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Fibonacci sequences and the golden section

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FIBONACCI SEQUENCES AND THE GOLDEN SECTION

A Thesis

Presented to

The Faculty of the Department of Mathematics and Computer Science

San Jose State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science

By

Medha. A. Bodas.

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ABSTRACT

FIBONACCI SEQUENCES AND THE GOLDEN SECTION

By Medha A. Bodas

In the year 1202, the Italian mathematician Leonardo of Pisa, also known as Fibonacci proposed the now famous rabbit problem and showed that the solution is given by a sequence of numbers, F_n defined recursively by

$$F_{n+2} = F_{n+1} + F_n, F_1 = 1, F_2 = 1, n \geq 1$$

We explore different topics related to these numbers. We start by discussing generalizations of the Fibonacci Sequence and prove some generalized identities. A general method to obtain series expansion of a given class of functions in terms of the Fibonacci numbers is given. We also find a formula for its coefficients. Some matrix techniques are used to obtain Fibonacci-type identities.

The positive root of the Fibonacci quadratic equation $x^2 - x - 1 = 0$ is $\alpha = \frac{1+\sqrt{5}}{2}$

called the golden section. Interestingly enough, it appears frequently in geometrical shapes such as triangles, circles, ellipses, and hyperbolas.

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INTRODUCTION

For centuries, mathematicians have been intrigued by the sequence of Fibonacci numbers and the closely related irrational numbers

$$\alpha = \frac{1+\sqrt{5}}{2} = 1.618034, \beta = \frac{1-\sqrt{5}}{2} = -0.61803 .$$

The main purpose of this thesis is to study the various topics in which Fibonacci numbers can be found and the appearance of the golden section in geometry. The Fibonacci sequence 1,1,2,3,5,8,13,..... is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, F_1 = 1, F_2 = 1, n \geq 1 .$$

Changing the initial values to $F_1 = 1, F_2 = 3$ and using the same recurrence relation gives another sequence 1,3,4,7,11,18,29,47, ... the Lucas sequence named after E.Lucas (1842-1891) whose terms are called the Lucas numbers.

Most of the articles referred to in this thesis can be found in one journal, "The Fibonacci Quarterly", which was started by Professor Verner. E. Hoggatt, Jr. of San Jose State University. The first issue was published in February 1963, with Editor Vern. E. Hoggatt, Jr. and Managing Editor Brother. U. Alfred. In addition to the Fibonacci Quarterly, Prof. Hoggatt supervised several theses related to the Fibonacci sequence. As a result, there are several theses in our department during the period (1960-1974) with many new results in the field. It was amazing to look at the amount of work he did and his deep insight and understanding of the subject. Therefore, some of the work done by Hoggatt was taken as a basis or looked upon as well established results about Fibonacci numbers.

Chapter I of H.T. Leonard, Jr.'s [1] thesis provides an introduction to the Fibonacci and Lucas sequences and the Binet forms of these sequences given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n \text{ where } \alpha, \beta \text{ are the roots of the equation } x^2 - x - 1 = 0.$$

In Chapter II, Leonard [1] demonstrates the method of H.W.Gould to find generating functions for the Fibonacci, Lucas, as well as Pell sequences. In this thesis we shall study various topics as follows:

Various generalizations of the Fibonacci sequence are discussed in Chapter I.

One of them is the sequence $\{w_n\} = \{w_n(a, b; p, q)\}$: $w_0 = a, w_1 = b$ defined by

$$w_n = pw_{n-1} - qw_{n-2}, n \geq 2, \text{ which was introduced by Horadam [3]. Along with the}$$

Fibonacci sequence, w_n generalizes various other sequences, which belong to the bigger family of sequences defined by linear recurrence relations. A generalization of various Fibonacci-type identities is obtained. Yet another generalization is given by the sequences x_n and y_n defined by the recurrence relations:

$$\begin{aligned} x_{n+2} &= y_{n+1} + y_n \\ y_{n+2} &= x_{n+1} + x_n, \end{aligned} \quad n \geq 0$$

$x_0 = a, x_1 = c, y_0 = b, y_1 = d$ where a, b, c, d are arbitrary real numbers.

Chapter II deals with developing the theory of $\{w_n(a, b, p, q)\}$ where the sequence is related to Chebyshev polynomials of the first and second kind. We use this relation to find some known as well as new trigonometric identities. Expansion of analytic functions in a series of polynomials related to the Fibonacci numbers is discussed. A

formula is given for determining the coefficients. Some examples are given which involve the Bessel functions and the modified Bessel functions. Extensive use of the known results and properties of these functions is made.

In Chapter III, we introduce the concept of a function of a matrix and use it to give a matrix perspective of the Fibonacci numbers. We use many ideas from Linear Algebra such as Sylvester's Interpolation Formula to find several identities involving the sums of the Fibonacci numbers. Matrix techniques like eigenvalues are used to obtain many Fibonacci-type identities.

Chapter IV contains interesting examples of the unexpected occurrence of $\alpha = \frac{1+\sqrt{5}}{2}$ in various places in geometry. Certain cases involving different geometrical shapes such as triangle, ellipse and circle are discussed. Some particular cases of the well-known Fibonacci identity, the Simon's formula $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$ which can be quite easily proved using the Binet form leads to a system of hyperbolas which involve α .

CHAPTER I

GENERALIZATIONS OF THE FIBONACCI SEQUENCE

1. The Generalized Sequence w_n And Other Related Sequences

The Fibonacci sequence 1,1,2,3,5,8,13,.. is defined recursively by $F_{n+2} = F_{n+1} + F_n, n \geq 1, F_1 = 1, F_2 = 1$. The same recurrence relation with different initial values $F_1 = 1, F_2 = 3$ gives the Lucas sequence in honor of the nineteenth century French mathematician E. Lucas; the sequence is usually denoted by L_n . A comparison of the two sequences F_n and L_n yields the relation $L_n = F_{n-1} + F_{n+1}$. More relations between the two sequences can be found in Hoggatt [2].

The roots of the quadratic equation

$$(1.1) \quad x^2 - x - 1 = 0$$

are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The positive root α is called the golden section.

These roots play an important role in the study of both F_n and L_n . It is obvious that

$\alpha - \beta = \sqrt{5}, \alpha\beta = -1, \alpha + \beta = 1$. Since α, β are the roots of (1.1), we have $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$.

The Binet form for the Fibonacci numbers, named after the French mathematician Jacques-Phillipie-Marie Binet (1786-1856) is given by

$$(1.2) \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad n = 1, 2, 3, \dots, \quad \text{where } F_n, \text{ is the } n^{\text{th}} \text{ Fibonacci number. This}$$

formula may be easily established by using mathematical induction. (1.1) is called the

Fibonacci quadratic equation. The Binet form for L_n , the n^{th} Lucas number is given by

$$L_n = \alpha^n + \beta^n.$$

Horadam [3] defined a generalized sequence $w_n = \{w_n(a, b; p, q)\}$ by

$$(1.3) \quad w_n = pw_{n-1} - qw_{n-2}, n \geq 2$$

where p, q are arbitrary integers, $w_0 = a, w_1 = b$, a, b are any numbers.

Let α_1, β_1 be roots of $x^2 - px + q = 0$. Hence,

$$\alpha_1 = \frac{p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \beta_1 = \frac{p - \sqrt{p^2 - 4q}}{2}. \quad \text{Let } d = \sqrt{p^2 - 4q}, \text{ which gives}$$

$$\alpha_1 = \frac{p+d}{2}, \quad \beta_1 = \frac{p-d}{2}, \quad \alpha_1 + \beta_1 = p, \quad \alpha_1 \beta_1 = q, \quad \alpha_1 - \beta_1 = d$$

The Binet form for w_n is given by

$$(1.4) \quad w_n = A\alpha_1^n + B\beta_1^n, \text{ where}$$

$$(1.5) \quad A = \frac{b - a\beta_1}{\alpha_1 - \beta_1}, \quad B = \frac{a\alpha_1 - b}{\alpha_1 - \beta_1}$$

$$A + B = a, A - B = \frac{2b - ap}{d} AB = \frac{e}{d^2}, \text{ where } e = pab - q^2 - b^2. \text{ Formula (1.4) may be}$$

proved by induction.

If $n = 0$, then $w_0 = A\alpha_1^0 + B\beta_1^0 = A + B = a..$

So the expression holds for $n = 0$. We now assume it holds for all $n \leq k$,

$w_k = A\alpha_1^k + B\beta_1^k$. We prove the result for $n = k + 1$, $w_{k+1} = pw_k - qw_{k-1}$. Now,

$$w_{k+1} = p(A\alpha_1^k + B\beta_1^k) - q(A\alpha_1^{k-1} + B\beta_1^{k-1})$$

$$w_{k+1} = A\alpha_1^{k-1}(p\alpha_1 - q) + B\beta_1^{k-1}(p\beta_1 - q)$$

$$w_{k+1} = A\alpha_1^{k-1}(\alpha_1(\alpha_1 + \beta_1) - \alpha_1\beta_1) + B\beta_1^{k-1}(\beta_1(\alpha_1 + \beta_1) - \alpha_1\beta_1).$$

Since $\alpha_1 + \beta_1 = p, \alpha_1\beta_1 = q$, then

$$w_{k+1} = A\alpha_1^{k-1}(\alpha_1^2) + B\beta_1^{k-1}(\beta_1^2) = A\alpha_1^{k+1} + B\beta_1^{k+1}.$$

Thus (1.4) holds for all positive integers n . It can be proved similarly for negative integers n .

Particular cases of w_n are:

$$(1.6) \quad u_n = \{w_n(1, p; p, q)\} = u_n(p, q)$$

$$(1.7) \quad v_n = \{w_n(2, p; p, q)\} = v_n(p, q)$$

$$(1.8) \quad h_n = \{w_n(r, r+s; 1, -1)\} = h_n(r, s)$$

$$(1.9) \quad F_n = \{w_n(1, 1; 1, -1)\} = u_n(1, -1) = h_n(1, 0).$$

We note that w_n generalizes the Fibonacci sequence given by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, n \geq 2 \text{ with the initial values } F_0 = a = 1, F_1 = b = 1.$$

$$(1.10) \quad L_n = \{w_n(2, 1; 1, -1)\} = v_n(1, -1) = h_n(2, -1)$$

From (1.4), (1.5), and (1.6) it follows that

$$u_n = \frac{(p - \beta_1)}{d} \alpha_1^n + \frac{(\alpha_1 - p)}{d} \beta_1^n$$

$$u_n = \frac{(\alpha_1) \alpha_1^n + (-\beta_1) \beta_1^n}{d}, \text{ since } \alpha_1 - p = -\beta_1.$$

Thus u_n has the Binet form

$$(1.11) \quad u_n = \frac{\alpha_1^{n+1} - \beta_1^{n+1}}{d}.$$

Similarly,

$$(1.12) \quad v_n = \alpha_1^n + \beta_1^n$$

$$(1.13) \quad h_n = \frac{(r+s-r\beta_1)\alpha_1^n - (r+s-r\alpha_1)\beta_1^n}{\sqrt{5}}.$$

$$F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}}, \quad L_n = \alpha^n + \beta^n \text{ where } \alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2} \text{ are the roots of the}$$

equation $x^2 - x - 1 = 0$. We want to show that

$$(1.14) \quad w_n = au_n + (b - pa)u_{n-1}.$$

This can be proved as follows:

$$w_n = A\alpha_1^n + B\beta_1^n$$

$$w_n = \left(\frac{b - a\beta}{d} \right) \alpha_1^n + \left(\frac{a\alpha_1 - b}{d} \right) \beta_1^n$$

$$w_n = b \left(\frac{\alpha_1^n - \beta_1^n}{d} \right) - a\alpha_1\beta_1 \left(\frac{\alpha_1^{n-1} - \beta_1^{n-1}}{d} \right)$$

$w_n = bu_{n-1} - aqu_{n-2}$. Thus w_n can also be written as

$$(1.15) \quad w_n = bu_{n-1} - aqu_{n-2}$$

Here we used (1.11). Thus we have $w_n = bu_{n-1} - a(pu_{n-1} - u_n)$, where we used the

recurrence relation for u_n . Hence $w_n = bu_{n-1} + au_n - apu_{n-1} = au_n + (b - pa)u_{n-1}$.

Substituting $a = 2, b = p$ in (1.14), (1.15) and using (1.7), we get

$$(1.16) \quad v_n = 2u_n - pu_{n-1}$$

$$(1.17) \quad v_n = pu_{n-1} - 2qu_{n-2}.$$

Putting $n=0$ in (1.14) and (1.15) requires us to define w_{-n} . Therefore replacing n by $-n$ in (1.14) gives,

$$w_{-n} = A\alpha_1^{-n} + B\beta_1^{-n}. \text{ In fact we will show that}$$

$$(1.18) \quad w_{-n} = q^{-n}(au_n - bu_{n-1}).$$

The above equation (1.18) can be proved as follows:

$$w_{-n} = \left(\frac{b - a\beta_1}{\alpha_1 - \beta_1} \right) \left(\frac{1}{\alpha_1} \right)^n + \left(\frac{a\alpha_1 - b}{\alpha_1 - \beta_1} \right) \left(\frac{1}{\beta_1} \right)^n$$

$$w_{-n} = \left(\frac{b - a\beta_1}{\alpha_1 - \beta_1} \right) \left(\frac{\beta_1}{q} \right)^n + \left(\frac{a\alpha_1 - b}{\alpha_1 - \beta_1} \right) \left(\frac{\alpha_1}{q} \right)^n$$

Since $\alpha_1\beta_1 = q$, $(\alpha_1\beta_1)^n = q^n$, $\left(\frac{1}{\alpha_1\beta_1} \right)^n = \left(\frac{1}{q} \right)^n$, we have

$$w_{-n} = \left(\frac{1}{q} \right)^n \frac{1}{\alpha_1 - \beta_1} (b\beta_1^n - a\beta_1^{n+1} + a\alpha_1^{n+1} - b\alpha_1^n)$$

$$w_{-n} = \left(\frac{1}{q} \right)^n \left[a \left(\frac{\alpha_1^{n+1} - \beta_1^{n+1}}{d} \right) - b \left(\frac{\alpha_1^n - \beta_1^n}{d} \right) \right]$$

$$w_{-n} = \left(\frac{1}{q} \right)^n (au_n - bu_{n-1}), \text{ where we used (1.11). Thus}$$

$$w_{-n} = q^{-n}(au_n - bu_{n-1}).$$

Substituting $a=1, b=p$ in (1.18) we have,

$$u_{-n} = q^{-n}(u_n - pu_{n-1}) = -q^{-n+1}u_{n-2} \text{ because } u_n - pu_{n-1} = -qu_{n-2} \text{ from the recurrence}$$

relation. Combining (1.14), (1.15) and (1.18), we arrive at

$$(1.19) \quad w_{-n} = q^{-n} \frac{(au_n - bu_{n-1})}{au_n + (b - pa)u_{n-1}} w_n$$

Putting $a = 2, b = p$ in (1.19) and using (1.7) gives

$$(1.20) \quad v_{-n} = q^{-n} \left(\frac{2u_n - pu_{n-1}}{2u_n - pu_{n-1}} \right) v_n, \text{ or } v_{-n} = q^{-n} v_n.$$

Similarly,

$$(1.21) \quad h_{-n} = (-1)^n \frac{\{r(u_n - u_{n-1}) - su_{n-1}\}}{ru_n + su_{n-1}} h_n.$$

Also putting $a = 0, p = b = 1$ in (1.19) gives $F_{-n} = (-1)^{n+1} F_n$ which is a well-known identity for F_n . In particular for $n = -1$,

$$\begin{aligned} w_{-1} &= A\alpha_1^{-1} + B\beta_1^{-1} = \frac{A}{\alpha_1} + \frac{B}{\beta_1} \\ w_{-1} &= \frac{(b - a\beta_1)}{(\alpha_1 - \beta_1)\alpha_1} + \frac{(a\alpha_1 - b)}{(\alpha_1 - \beta_1)\beta_1} \\ w_{-1} &= \frac{b\beta_1 - a\beta_1^2 + a\alpha_1^2 - b\alpha_1}{q(\alpha_1 - \beta_1)}, \text{ since } \alpha_1\beta_1 = q \\ w_{-1} &= \frac{(\alpha_1 - \beta_1)(-b - a(\alpha_1 + \beta_1))}{q(\alpha_1 - \beta_1)} \\ (1.22) \quad w_{-1} &= \frac{pa - b}{q}. \end{aligned}$$

Substituting different values for a, b, p, q in (1.22),

$$a = 1, b = p \text{ implies } u_{-1} = \frac{p - p}{q} = 0.$$

$$a = 2, b = p \text{ implies } v_{-1} = \frac{2p - p}{q} = \frac{p}{q}.$$

$$a = r, b = r + s, p = 1, q = -1 \text{ implies } h_{-1} = \frac{r - (r + s)}{-1} = \frac{-s}{-1} = s.$$

$$a = 1, b = 1, p = 1, q = -1 \text{ implies } F_{-1} = \frac{1 - 1}{-1} = 0.$$

$$a = 2, b = 1, p = 1, q = -1 \text{ implies } L_{-1} = \frac{2 - 1}{-1} = -1.$$

Many w_n can be expressed in terms of F_n besides the case $\{w_n(1, 1; 1, -1)\} = F_n$. Some of them are:

$$(1.23) \quad w_n(-1, 1; -1, -1) = (-1)^{n-1} F_n$$

Which can be proved by substituting $a = -1, b = 1, p = -1, q = -1$ in (1.4).

$$w_n = \left(\frac{b - a\beta_1}{\alpha_1 - \beta_1} \right) \alpha_1^n + \left(\frac{a\alpha_1 - b}{\alpha_1 - \beta_1} \right) \beta_1^n \text{ where } \alpha_1, \beta_1 \text{ are roots of } x^2 - px + q = 0.$$

