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The failure of GCH at a measurable cardinal

John F. Freno
San Jose State University

THE FAILURE OF *GCH* AT A MEASURABLE CARDINAL

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John F. Freno
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UMI
300 North Zeeb Road
Ann Arbor, MI 48103

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Dr. M.C. Stanley

Dr. Jon Pearce

Dr. Rudy Rucker

APPROVED FOR THE UNIVERSITY

ABSTRACT

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by John F. Freno

This thesis provides a new proof of the classical result that if the existence of a κ^{++}-supercompact cardinal is consistent with Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC), then the failure of the Generalized Continuum Hypothesis (GCH) at a measurable cardinal is also consistent with ZFC. The new proof differs from the original backwards Easton forcing proof due to Silver in that a much simpler forwards Easton forcing is used, although the role of supercompactness remains the same.

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1. Introduction

As part of the on-going research into the independence of the Generalized Continuum Hypothesis (GCH) from the axioms of standard Zermelo-Frankel set theory with the Axiom of Choice (ZFC), much work has gone into determining whether GCH is independent of all of the many large cardinal axioms as yet devised (inaccessibles, compact cardinals, measurables, etc.) or whether such axioms are inconsistent with either GCH or its negation.

The first definitive result on the relative consistency of GCH and "there exists a measurable cardinal" came with the construction of a model $L[U]$ of ZFC, in which both axioms held [Jech, pp. 359ff]. Hence, $CON(ZFC + GCH+$ There exists a measurable cardinal) was established. However, no similar model of $CON(ZFC + \neg GCH+$ There exists a measurable cardinal) was readily constructed.

The following theorem of Kunen's [Jech, pp. 450ff] provided some insights into the problem:

Theorem. *If there exists a measurable cardinal κ such that $2^\kappa > \kappa^+$, then for every ordinal number θ there exists a transitive model M of ZFC with at least θ measurable cardinals.*

Hence, the violation of GCH at a measurable cardinal was a much stronger hypothesis than the mere existence of a measurable cardinal. In fact, such a violation implied similar violations at many smaller inaccessible cardinals [Jech, pp. 319].

Theorem. *If κ is measurable, and $2^\alpha = \alpha^+$ for every $\alpha < \kappa$, then $2^\kappa = \kappa^+$.*

It was apparent that for $CON(ZFC + \neg GCH+$ There exists a measurable cardinal) to be established a stronger large cardinal assumption would be required. The following theorem of Silver's [Jech, sec. 36] provided the first "upper bound" on what sort of large cardinals would be needed:

Silver's Theorem. *Relative to the consistency of a cardinal κ that is $\kappa^{++}-$supercompact, it is consistent that there exists a measurable cardinal κ such that $2^\kappa = \kappa^{++}$.*

That is, given the consistency of the existence of a κ^{++}-supercompact cardinal, it is consistent that GCH fails at a measurable.

Beyond general questions about the consistency of $\neg GCH$, Silver's Theorem also has applications towards the well-known *Singular Cardinals Problem*: that is, the question of possible values of 2^λ for singular λ. Easton [Kanamori, pp. 122] proved the following theorem, which completely specified the possible values of 2^λ for regular λ:

Easton's Theorem. *Suppose that GCH holds and F is a class function from the class of regular cardinals to the class of cardinals such that for regular $\kappa \leq \lambda : F(\kappa) \leq F(\lambda)$ and $\kappa < cf(F(\kappa))$. Then there is a generic extension preserving cardinals and cofinalities in which $2^\kappa = F(\kappa)$ for every regular κ.*

That is, up to two simple requirements, we can find models of ZFC satisfying whatever regular cardinal exponentiation we desire. However, the corresponding theorem for singular cardinals has yet to be discovered.

Prikry [Kanamori, sec. 18] developed a version of the method of forcing based on the partial ordering:

$$P_F = [\kappa]^{<\omega} \times F,$$

where F is a filter over a cardinal κ, and P_F is ordered by:

$$(s, A) \leq (t, B) \leftrightarrow t \text{ is an initial segment of } s \text{ and } A \cup (s - t) \subseteq B.$$

The main application of Prikry forcing is given by the following theorem:

Prikry's Theorem. *Suppose that κ is measurable and U is a normal ultrafilter over κ. If G is P_U-generic, then $(V_\kappa)^V = (V_\kappa)^{V[G]}$ and the cardinals of V and $V[G]$ coincide, but $cf^{V[G]}(\kappa) = \omega$.*

That is, Prikry forcing turns measurable cardinals into singular cardinals of cofinality ω without collapsing cardinals. This technique, combined with Silver's Theorem, can be used to prove the consistency of the existence of a singular strong limit cardinal κ of cofinality ω such that GCH fails at κ. (Magidor later extended this result to singular strong limit cardinals of uncountable cofinality.) Thus, large singular cardinals were not automatically bound by GCH, but showed some level of flexibility similar to the regular cardinals.

In fact, Silver, building upon these results, later proved that if κ was a singular cardinal of uncountable cofinality with the property that $\{\alpha < \kappa : 2^\alpha = \alpha^+\}$ is stationary in κ, then $2^\kappa = \kappa^+$. Hence, one violation would trigger many smaller violations.

Therefore, with regards to the Singular Cardinals Problem, Silver's Theorem was crucial in demonstrating that naive hopes for easy strong limit singular cardinal exponentiation were unfounded. However, it has also led to significant consistency results for singular cardinals and provided some basic information on the nature of cardinal exponentiation.

Silver originally proved his theorem using a backwards Easton support iteration of Cohen product forcing that adds α^{++} many subsets to each inaccessible less than or equal to κ. The purpose of this paper is to give an alternative proof that avoids such a backwards Easton forcing in favor of a forwards Easton forcing that adds a κ-tree with κ^{++} many branches while preserving the measurability of κ. Only this part of the proof is different from Silver's: the use of supercompactness is essentially the same.

2. A Tour de Forcing

The proof of Silver's Theorem will utilize the mathematical technique known as the method of forcing. To preface the proof, we will first review basic concepts of the method of forcing, as well as develop the necessary technical results needed for the forcing construction used in the proof.

Notation. *In the theorems that follow, "CTM" will be used to abbreviate "countable transitive model of ZFC."*

Definition. *An ordering is a pair $(P, \prec_P)$, where P is a non-empty collection of sets, called **conditions**, and $\prec_P$ is a relation on P such that:*

For all distinct $p_1, p_2, p_3 \in P$, if $p_1 \prec_P p_2$ and $p_2 \prec_P p_3$ then $p_1 \prec_P p_3$.

Notation. *Throughout this paper, except where specific orderings are defined or ambiguity may occur, an ordering* $(P, \prec_P)$ *will be abbreviated as* P.

Notation. *Let* P *be an ordering.* $t \preceq_P t'$ *if* $t \prec_P t'$ *or* $t = t'$.

Notation. *Let* P *be an ordering.* t *is a* $\prec_P$ *-**predecessor** (**resp.*** $\prec_P$*-**successor**) of* t' *if* $t \prec_P t'$ *(resp.* $t' \prec_P t$*).*

Definition. *A **linear ordering*** $(P, \prec_P)$ *is an ordering such that for all distinct* $t, t' \in P$, $t \prec_P t'$ *or* $t' \prec_P t$.

Definition. *A **well-ordering** is a linear ordering* $(P, \prec_P)$ *such that for any* $S \subseteq P$ *there exists a **least element*** $m \in S$ *such that* $m \preceq_P t$ *for all* $t \in S$.

Definition. *A **tree(-ordering)** is an ordering* $(P, \prec_P)$ *such that every non-empty subset* S *is well-ordered.*

Definition. *Let* T *be a tree. If* $x \in T$, *then the **height** of* x *is* $ord(\{y \in T : y \prec_T x\})$. *For ordinal* α, *the* $\alpha-$ *th **level** of* T *is* $\{x \in T : height(x) = \alpha\}$. *The **height** of* T *is the least* α *such that the* α*-th level of* T *is empty. A **branch*** B *of* T *is a linearly ordered subset of* T *such that* $ord(\{height(t) : t \in B\})$ *is an initial segment of the ordinals. A **maximal branch** of* T *is a branch of ordertype height(*T*).*

Definition. *A partial ordering is a triple* $(P, \prec_P, 1_P)$, *where* $(P, \prec_P)$ *is an ordering and* $1_P \in P$ *such that for all* $p \in P$, $p \prec_P 1_P$.

