$, then

$$\begin{pmatrix} 1 & \frac{B}{A} \\ \frac{C}{A} & \frac{D}{A} \end{pmatrix} \quad (2.4)$$

is a nonzero multiple of H . We call this form of H a *normalized* Hermitian matrix. It is important to note that H and matrix (2.4) have the same “quadratic form”,

$(z \ 1) H \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0$. If we compare (2.4) with the matrix for the circle given by (2.3),

$\begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix}$, we can state that H represents the equation of a circle with center $-\frac{C}{A}$.

Furthermore, if we calculate the determinants of (2.3) and (2.4) and set them equal to one

another, we have $-r^2 = \frac{D}{A} - \frac{BC}{A^2}$, so that $-A^2r^2 = AD - BC = |H|$. Thus the circle

given by (2.4) has radius $r = \frac{1}{A}\sqrt{BC - AD} = \frac{1}{A}\sqrt{-|H|}$.

Next, we recall a theorem from the Euclidean geometry of circles which states: the product of the lengths of the segments from an exterior point to the points of intersection of a secant with a circle is equal to the square of the length of the tangent from the point to the circle. (Smart [17].) In Greenberg [6], this product is called the *power of a point* with respect to the circle. Taking the origin, O , as the point and a general circle, $C_{(\gamma, r)}$, we can visualize one case of this theorem as shown in Figure 2.1.

Figure 2.1

In this case, since $OA = |\gamma| - r$ and $OB = |\gamma| + r$, we have

$$(OP)^2 = OA \cdot OB = (|\gamma| - r)(|\gamma| + r) = |\gamma|^2 - r^2.$$

We notice that the value $|\gamma|^2 - r^2$ appears in the lower right corner of matrix (2.3). Thus, this entry represents the power of the origin with respect to the circle $C_{(\gamma, r)}$. If the origin lies outside the circle $C_{(\gamma, r)}$, as in Figure 2.1, then $|\gamma|^2 - r^2$ must be positive. If we have $|\gamma|^2 - r^2 = 0$, then the circle passes through the origin. If the origin lies inside the circle $C_{(\gamma, r)}$, then $|\gamma|^2 - r^2$ will be negative. Now, if we consider the normalized matrix of H as in (2.4), the power of the origin with respect to the circle is given by $\frac{D}{A}$.

The reader may have noticed that, for a Hermitian matrix H , $|H|$ may be positive or zero, giving a circle with complex radius or a radius of zero, respectively. Furthermore, in some Hermitian matrices $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the value of A may be zero. In order to have a correspondence between all Hermitian matrices and geometric objects, we must extend the complex plane to include the point at infinity, “ ∞ ”. The set of points $\mathbb{C} \cup \{\infty\}$ is called the *extended complex plane* or the *inversive plane*.

We can show a correspondence between all normalized Hermitian matrices H , i.e., quadratic forms $(z \ 1) H \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0$, and “circles” or geometric objects by the following classification:

1) If $A \neq 0$, $|H| < 0$, then H represents a real circle. The matrix in (2.2) satisfies all these conditions. We note that its determinant is $|\gamma|^2 - r^2 - |\gamma|^2 = -r^2$. Since r is a real number, we see that $-r^2 < 0$.

2) If $A \neq 0$, $|H| = 0$, then H represents a “point circle”. In this case, we can represent the complex point γ as a circle with radius equal to zero:

$$C_{(\gamma, 0)} = \begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 \end{pmatrix}.$$

3) If $A \neq 0$, $|H| > 0$, then H represents an “imaginary (or virtual) circle”. In such a case, the radius r must be an imaginary number so that the determinant, $-r^2 > 0$. An imaginary circle is usually drawn with center γ and a dashed circumference to indicate the absence of complex points on the circumference of the circle.

4) If $A = 0$, $|H| = -|B|^2 < 0$, then H represents a straight line. In this case, for

$$(z \ 1) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0,$$

the equation $Az\bar{z} + Bz + C\bar{z} + D = 0$ turns out to be a linear equation in x and y with real coefficients. In our classification scheme, we can think of a line as a “circle” of infinite radius. The point at infinity, given by $(1 \ 0)$ in homogeneous form, is always on such a circle because it satisfies the equation.

5) If $A = 0$, $|H| = -|B|^2 = 0$, then $A = B = C = 0$. We may consider this special case as the representation of a “point circle at infinity”. The only point on this circle is the point at infinity, $(1 \ 0)$.

Thus we have a correspondence between all Hermitian matrices and the “circles” or geometric objects described above. It is important to note that if λ is any real number except zero and $H_1 = \lambda H$, then H and H_1 represent the same circle, so the correspondence between 2×2 Hermitian matrices and circles in the inversive plane is unique up to nonzero multiples.

§ 2.2 Pencils of Circles

We now let C_1 and C_2 represent two different circles. That is, $C_1 \neq \lambda C_2$ for any real number λ . If $C_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$, $C_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$, and λ_1, λ_2 are real, not both zero, then $C = \lambda_1 C_1 + \lambda_2 C_2 = \begin{pmatrix} \lambda_1 A_1 + \lambda_2 A_2 & \lambda_1 B_1 + \lambda_2 B_2 \\ \lambda_1 C_1 + \lambda_2 C_2 & \lambda_1 D_1 + \lambda_2 D_2 \end{pmatrix}$ represents a one-parameter family of circles called a *pencil of circles*. Since C is the sum of Hermitian matrices, C is Hermitian itself and hence represents a circle.

All of the circles in a given pencil of circles can be written as real linear combinations of C_1 and C_2 . The circles C_1 and C_2 are called the *generators* of the pencil $\lambda_1 C_1 + \lambda_2 C_2$. We state without proof that any two circles of a given pencil may serve as the generators of the pencil. (See Schwerdtfeger [15], pp. 8-9.)

We note that for any pencil of circles, if $\lambda_1 = -A_2$ and $\lambda_2 = A_1$, then $-A_2 A_1 + A_1 A_2 = 0$. Thus the matrix of $-A_2 C_1 + A_1 C_2$ represents a line. Therefore, each pencil of circles contains exactly one line. This line is called the *radical axis* of the given pencil of circles.

There are three different types of pencils of circles and these are distinguished by the type of intersection or the non-intersection of circles C_1 and C_2 .

1) If C_1 and C_2 intersect at two points, z_1 and z_2 , then $C = \lambda_1 C_1 + \lambda_2 C_2$ is an *elliptic pencil of circles*. All of the circles in this pencil of circles will pass through z_1 and z_2 , including the radical axis. Furthermore, the centers of the circles in the pencil all lie along a particular line called the *line of centers*. (See Figure 2.2.) Notice that the line of centers is perpendicular to the radical axis.

2) If C_1 and C_2 intersect at one point, z_1 , (that is, C_1 and C_2 are tangent at z_1), then $C = \lambda_1 C_1 + \lambda_2 C_2$ is a *parabolic pencil of circles*. As we see in Figure 2.3, all of the

circles in the pencil are tangent at z_1 and the line of centers and the radical axis are perpendicular.

3) If C_1 and C_2 are non-intersecting, then $C = \lambda_1 C_1 + \lambda_2 C_2$ is a *hyperbolic pencil of circles*. In this case, none of the circles in the pencil intersect. We note further that in the hyperbolic pencil, there are two point circles, γ_1 and γ_2 , as in Figure 2.4, and there are virtual circles with centers between γ_1 and γ_2 (not shown). Hyperbolic pencils are the only types containing virtual circles. The radical axis in the hyperbolic pencil is the perpendicular bisector of the segment between γ_1 and γ_2 . It is possible, in the case of a hyperbolic pencil of circles, that virtual circles may serve as one or both generators of the pencil.

Elliptic Pencil of Circles

Figure 2.2

Parabolic Pencil of Circles

Figure 2.3

Hyperbolic Pencil of Circles

Figure 2.4

§ 2.3 Inversion in a Circle

We now turn our attention to the geometric transformation of inversion in a circle. These transformations generate Möbius transformations, and later we will see that they generate all motions in the hyperbolic plane. We will be concerned with inversions in the extended complex plane, that is $\mathbb{C} \cup \{\infty\}$, but most of the discussion of inversion can be demonstrated in a Euclidean context. The following definitions and theorems are adapted from Greenberg [6] and Smart [17].

Definition 2.1: Let $C_{(O,r)}$ be a circle with center O and radius r . For any point $P \neq O$, the *inverse* P' of P with respect to $C_{(O,r)}$ is the unique point P' on ray $\overrightarrow{OP}$ such that $(OP)(OP') = r^2$.

Figure 2.5

The properties listed below follow directly from the definition.

- 1) If P is inside circle $C_{(O,r)}$, then P' lies outside the circle.
- 2) If P is outside circle $C_{(O,r)}$, then P' lies inside the circle.
- 3) $P = P'$ if and only if P lies on the circle of inversion, $C_{(O,r)}$.

It is possible to justify the construction of the inverse point with the following theorems.

Theorem 2.2: Suppose P is inside $C_{(O,r)}$. Let $\overline{TU}$ be the chord of $C_{(O,r)}$ through P which is perpendicular to line $\overline{OP}$. Then the inverse P' of P is the pole of chord $\overline{TU}$, i.e., the point of intersection of the tangents to $C_{(O,r)}$ at T and U . (See Figure 2.6.)

Proof: Suppose the tangent to $C_{(O,r)}$ at T cuts line $\overline{OP}$ at point P' . Right triangle $\triangle OPT$ is similar to right triangle $\triangle OTP'$ since they have $\angle TOP$ in common and the angle sum is 180° . Hence, corresponding sides are proportional. Since $OT = r$, we have $\frac{OP}{r} = \frac{r}{OP'}$, so that $(OP)(OP') = r^2$. By definition, P' is inverse to P . Similarly, we see that the tangent to $C_{(O,r)}$ at U also passes through P' , so P' is indeed the pole of $\overline{TU}$. ♦

Figure 2.6

Theorem 2.3: If P is outside C_o , let Q be the midpoint of the segment $\overline{OP}$. Let C_Q be the circle with center Q and radius $OQ = QP$. Then C_Q cuts C_o in two points T and U , lines $\overline{PT}$ and $\overline{PU}$ are tangent to C_o , and the inverse P' of P is the intersection of $\overline{TU}$ and $\overline{OP}$. (See Figure 2.7.)

Proof: Since circle C_Q has one point inside and one point outside circle C_o , the two circles must intersect in two points, T and U . Since $\angle OTP$ and $\angle OUP$ are inscribed in semicircles of circle C_Q , they are right angles. Hence, $\overline{PT}$ and $\overline{PU}$ are tangent to C_o . If $\overline{TU}$ meets $\overline{OP}$ in a point P' , then P is the inverse of P' by Theorem 2.2. So P' is the inverse of P with respect to circle C_o . ♦

Figure 2.7

It is a fairly simple matter to perform the constructions discussed above using *Geometer's Sketchpad*. Construction tools are available on the side toolbar and also within the pull-down menus. If desired, measurements can be taken and calculations performed to verify the mathematical quantities, as well. One feature of *Geometer's Sketchpad* is the

ability to create “scripts” for constructions. Scripts allow a series of steps to be enacted after certain key objects have been drawn and selected. Since the construction of an inverse point is such a basic operation, scripts entitled Invert Interior Point and Invert Exterior Point have been created. These scripts and others are listed in the Appendix and included on the disk.

We now return to the context of analytic geometry in the extended complex plane. Consider inversion in the circle with center $\gamma = h + ki$ and radius r . If we let (x, y) represent the coefficients of a complex number $z = x + yi$, we can describe a point and its inverse, $z' = x' + y'i$, by the following equations:

$$x' - h = \frac{(x - h)r^2}{(x - h)^2 + (y - k)^2} \quad \text{and} \quad y' - k = \frac{(y - k)r^2}{(x - h)^2 + (y - k)^2}. \quad (2.5)$$

Consider, for example, the inversion of complex points with respect to the unit circle with center at the origin. If $z = a + bi$, then from the equations in (2.5), we have

$$a' = \frac{a}{a^2 + b^2} \quad \text{and} \quad b' = \frac{b}{a^2 + b^2} \quad \text{so that} \quad z' = \frac{a + bi}{a^2 + b^2} = \frac{a + bi}{(a + bi)(a - bi)} = \frac{1}{a - bi} = \frac{1}{\bar{z}}.$$

In general, if a complex point z is inverted with respect to any circle with center γ and radius r , then the image point z' must satisfy the quadratic form

$$(z' - \gamma) \begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix} \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0 \quad (2.6)$$

or the equivalent expression $z'\bar{z} - z'\bar{\gamma} - \bar{z}\gamma + |\gamma|^2 - r^2 = 0$. In this case, we must have

$$z' = \gamma + \frac{r^2}{\bar{z} - \bar{\gamma}}. \quad (2.7)$$

We recall that if a point P is inside the circle of inversion, then its inverse P' lies outside the circle, and vice versa. Thus the interior of the circle will be transformed to

points on the exterior, and the exterior points will all map to the interior. If we suppose $\gamma = h + ki$ is the center of the circle of inversion, then by the equations in (2.5) we find that the inverse of γ is given by $\gamma' = h' + k'i$, where $h' = h + \frac{0}{0^2 + 0^2}$ and $k' = k + \frac{0}{0^2 + 0^2}$.

This poses a problem since we cannot divide by zero. By convention, we agree that the center of the circle and the “point at infinity”, denoted ∞ , are inverses of each other.

With these basic facts in mind, we shall study the transformation of particular geometric objects under inversion. For the following theorems, let $C_{(O, r)}$ be the circle of inversion.

Theorem 2.4: The circle of inversion is an invariant under the transformation of inversion.

Proof: Since $P = P'$ if and only if P is on $C_{(O, r)}$, each point on the circle is its own image. Thus the circle of inversion is pointwise fixed. ♦

Theorem 2.5: Let ℓ be a line through O . The image of ℓ under inversion with respect to $C_{(O, r)}$ is ℓ itself. (See Figure 2.8.)

Proof: If line ℓ passes through O , then it intersects the circle at two points, say P and Q . Since P and Q are on $C_{(O, r)}$, then by definition, $P = P'$ and $Q = Q'$. Consider any other point on line ℓ , say R . Regardless of whether R is inside or outside the circle of inversion, the image R' lies on the line through $\overline{OR}$. Thus line ℓ remains fixed, but not pointwise fixed. ♦

Figure 2.8

Theorem 2.6: Let ℓ be a line not through O . The image of ℓ under inversion with respect to $C_{(O,r)}$ is a circle passing through O . (See Figure 2.9.)

Figure 2.9

Proof: Let P be the foot of the perpendicular from O to ℓ and let Q be any other point on ℓ . Let P and P' and Q and Q' be pairs of inverse points with respect to $C_{(O,r)}$. By the definition of inverse points, $OP \cdot OP' = OQ \cdot OQ'$ or $\frac{OP}{OQ} = \frac{OQ'}{OP'}$. This proportion implies that $\triangle OPQ$ is similar to $\triangle OQ'P'$ since they have $\angle POQ$ in common. Thus $\triangle OQ'P'$ is a

right triangle with its right angle at Q' . Further, Q' is a vertex of a right triangle inscribed in a semicircle, with $\overline{OP'}$ the diameter of the circle. ♦

It is interesting to see how the image of the line ℓ , not through O , changes as ℓ approaches the center of inversion. If ℓ does not intersect $C_{(O,r)}$, then its image does not intersect $C_{(O,r)}$ either. If ℓ is tangent to $C_{(O,r)}$, then its image is also tangent to $C_{(O,r)}$. If ℓ intersects $C_{(O,r)}$ at two points, then its image must intersect $C_{(O,r)}$ at the same two points. (See Figure 2.10.)

Figure 2.10

Theorem 2.7: The image of a circle passing through O , under the transformation of inversion with respect to $C_{(O,r)}$, is a straight line not through O . (see Figure 2.11)

Proof: Let $\overline{OP}$ be a diameter of the circle passing through O , and let Q be any other point on the circle. The argument is now similar to Theorem 2.6. ♦

Figure 2.11

Theorem 2.8: The image of a circle not passing through O , under the transformation of inversion with respect to $C_{(O, r)}$, is a circle not passing through O .

Proof: Let O' be the center of the circle which is to be inverted and assume, without loss of generality, that this circle lies entirely outside $C_{(O, r)}$. (See Figure 2.12.) Let ℓ be a line through O that cuts the circle at points P and Q . If P' and Q' are the inverse points for P and Q , then $OP \cdot OP' = OQ \cdot OQ' = c$, where c is the constant of inversion. By the definition of the power of a point, we also have $OP \cdot OQ = (OA)^2 = k$, where k is a constant equal to the square of the length of the tangent from O to the given circle. Then

$$\frac{OP \cdot OP'}{OQ \cdot OP} = \frac{OQ \cdot OQ'}{OQ \cdot OP} = \frac{c}{OQ \cdot OP} \text{ which can be simplified to } \frac{OP'}{OQ} = \frac{OQ'}{OP} = \frac{c}{k}.$$

Since $\frac{c}{k}$ is a constant, the situation is reduced to a dilation with center O . ◆

Figure 2.12

It is important to make several observations about the properties of inversion as studied thus far. First, we notice in Figure 2.12 that the points $A-P-Q$ are oriented in a clockwise direction on their common circle. The image points $A'-P'-Q'$, however, are oriented in a counterclockwise direction on the image circle. We see therefore that inversion reverses orientation.

Again, using Figure 2.12, we suppose that A and A' are inverse points with respect to the circle of inversion. The inversion takes A to its inverse point A' . If the same inversion is applied again, then the transformation takes A' to its image point A . We see that two successive applications of the inversion result in the identity. Thus, the inversion is called an *involutory transformation* or a *transformation of period two*.

Next, we notice that lines and circles are not invariant, that is to say, they do not always map to the same type of object. If we consider the matrix representation of lines and circles introduced earlier, we recall that both circles and lines can be represented by Hermitian matrices. In that context, we describe a line as a circle of infinite radius. It is appropriate to state therefore, that circles and lines invert to circles and lines.

Another important property is that inversion is *conformal*. Stated more precisely, the measure of the angle between two intersecting curves is an invariant under the transformation of inversion. There are various ways to prove this statement. For two different approaches, see Smart [17] and Greenberg [6]. For our purposes, it will be instructive to demonstrate this property using *Geometer's Sketchpad*. (See Figure 2.13.) First, we define the angle between two intersecting curves to be the angle between the tangents to the curves at the point of intersection. We are able to measure the angle between two curves, say C_1 and C_2 , then compare this measure with the angle between the image curves C_1' and C_2' . Notice again, that since inversion reverses orientation, the direction of the angle is reversed. (Some authors, for example, Schwerdtfeger [15], call this *anti-conformal* or *isogonal*, reserving the word “conformal” for angle preserving maps that also preserve orientation. The Möbius transformations are of this latter type.)

Figure 2.13

§ 2.4 Orthogonal Circles

The conformal property of inversion implies that two curves that meet at right angles have images that meet at right angles. These curves will play a very important part in our discussion of hyperbolic geometry, so we include the following definition and observations.

Definition 2.9: Two circles that meet at right angles are called *orthogonal circles*.

Figure 2.14

Consider Figure 2.14. The angle between C_1 and C_2 is shown as ω , the angle between the tangents to C_1 and C_2 at their point of intersection. If ω is a right angle, then δ is a right angle also since ω and δ are supplementary angles. By the Pythagorean Theorem, δ is a right angle if and only if $d^2 = r_1^2 + r_2^2$, where d is the distance between the centers γ_1 and γ_2 .

Since $d^2 = |\gamma_1 - \gamma_2|^2$, we have $|\gamma_1 - \gamma_2|^2 = r_1^2 + r_2^2$

$$|\gamma_1 - \gamma_2|^2 - r_1^2 - r_2^2 = 0$$

$$|\gamma_1|^2 - \gamma_1 \overline{\gamma_2} - \overline{\gamma_1} \gamma_2 + |\gamma_2|^2 - r_1^2 - r_2^2 = 0$$

$$|\gamma_2|^2 - r_2^2 + |\gamma_1|^2 - r_1^2 - \overline{\gamma_1} \gamma_2 - \gamma_1 \overline{\gamma_2} = 0 \quad (2.8)$$

The terms of equation (2.8) may be substituted from the matrices of circles C_1 and

C_2 , where we have: $C_1 = \begin{pmatrix} 1 & -\overline{\gamma_1} \\ -\gamma_1 & |\gamma_1|^2 - r_1^2 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 1 & -\overline{\gamma_2} \\ -\gamma_2 & |\gamma_2|^2 - r_2^2 \end{pmatrix}$. If we let

H_1 and H_2 represent the circles with the more general form of the Hermitian matrices:

$H_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and $H_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$, then equation (2.8) is $\frac{D_2}{A_2} + \frac{D_1}{A_1} - \frac{B_1 C_2}{A_1 A_2} - \frac{B_2 C_1}{A_1 A_2} = 0$

which reduces to $A_1 D_2 + A_2 D_1 - B_1 C_2 - B_2 C_1 = 0$. Thus, if we know the matrices of two circles, we can easily determine if the two circles are orthogonal circles.

Let us now examine several theorems involving orthogonal circles and inversion.

Theorem 2.10: Let C_O be the circle of inversion and let C_1 be any circle orthogonal to C_O . C_1 is an invariant set of points (but not pointwise fixed) under inversion. Further, the interior of C_1 maps onto itself under inversion. (See Figure 2.15)

Proof: If a line through O intersects circle C_1 at P and Q , then $OP \cdot OQ = (OR)^2$, where R is a point of intersection of the two circles. But since $\overline{OR}$ is also the radius of the circle of inversion, P and Q are by definition inverse points with respect to C_O . For each point of arc RQS , the image under inversion is a point of arc RPS , with the points of intersection R and S remaining fixed. Thus the circle C_1 is an invariant set of points.

Now, suppose that T is inside C_1 and T' is its inverse in C_O . Let r be the radius of C_O . Let the line $\overleftrightarrow{OT}$ cut C_1 at U and V . By the definition of inverse points, we have $(OT) \cdot (OT') = r^2$. By the argument above, U and V are inverse points with respect to C_O , so $(OU) \cdot (OV) = r^2$. Since T is between U and V , we have $OU < OT < OV$. We can substitute and rewrite this inequality as $\frac{r^2}{OV} < \frac{r^2}{OT'} < \frac{r^2}{OU}$. Since taking the reciprocal reverses the sense of the inequality, we have $\frac{OV}{r^2} > \frac{OT'}{r^2} > \frac{OU}{r^2}$, which is also the same as $OV > OT' > OU$. Thus T' lies between U and V and therefore is inside C_1 . ♦

Figure 2.15

We illustrate the next two theorems without proof.

Theorem 2.11: If P and P' are inverse points with respect to the circle of inversion C_O , then any circle through P and P' is orthogonal to C_O . (See Figure 2.16.)

Figure 2.16

Theorem 2.12: If two circles, C_1 and C_2 , are both orthogonal to the circle of inversion C_0 , and C_1 intersects C_2 at P and Q , then P and Q are inverse points with respect to C_0 . (See Figure 2.17.)