$p = -1, q = -1$ implies $x^2 - px + q = x^2 + x - 1 = 0$. Let m_1, m_2 be roots of

$$x^2 + x - 1 = 0 \text{ then, } m_1 = \frac{-1 + \sqrt{5}}{2} = \frac{-(1 - \sqrt{5})}{2} = -\beta \text{ and}$$

$$m_2 = \frac{-1 - \sqrt{5}}{2} = \frac{-(1 + \sqrt{5})}{2} = -\alpha, \text{ so,}$$

$$m_1 + m_2 = -\beta + (-\alpha) = -(\alpha + \beta) = -1,$$

$$m_1 - m_2 = -\beta - (-\alpha) = \alpha - \beta = \sqrt{5},$$

$$m_1 m_2 = -\beta(-\alpha) = \alpha\beta = -1.$$

$$w_n = \left(\frac{b - am_2}{m_1 - m_2} \right) m_1^n + \left(\frac{am_1 - b}{m_1 - m_2} \right) m_2^n$$

$$w_n = \left(\frac{1 + m_2}{\sqrt{5}} \right) m_1^n + \left(\frac{-m_1 - 1}{\sqrt{5}} \right) m_2^n$$

$$w_n = \frac{1}{\sqrt{5}} [(1 + m_2)m_1^n - (1 + m_1)m_2^n]$$

$$w_n = \frac{1}{\sqrt{5}} [(-m_1)m_1^n - (-m_2)m_2^n]$$

$$w_n = \frac{1}{\sqrt{5}} (-m_1^{n+1} + m_2^{n+1})$$

$$w_n = \frac{1}{\sqrt{5}} ((-\alpha)^{n+1} - (-\beta)^{n+1})$$

$$w_n = \frac{1}{\sqrt{5}} (-1)^{n-1} (\alpha^{n+1} - \beta^{n+1}), \text{ or}$$

$$w_n = (-1)^{n-1} F_n.$$

Similarly, it can be proved that,

$$(1.24) \quad w_n(1, -1; 1, -1) = F_{n-3}$$

$$(1.25) \quad w_n(1, 1; -1, -1) = (-1)^{n-1} F_{n-3} . \text{ Generally,}$$

$$(1.26) \quad w_n(a, b; 1, -1) = aF_{n-2} + bF_{n-1}$$

$$(1.27) \quad w_n(a, b; -1, -1) = (-1)^n (aF_{n-2} - bF_{n-1}) .$$

Now we will prove (1.27). If $p = q = -1$, then $w_n = Am_1^n + Bm_2^n$ where m_1, m_2 are roots of $x^2 + x - 1 = 0$, also, $m_1 = -\beta, m_2 = -\alpha$. Hence

$$w_n = \frac{1}{\sqrt{5}} [(b - am_2)m_1^n + (am_1 - b)m_2^n]$$

$$w_n = \frac{1}{\sqrt{5}} [a(m_1m_2^n - m_2m_1^n) + b(m_1^n - m_2^n)]$$

$$w_n = \frac{1}{\sqrt{5}} [a((- \beta)(- \alpha)^n - (- \alpha)(- \beta)^n) + b((- \beta)^n - (- \alpha)^n)]$$

$$w_n = \frac{1}{\sqrt{5}} [a((- \alpha)^{n-1} - (- \beta)^{n-1}) + ((- \beta)^n - (- \alpha)^n)], \text{ since } -\beta = \frac{1}{\alpha}, -\alpha = \frac{1}{\beta}$$

$$w_n = (-1)^n \left[a \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} \right) + \left(\frac{\beta^n - \alpha^n}{\sqrt{5}} \right) \right]$$

$$w_n = (-1)^n [aF_{n-2} + bF_{n-1}]$$

2. Basic Properties of w_n

We have the recurrence relation (1.3) $w_{n+2} = pw_{n+1} - qw_n$ which can also be written as

$$w_{n+2} = p(pw_n - qw_{n-1}) - qw_n = p^2w_n - pqw_{n-1} - qw_n = (p^2 - q)w_n - pqw_{n-1}, \text{ or}$$

$$(1.28) \quad w_{n+2} - (p^2 - q)w_n + pqw_{n-1} = 0$$

Using the recurrence relation in the same manner also leads to

$$(1.29) \quad pw_{n+2} - (p^2 - q)w_{n+1} + q^2w_{n-1} = 0$$

(1.28) and (1.29) are just linear relations in w_n . To find the sum of the first n terms of

w_n we use $qw_{k-1} = pw_k - w_{k+1}$, a different form of the recurrence relation (1.3). Hence we have

$$q \sum_{j=0}^{n-1} w_j = q(w_0 + w_1 + w_2 + \dots + w_{n-1})$$

$$q \sum_{j=0}^{n-1} w_j = qw_0 + qw_1 + qw_2 + \dots + qw_{n-1}.$$

Substituting from above for each $qw_j, j = 0, 1, 2, 3, \dots, n-1$ we get,

$$q \sum_{j=0}^{n-1} w_j = pw_1 - w_2 + pw_2 - w_3 + pw_3 - w_4 + \dots + pw_n - w_{n+1}$$

$$(1.30) \quad q \sum_{j=0}^{n-1} w_j = pw_1 + (p-1)w_2 + (p-1)w_3 + \dots + (p-1)w_n - w_{n+1}.$$

Some other summation results listed below can be obtained similarly

$$(1.31) \quad (p-q-1) \sum_{j=0}^{n-1} w_j = w_{n+1} - w_1 - (p-1)(w_n - w_0)$$

$$(1.32) \quad q \sum_{j=0}^{n-1} (-1)^j w_j = (p+1)(-w_2 + w_3 + \dots + (-1)^n w_n) + (-1)^n w_{n+1} + pw_1$$

$$(1.33) \quad (p-q+1) \sum_{j=0}^{n-1} (-1)^j w_j = (p+1)(w_0 + (-1)^n w_n) + w_1 + (-1)^{n+1} w_{n+1}$$

Let ,

$$(1.34) \quad \sigma = w_0 + w_2 + w_4 + \dots + w_{2n-2}$$

$$(1.35) \quad \rho = w_1 + w_3 + w_5 + \dots + w_{2n-1}$$

Replacing n by $2n$ in (1.30) and (1.32) and adding the two gives the sum of even indexed terms of w_n .

$$q \sum_{j=0}^{2n-1} w_j + q \sum_{j=0}^{2n-1} (-1)^j w_j = q \left(\sum_{j=0}^{2n-1} w_j + \sum_{j=0}^{2n-1} (-1)^j w_j \right) = q(2w_0 + 2w_2 + \dots + 2w_{2n-2}) = 2q\sigma,$$

where we used (1.34). Let $I = q \sum_{j=0}^{2n-1} w_j + q \sum_{j=0}^{2n-1} (-1)^j w_j$. Relations (1.30) and (1.32)

imply,

$$\begin{aligned} I &= (p-1)(w_2 + w_3 + \dots + w_{2n}) - w_{2n+1} + pw_1 + (p+1)(-w_2 + w_3 + \dots - w_{2n}) + w_{2n+1} + pw_1 \\ I &= 2pw_1 + 2p(w_3 + w_5 + \dots + w_{2n-1}) - 2(w_2 + w_4 + \dots + w_{2n}) \end{aligned}$$

Adding and subtracting the term $2w_0$ gives,

$$I = 2p\rho - 2\sigma + 2w_0 - 2w_{2n}. \text{ Hence } 2p\rho - 2\sigma + 2w_0 - 2w_{2n} = 2q\sigma, \text{ or}$$

$$2\sigma(1+q) = 2(p\rho - (w_{2n} - w_0)), \text{ canceling the factor 2 from both sides gives,}$$

$$(1.36) \quad (1+q)\sigma = p\rho - (w_{2n} - w_0)$$

Subtracting (1.30) from (1.32) yields,

$$(1.37) \quad (1+q)\rho = p\sigma + q(w_{2n-1} - w_{-1})$$

Suitable substitutions in the above results lead to identities of the related sequences, for

example: Putting $p = 1, q = -1$ in (1.30) yields $-\sum_{j=0}^{n-1} F_j = F_1 - F_{n+1}$. Hence

$$(1.38) \quad \sum_{j=0}^{n-1} F_j = F_{n+1} - 1, \text{ since } F_1 = 1.$$

This is a well-known identity for F_n . Replacing n by $2n$ in (1.38), we obtain

$\sum_{j=0}^{2n-1} F_j = F_{2n+1} - 1$, or $\sigma + \rho = F_{2n+1} - 1$. Using (1.14), another form of w_n , given by

$w_n = bu_{n-1} - aqu_{n-2}$, we get $w_{n+1} = bu_n - aqu_{n-1}$. By substituting $w_0 = a, w_1 = b$, we

obtain $w_{n+1} = w_1 u_n - q w_0 u_{n-1}$.

Now, $w_{n+2} = p w_{n+1} - q w_n = p(w_1 u_n - q w_0 u_{n-1}) - q(w_1 u_{n-1} - q w_0 u_{n-2})$

$$w_{n+2} = p w_1 u_n - p q w_0 u_{n-1} - q w_1 u_{n-1} + q^2 w_0 u_{n-2} = p w_1 u_n - q w_1 u_{n-1} - q w_0 (p u_{n-1} - q u_{n-2})$$

$$w_{n+2} = p w_1 u_n - q w_1 u_{n-1} - q w_0 u_n = u_n (p w_1 - q w_0) - q w_1 u_{n-1} = u_n (w_2) - q w_1 u_{n-1}.$$

Hence $w_{n+2} = w_2 u_n - q w_1 u_{n-1}$. Similarly we can write $w_{n+3} = w_3 u_n - q w_2 u_{n-1}$,

$w_{n+4} = w_4 u_n - q w_3 u_{n-1}$, In general,

$$(1.39) \quad w_{n+r} = w_r u_n - q w_{r-1} u_{n-1}$$

Using (1.4) and (1.14) we obtain,

$$(1.40) \quad \frac{w_{n+r} + q^r w_{n-r}}{w_n} = v_r$$

an expression independent of a and b . (1.40) can be easily verified as follows,

$$\begin{aligned} \frac{w_{n+r} + q^r w_{n-r}}{w_n} &= \frac{A\alpha^{n+r} + B\beta^{n+r} + q^r (A\alpha^{n-r} + B\beta^{n-r})}{A\alpha^n + B\beta^n} = \frac{A\alpha^{n+r} + B\beta^{n+r} + q \left(A \frac{\alpha^n}{\alpha^r} + B \frac{\beta^n}{\beta^r} \right)}{A\alpha^n + B\beta^n} \\ &= \frac{A\alpha^{n+r} + B\beta^{n+r} + q^r \left(\frac{A\alpha^n \beta^r + B\beta^n \alpha^r}{(\alpha\beta)^r} \right)}{A\alpha^n + B\beta^n}, \text{ since } \alpha\beta = q, (\alpha\beta)^r = q^r, \end{aligned}$$

Canceling the term q^r and grouping the like terms, we get,

$$= \frac{A\alpha^n(\alpha^r + \beta^r) + B\beta^n(\beta^r + \alpha^r)}{A\alpha^n + B\beta^n} = \frac{(A\alpha^n + B\beta^n)(\alpha^r + \beta^r)}{A\alpha^n + B\beta^n} = \alpha^r + \beta^r = v_r.$$

One other identity whose proof is similar is

$$(1.41) \quad aw_{m+n} + (b - pa)w_{m+n-1} = w_m w_n - qw_{m-1} w_{n-1}.$$

Some of these generalized identities like (1.40) and (1.41) will be used to derive identities about the functions associated with w_n in chapter II, section 1.

Another form of w_n can be found as follows, from (1.4),

$$w_n = A\alpha_1^n + B\beta_1^n = A\left(\frac{p+d}{2}\right)^n + B\left(\frac{p-d}{2}\right)^n, \therefore 2^n w_n = A(p+d)^n + B(p-d)^n,$$

Since $A = \frac{b - a\beta_1}{d}$, $B = \frac{a\alpha_1 - b}{d}$, we get,

$$2^n w_n = \frac{1}{d} \left[(b - a\beta_1)(p+d)^n + (a\alpha_1 - b)(p-d)^n \right]$$

$$2^n w_n = \frac{1}{d} \left[b(p+d)^n - a\beta_1(p+d)^n + a\alpha_1(p-d)^n - b(p-d)^n \right]$$

$$2^n w_n = \frac{1}{d} \left[b \left\{ (p+d)^n - (p-d)^n \right\} + \frac{a}{2} \left\{ (p+d)(p-d)^n - (p-d)(p+d)^n \right\} \right]$$

Let $J = \left\{ (p+d)^n - (p-d)^n \right\}$ and $K = \left\{ (p+d)(p-d)^n - (p-d)(p+d)^n \right\}$. So the above

expression becomes $2^n w_n = \frac{1}{d} \left[bJ + \frac{a}{2} K \right]$. We simplify J and K to obtain an expression

of w_n .

$$J = \left\{ \sum_{j=0}^n \binom{n}{j} d^j p^{n-j} - \sum_{j=0}^n (-1)^j \binom{n}{j} d^j p^{n-j} \right\} = \left\{ \sum_{j=0}^n \binom{n}{j} d^j p^{n-j} + \sum_{j=0}^n (-1)^{j+1} \binom{n}{j} d^j p^{n-j} \right\}$$

Terms with even indices cancel, but terms with odd indices remain.

$$J = 2 \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j+1} p^{n-2j-1}. \text{ Now simplifying the terms in } K, \text{ we get}$$

$$(p+d)(p-d)^n = (p+d) \sum_{j=0}^n (-1)^j \binom{n}{j} d^j p^{n-j} \text{ and}$$

$$-(p-d)(p+d)^n = (-p+d) \sum_{j=0}^n \binom{n}{j} d^j p^{n-j}. \text{ Adding the above two relations yields}$$

$$K = p \left[\sum_{j=0}^n (-1)^j \binom{n}{j} d^j p^{n-j} - \sum_{j=0}^n \binom{n}{j} d^j p^{n-j} \right] + d \left[\sum_{j=0}^n (-1)^j \binom{n}{j} d^j p^{n-j} + \sum_{j=0}^n \binom{n}{j} d^j p^{n-j} \right]$$

Even indices of d^j cancel out in the first parentheses and odd indices of d^j cancel out in the second parentheses to give,

$$K = p \left[-2 \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j+1} p^{n-2j-1} \right] + d \left[2 \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} \right]. \text{ Now replacing } J \text{ and } K, \text{ we}$$

obtain

$$2^n w_n = \frac{1}{d} \left[2b \left(\sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j+1} p^{n-2j-1} \right) + \frac{a}{2} \left(-2p \left(\sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j+1} p^{n-2j-1} \right) \right) \right. \\ \left. + 2d \left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} \right) \right]$$

Regrouping the terms, we get,

$$2^n w_n = (2b - pa) \left(\sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j} p^{n-2j-1} \right) + \frac{2d}{d} \cdot \frac{a}{2} \left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} \right), \text{ or}$$

$$(1.42) \quad 2^n w_n = a \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} + (2b - pa) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j} p^{n-2j-1}.$$

Putting $a = 1, b = p$ in (1.42) gives a result in u_n as follows

$$2^n u_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} + (2p - p) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j} p^{n-2j-1}$$

$$2^n u_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j} p^{n-2j}$$

$$(1.43) \quad 2^n u_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n+1}{2j+1} d^{2j} p^{n-2j},$$

where we used the well known combinatorial identity

$$\binom{n+1}{2j+1} = \binom{n}{2j+1} + \binom{n}{2j}.$$

Substituting $a = 2, b = p$ in (1.42) gives a result in v_n

$$2^n v_n = 2 \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j} + (2b - 2b) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} d^{2j} p^{n-2j-1}$$

$$2^n v_n = 2 \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j}, \text{ or}$$

$$(1.44) \quad 2^{n-1} v_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} d^{2j} p^{n-2j}.$$

Similarly using $a = 1, b = 1, p = 1, q = -1$ in (1.42) leads to a result in F_n

$$2^n F_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} (\sqrt{5})^{2j} (1)^{n-2j} + (2-1) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} (\sqrt{5})^{2j} (1)$$

$$(1.45) \quad 2^n F_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n+1}{2j+1} 5^j.$$

Also, $a = 2, b = 1, p = 1, q = -1, d = \sqrt{p^2 - 4q} = \sqrt{5}$ yields,

$$2^n L_n = 2 \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} (\sqrt{5})^{2j} + (2-2) \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2j+1} (\sqrt{5})^{2j} (1)$$

$$2^n L_n = 2 \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} (5)^j$$

$$(1.46) \quad 2^{n-1} L_n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} (5)^j.$$

Throughout the above discussion of w_n we assumed $p^2 \neq 4q$. We note that the case

$p^2 = 4q$ does not apply to F_n and L_n , the Fibonacci and Lucas sequences as for both of them, $p^2 - 4q = 5$ ($\neq 0$).

3. The sequences x_n and y_n .

We construct two sequences x_n and y_n , which again generalize the Fibonacci sequence F_n . Let $x_0 = a, x_1 = c, y_0 = b, y_1 = d$ be arbitrary real numbers. The sequences x_n and y_n are defined by

$$(1.47) \quad x_{n+2} = y_{n+1} + y_n$$

$$(1.48) \quad y_{n+2} = x_{n+1} + x_n \quad n \geq 0$$

$a = b$ and $c = d$ implies $x_0 = y_0 = a$ and $x_1 = y_1 = d$ and $x_2 = y_0 + y_1 = a + d$,

$y_2 = x_0 + x_1 = a + d$. Thus $x_2 = y_2$.

Similarly, we find $x_i = y_i$ $i = 0, 1, 2, \dots$. Therefore the sequences x_n and y_n coincide with each other and with the generalized Fibonacci sequence H_n with $H_0 = a, H_1 = d$.

Hogatt[2] in his book defines this sequence H_n . The following theorems are Fibonacci-type identities in x_n and y_n .

Theorem 1.

$$(a) \quad x_{3n} + y_0 = y_{3n} + x_0, \quad n \geq 0$$

$$(b) \quad x_{3n+1} + y_1 = y_{3n+1} + x_1, \quad n \geq 0$$

$$(c) \quad x_{3n+2} + x_0 + x_1 = y_{3n+2} + y_0 + y_1, \quad n \geq 0.$$

Proof of part (a) and part (c) is similar to part (b). Part (b) is proved as follows using induction.

$n = 0$ implies $x_1 + y_1 = y_1 + x_1$. The statement is true for $n = 0$. Assume that the statement is true $\forall n \leq k$. Hence

$$x_{3k+1} + y_1 = y_{3k+1} + x_1.$$

To prove the result for $n = k + 1$, we have to prove that $x_{3k+4} + y_1 = y_{3k+4} + x_1$. Now,

$$\begin{aligned}
 x_{3k+4} + y_1 &= y_{3k+3} + y_{3k+2} + y_1 \\
 &= x_{3k+2} + x_{3k+1} + y_{3k+2} + y_1 && \text{by (1.48)} \\
 &= x_{3k+2} + y_{3k+2} + x_{3k+1} + y_1 \\
 &= x_{3k+2} + y_{3k+2} + y_{3k+1} + x_1 && \text{by the induction hypothesis.} \\
 &= x_{3k+2} + x_{3k+3} + x_1 && \text{by (1.48)} \\
 &= y_{3k+4} + x_1 && \text{by (1.47)}
 \end{aligned}$$

So, the statement is true for all integers $n \geq 0$.

Theorem 2. For all integers $k \geq 0$, we have

$$\begin{aligned}
 \text{(a)} \quad x_{3k+2} &= \sum_{i=0}^{3k} y_i + y_1 & \text{(b)} \quad x_{3k+3} &= \sum_{i=0}^{3k+1} x_i + y_1 \\
 \text{(c)} \quad x_{3k+4} &= \sum_{i=0}^{3k+2} y_i + x_1 & \text{(d)} \quad y_{3k+2} &= \sum_{i=0}^{3k} x_i + x_1 \\
 \text{(e)} \quad y_{3k+3} &= \sum_{i=0}^{3k+1} y_i + x_1 & \text{(f)} \quad y_{3k+4} &= \sum_{i=0}^{3k+2} x_i + y_1
 \end{aligned}$$

We prove part (a) by induction, proofs of other parts are similar. $k = 0$ implies

$x_2 = y_0 + y_1$ which is true by (1.47). assume the identity is true for $k \leq r, r \geq 0$

$x_{3r+2} = \sum_{i=0}^{3r} y_i + y_1$. We have to prove the result for $k = r + 1$, we prove that,

$x_{3r+5} = \sum_{i=0}^{3r+3} y_i + y_1$. Now, we use (1.47) and (1.48) to write L.H.S as

$x_{3r+5} = y_{3r+4} + y_{3r+3} = y_{3r+4} + x_{3r+2} + x_{3r+1}$. We can replace the term x_{3r+2} by the induction hypothesis to obtain,

$x_{3r+5} = y_{3r+4} + x_{3r+1} + \sum_{i=0}^{3r} y_i + y_1$. We use the recurrence relation for y_n to get,

$x_{3r+5} = x_{3r+3} + x_{3r+2} + x_{3r+1} + \sum_{i=0}^{3r} y_i + y_1$. Since $x_{3r+3} = y_{3r+2} + y_{3r+1}$ and

$x_{3r+2} + x_{3r+1} = y_{3r+3}$, we get $x_{3r+5} = y_{3r+2} + y_{3r+1} + y_{3r+3} + \sum_{i=0}^{3r} y_i + y_1$

We then have $x_{3r+5} = \sum_{i=0}^{3r+3} y_i + y_1$. Hence (a) is true $\forall r \geq 0$.

Now we add the terms with even or odd subscripts of both x_n and y_n to come up with more Fibonacci-type results.