Notation. *In parallel to the above convention, throughout this paper, except where specific partial orderings are defined or where ambiguity may occur, a partial ordering* $(P, \prec_P, 1_P)$ *will be abbreviated as* P.

Definition. *A subset* D *of a partial ordering* P *is **dense** iff for all* $p \in P$ *there exists* $d \in D$ *such that* $d \preceq_P p$.

Definition. *A subset* A *of a partial ordering* P *is an **antichain** if its members are not pairwise comparable in the ordering.*

Definition. *A partial ordering P is $< \lambda$-closed if whenever $\nu < \lambda$ and $\langle p_\gamma : \gamma < \nu \rangle$ is a $\preceq_p$-decreasing sequence of elements of P, there exists $q \in P$ such that for all p_γ, $q \preceq_p p_\gamma$.*

Definition. *A filter over a partial ordering P is a subset F of P such that:*

(1) *If $p, q \in F$, then there exists $r \in F$ such that $r \preceq_P p$ and $r \preceq_P q$. That is, r witnesses that p and q are compatible.*
(2) *If $p, q \in P$, $p \prec_P q$, and $p \in F$, then $q \in F$.*

Definition. *A filter F over a partial ordering P is P-generic iff $F \cap D \neq \emptyset$ for all dense sets $D \subset P$.*

Definition. *Let M be a CTM. Let $P \subset M$ be a partial ordering. Then a P-term is any subset of $a \times P$, where a is any set in M.*

Definition. *Let M be a CTM. Let $P \subset M$ be a partial ordering and G a P-generic filter. If $t \subseteq a \times P$ is a P-term, then $I_G(t) = \{b \in a : \exists p \in G(< b, p > \in t)\}$.*

Theorem (Forcing Theorem). *Let M be a CTM, $P \in M$ a partial ordering, and $G \subseteq P$ an M-generic filter. Then there exists a CTM $M[G]$, such that $M \subseteq M[G]$ and $G \in M[G]$. In fact, $M[G] \subseteq N$ for all CTM's N such that $M \subseteq N$ and $G \in N$.*

Proposition. $ORD^M = ORD^{M[G]}$.

Definition. *Let M be a CTM, $P \in M$ a partial ordering, $p \in P$, $t_1, \ldots, t_n \in M$ P-terms, and $\phi(x_1, \ldots, x_n)$ a formula with free variables $x_1, \ldots, x_n$. Then p P-forces $\phi(x_1, \ldots, x_n)$ over M, denoted*

$$p \Vdash_M^P \phi(t_1, \ldots, t_n)$$

iff $M[G] \models \phi(I_G(t_1), \ldots, I_G(t_n))$, for every M-generic $G \subseteq P$ with $p \in G$.

Lemma (Truth Lemma). *Let M be a CTM, $P \in M$ a partial ordering, $G \subseteq P$ M-generic, $t_1, \ldots, t_n \in M$ P-terms, and $\phi(x_1, \ldots, x_n)$ a formula. Then $M[G] \models \phi(I_G(t_1), \ldots, I_G(t_n))$ iff there exists $p \in G$ such that $p \Vdash_M^P \phi(t_1, \ldots, t_n)$.*

Proposition. *Suppose that $e : G \to H$ is order-preserving and that the range of e is dense in H; where G, H are partial orderings. Then G and H are equivalent forcing properties. That is,*

$$M[G] \models \phi \text{ iff } M[H] \models \phi.$$

3. Large Cardinals and their Properties

Definition. *The Power Set $\mathcal{P}(S)$ of a set S is $\{S : S \subseteq S\}$.*

Definition. *A filter over a set S is a filter on the partial ordering $(\mathcal{P}(S), \subset, S)$. In particular, for $X, Y \in S$, the element of S witnessing that X and Y are compatible is $X \cap Y$.*

Definition. *A filter F over a set S is proper iff $F \neq S$ and $F \neq \mathcal{P}(S)$.*

Definition. *A proper filter F over a set S is non-principal iff $F \neq \{X \subseteq S : X \supseteq X_0\}$, for some subset X_0 of S.*

Definition. *A proper filter U over a set S is an ultrafilter iff for all $X \subseteq S$, either $X \in U$ or $S - X \in U$.*

Definition. *Let κ be a cardinal. Then a proper filter F is κ-complete iff $\forall A \subseteq F(|A| < \kappa \to \bigcap A \in F)$.*

Definition. *Let κ be an uncountable cardinal such that there exists a κ-complete non-principal ultrafilter over κ. Then κ is a measurable cardinal.*

Definition. *Let M_0, M_1 be models of ZFC, then an injective function $j : M_0 \to M_1$ is an elementary embedding iff for any formula $\phi(v_1, \ldots, v_n)$ in the language of ZFC and $x_1, \ldots, x_n \in M_0$,*

$$M_0 \models \phi[x_1, \ldots, x_n] \iff M_1 \models \phi[j(x_1), \ldots, j(x_n)].$$

Theorem. *If j is an elementary embedding between models of ZFC, then there exists an cardinal $crit(j)$, such that for all ordinals $\alpha < crit(j)$, $j(\alpha) = \alpha$, and $j(crit(j)) > crit(j)$.*

The following theorem gives an equivalent, and much simpler, characterization of measurable cardinals which will be used in the proof of Silver's theorem. The reason this characterization was not used as our primary definition of measurable cardinal lies in the fact that it involves models of ZFC and so is not formalizable inside ZFC itself.

Theorem. *If j is an elementary embedding from V into an inner model M, then $crit(j)$ is a measurable cardinal.*

Proof. Set $\delta = crit(j)$. $\delta > \omega$ since j is elementary and all ordinals $\leq \omega$ are definable.

Define U by:

$$x \in U \text{ iff } x \subseteq \delta \text{ and } \delta \in j(x).$$

We claim that U is a δ-complete non-principal ultrafilter over δ, and hence δ is a measurable cardinal.

Let $x, y \in U$. Since j is elementary, it follows that $j(x \cap y) = j(x) \cap j(y)$, and $j(x) \subseteq j(y)$ if $x \subseteq y$. Hence, $\delta \in j(x \cap y)$, so U is closed under pairwise intersection. Also, for $x \subseteq y$, $\delta \in j(x)$ implies $\delta \in j(y)$, and therefore $x \in U$ implies $y \in U$. So U is closed under supersets. Consequently, U is a filter.

In fact, U is an ultrafilter. Let $x \subseteq \delta$. By elementarity, $j(\delta - x) = j(\delta) - j(x)$. Hence, if $x \in U$, $(\delta - x) \notin U$, and vice versa.

That U is non-principal follows from the fact that for all $\alpha < \delta$ $j(\{\alpha\}) = \{\alpha\} \subset \delta$, and so $\{\alpha\} \notin U$.

Finally, U is δ-complete. Let $\gamma < \delta$ and let $A = < x_\alpha \in U : \alpha < \gamma >$. Now, $\delta \in \bigcap\{j(x_\alpha) : \alpha < \gamma\}$. By elementarity, for all $\alpha \leq \gamma$ $j(\alpha) = \alpha$. Hence, for all $\alpha < \gamma$ $j(A)(\alpha) = j(x_\alpha)$. Consequently,

$$\delta \in \bigcap\{j(x_\alpha) : \alpha < \gamma\} = \bigcap\{j(A)(\alpha) : \alpha < \gamma\} = j(\bigcap\{x_\alpha : \alpha < \gamma\}.$$

So, $\bigcap\{x_\alpha : \alpha < \gamma\} \in U$.