Figure 2.17

We recall from § 2.2 that there are three types of pencils of circles: elliptic, parabolic, and hyperbolic. These pencils of circles are generated by a pair of circles belonging to the pencils. Each type of pencil is distinguished by the type of intersection or non-intersection of its circles.

Consider a hyperbolic pencil of circles, $\mathcal{G}_H$, generated by two point circles C_1 and C_2 . As shown in §2.2, every circle in the pencil can be expressed in the form $C = \lambda_1 C_1 + \lambda_2 C_2$, where λ_1 and λ_2 are real. We can show that the centers of the point circles, γ_1 and γ_2 , are mutually inverse with respect to any circle of the pencil.

In matrix form, $C_1 = \begin{pmatrix} 1 & -\overline{\gamma_1} \\ -\gamma_1 & |\gamma_1|^2 \end{pmatrix}$ and $C_2 = \begin{pmatrix} 1 & -\overline{\gamma_2} \\ -\gamma_2 & |\gamma_2|^2 \end{pmatrix}$, respectively. Hence,

a circle in the pencil is given by $C_O = \lambda_1 C_1 + \lambda_2 C_2 = \begin{pmatrix} \lambda_1 + \lambda_2 & -(\lambda_1 \overline{\gamma_1} + \lambda_2 \overline{\gamma_2}) \\ -(\lambda_1 \gamma_1 + \lambda_2 \gamma_2) & \lambda_1 |\gamma_1|^2 + \lambda_2 |\gamma_2|^2 \end{pmatrix}$.

To show that γ_1 and γ_2 are mutually inverse with respect to C_O , we must show, as in

equation (2.6), that $(\gamma_1 \ 1)C_O \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} = 0$. We see

$$\begin{aligned} & (\gamma_1 \ 1) \begin{pmatrix} \lambda_1 + \lambda_2 & -(\lambda_1 \overline{\gamma_1} + \lambda_2 \overline{\gamma_2}) \\ -(\lambda_1 \gamma_1 + \lambda_2 \gamma_2) & \lambda_1 |\gamma_1|^2 + \lambda_2 |\gamma_2|^2 \end{pmatrix} \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} \\ &= (\lambda_1 \gamma_1 + \lambda_2 \gamma_1 - \lambda_1 \gamma_1 - \lambda_2 \gamma_2 \quad -\lambda_1 |\gamma_1|^2 - \lambda_2 \gamma_1 \overline{\gamma_2} + \lambda_1 |\gamma_1|^2 + \lambda_2 |\gamma_2|^2) \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} \\ &= (\lambda_2 \gamma_1 - \lambda_2 \gamma_2 \quad -\lambda_2 \gamma_1 \overline{\gamma_2} + \lambda_2 |\gamma_2|^2) \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} \\ &= \lambda_2 \gamma_1 \overline{\gamma_2} - \lambda_2 |\gamma_2|^2 - \lambda_2 \gamma_1 \overline{\gamma_2} + \lambda_2 |\gamma_2|^2 = 0, \text{ as desired.} \end{aligned}$$

Now we let line ℓ be the radical axis of the pencil. To show that γ_1 and γ_2 are mutually inverse with respect to ℓ , we consider the matrix for ℓ and apply equation (2.6)

again. As a line, the Hermitian matrix for ℓ must be of the form $\ell = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$. Thus we

see in this case, for $\ell = \lambda_1 C_1 + \lambda_2 C_2$, we must have $\lambda_1 + \lambda_2 = 0$. Hence, for $\lambda_1 = 1$ and

$\lambda_2 = -1$, the matrix for ℓ is given by $\ell = C_1 - C_2 = \begin{pmatrix} 0 & -\overline{\gamma_1} + \overline{\gamma_2} \\ -\gamma_1 + \gamma_2 & |\gamma_1|^2 - |\gamma_2|^2 \end{pmatrix}$. Applying

equation (2.6), we have

$$\begin{aligned} (\gamma_1 \ 1) \begin{pmatrix} 0 & -\overline{\gamma_1} + \overline{\gamma_2} \\ -\gamma_1 + \gamma_2 & |\gamma_1|^2 - |\gamma_2|^2 \end{pmatrix} \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} &= (-\gamma_1 + \gamma_2 \quad -|\gamma_1|^2 + \gamma_1\gamma_2 + |\gamma_1|^2 - |\gamma_2|^2) \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} \\ &= (-\gamma_1 + \gamma_2 \quad \gamma_1\gamma_2 - |\gamma_2|^2) \begin{pmatrix} \overline{\gamma_2} \\ 1 \end{pmatrix} = -\gamma_1\overline{\gamma_2} + |\gamma_2|^2 + \gamma_1\gamma_2 - |\gamma_2|^2 = 0, \text{ as desired.} \end{aligned}$$

In Figure 2.18, we illustrate the inversion of γ_1 with respect to C_O and its inverse point γ_2 . Notice that this is equivalent to the Euclidean reflection of γ_1 across the radical axis ℓ . (Recall that the radical axis is indeed the perpendicular bisector of segment $\overline{\gamma_1\gamma_2}$.)

Figure 2.18

We have shown that pairs of inverse points lie on a circle that is orthogonal to the circle of inversion, so γ_1 and γ_2 lie on such a circle with respect to C_O . (See Figure 2.19.) If C_P is any other real circle in the hyperbolic pencil $\mathcal{G}_H$, we know that γ_1 and γ_2 are inverse points with respect to C_P , and that γ_1 and γ_2 lie on a circle that is orthogonal to C_P and C_O . We notice that each of these circles through γ_1 and γ_2 is orthogonal to the radical axis ℓ , as well.

Figure 2.19

We know also that *any* circle through inverse points is orthogonal to the circle of inversion. So if we consider the pencil of all circles through γ_1 and γ_2 , we have an elliptic pencil $\mathcal{G}_E$ with the property that every circle in $\mathcal{G}_E$ is orthogonal to C_O . Thus, we have proved the claim: for every hyperbolic pencil of circles $\mathcal{G}_H$, there is a unique elliptic pencil

of circles $\mathcal{C}_E$, such that every circle of $\mathcal{C}_E$ is orthogonal to every circle of $\mathcal{C}_H$. (See Figure 2.20.)

Figure 2.20

Similarly, for every elliptic pencil of circles $\mathcal{G}_E$ there is a unique hyperbolic pencil of circles $\mathcal{G}_H$, such that every circle of $\mathcal{G}_H$ is orthogonal to every circle of $\mathcal{G}_E$. (Fig. 2.21)

Figure 2.21

And for every parabolic pencil of circles $\mathcal{G}_p$, there is a unique parabolic pencil of circles $\mathcal{G}_p$, such that every circle of $\mathcal{G}_p$ is orthogonal to every circle of $\mathcal{G}_p$. (Figure 2.22)

Figure 2.22

§ 2.5 Additional Properties Under Inversion

Let us consider the effect of inversion on the distance between two points. If C_O is the circle of inversion and $\overline{PQ}$ is a segment which lies entirely inside C_O , we wish to consider the length of segment $\overline{P'Q'}$. (See Figure 2.23.) Since P, P' and Q, Q' are pairs of inverse points, we know $OP \cdot OP' = r^2 = OQ \cdot OQ'$, where r is the radius of C_O . Thus we have the ratio $\frac{OP}{OQ} = \frac{OQ'}{OP'}$. And since $\triangle OPQ$ and $\triangle OQ'P'$ have a common angle at O , we can state that $\triangle OPQ$ and $\triangle OQ'P'$ are similar triangles. So now,

$$\frac{Q'P'}{PQ} = \frac{OP'}{OQ} = \frac{OP'}{OQ} \cdot \frac{OP}{OP} = \frac{r^2}{OP \cdot OQ}.$$

Therefore, $Q'P' = \frac{r^2}{OP \cdot OQ} \cdot PQ$, so we see that distance is not preserved under inversion.

Figure 2.23

We recall from Theorem 2.8, the image of a circle not passing through O is another circle not passing through O . We can show that the center of the first circle does not map to the center of the image circle. (See Figure 2.24.)

Figure 2.24

We assume that O , A , B , and C are collinear. By the previous discussion of

distance under inversion, we have $A'C' = \frac{r^2}{OA \cdot OC} \cdot AC$ and $C'B' = \frac{r^2}{OC \cdot OB} \cdot CB$. Since

AB is a diameter, $AC = CB$, so we can substitute to obtain $C'B' = \frac{r^2}{OC \cdot OB} \cdot AC$. Now

we can compare $A'C' = \frac{r^2 \cdot AC}{OA \cdot OC}$ with $C'B' = \frac{r^2 \cdot AC}{OC \cdot OB}$. Since $OA < OB$, we know

$\frac{1}{OA} > \frac{1}{OB}$, so we must have $A'C' > C'B'$.

We now introduce a definition which will allow us later to define the distance function for hyperbolic geometry. In the next chapter, we shall also see that this definition allows us to distinguish between direct and opposite motions in the hyperbolic plane.

Definition 2.13: Let z_1, z_2, z_3 , and z_4 be four points of the inversive plane, $\mathbb{C} \cup \{\infty\}$, such that no three are equal. We define the *cross ratio* of the two pairs $z_1, z_2; z_3, z_4$ by

$$(z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_4 - z_1} + \frac{z_3 - z_2}{z_4 - z_2} = \frac{z_3 - z_1}{z_4 - z_1} \cdot \frac{z_4 - z_2}{z_3 - z_2}.$$

It can be shown that the cross ratio of four points $(z_1, z_2; z_3, z_4)$ is real if and only if the four points lie on a common circle or on a straight line. For two different proofs of this theorem, see Schwerdtfeger [15], pp 36-37. The next theorem shows an important relationship between the cross ratio of four points and the cross ratio of their images under inversion.

Theorem 2.14: Let z'_j be the inverse point of z_j , for $j = 1, 2, 3, 4$, with respect to a circle C_O . Then $(z'_1, z'_2; z'_3, z'_4) = \overline{(z_1, z_2; z_3, z_4)}$.

Proof: Let circle C_O have center γ and radius r . Then by equation (2.7), we have

$$z'_j = \gamma + \frac{r^2}{\overline{z_j - \gamma}}. \text{ Furthermore, } z'_j - z'_k = \gamma + \frac{r^2}{\overline{z_j - \gamma}} - \left(\gamma + \frac{r^2}{\overline{z_k - \gamma}} \right) = r^2 \frac{\overline{z_k - z_j}}{(\overline{z_j - \gamma})(\overline{z_k - \gamma})}.$$

So we see

$$\begin{aligned} (z'_1, z'_2; z'_3, z'_4) &= \frac{r^2 \frac{\overline{z_1 - z_3}}{(\overline{z_3 - \gamma})(\overline{z_1 - \gamma})}}{r^2 \frac{\overline{z_1 - z_4}}{(\overline{z_4 - \gamma})(\overline{z_1 - \gamma})}} \cdot \frac{r^2 \frac{\overline{z_2 - z_4}}{(\overline{z_4 - \gamma})(\overline{z_2 - \gamma})}}{r^2 \frac{\overline{z_2 - z_3}}{(\overline{z_3 - \gamma})(\overline{z_2 - \gamma})}} \\ &= \frac{(\overline{z_1 - z_3})}{(\overline{z_1 - z_4})} \cdot \frac{(\overline{z_2 - z_4})}{(\overline{z_2 - z_3})} = \frac{(\overline{z_3 - z_1})}{(\overline{z_4 - z_1})} \cdot \frac{(\overline{z_4 - z_2})}{(\overline{z_3 - z_2})} = \overline{(z_1, z_2; z_3, z_4)}. \quad \blacklozenge \end{aligned}$$

We conclude this section with a return to an earlier concept. In Theorem 2.2 we defined the *pole* of chord $\overline{TU}$ as the point of intersection of the tangents to the circle of inversion at T and U . Let us consider an alternate description.

Definition 2.15: Let C_O be the circle of inversion with center O , and let P and P' be a pair of inverse points. The line through P' and perpendicular to line $\overline{OP}$ is called the *polar* of P , whereas the point P is called the *pole* of the line. A similar definition could be stated such that the line through P and perpendicular to line $\overline{OP'}$ is the polar line of P' , where P' is the pole of the line.

There is a reciprocal relation between poles and polars which is stated in the next theorem.

Theorem 2.16: If point Q' is on the polar line of point P , with respect to a given circle of inversion, then P is on the polar line of Q' with respect to the same circle.

Proof: Let C_O be the circle of inversion, with P and P' a pair of inverse points. If line $\overline{P'Q'}$ is the polar of P , then we must show that the polar of Q' passes through P . If we let point Q be the inverse point of Q' , then all we need to show is that line $\overline{QP}$ is perpendicular to line $\overline{OQ'}$. Since Q is the inverse point of Q' , then as previously shown in this section, $\triangle OP'Q'$ and $\triangle OQP$ are similar triangles. Since $\triangle OP'Q'$ has a right angle at P' , we see that $\triangle OQP$ has a right angle at Q , as desired. (See Figure 2.25.) ♦

It is of further interest to note that the points P , P' , Q , and Q' all lie on a circle, with PQ' as diameter. Thus, pairs of inverse points with respect to a common circle of inversion will have a real cross ratio.

Figure 2.25

Chapter 3

The Möbius Transformations

§ 3.1 Definitions and Notation

The transformations known as Möbius transformations, also called a linear fractional transformations, will be very important to our investigation of hyperbolic geometry in the Poincaré models. In fact, they will form the direct (orientation preserving) motions of translations, rotations and horolations in the models. The following theorems and examples are adapted from Schwerdtfeger [15], Greenleaf [7], and Hahn [8].

Definition 3.1: A Möbius transformation is given by

$$z' = f(z) = \frac{Az + B}{Cz + D}, \quad (3.1)$$

where A, B, C, D are any complex numbers with $AD - BC \neq 0$, and z may vary throughout $\mathbb{C} \cup \{\infty\}$. We may also represent the Möbius transformation in (3.1) by the matrix product

$$(z \ 1) \begin{pmatrix} A & C \\ B & D \end{pmatrix} = (Az + B \ Cz + D), \quad (3.2)$$

where the right-hand side of equation (3.2) is equivalent in homogeneous coordinates to

$$\begin{pmatrix} \frac{Az + B}{Cz + D} & 1 \end{pmatrix} = (z' \ 1). \text{ For convenience, we shall use both methods of notation to}$$

represent Möbius transformations throughout our discussion.

The matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ defines the Möbius transformation up to a constant factor, so

for any complex nonzero λ , $\lambda \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ represents the same Möbius transformation. Like

inversion, the Möbius transformation is a one-to-one and onto mapping from $\mathbb{C} \cup \{\infty\}$ to itself. We note that if $f(z) = (z \ 1) \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is one-to-one, then $AD - BC \neq 0$, and conversely.

We examine some particular values and their images under a general Möbius transformation, as in (3.1):

$$\text{a) } f(0) = \frac{A(0) + B}{C(0) + D} = \frac{B}{D};$$

$$\text{b) } f(\infty) = \frac{A(\infty) + B}{C(\infty) + D} = \lim_{x \rightarrow \infty} \left(\frac{Ax + B}{Cx + D} \right) = \lim_{x \rightarrow \infty} \frac{x}{x} \left(\frac{A + \frac{B}{x}}{C + \frac{D}{x}} \right) = \frac{A}{C};$$

$$\text{c) } f\left(-\frac{B}{A}\right) = \frac{A\left(-\frac{B}{A}\right) + B}{C\left(-\frac{B}{A}\right) + D} = 0;$$

$$\text{d) } f\left(-\frac{D}{C}\right) = \frac{A\left(-\frac{D}{C}\right) + B}{C\left(-\frac{D}{C}\right) + D} = \infty.$$

We call the complex number $z = -\frac{D}{C}$ the *pole* of the function f since $f\left(-\frac{D}{C}\right) = \infty$.

(This has nothing to do with the terms pole and polar discussed in Chapter 2.) We call the complex number $z = \frac{A}{C}$ the *inverse pole* of f since $f(\infty) = \frac{A}{C}$ or alternately, $f^{-1}\left(\frac{A}{C}\right) = \infty$.

We will be interested in knowing the *fixed points* of a Möbius transformation. We exclude here the case of the identity transformation which maps every point to itself. If a point z is a fixed point of a general Möbius transformation, then $f(z) = z$, or by equation

(3.1), $z = \frac{Az + B}{Cz + D}$. Solving for z , we have $Cz^2 + (D - A)z - B = 0$. Now, by the

quadratic formula,
$$z = \frac{A - D \pm \sqrt{(D - A)^2 + 4BC}}{2C}, \quad C \neq 0. \quad (3.3)$$

Thus, a Möbius transformation has, in general, two fixed points, call them γ_1 and γ_2 .

If a Möbius transformation has $C = 0$, then the fixed points must satisfy

$z = \frac{Az + B}{D}$. Solving for z , we have $z = \frac{B}{D - A}$. For such a case, we find that there is

just one fixed point. In this case, we say that the two fixed points coincide, with $\gamma_1 = \gamma_2$.

The fixed points of a Möbius transformation, γ_1 and γ_2 , the pole, $-\frac{D}{C}$, and the inverse

pole, $\frac{A}{C}$, form the vertices of the important fundamental parallelogram which will be

discussed later. (See Schwerdtfeger [15].)

§ 3.2 The Group of All Möbius Transformations

Theorem 3.2: The system of all Möbius transformations is a group with functional composition as group multiplication.

Proof: Let f_1 and f_2 be two Möbius transformations with matrices $M_1 = \begin{pmatrix} A_1 & C_1 \\ B_1 & D_1 \end{pmatrix}$ and

$M_2 = \begin{pmatrix} A_2 & C_2 \\ B_2 & D_2 \end{pmatrix}$, respectively. If we compose these transformations so that $f_2(f_1(z)) = z'$,

we see that the product under functional composition

$$f_2(f_1(z)) = \frac{A_2(f_1(z)) + B_2}{C_2(f_1(z)) + D_2} = \frac{(A_2A_1 + B_2C_1)z + A_2B_1 + B_2D_1}{(C_2A_1 + D_2C_1)z + C_2B_1 + D_2D_1} = z'$$

is also a Möbius transformation. The matrix of $f_2(f_1(z))$ is the product of M_1 and M_2 :

$$M_3 = M_2 M_1 = \begin{pmatrix} A_2 A_1 + B_2 C_1 & C_2 A_1 + D_2 C_1 \\ A_2 B_1 + B_2 D_1 & C_2 B_1 + D_2 D_1 \end{pmatrix}. \text{ Since } |M_1| \neq 0 \text{ and } |M_2| \neq 0, \text{ we have}$$

$$|M_3| = |M_2| |M_1| \neq 0.$$

The general Möbius transformation given by (3.1) is invertible. Given

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \text{ we know } M^{-1} = \frac{1}{|M|} \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \text{ Since we can omit the factor } \frac{1}{|M|}, \text{ we see}$$

that the inverse function $z = f^{-1}(z') = \frac{D(z') - B}{-C(z') + A}$ is also a Möbius transformation.

The identity transformation $z' = z$ is given by the unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. And

finally, since matrix multiplication is associative, we can state that the same is true for the composition of three or more Möbius transformations. ♦

§ 3.3 Simple Types of Möbius Transformations

1) Translation

Consider a Möbius transformation given by $f(z) = z + b$, where $b \neq 0$ is complex.

The matrix for such a Möbius transformation is given by the general form $M_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$.

This transformation may be recognized as the translation of every point in the complex

plane by the translation vector $\vec{b}$. If we consider $f(\infty) = \infty + b = \infty$, we see that the point at infinity remains a fixed point under translation.

We may illustrate the translation given by M_b as the product of two inversions.

Recall that reflection across a line corresponds to inversion in a circle. (See Figure 3.1.)

Figure 3.1

Notice that each straight line parallel to the translation vector remains fixed, but not pointwise fixed. The lines that are perpendicular to the translation vector constitute a pencil of lines that intersect only at the point at infinity, and as such may be viewed as a parabolic pencil. If we take two lines ℓ_1 and ℓ_2 perpendicular to $\vec{b}$, such that the distance between them is one-half the length of $\vec{b}$, then $z' = \ell_2 \circ \ell_1(z)$. The lines of every other pencil of parallel straight lines are mapped to lines in the same pencil according to the magnitude and direction of the translation vector, as well.

2) Rotation

Next, we consider rotation about the origin by the angle θ . In this case, $B = C = 0$ and $\frac{A}{D} = k$ where $|k| = 1$. Thus the matrix for this transformation is given by the general form $M_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$. Since $z' = e^{i\theta} z$, we see that 0 and ∞ are two fixed points under this rotation. Every rotation M_θ is the product of two inversions. (See Figure 3.2.)

Figure 3.2

We choose the real axis for the first inversion. Then we invert with respect to the line through 0 which makes the angle $\frac{1}{2}\theta$ with the real axis. Every circle with center 0 remains invariant, but not pointwise invariant. All straight lines through 0 are rotated by θ .

3) Dilation

The Möbius transformation given by $z' = kz$ represents dilation about the origin by the factor k , where $k > 0$. The matrix representation is given by $M_k = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. In this case, $B = C = 0$, and $\frac{A}{D} = k$ where $|k| \neq 1$. There are two fixed points under the given dilation, 0 and ∞ . We show that every dilation M_k is the product of two inversions. (See Figure 3.3.)

Figure 3.3

Let $f_1(z)$ be inversion with respect to the unit circle centered at the origin. Then $f_1(z) = \frac{1}{\bar{z}}$. For $f_2(z)$, choose the circle with radius $\sqrt{k}$ centered at the origin. Then

$z' = f_2(f_1(z)) = f_2\left(\frac{1}{\bar{z}}\right) = \frac{k}{\left(\frac{1}{\bar{z}}\right)} = kz$. We see that every line through the origin is invariant,

but not pointwise invariant. The circles centered at 0 map to other circles centered at 0.

4) Reciprocation

Finally, we consider the case when $A = D = 0$ and $B = C$. The matrix for this Möbius transformation may be reduced to the form $M_R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We see $z' = f(z) = \frac{1}{z}$, which is the transformation of reciprocation. The two fixed points of this Möbius transformation are 1 and -1 . Let f_1 be the inversion with respect to the unit circle centered at the origin. Let f_2 be reflection across the real axis, that is ordinary complex conjugation. Then we have $z' = f_2(f_1(z)) = f_2\left(\frac{1}{\bar{z}}\right) = \frac{1}{z}$. Also, $z' = f_1(f_2(z)) = f_1(\bar{z}) = \frac{1}{z}$. (Figure 3.4.)

Figure 3.4

Every circle through the points 1 and -1 is mapped onto itself. We recall that this is an example of an elliptic pencil of circles. Further, every circle of the orthogonal hyperbolic pencil is mapped onto itself.

Any composition of these simple types of Möbius transformations in any order yields another Möbius transformation. This is evident as the product of their matrix representations. A more useful result is presented in the following theorem.

Theorem 3.3: Every Möbius transformation can be written as a composition of at most seven simple types of Möbius transformations.