Theorem 3. For all integers $k \geq 0$, we have:

$$(a) \quad x_{6k+5} = \sum_{i=0}^{3k+3} y_{2i} - x_0 + y_1$$

$$(b) \quad x_{6k+6} = \sum_{i=1}^{3k+3} y_{2i-1} + x_0$$

$$(c) \quad x_{6k+7} = \sum_{i=0}^{3k+3} y_{2i} - y_0 + x_1$$

$$(d) \quad x_{6k+8} = \sum_{i=1}^{3k+4} y_{2i-1} + y_0$$

$$(e) \quad x_{6k+9} = \sum_{i=0}^{3k+4} y_{2i} - y_0 + y_1$$

$$(f) \quad x_{6k+10} = \sum_{i=1}^{3k+5} y_{2i-1} + x_0 + x_1 - y_1$$

$$(g) \quad y_{6k+5} = \sum_{i=0}^{3k+2} x_{2i} - y_0 + x_1$$

$$(h) \quad y_{6k+6} = \sum_{i=0}^{3k+3} x_{2i-1} + y_0$$

$$(i) \quad y_{6k+7} = \sum_{i=0}^{3k+3} x_{2i} - x_0 + y_1$$

$$(j) \quad y_{6k+8} = \sum_{i=0}^{3k+4} x_{2i-1} + x_0$$

$$(k) \quad y_{6k+9} = \sum_{i=0}^{3k+4} x_{2i} - x_0 + x_1$$

$$(l) \quad y_{6k+10} = \sum_{i=0}^{3k+5} x_{2i-1} + y_0 - x_1 + y_1$$

All the results (a) through (l) can be proved using induction. Proof of part (a) is as follows:

$$k = 0 \text{ implies } x_5 = \sum_{i=0}^2 y_{2i} - x_0 + y_1$$

$$R.H.S. = y_0 + y_2 + y_4 - x_0 + y_1$$

$$R.H.S. = y_0 - x_0 + y_4 + y_1 + y_2$$

$$R.H.S. = y_0 + y_1 + y_2 - x_0 + y_4$$

Since, $y_0 + y_1 = x_2$ and $y_2 = x_0 + x_1$, then $x_1 = y_2 - x_0$, and $x_5 = x_2 + x_1 + y_4$,

$$R.H.S. = y_3 + y_4 \quad \text{from (1.47)}$$

$$R.H.S. = x_5$$

Therefore the statement is true for $k = 0$. Assume that the statement is true $\forall$

$$r \leq k, r \geq 0. \quad x_{6r+5} = \sum_{i=0}^{3r+2} y_{2i} - x_0 + y_1.$$

We have to prove the result for $k = r + 1$. To prove that $x_{6r+11} = \sum_{i=0}^{3r+5} y_{2i} - x_0 + y_1$

$$x_{6r+11} = y_{6r+10} + y_{6r+9} \quad \text{from (1.47)}$$

$$= y_{6r+10} + y_{3(2r+3)}$$

$$= y_{6r+10} + x_{6r+9} + y_0 - x_0 \quad \text{from part (a) of Theorem 1.}$$

$$= y_{6r+10} + y_{6r+8} + y_{6r+7} + y_0 - x_0$$

$$= y_{6r+10} + y_{6r+8} + x_{6r+6} + x_{6r+5} + y_0 - x_0$$

as $x_{6r+5} = \sum_{i=0}^{3r+2} y_{2i} - x_0 + y_1$ by induction hypothesis. We also use part (a) of Theorem 1,

which gives $x_{6r+6} = x_{3(2r+2)} = y_{3(2r+2)} + x_0 - y_0$, to replace the term x_{6r+6} .

$$x_{6r+11} = y_{6r+10} + y_{6r+8} + y_{6r+6} + x_0 - y_0 + \sum_{i=0}^{3r+2} y_{2i} - x_0 + y_1 + y_0 - x_0$$

$$x_{6r+11} = y_{6r+10} + y_{6r+8} + y_{6r+6} + \sum_{i=0}^{3r+2} y_{2i} - x_0 + y_1$$

$$x_{6r+11} = \sum_{i=0}^{3r+5} y_{2i} - x_0 + y_1$$

Hence the result is true for all positive integers $k \geq 0$. Some more results can be obtained using Theorem 1 and Theorem 2.

Theorem 4. If $k \geq 0$, then

$$(a) \sum_{i=0}^{3k} (x_i - y_i) = x_0 - y_0$$

$$(b) \sum_{i=0}^{3k+1} (x_i - y_i) = y_2 - x_2$$

$$(c) \sum_{i=0}^{3k+2} (x_i - y_i) = 0$$

Proof of (a) through (c) is direct and similar. So only part (a) proof is shown below.

$$\sum_{i=0}^{3k} (x_i - y_i) = (x_0 - y_0) + (x_1 - y_1) + \cdots + (x_{3k} - y_{3k})$$

$$= \sum_{i=0}^{3k} x_i - \sum_{i=0}^{3k} y_i$$

From part (a) and part (d) of Theorem 2, we have $\sum_{i=0}^{3k} y_i = x_{3k+2} - y_1$ and

$\sum_{i=0}^{3k} x_i = y_{3k+2} - x_1$. Therefore replacing $\sum_{i=0}^{3k} x_i$ and $\sum_{i=0}^{3k} y_i$ and then using the relation from

part (c) of theorem 1 to replace y_{3k+2} , we get the following,

$\sum_{i=0}^{3k} (x_i - y_i) = x_{3k+2} + x_0 + x_1 - y_0 - y_1 - x_1 - x_{3k+2} + y_1$. We cancel the terms with

opposite sign to get the result $\sum_{i=0}^{3k} (x_i - y_i) = x_0 - y_0$.

The next theorem is one of the several expected relations between x_n, y_n and the Fibonacci sequence F_n .

Theorem 5: If $n \geq 0$, then $x_{n+2} + y_{n+2} = F_{n+1}(x_0 + y_0) + F_{n+2}(x_1 + y_1)$

Proof: We prove this by using induction on n .

$n = 0$ implies R.H.S = $F_1(x_0 + y_0) + F_2(x_1 + y_1)$. Since $F_1 = 1, F_2 = 1$, we use the recurrence relation to obtain R.H.S = $y_2 + x_2$. So the result is true for $n = 0$.

Assume the result is true for all $n \leq k, k \geq 0$.

$x_{k+2} + y_{k+2} = F_{k+1}(x_0 + y_0) + F_{k+2}(x_1 + y_1)$. To prove the result for $n = k + 1$, we have to show that $x_{k+3} + y_{k+3} = F_{k+2}(x_0 + y_0) + F_{k+3}(x_1 + y_1)$. Using the recurrence relations, we can write $x_{k+3} + y_{k+3} = y_{k+2} + y_{k+1} + x_{k+2} + x_{k+1} = (y_{k+2} + x_{k+2}) + (x_{k+1} + y_{k+1})$. By induction hypotheses we can replace the terms in parentheses to obtain

$$x_{k+3} + y_{k+3} = F_{k+1}(x_0 + y_0) + F_{k+2}(x_1 + y_1) + F_k(x_0 + y_0) + F_{k+1}(x_1 + y_1)$$

Group the terms together and use the Fibonacci recurrence relation to get,

$$x_{k+3} + y_{k+3} = F_{k+2}(x_0 + y_0) + F_{k+3}(x_1 + y_1).$$

Hence the result is true for all integers $n \geq 0$.

More identities or relations between x_n, y_n and the Fibonacci sequence F_n could be established. But we proceed on with the generalization route to construct yet another pair of sequences Γ_n and δ_n and go on to find formulas for x_n, y_n the n^{th} terms of the sequences x_n, y_n respectively in terms of the Fibonacci sequence F_n .

4. Additional Related Sequences

We express x_n and y_n , the n^{th} term of the sequences x_n and y_n respectively in terms of two new sequences Γ_n and δ_n . When $n \geq 0$, we define

$$(1.49) \quad x_n = \Gamma_n^1 a + \Gamma_n^2 b + \Gamma_n^3 c + \Gamma_n^4 d$$

$$(1.50) \quad y_n = \delta_n^1 a + \delta_n^2 b + \delta_n^3 c + \delta_n^4 d$$

where $x_0 = a, y_0 = b, x_1 = c, y_1 = d$. Incidentally we obtain eight sequences

$\Gamma_n^1, \Gamma_n^2, \Gamma_n^3, \Gamma_n^4, \delta_n^1, \delta_n^2, \delta_n^3, \delta_n^4, n = 0, 1, 2, 3, \dots$. We note that

$n = 0$ implies $x_0 = a$ which implies $\Gamma_0^1 = 1, \Gamma_0^j = 0, j = 2, 3, 4$. Also $y_0 = b$ implies

$\delta_0^2 = 1, \delta_0^j = 0, j = 1, 3, 4$. We also note that Γ_n and δ_n satisfy the same recurrence

relation as x_n and y_n . This can be easily seen as follows:

From (1.47) we have $x_{n+2} = y_{n+1} + y_n$. Using (1.49) and (1.50) we write,

$$x_{n+2} = \Gamma_{n+2}^1 a + \Gamma_{n+2}^2 b + \Gamma_{n+2}^3 c + \Gamma_{n+2}^4 d \text{ and } y_{n+1} = \delta_{n+1}^1 a + \delta_{n+1}^2 b + \delta_{n+1}^3 c + \delta_{n+1}^4 d.$$

Substituting in (1.47), we get

$$\Gamma_{n+2}^1 a + \Gamma_{n+2}^2 b + \Gamma_{n+2}^3 c + \Gamma_{n+2}^4 d = \delta_{n+1}^1 a + \delta_{n+1}^2 b + \delta_{n+1}^3 c + \delta_{n+1}^4 d + \delta_n^1 a + \delta_n^2 b + \delta_n^3 c + \delta_n^4 d$$

We may equate the coefficients of like terms to get

$$\Gamma_{n+2}^1 = \delta_{n+1}^1 + \delta_n^1, \Gamma_{n+2}^2 = \delta_{n+1}^2 + \delta_n^2, \Gamma_{n+2}^3 = \delta_{n+1}^3 + \delta_n^3, \Gamma_{n+2}^4 = \delta_{n+1}^4 + \delta_n^4. \text{ Thus in general,}$$

$$(1.51) \quad \Gamma_{n+2}^j = \delta_{n+1}^j + \delta_n^j, \quad j = 1, 2, 3, 4$$

Similarly we can verify that ,

$$(1.52) \quad \delta_{n+2}^j = \Gamma_{n+1}^j + \Gamma_n^j, \quad j = 1, 2, 3, 4$$

Since x_n and y_n are related to F_n , we can also find relations between these eight sequences and the Fibonacci numbers.

Theorem 6: For all $n \geq 0$, we have the following relations

$$(a) \quad \Gamma_n^1 + \delta_n^1 = F_{n-1}$$

$$(c) \quad \Gamma_n^3 + \delta_n^3 = F_n$$

$$(b) \quad \Gamma_n^2 + \delta_n^2 = F_{n-1}$$

$$(d) \quad \Gamma_n^4 + \delta_n^4 = F_n$$

Proof of all the parts (a) – (d) is similar. We prove part (c) by using induction on n .

$n = 0$ implies $\Gamma_0^3 + \delta_0^3 = 0 + 0 = 0$, since $\Gamma_0^j = 0, j = 2, 3, 4$, $\delta_0^j = 0, j = 1, 3, 4$. Also

$F_0 = 0$. Thus the identity is true for $n = 0$. Assume the result to be true for all

$n \leq k, k \geq 0$. $\Gamma_k^3 + \delta_k^3 = F_k$. To prove the result for $n = k + 1$, we need to prove that

$$\Gamma_{k+1}^3 + \delta_{k+1}^3 = F_{k+1}. \text{ We use (1.51) and (1.52) and write}$$

$$\Gamma_{k+1}^3 + \delta_{k+1}^3 = \delta_k^3 + \delta_{k-1}^3 + \Gamma_k^3 + \Gamma_{k-1}^3 = (\Gamma_k^3 + \delta_k^3) + (\Gamma_{k-1}^3 + \delta_{k-1}^3). \text{ By induction}$$

hypotheses we can substitute for both parentheses to obtain

$$\Gamma_{k+1}^3 + \delta_{k+1}^3 = F_k + F_{k-1} = F_{k+1}.$$

The next theorem is a set of relations between Γ_n^j and $\delta_n^j, j = 1, 2, 3, 4$.

Theorem 7: If $k \geq 0$, then

(a) $\Gamma_{3k}^1 = \delta_{3k}^1 + 1$	(g) $\Gamma_{3k}^3 = \delta_{3k}^3$
(b) $\Gamma_{3k+1}^1 = \delta_{3k+1}^1$	(h) $\Gamma_{3k+1}^3 = \delta_{3k+1}^3 + 1$
(c) $\Gamma_{3k+2}^1 = \delta_{3k+2}^1 - 1$	(i) $\Gamma_{3k+2}^3 = \delta_{3k+2}^3 - 1$
(d) $\Gamma_{3k}^2 = \delta_{3k}^2 - 1$	(j) $\Gamma_{3k}^4 = \delta_{3k}^4$
(e) $\Gamma_{3k+1}^2 = \delta_{3k+1}^2$	(k) $\Gamma_{3k+1}^4 = \delta_{3k+1}^4 - 1$
(f) $\Gamma_{3k+2}^2 = \delta_{3k+2}^2 + 1$	(l) $\Gamma_{3k+2}^4 = \delta_{3k+2}^4 + 1$

Parts (a) – (l) can be proved using induction. We prove part (k).

When $k = 0$, we get $\Gamma_1^4 = 0, \delta_1^4 = 1$, as $y_1 = d$, where $y_1 = \delta_1^1 a + \delta_1^2 b + \delta_1^3 c + \delta_1^4 d$

Therefore, $0 = \Gamma_1^4 = \delta_1^4 - 1 = 0$. Thus the result is true for $k = 0$.

Assume the result to be true for all $k \leq r, r \geq 0$. $\Gamma_{3r+1}^4 = \delta_{3r+1}^4 - 1$. To prove it for

$k = r + 1$, we have to show that $\Gamma_{3r+4}^4 = \delta_{3r+4}^4 - 1$. We use (1.51) and (1.52) to get

$\Gamma_{3r+1}^4 = \Gamma_{3r+2}^4 + \Gamma_{3r+1}^4 + \delta_{3r+2}^4$. We now write the next step using induction hypotheses

$\Gamma_{3r+1}^4 = \Gamma_{3r+2}^4 + \delta_{3r+1}^4 - 1 + \delta_{3r+2}^4$, using (1.51) and (1.52) gives

$\Gamma_{3r+1}^4 = \Gamma_{3r+2}^4 + \Gamma_{3r+3}^4 - 1 = \delta_{3r+1}^4 - 1$. Hence the result is true for all $k, k \geq 0$.

Theorem 8: If $n \geq 0$ then

(a) $\Gamma_n^1 + \Gamma_n^2 = \delta_n^1 + \delta_n^2$	(b) $\Gamma_n^3 + \Gamma_n^4 = \delta_n^3 + \delta_n^4$

We prove part (b) using induction. (a) can be proved similarly.

When $n = 0$ $\Gamma_0^3 + \Gamma_0^4 = 0 + 0 = 0$ and $\delta_0^3 + \delta_0^4 = 0 + 0 = 0$, as noted earlier we get values of Γ_n^j and δ_n^j , $j = 1, 2, 3, 4$ by putting $n = 0$ in (1.51) and (1.52).

The result is true for $n = 0$. Assume that the result is true for all $n \leq k, k \geq 0$.

$\Gamma_k^3 + \Gamma_k^4 = \delta_k^3 + \delta_k^4$. To prove the result for $n = k + 1$, we need to prove that

$\Gamma_{k+1}^3 + \Gamma_{k+1}^4 = \delta_{k+1}^3 + \delta_{k+1}^4$. Again using (1.51) we can write

$\Gamma_{k+1}^3 + \Gamma_{k+1}^4 = \delta_k^3 + \delta_{k-1}^3 + \delta_k^4 + \delta_{k-1}^4$ we group them to use induction hypotheses and get

$\Gamma_{k+1}^3 + \Gamma_{k+1}^4 = \Gamma_k^3 + \Gamma_{k-1}^3 + \Gamma_k^4 + \Gamma_{k-1}^4$, we use (1.52) to obtain the result

$\Gamma_{k+1}^3 + \Gamma_{k+1}^4 = \delta_{k+1}^3 + \delta_{k+1}^4$. Hence the result is true for all $n, n \geq 0$.

We need the next three theorems in order to find a formula for x_n and y_n in terms of the Fibonacci numbers. Identities proved above will be used to prove some of the results in these theorems.

Theorem 9: If $n \geq 0$ then

(a) $\delta_n^1 = \Gamma_n^2$	(e) $\Gamma_n^3 = \Gamma_{n+1}^2$
(b) $\delta_n^2 = \Gamma_n^1$	(f) $\Gamma_n^4 = \Gamma_{n+1}^1$
(c) $\delta_n^3 = \Gamma_n^4$	(g) $\delta_n^3 = \delta_{n+1}^2$
(d) $\delta_n^4 = \Gamma_n^3$	(h) $\delta_n^4 = \delta_{n+1}^1$

We prove (b) by induction. (a) – (h) can be verified in the same manner.

When $n = 0$ $\delta_0^2 = 0$ and $\Gamma_0^1 = 0$ which is obtained by putting $n = 0$ in (1.51) and (1.52).

Thus the result is true for $n = 0$. Assume it to be true for all $n \leq k, k \geq 0$. $\delta_k^2 = \Gamma_k^1$.

We need to prove $\delta_{k+1}^2 = \Gamma_{k+1}^1$. Use (1.51) and (1.52) to get,

$\delta_{k+1}^2 = \delta_{k-1}^2 + \delta_{k-2}^2 + \delta_{k-2}^2 + \delta_{k-3}^2$, by induction hypotheses we can write

$\delta_{k+1}^2 = \Gamma_{k-1}^1 + \Gamma_{k-2}^1 + \Gamma_{k-2}^1 + \Gamma_{k-3}^1$, we again use (1.52) and (1.51) to arrive at the result

$\delta_{k+1}^2 = \delta_k^1 + \delta_{k-1}^1 = \Gamma_{k+1}^1$. Therefore the result is true for all $n, n \geq 0$.

Combining parts (a) – (d) of theorem 6 and theorem 9 gives the next theorem.

Theorem 10: For all $n \geq 0$

$$(a) \delta_n^1 + \delta_n^2 = \Gamma_n^2 + \Gamma_n^1 = F_{n+1}$$

$$(b) \delta_n^3 + \delta_n^4 = \Gamma_n^4 + \Gamma_n^3 = F_n$$

Theorem 11: If $n \geq 2$, then

$$(a) \Gamma_n^1 = \Gamma_{n-1}^1 + \Gamma_{n-2}^1 + 3\left[\frac{n}{3}\right] - n + 1$$

$$(b) \Gamma_n^2 = \Gamma_{n-1}^2 + \Gamma_{n-2}^2 - 3\left[\frac{n}{3}\right] + n - 1$$

$$(c) \Gamma_n^1 = \Gamma_n^2 + 3\left[\frac{n}{3}\right] - n + 1$$

$$(d) \Gamma_n^3 = \Gamma_{n-1}^3 + \Gamma_{n-2}^3 - 3\left[\frac{n+1}{3}\right] + n$$

$$(e) \Gamma_n^4 = \Gamma_{n-1}^4 + \Gamma_{n-2}^4 + 3\left[\frac{n+1}{3}\right] - n$$

$$(f) \Gamma_n^3 = \Gamma_n^4 - 3\left[\frac{n+1}{3}\right] + n$$

We prove part (b). Parts (a) – (f) can be proved similarly. When $n = 2$ then $\Gamma_2^2 = 1$.

R.H.S of (b) becomes $\Gamma_1^2 + \Gamma_0^2 - 3\left[\frac{2}{3}\right] + 2 - 1 = 0 + 0 + 2 - 0 - 1 = 1$. Thus the result is

true for $n = 2$. Assume the result to be true for all $n \leq k, k \geq 2$,

$\Gamma_k^2 = \Gamma_{k-1}^2 + \Gamma_{k-2}^2 - 3\left[\frac{k}{3}\right] + k - 1$. We need to prove the result for $n = k + 1$, we have to

prove that $\Gamma_{k+1}^2 = \Gamma_k^2 + \Gamma_{k-1}^2 - 3\left[\frac{k+1}{3}\right] + (k+1) - 1$. We use (1.51) and (1.52) to write

$\Gamma_{k+1}^2 = \delta_k^2 + \delta_{k-1}^2 = (\Gamma_{k-1}^2 + \Gamma_{k-2}^2) + (\Gamma_{k-2}^2 + \Gamma_{k-3}^2)$. We replace the terms in parentheses

by induction hypotheses to get

$\Gamma_{k+1}^2 = \Gamma_k^2 + 3\left[\frac{k}{3}\right] - k + 1 + \Gamma_{k-1}^2 + 3\left[\frac{k-1}{3}\right] - (k-1) + 1$. Adding and subtracting the term

$3\left[\frac{k+1}{3}\right]$ in R.H.S of the above equation we obtain

$\Gamma_{k+1}^2 = \Gamma_k^2 + \Gamma_{k-1}^2 - 2k + 3\left\{\left[\frac{k}{3}\right] + \left[\frac{k-1}{3}\right] + \left[\frac{k+1}{3}\right]\right\} - 3\left[\frac{k+1}{3}\right] + 3$. We use the identity

$\left[\frac{n+1}{3}\right] + \left[\frac{n}{3}\right] + \left[\frac{n-1}{3}\right] = (n-1)$ to replace the terms in the parentheses and get

$\Gamma_{k+1}^2 = \Gamma_k^2 + \Gamma_{k-1}^2 - 2k - 3\left[\frac{k+1}{3}\right] + 3(k-1) + 3 = \Gamma_k^2 + \Gamma_{k-1}^2 - 3\left[\frac{k+1}{3}\right] + (k+1) - 1$.