Definition. *Let A be a set of cardinality at least κ. Let $P_\kappa(A) = \{P \subset A : |P| < \kappa\}$. For each $P \in P_\kappa(A)$, let $\hat{P} = \{Q \in P_\kappa(A) : P \subseteq Q\}$. Consider the filter $F = \{X \subseteq P_\kappa(A) : X \supseteq \hat{P}$ for some $P \in P_\kappa(A)\}$ generated by the sets $\hat{P}$ for all $P \in P_\kappa(A)$. Call U a fine measure on $P_\kappa(A)$ iff U is a κ-complete ultrafilter extending F.*

Definition. *A fine measure U on $P_\kappa(A)$ is normal iff whenever $f : P_\kappa(A) \to A$ is such that if $f(P) \in P$ on a set in U, then f is constant on a set in U.*

Definition. *A cardinal κ is supercompact iff for all A such that $|A| \geq \kappa$, there exists a normal measure on $P_\kappa(A)$.*

Proposition. *If κ is a supercompact cardinal, then it is also a measurable cardinal.*

As with measurable cardinals, there is an equivalent, and much simpler, definition of supercompact cardinals than the primary one given. However, as before, the auxilliary definition depends on the ZFC-undefinable concept of "model of ZFC," and is therefore unsuitable for a basic definition. The second definition of supercompact cardinal is provided by the following theorem:

Theorem. *Let $\lambda \geq \kappa$. A normal measure on $P_\lambda(A)$ exists if and only if there is an elementary embedding $j : V \to M$ such that*

(1) *$j(\gamma) = \gamma$ for all $\gamma < \kappa$;*

(2) *$j(\kappa) > \lambda$;*

(3) *${}^\lambda M \subseteq M$, where ${}^\lambda M$ is the set of λ-sequences of elements of M.*

In this paper, full supercompactness will not be used. Instead, the weaker related notion of κ^{++}-supercompactness will be sufficient. The definition of κ^{++}-supercompactness is identical to the above theorem except that $\lambda \leq \kappa^{++}$.

Elementary embeddings, with their connections to measurable and supercompact cardinals, play a crucial role in the final steps of the proof of Silver's theorem.

One final result, used later:

Definition. *A collection A of sets is called a Δ-System if there exists a fixed set r such that $x \cap y = r$ for all $x, y \in A$.*

Δ-System Lemma. *Let κ be any infinite cardinal. Let $\theta > \kappa$ be regular and satisfy $\forall\alpha < \theta(\alpha^{<\kappa}| < \theta$. Assume $|A| \geq \theta$ and $\forall x \in A(|x| < \kappa)$, then there is a $B \subseteq A$ such that $|B| = \theta$ and B forms a Δ-system.*

4. A Partial Ordering on a Pair-Tree

Notation. *For an inaccessible cardinal α, let α^* denote the least inaccessible greater than α, should such a cardinal exist.*

Fix an inaccessible cardinal κ and declare that $p = (E_p, \prec_p) \in \mathbb{P}^\kappa$ iff:

(1) E_p is an Easton set of inaccessible cardinals less than κ.

(2) $\prec_p$ is a tree ordering on $\bigcup_{\alpha \in E_p \cup \{\kappa\}} \{\alpha\} \times \alpha^{++}$.

Definition. $FLD_p = \{t \in \bigcup_{\alpha \in E_p \cup \{\kappa\}} \{\alpha\} \times \alpha^{++}$: *there exists t' such that $t \prec_p t'$ or $t' \prec_p t$*$\}$.

Notation. *If $t \in \{\alpha\} \times \alpha^{++}$, let $ht(t) = \alpha$. For all $\alpha \in E_p \cup \{\kappa\}$, let $p_\alpha = \{t \in FLD_p : ht(t) = \alpha\}$.*

Let $t, t' \in FLD_p$.

(3) $t \prec_p t' \rightarrow ht(t) < ht(t')$

(4) If $t \prec_p t'$ and $\alpha \in (ht(t), ht(t')) \cap E_p$, then there exists a t'' such that $ht(t'') = \alpha$ and $t \prec_p t'' \prec_p t'$.

(5) If $t \in FLD_p$, then there exists $t' \in FLD_p$ such that $t \preceq_p t'$ and $ht(t') = \kappa$.

(6) If α' is the least inaccessible in $E_p \cup \{\kappa\}$ greater than $\alpha \in E_p$, then $|\{t' \in p_{\alpha'} : \exists t \in p_\alpha(t \prec_p t')\}| < \alpha^*$.

(7) If $E_p \cap \alpha \neq \emptyset$ and $\alpha = \sup(E_p \cap \alpha)$, then $\alpha^* \in E_p$.

(8) $\forall \alpha \in E_p(p_\alpha \neq \emptyset)$.

Declare that $\bar{p} \geq p$ iff $E_{\bar{p}} \subseteq E_p$ and $\prec_{\bar{p}} \subseteq \prec_p$.

Definition. *If $p \in \mathbb{P}^\kappa$, then p is a condition.*

The conditions are intended to represent fragments of a generic κ-tree, whose underlying set will be $\bigcup_{p \in G} FLD_p$, where G is the corresponding generic filter on

$\mathbb{P}^\kappa$. The levels of the κ-tree will be subsets of the sets $\{\alpha\} \times \alpha^{++}$, where α is an inaccessible $\leq \kappa$.

The following "workhorse" lemma will be used throughout the development of the κ-tree.

Extension Lemma. *Suppose that $\bar{p} \in \mathbb{P}^\kappa$. Let $\beta < \kappa$ be inaccessible.*

(1) *There exists a condition p extending $\bar{p}$ such that $\beta \in E_p$.*

(2) *If $u \in \bar{p}_\kappa$, then there exists a condition p extending $\bar{p}$ such that $t \prec_p u$, for some $t \in p_\beta$.*

Proof of (1). Suppose $\beta \notin E_{\bar{p}}$. If $E_{\bar{p}} = \emptyset$, then simply pick $u \in \{\kappa\} \times \kappa^{++}$ and $t \in \{\beta\} \times \beta^{++}$, and extend $\bar{p}$ to p by defining $E_p = \{\beta\}$ and $\prec_p$ to be the ordering consisting of the single relation: $t \prec_p u$. By inspection, p satisfies the Condition Axioms.

If $\beta < \min(E_{\bar{p}})$, choose $t \in \{\beta\} \times \beta^{++}$. Let $t' \in \bar{p}_{\min(E_{\bar{p}})}$. Extend $\bar{p}$ to p by declaring $E_p = E_{\bar{p}} \cup \{\beta\}$ and adding relations as follows:

$$(t \prec_p t''), \text{ for all } t'' \succeq_{\bar{p}} t'.$$

Essentially, p is just $\bar{p}$ with one maximal branch extended by the addition of a new least element. That p is a condition is clear.

Otherwise, let $\beta > \min(E_{\bar{p}})$. We claim that there exists a largest inaccessible in $\beta \cap E_{\bar{p}}$. Suppose not. Let $\lambda = \sup(\beta \cap E_{\bar{p}})$. Since $\bar{p}$ is a condition, $E_{\bar{p}}$ is an Easton set. It follows that $\lambda < \beta$, since β is an inaccessible and so is regular. However, $\lambda = \sup(\lambda \cap E_{\bar{p}})$, and so $\lambda^* \in E_{\bar{p}}$ by Axiom 7. But since $\lambda < \beta$, then $\lambda^* \leq \beta$, which means that either $\lambda^* < \sup(\beta \cap E_{\bar{p}}) = \lambda$, which is clearly impossible, or $\lambda^* = \beta$, which contradicts $\beta \notin E_{\bar{p}}$. Hence, there exists a largest inaccessible in $\beta \cap E_{\bar{p}}$. Call it θ. For each $t \in \bar{p}_\theta$ pick a different $s_t \in \{\beta\} \times \beta^{++}$.

Extend $\bar{p}$ to p by declaring $E_p = E_{\bar{p}} \cup \{\beta\}$ and adding relations as follows:

For all $t \in \bar{p}_\theta$,

$$t \prec_p s_t.$$

$$s_t \prec_p u, \text{ if } t \prec_{\bar{p}} u.$$

$$u \prec_p s_t, \text{ if } u \prec_{\bar{p}} t.$$

By inspection, Axioms 1,3,4,5,8 hold for p.

Axiom 2 holds as well. Suppose there exists $u \in FLD_p$ such that the $\prec_p$-predecessors of u are not well-ordered. If the predecessors of u are linearly ordered, then by Axiom 3, there exists an order-preserving bijection $ht(\,)$ from the set of predecessors into the ordinals. Consequently, the set of predecessors will be well-ordered. So, assume the predecessors of u are not linearly ordered. If $ht(u) < \beta$, then u has the same predecessors that it had in $\bar{p}$, so they must be linearly ordered since $\bar{p}$ is a condition.