Proof: If $C \neq 0$, then we can write $f(z) = \frac{Az + B}{Cz + D} = \frac{A}{C} - \frac{AD - BC}{C} \cdot \frac{1}{Cz + D}$, which

expresses f in the form $f = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, where $f_1(z) = Cz$, $f_2(z) = z + D$,

$f_3(z) = \frac{1}{z}$, $f_4(z) = -\left(\frac{AD - BC}{C}\right)z$, and $f_5(z) = z + \frac{A}{C}$. But for both f_1 and f_4 , if the

coefficients of z are complex (nonreal), then each may be written as a dilation about 0 composed with a rotation about 0. Hence $f(z)$ is a composition of at most seven simple types of Möbius transformations.

If $C = 0$, then $f(z) = \frac{A}{D}z + \frac{B}{D}$. In this case, $f = f_2 \circ f_1$, where $f_1(z) = \frac{A}{D}z$ and

$f_2(z) = z + \frac{B}{D}$. As noted above, f may be written as the composition of a dilation about 0

and a rotation about 0. So in this case, $f(z)$ is a composition of at most three simple types of Möbius transformations. ◆

§ 3.4 Mapping Properties of Möbius Transformations

Theorem 3.4: Every Möbius transformation is a conformal mapping of $\mathbb{C} \cup \{\infty\}$ onto itself. It maps circles and lines to circles and lines.

Proof: We have shown that every simple type of Möbius transformation can be written as the composition of two inversions. By Theorem 3.3, every Möbius transformation can be written as the composition of the simple types. Since a single inversion preserves angles between curves but reverses orientation, pairs of inversions preserve both angles between curves and orientation. Thus every Möbius transformation is a conformal mapping. Similarly, by the theorems in § 2.3, inversions map circles and lines to circles and lines, so the same is true for Möbius transformations. ♦

We now consider a more detailed approach utilizing the analytic geometry of circles from § 2.1. We recall that if H is the Hermitian matrix representing a generalized circle (i.e., a real circle, line, imaginary circle, or point circle) then the set of complex points z on the circle belong to the set $\left\{ z : (z \ 1) H \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0 \right\}$. If M is the matrix of a Möbius transformation, then for any point z in the set above, $(z \ 1)M = \lambda(z' \ 1)$. We can adjust the constant in M so that $(z \ 1)M = (z' \ 1)$.

We solve for $(z \ 1)$: $(z \ 1)MM^{-1} = \lambda(z' \ 1)M^{-1}$ which gives

$(z \ 1) = \lambda(z' \ 1)M^{-1}$. Furthermore, solving for $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}$, we have $\overline{(z \ 1)}^T = \bar{\lambda} \overline{(z' \ 1)M^{-1}}^T$,

so that $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = \bar{\lambda} \overline{M^{-1}}^T \begin{pmatrix} \bar{z}' \\ 1 \end{pmatrix}$. Substituting these expressions into the quadratic form for H

above, we have $(z \ 1) H \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = |\lambda|^2 (z' \ 1) M^{-1} H \overline{M^{-1}}^T \begin{pmatrix} \bar{z}' \\ 1 \end{pmatrix} = 0$. Thus, the image of the circle given by H , is $H' = M^{-1} H \overline{M^{-1}}^T$. We notice that if we take the conjugate transpose of H' , we have $\overline{M^{-1} H \overline{M^{-1}}^T}^T = M^{-1} \overline{H}^T \overline{M^{-1}}^T = M^{-1} H \overline{M^{-1}}^T$. Thus H' is Hermitian and represents another generalized circle.

If H is written in the form $\begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix}$, we recall that $|H| = -r^2$ determines the type of circle (or line) represented by H . For comparison, we calculate $|H'|$:

$$|H'| = \left| M^{-1} H \overline{M^{-1}}^T \right| = |M^{-1}| |H| \left| \overline{M^{-1}}^T \right| = |M^{-1}|^2 |H|.$$

Now, even though $|M^{-1}|$ and $\left| \overline{M^{-1}}^T \right|$ may be complex numbers, $|M^{-1}|^2$ represents a positive real number. Hence, H and H' represent the same type of circle (or line.) To be precise, a Möbius transformation maps real circles and lines to real circles and lines, imaginary circles to imaginary circles, and point circles to point circles.

Theorem 3.5: Every Möbius transformation maps pencils of circles to pencils of circles.

Proof: Let $H = \lambda_1 H_1 + \lambda_2 H_2$ be a general element of a pencil of circles. The image of H under the Möbius transformation given by M is given by

$$H' = M^{-1} H \overline{M^{-1}}^T = \lambda_1 \left(M^{-1} H_1 \overline{M^{-1}}^T \right) + \lambda_2 \left(M^{-1} H_2 \overline{M^{-1}}^T \right) = \lambda_1 H'_1 + \lambda_2 H'_2.$$

Since H' can be written in the appropriate form, we see that Möbius transformations preserve pencils. ◆

Theorem 3.6: Every Möbius transformation maps pairs of inverse points to pairs of inverse points.

Proof: Let z and z^* be inverse points with respect to the circle given by matrix H . By

equation (2.6) in § 2.3, $(z^* \ 1) H \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = 0$. If M is the matrix of a Möbius transformation,

then the image of z^* under M is given by $(z^* \ 1)M = \lambda(z'^* \ 1)$. We solve for $(z'^* \ 1)$:

$(z'^* \ 1) = \lambda(z^* \ 1)M^{-1}$. The image of z under M is given by $(z \ 1)M = \mu(z' \ 1)$.

Solving for $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}$, we have $\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = \bar{\mu} \overline{M^{-1}}^T \begin{pmatrix} \bar{z}' \\ 1 \end{pmatrix}$. Substituting into the quadratic form above,

we have $0 = (z^* \ 1) H \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = \lambda(z'^* \ 1) M^{-1} H \overline{M^{-1}}^T \begin{pmatrix} \bar{z}' \\ 1 \end{pmatrix} \bar{\mu}$, so $(z'^* \ 1) H' \begin{pmatrix} \bar{z}' \\ 1 \end{pmatrix} = 0$, where

$H' = M^{-1} H \overline{M^{-1}}^T$. Thus, z' and z'^* are inverses with respect to H' , as desired. ♦

Theorem 3.7: Let z_1, z_2, z_3 , and z_4 be any four distinct points in $\mathbb{C} \cup \{\infty\}$. Let $f(z)$ be a Möbius transformation and $w_j = f(z_j)$ for $j = 1, 2, 3, 4$. Then the cross ratio is invariant. That is, $(w_1, w_2; w_3, w_4) = (z_1, z_2; z_3, z_4)$.

Proof: If the four points z_j are transformed into w_j by an inversion, then their cross ratio is taken into its conjugate value by Theorem 2.14. Thus the cross ratio is invariant under a product of two or any even number of inversions. ♦

Theorem 3.8: A Möbius transformation is uniquely determined by any three points and their images in $\mathbb{C} \cup \{\infty\}$.

Proof: Let z_1, z_2, z_3 be three distinct points in $\mathbb{C} \cup \{\infty\}$ and let w_1, w_2, w_3 be three distinct points in $\mathbb{C} \cup \{\infty\}$. The Möbius transformation $f(z)$, by which $w_j = f(z_j)$ for $j = 1, 2, 3$ may be determined by solving $(z_1, z_2; z_3, z) = (w_1, w_2; w_3, w)$ for w with respect to z .

We have: $\frac{z_3 - z_1}{z - z_1} \cdot \frac{z - z_2}{z_3 - z_2} = \frac{w_3 - w_1}{w - w_1} \cdot \frac{w - w_2}{w_3 - w_2}$ which simplifies to

$$k \cdot \frac{z - z_2}{z - z_1} = c \cdot \frac{w - w_2}{w - w_1} \quad \text{where } k = \frac{z_3 - z_1}{z_3 - z_2} \text{ and } c = \frac{w_3 - w_1}{w_3 - w_2}.$$

$$\text{Thus } w = \frac{(kw_1 - cw_2)z + (cw_2z_1 - kw_1z_2)}{(k - c)z + cz_1 - kz_2} = \frac{Az + B}{Cz + D}.$$

In order to show we have a Möbius transformation, we must show that $AD - BC \neq 0$.

$$\begin{aligned} \text{We calculate } AD - BC &= (kw_1 - cw_2)(cz_1 - kz_2) - (cw_2z_1 - kw_1z_2)(k - c) \\ &= ck(w_1z_1 + w_2z_2 - w_2z_1 - w_1z_2) \\ &= ck(w_1 - w_2)(z_1 - z_2). \end{aligned} \quad (3.4)$$

Since each of the w_i and each of the z_i for $i = 1, 2, 3$, are distinct, each of the factors in (3.4) are nonzero. Hence $AD - BC \neq 0$, and the mapping is a Möbius transformation. ♦

Corollary 3.9: Every transformation $z' = f(z)$ of $\mathbb{C} \cup \{\infty\}$ onto itself, that has the cross ratio as a four point invariant, is necessarily a Möbius transformation. ♦

Theorem 3.10: A Möbius transformation with three or more fixed points must be the identity transformation.

Proof: If a Möbius transformation f has three fixed points, the identity has the same three fixed points. Therefore, by Theorem 3.8, since f is uniquely determined, f is equivalent to the identity transformation. ♦

§ 3.5 Involutions

Definition 3.11: Let $i(z)$ represent the identity transformation. If $f(z)$ is a Möbius transformation such that $f \neq i$ but $f^2 = i$, then we call $f(z)$ an *involution*.

The Möbius transformation given by the matrix $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is an involution if

and only if $(z \ 1) \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \lambda(z \ 1)$ for all z in the extended complex plane. This is true if and only if the product

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A^2 + CB & AC + CD \\ BA + BD & BC + D^2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix};$$

and this implies both $(A + D)C = 0$ and $(A + D)B = 0$. We consider the cases:

1) $A + D = 0$ and 2) $A + D \neq 0$ and $B = C = 0$.

In Case 2, if $B = C = 0$, then $\begin{pmatrix} A^2 & 0 \\ 0 & D^2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, which implies $A^2 = D^2$.

Then we have $(A + D)(A - D) = 0$ so that either $A = -D$, as in Case 1, or $A = D$. Now if

$A = D$ then the original matrix is $M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. But, by definition, the involution f is not

the identity. So M represents the matrix of an involution only when $A = -D$. The converse is easy to see. Thus we have the following theorem.

Theorem 3.12: The Möbius transformation f is an involution if and only if $A = -D$.

Thus the matrix of an involution is of the form $M = \begin{pmatrix} A & C \\ B & -A \end{pmatrix}$.

Suppose that f is an involution with matrix $M = \begin{pmatrix} A & C \\ B & -A \end{pmatrix}$. To determine the fixed points we consider $Cz^2 - 2Az - B = 0$. Solving for z , we have $z = \frac{2A \pm \sqrt{4A^2 + 4BC}}{2C}$.

Since the discriminant, $-4|B|$, is not zero, this equation must have two roots. Thus an involution must always have two distinct fixed points.

Theorem 3.13: Suppose f is a Möbius transformation that interchanges two distinct points. That is to say, $f(z_1) = z_2$ and $f(z_2) = z_1$ where $z_1 \neq z_2$. Then f is an involution.

Proof: Let $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ be the matrix representation of f . Then we have

$$\begin{pmatrix} z_1 & 1 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} \lambda z_2 & \lambda \\ \mu z_1 & \mu \end{pmatrix} \text{ for some } \lambda \neq 0, \mu \neq 0. \text{ To solve for } M, \text{ we multiply by the}$$

inverse of $\begin{pmatrix} z_1 & 1 \\ z_2 & 1 \end{pmatrix}$ as shown:

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \frac{1}{z_1 - z_2} \begin{pmatrix} 1 & -1 \\ -z_2 & z_1 \end{pmatrix} \begin{pmatrix} \lambda z_2 & \lambda \\ \mu z_1 & \mu \end{pmatrix} = \frac{1}{z_1 - z_2} \begin{pmatrix} \lambda z_2 - \mu z_1 & \lambda - \mu \\ -\lambda z_2^2 + \mu z_1^2 & -\lambda z_2 + \mu z_1 \end{pmatrix}.$$

Since $A + D = \lambda z_2 - \mu z_1 - \lambda z_2 + \mu z_1 = 0$, f must be an involution. ◆

We recall from Theorem 3.7 that the cross ratio is a four point invariant for a given Möbius transformation. Suppose that f is an involution with $f(z_1) = z_1$, $f(z_2) = z_2$, and $f(z_3) = z_4$ for four distinct points z_1, z_2, z_3 , and z_4 . By Theorem 3.7 we have

$$(z_1, z_2; z_3, z_4) = (z_1, z_2; z_4, z_3). \text{ If we let } (z_1, z_2; z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1} = k, \text{ then we see}$$

$$\text{that } (z_1, z_2; z_4, z_3) = \frac{z_4 - z_1}{z_4 - z_2} \cdot \frac{z_3 - z_2}{z_3 - z_1} = \frac{1}{k}.$$

Now $k = \frac{1}{k}$ if and only if $k^2 = 1$, which implies that $k = \pm 1$. If $k = 1$, then we must have $z_1 = z_2$ or $z_3 = z_4$ which violates the requirement that these four points are distinct. Thus $k = -1$ is the only possible value for the common cross ratio. Any set of four distinct points z_1, z_2, z_3 , and z_4 with $(z_1, z_2; z_3, z_4) = -1$ is called a *harmonic set*. The points z_3 and z_4 are called *harmonic conjugates* with respect to z_1 and z_2 .

From these results we state the following theorems:

Theorem 3.14: If f is an involution with distinct fixed points z_1 and z_2 , then for any other point z_3 , we have $f(z_3) = z'_3 \neq z_3$ and $(z_1, z_2; z_3, z'_3) = -1$.

Theorem 3.15: Given any harmonic set of distinct points z_1, z_2, z_3 , and z_4 , there exists a unique involution f such that $f(z_1) = z_1$, $f(z_2) = z_2$, $f(z_3) = z_4$, and $f(z_4) = z_3$.

Theorem 3.16: Any Möbius transformation that is not an involution may be written as the product of two involutions.

Proof: Suppose a Möbius transformation f is not an involution and not the identity transformation. Let $f(z_1) = z_2$, and $f(z_2) = z_3$, where $z_3 \neq z_1$. Let h be an involution such that $h(z_2) = z_2$, $h(z_1) = z_3$, and $h(z_3) = z_1$. Then we have $h \circ f(z_1) = h(z_2) = z_2$ and $h \circ f(z_2) = h(z_3) = z_1$. Thus $h \circ f$ interchanges z_1 and z_2 , so $h \circ f$ is an involution. If we let $h \circ f = j$, then $f = h^{-1} \circ j = h \circ j$. Thus f is a product of involutions. ♦

§ 3.6 Similarity and Classification of the Möbius Transformations

Let matrices A and B be two elements of $M_{2 \times 2}(\mathbb{C})$. We say that A and B are similar matrices if there exists an invertible matrix Q in $M_{2 \times 2}(\mathbb{C})$ such that $Q A Q^{-1} = B$. In terms of group theory, we recognize the elements A and B as conjugates. That is, if matrices A and B are similar, then we can state that they are in the same conjugacy class.

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix.

The process of diagonalizing the matrix $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ of a general Möbius transformation

involves the solution of the characteristic polynomial $\lambda^2 - (A + D)\lambda + AD - BC = 0$. We

notice that this equation can be restated as $\lambda^2 - (\text{trace } M)\lambda + |M| = 0$. In this form, we

solve for λ : $\lambda = \frac{\text{trace } M \pm \sqrt{(\text{trace } M)^2 - 4|M|}}{2}$. We know that there are two distinct

roots if and only if $(\text{trace } M)^2 - 4|M| \neq 0$. We define $\sigma(M) = \frac{(\text{trace } M)^2}{|M|} - 4 \neq 0$ to be the

magnitude of the matrix M . The value $\sigma(M)$ is an invariant under the conjugacy class of M .

It is important to require the existence of two distinct roots if M is similar to a nontrivial diagonal matrix D . The roots of the characteristic polynomial are the eigenvalues which appear along the diagonal of D . Let us suppose that there are two distinct roots of the characteristic polynomial, say λ_1 and λ_2 . The matrix M is similar to the diagonal

matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ which, as a Möbius transformation, is equivalent to $\begin{pmatrix} \frac{\lambda_1}{\lambda_2} & 0 \\ 0 & 1 \end{pmatrix}$. We

call the ratio $\frac{\lambda_1}{\lambda_2} = k$ the *characteristic constant*. Using this value k in the matrix above, we

see that M is similar to $D = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. We calculate $\sigma(M)$ in term of k :

$$\sigma(M) = \sigma(D) = \frac{(k+1)^2}{k} - 4 = \frac{k^2 + 2k + 1}{k} - 4 = k + \frac{1}{k} - 2.$$

We define the conjugacy (or equivalence) classes of Möbius transformations according to the value of k as follows:

1) Proper Hyperbolic

If k is real and $k > 0$, then M is similar to $D = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. (Note that $k \neq 1$ since this is the identity.) For positive real values of $k \neq 1$, we have $\sigma(M) = \sigma(D) = k + \frac{1}{k} - 2 > 0$.

Thus a proper hyperbolic Möbius transformation is similar to a dilation as described in §3.3, (3).

2) Improper Hyperbolic

If k is real and $k < 0$, then M is similar to $D = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. In this case,

$\sigma(M) = \sigma(D) = k + \frac{1}{k} - 2 \leq -4$. We consider in particular the case $k = -1$ for which

$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\text{trace } D = -1 + 1 = 0$, by Theorem 3.12, this Möbius transformation

is an involution. Except for the case in which $k = -1$, the Improper Hyperbolic Möbius transformations will not appear in the models for hyperbolic geometry.

3) Elliptic

If k is imaginary and $k = e^{i\theta}$, then M is similar to $D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$. For these values,

with $\sigma(M) = \sigma(D)$, we can verify that $-4 < \sigma(M) < 0$. We see that an elliptic Möbius transformation is similar to a rotation as described in § 3.3 (2).

4) Loxodromic

If k is imaginary and $k = re^{i\theta}$, where $r \neq \pm 1$, then M is similar to $D = \begin{pmatrix} re^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}$.

For these values of k , the magnitude, $\sigma(M) = \sigma(D)$, is not real. We see also that

$$D = \begin{pmatrix} re^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}. \text{ Thus a loxodromic Möbius}$$

transformation may be written as the composition of a proper hyperbolic and elliptic transformation, with these two factors being commutative.

5) Parabolic

If the magnitude $\sigma(M) = \frac{(\text{trace } M)^2}{|M|} - 4 = 0$, and $M \neq aI$, then M is not equivalent

(as a Möbius transformation) to the identity. We know that there is only one solution to the

equation $\lambda = \frac{\text{trace } M \pm \sqrt{(\text{trace } M)^2 - 4|M|}}{2}$. This corresponds to the existence of one

fixed point for the Möbius transformation given by M . In this case, we cannot diagonalize

the matrix M , but we can reduce it to a lower triangular matrix $T = \begin{pmatrix} \lambda_1 & 0 \\ x & \lambda_1 \end{pmatrix}$ which is

equivalent (as a Möbius transformation) to $\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$. Referring again to the simple types of

Möbius transformations in §3.2, we recognize this form as a translation. Thus every parabolic Möbius transformation is similar to a translation.

6) Identity

We shall consider the identity transformation given by $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or any nonzero multiple of M to be in a distinct equivalence class. We note that for the identity matrix,

$$\sigma(M) = \frac{(1+1)^2}{1} - 4 = 0.$$

We recall some important facts about equivalence classes. All transformations that are similar to a particular matrix are similar to each other. Every element in a class can be obtained from any one element in the class by means of a suitable similarity transformation. Every Möbius transformation is contained in one and only one class. Two different classes have no elements in common.

§ 3.7 The Characteristic Parallelogram

Suppose $f(z) = \frac{Az + B}{Cz + D}$ is a Möbius transformation. We have already shown that a Möbius transformation f other than the identity has at most two fixed points, γ_1 and γ_2 . As shown in §3.1, if we assume $C \neq 0$, then

$$\gamma_1, \gamma_2 = \frac{A - D \pm \sqrt{(D - A)^2 + 4BC}}{2C}.$$

(Recall that if $C = 0$, then $f(z) = \frac{Az + B}{D}$. In this case, f is a translation with one fixed point, $\gamma_1 = \gamma_2 = \infty$.)

Let us assume that $C \neq 0$ and that f has two distinct fixed points. Consider

$$\gamma_1 + \gamma_2 = \frac{A - D + \sqrt{(D - A)^2 + 4BC}}{2C} + \frac{A - D - \sqrt{(D - A)^2 + 4BC}}{2C} = \frac{A - D}{C}. \quad (3.5)$$

Next, we recall that the pole of a Möbius transformation is the complex point $z = -\frac{D}{C}$. We

shall denote the pole by z_∞ . The inverse pole of a Möbius transformation is the complex

point $z = \frac{A}{C}$, which we shall denote by Z_∞ . Now consider

$$z_\infty + Z_\infty = -\frac{D}{C} + \frac{A}{C} = \frac{A - D}{C} \quad (3.6)$$

By (3.5) and (3.6), we have $\gamma_1 + \gamma_2 = z_\infty + Z_\infty$.

These results allow us to state $\frac{\gamma_1 + \gamma_2}{2} = \frac{z_\infty + Z_\infty}{2} = \frac{A - D}{2C}$. Thus the point $\frac{A - D}{2C}$ is the midpoint for segments $\overline{\gamma_1 \gamma_2}$ and $\overline{z_\infty Z_\infty}$. We view the two pairs γ_1, γ_2 and z_∞, Z_∞ as pairs of opposite vertices of a parallelogram. In this manner, we can characterize a Möbius transformation by the parallelogram generated by the four points $\gamma_1, \gamma_2, z_\infty$, and Z_∞ . Each type of “characteristic parallelogram” corresponds to a particular equivalence class of the Möbius transformations. We will show the relationships by considering the example of a one-parameter family of Möbius transformations. This example is adapted from Derrick and Eidswick [3].

Let $f(z) = \frac{2z - a}{z}$. The matrix of f is given by $M = \begin{pmatrix} 2 & 1 \\ -a & 0 \end{pmatrix}$. The fixed points of f are given by $\gamma_1, \gamma_2 = \frac{2 - 0 \pm \sqrt{(0 - 2)^2 + 4(-a)(1)}}{2(1)} = 1 \pm \sqrt{1 - a}$. The pole and inverse

pole of f are given by $z_{\infty} = -\frac{D}{C} = 0$ and $Z_{\infty} = \frac{A}{C} = 2$, respectively. The magnitude of M is given by $\sigma(M) = \frac{(2)^2}{a} - 4 = \frac{4-4a}{a}$.

Now as a is allowed to vary over the complex numbers, we have the following:

Case 1: $a = 0$. $\gamma_1, \gamma_2 = 1 \pm \sqrt{1-0} = 2, 0$ $\sigma(M) = \frac{(2+0)^2}{0} - 4 = \infty$

Characteristic Parallelogram:

Classification: When $a = 0$, the function becomes $f(z) = 2$, a constant function. This is not a Möbius transformation.