Hence the result is true for all $n, n \geq 2$.

Adding part (a) of Theorem 10 and Theorem 11 gives an expression for Γ_{n+2}^1 in terms of the Fibonacci numbers. We can write part (a) of Theorem 10 as

$\Gamma_{n+2}^1 + \Gamma_{n+2}^2 = F_{n+1}$, and part (a) of Theorem 11 can be written as

$\Gamma_{n+2}^1 = \Gamma_{n+1}^1 + \Gamma_n^1 + 3\left[\frac{n+2}{3}\right] - n - 1$. Add the two to get

$2\Gamma_{n+2}^1 = F_{n+1} - \Gamma_{n+2}^2 + \Gamma_{n+1}^1 + \Gamma_n^1 + 3\left[\frac{n+2}{3}\right] - n - 1$, since $\Gamma_{n+1}^1 + \Gamma_n^1 = \delta_{n+2}^1$ we get

$\Gamma_{n+2}^1 = \frac{1}{2}\left(F_{n+1} - \Gamma_{n+2}^2 + \delta_{n+2}^1 + 3\left[\frac{n+2}{3}\right] - n - 1\right)$, part (a) of Theorem 9 is $\delta_n^1 = \Gamma_n^2$, we use

the relation to cancel the terms and get the result

$\Gamma_{n+2}^1 = \frac{1}{2}\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right) = \delta_{n+2}^2$. We write the last equality using part (b) of

theorem 9. Similarly we obtain the following results:

$$\Gamma_{n+2}^2 = \frac{1}{2}\left(F_{n+1} - 3\left[\frac{n+2}{3}\right] + n + 1\right) = \delta_{n+2}^1.$$

$$\Gamma_{n+2}^3 = \frac{1}{2}\left(F_{n+2} - 3\left[\frac{n}{3}\right] + n - 1\right) = \delta_{n+2}^4.$$

$$\Gamma_{n+2}^4 = \frac{1}{2}\left(F_{n+2} + 3\left[\frac{n}{3}\right] - n + 1\right) = \delta_{n+2}^3.$$
 Substituting this expression for Γ_{n+2}^j and

$\delta_{n+2}^j, j = 1, 2, 3, 4$ in (1.49) and (1.50) we get the explicit formula for x_{n+2} and y_{n+2}

in terms of the Fibonacci numbers:

$$x_{n+2} = \frac{1}{2}\left\{\left(F_{n+1} + 3\left[\frac{n+2}{3}\right] - n - 1\right)a + \left(F_{n+1} - 3\left[\frac{n+2}{3}\right] + n + 1\right)b + \right. \\ \left. \left(F_{n+2} - 3\left[\frac{n}{3}\right] + n - 1\right)c + \left(F_{n+2} + 3\left[\frac{n}{3}\right] - n + 1\right)d\right\};$$

$$\begin{aligned}
y_{n+2} &= \frac{1}{2} \left\{ \left(F_{n+1} - 3 \left\lfloor \frac{n+2}{3} \right\rfloor + n + 1 \right) a + \left(F_{n+1} + 3 \left\lfloor \frac{n+2}{3} \right\rfloor - n - 1 \right) b + \right. \\
&\quad \left. \left(F_{n+2} + 3 \left\lfloor \frac{n}{3} \right\rfloor - n + 1 \right) c + \left(F_{n+2} - 3 \left\lfloor \frac{n}{3} \right\rfloor + n - 1 \right) d \right\}; \\
\text{or, } x_{n+2} &= \left\{ F_{n+1}(a+b) + F_{n+2}(c+d) + \left(3 \left\lfloor \frac{n+2}{3} \right\rfloor - n - 1 \right) (a-b) + \left(n - 3 \left\lfloor \frac{n}{3} \right\rfloor - 2 \right) (c-d) \right\} \\
y_{n+2} &= \left\{ F_{n+1}(a+b) + F_{n+2}(c+d) + \left(3 \left\lfloor \frac{n+2}{3} \right\rfloor - n - 1 \right) (b-a) + \left(n - 3 \left\lfloor \frac{n}{3} \right\rfloor - 1 \right) (d-c) \right\}.
\end{aligned}$$

x_n and y_n are one of the examples of many sequences that can be constructed to generalize the Fibonacci sequence. Atanassov [4] hints at two more pairs that could be constructed and the same procedure followed to generalize the Fibonacci sequence.

CHAPTER II

EXPANSIONS OF ANALYTIC FUNCTIONS

Some Functions Associated With w_n

The relation (1.6) of chapter I is given by,

$u_n(p, q) = pu_{n-1} - qu_{n-2}, u_0 = 1, u_1 = p$. We replace n by $n+1$ and write the above relation

as, $u_{n+1} = pu_n - qu_{n-1}, n \geq 2, u_0 = 1, u_1 = p$ Then we have

$$u_0 = 1, u_1 = p, u_2 = p^2 - q, u_3 = p^3 - 2pq, u_4 = p^4 - 3p^2q + q^2, u_5 = p^5 - 4p^3q + 3pq^2, \dots$$

and so on. One can show that the above terms could be written as

$$(2.1) \quad u_n(p, q) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} p^{n-2k} q^k$$

Similarly from (1.7) we have $\{w_n(2, p; p, q)\} = \{v_n(p, q)\}$ which satisfies the recurrence

relation $v_{n+2} = pv_{n+1} - qv_n, n \geq 0, v_0 = 2, v_1 = p$. The first few terms of v_n are:

$$v_0 = 2, v_1 = p, v_2 = p^2 - 2q, v_3 = p^3 - 3pq, v_4 = p^4 - 4p^2q + 2q^2 \text{ and so on. The general}$$

term for v_n may be written in the form

$$(2.2) \quad v_n(p, q) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} p^{n-2k} q^k$$

The above equation could be easily verified. It is well-known that $u_n(1, -1) = F_n$,

substituting $p = 1, q = -1$ in the equation (2.1) gives $F_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (-1)^k$ or,

$$(2.3) \quad F_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k}$$

Notice that $v_n(1, -1) = L_n$, putting $p = 1, q = -1$ in (2.2) gives,

$$(2.4) \quad L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} \frac{n}{n-k}$$

We have the relation (1.25), $w_n = au_n + (b - pa)u_{n-1}$. Using (2.1) w_n can be expressed in the form,

$$(2.5) \quad w_n(a, b; p, q) = a \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} p^{n-2k} q^k + (b - pa) \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-1-k}{k} p^{n-1-2k} q^k$$

Chebyshev polynomials of the first kind are defined by

$$(2.6) \quad T_n(x) = \cos n\theta, x = \cos \theta, n \geq 0, 0 \leq \theta \leq \pi. T_n(x) \text{ is defined on the interval}$$

$-1 \leq x \leq 1$. $T_n(x)$ is called the Chebyshev polynomial of the first kind, and

$$(2.7) \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin \theta}, x = \cos \theta, n \geq 0. U_n(x) \text{ is called the Chebyshev polynomial}$$

of the second kind. With $T_0(x) = U_0(x) = 1$, it can be shown that $T_n(x)$ and

$U_n(x)$ satisfy the recurrence relation

$$(2.8) \quad \begin{aligned} T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x) \end{aligned}$$

Write $x = \cos \theta$ and define $w_n = w_n(a, 2x; 2x, 1)$ as follows:

$$(2.9) \quad w_n = w_n(a, 2x; 2x, 1) = a \cos n\theta + (2 - a) \sin n\theta \cot \theta.$$

We show that this definition of w_n is consistent with the earlier definition of w_n which was defined by the recurrence relation $w_{n+2} = pw_{n+1} - qw_n$. We justify (2.9) by showing that (2.9) indeed satisfies the recurrence relation for w_n with $q = 1$. For this, we use the identities

$$(2.10) \quad \begin{aligned} \sin(n+2)\theta &= 2\cos\theta \sin(n+1)\theta - \sin n\theta \\ \cos(n+2)\theta &= 2\cos\theta \cos(n+1)\theta - \cos n\theta \end{aligned}$$

The above pair of identities could be proved by using the recurrence relations (2.8) and the definitions (2.6) and (2.7). By (2.9) we write w_{n+2} as,

$$w_{n+2} = a\cos(n+2)\theta + (2-a)\sin(n+2)\theta \cot\theta. \text{ Substituting from (2.10) we get}$$

$$w_{n+2} = a(2\cos\theta \cos(n+1)\theta - \cos n\theta) + (2-a)\cot\theta(2\cos\theta \sin(n+1)\theta - \sin n\theta).$$

Substitute $x = \cos\theta$, $p = 2x$ and collect the terms together to get,

$$w_{n+2} = 2x\{a\cos(n+1)\theta + (2-a)\sin(n+1)\theta \cot\theta\} - \{a\cos n\theta + (2-a)\sin n\theta \cot\theta\}$$

The terms in parentheses are w_{n+1} and w_n respectively by the definition in (2.9). Since $q = 1$, we have proved that $w_{n+2} = pw_{n+1} - w_n$.

Then, substituting $a = 1$ in (2.9) gives,

$$w_n = w_n(1, 2x; 2x, 1) = \cos n\theta + \sin n\theta \cot\theta = \cos n\theta + \frac{\sin n\theta \cos\theta}{\sin\theta}$$

$$w_n = w_n(1, 2x; 2x, 1) = \frac{\sin n\theta \cos\theta + \cos n\theta \sin\theta}{\sin\theta}. \text{ We get,}$$

$$w_n = w_n(1, 2x; 2x, 1) = \frac{\sin(n+1)\theta}{\sin\theta}, \text{ which is } U_n(x), \text{ the Chebyshev polynomial of second}$$

kind. Since $u_n(p, q) = w_n(a, b; p, q)$, we get $u_n(2x, 1) = w_n(a, b; 2x, 1) = U_n(x)$.

Using (2.1) we then have

$$(2.11) \quad u_n(2x,1) = U_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \quad \text{which is a known series}$$

representation of $U_n(x)$. Similarly putting $a = 2$ in (2.9) we obtain

$w_n(2,2x;2x,1) = 2 \cos n\theta$, which gives $v_n(2x,1) = 2 \cos n\theta = 2T_n(x)$. So, from (2.2) we get the series representation of $T_n(x)$.

$$(2.12) \quad v_n(2x,1) = 2T_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}$$

Thus w_n generalizes the Chebyshev polynomials. Since w_n is a generalization of the Fibonacci sequence, we pause here to note the relation between the Fibonacci and the Lucas numbers and the Chebyshev polynomials which is given by,

$$(2.12) \quad 2i^{-n} T_n\left(\frac{i}{2}\right) = L_n \quad \text{and}$$

$$(2.13) \quad i^{-n} U_n\left(\frac{i}{2}\right) = F_{n+1} . \quad (2.13) \quad \text{can be obtained by considering the recurrence}$$

relation $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), n \geq 0$ with initial values $U_0(x) = 1, U_1(x) = 2x$.

Replacing x by $\frac{i}{2}$, we get, $U_0\left(\frac{i}{2}\right) = 1, i^{-1}U_1\left(\frac{i}{2}\right) = i^{-1}2\frac{i}{2} = 1$ which agree with the initial

values of the Fibonacci sequence. Replace x by $\frac{i}{2}$ in the recurrence relation for $U_n(x)$

and use the relation (2.13) to deduce that

$$i^{-n-1}U_{n+1}\left(\frac{i}{2}\right)=i^{-n}U_n\left(\frac{i}{2}\right)+i^{-n+1}U_{n-1}\left(\frac{i}{2}\right) \text{ with initial values } U_0\left(\frac{i}{2}\right)=1, i^{-1}U_1\left(\frac{i}{2}\right)=1 \text{ is}$$

same as the recurrence relation for the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$ with initial values $F_1 = 1, F_2 = 1$. By (1.6) we can write recurrence relations for both

$$u_n(2x,1)=U_n(x) \text{ and } v_n(2x,1)=2T_n(x)$$

$$(2.14) \quad u_n(2x,1)=U_n(x)=2xU_{n-1}(x)-U_{n-2}(x) \quad \text{and}$$

$$(2.15) \quad v_n(2x,1)=2T_n(x)=2(xU_{n-1}(x)-U_{n-2}(x))$$

We obtain some special cases using (2.14) and (2.15) ,

$$(2.16) \quad w_n(1,1;1,1)=u_n(1,1)=U_n\left(\frac{1}{2}\right)=U_{n-1}\left(\frac{1}{2}\right)-U_{n-2}\left(\frac{1}{2}\right)$$

$$(2.17) \quad w_n(2,1;1,1)=v_n(1,1)=2T_n\left(\frac{1}{2}\right)=U_{n-1}\left(\frac{1}{2}\right)-U_{n-2}\left(\frac{1}{2}\right)$$

$$\text{Generally, } w_n(a,b;2x,1)=bU_{n-1}(x)-aU_{n-2}(x)$$

Since $x = \cos \theta, p = 2x, q = 1$ from (2.9), we have $d = \sqrt{p^2 - 4q}$. Substituting for p, q, x , we get

$$(2.18) \quad d = \sqrt{4(-\sin^2 \theta)} = 2i \sin \theta.$$

We also have $e = pab - qa^2 - b^2$, with $x = \cos \theta, p = b = 2x, q = 1$, we get

$$(2.19) \quad e = 4(a-1)\cos^2 \theta - a^2.$$

Since $w_n(1, p; p, q) = u_n(p, q)$ and $w_n(2, p; p, q) = v_n(p, q)$. We substitute $a = 1$ and

$a = 2$ in (2.19) to get e for u_n and v_n respectively. Thus we get

$$(2.20) \quad e = -1 \text{ for } u_n, \text{ and}$$

$$(2.21) \quad e = 4(2-1)\cos^2 \theta - 4 = 4\cos^2 \theta - 4 = -4\sin^2 \theta \quad \text{for } v_n.$$

We make use of some of the identities proved in section 1 and section 2 of chapter I to obtain identities involving trigonometric functions. We write the formula given below, which was proved by Horadam [6], using the definition (2.9) of w_n

$$(2.22) \quad w_{n+r}w_{n-r} - w_n^2 = eq^{n-r}u_{r-1}^2$$

Substitute $q = 1$ in the above equation (2.22) we get

$$(a \cos(n+r)\theta + (2-a)\sin(n+r)\theta \cot \theta)(a \cos(n-r)\theta + (2-a)\sin(n-r)\theta \cot \theta) - (a \cos n\theta + (2-a)\sin n\theta \cot \theta)^2 = e \frac{\sin^2 r\theta}{\sin \theta}$$

Rearranging the terms, we can write the above equation as

$$a^2 \{ \cos(n+r)\theta \cos(n-r)\theta - \cos^2 \theta \} + (2-a)^2 \cot^2 \theta \{ \sin(n+r)\theta \sin(n-r)\theta - \sin^2 n\theta \} = e \frac{\sin^2 r\theta}{\sin^2 \theta},$$

where e is given by (2.20) and (2.21) for u_n and v_n respectively.

From (2.7) $u_{n+r}(x) = \frac{\sin(n+r+1)\theta}{\sin \theta}$. Hence, (2.22) becomes

$$u_{n+r}(x)u_{n-r}(x) - u_n^2(x) = -lu_{r-1}^2(x), \text{ which is}$$

$$\frac{\sin(n+r+1)\theta}{\sin \theta} \frac{\sin(n-r+1)\theta}{\sin \theta} - \frac{\sin^2(n+1)\theta}{\sin^2 \theta} = -\frac{\sin^2 r\theta}{\sin^2 \theta}. \text{ Cancel the common}$$

denominator from both sides to get

$$(2.23) \quad \{ \sin(n+r+1)\theta \sin(n-r+1)\theta - \sin^2(n+1)\theta \} = -\sin^2 r\theta$$

Similarly, $v_n(2x,1) = 2T_n(x) = 2\cos n\theta$ implies $v_{n+r}v_{n-r} - v_n^2 = -4\sin^2 \theta u_{r-1}^2$, or

$$(2.24) \quad \{ \cos(n+r)\theta \cos(n-r)\theta - \cos^2 n\theta \} = -\sin^2 r\theta$$

The result $w_n^2 + eu_{n-1}^2 = aw_{2n}$ gives the well-known trigonometric identity with

$v_n = 2T_n(x) = 2\cos n\theta$ which is as follows:

$$(2.25) \quad \cos^2 n\theta - \sin^2 n\theta = \cos 2n\theta$$

We continue with our examples of other trigonometrical identities hidden in the identities

w_n . The identity (1.29) $pw_{n+2} - (p^2 - q)w_{n+1} + q^2w_{n-1} = 0$ leads to two other

trigonometric identities. $w_n = u_n = \frac{\sin(n+1)\theta}{\sin \theta}$ implies

$$2x \frac{\sin(n+3)\theta}{\sin \theta} - (4x^2 - 1) \frac{\sin(n+2)\theta}{\sin \theta} + \frac{\sin n\theta}{\sin \theta} = 0. \text{ Since } x = \cos \theta \text{ from (2.9), we have}$$

$$(2.26) \quad 2\cos \theta \sin(n+3)\theta - (4\cos^2 \theta - 1)\sin(n+2)\theta + \sin n\theta = 0$$

and $w_n(x) = v_n(x) = 2T_n(x) = 2\cos n\theta$

$$2x\cos(n+2)\theta - (4x^2 - 1)\cos(n+1)\theta + \cos(n-1)\theta = 0. \quad x = \cos \theta \text{ implies}$$

$$(2.27) \quad 2\cos \theta \cos(n+2)\theta - (4\cos^2 \theta - 1)\cos(n+1)\theta + \cos(n-1)\theta = 0$$

Another identity (1.54) which was proved in chapter I $\frac{w_{n+r} + q^r w_{n-r}}{w_n} = v_r$, with

$$w_n = u_n(x) = \frac{\sin(n+1)\theta}{\sin \theta} \text{ gives}$$

$$(2.28) \quad \frac{\sin(n+r+1)\theta + \sin(n-r+1)\theta}{\sin(n+1)\theta} = 2\cos r\theta$$

$w_n = v_n(x) = 2\cos n\theta$ implies

$$(2.29) \quad \frac{\cos(n+r)\theta + \cos(n-r)\theta}{\cos n\theta} = 2\cos r\theta$$

Putting $p = 2x, q = 1, x = \cos \theta$ in the summation formula (1.30) proved in chapter I

gives an expression for $\sum_{j=0}^{n-1} \cos j\theta$ and $\sum_{j=0}^{n-1} \sin(j+1)\theta$ when w_j is first replaced by

$u_j(2x,1) = \frac{\sin(j+1)\theta}{\sin \theta}$ and then by $v_j(2x,1) = 2 \cos j\theta$. The formula (1.30) is

$q \sum_{j=0}^{n-1} w_j = (p-1)(w_2 + w_3 + w_4 + \dots + w_n) - w_{n+1} + pw_1$. Replacing w_j by u_j we get

$\sum_{j=0}^{n-1} u_j = (2x-1)(u_2 + u_3 + u_4 + \dots + u_n) - u_{n+1} + 2xu_1$. Therefore,

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{\sin(j+1)\theta}{\sin \theta} &= \frac{1}{\sin \theta} \sum_{j=0}^{n-1} \sin(j+1)\theta \\ &= \frac{1}{\sin \theta} \left\{ (2 \cos \theta - 1)(\sin 3\theta + \sin 4\theta + \dots + \sin(n+1)\theta) - \sin(n+2)\theta + 2 \cos \theta \sin \theta \right\} \end{aligned}$$

Cancel the term $\frac{1}{\sin \theta}$ from both the sides and use the identity

$2 \cos a \sin b = \sin(a+b) - \sin(a-b)$ to write the the above equation as

$$(2.30) \quad \sum_{j=0}^{n-1} \sin(j+1)\theta = \left\{ (2 \cos \theta - 1)(\sin 3\theta + \sin 4\theta + \dots + \sin(n+1)\theta) - \sin(n+2)\theta \right\} + \sin 3\theta + \sin \theta$$

Similarly, we obtain

$$(2.31) \quad \sum_{j=0}^{n-1} \cos j\theta = \left\{ (2 \cos \theta - 1)(\cos \theta + \cos 3\theta + \dots + \cos n\theta) - \cos(n+1)\theta \right\} + 2 \cos^2 \theta$$

We continue with the idea of finding other functions related to the Fibonacci-type sequences. In the next section, we define the Fibonacci polynomials and consider expansion of certain functions in a series of these polynomials.