Otherwise, if $ht(u) = \beta$, then $u = s_t$, for some $t \in p_\theta$, and so by definition of $\prec_p$, the predecessors of u are t and its $\prec_{\bar{p}}$-predecessors. Since $\bar{p}$ was a condition, these are clearly linearly ordered.

If $ht(u) > \beta$, then the predecessors of u are its $\prec_{\bar{p}}$-predecessors plus the unique $s_t \in p_\beta$ corresponding to its $\prec_{\bar{p}}$-predecessor $t \in \bar{p}_\theta = p_\theta$. Using the definition of p, it follows that for all t', t'' such that $t' \prec_{\bar{p}} t$ and $t \prec_{\bar{p}} t'' \prec_{\bar{p}} u$,

$$t' \prec_p t \prec_p s_t \prec_p t'' \prec_p u.$$

Hence, the $\prec_p$-predecessors of u are linearly ordered, and so well-ordered. Therefore, since $u \in FLD_p$ was arbitrary, $\prec_p$ is a tree-ordering.

For Axiom 7, suppose that $\lambda = \sup(E_p \cap \lambda)$. By the definition of supremum, $E_p \cap \lambda$ is unbounded in λ. Since E_p is an Easton set, it follows that λ cannot be regular, and is consequently not an inaccessible. Hence, λ is a singular limit cardinal not in E_p. Now, $E_p \backslash E_{\bar{p}} = \{\beta\}$, so $\lambda = \sup(E_{\bar{p}} \cap \lambda)$, since the addition of a single cardinal cannot create a sequence unbounded in a limit cardinal. Hence, λ^* is already in $\bar{p}$.

Finally, Axiom 6 holds. Let α be the least inaccessible in $E_{\bar{p}} \cup \{\kappa\}$ such that $\alpha > \beta$. Since by definition of $\prec_p$, to each $s_t \in \{\beta\} \times \beta^{++}$, there corresponds a single $t \in \bar{p}_\theta = p_\theta$, and $s_t \prec_p u$ iff $t \prec_p u$, then

$$\{u \in p_\alpha : \exists s_t \in p_\beta \text{ such that } s_t \prec_p u\} = \{u \in p_\alpha : \exists t \in p_\theta \text{ such that } t \prec_p u\}$$
$$= \{u \in \bar{p}_\alpha : \exists t \in \bar{p}_\theta \text{ such that } t \prec_{\bar{p}} u\}.$$

Hence, since $\bar{p}$ is a condition, the latter set has cardinality less that θ^*, and hence has cardinality less than β^*. Therefore, Axiom 6 holds for θ, β and so in general for p.

Hence, p is a condition extending $\bar{p}$ such that $\beta \in E_p$.

Proof of (2). We can assume that $\beta \in E_{\bar{p}}$, since otherwise by (1), we can always extend $\bar{p}$ to an intermediate condition p' such that $\beta \in E_{p'}$.

So, since $\beta \in E_{\bar{p}}$, there exist $t \in \bar{p}_\beta$ and $\bar{t} \in \bar{p}_\kappa$ such that $t \prec_{\bar{p}} \bar{t}$. Let $\alpha \in E_p$ be least such that there exists $t_\alpha \in \bar{p}_\alpha$ such that $t_\alpha \preceq_{\bar{p}} u$. By Axiom (4) of the definition of $\mathbb{P}^\kappa$, for all $\gamma \in (\alpha, \kappa) \cap E_{\bar{p}}$, there exists t_γ such that $ht(t_\gamma) = \gamma$ and $t_\alpha \prec_{\bar{p}} t_\gamma \prec_{\bar{p}} u$. Consequently, $\beta < \alpha$. Again by Axiom (4), since $t \prec_{\bar{p}} \bar{t}$, for all $\delta \in (\beta, \kappa) \cap E_{\bar{p}}$, and so for all $\delta \in (\beta, \alpha) \cap E_{\bar{p}} \subset (\beta, \kappa) \cap E_{\bar{p}}$, there exists $s_\delta \in \bar{p}_\delta$ such that $t \prec_{\bar{p}} s_\delta \prec_{\bar{p}} \bar{t}$. Now, extend $\bar{p}$ to p by adding relations:

$$t \prec_p u;$$

and for all $\gamma \in [\alpha, \kappa) \cap E_{\bar{p}}$ and $\delta \in (\beta, \alpha) \cap E_{\bar{p}}$:

$$t \prec_p t_\gamma;$$

$$s_\delta \prec_p t_\gamma;$$

$$s_\delta \prec_p u;$$

$$t' \prec_p t_\gamma, \text{ if } t' \prec_p t;$$

$$t' \prec_p u, \text{ if } t' \prec_p t.$$

By inspection, Axioms 1,3,7,8 hold for p.

Axiom 2 holds since the sets $\{t\} \cup \{s_\delta : \delta \in (\beta, \alpha) \cap E_p\}$, $\{t_\gamma : \gamma \in [\alpha, \kappa) \cap E_p\} \cup \{u\}$ are both well-ordered in $\bar{p}$ and the elements of the former set are defined as strictly preceeding those of the latter.

Axiom 4 holds since $t \in p_\beta$, for all $\delta \in (\beta, \alpha) \cap E_p$ there exists $s_\delta \in p_\delta$, for all $\gamma \in [\alpha, \kappa) \cap E_p$ there exists $t_\gamma \in p_\gamma$, $u \in p_\kappa$ and all these elements are well-ordered by Axiom 2.

For Axiom 5, simply note that all elements s of FLD_p are elements of $FLD_{\bar{p}}$ and so inherit from $\bar{p}$ an element $u' \in \bar{p}_\kappa$ such that $s \prec_{\bar{p}} u'$. Hence $s \prec_p u'$.

Finally, for Axiom 6, note that for all $t' \in FLD_p$ such that $ht(t') \geq \alpha$, the $\prec_p$-successors of t' are the same as its $\prec_{\bar{p}}$-successors. Hence, since $\bar{p}$ is a condition, Axiom 6 holds in this case. For t' such that $ht(t') < \alpha$, we need only check that it holds for $t' \in \{t\} \cup \{s_\delta : \delta \in (\beta, \alpha) \cap E_p\}$. However, since for all t' in this set, t' gains no new $\prec_p$-predecessors of $ht() < \alpha$, we need only show that if t' has $< \delta^*$ many successors in $\bar{p}_\alpha$, then it still does in p_α. Since by construction only a single $t_\alpha \in \bar{p}_\alpha$ becomes a new $\prec_p$-successor of t' in p_α, then the result follows trivially.

Hence, p is a condition extending $\bar{p}$ such that there exists $t \in p_\beta$ such that $t \prec_p u$. □

The above theorem can be stately loosely as follows: The first clause states that if a given condition has no elements of height β, one may be added to a branch of the tree ordering without penalty. The second clause states that all pre-existing branches in the tree ordering may have elements of height β added to them.

The following proposition and lemma show that a generic filter over $\mathbb{P}^\kappa$ does indeed produce a κ-tree conforming to our intuitive picture.

Definition. *Suppose that G is a filter on $\mathbb{P}^\kappa$. Let $T = (E_T, \preceq_T)$, where $E_T = \bigcup_{p \in G} E_p$ and $\preceq_T = \bigcup_{p \in G} \preceq_p$. Also, let $FLD_T = \bigcup_{p \in G} FLD_p$.*

Tree Lemma. *Suppose that κ is an inaccessible limit of inaccessibles and that G is $\mathbb{P}^\kappa$ generic over v. Then T is a κ-tree with κ^{++}-many branches.*

Proof. Suppose T is not tree-ordered. Then there exists $t \in FLD_T$ such that the $\prec_T$-predecessors of t are not well-ordered. Assume that the $\prec_T$-predecessors of t are not even linearly ordered.

By arguments similar to the paragraph which directly follows, $\prec_T$ can be shown to be reflexive and transitive. Hence, it is at least an ordering.

Note that the predecessors of a given node in T are pairwise comparable. Indeed, if $t' \prec_T t$ and $t'' \prec_T t$, then there exist conditions $p', p'' \in G$ such that $t' \prec_{p'} t$ and $t'' \prec_{p''} t$. Since G is generic, there exists a condition $p \in G$ extending both p' and p''. Then t' and t'' must be $\prec_p$-comparable. Hence, t' and t'' are $\prec_T$-comparable.