Case 2: $0 < a < 1$. For example, let $a = \frac{1}{4}$.

$$\gamma_1, \gamma_2 = 1 \pm \sqrt{1 - \frac{1}{4}} = 1 \pm \frac{\sqrt{3}}{2} \quad \sigma(M) = \frac{(2+0)^2}{\frac{1}{4}} - 4 = 12$$

Characteristic Parallelogram:

Classification: *Proper Hyperbolic*

Case 3: $a = 1$. $\gamma_1, \gamma_2 = 1 \pm \sqrt{1-1} = 1$ $\sigma(M) = \frac{(2+0)^2}{1} - 4 = 0$

Characteristic Parallelogram:

Classification: *Parabolic*

Case 4: $a > 1$. For example, let $a = 2$.

$$\gamma_1, \gamma_2 = 1 \pm \sqrt{1-2} = 1 \pm i$$

$$\sigma(M) = \frac{(2+0)^2}{2} - 4 = -2$$

Characteristic Parallelogram:

Classification: *Elliptic*

Case 5: $a < 0$. For example, let $a = -1$.

$$\gamma_1, \gamma_2 = 1 \pm \sqrt{1-(-1)} = 1 \pm \sqrt{2}$$

$$\sigma(M) = \frac{(2+0)^2}{-1} - 4 = -8$$

Characteristic Parallelogram:

Classification: *Improper Hyperbolic*

Case 6: a is not real. For example, let $a = i$.

$$\gamma_1, \gamma_2 = 1 \pm \sqrt{1-i}$$

$$\sigma(M) = \frac{(2+0)^2}{-i} - 4 = -4 + 4i$$

Classification: *Loxodromic*

We have concentrated on a single example in this case in order to utilize the *Geometer's Sketchpad* to illustrate the characteristic parallelograms. It is straightforward to prove that the proper and improper hyperbolic Möbius transformations always have characteristic parallelograms that are degenerate, the elliptic Möbius transformations have characteristic parallelograms in the shape of a rhombus, the parabolic transformations have the degenerate type shown, and the loxodromic Möbius transformations always have general, nontrivial characteristic parallelograms.

§ 3.8 Factorization Methods

In § 3.3 we proved that every Möbius transformation can be written as the composition of four simple types of Möbius transformations. Since each simple type is a product of two inversions, every Möbius transformation can be written as a composition of inversions. We now obtain a means to factor any given Möbius transformation into the composition of two or four inversions, depending on its equivalence class.

We shall derive two different methods of factorization. In both methods, we find that pairs of inversions correspond to pairs of circles which belong to a common pencil of circles. The equivalence class of a given Möbius transformation determines the type of pencil. Specifically, a hyperbolic Möbius transformation, proper or improper, can be factored into two inversions with respect to circles of a hyperbolic pencil. Similarly, an elliptic Möbius transformation can be factored into two inversions with respect to circles of

an elliptic pencil, and a parabolic Möbius transformation can be factored into two inversions with respect to circles of a parabolic pencil. Since a loxodromic Möbius transformation is the composition of a hyperbolic and an elliptic Möbius transformation, in any order, every loxodromic Möbius transformation can be factored into four inversions with respect to pairs of circles in the appropriate pencils.

The first method, the *standard factorization* of a Möbius transformation f will be defined as $f = f_2 \circ f_1$, where f_1 is inversion with respect to a circle and f_2 is inversion with respect to the radical axis, both within a given pencil of circles. If f is a loxodromic Möbius transformation, then the standard factorization is written $f = f_4 \circ f_3 \circ f_2 \circ f_1$. In this case, the pairs of inversions occur within orthogonal pencils of circles as above: inversion in a circle first, followed by inversion in the radical axis. The characteristic parallelogram discussed in § 3.7 will be essential to the factorization procedure.

Case 1: $f = f_2 \circ f_1$

For any Möbius transformation with matrix M , we calculate $\sigma(M)$ to determine the equivalence class of the transformation. We determine the points γ_1 , γ_2 , z_∞ , and Z_∞ by methods previously shown to obtain the characteristic parallelogram of f . For each equivalence class, we determine the radical axis, ℓ , as follows:

Hyperbolic: ℓ is the perpendicular bisector of $\overline{\gamma_1\gamma_2}$;

Elliptic: ℓ is the line containing $\overline{\gamma_1\gamma_2}$;

Parabolic: ℓ is the line through $\gamma_1 = \gamma_2$, perpendicular to $\overline{z_\infty Z_\infty}$.

In each case, line ℓ passes through the point $R = \frac{A-D}{2C}$, which is the midpoint of $\overline{z_\infty Z_\infty}$.

We note that f_2 corresponds to inversion with respect to ℓ .

By definition, f_1 is an inversion with respect to a circle C . The point z_∞ must be the center of C since $f(z_\infty) = f_2(f_1(z_\infty)) = f_2(\infty) = \infty$. If we consider the image of C under f , we see that $f(C) = f_2(f_1(C)) = f_2(C) = C'$, where C' is the reflection of C across ℓ .

If C is a real circle, then the line containing $\overline{z_\infty Z_\infty}$ intersects C in two points, P and Q . By following $f_2 \circ f_1(P, Q)$, we see that these two points map to their reflections across R , hence these two points satisfy $zM = 2\left(\frac{A-D}{2C}\right) - z$. That is to say, $\frac{z' + z}{2} = \frac{A-D}{2C}$.

If we solve $zM = 2\left(\frac{A-D}{2C}\right) - z$ for z , we have

$$\frac{Az + B}{Cz + D} = 2\left(\frac{A-D}{2C}\right) - z = \frac{A-D-zC}{C}$$

so that

$$C(Az + B) = (Cz + D)(A - D - zC)$$

$$ACz + BC = ACz - 2CDz + AD - D^2 - C^2z^2$$

$$C^2z^2 + 2CDz + D^2 = AD - BC$$

$$(Cz + D)^2 = AD - BC$$

$$Cz + D = \pm\sqrt{AD - BC}$$

$$z = -\frac{D}{C} \pm \frac{\sqrt{AD - BC}}{C} = z_\infty \pm \frac{\sqrt{AD - BC}}{C}.$$

$$\therefore P = z_\infty + \frac{\sqrt{AD - BC}}{C}, \quad Q = z_\infty - \frac{\sqrt{AD - BC}}{C}.$$

Now suppose C is an imaginary circle with center z_∞ . We recall that C belongs to a hyperbolic pencil of circles with radical axis ℓ . We claim that the points P and Q which

satisfy $zM = 2\left(\frac{A-D}{2C}\right) - z$ determine the radius of C . In this case, P and Q are on the line through z_∞ that is perpendicular to $\overline{z_\infty Z_\infty}$.

We reconsider the one-parameter family of Möbius transformations studied in § 3.7 in order to illustrate these factorizations. Recall that f is given by the matrix $M = \begin{pmatrix} 2 & 1 \\ -a & 0 \end{pmatrix}$ and that a is allowed to vary over the complex numbers. In each case, we also illustrate the *orbit* generated by a general point z in the complex plane as the Möbius transformation is repeated. The orbits of a Möbius transformation are precisely the circles belonging to the pencil orthogonal to the circles of inversion. In Figures 3.5 through 3.8, the orbits are the dashed lines containing $z' = f(z)$ and $z'' = f(z')$.

Proper Hyperbolic: $M = \begin{pmatrix} 2 & 1 \\ -1/4 & 0 \end{pmatrix} \quad \sigma(M) > 0 \quad P, Q = \pm 1/2$

Figure 3.5

Parabolic: $M = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma(M) = 0 \quad P, Q = \pm 1$

Figure 3.6

Elliptic: $M = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} \quad -4 \leq \sigma(M) < 0 \quad P, Q = \pm\sqrt{2}$

Figure 3.7

Improper Hyperbolic: $M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma(M) < -4$ $P, Q = \pm i$

Figure 3.8

Case 2: $f = f_4 \circ f_3 \circ f_2 \circ f_1$

In this case, f is Loxodromic. We obtain the characteristic parallelogram as before.

We determine a pair of lines ℓ_H , the perpendicular bisector of $\overline{\gamma_1 \gamma_2}$, and ℓ_E , the line containing $\overline{\gamma_1 \gamma_2}$. These lines are the radical axes of orthogonal pencils of circles.

We recall that a loxodromic Möbius transformation may be written in the form $f = H \circ E = E \circ H$, where H and E are hyperbolic and elliptic Möbius transformations, respectively. Thus, the circle of inversion for f_1 may be either a member of a hyperbolic pencil or a member of an elliptic pencil. Let $H = \ell_H \circ C_H$ and $E = \ell_E \circ C_E$ represent standard factorizations of H and E . Then for $f = E \circ H$, f_1 is inversion in C_H , f_2 is inversion in ℓ_H , f_3 is inversion in C_E , and f_4 is inversion in ℓ_E . This conveniently puts the factors ℓ_H , C_H , and ℓ_E , C_E in orthogonal pencils of circles.

It is straightforward to show that the points P and Q that satisfy the equation

$$zM = 2\left(\frac{A-D}{2C}\right) - z \quad \text{are the common points of intersection for the circles } C_H \text{ and } C_E.$$

Furthermore, the center of C_H lies on ℓ_E and the center of C_E lies on ℓ_H . If we construct ρ , the perpendicular bisector of $\overline{PQ}$, then ρ meets ℓ_E at S , the center of C_H , and ρ meets ℓ_H at T , the center of C_E . (Alternately, the circle through P , Q , and R intersects the radical axes at the desired points, S and T .) This construction is illustrated in Figure 3.9.

An alternate construction to obtain the points S and T is shown in Figure 3.10. In this case, we construct circle C_O through γ_1 , γ_2 , and z_∞ . Circle C_O meets ℓ_H at T . We then construct the line $\overrightarrow{Tz_\infty}$ which meets ℓ_E at S . (This corresponds to line ρ in Figure 3.9.) We have omitted the illustration of orbits in the Loxodromic factorizations since these orbits are spiral in shape and are not easily drawn with *Geometer's Sketchpad*.

Loxodromic: $M = \begin{pmatrix} 2 & 1 \\ -i & 0 \end{pmatrix}$ $\sigma(M)$ is not real $P, Q = \pm\sqrt{i}$

Figure 3.9

Loxodromic: $M = \begin{pmatrix} 2 & 1 \\ -i & 0 \end{pmatrix}$ $\sigma(M)$ is not real $P, Q = \pm\sqrt{i}$

Figure 3.10

Once we have obtained the standard factorization of a Möbius transformation, we are also able to find a *symmetric factorization*. For each type of Möbius transformation (except improper hyperbolic), the symmetric factorization consists of inversions in two circles that are symmetric about the radical axis obtained in the standard factorization. We describe and illustrate using *Geometer's Sketchpad*, the construction methods for each type of symmetric factorization (excluding improper hyperbolic.)

We have omitted a proof showing that the standard and symmetric factorizations of a Möbius transformation do indeed result in the same motion. However, by following any three points to their images, we can see that the two factorizations determine the same Möbius transformation. Again, although we are working with one example of say, a proper hyperbolic Möbius transformation, all Möbius transformations in the equivalence class would look the same and factor in the same way.

Proper Hyperbolic:

The standard factorization of the proper hyperbolic Möbius transformation produces the circle C_H , radical axis ℓ_H , and points P and Q . Using the characteristic parallelogram and point R , construct the circle C_R with center R and radius $R\gamma_1$. The perpendicular to $\overline{\gamma_1\gamma_2}$ at P meets C_R at point X . Construct circle C_1 with center Q and radius QX . Reflect circle C_1 across the radical axis ℓ_H to obtain circle C_2 (with center Q' .) Thus $f = f_2 \circ f_1$ where f_1 is inversion in C_1 and f_2 is inversion in C_2 . (See Figure 3.11.) (It is also possible to obtain a symmetric factorization with virtual circles centered at P and P' .)

Figure 3.11

Parabolic:

The standard factorization of the parabolic Möbius transformation produces the circle C_p , the radical axis ℓ_p , and points $P = R$ and Q . Construct circle C_1 with center Q and radius QP . Reflect C_1 across the radical axis ℓ_p to obtain circle C_2 (with center Q' .) Again, $f = f_2 \circ f_1$ where f_1 is inversion in C_1 and f_2 is inversion in C_2 . (See Figure 3.12.)

Figure 3.12

Elliptic:

The standard factorization of the elliptic Möbius transformation produces the circle C_E , radical axis ℓ_E , and points P and Q . There are two symmetric factorizations for the elliptic Möbius transformation. The first consists of the circles C_1 and C_2 , centered at Q and Q' , respectively. (The point Q' is the image of Q with respect to reflection across ℓ_E .) Both circles pass through the fixed points γ_1 and γ_2 . In this case, $f = f_2 \circ f_1$ where f_1 is inversion in C_1 and f_2 is inversion in C_2 . The second symmetric factorization consists of

the circles C'_1 and C'_2 , centered at P and P' , respectively, where the point P' is the image of P with respect to reflection across ℓ_E . Again, both circles pass through the fixed points γ_1 and γ_2 . In this case, $f = f_2 \circ f_1$ where f_1 is inversion in C'_1 and f_2 is inversion in C'_2 . (See Figure 3.13.)

Figure 3.13

Loxodromic:

The standard factorization of the loxodromic Möbius transformation produces the circle C_H , radical axis ℓ_H , circle C_E , radical axis ℓ_E , and points P , Q , R , S , and T . Circle C_H meets radical axis ℓ_E at points P_H and Q_H . (See Figure 3.14) We use these new points to construct the symmetric circles C_{H1} and C_{H2} according to the methods given in the proper hyperbolic construction. Circle C_E meets radical axis ℓ_H at points P_E and Q_E . We use these points to construct the symmetric circles C_{E1} and C_{E2} according to the methods given in the elliptic construction.

One of the interesting features of Figure 3.14 is the presence of three distinct, but related characteristic parallelograms. The black parallelogram is the characteristic parallelogram for the loxodromic Möbius transformation. The solid pink degenerate parallelogram corresponds to the proper hyperbolic Möbius transformation, and the dashed pink rhombus corresponds to the elliptic Möbius transformation.

Figure 3.14

Chapter 4

Models of Hyperbolic Geometry

§ 4.1 The Poincaré Models

Henri Poincaré (1854-1912) made important contributions in several areas of mathematics including differential equations, probability, and topology. He was one of several mathematicians in the late nineteenth century who created Euclidean models to visualize hyperbolic geometry. One can prove that all possible models of hyperbolic geometry are isomorphic to one another. The advantage of the Poincaré models is that they represent angles accurately, while some other models do not.

The axioms of hyperbolic geometry consist of the axioms of Euclidean geometry with the hyperbolic parallel postulate replacing the parallel postulate. The hyperbolic parallel postulates states “If point P is off line ℓ , then there exist at least two lines through P that are parallel to ℓ .” Any model of hyperbolic geometry must satisfactorily demonstrate these axioms.

In this chapter, we will emphasize the use of *Geometer's Sketchpad* (and our previous work) to illustrate and verify properties of hyperbolic geometry in the Poincaré models. While we cite the caveat that “a picture is not a proof,” our goal here is to present convincing and intuitive illustrations. Thus the proofs have been omitted.

The Poincaré Disk Model. The points of the hyperbolic plane are represented by the points interior to a Euclidean circle γ . While this circle may have a radius of arbitrary length, the usual convention is to choose the unit circle. In this case, the points of the hyperbolic plane belong to the set $\mathbb{H} = \{z \in \mathbb{C} : |z| < 1\}$. We shall indicate the convenience of this choice in subsequent discussions.

Straight lines in the model are represented by arcs of circles orthogonal to the unit circle. We note that all diameters of the unit circle also represent lines in the model. One

can verify that the model satisfies the axioms of hyperbolic geometry. For example, if P and Q are two distinct points in $\mathcal{U}$ we construct the line that passes through P and Q by finding the inverse point P' (or Q') with respect to the unit circle. The circle through P , Q , P' , and Q' is orthogonal to the unit circle and thus represents the unique line through P and Q .

To illustrate the hyperbolic parallel postulate, let ℓ be a line in the model and let P be a point, not on ℓ . We can construct lines m and n that meet ℓ on the unit circle. The points on the unit circle do not belong to the set $\mathcal{U}$; we consider them to be a boundary of points at infinity. (Some authors refer to these points as *ideal* points.) If we define parallel lines to be lines that do not meet in the model, then we see that m and n are parallel to ℓ .

Martin [13] uses the term *critically parallel* for lines such as m and n . For if we describe parallel lines as nonintersecting lines, we see that there are infinitely many lines through P that do not intersect ℓ . Thus in Figure 4.1, the lines m and n are critically parallel to ℓ and the line q is parallel to ℓ . There are infinitely many parallel lines to ℓ through P obtained by choosing point Q anywhere in the space between lines m and n . Further, we see that lines r and s also represent parallel lines; while t and u are examples of lines that intersect ℓ .

(The unlabeled point on the circumference of the disk in Figure 4.1 is used in *Geometer's Sketchpad* to adjust the radius of the circle. The point will appear in all of the disk model figures. It has no meaning in the disk model of hyperbolic geometry and should be ignored by the reader.)

Figure 4.1

The *distance* between two points P and Q in the model is defined by

$$d(P, Q) = k |\ln(PQ, RS)|, \quad (k \text{ is a positive, real constant})$$

where R and S are the intersections of the line through P, Q and the unit circle. In this definition, the expression (PQ, RS) refers to the complex cross ratio defined in § 2.5. We let $k = 1$ to simplify future calculations. The base of the logarithm is an arbitrary choice; it simply results in a change of scale. It can be shown that the distance function satisfies the metric axioms:

- 1) $d(P, Q) \geq 0$ for all $P, Q \in \mathcal{H}$. Equality holds if and only if $P = Q$.
- 2) $d(P, Q) = d(Q, P)$ for all $P, Q \in \mathcal{H}$.
- 3) The triangle inequality: $d(P, Q) + d(Q, R) \geq d(P, R)$ for all $P, Q, R \in \mathcal{H}$.

The measure of an angle between two lines in the model is given by the usual Euclidean angle between a pair of intersecting circles. We recall that this is the angle between the tangents to the circles at the point of intersection. We note that the Euclidean

centers of a pair of circles (i.e., lines in the model) will necessarily lie outside the Poincaré disk.

To begin a construction in the Poincaré disk model using *Geometer's Sketchpad*, the user must first choose the Poincaré startup screen. The sketch environment includes a red circle representing the boundary points of the hyperbolic plane. The disk center and a point on the circumference allow adjustment of the position and size of the circle on the screen. (It is best to copy and paste all parts of the figure onto a new sketch. Doing so allows the user to have the startup screen available for future sketches.)

The Poincaré tools are recorded scripts that perform useful but sometimes complicated constructions. The Poincaré tools include the following constructions: line, perpendicular to a line through a point (not on the line), perpendicular to a line through a point on the line, segment, measure distance, perpendicular bisector, measure angle, angle bisector, circle by center and point on the radius, circle by center and radius length. The utility of these tools may be fully appreciated by reading the scripts involved.

The Poincaré Half-Plane Model. The points of the hyperbolic plane are now represented by all the points of the upper half of the Euclidean plane. In this case, the x -axis becomes the set of boundary points at infinity; and the point $(0, 1)$ on the y -axis provides the distance scale. The straight lines in the model are represented by arcs of Euclidean circles that are centered on the x -axis, and by Euclidean straight lines that are perpendicular to the x -axis.

The hyperbolic parallel postulate is illustrated in Figure 4.2. Again, we see that given a line ℓ in the model and a point P , not on ℓ , the lines m and n are critically parallel to ℓ . The line q through points P and Q is nonintersecting or parallel to ℓ . We see again, in this model, that lines r and s represent parallel lines; while t and u are examples of lines that intersect ℓ .

Figure 4.2

The construction of lines in the half-plane model utilizes the Euclidean tools of the *Geometer's Sketchpad*. Since the construction of semicircles centered on the x-axis requires several steps, we have included scripts to simplify this task.

The distance function in the model is analogous to that in the disk model. The distance between points P and Q is defined as

$$d(P, Q) = k |\ln(PQ, RS)|, \text{ (again let } k = 1 \text{)}$$

where R and S are the intersections of the line through P , Q and the x-axis. If the segment $\overline{PQ}$ is on a straight Euclidean line with $P = (a, y_1)$ and $Q = (a, y_2)$, $0 < y_1 \leq y_2$, then

$$d(P, Q) = \ln \frac{y_2}{y_1}.$$

An alternate definition for the length of segment $\overline{PQ}$ may be obtained from a proof in Stahl [18]. If P and Q are points on a semicircle with center C , we let radii $\overline{CP}$ and $\overline{CQ}$

make angles α and β ($\alpha < \beta$) respectively, with the positive x-axis. The distance of the segment $\overline{PQ}$ is given by $d(P, Q) = \ln \frac{\csc \beta - \cot \beta}{\csc \alpha - \cot \alpha}$. Figure 4.3 illustrates the fact that the two definitions provide equal results. Scripts for both methods are included in the Appendix.

Figure 4.3

The measure of an angle between two lines is given by the Euclidean angle between a pair of intersecting circles. The construction for this measure in the half-plane model is necessarily different than in the disk model. Consequently, a script for angle measure in the half-plane model is included in the Appendix.

§ 4.2 Geometric Figures and Structures

We are interested in certain geometric figures in the hyperbolic plane and how they compare with their Euclidean counterparts. The triangle, Saccheri quadrilateral, and Lambert quadrilateral will be most important for further investigations within the Poincaré models. In addition, we consider hyperbolic circles and two other structures, horocircles and hypercircles, that do not have Euclidean counterparts. Finally, we describe the pencils of lines that occur in the hyperbolic plane.

It can be shown that Euclid's Fifth Postulate is equivalent to the statement that the angle sum of every triangle is equal to π . We have an alternate parallel postulate in the hyperbolic plane; consequently, the angle sum of a hyperbolic triangle is not equal to π . In fact, the angle sum of a hyperbolic triangle is strictly less than π .

We illustrate in Figures 4.4 and 4.5 the angle sum property of triangles in both Poincaré models. In each case, triangle ABC represents a general hyperbolic triangle. Triangle PQR illustrates a hyperbolic triangle with one side on a "diameter" in the model. (The straight Euclidean lines in the half-plane model correspond to the diameters in the disk model.) We say that triangle XYZ represents a right triangle in standard position within each model when the right angle at Z is at the origin in the disk model and the right angle at Z is at $(0, 1)$ in the half-plane model.

Next, we consider a general quadrilateral. If we construct one of its diagonals, we see that the quadrilateral may be viewed as the union of two triangles that share a common side. We will prove later that the angle sum of the quadrilateral is equal to the sum of the angle sums of its component triangles. For the moment, we consider the consequence of this statement in the hyperbolic plane. Since the angle sum of any hyperbolic triangle is strictly less than π , the angle sum of a hyperbolic quadrilateral is strictly less than 2π .