1. Fibonacci Polynomials

Byrd [7] considered the expansion of analytic functions as a series in the Fibonacci polynomials. Bessel functions are involved in finding a formula for its coefficients.

By definition, a real analytic function is a function which is infinitely differentiable at each point of its domain. We later consider functions that are analytic everywhere. We shall expand these functions as series in the Fibonacci polynomials. We start out by defining the Fibonacci polynomials.

Fibonacci polynomials: The sequence of polynomials $\phi_k(x), k = 0, 1, 2, 3, \dots$ which satisfies the recurrence relation

$$(2.32) \quad \phi_{k+2}(x) - 2x\phi_{k+1}(x) - \phi_k(x) = 0, -\infty < x < \infty \text{ with initial conditions}$$

$$\phi_0(x) = 0, \phi_1(x) = 1 \text{ are called the Fibonacci Polynomials.}$$

When $x = \frac{1}{2}$, (2.32) becomes, $\phi_{k+2}(x) - \phi_{k+1}(x) - \phi_k(x) = 0, \phi_0(x) = 0, \phi_1(x) = 1$ which is the relation of the Fibonacci numbers. Therefore,

$$(2.33) \quad \phi_k\left(\frac{1}{2}\right) = F_k$$

We introduce the generating function, which is used to find an explicit formula for $\phi_{k+1}(x)$.

The generating function concept: Consider a function $F(x, s)$ which has a power series

expansion in t : $F(x, s) = \sum_{n=0}^{\infty} f_n(x)s^n$. The coefficient of s^n is, in general a function of

x . We say that the expansion of $F(x, s)$ has generated the set $f_n(x)$ and that $F(x, s)$ is a generating function for the set $f_n(x)$.

Leonard [1] demonstrates the method of H.W. Gould to find the generating function.

Here, we demonstrate a direct method for finding a generating function from the recurrence relation.

Example: Consider the Fibonacci recurrence

$$(2.34) \quad F_{n+2} = F_n + F_{n+1}, \quad n \geq 0 \quad F_0 = 0, F_1 = 1$$

To find a generating function for the Fibonacci numbers, let us name the generating

function $F(s)$ and write it as $F(s) = \sum_{n=0}^{\infty} F_n s^n$

$$F(s) = (F_0 + F_1 s + F_2 s^2 + \dots) = (0 + s + F_2 s^2 + \dots)$$

Multiply (2.34) by s^n and sum over $n \geq 0$. Then,

$$\sum_{n \geq 0} F_{n+2} s^n = \sum_{n \geq 0} F_{n+1} s^n + \sum_{n \geq 0} F_n s^n$$

$(F_2 + F_3 s + F_4 s^2 + \dots) = (F_1 + F_2 s + F_3 s^2 + \dots) + F(s)$. Write each sum in terms of

$F(s)$ to get

$$\frac{(F(s) - F_0 - F_1 s)}{s^2} = \frac{(F(s) - F_0)}{s} + F(s). \text{ Since } F_0 = 0, F_1 = 1, \text{ we get}$$

$$\frac{F(s) - s}{s^2} = \frac{F(s)}{s} + F(s). \text{ Collect all the terms with } F(s) \text{ on one side so that,}$$

$$F(s) \left(\frac{1}{s^2} - \frac{1}{s} - 1 \right) = \frac{1}{s}. \text{ Therefore, } F(s) = \frac{s}{1 - s - s^2}.$$

Following the same method as in the previous example, we proceed to find a generating function defining the polynomials $\phi_k(x)$.

Let $F(x, s) = \sum_{k=0}^{\infty} \phi_k(x) s^k = (\phi_0(x) + \phi_1(x)s + \phi_2(x)s^2 + \dots)$. The recurrence relation

$$(2.32) \text{ is } \phi_{k+2}(x) - 2x\phi_{k+1}(x) - \phi_k(x) = 0 \quad , \quad \phi_0(x) = 0, \phi_1(x) = 1$$

We multiply (2.32) by s^k and sum from $k = 0$ to $k = \infty$. Therefore,

$$\sum_{k=0}^{\infty} \phi_{k+2}(x) s^k - 2x \sum_{k=0}^{\infty} \phi_{k+1}(x) s^k - \sum_{k=0}^{\infty} \phi_k(x) s^k = 0. \text{ Expanding the sum,}$$

$$(2.35) \quad (\phi_2(x) + \phi_3(x)s + \phi_4(x)s^2 + \dots) - 2x(\phi_1(x)s + \phi_2(x)s^2 + \phi_3(x)s^3 + \dots) - F(x, s) = 0$$

We shall write both the parentheses in terms of $F(x, s)$. First, the left hand side of the above relation may be written in the form

$$\begin{aligned} (\phi_2(x) + \phi_3(x)s + \phi_4(x)s^2 + \dots) &= \frac{(\phi_0 + \phi_1(x)s + \phi_2(x)s^2 + \dots) - (\phi_0 + \phi_1(x)s)}{s^2} = \frac{F(x, s) - 1}{s^2} \\ \text{and, } (\phi_1(x)s + \phi_2(x)s^2 + \phi_3(x)s^3 + \dots) &= \frac{(\phi_0 + \phi_1(x)s + \phi_2(x)s^2 + \dots) - (\phi_0(x))}{s^2} = \frac{F(x, s)}{s^2} \end{aligned}$$

Substituting the above results in the equation (2.35), we get

$$\frac{F(x, s) - 1}{s^2} - 2x \frac{F(x, s)}{s} - F(x, s) = 0. \text{ Collecting all the terms with } F(x, s) \text{ on one side}$$

and the rest of the terms on the other side, we get

$$F(x, s) \left(\frac{1}{s^2} - \frac{2x}{s} - 1 \right) = \frac{1}{s^2}. \text{ And so, } F(x, s) = \frac{s}{1 - 2xs - s^2}. \text{ Therefore,}$$

$$(2.36) \quad \frac{s}{1 - 2xs - s^2} = \sum_{k=0}^{\infty} \phi_k(x) s^k \text{ is a generating function of the polynomials } \phi_k(x).$$

Formula for $\phi_{k+1}(x)$: To obtain this formula we expand the left hand side of (2.36)

$$\frac{s}{1-2xs-s^2} = s \left(\frac{1}{1-2xs-s^2} \right) = s \left(\frac{1}{1-(2xs+s^2)} \right). \text{ Using the identity}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots + z^n + \dots = \sum_{k=0}^{\infty} z^k \text{ gives}$$

$$s \left(\frac{1}{1-(2xs+s^2)} \right) = s \sum_{n=0}^{\infty} (2xs+s)^n, \quad |2xs+s^2| \leq 1$$

Let us now expand $(2xs+s^2)^n$ using the binomial theorem $(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i$. Thus

we have

$$s \left(\frac{1}{1-(2xs+s^2)} \right) = s \sum_{n=0}^{\infty} (2xs+s)^n = s \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (2xs)^{n-k} (s^2)^k$$

Now we use the series manipulating technique $\sum_{n=0}^{\infty} \sum_{k=0}^n c(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c(k, n-k)$ to obtain

$$\frac{s}{1-2xs-s^2} = s \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (2xs)^{n-2k} s^{2k}$$

$$\frac{s}{1-2xs-s^2} = s \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (2x)^{n-2k} s^n = \sum_{n=0}^{\infty} s^{n+1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (2x)^{n-2k}. \text{ We now replace } n$$

by k and k by m in the above equation so that, by (2.36), we have

$$\sum_{k=0}^{\infty} \phi_k s^k = \sum_{k=0}^{\infty} s^{k+1} \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-m}{m} (2x)^{k-2m}. \text{ Therefore, equating the coefficients of } s^{k+1}, \text{ we get}$$

$$(2.37) \quad \phi_{k+1}(x) = \sum_{m=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-m}{m} (2x)^{k-2m}$$

We can express $\phi_k(x)$ in another form, this time using the exponential generating function defined by

$$(2.38) \quad Y(s, x) = \sum_{k=0}^{\infty} \phi_k(x) \frac{s^k}{k!}.$$

We transform the recurrence relation (2.32) into a differential equation and then solve the differential equation to obtain $Y(s, x)$. The recurrence relation (2.32) is given by

$$\phi_{k+2}(x) - 2x\phi_{k+1}(x) - \phi_k(x) = 0. \text{ Multiply the terms in the above equation by } \frac{s^k}{k!} \text{ and then}$$

take the sum from $k = 0$ to $k = \infty$. Hence we get

$$\sum_{k=0}^{\infty} \phi_{k+2}(x) \frac{s^k}{k!} - 2x \sum_{k=0}^{\infty} \phi_{k+1}(x) \frac{s^k}{k!} - \sum_{k=0}^{\infty} \phi_k(x) \frac{s^k}{k!} = 0. \text{ Now, } Y = Y(s, x) = \sum_{k=0}^{\infty} \phi_k(x) \frac{s^k}{k!}.$$

Differentiating Y with respect to s , we obtain

$$\frac{dY}{ds} = \sum_{k=1}^{\infty} \phi_k(x) \frac{k s^{k-1}}{k!} = \sum_{k=1}^{\infty} \phi_k(x) \frac{s^{k-1}}{(k-1)!}. \text{ It is clear that we can write}$$

$$\frac{dY}{ds} = \sum_{k=0}^{\infty} \phi_{k+1}(x) \frac{s^k}{k!} \text{ by shifting the index. For the second derivative, we differentiate } \frac{dY}{ds}$$

with respect to s . Thus we have

$$\frac{d^2 Y}{ds^2} = \sum_{k=2}^{\infty} \phi_k(x) \frac{(k-1)s^{k-2}}{(k-1)!} = \sum_{k=2}^{\infty} \phi_k(x) \frac{s^{k-2}}{(k-2)!}. \text{ Shifting the index yields,}$$

$$\frac{d^2 Y}{ds^2} = \sum_{k=0}^{\infty} \phi_{k+2}(x) \frac{s^k}{k!}. \text{ Therefore the recurrence relation (2.32) is transformed into the}$$

differential equation given by

$$(2.39) \quad \frac{d^2 Y}{ds^2} - 2x \frac{dY}{ds} - Y = 0 \text{ with the initial conditions } Y(0, x) = 0 \text{ and } \left. \frac{dY}{ds} \right|_{s=0} = 1$$

The differential equation (2.39) is a linear homogenous second order differential

equation. To solve it, we consider its auxiliary equation $r^2 - 2xr - 1 = 0$ whose roots are

$$\text{given by } r_1, r_2 = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2x \pm \sqrt{4(x^2 + 1)}}{2}, \text{ where } r_1 = x + \sqrt{1 + x^2} \text{ and}$$

$$r_2 = x - \sqrt{1 + x^2}. \quad r_1 - r_2 = 2\sqrt{1 + x^2}. \text{ Therefore the solution to (2.39) is given by}$$

$$Y(s, x) = c_1 e^{r_1 s} + c_2 e^{r_2 s}. \text{ To find } c_1 \text{ and } c_2 \text{ we make use of the initial conditions:}$$

$$Y(0, x) = c_1 e^0 + c_2 e^0 = c_1 + c_2. \text{ Since } Y(0, x) = 0, \text{ we have } c_1 + c_2 = 0 \text{ or } c_1 = -c_2$$

$$\text{and } \left. \frac{dY}{ds} \right|_{s=0} = c_1 r_1 + c_2 r_2, \text{ since } \left. \frac{dY}{ds} \right|_{s=0} = 1, \text{ we have } c_1 r_1 + c_2 r_2 = 1. \text{ Solving the two}$$

$$\text{relations in } c_1, c_2 \text{ yields } c_1 = \frac{1}{r_1 - r_2} = \frac{1}{2\sqrt{1 + x^2}} \text{ and } c_2 = -c_1 = -\frac{1}{2\sqrt{1 + x^2}}. \text{ Therefore,}$$

the solution $Y(s, x)$ is given by

$$(2.40) \quad Y(s, x) = \frac{1}{2\sqrt{1 + x^2}} (e^{r_1 s} - e^{r_2 s}), \text{ where } r_1 = x + \sqrt{1 + x^2} \text{ and } r_2 = x - \sqrt{1 + x^2}.$$

If we let $x = \sinh \omega$, then $\sqrt{1 + x^2} = \cosh \omega$ and therefore, $r_1 = (\sinh \omega + \cosh \omega)$ and

$r_2 = (\sinh \omega - \cosh \omega)$. We apply the inverse transformation

$$\phi_k(x) = \left. \frac{d^k Y}{ds^k} \right|_{s=0}, k = 0, 1, 2, 3, 4, \dots \text{ to obtain } \phi_0(x) = 0. \text{ We compute } \phi_2(x) \text{ as follows:}$$

$$\phi_2(x) = \left. \frac{d^2 Y}{ds^2} \right|_{s=0} \text{ which after differentiating the equation (2.40) twice can be written as}$$

$\phi_2(x) = \frac{1}{2 \cosh \omega} \left(r_1^2 e^{r_1 s} - r_2^2 e^{r_2 s} \right) \Big|_{s=0}$. We now substitute for $r_1 = (\sinh \omega + \cosh \omega)$ and

$r_2 = (\sinh \omega - \cosh \omega)$ and simplify to get

$$\phi_2(x) = \frac{1}{2 \cosh \omega} \left((\cosh 2\omega + \sinh 2\omega) e^{r_1 s} - (\cosh 2\omega - \sinh 2\omega) e^{r_2 s} \right) \Big|_{s=0} = \frac{\sinh 2\omega}{\cosh \omega}.$$

We can use the recurrence relation to find $\phi_3(x)$ which is equal to

$$\phi_3(x) = 2x\phi_2(x) + \phi_1(x) = 2x \left(\frac{\sinh \omega}{\cosh \omega} \right) + 1. \text{ We simplify this to obtain}$$

$$\phi_3(x) = \frac{\cosh(2k+1)\omega}{\cosh \omega}. \text{ Similarly we can write } \phi_k(x) \text{ alternately as}$$

$$\begin{aligned} \phi_{2k}(x) &= \frac{\sinh 2k\omega}{\cosh \omega} \\ \phi_{2k+1}(x) &= \frac{\cosh(2k+1)\omega}{\cosh \omega} \end{aligned} \quad (k = 0, 1, 2, \dots).$$

Above equations are just one more way of writing $\phi_k(x)$ using another generating function and in terms of hyperbolic functions.

3. Relation to Chebyshev Polynomials.

The polynomials $\phi_m(x)$ are related to Chebyshev polynomials $U_m(x)$ of the second kind defined by $U_m(x) = \frac{\sin(m+1)\theta}{\sin \theta}$, $x = \cos \theta$. These polynomials were previously introduced in chapter I. It is known that one of the generating functions of $U_n(x)$ is given by

$$(2.41) \quad \frac{1}{1 - 2xs + s^2} = \sum_{n=0}^{\infty} U_n(x) s^n$$

We justify the relation $\phi_{n+1}(x) = (-i)^n U_n(ix)$ between the Fibonacci polynomials and the Chebyshev polynomials using the generating functions.

Replacing s by $(-is)$ and x by ix in (2.41), we have

$$\frac{1}{1 - 2(ix)(-is) + (-is)^2} = \sum_{n=0}^{\infty} U_n(ix)(-is)^n$$

Simplifying we get,

$$\frac{1}{1 - 2xs + s^2} = \sum_{n=0}^{\infty} (-i)^n U_n(ix)s^n$$

Multiplying throughout by s , yields

$$\frac{s}{1 - 2xs + s^2} = \sum_{n=0}^{\infty} (-i)^n U_n(ix)s^{n+1}$$

But the left hand side of the above equation is equal to $\sum_{k=0}^{\infty} \phi_k(x)s^k$ from (2.36).

Therefore we get, $\sum_{n=0}^{\infty} \phi_n(x)s^n = \sum_{n=0}^{\infty} (-i)^n U_n(ix)s^{n+1}$. Equating the coefficients of s^{n+1} on

both sides of the above equation we have

$$(2.42) \quad \phi_{n+1} = (-i)^n U_n(ix)$$

A set of polynomials $\{P_n(x)\}$ $n = 0, 1, 2, \dots$ is called a simple set if $P_n(x)$ is of degree precisely n in x , so that the set contains one polynomial of each degree. Upon expanding the series expression (2.37) for $\phi_{k+1}(x)$ we see that,

$$\phi_{k+1}(x) = \binom{k}{0}(2x)^k + \binom{k-1}{1}(2x)^{k-2} + \binom{k-2}{2}(2x)^{k-4} + \dots + \binom{k - \left\lfloor \frac{k}{2} \right\rfloor}{\left\lfloor \frac{k}{2} \right\rfloor} (2x)^{k-2\left\lfloor \frac{k}{2} \right\rfloor}$$

$\phi_{k+1}(x)$ is a polynomial of degree k in x . So by the above definition, $\phi_{k+1}(x)$ is a simple set.

Theorem 53 of Rainville [8] suggests that any polynomial can be expressed linearly in terms of the elements of $P_n(x)$, where $\{P_n(x)\}$ is a simple set of polynomials.

We cite theorem 53 below as:

Theorem: If $\{P_n(x)\}$ is a simple set of polynomials and if $P(x)$ is a polynomial of degree

m , there exists constants C_k such that $P(x) = \sum_{k=0}^m C_k P_k(x)$, C_k are functions of k and of

any parameters involved in $P(x)$. Therefore, we replace $P_n(x)$ by $\{\phi_{k+1}(x)\}$ in the

statement of the theorem to get $P(x) = \sum_{k=0}^m C_k \phi_{k+1}(x)$, where $P(x)$ is any polynomial of

degree m . We therefore can have x^n in a series of the set $\{\phi_{k+1}(x)\}$. To prove this, we

make use of the Chebyshev reciprocal relations and the relation (2.42), which relates the

Fibonacci polynomials $\phi_{n+1}(x)$ to the Chebyshev polynomials $U_n(x)$. Riordan [5] gives

the Chebyshev inverse relation stated below:

If $b_n = \sum_{k=0}^m (-1)^k \binom{n-k}{k} a_{n-2k}$ $m = \left\lfloor \frac{n}{2} \right\rfloor$ then

$a_n = \sum_{k=0}^m \left[\binom{n}{k} - \binom{n}{k-1} \right] b_{n-2k}$. We use the known identity $\binom{n}{k} - \binom{n}{k-1} = \frac{n+1-2k}{n+1-k} \binom{n}{k}$ to

write the inverse relation as

$$(2.43) \quad \text{If } b_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} a_{n-2k} \quad \text{then} \quad a_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n+1-2k}{n+1-k} \binom{n}{k} b_{n-2k}$$

The sign factor may be moved from one pair to the other by substitution $A_n = (-1)^m a_n$

and $B_n = (-1)^m b_n$ where $m = \left\lfloor \frac{n}{2} \right\rfloor$. A proof of (2.43) is available in Riordan [5]. To

use (2.43), we first show the equivalence of b_n and $\phi_{n+1}(x)$.

The expansion of $U_n(x)$ as a polynomial is known to be $U_n\left(\frac{x}{2}\right) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}$.

The relation (2.42) is $\phi_{n+1}(x) = (-i)^n U_n(ix)$ which can be written as

$\phi_{n+1}(x) = (-1)^n U_n\left(\frac{2ix}{2}\right)$. The above relations yield

$$\phi_{n+1}(x) = (-i)^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} (2ix)^{n-2k} = (-i)^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \frac{i^n}{i^{2k}},$$

$$\phi_{n+1}(x) = (-i)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \frac{i^n}{(-1)^k}. \text{ Canceling the common factor } (-1)^k \text{ and}$$

combining the powers of i , we get $\phi_{n+1}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n-k}{k} (2x)^{n-2k}$ which agrees with

(2.32). Therefore, from (2.42) we can write the inverse relation as

$$(2x)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k} \left(\frac{n+1-2k}{n+1-k} \right) \phi_{n+1-2k}(x)$$

x^n can be expressed as follows:

$$(2.44) \quad x^n = \left(\frac{1}{2} \right)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n}{k} \left(\frac{n+1-2k}{n+1-k} \right) \phi_{n+1-2k}(x)$$

Thus, we have x^n in a series of the set $\{\phi_{k+1}(x)\}$.