Consequently, by Axiom 3, ht is an order-preserving injection from the predecessors of a given node in T into the ordinals. Hence, the predecessors of nodes in T are well-ordered, and T is a tree-ordering.

Now, let $t \in FLD_T$. Suppose that for some inaccessible $\alpha < \kappa$ there does not exist t' such that $ht(t') = \alpha$ and t, t' are comparable in $\prec_T$. Since $t \in FLD_T$, there exists a condition $\bar{p} \in G$ such that $t \in FLD_{\bar{p}}$. Since $\bar{p}$ is a condition, there exists $u \in \bar{p}_\kappa$ such that $t \prec_{\bar{p}} u$. By the second part of the Extension Lemma, there exists an extension p of $\bar{p}$ with $t' \in p_\alpha$ such that $t' \prec_p u$. Since $t, t' \prec_p u$ and $\prec_p$ is a tree ordering, it follows that t, t' are comparable.

Let $t \in FLD_T$ be such that $ht(t) = \alpha$, and let β be an inaccessible less than α. By the last paragraph, there must exist $t' \in FLD_T$ such that $ht(t') = \beta$ and t, t' are comparable. By Axiom 3, t' must be a $\prec_T$-predecessor of t. In fact, it must be the unique $\prec_T$-predecessor of $ht(t') = \beta$ since if there existed $t'' \neq t'$ such that $ht(t'') = \beta$, then by Axiom 3 they would be incomparable. Hence, for each inaccessible $\beta < \alpha$, there exists a unique t' such that $t' \prec_T t$. Hence, the set of $\prec_T$-predecessors of t has ordertype $ord(\{\text{inaccessibles} < \alpha\})$. Hence, t's level in the tree ordering T is $ord(\{\text{inaccessibles} < \alpha\})$.

By the definition of $ht()$, it follows that each level of T is T_α, where α is the unique $ht(t)$ for all t in the level. Since for every inaccessible $\alpha < \kappa$ there exists an $t \in FLD_T$ such that $ht(t) = \alpha$, there are as many levels in the tree ordering of T as there are inaccessibles $\leq \kappa$. Since κ is an inaccessible limit of inaccessibles, there are therefore κ many such levels. Since each level T_α has cardinality less than $\{\alpha\} \times \alpha^{++} = \alpha^{++} < \kappa$, then T is a tree ordering of height κ, whose levels have cardinality less than κ. By definition, T is a κ-tree.

All that is left is to prove that T has κ^{++}-many branches.

Let $\bar{p} \in \mathbb{P}^\kappa$. Suppose $u \neq u'$ lie in $\bar{p}_\kappa$. We can assume that there exists a largest inaccessible η in $E_{\bar{p}} \cap \kappa$. If not, by Axiom 7, $\sup(E_{\bar{p}} \cap \kappa)^* \in E_p$, in which case either $\sup(E_{\bar{p}} \cap \kappa)^* = \kappa$, which violates $E_{\bar{p}}$ being an Easton set, or worse, $\sup(E_{\bar{p}} \cap \kappa)^* < \sup(E_{\bar{p}} \cap \kappa)$. Choose an inaccessible β such that $\eta < \beta < \kappa$. Now, $|\bar{p}_\kappa| < \eta^* \leq \beta$, so there exists a one-to-one function $f : \bar{p}_\kappa \to \{\beta\} \times \beta$. Let p be identical with $\bar{p}$, except that $E_p = E_{\bar{p}} \cup \{\beta\}$ and the following relations are added:

For all $u \in \bar{p}_\kappa$:

$$f(u) \prec_p u.$$

$$t \prec_p f(u), \text{ if } t \prec_{\bar{p}} u.$$

By inspection, Axioms 1,2,3,4,5,8 hold in p.

For Axiom 7, suppose that $\lambda = \sup(E_p \cap \lambda)$. If $\lambda < \beta$, then since $\bar{p}$ is a condition and $E_{\bar{p}} = E_p \cap \beta$, Axiom 7 holds for $E_p \cap \beta$. Hence, suppose $\beta = \lambda$. Then $\beta = \sup(E_p \cap \beta) = \eta$. A contradiction. Therefore, $\beta \neq \sup(E_p \cap \beta)$ and so Axiom 7 holds in p.

For Axiom 6, note first that by definition of p, the set $p_\kappa = \bar{p}_\kappa)$ has cardinality $< \eta^* \leq \beta < \beta^*$. Hence,

$$|\{u \in p_\kappa : \exists t \in p_\beta (t \prec_p u)\}| < \beta^*$$

and

$$|\{u \in p_\kappa : \exists t \in p_\eta (t \prec_p u)\}| < \eta^*.$$

By the first inequality, Axiom 6 holds for β. By the second inequality, since p_η has $< \eta^*$ successors in p_κ and p is a tree ordering, then it must have $< \eta^*$ successors in p_β, that is:

$$|\{t \in p_\beta : \exists t' \in p_\eta (t' \prec_p t)\}| < \eta^*.$$

Hence, Axiom 6 holds for η, and so in general for p.

Now, in p, $f(u) \prec_p u$ and $f(u') \prec_p u'$. Since f is one-to one, $f(u) \neq f(u')$. Hence, since both $f(u), f(u')$ lie on the same level of the tree ordering, $f(u) \not\prec_q u'$ and $f(u') \not\prec_q u$ for all $q \leq p$. Since $u, u' \in \bar{p}_\kappa$ were arbitrary, p is a condition extending

$\bar{p}$ such that $p \Vdash$ " $\preceq_{\dot{T}}$-predecessors of two distinct $u, u' \in \bar{p}_\kappa$ are themselves distinct". Hence, in $\prec_T$, for any two distinct $u, u' \in \{\kappa\} \times \kappa^{++}$, the predecessors of u, u' are distinct. Therefore, T has κ^{++} many branches. $\square$

Notation. *For $u \in p_\kappa$, define $p_\alpha(u) = t$, where $t \in p_\alpha$ is such that $t \prec_p u$, if such a t exists. Also, let $Dom(p_\alpha) = \{u \in p_\kappa : \exists t \in p_\alpha$ such that $t \prec_p u\}$.*

The following Antichain, Closure, Embedding and Factor Lemmas provide details about $\mathbb{P}^\kappa$ needed for cardinal arithmetical calculations in $\mathbb{P}^\kappa$-generic extensions.

Notation. *Suppose that $X \subseteq \{\kappa\} \times \kappa^{++}$. Then*

$$\mathbb{P}^\kappa(X) = \{p \in \mathbb{P}^\kappa : p_\kappa \subseteq X\}.$$

Antichain Lemma. *Suppose that κ is inaccessible, but not the least inaccessible cardinal. Set $\lambda = \sup\{\nu^* : \nu < \kappa$ is inaccessible$\}$. Then $\mathbb{P}^\kappa(X)$ has antichains of cardinality at most $\lambda^{<\lambda}$.*

Proof. If κ is inaccessible, but not the least inaccessible, then either $\lambda = \kappa$ or λ is a singular strong limit cardinal. If $\lambda = \kappa$, then $\lambda^{<\lambda} = \kappa$. Otherwise, $\lambda^{<\lambda} = \lambda^\lambda = 2^\lambda = \lambda^{cf(\lambda)}$ because $\lambda^{<\lambda} \leq \lambda^\lambda \leq 2^\lambda$, for all cardinals; $\lambda^{cf(\lambda)} \leq \lambda^{<\lambda}$, since λ is singular; and $2^\lambda \leq \lambda^{cf(\lambda)}$, since λ is a strong limit cardinal. Suppose $A \subseteq \mathbb{P}^\kappa(X)$ has cardinality $(\lambda^{<\lambda})^+$. If $\lambda = \kappa$, then λ has at most $\lambda^{<\lambda}$ bounded subsets, and so at most $\lambda^{<\lambda}$ Easton subsets. Otherwise, if λ is a singular strong limit cardinal, then it has at most $2^\lambda = \lambda^{<\lambda}$ subsets, and so at most $\lambda^{<\lambda}$ Easton subsets. So, in general, it has at most $\lambda^{<\lambda}$ Easton subsets. Since $|A| = (\lambda^{<\lambda})^+$ and only $\lambda^{<\lambda}$ Easton subsets exist, we can assume there exists a fixed $D \subseteq \lambda$ such that $E_p = D$ for all $p \in A$. Set $\mathcal{F} = \{p_\kappa : p \in A\}$. Next, note that we can assume that $\mathcal{F}$ forms a Δ-system. Certainly, $|\mathcal{F}| \leq (\lambda^{<\lambda})^+$, since $|A| = (\lambda^{<\lambda})^+$. If $|\mathcal{F}| < (\lambda^{<\lambda})^+$, then we can assume that $|\mathcal{F}| = 1$, by thinning A if necessary. So, suppose that $|\mathcal{F}| = (\lambda^{<\lambda})^+$. Now,

$$|p_\kappa| \leq \sup_{\nu \in E_p} \nu^* = \sup_{\nu \in D} \nu^* \leq \lambda.$$