$\angle A = 89.2^\circ$	$\angle P = 20.3^\circ$	$\angle X = 32.4^\circ$
$\angle B = 15.8^\circ$	$\angle Q = 83.5^\circ$	$\angle Y = 30.6^\circ$
$\angle C = 20.4^\circ$	$\angle R = 45.8^\circ$	$\angle Z = 90.0^\circ$
$\angle A + \angle B + \angle C = 125.4^\circ$	$\angle P + \angle Q + \angle R = 149.6^\circ$	$\angle X + \angle Y + \angle Z = 152.9^\circ$

Figure 4.4

The reader will notice that the computations above do not always sum correctly. Angle measure and lengths in the *Sketchpad* are estimated to the hundredth of a degree or unit, then rounded to the nearest tenth. Consequently, when calculations are performed, the results may vary slightly in the tenth place value.

Figure 4.5

A quadrilateral $ABCD$ that has right angles at B and C with $AB = CD$ is defined to be a *Saccheri quadrilateral*. One of the properties of a Saccheri quadrilateral is that the remaining angles at A and D are equal and acute. A *Lambert quadrilateral* is defined to be a quadrilateral with three right angles. In Figures 4.6 and 4.7, we illustrate Saccheri quadrilateral $ABCD$ and Lambert quadrilateral $CDEF$, both in standard position, in the Disk and Half Plane models, respectively.

$\angle A = 45.1^\circ$		$\angle C = 90.0^\circ$
$\angle B = 90.0^\circ$	$AB = 1.16$	$\angle D = 45.1^\circ$
$\angle C = 90.0^\circ$	$CD = 1.16$	$\angle E = 90.0^\circ$
$\angle D = 45.1^\circ$		$\angle F = 90.0^\circ$

Figure 4.6

Figure 4.7

The general definition of a circle may be given as the locus of points that are equidistant from a given point, its center. A *circle* in the Poincaré models is a Euclidean circle that lies entirely within the boundary of the model. We notice however, that because of the hyperbolic distance function, the hyperbolic center of a circle does not generally coincide with its Euclidean center.

In Figure 4.8, we represent two sets of concentric circles in the Poincaré disk model. The set $\mathcal{C}_1$ is a set in which the Euclidean and the hyperbolic centers coincide. The set $\mathcal{C}_2$ demonstrates the alternate case. We recognize that the set $\mathcal{C}_1$ is one half of the (real) circles in a hyperbolic pencil of circles. The hyperbolic center, the point C , corresponds to

one of the point circles. It is not difficult to see that the Euclidean centers of the circles lie on a diameter of the model.

Figure 4.8

There is only one way to visualize a set of concentric circles in the Poincaré half-plane model. The circles must lie entirely above the x -axis. We see in Figure 4.9 that the hyperbolic center of the circles corresponds to the point circle of a hyperbolic pencil of circles. In this case, the radical axis of the pencil coincides with the x -axis.

Figure 4.9

Consider the point C as the center of a concentric set of circles. Suppose that we let C approach the boundary. As long as C is within the boundary, the circles are still circles. But if C becomes a point on the boundary, i.e., a point at infinity, we can no longer use the term “circles” for the associated structures. We call such a structure a *horocircle*.

Intuitively, a horocircle is defined to be the locus of points that are equidistant from a point at infinity. There is no corresponding structure in the Euclidean plane.

We illustrate a set of concentric horocircles in Figures 4.10 and 4.11. We may view them as part of a parabolic pencil of circles. A curious fact about horocircles may be deduced if we consider the distance from a point on any horocircle to its center. This distance will always be infinitely long; so we may deduce that any two horocircles are congruent. We will establish the means to determine distinct horocircles in § 4.3 when we discuss invariant motions in the hyperbolic plane.

Figure 4.10

Figure 4.11

Next, we consider the structures in the hyperbolic plane that might be associated with an elliptic pencil of circles. We superimpose an elliptic pencil on the model so that the common points of intersection lie on the boundary. We find that there is exactly one Euclidean circle that corresponds to a line in the model. We call the line c . Each of the remaining arcs of Euclidean circles in the pencil is called a *hypercircle*.

The *radius* of a hypercircle is the length of a perpendicular segment drawn from a point P on the hypercircle to line c . For a given hypercircle, this distance is the same regardless of the choice of P . Thus, some authors use the term *equidistant curve* for a hypercircle. Since each hypercircle is equidistant to its related line c , we may consider line c to be the “center” of a set of concentric hypercircles. Two hypercircles are congruent if and only if they have the same radius.

We find that hypercircles have no counterparts in the Euclidean plane. This becomes apparent when we realize that the Euclidean structure equidistant from a given line is another (parallel) line. In Figures 4.12 and 4.13, we illustrate a set of concentric hypercircles, the center c , and a pair of congruent hypercircles, h_1 and h_2 .

$$PQ = 0.87 \quad RS = 0.87 \quad ST = 0.87$$

Figure 4.12

$$\ln \left(\frac{(\overline{m} \overline{OQ}) \cdot (\overline{m} \overline{PS})}{(\overline{m} \overline{OP}) \cdot (\overline{m} \overline{QS})} \right) = 0.83 \quad \ln \left(\frac{(\overline{m} \overline{OR}) \cdot (\overline{m} \overline{QS})}{(\overline{m} \overline{OQ}) \cdot (\overline{m} \overline{RS})} \right) = 0.83$$

Figure 4.13

In § 2.2 we defined pencils of circles in the Euclidean plane according to the common points of intersection or the absence of such points. These pencils of circles correspond to pencils of lines in the hyperbolic models. We consider the three possible types.

Let ℓ_1 and ℓ_2 be two lines in the model. If ℓ_1 meets ℓ_2 at a point P in the model, then the lines belong to a *pencil* of lines through P . The lines that form a pencil in the model may be recognized as circles belonging to an elliptic pencil of circles in the Euclidean context. The associated orthogonal circles form a set of concentric circles in the model. Figures 4.14 and 4.15 illustrate a pencil and its associated set of concentric circles in the disk model and the half-plane model, respectively.

Figure 4.14

Figure 4.15

If ℓ_1 and ℓ_2 are critically parallel (intuitively, they “meet at infinity”), then they belong to a *horopencil*. In this case, we see that the lines in a horopencil correspond to a parabolic pencil of circles in the Euclidean plane. In Figures 4.16 and 4.17, we show a horopencil and the related orthogonal set of concentric horocircles.

Figure 4.16

Figure 4.17

If ℓ_1 and ℓ_2 do not meet, then there exists a unique line c that is the common perpendicular to both ℓ_1 and ℓ_2 . In this case, we say that ℓ_1 and ℓ_2 belong to a *hyperpencil* with center c . The lines in a hyperpencil correspond to the circles in a hyperbolic pencil of circles. Figures 4.18 and 4.19 demonstrate a hyperpencil with center c and the orthogonal set of concentric hypercircles.

Figure 4.18

Figure 4.19

§ 4.3 Sketchpad Illustrations of Theorems in Hyperbolic Geometry

In this section, we present selected illustrations of theorems in hyperbolic geometry that provide interesting comparisons with Euclidean geometry. Since our purpose is to show the power of *Geometer's Sketchpad*, we do not include proofs of the theorems. The reader may find these in Martin [13].

We adopt some additional terminology from Martin which allows us to generalize certain results. The term *brush* refers to any one of a pencil, a horopencil, or a hyperpencil in the model. Rotation by the angle $\theta = \pi$ is called a *halfturn*. It can be shown that every halfturn is an involution. Furthermore, all of the motions which we will discuss in § 4.4 can be written as the product of halfturns. We illustrate most theorems in both the Poincaré disk model and the Poincaré half-plane model.

Theorem 4.1: If two triangles have three angles of one congruent to the three angles of the other, then the triangles are congruent.

$\angle A = 32.2^\circ$	$AB = 1.69$	$\angle A' = 32.2^\circ$	$A'B' = 1.69$
$\angle B = 47.5^\circ$	$BC = 1.27$	$\angle B' = 47.5^\circ$	$B'C' = 1.27$
$\angle C = 57.9^\circ$	$CA = 1.56$	$\angle C' = 57.9^\circ$	$C'A' = 1.56$

Figure 4.20

This theorem is in direct contrast to Euclidean geometry. If two Euclidean triangles have corresponding angles congruent, then the triangles are said to be *similar*. We find therefore, that in hyperbolic geometry, there are no similar triangles.

Theorem 4.2: The three perpendicular bisectors of the sides of a triangle are in a unique pencil.

Case A: Perpendicular bisectors x , y , and z belong to the Pencil with center C . Notice that C is the circumcenter of $\triangle XYZ$.

Figure 4.21

Theorem 4.2: The three perpendicular bisectors of the sides of a triangle are in a unique brush.

Case B: Perpendicular bisectors x , y , and z belong to the Horopencil approaching C .

We say that C is the “circumcenter” of $\triangle XYZ$ because it is equidistant from vertices X , Y , and Z , even though that distance is infinite.

Figure 4.22

Theorem 4.2: The three perpendicular bisectors of the sides of a triangle are in a unique brush.

$$XX' = 2.09 \quad YY' = 2.09 \quad ZZ' = 2.08$$

Case C: Perpendicular bisectors x , y , and z belong to the Hyperpencil with center c .

We say that line c is the “circumcenter” of $\triangle XYZ$ because it is equidistant from the vertices X , Y , and Z .

Figure 4.23

Theorem 4.2: The three perpendicular bisectors of the sides of a triangle are in a unique pencil.

Case A: Perpendicular bisectors x , y , and z belong to the Pencil with center C . Notice that C is the circumcenter of $\triangle XYZ$.

Figure 4.24

Theorem 4.2: The three perpendicular bisectors of the sides of a triangle are in a unique brush.

Case B: Perpendicular bisectors x , y , and z belong to the Horopencil approaching C .
Notice that C is the “circumcenter” of $\triangle XYZ$.

Figure 4.25

Theorem 4.2: The three perpendicular bisectors of the sides of a triangle are in a unique brush.

Case C: Perpendicular bisectors x , y , and z belong to the Hyperpencil with center c .

Notice that line c is the “circumcenter” of $\triangle XYZ$.

Figure 4.26

While the next theorem may be illustrated in each of three cases, depending on the type of brush, we include only the illustrations for the Pencil in both the disk model and the half-plane model. Illustrations of the remaining cases are included on the disk. Following Theorem 4.3, we include two theorems illustrating some of the relationships between involutions, halfturns, and the motions in the Poincaré models.

Case A: Lines ℓ_1 , ℓ_2 , and ℓ_3 are in the Pencil with center C . We see $\ell_3(\ell_2(\ell_1(A))) = B$ and $m(A) = B$.

Figure 4.27

Theorem 4.3: The product of three reflections in lines of a brush is a reflection in a line of that brush.

Case A: Lines ℓ_1 , ℓ_2 , and ℓ_3 are in the Pencil with center C . We see $\ell_3(\ell_2(\ell_1(A))) = B$ and $m(A) = B$.

Figure 4.28

Theorem 4.4: For lines ℓ_1 and ℓ_2 , $\ell_2 \circ \ell_1 = \ell_1 \circ \ell_2$ if and only if $\ell_1 = \ell_2$ or ℓ_1 is perpendicular to ℓ_2 . Furthermore, $\ell_2 \circ \ell_1$ is an involution if and only if lines ℓ_1 and ℓ_2 are perpendicular.

Figure 4.29

Theorem 4.5: The product of two distinct halfturns is a translation. Conversely, every translation is the product of two halfturns.

In this example, we see that $m_4(m_3(m_2(m_1(A)))) = B$ and $\ell_2(\ell_1(A)) = B$.

Figure 4.30

There are fourteen additional illustrations included on the computer disks which the reader may explore. These illustrated theorems are contained in the folder entitled "Chapter 4: More Theorems." They continue to develop the important relationships that will lead to the classification of motions in the Poincaré model.

§ 4.4 The Group of Motions in the Poincaré Model

A transformation T of the entire plane onto itself is called a *motion* or an *isometry* if length is invariant under T . In this section, our goal is to describe the invariant motions in the hyperbolic plane. For this purpose, we recall our earlier discussion of the Möbius transformations. We focus initially on the Poincaré disk model as the set of complex points z such that $|z| < 1$. In particular, we need to define the group of Möbius transformations that map the interior of the unit disk to itself. We will show that they are a subgroup of the Möbius transformations that map the unit circle to itself.

Let f be a Möbius transformation given by the matrix $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ and let

$H = \begin{pmatrix} 1 & -\bar{\gamma} \\ -\gamma & |\gamma|^2 - r^2 \end{pmatrix}$ represent a circle. The image of the circle with respect to f is given

by $H' = M^{-1}H\overline{M}^{-1^T}$. (For a complete explanation, see Schwerdtfeger [15].) The unit

circle is given by the matrix $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. If a Möbius transformation maps the unit

circle to itself, then $f(H) = \lambda H$ for some nonzero, real λ . Thus we have

$\lambda H = M^{-1}H\overline{M}^{-1^T}$. We can solve for M using $\lambda HM = H\overline{M}^{-1^T}$ and the following substitutions:

$$\begin{aligned} \lambda HM &= H\overline{M}^{-1^T} \\ \lambda \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{D} & -\bar{B} \\ -\bar{C} & \bar{A} \end{pmatrix} \\ \lambda \begin{pmatrix} A & -C \\ B & -D \end{pmatrix} &= \begin{pmatrix} \bar{D} & -\bar{B} \\ \bar{C} & -\bar{A} \end{pmatrix}. \end{aligned} \tag{4.1}$$

From equation (4.1) we have the following set of conditions:

$$\lambda A = \bar{D} \quad (4.2)$$

$$-\lambda C = -\bar{B}, \text{ or equivalently } \lambda C = \bar{B} \quad (4.3)$$

$$\lambda B = \bar{C} \quad (4.4)$$

$$-\lambda D = -\bar{A}, \text{ or equivalently } \lambda D = \bar{A} \quad (4.5)$$

If we take the conjugate of equation (4.5), we have $\lambda \bar{D} = A$. Substituting into (4.2), we see

$$\lambda^2 \bar{D} = \bar{D}. \quad (4.6)$$

We deduce from (4.6) that either $\lambda = \pm 1$ or $\bar{D} = 0$. Using a similar approach to (4.3) and (4.4), we see that $\lambda^2 \bar{B} = \bar{B}$, so that either $\lambda = \pm 1$ or $\bar{B} = 0$. If $\lambda \neq \pm 1$, then $\bar{D} = A = 0$ and $\bar{B} = C = 0$. In this case, M is the zero matrix which is not a Möbius transformation. (Recall $AD - BC \neq 0$.) Thus $\lambda = \pm 1$, and it follows that $D = \pm \bar{A}$ and $C = \pm \bar{B}$.

We conclude that the Möbius transformations that fix the unit circle must be of the form $M = \begin{pmatrix} A & \pm \bar{B} \\ B & \pm \bar{A} \end{pmatrix}$. The determinant $\pm |A|^2 - (\pm |B|^2) \neq 0$ if and only if $|A| \neq |B|$. The group of Möbius transformations that satisfy these conditions is called the group of the unit circle.

We consider next the image of the origin under these transformations. If f is given by $M = \begin{pmatrix} A & \bar{B} \\ B & \bar{A} \end{pmatrix}$, then $f(0) = \frac{B}{A}$. If the origin is mapped to a point in the interior of the unit circle, then we must have $\left| \frac{B}{A} \right| < 1$. This will be true if and only if $|B| < |A|$.

Similarly, if f is given by $M = \begin{pmatrix} A & -\bar{B} \\ B & -\bar{A} \end{pmatrix}$, then $f(0) = \frac{B}{-\bar{A}}$. We require $\left| \frac{B}{-\bar{A}} \right| < 1$ which is true if and only if $|B| < |A|$.

Thus, we define the *Proper group of the unit circle* to be the subgroup of Möbius transformations that fix the unit circle and map the interior to itself. We represent the group

$$\mathcal{U}_c = \left\{ \begin{pmatrix} A & \pm \overline{B} \\ B & \pm \overline{A} \end{pmatrix} : |B| < |A| \right\}.$$

These consist precisely of the identity and the elliptic, parabolic, and proper hyperbolic Möbius transformations in $\mathcal{U}_c$. Since each of these transformations can be factored into two inversions, we refer to this group as the *even* isometries of the hyperbolic plane. The circles of inversion must belong as a pair to an appropriate pencil, horopencil, or hyperpencil.

We consider now any line ℓ in the model, and we recall that each line is orthogonal to the unit circle. If we invert with respect to ℓ , then by Theorem 2.10, the unit circle stays fixed and the interior of the unit circle is mapped to itself. Thus inversions with respect to a line in the model provide another type of isometry in the hyperbolic plane. The composition of an inversion and a Möbius transformation is equivalent to three inversions. We will demonstrate later that three inversions result in an isometry, as well. Furthermore, it can be shown that a transformation composed of an odd number of inversions can be written as the product of one or three inversions. We refer to these as the *odd* isometries.

While this development has focused on the unit circle and the Poincaré disk model, the results are applicable to the Poincaré half-plane model, as well. It can be shown that the proper group of the unit circle is isomorphic to the group of transformations that fix the real axis and map the upper half-plane to itself. Inversion with respect to a circle orthogonal to the real axis similarly preserves the upper half-plane and the real axis. Thus the even and odd isometries occur appropriately in the Poincaré half-plane model.

We present the following classification of the isometries of the hyperbolic plane, and we will illustrate selected examples. There are many wonderful proofs that introduce

and develop this system of classification thoroughly which cannot be included here. The reader is referred to Martin [13] and Singer [16]. We include a description of the transformation, the fixed points, and the orbit of a point under repeated iteration of the transformation.

An isometry of the hyperbolic plane is exactly one of the following:

- 1) The Identity: The identity consists of two inversions with respect to any line within the model. All points remain fixed. The orbit of a point consists of the point itself.
- 2) Reflection: A reflection consists of one inversion with respect to any line ℓ in the model. The two points where line ℓ meets the boundary at infinity are fixed. The line ℓ remains pointwise fixed. The orbit of a point not on ℓ consists of two distinct points, the point and its inverse with respect to ℓ .
- 3) Rotation: A rotation consists of two inversions with respect to two lines in a pencil through a point P . The point P remains fixed. The orbits lie on circles that are orthogonal to the pencil. If the angle of rotation is π , then the orbit of a point consists of two distinct points. If the angle of rotation is a rational multiple of 2π , then the orbit of a point consists of finitely many points equally spaced on a circle. If the angle of rotation is not a rational multiple of 2π , then the orbit of a point consists of infinitely many points on a circle.
- 4) Horolation: A horolation consists of two inversions with respect to two lines in a horopencil through a point P at infinity. The point P remains fixed. The orbit of a point consists of infinitely many points on a horocircle that is orthogonal to the horopencil.
- 5) Translation: A translation consists of two inversion with respect to two lines in a hyperpencil with center c . The two points where c meets the boundary at infinity remain fixed. The line c remains fixed, but not pointwise fixed. The orbit of a point not on c consists of infinitely many points on a hypercircle that is orthogonal to the hyperpencil.

Figure 4.31

We recall that a transformation is uniquely determined by three points and their images. The following illustrations demonstrate (in the disk model) each type of isometry with respect to a triangle ABC and its image under transformation. It is easy to see in each case that the odd isometries are orientation reversing, while the even isometries are orientation preserving.

In Figure 4.32, we illustrate $\triangle ABC$ reflected across line ℓ and its image, $\triangle A'B'C'$. This corresponds exactly to the inversion of points A , B , and C with respect to the Euclidean circle with center Q and circumference coincident with the hyperbolic line ℓ . As an odd isometry, we see that while vertices $A-B-C$ are oriented in a counterclockwise direction, the orientation of vertices $A'-B'-C'$ is clockwise.

Figure 4.32

Next, in Figure 4.33, we see the image of $\triangle ABC$ under rotation as it is reflected across lines ℓ_1 and ℓ_2 . The lines ℓ_1 and ℓ_2 belong to a Pencil of lines with center P . Since a rotation is an even isometry, the orientation of the vertices must be preserved. We see that both sets of vertices (A - B - C and A'' - B'' - C'') are oriented in a counterclockwise direction.

Figure 4.33

In Figure 4.34, we show a horolation. In this case, lines ℓ_1 and ℓ_2 belong to a Horopencil approaching P . Again, since this is an even isometry, the orientation of the vertices is preserved.

Figure 4.34

Figure 4.35 illustrates translation across lines ℓ_1 and ℓ_2 which belong to the Hyperpencil with center c . Translation is an even isometry, so we see that orientation is preserved.

Figure 4.35

Finally, in Figure 4.36, we see the glide reflection consisting of translation across lines ℓ_1 and ℓ_2 followed by reflection across line c . All the lines belong to the Hyperpencil with center c . As the product of three reflections, the glide reflection is an odd isometry. As we expect, the orientation of the vertices ($A-B-C$ and $A'''-B'''-C'''$) is reversed.

Figure 4.36

The reader may recognize the correspondence between some of these motions and the Möbius transformations discussed in earlier chapters. The rotation corresponds to the elliptic Möbius transformation, the horolation to the parabolic, and the translation to the proper hyperbolic. As noted previously, we do not see examples of the improper hyperbolic or loxodromic transformations in the hyperbolic models.

§ 4.5 Factorization Methods in the Poincaré Disk Model

We now take a different approach to the motions in the disk model. We assume that we are given a triangle, $\triangle ABC$ and its image, $\triangle A'B'C'$. (Recall that length is invariant and angle measures are preserved by any isometry in the model.) Our task will be to find the factorization of the isometry represented in each figure. Specifically, we must find the line or lines in the model that properly relate the two triangles by an inversion or by a composition of inversions. We provide instructions for each of the factorizations. For the even isometries (i.e., the rotation, horolation, and the translation) and for the glide reflection, we will utilize our definitions of the standard factorization and the symmetric factorization as presented in Chapter 3, and provide instructions for both factorizations.

As previously stated, the Identity may be represented by two inversions with respect to any single line in the model. As a trivial example, an illustration of the identity is not included. In Figure 4.37, we illustrate the factorization of a reflection. In order to find the line ℓ that relates $\triangle ABC$ and its image, we perform the following construction:

Factorization of a Reflection: Given the P-disk with center O , $\triangle ABC$, and $\triangle A'B'C'$.

- 1) Construct (Euclidean) rays $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, and $\overrightarrow{CC'}$. The rays meet at point P .
- 2) Construct $\overline{OP}$ and midpoint M .
- 3) Construct circle $(M, \overline{MO})$. The circle meets the P-disk at points Q and R .
- 4) Construct circle $(P, \overline{PQ})$.
- 5) Construct the hyperbolic line that lies on circle $(P, \overline{PQ})$: Construct two points on circle $(P, \overline{PQ})$. Drag points to the interior of the P-disk, if necessary. Use the Poincaré line tool to construct the hyperbolic line ℓ .

The reflection is inversion in ℓ .

Standard Factorization of a Rotation: Given the P-disk, center O , ΔABC and $\Delta A''B''C''$.