4. Formula For the Coefficients c_k

We consider expansion of analytic functions having a power series representation and

whose radius of convergence R is at least $\frac{1}{2}$ or, $R \geq \frac{1}{2}$. So, we assume our given

function has a power series representation,

$$(2.45) \quad f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ and has a radius of convergence, } R \geq \frac{1}{2}.$$

The coefficients a_n are expressed by, $a_n = \frac{f^{(n)}(0)}{n!}$, $n \geq 0$.

We want to represent $f(x)$ as follows:

$$(2.46) \quad f(x) = \sum_{k=0}^{\infty} c_k \phi_{k+1}(x)$$

and it is the formula for this c_k is that we seek. Substituting (2.44) into (2.45) yields

$$f(x) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{2}\right)^n \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \binom{n}{r} \binom{n+1-2r}{n+1-r} \phi_{n+1-2r}(x), \quad n \geq 0$$

We use the following lemma to rewrite the above summation.

Lemma:
$$\sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} B(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B(k, n+2k)$$

The proof of this lemma could be found in Rainville [8]. We use the lemma to write

$$f(x) = \sum_{n=0}^{\infty} \sum_{r=0}^{\left[\frac{n}{2}\right]} A(r, n) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} A(r, n+2r)$$

We replace n by $n+2r$ to get

$$f(x) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r a_{n+2r} \left(\frac{1}{2}\right)^{n+2r} \binom{n+2r}{r} \binom{n+1}{n+r+1} \phi_{n+1}(x) \quad \text{Replacing } n \text{ by } k, \text{ we obtain}$$

$$(2.47) \quad f(x) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{\infty} (-1)^r a_{k+2r} \left(\frac{1}{2}\right)^{k+2r} \binom{k+2r}{r} \frac{k+1}{k+r+1} \right) \phi_{k+1}(x)$$

We compare the right hand side of (2.46) and (2.47) and replace r by j to obtain

$$c_k = \sum (-1)^j \left(\frac{1}{2}\right)^{2j+k} \binom{2j+k}{j} \binom{k+1}{k+j+1} a_{2j+k}. \quad \text{The relation may be rewritten as}$$

$$(2.48) \quad c_k = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j+k} (j+k+1)} \binom{2j+k}{j} a_{2j+k}$$

We now introduce a parameter 2μ such that $|2\mu| \leq R$, and we have

$$f(2\mu x) = \sum_{n=0}^{\infty} a_n (2\mu x)^n = \sum_{n=0}^{\infty} (2^n \mu^n a_n) x^n = \sum_{n=0}^{\infty} A_n x^n \quad \text{where } A_n = \frac{1}{n!} \frac{d^n}{dx^n} f(2\mu x) \text{ at } x=0.$$

The expansion (2.46) then becomes

$$(2.49) \quad f(2\mu x) = \sum_{k=0}^{\infty} \beta_k \phi_{k+1}(x)$$

The coefficients β_k are now functions of μ . To find β_k , we replace a_n by A_n in the

$$\text{equation (2.48) to get } \beta_k(\mu) = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2j+k} (j+k+1)} \binom{2j+k}{j} A_{2j+k}. \text{ But}$$

$A_n = 2^n \mu^n a_n$, we substitute for A_n in the above relation to get

$$(2.50) \quad \beta_k(\mu) = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \mu^{2j+k}}{(j+k+1)} \binom{2j+k}{j} a_{2j+k}$$

Since $\phi_k\left(\frac{1}{2}\right) = F_k$, we put $x = \frac{1}{2}$ in (2.49) to obtain the expansion of $f(x)$ in terms of

$$\text{Fibonacci numbers } f(2\mu x) = f(\mu) = \sum_{k=0}^{\infty} \beta_k(\mu) \phi_{k+1}\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \beta_k(\mu) F_{k+1}.$$

Thus, (2.48) and (2.49) give formulas for the coefficients c_k and β_k . We now consider

some examples. As our examples involve Bessel functions and modified Bessel

functions, we pause here to note their definition and some basic representations. To

define Bessel functions, Watson [9] considers the differential equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - n^2) y = 0 \quad \text{where } n \text{ is not necessarily an integer, } n \text{ can be any real}$$

or complex number, and defines Bessel function of order n to be a certain solution to this equation which is given by

$$(2.51) \quad J_n(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{n+2m}}{m! \Gamma(n+m+1)}$$

where $\Gamma(n)$ is the well-known Gamma function, which satisfies the relation

$$(2.52) \quad \begin{aligned} \Gamma(n+1) &= n\Gamma(n) \\ \Gamma(n+1) &= n! \end{aligned} \quad \text{if } n = 0, 1, 2, 3, \dots$$

We make use of some of the known results about Bessel functions in the examples.

Proofs of the results used, as well as many more results, could be found in Watson [9].

5.Examples

Example 1: Consider the expansion of $f(x) = e^{2\mu x}$, $0 < |\mu| < \infty$. We want to express

$$f(x) = e^{2\mu x} = \sum_{k=0}^{\infty} c_k \phi_{k+1}(x). \text{ Power series representation of } e^{2\mu x} \text{ is}$$

$$e^{2\mu x} = \sum_{n=0}^{\infty} \frac{(2\mu x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n \mu^n}{n!} x^n = \sum_{n=0}^{\infty} a_n x^n, \quad \text{where } a_n = \frac{2^n \mu^n}{n!}.$$

To find c_k , we substitute for a_n in (2.48). Therefore, we get

$$c_k = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \mu^{2j+k}}{(2j+k)!(j+k+1)} \binom{2j+k}{j}. \text{ Simplifying the factorial gives}$$

$$c_k = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+k)!(j+k+1)} \frac{(2j+k)!}{j!(j+k)!} \mu^{2j+k}. \text{ Canceling the term } (2j+k)! \text{ and}$$

$$\text{combining } (j+k)!(j+k+1) \text{ yields } c_{k+1} = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \mu^{2j+k}}{j!(j+k+1)!}.$$

Comparing the above formula for c_{k+1} with (2.51) gives

$c_{k+1} = \frac{k+1}{\mu} J_{k+1}(2\mu)$. Thus we get a formula for $e^{2\mu x}$ in terms of the Fibonacci

polynomials: $e^{2\mu x} = \sum_{k=0}^{\infty} \frac{k+1}{\mu} J_{k+1}(2\mu) \phi_{k+1}(x)$. Replacing $(k+1)$ by m gives

$$(2.53) \quad e^{2\mu x} = \frac{1}{\mu} \sum_{m=1}^{\infty} m J_m(2\mu) \phi_m(x)$$

Substituting (2.53) into the well-known results $\cosh 2\mu x = \frac{e^{2\mu x} + e^{-2\mu x}}{2}$ and

$\sinh 2\mu x = \frac{e^{2\mu x} - e^{-2\mu x}}{2}$, we obtain the two functions $\cosh 2\mu x$ and $\sinh 2\mu x$ in terms of

the Fibonacci polynomials $\phi_m(x)$. We have

$e^{2\mu x} = \frac{1}{\mu} \sum_{m=1}^{\infty} m J_m(2\mu) \phi_m(x)$ and $e^{-2\mu x} = -\frac{1}{\mu} \sum_{m=1}^{\infty} m J_m(-2\mu) \phi_m(x)$. Using the result

$J_m(-z) = (-1)^m J_m(z)$, we write $e^{-2\mu x}$ as, $e^{-2\mu x} = \frac{1}{\mu} \sum_{m=1}^{\infty} (-1)^{m+1} m J_m(2\mu) \phi_m(x)$. Therefore

$$\cosh 2\mu x = \frac{e^{2\mu x} + e^{-2\mu x}}{2} = \frac{1}{2\mu} \left[\sum_{m=1}^{\infty} m J_m(2\mu) \phi_m(x) + \sum_{m=1}^{\infty} (-1)^{m+1} m J_m(2\mu) \phi_m(x) \right]. \text{ Terms}$$

with even index cancel and odd index double to give

$$(2.54) \quad \cosh 2\mu x = \frac{1}{\mu} \left[\sum_{m=1}^{\infty} (2m-1) J_{2m-1}(2\mu) \phi_{2m-1}(x) \right]$$

Similarly, we get

$$(2.55) \quad \sinh 2\mu x = \frac{1}{\mu} \left[\sum_{m=1}^{\infty} 2m J_m(2\mu) \phi_m(x) \right]$$

Putting $x = \frac{1}{2}$ in (2.53), (2.54) and (2.55), we obtain the expansions of e^μ , $\cosh \mu$, $\sinh \mu$

in terms of Fibonacci numbers. Therefore, we have

$$(2.56) \quad e^\mu = \frac{1}{\mu} \sum_{m=1}^{\infty} m J_m(2\mu) F_m,$$

$$(2.57) \quad \cosh \mu = \frac{1}{\mu} \sum_{m=1}^{\infty} (2m-1) J_{2m-1}(2\mu) F_{2m-1},$$

$$(2.58) \quad \sinh \mu = \frac{1}{\mu} \sum_{m=1}^{\infty} (2m) J_{2m}(2\mu) F_{2m}.$$

The modified Bessel function as defined by Watson [9] is $I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m! \Gamma(m+2)}$.

Putting $\nu = 1$ in $I_\nu(z)$ we obtain

$$(2.59) \quad I_1(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+1}}{\Gamma(m+1) \Gamma(m+2)}$$

Example2: We now consider the expansion of this $I_1(z)$ in terms of $\phi_{k+1}(x)$. We start by writing,

$$I_1(2\mu x) = \sum_{m=0}^{\infty} \frac{\mu^{2m+1} x^{2m+1}}{\Gamma(m+1) \Gamma(m+2)}. \text{ Replacing } (2m+1) \text{ by } n, \text{ the expression reduces to}$$

$$(2.60) \quad I_1(2\mu x) = \sum_{n=1}^{\infty} \frac{\mu^n x^n}{\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n+3}{2}\right)}.$$

We want to express $f(x) = I_1(2\mu x) = \sum_{k=0}^{\infty} c_k \phi_{k+1}(x)$ where c_k is given by (2.48).

Since $I_1(z)$ is an odd function of z , the coefficients with even indices are zeros, that is $a_{2m} = 0$. Therefore all the c_{2k} that contain a_{2j+2k} are zeros. Hence only the coefficients with odd indices are nonzeros, which means only c_{2k+1} exists. Therefore,

$$f(x) = I_1(2\mu x) = \sum_{k=0}^{\infty} c_{2k+1} \phi_{2k+2}(x). \text{ Power series representation of } I_1(2\mu x) = \sum_{n=0}^{\infty} a_n x^n \text{ is}$$

$$\text{given by (2.60), where } a_n = \frac{\mu^n}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+3}{2}\right)}. \text{ Using the formula for } c_{2k+1} \text{ (2.48) and}$$

substituting for a_n gives

$$c_{2k+1} = (2k+2) \sum_{j=0}^{\infty} \frac{(-1)^j \mu^{2j+2k+1}}{2^{2j+2k+1} \Gamma\left(\frac{2j+2k+1}{2}\right) \Gamma\left(\frac{2j+2k}{2} + 2\right) (j+2k+2)} \binom{2j+2k+1}{j}$$

Multiply the numerator by the term $\left(\frac{2}{\mu}\right)$ and simplify. We get

$$c_{2k+1} = 2(k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\mu}{2}\right)^{2j+2k+1} \left(\frac{2}{\mu}\right) (2j+2k+1)!}{j!(j+2k+1)! \left(\frac{2j+2k}{2}\right)! \Gamma(j+k+2) (j+2k+2)}$$

Combining terms $(j+2k+1)!(j+2k+2)$ and writing $\Gamma(j+k+2) = (j+k+1)!$ we have

$$c_{2k+1} = 2(k+1) \frac{2}{\mu} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\mu}{2}\right)^{2j+2k+2} (2j+2k+1)!}{j!(j+2k+2)!(j+k)!(j+k+1)!}$$

Multiplying and dividing by the term $(2j+2k+2)$ we obtain

$$c_{2k+1} = 2(k+1) \frac{2}{\mu} \frac{1}{2} \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\mu}{2}\right)^{2j+2k+2} (2j+2(k+1))!}{j!(j+2(k+1))!(j+k)!(j+k+1)!(j+k+1)}$$

Since $(j+k)!(j+k+1) = (j+k+1)!$, we get

$$(2.61) \quad c_{2k+1} = \frac{2(k+1)}{\mu} \left(\sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\mu}{2}\right)^{2j+2k+2} (2j+2(k+1))!}{j!(j+2(k+1))![(j+k+1)!]^2} \right).$$

Formula for $J_n^2(\mu)$ as given in Watson [9] is $J_n^2(\mu) = \sum_{m=0}^{\infty} \frac{(-1)^m (2n+2m)! \left(\frac{\mu}{2}\right)^{2n+2m}}{m!(2n+m)![(n+m)!]^2}$.

Comparing the above formula for $J_n^2(\mu)$ with (2.61) yields

$$c_{2k+1} = \frac{2(k+1)}{\mu} J_{k+1}^2(\mu). \text{ Thus, we have } I_1(2\mu x) = \frac{2}{\mu} \sum_{k=0}^{\infty} (k+1) J_{k+1}^2(\mu) \phi_{2k+2}(x).$$

Replacing $(k+1)$ by m in the above expression gives the desired formula for $I_1(2\mu x)$:

$$(2.62) \quad I_1(2\mu x) = \frac{2}{\mu} \sum_{m=1}^{\infty} m J_m^2(\mu) \phi_{2m}(x)$$

Putting $x = \frac{1}{2}$ in (2.63) to get the expansion of $I_1(\mu)$ in terms of Fibonacci numbers:

$$(2.63) \quad I_1(\mu) = \frac{2}{\mu} \sum_{m=1}^{\infty} m J_m^2(\mu) F_m$$

The coefficients in the above examples involve Bessel functions, but this may not be the case in general.

The coefficients in the series expansions could also be obtained in another way, one of them by using certain known properties of orthogonal polynomials.

CHAPTER III

A MATRIX APPROACH

1. Functions Of Matrices

We are going to consider the n^{th} power of a 2×2 matrix X and establish its connection with the polynomials $u_n(p, q)$ and $v_n(p, q)$ defined in chapter II. Let X be a 2×2 matrix. We write X as

$$(3.1) \quad X = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with $\text{trace } X = p = a_{11} + a_{22}$ and, $\text{determinant } X = q = a_{11}a_{22} - a_{12}a_{21}$. Then the characteristic equation of X is given by $\lambda^2 - p\lambda + q = 0$. By the well-known Cayley-Hamilton theorem of Linear Algebra, every matrix satisfies its characteristic equation. Hence we can replace λ by X in the characteristic equation. Thus we have,

$$(3.2) \quad X^2 = pX - qI \quad \text{where } I \text{ is a } 2 \times 2 \text{ identity matrix.}$$

Multiplying (3.2) by X we get, $X^3 = pX^2 - qX$. Substituting for X^2 from (3.2) we obtain, $X^3 = p(pX - qI) - qX = (p^2 - q)X - pqI$. Continuing this process leads to $X^4 = (p^3 - 2pq)X - q(p^2 - q)I$, $X^5 = (p^4 - 3p^2q + q^2)X - q(p^3 - 2pq)I$ and so on.

We will see that the polynomial coefficients of X and I are the polynomials $u_n(p, q)$, which were defined in chapter II as satisfying the recurrence relation $u_{n+1} = pu_n - qu_{n-1}$, $n \geq 1$ with initial conditions $u_0 = 1, u_1 = p$. We replace the subscript $n+1$ by $n+2$ in the above relation. This gives,

$$(3.3) \quad u_{n+2} = pu_{n+1} - qu_n, \quad n \geq 0$$

With different initial conditions $u_0 = 0, u_1 = 1$, the recurrence relation (3.3) gives the terms $u_2 = p$, $u_3 = p^2 - q$, $u_4 = p^3 - 2pq, \dots$. So we can write $X^2 = pX - qI$ as $X^2 = u_2X - qu_1I$ and $X^3 = (p^2 - q)X - pqI = u_3X - qu_2I$. In general, we can write,

$$(3.4) \quad X^n = u_n(p, q)X - qu_{n-1}(p, q)I.$$

Barakat [11] showed that we could express X^n in terms of u_n for unrestricted values of n . For this, we first establish the relation for negative-valued subscripts of u_n .

$u_0 = 0, u_1 = 1$ and the recurrence relation (3.3) implies $u_1 = pu_0 - qu_{-1}$ which in turn implies $u_{-1} = -\frac{u_1}{q}$. Therefore, $u_{-1} = -q^{-1}u_1$. Similarly, by considering the relation

$u_0 = pu_{-1} - qu_{-2}$, we get $0 = pu_{-1} - qu_{-2}$, so $u_{-2} = \frac{p}{q}u_{-1}$ but we have, $u_{-1} = -q^{-1}u_1$.

Therefore, $u_{-2} = \frac{p}{q} \left(-\frac{1}{q} \right) = -q^{-2}p$, since $u_2 = p$, we find that $u_{-2} = -q^{-2}u_2$. In fact it is

straightforward to prove the relation (3.5) by induction

$$(3.5) \quad u_{-n} = -q^{-n}u_n$$

To find an expression for X^{-n} , $n \geq 0$. We consider the inverse of X

$$X^{-1} = \frac{1}{q} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{q} Y, \quad q \neq 0. \text{ So } qX^{-1} = Y. \text{ Then}$$

$(qX^{-1})^n = Y^n = u_n(p, q)Y - qu_{n-1}(p, q)I$. We can now substitute for Y to get

$$(qX^{-1})^n = qu_n(p, q)X^{-1} - qu_{n-1}(p, q)I. \text{ So, } q^n X^{-n} = qu_n(p, q)X^{-1} - qu_{n-1}(p, q)I.$$

Dividing the last equation throughout by $q^n (q \neq 0)$, we get

$X^{-n} = q^{-n+1}u_n(p, q)X^{-1} - q^{-n+1}u_{n-1}(p, q)I$, so

$X^{-n} = q(q^{-n}u_n(p, q))X^{-1} - (q^{-(n-1)}u_{n-1}(p, q))I$. Using (3.5) in this relation yields

$$(3.5) \quad X^{-n} = q(-u_{-n}(p, q))X^{-1} + (u_{-n+1}(p, q))I \text{ or}$$

$$X^{-n} = -qu_{-n}(p, q)X^{-1} + u_{-n+1}(p, q)I$$

Barakat [11] initially obtained summation formulas for $u_n(p, q)$ and $v_n(p, q)$ by

considering the matrix exponential e^x . We go along the same lines, but we consider the matrix $\sin e$ and $\cosine$. It is known that for every square matrix X , the $\sin e$ of X and the $\cosine$ of X are defined by the power series

$$(3.7) \quad \sin X = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n+1}}{(2n+1)!} \quad \text{and}$$

$$(3.8) \quad \cos X = \sum_{n=0}^{\infty} \frac{(-1)^n X^{2n}}{(2n)!}$$

Substituting (3.4) into (3.7) we get, $\sin X = \sum_{n=0}^{\infty} \frac{(-1)^n (u_{2n+1}X - qu_{2n}I)}{(2n+1)!}$. Thus, we have

$$(3.9) \quad \sin X = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1} \right) X - q \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n} \right) I$$

We now want to simplify (3.9). In order to accomplish this, we make use of the known

Sylvester's Interpolation Formula. Walton [10] refers to this formula which is as follows:

Let X be a $N \times N$ matrix. If $f(t)$ is a polynomial of degree $\leq N-1$ and if

$\lambda_1, \lambda_2, \dots, \lambda_N$ are the N distinct eigenvalues of X , then

$f(X) = \sum_{i=1}^N f(\lambda_i) \prod_{\substack{1 \leq j \leq N \\ i \neq j}} \left[\frac{X - \lambda_i I}{\lambda_i - \lambda_j} \right]$. Since we are dealing with a 2×2 matrix X , by the

formula, if λ_1 and λ_2 are distinct eigenvalues of X , then

$$(3.10) \quad f(X) = f(\lambda_1) \left(\frac{X - \lambda_2 I}{\lambda_1 - \lambda_2} \right) + f(\lambda_2) \left(\frac{X - \lambda_1 I}{\lambda_2 - \lambda_1} \right)$$

which can also be written as

$$f(X) = \frac{1}{\lambda_1 - \lambda_2} ((f(\lambda_1) - f(\lambda_2))X - (\lambda_1 f(\lambda_1) - \lambda_2 f(\lambda_2))I)$$