If $\lambda < \kappa$, then $\mathcal{F}$ is a family of at most $(2^\lambda)^+$ many sets of cardinality less than λ^+. On the other hand, if $\lambda = \kappa$, then $\sup_{\nu \in D} \nu^* < \kappa$, so $\mathcal{F}$ is a family of κ^+-many sets of cardinality less than κ. In either case, we can apply the Δ-System Lemma. Say the root of $\mathcal{F}$ is r. We may assume that $p_\alpha = {p'}_\alpha$ for all $p, p' \in A$ and all $\alpha \in D$. Finally, if $p, p' \in A$, we may assume that there exists a bijection $\epsilon : p_\kappa \to p'_\kappa$ such that

$$\epsilon \restriction r = id \restriction r;$$

$$u \in Dom(p_\alpha) \iff \epsilon(u) \in Dom(p'_\alpha), \forall \alpha \in D, u \in p_\kappa; and$$

$$p_\alpha(u) = p'_\alpha(\epsilon(u)), \forall \alpha \in D, u \in Dom(p_\alpha).$$

(Consider isomorphism types of $p \in A$ in a language with function symbols for each p_α and constant symbols for each element of r and of $\bigcup_{\alpha \in D} p_\alpha$).) Call such an ϵ an *isomorphism* from p to p'.

Suppose that $p, p' \in A$. Let $E_q = D$ and also set $\prec_q = \prec_p \cup \prec_{\dot{p}}$. We maintain that $q = (E_q, \prec_q)$ is a condition extending both p and p'.

By inspection, Axioms 1,3,4,5,7,8 hold for q.

Axiom 2 holds as well. To see that when $\alpha \in D$, the predecessors of $u \in p_\kappa$ are well ordered, let ϵ be an isomorphism from p to p'. Suppose that there exists $t \in FLD_p$ and $t' \in FLD_{p'}$ such that $t, t' \prec_q u \in q_\kappa$, then $u \in p_\kappa, p'_\kappa$ and so $u \in r$. Hence, $u = \epsilon(u)$ and so for all $\alpha \in E_q$, $p_\alpha(u) = p'_\alpha(\epsilon(u)) = p'_\alpha(u)$. Hence, $t, t' \in FLD_p, FLD_{p'}$. Consequently, t, t' must be comparable in p, p' and so in q. It follows by Axiom 3 that they are well-ordered.

Finally, Axiom 6 holds. Since $p_\alpha = p'_\alpha$ for all $\alpha \in D = E_q$, we need only check that it holds at the κ-th level. Since D is Easton and κ is regular, there exists a largest inaccessible η in D less than κ by Axiom 7. Since p and p' are conditions, the following inequalities hold:

$$|\{t' \in p_\kappa : \exists t \in p_\eta (t \prec_p t')\}| < \eta^*.$$

$$|\{t' \in p'_\kappa : \exists t \in p'_\eta (t \prec_{p'} t')\}| < \eta^*$$

Hence, since $\prec_q = \prec_p \cup \prec_{p'}$,

$$\{t' \in q_\kappa : \exists t \in q_\eta (t \prec_q t')\}$$

is the union of the two above sets, and therefore has cardinality $< \eta^*$.

Hence, q is a condition extending both p and p'. Since they are therefore compatible, by contradiction, $\mathbb{P}^\kappa(X)$ does not contain an antichain of cardinality $(\lambda^{<\lambda})^+$. □

Definition. *Suppose that $\nu < \kappa$ and ν is inaccessible. Set*

$$\mathbb{P}^\kappa_\nu = \{p \in \mathbb{P}^\kappa : E_p \cap \nu = \emptyset\}.$$

Closure Lemma. *Suppose that $\nu < \kappa$ are inaccessible. Then*

$$\mathbb{P}^\kappa_\nu \text{ is } {<}\nu^* - \text{closed.}$$

Proof. Suppose $\beta < \nu^*$. Let $\langle p_\alpha : \alpha < \beta \rangle$ be a decreasing sequence in $\mathbb{P}^\kappa_\nu$: that is, $\alpha_1 < \alpha_2 \rightarrow p_{\alpha_1} \geq p_{\alpha_2}$.

Define $E_q = \bigcup_{\alpha<\beta} E_{p_\alpha}$, $\prec_q = \bigcup_{\alpha \in E_q} \prec_{p_\alpha}$. Let $q = (E_q, \prec_q)$.

By inspection, Axioms 1,2,3,4,5,8 hold for q.

Axiom 6 holds as well. Let $\gamma \in E_q$. Let γ' be the least inaccessible in $(E_p \cup \{\kappa\}) \backslash \gamma^*$. Since all the p_α are conditions, for any p_α such that $\gamma \in E_{p_\alpha}$, the cardinality of the set of successors of elements of $(p_\alpha)_\gamma$ in $(p_\alpha)_{\gamma'}$ is less than γ^*. Hence, the set of successors of elements of q_γ in $q_{\gamma'}$ has cardinality less than $\beta \cdot \gamma^* \leq \nu^* \cdot \gamma^* \leq \gamma^* \cdot \gamma^* = \gamma^*$, and so Axiom 6 holds for γ and so in general.

Unfortunately, Axiom 7 could fail in q. Suppose λ is a singular strong limit cardinal with cofinality $\gamma \leq \beta$ and $\langle \delta_i : i < \gamma \rangle$ is a sequence of inaccessibles cofinal in λ. If for all i, there exists p_α such that $\delta_i \in E_{p_\alpha}$, then $\langle \delta_i : i < \gamma \rangle \subseteq E_q$ and so $\lambda = \sup(E_q \cap \lambda)$. However, unless $\lambda^* \in p_\alpha$ for some $\alpha < \beta$, λ^* will not be in E_q.

Suppose Axiom 7 does fail, that is, $\lambda = \sup(E_q \cap \lambda)$ and $\lambda^* \notin E_q$. To correct the failure of Axiom 7, let $\delta \in E_q \cup \{\kappa\}$ be the least such that $\delta > \lambda^*$, and let

$$U = \{u \in q_\delta : \exists t \text{ such that } ht(t) < \lambda^* \text{ and } t \prec_q u\}.$$

For $u, u' \in U$, declare $u \sim u'$ if for all sufficiently large $\gamma < \lambda^*$ there exists $t \in q_\gamma$ such that $t \prec_q u, u'$. That $\sim$ is an equivalence relation is easily verified. Now,

$|U| < \lambda^*$ because $U = \bigcup_{\alpha<\beta} U_\alpha$, where $U_\alpha = \{u \in (p_\alpha)_\delta : u$ is not $\prec_{p_\alpha}$-minimal$\}$ and $|U_\alpha| < \eta_\alpha{}^*$, where $\eta_\alpha = \sup(E_{p_\alpha} \cap \lambda) < \lambda$. So there exists a function $f : U \to \{\lambda^*\} \times (\lambda^*)^{++}$ such that $f(u) = f(u')$ if and only if $u \sim u'$. Let q' be identical to q except that $E_{q'} = E_q \cup \{\lambda^*\}$ and the following relations are added:

For all $u \in U$:

$$f(u) \prec_{q'} u;$$

$$f(u) \prec_{q'} u', \text{ if } u \prec_q u';$$

$$u' \prec_{q'} f(u), \text{ if } u' \prec_q u.$$

Obviously, Axiom 7 holds in q' when this correction is made for all offending λ. By inspection, Axioms 1,2,3,4,5,6,8 also hold in q'.