- 1) Construct perpendicular bisectors of $\overline{AA''}$ and $\overline{BB''}$. (You may use $\overline{CC''}$ for one of the choices.) The perpendicular bisectors j and k meet at point Q .
- 2) Construct the diameter of the P-disk through $\overline{OQ} = \ell_2$.
- 3) Reflect $\Delta A''B''C''$ with respect to ℓ_2 to obtain $\Delta A'B'C'$.
- 4) Construct (Euclidean) rays $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ (or use $\overrightarrow{CC'}$.) The rays meet at point P .
- 5) Construct a (Euclidean) circle $(P, \overline{PQ})$. The circle meets the P-disk at points R and S .
- 6) Construct the hyperbolic line that lies on circle $(P, \overline{PQ})$. This is line ℓ_1 .

The rotation is a product of inversions in ℓ_1 then in ℓ_2 . See Figure 4.38.

Symmetric Factorization of a Rotation: Given the P-disk, center O , ΔABC and $\Delta A''B''C''$.

- 1) Obtain point P by steps 1-6 above.
- 2) Construct the line through $\overline{OQ}$. (This is the radical axis of an elliptic pencil of circles.)
- 3) Construct the perpendicular to $\overline{OQ}$ through point P .
- 4) The perpendicular meets circle $(P, \overline{PQ})$ at two points. Choose one of these points.

Call it V .

- 5) Reflect V in $\overline{OQ}$ to construct point W .
- 6) Construct circles $(V, \overline{VQ})$, $(W, \overline{WQ})$ and their points of intersection with the P-disk.
- 7) Construct the hyperbolic lines that lie on each of these circles. These are the two lines, ℓ_1 (with center V) and ℓ_2 (with center W), that are the factors of the rotation.

The rotation is a product of inversions in ℓ_1 then in ℓ_2 . See Figure 4.39.

Figure 4.38

Figure 4.39

Standard Factorization of a Horolation: Given the P-disk, center O , $\triangle ABC$ and $\triangle A''B''C''$.

- 1) Construct perpendicular bisectors of hyperbolic segments $\overline{AA''}$ and $\overline{BB''}$. (You may use $\overline{CC''}$ for one of the choices.) The perpendicular bisectors j and k meet at point Q .
- 2) Construct the diameter of the P-disk through $\overline{OQ} = \ell_2$.
- 3) Reflect $\triangle A''B''C''$ with respect to ℓ_2 to obtain $\triangle A'B'C'$.
- 4) Construct (Euclidean) rays $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ (or use $\overrightarrow{CC'}$.) The rays meet at point P .
- 5) Construct a (Euclidean) circle $(P, \overline{PQ})$.
- 6) Construct the hyperbolic line that lies on circle $(P, \overline{PQ})$. This is line ℓ_1 .

The horolation is a product of inversions in ℓ_1 then in ℓ_2 . See Figure 4.40.

Symmetric Factorization of a Horolation: Given P-disk, center O , $\triangle ABC$ and $\triangle A''B''C''$.

- 1) Obtain point P by steps 1-6 above.
- 2) Construct the perpendicular to $\overline{OQ}$ through point P .
- 3) The perpendicular meets circle $(P, \overline{PQ})$ at point V .
- 4) Construct a (Euclidean) circle $(V, \overline{VQ})$.
- 5) Construct the hyperbolic line on circle $(V, \overline{VQ})$. This is line ℓ_1 .
- 6) Reflect ℓ_1 with respect to $\overline{OQ}$. This is line ℓ_2 . (Alternately, construct point W by reflection in $\overline{OQ}$ and repeat steps 4 and 5.)

The horolation is a product of inversions in ℓ_1 then in ℓ_2 . See Figure 4.41.

Figure 4.40

Figure 4.41

Standard Factorization of a Translation: Given P-disk, center O , ΔABC and $\Delta A''B''C''$.

- 1) Construct perpendicular bisectors of hyperbolic segments $\overline{AA''}$ and $\overline{BB''}$. (You may use $\overline{CC''}$ for one of the choices.) Call the perpendicular bisectors j and k .
- 2) Find the Euclidean centers of the hyperbolic lines j and k . Call them J and K .

- 3) Construct the (Euclidean) line $\overleftrightarrow{JK}$ which meets the P-disk at points T and U .
- 4) Construct the midpoint of the (Euclidean) segment $\overline{TU}$. Call it M .
- 5) Construct the diameter of the P-disk through $\overline{OM}$. This line is ℓ_2 .
- 6) Reflect $\Delta A''B''C''$ with respect to ℓ_2 to obtain $\Delta A'B'C'$.
- 7) Construct (Euclidean) ray $\overrightarrow{AA'}$ ($\overrightarrow{BB'}$ or $\overrightarrow{CC'}$.) The ray meets $\overleftrightarrow{JK}$ at point P .
- 8) Construct the midpoint of (Euclidean) segment $\overline{OP}$. Call it N .
- 9) Construct a (Euclidean) circle $(P, \overline{PN})$. The circle meets the P-disk at points R and S .
- 10) Construct a (Euclidean) circle $(P, \overline{PR})$.
- 11) Construct the hyperbolic line that lies on circle $(P, \overline{PR})$. This is line ℓ_1 .

(To find line c , the center of the hyperpencil, invert point M with respect to the P-disk. The inverse point is Q , with line c on the circle $(Q, \overline{QT})$.)

The translation is a product of inversions in ℓ_1 then in ℓ_2 . See Figure 4.42.

Symmetric Factorization of a Translation: Given P-disk, center O , ΔABC and $\Delta A''B''C''$.

- 1) Complete the steps above.
- 2) Circle $(P, \overline{PR})$ meets $\overleftrightarrow{JK}$ at two points. Call the point farther from M point V .
(A different construction may be obtained with the remaining point.)
- 3) Construct (Euclidean) segment $\overline{OV}$ and its midpoint N' .
- 4) Construct (Euclidean) circle $(N', \overline{N'O})$. The circle meets the P-disk at X' and Y' .
- 5) Construct (Euclidean) circle $(V, \overline{VX'})$, and construct the hyperbolic line that lies on circle $(V, \overline{VX'})$. This is line ℓ_1 .

- 6) Reflect ℓ_1 with respect to $\overline{OM}$. This is line ℓ_2 . (Alternately, construct point W by reflection in $\overline{OM}$ and repeat steps 4 and 5.)

The translation is a product of inversions in ℓ_1 then in ℓ_2 . See Figure 4.43.

Figure 4.42

Figure 4.43

Standard Factorization of a Glide Reflection: Given P-disk, center O , $\triangle ABC$ and $\triangle A'''B'''C'''$.

- 1) Construct (Euclidean) rays $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, and $\overrightarrow{CC'}$. If the rays do not meet at a single point P , proceed with the following steps.
- 2) Construct hyperbolic segments $\overline{AA''}$ and $\overline{BB''}$ (or $\overline{CC''}$.)
- 3) Construct the midpoints of segments $\overline{AA''}$ and $\overline{BB''}$. Call them P and Q , respectively.

- 4) Construct the hyperbolic line $\overleftrightarrow{PQ}$. This is line c , center of the Hyperpencil we need.
- 5) Line c meets the P-disk at points X and Y . Construct the (Euclidean) segment $\overline{XY}$ and its midpoint, M .
- 6) Construct the diameter of the P-disk through $\overline{OM}$. This is line ℓ_2 .
- 7) Find the Euclidean center of line c . Call it R .
- 8) Invert $\Delta A'''B'''C'''$ with respect to (Euclidean) circle $(R, \overline{RX})$ to obtain $\Delta A''B''C''$.
- 9) Reflect $\Delta A''B''C''$ with respect to ℓ_2 to obtain $\Delta A'B'C'$.
- 10) Construct (Euclidean) rays $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ (or use $\overrightarrow{CC'}$.) The rays meet at point S .
- 11) Construct (Euclidean) segment $\overline{OS}$ and its midpoint, N .
- 12) Construct (Euclidean) circle $(N, \overline{NO})$. The circle meets the P-disk at points T and U .
- 13) Construct (Euclidean) circle $(S, \overline{ST})$ and construct the hyperbolic line that lies on $(S, \overline{ST})$. This is line ℓ_1 .

The glide reflection is a product of inversions in ℓ_1 , ℓ_2 , and then in c . See Figure 4.44.

Symmetric Factorization of a Glide Reflection: Given P-disk, center O , ΔABC and $\Delta A'''B'''C'''$.

- 1) Complete the steps above.
- 2) Construct the (Euclidean) line $\overleftrightarrow{XY}$. Circle $(N, \overline{NO})$ meets $\overleftrightarrow{XY}$ at two points. Call the point farther from M point V .
- 3) Construct (Euclidean) segment $\overline{OV}$ and its midpoint N' .
- 4) Construct (Euclidean) circle $(N', \overline{N'O})$. The circle meets the P-disk at X' and Y' .

- 5) Construct (Euclidean) circle $(V, \overline{VX'})$, and construct the hyperbolic line that lies on circle $(V, \overline{VX'})$. This is line ℓ_1 .
- 6) Reflect ℓ_1 with respect to $\overline{OM}$. This is line ℓ_2 . (Alternately, construct point W by reflection in $\overline{OM}$ and repeat steps 4 and 5.)

The glide reflection is a product of inversions in ℓ_1 , ℓ_2 , and then in c . See Figure 4.45.

Figure 4.44

Figure 4.45

So now, we can determine the type of motion that relates any two congruent triangles in the Poincaré disk. First, we consider the orientation of each set of vertices to identify an even or odd isometry. Then, we can use the construction methods described in this section to obtain a suitable factorization of the motion. It should be clear to the reader that, for all isometries except the reflection, these factorizations are not unique.

Chapter 5

Hyperbolic Trigonometry

§ 5.1 Trigonometric Functions in the Poincaré Model

Trigonometry is the study of the relationship among the sides and angles of a triangle. In Euclidean geometry, the *sine* of an acute angle is defined to be the ratio of the opposite side to the hypotenuse in any right triangle having that acute angle as one of its angles. We have similar definitions for *cosine*, *tangent*, and the other trigonometric functions. These definitions inherently utilize the similarity of triangles in the Euclidean plane. We demonstrated in Figure 4.20 that there are no similar triangles in the hyperbolic plane. So we need to take a different approach if we wish to develop suitable definitions for trigonometry in hyperbolic geometry.

In Calculus, we are introduced to the hyperbolic functions:

$$\text{hyperbolic sine:} \quad \sinh x = \frac{e^x - e^{-x}}{2};$$

$$\text{hyperbolic cosine:} \quad \cosh x = \frac{e^x + e^{-x}}{2};$$

$$\text{hyperbolic tangent:} \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}};$$

and their multiplicative inverses $\operatorname{csch} x$, $\operatorname{sech} x$, and $\operatorname{coth} x$, respectively. These functions are analogous in many ways to the Euclidean trigonometric functions. The name “hyperbolic functions” comes from the identity

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1. \quad (5.1)$$

For the parametric equations $x = \cosh \theta$ and $y = \sinh \theta$, the identity (5.1) gives one branch of the hyperbola $x^2 - y^2 = 1$ in the Cartesian plane. (The analogous identity for the circular

functions is $\cos^2 x + \sin^2 x = 1$. In this case, the parametric equations $x = \cos \theta$ and $y = \sin \theta$ give the Cartesian unit circle $x^2 + y^2 = 1$.)

We list the identities for hyperbolic and circular functions:

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos^2 x + \sin^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$1 + \tan^2 x = \sec^2 x$$

$$\coth^2 x + 1 = \operatorname{csch}^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2}$$

$$\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$$

$$\cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}$$

$$\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$$

$$\tanh^2 \frac{x}{2} = \frac{\cosh x - 1}{\cosh x + 1}$$

$$\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$$

$$\sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$\sinh x \pm \sinh y = 2 \sinh \frac{x \pm y}{2} \cosh \frac{x \mp y}{2}$$

$$\sin x \pm \sin y = 2 \sin \frac{x \pm y}{2} \cos \frac{x \mp y}{2}$$

$$\cosh x + \cosh y = 2 \cosh \frac{x + y}{2} \cosh \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cosh x - \cosh y = 2 \sinh \frac{x + y}{2} \sinh \frac{x - y}{2}$$

$$\cos x - \cos y = 2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

These identities provide a nice correlation between the hyperbolic and circular functions, but we still do not have definitions of the hyperbolic trigonometric functions in terms of a hyperbolic triangle.

Recall that, given any line ℓ and any point P not on ℓ , there are two lines through P that are critically parallel to ℓ . In Figure 5.1, we illustrate lines m and n through P that are critically parallel to ℓ . Furthermore, if $\overline{PQ}$ is the perpendicular from P to ℓ and points X and Y are on m and n , respectively, we see that $\angle XPQ \cong \angle YPQ$. Thus, either of these angles can be called the *critical angle* for P with respect to ℓ .

Figure 5.1

It can be shown that the number α of radians in the critical angle depends only on the distance d from P to Q , not on the particular line ℓ or on the particular point P . We define the *critical function* Π from the positive reals into the positive reals by

$$\Pi(PQ) = \alpha. \quad (5.2)$$

As P varies, $\Pi(PQ)$ takes on all possible values between 0 and $\pi/2$ radians.

The following formula relating α and $d = d(P, Q)$ was discovered by J. Bolyai and

Lobachevsky:
$$\tan \frac{\alpha}{2} = e^{-\frac{d}{k}} \quad (5.3)$$

where k is the constant chosen for the distance scale. (Proofs of this formula can be found in Kulczycki [12] and in Borsuk and Szmielew [3].) Recall that in the Poincaré model, we let $k = 1$. Thus, the fundamental formula becomes

$$\tan \frac{\alpha}{2} = e^{-d}. \quad (5.4)$$

An equivalent expression for α as a function of the real number x is given by

$$\Pi(x) = 2 \arctan e^{-x}. \quad (5.5)$$

We are now in a position to prove a most useful theorem. The relationships in this theorem establish a fundamental correlation between the Euclidean and hyperbolic lengths of a segment. We shall prove part (b); parts (a) and (c) may be proven in a similar manner.

Theorem 5.1: For all real x , the following hold:

a) $\sin \Pi(x) = \operatorname{sech} x = \frac{1}{\cosh x}$

b) $\cos \Pi(x) = \tanh x$

c) $\tan \Pi(x) = \operatorname{csch} x = \frac{1}{\sinh x}$

Proof: In part (b), let x be a real number. Without loss of generality, let x be the length of a hyperbolic segment $\overline{OZ}$ in the Poincaré disk model such that $O = (0, 0)$ and $Z = (0, z)$ for $-1 < z < 1$. (To be precise, $z = OZ$ in Euclidean measure and $x = |\ln(0, z, -1, 1)|$ in hyperbolic measure.) By definition (5.2), $\Pi(x)$ is the critical angle for Z with respect to the real axis. Let ℓ be the critically parallel line through Z and $B = (1, 0)$. Line ℓ is an arc of a circle δ that is orthogonal to the unit circle. Let P be the center of δ , and let the tangent to

δ at Z meet the real axis at point A . The critical angle $\Pi(x)$ is equivalent to the angle $\theta = \angle OZA$. (See Figure 5.2.)

Consider the two triangles ΔPAZ and ΔPAB . We have $PZ = PB$, $PA = PA$, and $m\angle PZA = m\angle PBA = \pi$. Thus, by the hypotenuse-leg theorem for right triangles, we have $\Delta PAZ \cong \Delta PAB$. Let $c = AZ = AB$. Then $OA = 1 - c$.

Figure 5.2

Now, for right triangle ΔAZO , with $z = OZ$, by the Pythagorean theorem, we have

$$z^2 + (1 - c)^2 = c^2$$

$$z^2 + 1 - 2c + c^2 = c^2$$

$$1 + z^2 = 2c$$

$$c = \frac{1 + z^2}{2}.$$

So we have $\cos \Pi(x) = \cos \theta = \frac{z}{c} = \frac{z}{\frac{1+z^2}{2}} = \frac{2z}{1+z^2}.$

Now, we consider the hyperbolic measure $x = |\ln(0, z, -1, 1)|$. Since z may be considered to be positive, we may disregard the absolute value. Calculating the cross ratio, we have $x = \ln \left(\frac{(1-0)(z-(-1))}{(z-1)(-1-0)} \right) = \ln \left(\frac{1+z}{1-z} \right)$. Exponentiating gives $e^x = \frac{1+z}{1-z}$.

So for $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, we have $\tanh x = \frac{\frac{1+z}{1-z} - \left(\frac{1-z}{1+z} \right)}{\frac{1+z}{1-z} + \left(\frac{1-z}{1+z} \right)} = \frac{(1+z)^2 - (1-z)^2}{(1+z)^2 + (1-z)^2} = \frac{2z}{1+z^2},$

as desired. ◆

To summarize the results of Theorem 5.1, given $x = d(O, Z)$ and $z = OZ$, we have

the relations $\sinh x = \frac{2z}{1-z^2} \quad \cosh x = \frac{1+z^2}{1-z^2} \quad \tanh x = \frac{2z}{1+z^2}.$

As stated earlier, a Euclidean trigonometric function of an *angle* is defined as a ratio of the sides of a right triangle containing that angle. In contrast, a hyperbolic function of a hyperbolic *length* may be stated as a ratio involving the corresponding Euclidean length. Thus, for a hyperbolic triangle, the hyperbolic functions are defined in terms of the hyperbolic length, x , of a given side (and the associated critical angle and corresponding Euclidean length, z .)

We now consider a useful application of Theorem 5.1 and Figure 5.2. We have

$\sin \theta = \frac{1}{\cosh x} = \frac{1+z^2}{1-z^2}$ and $\cos \theta = \tanh x = \frac{2z}{1+z^2}.$ We utilize the appropriate circular

functions and substitute to obtain the following expressions:

$$\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2} = \frac{1 - \frac{2z}{1+z^2}}{2} = \frac{(1-z)^2}{2(1+z^2)} \quad \text{and} \quad \cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2} = \frac{1 + \frac{2z}{1+z^2}}{2} = \frac{(1+z)^2}{2(1+z^2)}.$$

So now, since $\tan^2 \frac{\theta}{2} = \frac{\sin^2 \frac{\theta}{2}}{\cos^2 \frac{\theta}{2}} = \frac{(1-z)^2}{(1+z)^2}$, we have $\tan \frac{\theta}{2} = \frac{1-z}{1+z} = e^{-x}$. Thus, we are

able to derive the fundamental formula (5.4) directly from the model.

§ 5.2 Sketchpad Illustrations of Theorems in Hyperbolic Trigonometry

In this section, we present illustrations of several theorems of hyperbolic trigonometry, utilizing the Poincaré disk model. The proofs of these theorems are found in Greenberg [6] and Martin [13]. Another approach to this material is found in the article by Eves and Hoggatt [5]. One of the interesting conclusions derived from the study of hyperbolic trigonometry is that for very small triangles, hyperbolic trigonometry is Euclidean.

We include theorems for right triangles, general triangles, and Lambert quadrilaterals. It is interesting to note that some of these theorems have Euclidean analogues, while others do not. Since a general figure may be brought into standard position by a suitable transformation, each of the figures is illustrated in standard position. Values are provided so that the formulas of each theorem may be verified. The accuracy of *Sketchpad* measurements are to the tenth of a degree and to the hundredth in length. Hence, some of the values obtained in the formulas may not be exact.

Theorem 5.2: Given any right triangle ABC , with right angle at C , in the hyperbolic plane, then

$$\cosh c = \cosh a \cosh b.$$

This is the hyperbolic analogue of the Pythagorean Theorem.

$\angle A = 34.1^\circ$	$a = 1.29$	$\cosh a = 1.954$
$\angle B = 25.1^\circ$	$b = 1.06$	$\cosh b = 1.616$
$\angle C = 90^\circ$	$c = 1.82$	$\cosh c = 3.167$

Figure 5.3

Since the values of a , b , and c are estimated by *Sketchpad*, the measurements and computations in these examples are always estimations. In this case,

$$\cosh a \cdot \cosh b = (1.954)(1.616) = 3.158,$$

which does not correspond to the calculated value for $\cosh c$ as shown above.

Theorem 5.3: Given any right triangle ABC , with right angle at C , in the hyperbolic plane, then

$$\sin A = \frac{\sinh a}{\sinh c} \quad \sin B = \frac{\sinh b}{\sinh c} \quad \cos A = \frac{\tanh b}{\tanh c} \quad \cos B = \frac{\tanh a}{\tanh c}$$

$$\cosh a = \frac{\cos A}{\sin B} \quad \cosh b = \frac{\cos B}{\sin A} \quad \cosh c = \cot A \cot B .$$

$a = 1.29$	$\sinh a = 1.68$	$\cosh a = 1.95$	$\tanh a = 0.86$
$b = 1.06$	$\sinh b = 1.27$	$\cosh b = 1.62$	$\tanh b = 0.79$
$c = 1.82$	$\sinh c = 3.01$	$\cosh c = 3.17$	$\tanh c = 0.95$
$\angle A = 34.1^\circ$	$\sin A = 0.56$	$\cos A = 0.83$	$\cot A = 1.48$
$\angle B = 25.1^\circ$	$\sin B = 0.42$	$\cos B = 0.91$	$\cot B = 2.17$
$\angle C = 90^\circ$	$\sin C = 1.00$	$\cos C = 0.00$	

Figure 5.4

Theorem 5.4: For any triangle ABC in the hyperbolic plane

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}.$$

This is the hyperbolic Law of Sines.

$\angle A = 29.7^\circ$	$\sin A = 0.50$
$\angle B = 50.0^\circ$	$\sin B = 0.77$
$\angle C = 42.7^\circ$	$\sin C = 0.68$
$a = 1.60$	$\sinh a = 2.38$
$b = 2.01$	$\sinh b = 3.66$
$c = 1.89$	$\sinh c = 3.23$

Figure 5.5

Theorem 5.5: For any triangle ABC in the hyperbolic plane

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A$$

$$\cosh b = \cosh a \cosh c - \sinh a \sinh c \cos B$$

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C .$$

This is the hyperbolic Law of Cosines.

$$\angle A = 29.7^\circ \quad \cos A = 0.87$$

$$\angle B = 50.0^\circ \quad \cos B = 0.64$$

$$\angle C = 42.7^\circ \quad \cos C = 0.73$$

$a = 1.60$	$\sinh a = 2.38$	$\cosh a = 2.58$
$b = 2.01$	$\sinh b = 3.66$	$\cosh b = 3.80$
$c = 1.89$	$\sinh c = 3.23$	$\cosh c = 3.38$

Figure 5.6

Theorem 5.6: For any triangle ABC in the hyperbolic plane

$$\cosh c = \frac{\cos A \cos B + \cos C}{\sin A \sin B}$$

$$\cos C = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}.$$

These formulas do not have Euclidean analogues.