With $f(X) = \sin X$, the above equation can be written as

$$(3.11) \quad \sin X = \frac{1}{\lambda_1 - \lambda_2} ((\sin \lambda_1 - \sin \lambda_2)X - (\lambda_1 \sin \lambda_1 - \lambda_2 \sin \lambda_2)I)$$

Since the characteristic equation of X is given by $\lambda^2 - p\lambda + q = 0$ and λ_1, λ_2 satisfy this characteristic equation, we have

$$(3.12) \quad \lambda_1 = \frac{p+\partial}{2}, \lambda_2 = \frac{p-\partial}{2} \quad \text{where } \partial = \sqrt{p^2 - 4q}$$

$$\lambda_1 + \lambda_2 = p, \lambda_1 - \lambda_2 = \partial$$

Simplifying the coefficients of X in (3.11) we get

$$(3.13) \quad \sin X = \frac{1}{\lambda_1 - \lambda_2} \left(\left(2 \cos \left(\frac{\lambda_1 + \lambda_2}{2} \right) \sin \left(\frac{\lambda_1 - \lambda_2}{2} \right) \right) X - (\lambda_1 \sin \lambda_1 - \lambda_2 \sin \lambda_2) I \right)$$

Substituting (3.12) into (3.13) we get

$$\sin X = \frac{1}{\partial} \left(\left(2 \cos \frac{p}{2} \sin \frac{\partial}{2} \right) X - \left(\left(\frac{p+\partial}{2} \right) \sin \left(\frac{p+\partial}{2} \right) - \left(\frac{p-\partial}{2} \right) \sin \left(\frac{p-\partial}{2} \right) \right) I \right)$$

Use the formula $\sin(a \pm b)$ for $\sin\left(\frac{p \pm \partial}{2}\right)$ and rearrange the terms to get

$$\sin X = \frac{1}{\partial} \left(\left(2 \cos \frac{p}{2} \sin \frac{\partial}{2} \right) X - \left(\sin \frac{p}{2} \cos \frac{\partial}{2} \left(\left(\frac{p+\partial}{2} \right) - \left(\frac{p-\partial}{2} \right) \right) + \cos \frac{p}{2} \sin \frac{\partial}{2} \left(\frac{p+\partial}{2} + \frac{p-\partial}{2} \right) \right) I \right)$$

Dividing throughout by ∂ , we get

$$(3.14) \quad \sin X = \left(\left(\frac{2}{\partial} \cos \frac{p}{2} \sin \frac{\partial}{2} \right) X - \left(\frac{p}{\partial} \cos \frac{p}{2} \sin \frac{\partial}{2} + \sin \frac{p}{2} \cos \frac{\partial}{2} \right) I \right)$$

Comparing the coefficients of X and I in equations (3.9) and (3.14) we get

$$(3.15) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1} = \frac{2}{\partial} \sin \frac{\partial}{2} \cos \frac{p}{2} \quad \text{and}$$

$$(3.16) \quad q \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n} = \frac{p}{\partial} \sin \frac{\partial}{2} \cos \frac{p}{2} + \sin \frac{p}{2} \cos \frac{\partial}{2}.$$

We use the following relations to find the sum of series involving $v_n(p, q)$ which were also defined in chapter II.

$$(3.17) \quad v_{2n+1} = pu_{2n+1} - 2qu_{2n} \quad \text{and}$$

$$(3.18) \quad v_{2n} = 2u_{2n+2} - pu_{2n+1}.$$

Both (3.17) and (3.18) can be proved using induction, of which, we will prove (3.17) as follows:

Putting $n = 0$ in (3.17), we get $v_1 = p$ and the right hand side $pu_1 - 2qu_0 = p$, since

$u_1 = 1, u_0 = 0$. Therefore the result is true for $n = 0$. Assume the result to be true for all

$n \leq k, k \geq 0$, $v_{2k+1} = pu_{2k+1} - 2qu_{2k}$. We need to prove that $v_{2k+3} = pu_{2k+3} - 2qu_{2k+2}$.

Now $v_{2k+3} = pv_{2k+2} - qv_{2k+1}$ by the recurrence relation. Using the relation again gives

$$v_{2k+3} = p(pv_{2k+1} - qv_{2k}) - qv_{2k+1} = (p^2 - q)v_{2k+1} - pqv_{2k}.$$

Using the induction hypotheses to replace v_{2k+1} and v_{2k} yields $v_{2k+3} = (p^2 - q)(pu_{2k+1} - 2qu_{2k}) - pq(pu_{2k} - 2qu_{2k-1})$. So

$$v_{2k+3} = p^3u_{2k+1} - 2p^2qu_{2k} - pqv_{2k+1} + 2q^2u_{2k} - p^2qu_{2k} + 2pq^2u_{2k-1}.$$

Rearranging the terms, we obtain $v_{2k+3} = p^2(pu_{2k+1} - qu_{2k}) - 2pq(pu_{2k} - qu_{2k-1}) - pqv_{2k+1} + 2q^2u_{2k}$.

Using the recurrence relation for u_n , we write the terms in parentheses as follows:

$$v_{2k+3} = p^2u_{2k+2} - 2pqu_{2k+1} - pqv_{2k+1} + 2q^2u_{2k}.$$

We again group the terms together and use the recurrence relation to get the required result:

$$v_{2k+3} = p(pu_{2k+2} - qu_{2k+1}) - 2q(pu_{2k+1} - qu_{2k}) \text{ and thus, } v_{2k+3} = pu_{2k+3} - 2qu_{2k+2}.$$

Using (3.17) we arrive at
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} v_{2n+1} = p \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1} - 2q \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n}.$$

Employing (3.15), (3.16) and substituting in the right hand side we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} v_{2n+1} = p \left(\frac{2}{\partial} \sin \frac{\partial}{2} \cos \frac{p}{2} \right) - 2 \left(\frac{p}{\partial} \sin \frac{\partial}{2} \cos \frac{p}{2} + \sin \frac{p}{2} \cos \frac{\partial}{2} \right).$$

Simplifying, we get

$$(3.19) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} v_{2n+1} = -2 \sin \frac{p}{2} \cos \frac{\partial}{2}.$$

Let us write (3.18) in the form
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} v_{2n+1} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+2} - p \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1}.$$

Rearranging the terms we get
$$2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} v_{2n+1} + p \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1}.$$

Substituting the results of (3.15) and (3.19) in the right hand side of the above relation yields

$$(3.20) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+2} = \frac{p}{\partial} \sin \frac{\partial}{2} \cos \frac{p}{2} - \sin \frac{p}{2} \cos \frac{\partial}{2}.$$

Now, we consider the $\cos$ function of X . Substituting (3.4) into (3.8) we get

$$(3.21) \quad \cos X = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (u_{2n} X - q u_{2n-1} I).$$

Replacing $\sin X$ by $\cos X$ in (3.11) gives

$$(3.22) \quad \cos X = \frac{1}{\lambda_1 - \lambda_2} ((\cos \lambda_1 - \cos \lambda_2) X - (\lambda_1 \cos \lambda_1 - \lambda_2 \cos \lambda_2) I).$$

In a similar fashion as we did in deriving formula (3.13) for $\sin X$, we simplify (3.22)

$$\cos X = \frac{1}{\partial} \left(\left(-2 \sin \left(\frac{\lambda_1 + \lambda_2}{2} \right) \sin \left(\frac{\lambda_1 - \lambda_2}{2} \right) \right) X - \left(\left(\frac{p + \partial}{2} \right) \cos \left(\frac{p + \partial}{2} \right) - \left(\frac{p - \partial}{2} \right) \cos \left(\frac{p - \partial}{2} \right) \right) I \right)$$

Simplifying the terms in the parentheses we get

$$\cos X = \frac{1}{\partial} \left(\left(-2 \sin \frac{p}{2} \sin \frac{\partial}{2} \right) X - \left(\partial \cos \frac{p}{2} \cos \frac{\partial}{2} - p \sin \frac{p}{2} \sin \frac{\partial}{2} \right) I \right)$$

Dividing throughout by ∂ , we have

$$(3.23) \quad \cos X = \left(\left(-\frac{2}{\partial} \sin \frac{p}{2} \sin \frac{\partial}{2} \right) X - \left(\cos \frac{p}{2} \cos \frac{\partial}{2} - \frac{p}{\partial} \sin \frac{p}{2} \sin \frac{\partial}{2} \right) I \right)$$

Comparing the coefficients of X and I in (3.21) and (3.23) we get

$$(3.24) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} u_{2n} = \frac{2}{\partial} \sin \frac{p}{2} \sin \frac{\partial}{2} \quad \text{and}$$

$$(3.25) \quad q \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} u_{2n-1} = \frac{p}{\partial} \sin \frac{p}{2} \sin \frac{\partial}{2} - \cos \frac{p}{2} \cos \frac{\partial}{2}.$$

We can write the relation (3.17) as

$$(3.26) \quad v_{2n} = pu_{2n} - 2qu_{2n-1}.$$

As we did earlier, (3.24), (3.25) and (3.26) imply

$$(3.27) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} v_{2n} = 2 \cos \frac{p}{2} \cos \frac{\partial}{2}.$$

Let us rearrange the relation (3.18) as

$$(3.28) \quad 2u_{2n+1} = v_{2n} + pu_{2n}.$$

We write (3.28) in the form,
$$2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} u_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} v_{2n} + p \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} u_{2n}.$$

Substituting (3.24) and (3.27) in the above equation we obtain

$$(3.29) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} u_{2n+1} = \cos \frac{p}{2} \cos \frac{\partial}{2} + \frac{p}{\partial} \sin \frac{p}{2} \sin \frac{\partial}{2}.$$

2. Summation Formulas

We use the relations between $u_n(p, q)$, $v_n(p, q)$ and $U_n(x)$ and $T_n(x)$ the Chebyshev polynomials of the second kind and first kind respectively to obtain some more summations. If we let $x = \cos \theta$, $p = 2x$ and $q = 1$, then the recurrence relation (3.3)

becomes $u_{n+2} = 2xu_{n+1} - u_n$, $n \geq 0$ with initial conditions $u_0 = 0, u_1 = 1$ and the recurrence relation for the Chebyshev polynomials of the second kind as noted in chapter II is

$U_{n+1} = 2xU_n - U_{n-1}$, $n \geq 0$ with initial conditions $U_0 = 1, U_1 = p$ Thus, comparing the two recurrence relations, for all $n \geq 1$ we have

$$(3.30) \quad u_n(2x, 1) = U_{n-1}(x) = \frac{\sin n\theta}{\sin \theta} \quad n \geq 0, \quad U_0(x) = 1, U_1(x) = 2x.$$

Similarly $v_n(p, q)$, with $x = \cos \theta$, $p = 2x$ and $q = 1$ are the Chebyshev polynomials of the first kind $v_n(2x, 1) = 2T_n(x) = 2\cos n\theta (n \geq 0)$. $v_n(2x, 1)$ satisfies the recurrence relation $v_{n+2} = 2xv_{n+1} - v_n, n \geq 0$ with initial conditions $v_0 = 2, v_1 = 2x$. This translates to the Chebyshev recurrence relation with the substitution $v_n = 2T_n, T_{n+2} = 2xT_{n+1} - T_n, n \geq 0$ with initial conditions $T_0 = 2, T_1 = 2x$. Putting $p = 2x$ and $q = 1$, we can replace u_n by

U_n the Chebyshev polynomials, $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} U_{2n}(x)$. Thus, we get

$$(3.31) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} u_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\sin(2n+1)\theta}{\sin \theta}.$$

We also use the fact that $v_n(2x, 1) = 2T_n(x)$ to rewrite the left hand side of the following results. Proofs of all these results (3.32)-(3.35) are similar. We demonstrate here a proof of (3.34). The results are as follows:

$$(3.32) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{\sin(2n+1)\theta}{\sin \theta} = \frac{\cos(\cos \theta) \sinh(\sin \theta)}{\sin \theta}$$

$$(3.33) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cos(2n+1)\theta = \sin(\cos \theta) \cosh(\sin \theta)$$

$$(3.34) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{\sin(\cos \theta) \sinh(\sin \theta)}{\sin \theta}$$

$$(3.35) \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \cos 2n\theta = -\cos(\cos \theta) \cosh(\sin \theta)$$

Equation (3.34) can be proved as follows:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} u_{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{\sin \theta} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \sin 2n\theta \right)$$

We can substitute for $\sin 2n\theta$ and write

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{\sin \theta} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \left(\frac{e^{i2n\theta} - e^{-i2n\theta}}{2i} \right) \right)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{2i \sin \theta} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} e^{i2n\theta} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} e^{-i2n\theta} \right)$$

Using the fact that $\cos X = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} X^{2n}$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{2i \sin \theta} \left(-2 \sin \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right) \sin \left(\frac{e^{-i\theta} - e^{i\theta}}{2} \right) \right)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{2i \sin \theta} \left(-2 \sin(\cos \theta) \sin \left(- \left(\frac{e^{i\theta} - e^{-i\theta}}{2} \right) \right) \right)$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{2i \sin \theta} (-2 \sin(\cos \theta) \sin(-i \sin \theta))$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{2i \sin \theta} (-2 \sin(\cos \theta) (-\sin(i \sin \theta)))$$

Using the relation $-\sin iu = \frac{\sinh u}{i}$ to write the second parentheses we get

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{1}{2i \sin \theta} \left(-2 \sin(\cos \theta) \frac{\sinh(\sin \theta)}{i} \right)$$

Canceling the common factor 2 and since $i^2 = -1$ cancels the minus sign, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} \frac{\sin 2n\theta}{\sin \theta} = \frac{\sin(\cos \theta) \sinh(\sin \theta)}{\sin \theta}$$

3. Identities Using The Matrix Q

We are going to use functions of the 2×2 matrix Q to obtain some identities. Q is a square matrix defined as follows:

$$(3.36) \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{with} \quad \det Q = -1.$$

The relation between the n^{th} power of Q and the Fibonacci number is given by:

$$(3.37) \quad Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

The above result can be easily proved by induction and its proof is found in Hoggatt [2].

First, we make note of a familiar term.

Spectrum of a matrix: Let M be a square matrix of arbitrary order k . The set

$\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ of eigenvalues of M is called the spectrum of M . We illustrate below a method to find the function of a matrix using its characteristic equation when the function is a polynomial.

Example: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$, we want to find $f(A)$, where $f(\lambda) = \lambda^4 + 6\lambda$ is a

polynomial. In this special case, it is known that we can replace λ by A to get

$f(A) = A^4 + 6A$. We wish to find the value of $f(A)$ using the characteristic equation of

A . The characteristic equation of A is found by expanding $\det(A - \lambda I) = 0$, which is

given by $g(\lambda) = \lambda^2 - 2\lambda + 10 = 0$. Replacing λ by A in the polynomial we get,

$g(A) = A^2 - 2A + 10I = 0$. Since each matrix satisfies its own characteristic equation, we can replace λ by A in the polynomial. Now, $f(A)$ can be written as

$$(A^4 + 6A) = (A^2 - 2A + 10I)(A^2 + 2A - 6I) - 26A + 60I, \text{ or } f(A) = g(A)h(A) + r(A).$$

Since $g(A)$ is identically zero, we have $f(A) = r(A)$, $A^4 + 6A = -26A + 60I$, where

$r(\lambda)$ is the remainder when $f(\lambda)$ is divided by $g(\lambda)$. Suppose now, we wish to find

$f(M)$, where M is a square matrix of arbitrary order k , when $f(\lambda)$ is not just a

polynomial as in the above example, but f is any function which is analytic and is

defined on spectrum of M . F. R. Gantmacher [12] explains how $f(M)$ can be

rewritten as a polynomial in M of degree $\leq k - 1$. The domain of f now includes the

spectrum of M . Then, if f is represented by a power series expansion in a circle

containing the eigenvalues of M , this expansion remains valid when the scalar argument

is replaced by the matrix M . Below we cite a theorem 2.25 from Friedman [13] as

follows:

Theorem 2.25: If A is a matrix whose eigenvalues arranged in order of increasing

absolute values are $\lambda_1, \lambda_2, \dots, \lambda_n$ and if $f(\lambda)$ is an analytic function of λ in a circle

around the origin with radius greater than $|\lambda_n|$, then $f(A)$ equals $r(A)$, the polynomial

of degree $n - 1$ for which $f(\lambda_k) = r(\lambda_k)$, $k = 1, 2, \dots, n$.

The proof of this theorem is found in Friedman [13], it justifies our replacing λ by M

when f is analytic and is defined on the spectrum of M .

To obtain identities using the matrix Q we will generally proceed as follows:

Step I: We determine an expression of the entries a_{ij} of any function $f(Q) = A = [a_{ij}]$ based on the polynomial representation of f .

Step II: We find $f(Q)$ by means of a power series expansion $f(Q) = B = [b_{ij}]$.

Step III: Equate a_{ij} and b_{ij} .

We will illustrate the above procedure by introducing, as examples, some of the elementary functions like the square-root function, the exponential, circular and hyperbolic functions. We note that all of the above mentioned functions are analytic everywhere in the real plane and, certainly, analytic on the distinct eigenvalues of Q ,

which are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We know that the spectrum of Q is $\{\alpha, \beta\}$.

The polynomial representation of any analytic function defined on the spectrum of Q is given by

$$(3.38) \quad f(Q) = \sum_{i=0}^1 c_i Q^i = c_0 I + c_1 Q.$$

The coefficients c_0 and c_1 are found by solving the system of equations

$$(3.39) \quad \begin{aligned} f(\alpha) &= c_0 + c_1 \alpha \\ f(\beta) &= c_0 + c_1 \beta \end{aligned}$$

In fact one can solve (3.39) to obtain

$$(3.40) \quad c_0 = \frac{(\alpha f(\beta) - \beta f(\alpha))}{\sqrt{5}} \quad \text{and} \quad c_1 = \frac{f(\alpha) - f(\beta)}{\sqrt{5}}.$$

Using (3.38) we can write $f(Q) = c_0 I + c_1 Q = c_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$,

$f(Q) = \begin{bmatrix} c_0 + c_1 & c_1 \\ c_1 & c_0 \end{bmatrix}$. Substituting for c_0 and c_1 from (3.40) we get

$$(3.41) \quad f(Q) = A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha f(\alpha) - \beta f(\beta) & f(\alpha) - f(\beta) \\ f(\alpha) - f(\beta) & \alpha f(\beta) - \beta f(\alpha) \end{bmatrix}.$$

The above equation completes step I. We now begin our examples.

Example 1: $f(x) = e^x$

Step I: Substituting for f in (3.41) yields

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha e^\alpha - \beta e^\beta}{\sqrt{5}} & \frac{e^\alpha - e^\beta}{\sqrt{5}} \\ \frac{e^\alpha - e^\beta}{\sqrt{5}} & \frac{\alpha e^\beta - \beta e^\alpha}{\sqrt{5}} \end{bmatrix}$$

Step II: Another way of finding $f(Q)$ is $f(Q) = e^Q = \sum_{n=0}^{\infty} \frac{Q^n}{n!}$. From (3.37) we have

$$f(Q) = B = \sum_{n=0}^{\infty} \frac{\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}}{n!}. \text{ Therefore, } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} & \sum_{n=0}^{\infty} \frac{F_n}{n!} \\ \sum_{n=0}^{\infty} \frac{F_n}{n!} & \sum_{n=0}^{\infty} \frac{F_{n-1}}{n!} \end{bmatrix}.$$

Step III: Equating the corresponding terms of A and B from step I and step II above, we get

$$(3.42) \quad a_{11} = b_{11} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} = \frac{\alpha e^\alpha - \beta e^\beta}{\sqrt{5}}.$$

$$(3.43) \quad a_{12} = b_{21} = b_{12} = a_{21} \text{ implies } \sum_{n=0}^{\infty} \frac{F_n}{n!} = \frac{e^\alpha - e^\beta}{\sqrt{5}}.$$

$$(3.44) \quad a_{22} = b_{22} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} = \frac{\alpha e^{\beta} - \beta e^{\alpha}}{\sqrt{5}}.$$

We can also rewrite some of these identities in terms of Lucas numbers. As an example, Hoggatt [2] has the identity $F_{n+1} + F_{n-1} = L_n$. We can combine (3.42) and (3.44) to get

$$(3.45) \quad \sum_{n=0}^{\infty} \frac{L_n}{n!} = e^{\alpha} + e^{\beta}.$$

We continue with more identities using Q .