Hence, q' is a condition extending q, which was the union of all the p_α. So, if $\{p_\alpha : \alpha < \beta\}$ is a decreasing sequence in $\mathbb{P}^\kappa_\nu$, where $\beta < \nu^*$, there exists a condition q' which extends all the p_α. Therefore, $\mathbb{P}^\kappa_\nu$ is $< \nu^*$-closed.

Definition. *If $p \in \mathbb{P}^\kappa$ and $\nu < \kappa$ is inaccessible, set $(p)_\nu = (E, \prec)$, where $E = E_p \setminus \nu$ and $\prec = \{(t, t') \in \prec_p : ht(t) \geq \nu\}$. Note that $(p)_\nu \in \mathbb{P}^\kappa_\nu$.*

Definition. *Say that $p \in \mathbb{P}^\kappa$ is $\nu-$completed if $\nu \in E_p$ and $p'_\nu = p_\nu$, for all p' extending $(p)_\nu$ in $\mathbb{P}^\kappa_\nu$.*

Corollary. *Suppose that $\nu < \kappa$ is inaccessible. Then any condition in $\mathbb{P}^\kappa$ has a $\nu-$completed extension.*

Proof. Suppose that $\bar{p} \in \mathbb{P}^\kappa$. We may assume that $\nu \in E_{\bar{p}}$. Note that for each $t \in \{\nu\} \times \nu^{++}$ the set of conditions

$$D_t = \{p \in \mathbb{P}^\kappa_\nu : \text{either } t \in p_\nu \text{ or } t \notin p'_\nu, \text{ for all } p' \leq p\}$$

is dense and open. By the Closure Lemma, there exists a condition $p \leq (\bar{p})_\nu$ such that $p \in \bigcap_{t \in \{\nu\} \times \nu^{++}} D_t$. Set $p' = (E, \prec)$, where $E = (E_{\bar{p}} \cap \nu)$ and $\prec = \prec_{\bar{p}} \cup \prec_p$. Then p' extends p in $\mathbb{P}^\kappa$ and is ν-completed. $\square$

Definition. *Fix inaccessible cardinals* $\nu < \kappa$. *Define a function* $e_{\nu\kappa}$ *such that* $dom(e_{\nu\kappa}) = \mathbb{P}^\kappa$ *by setting* $e_{\nu\kappa}(p) = ((p)^\nu, (p)_\nu)$. *where* $(p)_\nu$ *is as before, and where* $(p)^\nu = (E, \prec)$, $E = E_p \cap \nu$, *and* $\prec = (t, t') \in \prec_p$: $ht(t') \leq \nu$.

Embedding Lemma. *Assume that* $\nu < \kappa$ *are inaccessible. Set* $e = e_{\nu\kappa}$. *Suppose that* $\bar{p} \in \mathbb{P}^\kappa$ *is* ν*-completed.*

(1) *If* $p, p' \leq \bar{p}$ *in* $\mathbb{P}^\kappa$, *then* $p' \geq p$ *iff* $e(p') \geq e(p)$ *in* $\mathbb{P}^\nu \times \mathbb{P}^\kappa_\nu$.

(2) *The range of* e *is dense below* $e(\bar{p})$ *in* $\mathbb{P}^\nu(\bar{p}_\nu) \times \mathbb{P}^\kappa_\nu$.

Proof of (1). Assume first that $p' \geq p$. Set $e(p') = (q', r')$ and $e(p) = (q, r)$. We must see that $(q', r') \geq (q, r)$. By definition of extension, $E_{p'} \subseteq E_p$ and $\prec_{p'} \subseteq \prec_p$. Hence,

$$E_{r'} = E_{p'} \cap [\nu, \kappa) \subseteq E_p \cap [\nu, \kappa) = E_r \text{ and } E_{q'} = E_{p'} \cap \nu \subseteq E_p \cap \nu = E_q.$$

Also,

$$\prec_{r'} = \prec_{\bar{p}} \setminus \{(t, t') : ht(t) < \nu \text{ or } ht(t') < \nu\} \subseteq$$

$$\prec_p \setminus \{(t, t') : ht(t) < \nu \text{ or } ht(t') < \nu\} = \prec_r \text{ and}$$

$$\prec_{q'} = \prec_{p'} \setminus \{(t, t') : ht(t) > \nu \text{ or } ht(t') > \nu\} \subseteq$$

$$\prec_p \setminus \{(t, t') : ht(t) > \nu \text{ or } ht(t') > \nu\} = \prec_q .$$

It follows that $q' \geq q$ and $r' \geq r$, and so $(q', r') \geq (q, r)$.

Conversely, suppose that $(q', r') \geq (q, r)$, where $e(p') = (q', r')$ and $e(p) = (q, r)$. Obviously, $E_{p'} = E_{q'} \cup E_{r'} \subseteq E_q \cup E_r = E_p$. Let $t, t' \in FLD_{p'}$ be such that $t' \prec_{p'} t$. If $ht(t') < ht(t) \leq \nu$, then $t, t' \in FLD_{q'} \subseteq FLD_q \subseteq FLD_p$. Hence, $t' \prec_p t$. Similarly, if $\nu \leq ht(t') < ht(t)$, then $t, t' \in FLD_{r'} \subseteq FLD_r \subseteq FLD_p$, and so $t' \prec_p t$. Now, if $ht(t') < \nu \leq ht(t)$, then $t' \in FLD_{q'}$ and $t \in FLD_{r'}$. Since $\nu \in E_{p'}$, there must exist an element $t'' \in p'_\nu$ such that $t' \prec_{p'} t'' \prec_{p'} t$. By definition of q' and r', since $ht(t'') = \nu$, then $t' \prec_{q'} t''$ and $t'' \prec_{r'} t$. Hence, $t' \prec_q t''$ and $t'' \prec_r t$. So, $t' \prec_p t'' \prec_p t$ and more succinctly, $t' \prec_p t$. Consequently, $\prec_{p'} \subseteq \prec_p$, and so $p' \geq p$.

Proof of (2). Suppose that $(q, r) \leq e(\bar{p})$ in $\mathbb{P}^\nu(\bar{p}_\nu) \times \mathbb{P}^\kappa_\nu$. Then $\bar{p}_\nu \subseteq q_\nu$ and $\bar{p}_\nu \subseteq r_\nu$, since $(q, r) \leq e(\bar{p})$, and $q_\nu \subseteq \bar{p}_\nu$, since $q \in \mathbb{P}^\nu(\bar{p}_\nu)$. Also $r_\nu \subseteq \bar{p}_\nu$, since $\bar{p}$ is ν-completed. Thus $q_\nu = r_\nu = \bar{p}_\nu$. Define $p \in \mathbb{P}^\kappa$ as follows: Let $E_p = E_q \cup E_r$, and also $p_\nu = \bar{p}_\nu$. (Note that if α is the greatest inaccessible in E_q, then the set of $\prec_p$-successors in p_ν has cardinality $< \alpha^*$ since q is a condition in $\mathbb{P}^\nu$.) In addition, let

$$\prec_p = \prec_q \cup \prec_r \cup \{(t, t') : t \prec_q t'' \prec_r t' \text{ for some } t'' \in p_\nu\}.$$

Then $p = (E_p, \prec_p)$ is a condition and $e(p) = (q, r)$.

Definition. *If $\mathbb{Q}$ is a partial ordering and $q \in \mathbb{Q}$, let $\mathbb{Q} \restriction q = \{q' \in \mathbb{Q} : q' \leq q\}$.*

Factor Lemma. *Suppose that $\nu < \kappa$ are inaccessible and that $\bar{p} \in \mathbb{P}^\kappa$. If $\bar{p}$ is ν-completed, then $\mathbb{P}^\kappa \restriction \bar{p}$ is equivalent to $(\mathbb{P}^\nu(\bar{p}_\nu) \restriction (\bar{p})^\nu) \times (\mathbb{P}^\kappa_\nu \restriction (\bar{p})_\nu)$.*

Proof. $e_{\nu\kappa}$ is a dense embedding of $\mathbb{P}^\kappa \restriction \bar{p}$ into $(\mathbb{P}^\nu(\bar{p}_\nu) \restriction (\bar{p})^\nu) \times (\mathbb{P}^\kappa_\nu \restriction (\bar{p})_\nu)$. □

Cardinal Preservation Lemma. *Assume the GCH in the ground model. If κ is an inaccessible limit of inaccessibles and α is a regular cardinal, then in a $\mathbb{P}^\kappa$ generic extension the range of each α-sequence is covered by a ground model set of ground model cardinality α. Consequently, $\mathbb{P}^\kappa$ is cardinal-preserving.*

Proof. Fix α. Let $\lambda = \sup\{\nu^* : \nu < \kappa \text{ is inaccessible}\}$. Since $\mathbb{P}^\kappa$ satisfies the $(\lambda^{<\lambda})^+ = \kappa^+$-chain condition, the lemma is clear if $\alpha \geq \kappa$. Suppose that $\alpha < \kappa$.