$\angle A = 29.7^\circ$	$\sin A = 0.50$	$\cos A = 0.87$
$\angle B = 50.0^\circ$	$\sin B = 0.77$	$\cos B = 0.64$
$\angle C = 42.7^\circ$	$\sin C = 0.68$	$\cos C = 0.73$
$a = 1.60$	$\sinh a = 2.38$	$\cosh a = 2.58$
$b = 2.01$	$\sinh b = 3.66$	$\cosh b = 3.80$
$c = 1.89$	$\sinh c = 3.23$	$\cosh c = 3.38$

Figure 5.7

Theorem 5.7: Given Lambert quadrilateral $ABCD$, let $u = AD$, $v = AB$, $w = CD$, and $z = BC$. Then

$$\text{a) } \sinh w = \sinh v \cosh z \quad \text{a') } \sinh z = \sinh u \cosh w$$

$$\text{b) } \tanh w = \cosh u \tanh v \quad \text{b') } \tanh z = \cosh v \tanh u$$

$$\text{c) } \sin C = \frac{\cosh v}{\cosh w} = \frac{\cosh u}{\cosh z}$$

$$\text{d) } \cos C = \sinh u \sinh v = \tanh w \tanh z$$

$$\text{e) } \cot C = \tanh u \sinh w = \tanh v \sinh z.$$

$u = 1.15$	$\sinh u = 1.42$	$\cosh u = 1.74$	$\tanh u = 0.82$
$v = 0.55$	$\sinh v = 0.58$	$\cosh v = 1.16$	$\tanh v = 0.50$
$w = 1.34$	$\sinh w = 1.78$	$\cosh w = 2.04$	$\tanh w = 0.87$
$z = 1.78$	$\sinh z = 2.88$	$\cosh z = 3.05$	$\tanh z = 0.95$
$\angle C = 34.7^\circ$	$\sin C = 0.57$	$\cos C = 0.82$	$\cot C = 1.44$

Figure 5.8

Theorem 5.8: We define similar Lambert quadrilaterals $ABCD$ and $AB'C'D'$ such that the acute angles at C and C' are congruent. Then for similar Lambert quadrilaterals $ABCD$ and $AB'C'D'$, we have

$$\frac{\sinh u}{\sinh u'} = \frac{\sinh v'}{\sinh v}.$$

Figure 5.9

We conclude this section with a discussion that will be useful in Chapter 6. In Figure 5.10, we consider right triangles $\triangle PQO$ and $\triangle PRO$. By Theorem 5.2, we have $\cosh PQ = \cosh OQ \cdot \cosh OP$ and $\cosh PR = \cosh OR \cdot \cosh OP$. Since

$$\cosh OP = \frac{\cosh PQ}{\cosh OQ} = \frac{\cosh PR}{\cosh OR}, \text{ we have } \cosh PQ \cdot \cosh OR = \cosh PR \cdot \cosh OQ.$$

We know $OQ < OR$, and since $f(x) = \cosh x$ is strictly increasing for $x > 0$, we have $\cosh OQ < \cosh OR$. Thus we must have $\cosh PQ < \cosh PR$, so that $PR > PQ$. Now since $f(x) = \cosh x$ is continuous, we can state that for any $x > PQ$, there is a point Z on $\overleftrightarrow{XY}$ (between Q and Y) such that $PZ = x$.

Figure 5.10

Chapter 6

Hyperbolic Area

§ 6.1 Defect and Triangulation of a Polygonal Region

The concept of area in the Euclidean plane is familiar to us as the two-dimensional measure of the interior of some figure. We recall the formulas for the area of various polygons and note that associated with each one is the appropriate square units. That is to say, we calculate the area of a polygon and determine how many square units fill the interior of the polygon. But as we have seen in the hyperbolic plane, it is not possible to construct a quadrilateral with four right angles, so squares do not exist. Hence, the concept of square units as a measure of area cannot be applied in the hyperbolic plane.

How then can we define the concept of “area” in hyperbolic geometry? If we wish to define a corresponding function in the hyperbolic plane, we should at least preserve two important properties of area: 1) invariance under congruence and 2) additivity. For example, congruent triangles must have the same area; and if a triangle $\triangle ABC$ is split into two triangles, $\triangle ABD$ and $\triangle ACD$, then the area of $\triangle ABC$ is equal to the sum of the areas of $\triangle ABD$ and $\triangle ACD$. The quantity or function which satisfies these properties and which leads to a satisfying definition of area in the hyperbolic plane is the *defect*. The following definitions and theorems related to defect and hyperbolic area have been adapted from Martin [13] and Moise [14]. We illustrate all of the theorems in the Poincaré disk model.

Definition 6.1: The *defect* of a convex polygon P with n sides is the positive difference between $(n - 2)\pi$ and the sum of the measures of the n angles of P . Given a polygon with vertices $P_1, P_2, \dots, P_n$, suppose that for each pair of successive vertices P_i, P_{i+1} all other vertices lie on the same side of $\overline{P_i P_{i+1}}$. Then the polygon is *convex*. The union of a

convex polygon and its interior is a *convex polygonal region* having the same *vertices*, *sides*, *interior*, and *defect* as the convex polygon. If Q is a convex polygon or a convex polygonal region, then $\text{int}(Q)$ and δQ are the interior of Q and the defect of Q , respectively. The union of a triangle and its interior is a *triangular region*. In particular, the defect of a triangle is $180^\circ - \text{angle sum}$. A *polygonal region* R is the union of a positive number of triangular regions $T_1, T_2, \dots, T_n$ such that $\text{int}(T_i)$ and $\text{int}(T_j)$ are disjoint for $i \neq j$. Further, if $T = \{T_1, T_2, \dots, T_n\}$ and δT is the sum of the defects of the triangular regions in T , then the set T is a *triangulation* of R and the number δT is the *defect* of the triangulation T . If S and T are triangulations of the same polygonal region such that each triangular region in S lies in some triangular region of T , then S is a *subtriangulation* of T .

The sides of a convex polygonal region and any point in the interior determine triangular regions that can be used to show a convex polygonal region is indeed a polygonal region. Such a configuration (see Figure 6.1) is called a *star triangulation*.

Figure 6.1

Theorem 6.2: If T_1 and T_2 are star triangulations of the same convex polygonal region R , then $\delta T_1 = \delta T_2$. (See Figure 6.2.)

Figure 6.2

Proof: The reason is that the total defect in each star triangulation is

$$\begin{aligned} & (n)180^\circ - (m\angle P_1 + m\angle P_2 + \dots + m\angle P_n + 360^\circ) \\ &= (n-2)180^\circ - (m\angle P_1 + m\angle P_2 + \dots + m\angle P_n). \quad \blacklozenge \end{aligned}$$

Since, by definition, this is equal to the defect of R itself, we state the following corollary.

Corollary 6.3: The defect δR of a convex polygonal region R is the number which is the defect of every star triangulation of R . $\blacklozenge$

In fact, any triangulation of a convex polygonal region R is not unique. But any two triangulations of the same convex polygonal region have the same defect, and these are equal to the defect of the convex polygonal region, δR . We shall consider these ideas more carefully in the following theorems.

Theorem 6.4: If a line ℓ with halfplanes H and K intersects the interior of a convex polygonal region R , then $R \cap (H \cup \ell)$ and $R \cap (K \cup \ell)$ are convex polygonal regions whose union is R and whose defects add up to δR . (See Figure 6.3.)

Figure 6.3

Proof: Let R be a convex polygonal region with m sides. Then by the definition above, $\delta R = (m - 2)\pi - (\text{angle sum of } R)$. If line ℓ intersects R , as in Figure 6.3, then ℓ cuts R into two convex polygonal regions as shown. Further, $R_1 = R \cap (H \cup \ell)$ has $m_1 + 1$ sides, and $R_2 = R \cap (K \cup \ell)$ has $m_2 + 1$ sides, with $m_1 + m_2 = m + 2$. We see that the defect of R_1 , $\delta R_1 = (m_1 + 1 - 2)\pi - (\text{angle sum of } R_1) = (m_1 - 1)\pi - (\text{angle sum of } R_1)$, and the defect of R_2 , $\delta R_2 = (m_2 + 1 - 2)\pi - (\text{angle sum of } R_2) = (m_2 - 1)\pi - (\text{angle sum of } R_2)$. Thus $\delta R_1 + \delta R_2 = (m_1 + m_2 - 2)\pi - [(\text{angle sum of } R_1) + (\text{angle sum of } R_2)]$. And since supplementary angles occur between the two pairs of adjacent vertices of R_1 and R_2 , we have $\delta R_1 + \delta R_2 = (m_1 + m_2 - 2)\pi - 2\pi - (\text{angle sum of } R) = (m - 2)\pi - (\text{angle sum of } R) = \delta R$, as desired. ♦

We note that there are two other cases to consider as shown in Figures 6.4 and 6.5. For Figure 6.4, $m_1 + m_2 = m + 1$; for Figure 6.5, we have $m_1 + m_2 = m + 1$. The proofs for these cases are similar to Theorem 6.4.

Figure 6.4

Figure 6.5

Theorem 6.5: Any two triangulations of a polygonal region have a common subtriangulation. That is, if S and T are triangulations of R , then there is a triangulation U of R which is a subtriangulation of both S and T .

Proof: (In Figure 6.6, the edges of S are drawn in pink and the edges of T are drawn in green.) Let $\ell_1, \ell_2, \dots, \ell_n$ be the lines which contain either an edge of S or an edge of T . (For Figure 6.6, we would have $n = 9$.)

Figure 6.6

Now ℓ_1 decomposes each S_i in S (and each T_j in T) into two convex regions if ℓ_1 intersects the interiors of these sets at all:

Figure 6.7

By induction, it now follows that the union of all of the ℓ_i 's decomposes R into a finite collection, $C = \{C_1, C_2, \dots, C_n\}$, of convex polygonal regions. (See Figure 6.8.)

Figure 6.8

Evidently every C_i lies in some *one* $S_j \in S$ and in some *one* $T_k \in T$. For each C_i we take a star triangulation. These fit together to give our common subtriangulation U . ♦

Theorem 6.6: Any two triangulations of the same polygonal region have the same defect.

Proof: Let S and T be two triangulations of the same polygonal region R such that U is a common subtriangulation of S and T . By Theorem 6.5, U is obtained by taking star triangulations of a finite collection $C = \{C_1, C_2, \dots, C_n\}$, of convex polygonal regions. By Corollary 6.3, the defect of each C_i is equal to the defect of its star triangulation. By a finite number of applications of Theorem 6.4, we have $\delta U = \delta C_1 + \delta C_2 + \dots + \delta C_n$. Now, since every C_i lies in some *one* $S_j \in S$ and in some *one* $T_k \in T$, we have $\delta U = \delta S$ and $\delta U = \delta T$. Therefore $\delta S = \delta T$, as desired. ♦

§ 6.2 Equivalence by Triangulation

Definition 6.7: If $S = \{S_1, S_2, \dots, S_n\}$ and $T = \{T_1, T_2, \dots, T_n\}$ are triangulations of polygons R_1 and R_2 , respectively, such that S_i is congruent to T_i for $i = 1, 2, \dots, n$, then we say R_1 and R_2 are *equivalent by triangulation* and write $R_1 \equiv R_2$. By convention, two congruent polygons are equivalent by triangulation.

Theorem 6.8: Equivalence by triangulation is an equivalence relation in the set of polygons.

Proof: Trivially, $\equiv$ is reflexive and symmetric. We must now show that if $R_1 \equiv R_2$ and $R_2 \equiv R_3$, then $R_1 \equiv R_3$. Since $R_1 \equiv R_2$, then R_1 and R_2 have triangulations S and T , respectively, such that $S = \{S_1, S_2, \dots, S_n\}$ and $T = \{T_1, T_2, \dots, T_n\}$ with $S_i \cong T_i$ for $i = 1, 2, \dots, n$. Since $R_2 \equiv R_3$, then R_2 and R_3 have triangulations T' and U such that $T' = \{T'_{n+1}, T'_{n+2}, \dots, T'_{n+m}\}$ and $U = \{U_1, U_2, \dots, U_m\}$, respectively, such that $T'_{n+j} \cong U_j$ for $j = 1, 2, \dots, m$. Let V be a common subtriangulation of T and T' . Suppose $V = \{V_1, V_2, \dots, V_t\}$. It follows that R_1 has a triangulation $\{S'_1, S'_2, \dots, S'_t\}$ such that $S'_i \cong V_i$. Likewise, it follows that R_3 has a triangulation $\{U'_1, U'_2, \dots, U'_t\}$ such that $U'_i \cong V_i$. Since congruence of triangles is a transitive relation, we must have $S'_i \cong U'_i$ for $i = 1, 2, \dots, t$. Hence $R_1 \equiv R_3$. ◆

Corollary 6.9: If two polygonal regions are equivalent by triangulation, then the polygonal regions have the same defect. In particular, two triangular regions that are equivalent by triangulation have the same defect. ◆

We have shown that if a triangular region T with defect δT equivalent by triangulation to another triangular region T' , then T' must also have defect δT . The following theorems will show that the converse is also true: two triangular regions with the same defect are equivalent by triangulation. The development of these proofs may seem somewhat circuitous since they rely on the use of an intermediate figure - the Saccheri quadrilateral. But the advantage of this approach can be appreciated when one considers the difficulty of finding suitable triangulations and subtriangulations for any pair of triangles with a given defect.

Theorem 6.10: Given $\triangle ABC$, let D and E be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let G and H be the feet of the perpendiculars to $\overline{DE}$ from B and C , respectively. Then $GBCH$ is a Saccheri quadrilateral. Further, if T is the union of $\triangle ABC$ and its interior and S is the union of $GBCH$ and its interior, then $T \equiv S$. So $\delta GBCH = \delta \triangle ABC$.

Proof: Let F be the foot of the perpendicular from A to $\overline{DE}$. Either D, G, F are distinct or coincident. If D, G, F are distinct then $\triangle BDG \cong \triangle ADF$ by SAA. In either case, $BG = AF$ and $DG = DF$. Similarly, either E, H, F are distinct or coincident. If E, H, F are distinct then $\triangle AEF \cong \triangle CEH$ by SAA. In either case, $AF = CH$ and $EF = EH$. Then $BG = AF = CH$ and $GBCH$ is a Saccheri quadrilateral. (See Figure 6.9.) Further,

$$\delta \triangle ABC = 180^\circ - (m\angle A + m\angle B + m\angle C).$$

$$\begin{aligned} \text{And} \quad \delta GBCH &= 360^\circ - (m\angle CBG + m\angle BGH + m\angle GHC + m\angle HCB) \\ &= 360^\circ - (m\angle CBA + m\angle DBG + 90^\circ + 90^\circ + m\angle HCE + m\angle ACB). \end{aligned}$$

$$\text{Since} \quad m\angle A = m\angle DAF + m\angle FAE = m\angle DBG + m\angle HCE,$$

$$m\angle B = m\angle CBA, \text{ and } m\angle C = m\angle ACB,$$

we see

$$\delta GBCH = 180^\circ - (m\angle A + m\angle B + m\angle C) = \delta\Delta ABC.$$

◆

$$\begin{array}{llll} AD = 1.50 & AE = 1.39 & GB = 1.33 & AF = 1.33 \\ BD = 1.50 & CE = 1.39 & HC = 1.33 & AFD = 89.9^\circ \end{array}$$

$$CAB = 26.7^\circ \quad ABC = 22.0^\circ \quad BCA = 28.0^\circ$$

$$180 - (CAB + ABC + BCA) = 103.27$$

$$BGH = 90.0^\circ \quad GBC = 38.3^\circ \quad DBG = 16.4^\circ \quad HCE = 10.3^\circ$$

$$CHG = 90.0^\circ \quad HCB = 38.3^\circ \quad DAF = 16.4^\circ \quad FAE = 10.3^\circ$$

$$360 - (BGH + GBC + HCB + CHG) = 103.2^\circ$$

Figure 6.9

While our main goal is to prove that any two triangles with the same defect are equivalent by triangulation, the next several theorems will focus on pairs of triangles with at least one pair of congruent sides. These theorems will lead to some interesting and important facts. We begin with a lemma.

Lemma 6.11: Any segment with endpoints on a pair of corresponding hypercircles in a hyperpencil has its midpoint on the line that is the center of the hyperpencil.

Proof: For points X and Y at infinity, let $\overleftrightarrow{XY}$ be a line in the model and let P and R be any two points such that XPY and XRY are corresponding hypercircles. Since any line in the model can be mapped to a diameter of the disk, we can assume that diameter $\overleftrightarrow{XY}$ is the center of the hyperpencil, with XPY and XRY equidistant from $\overleftrightarrow{XY}$. (See Figure 6.10.)

Let the line $\overleftrightarrow{PR}$ intersect $\overleftrightarrow{XY}$ at Q . Then if $\overline{PM}$ and $\overline{RN}$ are perpendicular to $\overleftrightarrow{XY}$, $\angle PMQ \cong \angle RNQ$ and $PM = RN$. Since $\angle PQM$ and $\angle RQN$ are vertical angles, we see that $\angle PQM \cong \angle RQN$. By SAA, $\triangle PMQ \cong \triangle RNQ$. Thus, we see that $PQ = QR$ and Q is the midpoint of $\overleftrightarrow{PR}$, as desired. ♦

$PMQ = 90.0^\circ$	$RNQ = 90.6^\circ$
$PM = 1.21$	$RN = 1.21$
$PQM = 45.9^\circ$	$RQN = 45.9^\circ$
$PQ = 1.50$	$QR = 1.50$

Figure 6.10

Theorem 6.12: Given $\triangle ABC$ and $x > AB$, there exists a point P such that $PB = x$ and the union of $\triangle PBC$ and its interior is equivalent by triangulation to the union of $\triangle ABC$ and its interior.

Proof: Let T be the triangular region determined by $\triangle ABC$. By Theorem 6.10, we can view $\triangle ABC$ as shown in Figure 6.9, so that D and E are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let F , G , and H be the feet of the perpendiculars to $\overline{DE}$ from A , B , and C , respectively. Then by Theorem 6.10, we know that $\overline{AF} = \overline{GB} = \overline{HC}$. Thus, we see that

vertices A, B , and C are on corresponding hypercircles with center line $\overleftrightarrow{DE}$. (See Figure 6.11.) By Lemma 6.11, we know that any segment with endpoints on hypercircles XAY and XBY will have its midpoint on $\overleftrightarrow{DE}$. Since D is the midpoint of $\overline{AB}$, $DB = \frac{AB}{2}$. As discussed in Chapter 5, we can find a point M on $\overleftrightarrow{DE}$ such that $BM = \frac{x}{2}$. If we extend $\overline{BM}$, then P is the point on hypercircle XAY such that $PB = x$. Now, since P and C are on corresponding hypercircles, the midpoint N of $\overline{PC}$ is on center line $\overleftrightarrow{DE}$. Let R be the triangular region determined by $\triangle PBC$. Let F' be the foot of the perpendicular from P to $\overleftrightarrow{MN}$. As in Theorem 6.10, if M, G, F' are distinct, then $\triangle BMG \cong \triangle PMF'$ by SAA. Similarly, if N, H, F' are distinct, then $\triangle CNH \cong \triangle PNF'$ by SAA. So we have $R \cong S$. And since $T \cong S$, by Theorem 6.10, we have $T \cong R$, as desired. $\blacklozenge$

We recall that $T \cong R$ implies $\delta\triangle ABC = \delta\triangle PBC$. A natural question arises: what is the locus of points P such that $\delta\triangle ABC = \delta\triangle PBC$? Figure 6.11 suggests an answer to our question: the locus of points P satisfying the equation above is precisely the hypercircle containing A with center line $\overleftrightarrow{DE}$.

$BM = 1.52$	$PN = 1.38$	$PF'M = 90.0^\circ$
$MP = 1.52$	$NC = 1.38$	

Figure 6.11

Theorem 6.13: Triangular regions with the same defect are equivalent by triangulation.

Proof: Since triangular regions determined by congruent equilateral triangles are obviously equivalent by triangulation, suppose R and S are the triangular regions determined by $\triangle ABC$ and $\triangle DEF$, respectively, where $DE > AB$. (See Figure 6.12.) Suppose R and S both have defect d_0 . By the preceding theorem, there exists a triangular region T determined by $\triangle PBC$ such that $PB = DE$ and $R \equiv T$. So $\delta T = d_0$. Now, by Theorem 6.10, each of T and S is equivalent by triangulation to a polygonal region determined by a Saccheri quadrilateral with defect d_0 and upper base of length DE . Since any two such Saccheri quadrilaterals are congruent, $T \equiv S$. Therefore $R \equiv S$. ♦

Corollary 6.14: If two triangles have the same defect, share common side $\overline{BC}$, and midpoint line $\overline{DE}$, then the third vertices lie on the hypercircle that is the mirror image, with respect to line $\overline{DE}$, of the hypercircle containing B and C . ♦

A simple demonstration of this corollary can be shown with the Geometer's Sketchpad using Figure 6.13. The user can construct the triangles as shown, calculate the angle sums and defect for each of the triangles, then drag point P along hypercircle XAY . While the angle measures change for the angles of $\triangle PBC$, the defect remains the same, so that $\delta\triangle ABC = \delta\triangle PBC$ for all points P along hypercircle XAY . It seems natural to wonder and make the conjecture that perhaps any two triangles with vertices on corresponding hypercircles have the same defect. It is very easy to construct a counterexample with GSP and determine that two triangles with vertices on corresponding hypercircles have the same defect if and only if they have the same length base. (See Figure A.6.14.)

$AB = 1.33$	$PB = 2.90$	$DE = 2.89$
$\angle A = 84.0^\circ$		$\angle D = 18.6^\circ$
$\angle B = 32.1^\circ$		$\angle E = 18.1^\circ$
$\angle C = 11.2^\circ$		$\angle F = 90.4^\circ$
$180 - (\angle A + \angle B + \angle C) = 52.80$		$180 - (\angle D + \angle E + \angle F) = 52.81$

Figure 6.12

$$CAB = 29.7^\circ$$

$$CPB = 26.2^\circ$$

$$ABC = 36.5^\circ$$

$$PBC = 23.9^\circ$$

$$BCA = 30.4^\circ$$

$$BCP = 46.5^\circ$$

$$180 - (CAB + ABC + BCA) = 83.37$$

$$180 - (CPB + PBC + BCP) = 83.33$$

Figure 6.13

$$BC = 3.36$$

$$YZ = 2.77$$

$$CAB = 22.5^\circ$$

$$ZXY = 17.5^\circ$$

$$ABC = 22.5^\circ$$

$$XYZ = 24.7^\circ$$

$$BCA = 19.1^\circ$$

$$YZX = 31.2^\circ$$

$$180 - (CAB + ABC + BCA) = 115.87$$

$$180 - (ZXY + XYZ + YZX) = 106.64$$

Figure 6.14

§ 6.3 Continuity of the Defect

We wish to show that given ΔABC and $0 < x < \delta\Delta ABC$, there exists a unique point P on $\overline{AC}$ such that $\delta\Delta ABP = x$. Such a statement implies that the defect of ΔABP is a function of $m\angle ABP$ and, in fact, that this function is continuous. In order to prove that the defect is continuous, we begin with two lemmas.