Case 1. There exists a largest inaccessible $\nu \leq \alpha$. Suppose $\bar{p} \in \mathbb{P}^\kappa$. Let $p \leq \bar{p}$ be ν-completed and let $e_{\nu\kappa}$ be defined as before. Say $e(p) = (q, r)$. Then $\mathbb{P}^\kappa \restriction p$ is equivalent to $(\mathbb{P}_\nu(p_\nu) \restriction q) \times \mathbb{P}^\kappa_\nu \restriction r$, where $\mathbb{P}^\kappa_\nu \restriction r$ has antichains of size at most ν. Thus, p forces that the range of every α-sequence is covered by a ground model set of cardinality α. Since the collection of ν-completed p is dense in $\mathbb{P}^\kappa$, the claim follows.

Case 2. There does not exist a largest inaccessible less than or equal to α. Set $\lambda = \sup\{\xi^* < \alpha : \xi \text{ is inaccessible}\}$. Then λ is singular or $\lambda = 0$, so $\lambda < \alpha$. Let ν be the least inaccessible greater than α. As before, suppose $\bar{p} \in \mathbb{P}^\kappa$. Let $p \leq \bar{p}$ be ν-completed and let $e_{\nu\kappa}$ be as previously defined. Say $e(p) = (q, r)$. Then $\mathbb{P}^\kappa \restriction p$

is equivalent to $(\mathbb{P}_\nu(p_\nu) \restriction q) \times \mathbb{P}^\kappa_\nu \restriction r$, and $\mathbb{P}^\kappa_\nu \restriction r$ is α-closed. If $\mathbb{P}_\nu(p_\nu) \restriction q$ is non-trivial, then by the Antichain Lemma, $\mathbb{P}_\nu(p_\nu) \restriction q$ has antichains of size at most $\lambda^{<\lambda} = 2^\lambda = \lambda^+ \leq \alpha$. Again, the claim follows. □

Strong Limit Lemma. *Suppose that κ is an inaccessible limit of inaccessibles. Then $\mathbb{P}^\kappa$ forces that κ is a strong limit cardinal.*

Proof. : Suppose that α is a cardinal less than κ and that $\bar{p} \in \mathbb{P}^\kappa$. Let $\nu < \kappa$ be an inaccessible cardinal greater than α. Extend $\bar{p}$ to a condition p that is ν-completed. Then $\mathbb{P}^\kappa \restriction p$ is equivalent to $(\mathbb{P}^\nu(p_\nu) \restriction (p)^\nu) \times (\mathbb{P}^\kappa_\nu \restriction (p)_\nu)$. Now $\mathbb{P}^\kappa_\nu \restriction (p)_\nu$ is ν^*-closed, hence α-closed. And $|\mathbb{P}^\nu(p_\nu)| \leq \nu^{++}$ and has antichains of size at most ν. Thus, if G is $\mathbb{P}^\kappa$ generic and $p \in G$, then

$$(2^\alpha)^V[G] \leq \left((\nu^{++})^{\nu \cdot \alpha}\right)^V = \nu^{++} < \kappa. \quad \square$$

5. Silver's Theorem

Now, with all this in hand, we are at last ready to finish the proof of Silver's Theorem.

Proof of Silver's Theorem.

Proof. Suppose that κ is λ-supercompact, where $\lambda = \kappa^{++}$. Suppose that $j : V \to M$ is elementary, where κ is the critical point of j and ${}^\lambda M \subseteq M$. Set $\mathbb{P} = \mathbb{P}^\kappa$.

Since $\mathbb{P} \subseteq H_\lambda$, $|\mathbb{P}| = \lambda$, and ${}^\lambda M \subseteq M$, it follows that $\mathbb{P} \in M$. Furthermore, $j(\mathbb{P})^\kappa = \mathbb{P}$. Set $\mathbb{Q} = \left(j(\mathbb{P})^{j(\kappa)}_\kappa\right)^M$. Then "$\mathbb{P} \times \mathbb{Q}$ is equivalent to $j(\mathbb{P})$" holds in M, hence in V. Because "$\mathbb{Q}$ is $\leq \lambda < j(\kappa)^*$-closed" holds in M and ${}^\lambda V \subseteq {}^\lambda M \subseteq M$, it follows that $\mathbb{Q}$ is, in fact, $\leq \lambda$-closed in V.

Define the master condition $\hat{q} \in \mathbb{Q}$ as follows: Set $E_{\hat{q}} = \{\kappa\}$ and declare $\prec_{\hat{q}}$ to consist of the relations:

$$t \prec_{\hat{q}} j(t), \text{ for all } t \in \{\kappa\} \times \kappa^{++}.$$

Note that $\hat{q}$ is κ-completed.

Let $e = e_{\kappa j(\kappa)} : \mathbb{P} \times \mathbb{Q} \restriction \hat{q} \to j(\mathbb{P})$ be the embedding defined previously. Hence, the range of e is dense below $\hat{q}$. Note that $e(p, \hat{q})$ is the greatest lower bound of $j(p)$ and $\hat{q}$ in $j(\mathbb{P})$.

Suppose that G is $\mathbb{P}$-generic over V. Choose H to be $\mathbb{Q}$-generic over $V[G]$ with $\hat{q} \in H$. Set

$$K = \{r \in j(\mathbb{P}) : r \geq e(p, q), \text{ for some } (p, q) \in G \times H \text{ such that } q \leq \hat{q}\}.$$

Then K is $j(\mathbb{P})$-generic over V, hence over M, and $\hat{q} \in K$. Now,

$$p \in G \leftrightarrow (p, \hat{q}) \in G \times H \leftrightarrow e(p, \hat{q}) \in K \leftrightarrow j(p) \in K.$$

(The last equivalence holds because $e(p, \hat{q})$ is the greatest lower bound of $j(p)$ and $\hat{q}$.)

It follows that j extends to an elementary $\hat{j} : V[G] \to M[K]$. (Set $\hat{j}(\dot{x}^{V[G]}) = j(\dot{x})^{M[K]}$ and note that

$$\begin{aligned} V[G] \models \varphi(\dot{x}) &\Rightarrow V \models p \Vdash_{\mathbb{P}} \varphi(\dot{x}), \text{ for some } p \in G \\ &\Rightarrow M \models j(p) \Vdash_{j(\mathbb{P})} \varphi(j(\dot{x})), \text{ for some } p \in G \\ &\Rightarrow M[K] \models \varphi(j(\dot{x})). \end{aligned}$$

Working in $V[K]$, define an ultrafilter U on κ by

$$X \in U \iff X \subset \kappa \text{ and } \kappa \in \hat{j}(X).$$

Then U is a κ-complete non-principal ultrafilter. But $|U| = \kappa^{++} = \lambda$ and $\mathbb{Q}$ is $\leq \lambda$-closed, so $U \in V[G]$.

Hence, κ is measurable, and $2^\kappa = \kappa^{++}$. $\square$

6. Afterword

Since the publication of Silver's original proof, Hugh Woodin has extended the result to the following theorem [Foreman and Woodin, pp. 1-35] the conclusion of which answers quite conclusively the question of how badly a model of ZFC can violate GCH.

Theorem. *If there exists a supercompact cardinal* κ, *then it is consistent that for all cardinals* λ, $2^\lambda = \lambda^{++}$.

That is, given the existence of a supercompact cardinal, a hypothesis stronger than that used in this paper, it is consistent that GCH fails at *all* cardinals.

On the other hand, Woodin also managed to prove the conclusion of Silver's theorem using weaker hypotheses [Foreman and Woodin, pp. 34].

Theorem. *If there exists a* $\mathcal{P}^2(\kappa)$*-hypermeasurable cardinal* κ, *then it is consistent that there exists a measurable cardinal* λ *such that* $2^\lambda = \lambda^{++}$.

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