Lemma 6.15: Given ΔABC with $\overline{BC} > \overline{BA}$, and points P, E on line $\overline{AC}$ such that P is between A and C , and C is between P and E . Then $BP < BE$ for every point P of $\overline{AC}$. (See Figure 6.15)

Proof: Construct the altitude of ΔABC , so that $\overline{BQ}$ is perpendicular to $\overline{AC}$. (Notice that in Figure 6.15, Q satisfies the requirement for one of our points P above. It may also be the case that A is between Q and C .) Consider ΔQBP and let $\lambda = m\angle BPQ$, $a = \overline{BQ}$, and $c = \overline{BP}$. Similarly, consider ΔQBE and let $\lambda' = m\angle BEQ$ and $c' = \overline{BE}$. From previous theorems, we know $\sin \lambda = \frac{\sinh a}{\sinh c}$ and $\sin \lambda' = \frac{\sinh a}{\sinh c'}$. Now, we must show that

$$\lambda' < \lambda < 90^\circ.$$

SUBPROOF: Since the angle sum of any triangle is less than 180° and $m\angle BQP = 90^\circ$, we see that $\lambda = m\angle BPQ < 90^\circ$. Let $\alpha = m\angle BPE$ and $\beta = m\angle PBE$. We see also that $\alpha + \lambda = 180^\circ$ so that $\lambda = 180^\circ - \alpha$. Since $\alpha + \beta + \lambda' < 180^\circ$, we have $\beta + \lambda' < 180 - \alpha = \lambda$. Further, since $\lambda' < \beta + \lambda'$, we have $\lambda' < \beta + \lambda' < \lambda < 90^\circ$, as desired.

Now, since the sine function is increasing on the interval $(0, 90^\circ)$, we have $\sin \lambda' < \sin \lambda$.

Substituting from above, we have $\frac{\sinh a}{\sinh c'} < \frac{\sinh a}{\sinh c}$, so $\sinh c < \sinh c'$. We recall that the

hyperbolic sine function is monotonic increasing, so $\sinh c < \sinh c'$ if and only if $c < c'$.

Thus, $BP < BE$ for every point P of $\overrightarrow{AC}$, as desired. $\blacklozenge$

$BC = 2.32$	$BP = 1.93$
$BA = 1.97$	$BE = 3.85$

Figure 6.15

Lemma 6.16: Given $\triangle ABC$, and given $\varepsilon > 0$. There exists points E and F on $\overrightarrow{AC}$ such that C is between A and E , and C is between A and F , with $\delta\triangle EBF < \varepsilon$.

Proof: Consider $\triangle ABC$, with altitude $\overline{BQ}$, and point D_1 on $\overrightarrow{AC}$ such that C is between A and D_1 . Let $\overline{CD_1} = m$. There exists a point D_2 on $\overrightarrow{AC}$ such that D_2 is between C and D_1 , with $\overline{CD_2} = \frac{m}{2}$. Similarly, there exists a point D_3 on $\overrightarrow{AC}$ such that D_3 is between C and D_2 , with $\overline{CD_3} = \frac{m}{3}$. Continuing in this manner, there exists a point D_n on $\overrightarrow{AC}$ such that D_n is between C and D_{n-1} , with $\overline{CD_n} = \frac{m}{n}$. Thus we get a sequence of triangles, $\triangle CBD_1$, $\triangle CBD_2$, (See Figure 6.16.) Let d_n be the defect of the n th triangle, $\triangle D_n B D_{n-1}$, and let $d_0 = \delta\triangle CBD_1$. Then $d_1 + d_2 + \dots + d_n < d_0$ for every n . And the infinite series $d_1 + d_2 + \dots + d_n + \dots$, is convergent. Therefore, $\lim_{n \rightarrow \infty} d_n = 0$, so $d_n < \varepsilon$ for some n . ♦

$BC = 1.83$	$CD1 = 1.82$	$CD3 = 0.60$
$BA = 1.80$	$CD2 = 0.91$	

Figure 6.16

Theorem 6.17: Given $\triangle ABC$, if P_r is any point on $\overrightarrow{AC}$, such that P_r is between A and C, and $r = m\angle ABP_r$, then the function $f(r) = \delta\triangle ABP_r$ is continuous.

Proof: Given $0 \leq k \leq m\angle ABC$, we need to show that $\lim_{r \rightarrow k} f(r) = f(k)$. By definition, this means the following condition holds:

- (1) Given $0 \leq k \leq m\angle ABC$. For every $\varepsilon > 0$ there is a $d > 0$ such that if

$$|r - k| < d, \text{ then } |f(r) - f(k)| < \varepsilon.$$

For the geometric interpretation, refer to Figure 6.17. Notice that we have

$|r - k| = m\angle P_k BP_r$. If $r < k$, then we need the absolute value signs to make this formula correct. Using the definition of the function f , and absolute value in case $r < k$, we can also write $|f(r) - f(k)| = |\delta\Delta ABP_r - \delta\Delta ABP_k| = \delta\Delta P_k BP_r$. Using these terms, we can restate Condition (1) as follows:

(2) Given $0 \leq k \leq m\angle ABC$. For every $\varepsilon > 0$ there is a $d > 0$ such that if

$$m\angle P_k BP_r < d, \text{ then } \delta\Delta P_k BP_r < \varepsilon.$$

We shall prove (2). First, let us suppose that $BC > BA$ as the figure suggests. By Lemma 6.15, if E is on $\overrightarrow{AC}$, with C between A and E , then $BP < BE$ for every point P of $\overrightarrow{AC}$.

Further, by Lemma 6.16, we can assert that there are points E and F on $\overrightarrow{AC}$, such that C is between A and E , and C is between A and F , with $\delta\Delta EBF < \varepsilon$. Given such an E and F , we let $d = m\angle EBF$. Now, for P_r and P_k on $\overrightarrow{AC}$, if $m\angle P_k BP_r < m\angle EBF$, then ΔEBF contains a congruent copy $\Delta P_k' BP_r'$ of $\Delta P_k BP_r$. Evidently, $\delta\Delta P_k BP_r = \delta\Delta P_k' BP_r' < \delta\Delta EBF$. Thus for $m\angle P_k BP_r < d$, we have $\delta\Delta P_k BP_r < \varepsilon$. ♦

In addition to the fact that the function $f(r) = \delta\Delta ABP_r$ is continuous, it can be noted that the additivity of the defect makes $f(r)$ strictly monotonic. While the previous proof presents the continuity of the function in a geometric sense (as in Moise), Martin proves the continuity of the defect directly from the trigonometric functions. For a right triangle ABC with right angle at C , $\tan \frac{\delta\Delta ABC}{2} = \tanh \frac{a}{2} \tanh \frac{b}{2}$. Since any triangle can be decomposed into two right triangles, we see that the defect is the composition of continuous, strictly monotonic increasing functions.

$BA = 2.00$	$BP_k = 1.51$	$BP_{k'} = 1.51$
$BC = 2.02$	$BP_r = 1.54$	$BP_{r'} = 1.54$
	$\text{Theta} = 2.0^\circ$	$\text{Theta} = 2.0^\circ$

Figure 6.17

§ 6.4 The Hyperbolic Area Function

It was stated earlier that an area function in hyperbolic geometry should at least preserve the properties of invariance under congruence and additivity. If we wish to be more complete, we should state that an area function α of a polygonal region R satisfies:

$\alpha: R \rightarrow \mathcal{R}$ such that 1) $\alpha R > 0$ for every R ;

2) if R_1 and R_2 intersect only in edges and vertices, then

$$\alpha(R_1 \cup R_2) = \alpha R_1 + \alpha R_2;$$

3) if $R_1 \equiv R_2$, then $\alpha R_1 = \alpha R_2$.

We know that the defect, as we have shown, satisfies these conditions. We now will show that the area function is proportional to the defect. That is, for any polygonal region R , there exists $g > 0$ such that $\alpha R = g\delta R$. It will be sufficient to prove this statement whenever R is a triangular region since any convex polygonal region may be subdivided by a star triangulation into disjoint triangles. The general formula will then follow by addition.

Theorem 6.18: For every two triangles $\triangle ABC$ and $\triangle DEF$, we have

$$\frac{\alpha \triangle DEF}{\alpha \triangle ABC} = \frac{\delta \triangle DEF}{\delta \triangle ABC}. \text{ If this holds, then we have } \frac{\alpha \triangle ABC}{\delta \triangle ABC} = \frac{\alpha \triangle DEF}{\delta \triangle DEF}, \text{ and the desired } g$$

is the fraction on the left; for every $\triangle DEF$, we have $\alpha \triangle DEF = g \delta \triangle DEF$.

Proof: For the case when $\delta \triangle ABC = \delta \triangle DEF$, we have $\triangle ABC \equiv \triangle DEF$ by Corollary 6.9. Now by condition (3) it follows that $\alpha \triangle ABC = \alpha \triangle DEF$. Therefore the theorem holds.

We may assume therefore that $\delta \triangle ABC > \delta \triangle DEF$. By Theorem 6.17, there exists a point P between A and C such that $\delta \triangle ABP = \delta \triangle DEF$. Since the defect is continuous, we

can take a sequence $B = P_0, P_1, \dots, P_i, \dots, P_q = C$, of points, in the stated order on $\overline{BC}$ (see Figure 6.18) such that $\delta\Delta AP_i P_{i+1} = \frac{1}{q}\delta\Delta ABC$ for every i . Thus the segments $\overline{AP_i}$ cut ΔABC into q triangles with the same defect. Applying Corollary 6.9, and conditions (2) and (3), we can see that each of the following conditions is equivalent to the next:

- a) $\frac{i}{q} < \frac{\alpha\Delta DEF}{\alpha\Delta ABC},$
 - b) $\frac{i}{q} < \frac{\alpha\Delta ABP}{\alpha\Delta ABC},$ (since P was chosen so that $\delta\Delta ABP = \delta\Delta DEF$)
 - c) $\frac{i}{q}\alpha\Delta ABC < \alpha\Delta ABP,$ (note that $\alpha\Delta ABP_i = \frac{i}{q}\alpha\Delta ABC$)
 - d) P_i is between B and P , (otherwise $\alpha\Delta ABP_i > \alpha\Delta ABP$)
 - e) $\frac{i}{q}\delta\Delta ABC < \delta\Delta ABP,$ (by the monotonicity of the defect)
- (note also that $\delta\Delta ABP_i = \frac{i}{q}\delta\Delta ABC$)
- f) $\frac{i}{q} < \frac{\delta\Delta ABP}{\delta\Delta ABC},$
 - g) $\frac{i}{q} < \frac{\delta\Delta DEF}{\delta\Delta ABC}$ (by the assumption above).

The additivity of the area and the defect tells us that $\frac{\alpha\Delta DEF}{\alpha\Delta ABC} < 1$ and $\frac{\delta\Delta DEF}{\delta\Delta ABC} < 1$.

By the Comparison Theorem, the theorem follows.

The Comparison Theorem: Let x and y be real numbers. Suppose that every rational

number $\frac{i}{q} < x$ if and only if $\frac{i}{q} < y$. Then $x = y$.

SUBPROOF: Suppose that $x < y$. Then there is a rational number $\frac{i}{q}$ between x and y .

Thus $x < \frac{i}{q} < y$, so that $\frac{i}{q}$ is less than y , but not less than x , which contradicts our

supposition. Similarly, if $y < x$, we have $y < \frac{i}{q} < x$ for some rational number $\frac{i}{q}$. And

now, $\frac{i}{q}$ is less than x , but not less than y , which contradicts our supposition. Therefore

$x = y$, which was to be proved. ◆

Figure 6.18

Hence area must be directly proportional to the defect. Since area should be related to the distance scale k , and to make certain calculations easier, it is convenient to choose g to be k^2 . Thus we state the following definition of hyperbolic area:

Definition 6.19: The *area* αR of a polygonal region R is $k^2 \delta T$ where k is the distance scale and T is any triangulation of R . Recall that in the Poincaré disk model $k=1$, hence $k^2=1$. Thus, we conveniently have $\alpha R = \delta R$ for any polygonal region R .

In Euclidean geometry every polygonal region is equivalent by triangulation to a triangular region. In hyperbolic geometry, however, this cannot be the case since the defect of a polygonal region may be arbitrarily large but, as we have shown, the defect of a triangular region is always less than 180° . The main result we can prove for polygonal regions in hyperbolic geometry is the following:

Theorem 6.20 - Bolyai's Theorem: Two polygonal regions are equivalent by triangulation if and only if the polygonal regions have the same area.

Proof: Let $S = \{S_1, S_2, \dots, S_m\}$ and $T = \{T_1, T_2, \dots, T_n\}$ be any two triangulations of R_1 and R_2 , respectively. Suppose $\alpha R_1 = \alpha R_2$. Some one of the triangular regions in S and T must have minimum defect. We can suppose without loss of generality that S_1 in S has this property. Let T_1 be an element of T . If it happens that $\delta S_1 = \delta T_1$, we delete S_1 from S and T_1 from T . This gives new sets S' and T' with fewer elements than S and T . Let R'_1 and R'_2 be the corresponding regions. If $R'_1 \equiv R'_2$, then $R_1 \equiv R_2$.

If $\delta S_1 < \delta T_1$, then we know by Theorem 6.18 that T_1 can be subdivided into two triangular regions U and V such that $\delta V = \delta S_1$. We now delete S_1 from S , and we replace

T_1 by U in T . If the resulting regions are equivalent by triangulation, then so are R_1 and R_2 . (See Figure 6.19.)

Thus in either case we can reduce our theorem to a case in which the total number of triangular regions is smaller than it was to start with. In a finite number of such steps, we can reduce the theorem to the case of two triangular regions, for which the theorem is known to be true.

Now if we assume $R_1 \equiv R_2$, then by Corollary 6.9 $\delta R_1 = \delta R_2$ and $\alpha R_1 = \alpha R_2$. ♦

$$540 - (\angle EAB + \angle ABC + \angle BCD + \angle CDE + \angle DEA) = 45.1^\circ$$

$$360 - (\angle MJK + \angle JKL + \angle KLM + \angle LMJ) = 45.1^\circ$$

$$180 - (\angle PAB + \angle ABP + \angle BPA) = 7.50$$

$$180 - (\angle RJQ + \angle JQR + \angle QRJ) = 7.56$$

Figure 6.19

Conclusion

Throughout the course of this investigation, our purpose has been to illustrate the basic theory of hyperbolic geometry using *Geometer's Sketchpad*. The Poincaré tools have provided the necessary means to visualize hyperbolic geometry in the Poincaré disk model, while the Euclidean construction tools and several new scripts have been sufficient to study the Poincaré half-plane model.

This thesis only begins the comparative illustration between the Euclidean theorems and their hyperbolic counterparts or counterexamples. The reader can easily select almost any theorem of Euclidean geometry and discover its hyperbolic representation using his or her favorite Poincaré model. Because of the distinctive properties of parallelism and distance in hyperbolic geometry, it would be especially interesting to study the hyperbolic counterparts of curves such as the cycloid and catenary.

There are opportunities for further study of topics already presented in this thesis as well. The factorization methods developed in Chapter 3 provide a geometric means to find the points S and T , the centers of the circles utilized in the standard factorization of a loxodromic Möbius transformation. It would be desirable to find a means to derive these points directly from the matrix of the Möbius transformation.

In § 6.3 we discussed the continuity of the defect with regard to the function $f(r) = \delta\Delta ABP_r$ for a variable angle r in ΔABC . Using *Geometer's Sketchpad*, it would be possible to gather suitable data to plot the function. Furthermore, one could attempt to identify a “best fit” using appropriate regression techniques.

As mentioned in § 3.7, the one-parameter family of Möbius transformations used to study the characteristic parallelogram was adapted from Derrick and Eidswick [3]. This article examines the concepts of periodicity and chaos in the context of continued fractions.

It is possible to show a correspondence between certain periodic points of continued fractions and particular elliptic Möbius transformations. Schwerdtfeger [15] also discusses periodic continued fractions and their relationship to Möbius transformations. The dynamic tools of *Geometer's Sketchpad* provide the means to explore the properties of continued fractions that result in either periodicity or chaos.

Under the general study of Möbius transformations, we find the *discrete groups*.

Beardon [2] defines the *Modular group* of all 2×2 complex matrices $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ with $AD - BC = 1$ and A, B, C, D all integers. More generally, *Picard's group* consists of all 2×2 complex matrices $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ with $AD - BC = 1$ and A, B, C, D all Gaussian integers (that is, $m + in$ where m and n are integers). Schwerdtfeger [15, page 71] shows that a Möbius transformation with finite fixed points γ_1, γ_2 and finite poles z_∞, Z_∞ may be represented by the matrix $M = \begin{pmatrix} Z_\infty & 1 \\ -\gamma_1\gamma_2 & -z_\infty \end{pmatrix}$. Using this form, we observe that for the Modular group and Picard's group, the trace of the matrix indicates the classification (i.e., Elliptic, Proper Hyperbolic, Improper Hyperbolic, Parabolic, Loxodromic) of the Möbius transformation. A systematic study of these restricted groups of Möbius transformations provides some nice applications of the characteristic parallelogram and factorization methods which we have discussed.

In his article "A New Characterization of Möbius Transformations," Haruki [9] proves that the symmedian of a triangle is preserved under a Möbius transformation. His proof utilizes an inscribed equilateral triangle. The *Geometer's Sketchpad* could be easily used to illustrate his proof, as well as demonstrate the results with more general examples of triangles.

We have omitted the study of symmetry and frieze groups in our discussion of hyperbolic geometry. Martin [13] presents a formal discussion of these topics.

Supplementary texts are available for *Geometer's Sketchpad* that explore symmetry and tessellations in Euclidean geometry. Creating examples using the Poincaré tools for hyperbolic geometry would be a fun and instructive project.

There are several tools available on *Geometer's Sketchpad* which we have not been able to utilize and incorporate into this thesis. The standard tools include an *animation* command, a *locus* command, *trace*, and others. These tools could add so much more to the dynamic quality of the material we have discussed, as well as provide further insight into the theory.

In addition, there are other computer programs available which can be explored. *Cabri* is a geometry program that is similar to *Geometer's Sketchpad*. It is available through many schools and is also part of the programming included on the Texas Instruments TI-92 calculator. *Cabri* users may contact cabri-forum@imag.fr on the Internet for an updated set of hyperbolic constructions and macros. Another program called *NonEuclid* is specifically designed to investigate hyperbolic geometry. The article by Austin, et al. [1] provides an overview of *NonEuclid* as well as some empirical results and algorithms used in the program.

In conclusion, this is an exciting time to investigate the theory of hyperbolic geometry. With so many computer tools available, students have an opportunity to visualize and understand this area of mathematics with relative ease. Hopefully, the material presented in this thesis will encourage students to explore hyperbolic geometry so that we all benefit from its further development and have greater understanding of its applications.

Appendix

You will need *Geometer's Sketchpad 3.0* and the Poincaré tools to view and manipulate the figures and scripts listed below. Becoming familiar with the basic use of *Sketchpad* will be time well spent, but here are a few tips on using the contents of the disks.

- 1) After you have opened a sketch, use the "Save As..." command so that you can alter the figure, but still have a copy of the original figure.
- 2) You will not be able to manipulate every part of a particular figure, depending on how it was created during the original construction. Consult the instruction manual for advice on "parents" and "children." Then try to reconstruct the figure so that the part you wish to move is free to do so.
- 3) If something goes wrong, just use the "Undo" command. Often you can undo or redo many steps in a sequence and save yourself some time and extra work.
- 4) You can view the script for a particular construction in a step-by-step mode by using the "Step" button in the Script window. Sometimes this makes it easier to understand what is happening, especially since the script text assigns labels to the construction that do not necessarily match the diagram.
- 5) Explore and have fun!

There are four disks included with the following contents:

Figures 1: Figure 2.1 - Figure 4.20;

Figures 2: Figure 4.21 - Figure 4.45;

Figures 3: Figure 5.1 - Figure 6.15;

Figures 4: Figure 6.16 - Figure 6.19; Chapter 4: More Theorems; Scripts.

References

1. J. D. Austin, et al., An Empirical Exploration of the Poincaré Model for Hyperbolic Geometry, *Mathematics and Computer Education*, 27 (1993) 51-68.
2. A. F. Beardon, *The Geometry of Discrete Groups*, Springer-Verlag, New York, 1983.
3. K. Borsuk, W. Szmielew, *Foundations of Geometry*, North Holland, Amsterdam, 1960.
4. W. Derrick, J. Eidswick, Continued Fractions, Chebychev Polynomials, and Chaos, *The American Mathematical Monthly*, 102 (1995) 337-344.
5. H. Eves, V. E. Hoggatt, Jr, Hyperbolic Geometry Derived From the Poincaré Model, *The American Mathematical Monthly*, 58 (1951) 469-474.
6. M. J. Greenberg, *Euclidean and Non-Euclidean Geometries*, Freeman, New York, 1993.
7. F. P. Greenleaf, *Introduction to Complex Variables*, W. B. Saunders Company, Philadelphia, 1972.
8. L. Hahn, *Complex Numbers and Geometry*, The Mathematical Association of America, Washington, DC, 1994.
9. H. Haruki, A New Characterization of Möbius Transformations, *Mathematicae Notae*, 34 (1990) 55-63.
10. B. Juraschek, Investigating Circles in the Poincaré Disk Using *Geometer's Sketchpad*, *The College Mathematical Journal*, 25 (1994) 145-155.
11. E. L. Kendrick, *Hyperbolic Geometry Within the Poincaré Model*, Master's thesis, San Jose State University, San Jose, CA, 1977.
12. S. Kulczycki, *Non-Euclidean Geometry*, Pergamon, Elmsford, NY, 1961.
13. G. E. Martin, *The Foundations of Geometry and the Non-Euclidean Plane*, Springer-Verlag, New York, 1975.
14. E. E. Moise, *Elementary Geometry from an Advanced Standpoint*, 3rd Edition, Addison-Wesley Publishing Co., Reading, Mass, 1990.
15. H. Schwerdtfeger, *Geometry of Complex Numbers*, Dover Publications, New York, 1979.
16. D. A. Singer, Isometries of the Plane, *The American Mathematical Monthly*, 102 (1995) 628-631.

17. J. R. Smart, *Modern Geometries*, Brooks/Cole Publishing Co., Pacific Grove, CA, 1988.
18. S. Stahl, *The Poincaré Half-Plane*, Jones and Bartlett Publishers, Boston, 1993.

Computer Software:

19. B. Finzer, N. Jackiw, (1995), *Sketchpad 3.0 Gallery - Poincaré* [On-line software], Available Internet:
<http://forum.swarthmore.edu/sketchpad/gsp.gallery/poincare/poincare.html>.
20. *The Geometer's Sketchpad, Version 2* [software manual], Key Curriculum Press, Berkeley, CA, 1992.
21. *The Geometer's Sketchpad, Version 3.0* [software manual], Key Curriculum Press, Berkeley, CA, 1995.
22. *The Geometer's Sketchpad, Version 3.0* [computer software], (1995) Key Curriculum Press, Berkeley, CA.

Please Note

The diskettes are not included in this material. They are however, available for consultation at this author's graduate School library.

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