Example 2: $f(x) = \sin x$

Step I: Substituting for f in (3.41), we have

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha \sin \alpha - \beta \sin \beta}{\sqrt{5}} & \frac{\sin \alpha - \sin \beta}{\sqrt{5}} \\ \frac{\sin \alpha - \sin \beta}{\sqrt{5}} & \frac{\alpha \sin \beta - \beta \sin \alpha}{\sqrt{5}} \end{bmatrix}.$$

Step II: $f(Q) = B$ can be alternatively obtained as $\sin Q = \sum_{n=0}^{\infty} (-1)^n \frac{Q^{2n+1}}{(2n+1)!}$.

$$\text{Using (3.37) gives } f(Q) = B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+2}}{(2n+1)!} & \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n+1)!} \\ \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n+1)!} & \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n+1)!} \end{bmatrix}.$$

Step III: Equating the entries of A and B we obtain

$$(3.46) \quad a_{11} = b_{11} \text{ implies } \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+2}}{(2n+1)!} = \frac{\alpha \sin \alpha - \beta \sin \beta}{\sqrt{5}}.$$

$$(3.47) \quad a_{12} = b_{21} = b_{12} = a_{21} \text{ implies } \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n+1)!} = \frac{\sin \alpha - \sin \beta}{\sqrt{5}}.$$

$$(3.48) \quad a_{22} = b_{22} \text{ implies } \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n+1)!} = \frac{\alpha \sin \beta - \beta \sin \alpha}{\sqrt{5}}.$$

Repeating the above example 2 with $f(x) = \cos x$, $f(Q) = \cos Q = \sum_{n=0}^{\infty} (-1)^n \frac{Q^{2n}}{(2n)!}$ leads

us to some more identities

$$(3.49) \quad a_{11} = b_{11} \text{ implies } \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n)!} = \frac{\alpha \cos \alpha - \beta \cos \beta}{\sqrt{5}}.$$

$$(3.50) \quad a_{12} = b_{21} = b_{12} = a_{21} \text{ implies } \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n)!} = \frac{\cos \alpha - \cos \beta}{\sqrt{5}}.$$

$$(3.51) \quad a_{22} = b_{22} \text{ implies } \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n-1}}{(2n)!} = \frac{\alpha \cos \beta - \beta \cos \alpha}{\sqrt{5}}.$$

Example 3: $f(x) = \cosh x$

Step I: Substituting for f in (3.41), we have

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha \cosh \alpha - \beta \cosh \beta}{\sqrt{5}} & \frac{\cosh \alpha - \cosh \beta}{\sqrt{5}} \\ \frac{\cosh \alpha - \cosh \beta}{\sqrt{5}} & \frac{\alpha \cosh \beta - \beta \cosh \alpha}{\sqrt{5}} \end{bmatrix}.$$

Step II: $f(Q) = B$ can be alternatively obtained as $\cosh Q = \sum_{n=0}^{\infty} \frac{Q^{2n}}{(2n)!}$.

Substituting Q^{2n} from (3.37) we get

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n)!} & \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n)!} \\ \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n)!} & \sum_{n=0}^{\infty} (-1)^n \frac{F_{2n-1}}{(2n+1)!} \end{bmatrix}.$$

Step III: Equating the entries of A and B we get

$$(3.52) \quad a_{11} = b_{11} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{2n+1}}{(2n)!} = \frac{\alpha \cosh \alpha - \beta \cosh \beta}{\sqrt{5}}.$$

$$(3.53) \quad a_{12} = b_{21} = b_{12} = a_{21} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{2n}}{(2n)!} = \frac{\cosh \alpha - \cosh \beta}{\sqrt{5}}.$$

$$(3.54) \quad a_{22} = b_{22} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{2n-1}}{(2n)!} = \frac{\alpha \cosh \beta - \beta \cosh \alpha}{\sqrt{5}}.$$

Similarly, with $f(x) = \sinh x$ we have $f(Q) = \sinh Q = \sum_{n=0}^{\infty} \frac{Q^{2n+1}}{(2n+1)!}$.

We get some more identities as follows:

$$(3.55) \quad a_{11} = b_{11} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{2n+2}}{(2n+1)!} = \frac{\alpha \sinh \alpha - \beta \sinh \beta}{\sqrt{5}}.$$

$$(3.56) \quad a_{12} = b_{21} = b_{12} = a_{21} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{2n+1}}{(2n+1)!} = \frac{\sinh \alpha - \sinh \beta}{\sqrt{5}}.$$

$$(3.57) \quad a_{22} = b_{22} \text{ implies } \sum_{n=0}^{\infty} \frac{F_{2n}}{(2n+1)!} = \frac{\alpha \sinh \beta - \beta \sinh \alpha}{\sqrt{5}}.$$

We could obtain many more identities apart from the ones mentioned, all of them having the general form as follows:

When $f(x) = \sum_{n=0}^{\infty} c_n x^n$, we have $f(Q)$ as $f(Q) = \sum_{n=0}^{\infty} c_n Q^n$. So

$$f(Q) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} c_n F_{n+1} & \sum_{n=0}^{\infty} c_n F_n \\ \sum_{n=0}^{\infty} c_n F_n & \sum_{n=0}^{\infty} c_n F_{n-1} \end{bmatrix} \text{ which give us identities of the form:}$$

$$(3.58) \quad \sum_{n=0}^{\infty} c_n F_n = \frac{f(\alpha) - f(\beta)}{\sqrt{5}}.$$

$$(3.59) \quad \sum_{n=0}^{\infty} c_n F_{n+1} = \frac{(\alpha f(\alpha) - f(\beta)\beta)}{\sqrt{5}}.$$

$$(3.60) \quad \sum_{n=0}^{\infty} c_n F_{n-1} = \frac{(\alpha f(\beta) - f(\alpha)\beta)}{\sqrt{5}}.$$

As a last example, we consider the square-root function $f(x) = \sqrt{x}$. Since this function is not analytic in a circle containing the origin, we justify this example by using a theorem from Lancaster [19] which is as follows:

Theorem 5.3.1: If f is a function defined on the spectrum of A and g is the interpolatory polynomial of minimum degree determined by the values of f on the spectrum of A , then $f(A) = g(A)$.

Example 4: $f(x) = \sqrt{x}$.

Step I: Putting f as the square-root function in (3.41), we get

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha\sqrt{\alpha} - \beta\sqrt{\beta}}{\sqrt{5}} & \frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{5}} \\ \frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{5}} & \frac{\alpha\sqrt{\beta} - \beta\sqrt{\alpha}}{\sqrt{5}} \end{bmatrix}. \text{ We know that } \alpha\beta = -1 \text{ which implies}$$

$\alpha\beta = i^2, \sqrt{\alpha\beta} = \pm i$. We take $\sqrt{\alpha} > 0$ and $\sqrt{\beta} = \frac{i}{\sqrt{\alpha}}$ and $\sqrt{\alpha} = \frac{i}{\sqrt{\beta}}$. So we can write

$$\text{the entries of } A, \text{ all in terms of } \alpha, \text{ as } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha\sqrt{\alpha} + i\sqrt{\frac{1}{\alpha^3}}}{\sqrt{5}} & \frac{\sqrt{\alpha} - i\sqrt{\frac{1}{\alpha}}}{\sqrt{5}} \\ \frac{\sqrt{\alpha} - i\sqrt{\frac{1}{\alpha}}}{\sqrt{5}} & \frac{\sqrt{\frac{1}{\alpha}} + i\sqrt{\alpha}}{\sqrt{5}} \end{bmatrix}.$$

Step II: Another way to find $f(Q) = \sqrt{Q} = B$ is by squaring both sides of the equation

$\sqrt{Q} = B$ to get $Q = B^2$, $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Upon expanding we get,

$$\begin{bmatrix} b_{11}^2 + b_{12}b_{21} & b_{11}b_{12} + b_{12}b_{22} \\ b_{21}b_{11} + b_{22}b_{21} & b_{21}b_{12} + b_{22}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ which implies the equality } b_{12} = b_{21}, \text{ since}$$

we have the equation $b_{12}(b_{11} + b_{22}) = 1 = b_{21}(b_{11} + b_{22})$ obtained by equating the

corresponding entries of B^2 and Q . Put $b_{12} = b_{21}$ in $b_{12}b_{21} + b_{22}^2 = 0$ to get

$b_{12}^2 = b_{21}^2 = -b_{22}^2$ which implies $b_{12} = b_{21} = \pm ib_{22}$. The equation $b_{11}^2 + b_{12}^2 = 1$ becomes

$b_{11}^2 \pm ib_{22}^2 = 1$. Now the equations, $b_{11}^2 \pm ib_{22}^2 = b_{11} \pm ib_{22} + ib_{22}^2 = 1$ implies $b_{11} = (1 \pm i)b_{22}$.

Therefore, so far we have found that

$$(3.61) \quad b_{11} = (1 \pm i)b_{22} \text{ and } b_{12} = b_{21} = \pm ib_{22}.$$

We find b_{22} as follows: We make use of (3.61) and the equation $b_{12}(b_{11} + b_{22}) = 1$

which can be written using (3.61) :

$$\pm ib_{22}((1 \pm i)b_{22} + b_{22}) = 1, \text{ so } \pm i(1 \pm i)b_{22}^2 + (\pm i)b_{22}^2 = 1, \quad b_{22}^2 \pm (-1 \pm i \pm i) = 1,$$

$$b_{22}^2(-1 \pm 2i) = 1. \text{ Therefore, } b_{22}^2 = \frac{1}{\pm(-1 \pm 2i)}. \text{ We rationalize the denominator to get}$$

$$b_{22}^2 = \frac{\pm(-1 \mp 2i)}{5} \text{ which gives}$$

$$(3.62) \quad b_{22}^2 = \pm \sqrt{\frac{(-1 \mp 2i)}{5}}.$$

We will show that $-1 \pm 2i = \sqrt{5}e^{i(\pi \mp \arctan 2)}$. We will prove that $-1 + 2i = \sqrt{5}e^{i(\pi - \arctan 2)}$.

$$e^{i(\pi - \arctan 2)} = e^{i\pi} e^{-i(\arctan 2)} = -[\cos(\arctan 2) - i \sin(\arctan 2)]$$

$$= (-1) \left(\frac{1}{\sqrt{1+4}} \right) + i \left(-(-1) \frac{2}{\sqrt{1+4}} \right) = \frac{-1+2i}{\sqrt{5}}$$

Now $-1 \pm 2i = \sqrt{5} e^{i(\pi \mp \arctan 2)}$. Taking the square-root of both sides, we have

$$\sqrt{-1 \pm 2i} = \left(\frac{1}{5} \right)^{\frac{1}{4}} \sqrt{e^{i\theta}}, \text{ where } \theta = \pi \mp \arctan 2 \text{ or } \sqrt{-1 \pm 2i} = \left(\frac{1}{5} \right)^{\frac{1}{4}} e^{i\frac{\theta}{2}}. \text{ Thus}$$

$$b_{22} = \left(\frac{1}{5} \right)^{\frac{1}{4}} e^{\frac{i(\pi \pm \arctan 2)}{2} + ik\pi}, \quad k = 0, 1.$$

Step III: Equate the real parts of a_{22} and b_{22} to get an identity in α .

Real part of a_{22} is $\text{Re}(a_{22}) = \sqrt{\frac{1}{5\alpha}}$ and real part of b_{22} is

$$\text{Re}(b_{22}) = (-1)^k \left(\frac{1}{5} \right)^{\frac{1}{4}} \cos \left(\frac{\pi \pm \arctan 2}{2} \right). \text{ Expanding } \cos \left(\frac{\pi \pm \arctan 2}{2} \right) \text{ by using the}$$

identities from trigonometry to get

$$\cos \left(\frac{\pi \pm \arctan 2}{2} \right) = \cos \frac{\pi}{2} \cos \left(\frac{\arctan 2}{2} \right) \pm \sin \frac{\pi}{2} \sin \left(\frac{\arctan 2}{2} \right). \text{ Since } \cos \frac{\pi}{2} = 0 \text{ and}$$

$$\sin \frac{\pi}{2} = +1, \text{ we have } \text{Re}(b_{22}) = (-1)^k \left(\frac{1}{5} \right)^{\frac{1}{4}} \sin \left(\mp \frac{\sin \arctan 2}{2} \right).$$

Equating $\text{Re}(a_{22})$ and $\text{Re}(b_{22})$ and squaring them, we obtain

$$(3.63) \quad \frac{1}{5\alpha} = \left(\frac{1}{5} \right)^{\frac{1}{2}} \sin^2 \frac{\arctan 2}{2} \text{ which leads to the trigonometric identity}$$

$$(3.64) \quad \frac{1+\sqrt{5}}{2} = \frac{1}{\left(\sqrt{5 \sin^2 \frac{\arctan 2}{2}} \right)}.$$

Similarly by equating the imaginary parts of a_{22} and b_{22} , we obtain another identity

$$(3.65) \quad \frac{1+\sqrt{5}}{2} = \sqrt{5} \cos^2 \frac{\arctan 2}{2}.$$

We stop here with the quest of identities and take on a little different route. We look into the more general 2×2 matrix P which is related to the sequence u_n which was defined in the previous chapters. $u_n, n = 0, 1, 2, 3, \dots$ is defined by the recurrence relation

$$u_{n+2} = pu_{n+1} - qu_n \quad u_0 = 0, u_1 = 1. \text{ With } q = -1, \text{ this relation becomes}$$

$$(3.66) \quad u_{n+2} = pu_{n+1} + u_n, \quad u_0 = 0, u_1 = 1.$$

We shall define the matrix P and as with Q , the n^{th} power of P will have its entries u_n which is defined by (3.66).

$$(3.67) \quad \text{Let } P = \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \text{ where } p \text{ is an arbitrary real number.}$$

Note that $\det P = \det Q = -1$. Also with $p = 1$ the matrix P is nothing but the matrix Q . Thus Q is a special case of P . P has the property that

$$(3.68) \quad P^n = \begin{bmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{bmatrix}.$$

The above result can be proved using induction on n . Put $n = 1$ in (3.68) and we get

$$P = \begin{bmatrix} u_2 & u_1 \\ u_1 & u_0 \end{bmatrix}. \text{ By the recurrence relation (3.66), we have } u_2 = pu_1 + u_0 = p(1) + 0 = p.$$

Thus $P = \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix}$ which agrees with the definition of P in (3.67).

Assume the result to be true for all $n \leq k$. $P^k = \begin{bmatrix} u_{k+1} & u_k \\ u_k & u_{k-1} \end{bmatrix}$. We need to prove the

result for $n = k + 1$. We have to prove that $P^{k+1} = \begin{bmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{bmatrix}$.

Now $P^{k+1} = P^k P = \begin{bmatrix} u_{k+1} & u_k \\ u_k & u_{k-1} \end{bmatrix} \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} pu_{k+1} + u_k & u_{k+1} \\ pu_k + u_{k-1} & u_k \end{bmatrix}$. Using the relation (3.66)

we can write the entries of P^{k+1} as $P^{k+1} = \begin{bmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{bmatrix}$.

The characteristic equation of P is given by $\lambda^2 - p\lambda - 1 = 0$. Its roots are

$\frac{p \pm \sqrt{p^2 + 4}}{2}$. Let $\Delta = \sqrt{p^2 + 4}$. Let α_p, β_p be the eigenvalues of P , then α_p, β_p are

given by $\alpha_p = \frac{p + \Delta}{2}$ and $\beta_p = \frac{p - \Delta}{2}$. Note that $\alpha_p + \beta_p = p$, $\alpha_p - \beta_p = \Delta$,

$\alpha_p \beta_p = -1$ which implies $\beta_p = \frac{-1}{\alpha_p}$. All of the work done with Q can now be done

with P just by replacing Q with P . Since we intended to find identities involving

Fibonacci numbers, we went here from particular Q to the more general P . We can

parallel the argument for Fibonacci numbers as in example 4 and get identities

corresponding to (3.64) and (3.65) which are:

$$(3.69) \quad \alpha_p = \frac{1}{\left(\Delta \sin^2 \left(\frac{\arctan\left(\frac{2}{p}\right)}{2} \right) \right)}.$$

$$(3.70) \quad \alpha_p = \Delta \cos^2 \left(\frac{\arctan \left(\frac{2}{p} \right)}{2} \right).$$

Finally note that by putting $p = 1$ in (3.69) and (3.70) we obtain the Fibonacci identities (3.63) and (3.64). For more identities using matrices, see Filipponi and Horadam [14].

CHAPTER IV

GOLDEN SECTION IN GEOMETRY

The irrational number $\alpha = \frac{1+\sqrt{5}}{2}$ is also called the Golden mean or Golden ratio besides the term Golden section. We explain our choice of the term. Hoggatt [2] has this interpretation of α .

Given a line segment AB , we want to find a point C on it (between A and B) such that

$\frac{AB}{AC} = \frac{AC}{CB}$ where A, B, C are all distinct. We refer to the figure below.

Figure 1.

Let $\frac{AB}{AC} = x$, $x > 0$. Then, $\frac{AB}{AC} = \frac{AC + CB}{AC} = 1 + \frac{CB}{AC}$. So,

$\frac{AB}{AC} = 1 + \frac{1}{\frac{AC}{CB}} = 1 + \frac{1}{x}$. Therefore, $x = 1 + \frac{1}{x}$ which implies $x^2 = x + 1$. This is the

Fibonacci quadratic equation whose roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$,

$\alpha = 1.618(\text{approx})$ and $\beta = -0.618(\text{approx})$. We take for the ratio positive root α .

Therefore, $\frac{AB}{AC} = \alpha$. Using this value of α and by construction it can be shown that C

is located between A and B such that C divides segment AB in the above ratio: C divides segment AB in the Golden section.

We are interested in the unexpected appearance of this α in various geometrical examples.

1. Smallest Triangle Circumscribing A Semicircle

We are looking for an isosceles triangle of the smallest perimeter, which circumscribes a semicircle of radius 1. Let ABC be such an isosceles triangle circumscribing a semicircle of radius 1. The base BC contains the diameter of the semicircle as shown in figure 2. AB is tangent to the circle at a point E . $\triangle ABC$ may be parameterized by the base angle θ as shown in the figure 2.

Figure 2.

The perimeter s of ΔABC is given by

$$(4.1) \quad s = 2(AE + EB + BD) = 2(\tan \theta + \cot \theta + \csc \theta)$$

Differentiating (4.1) with respect to θ we get

$$(4.2) \quad s' = 2(\sec^2 \theta - \csc^2 \theta - \csc \theta \cot \theta)$$

which can be written in terms of $\sin \theta$ and $\cos \theta$ as

$$(4.3) \quad s' = \frac{2(1 - \cos \theta - \cos^2 \theta)(1 + \cos \theta)}{\cos^2 \theta \sin^2 \theta}.$$

To find the triangle of smallest perimeter we consider $s' = 0$.

$s' = 0$ implies $2(1 - \cos \theta - \cos^2 \theta)(1 + \cos \theta) = 0$. In the range $0 < \theta < \frac{\pi}{2}$, $s' = 0$

translates to

$$(4.4) \quad 1 - \cos \theta - \cos^2 \theta = 0.$$

If we let $m = \cos \theta$, then (4.4) becomes $1 - m - m^2 = 0$ or $m^2 = m - 1$ whose positive

root is given by $m_1 = \frac{-1 + \sqrt{5}}{2}$. The root m_1 is the reciprocal of the golden section.

Thus $m_1 = \frac{1}{\alpha}$. So, $s' = 0$ has just one solution in $0 < \theta < \frac{\pi}{2}$ which is given by

$$(4.5) \quad m_1 = \cos \theta = \frac{1}{\alpha}.$$

Since $\sin^2 \theta = 1 - \cos^2 \theta$, we have $\sin^2 \theta = 1 - \frac{1}{\alpha^2}$, $\sin^2 \theta = \frac{\alpha^2 - 1}{\alpha^2}$, since $\alpha^2 = \alpha + 1$,

we have $\sin^2 \theta = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}$. So we get $\operatorname{cosec} \theta = \sqrt{\alpha}$ and $\cot \theta = \frac{1}{\sqrt{\alpha}}$.

Substituting these vales in (4.1) we obtain

$$(4.6) \quad s = 2 \left(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}} + \sqrt{\alpha} \right) = 2\sqrt{\alpha} \left(2 + \frac{1}{\alpha} \right)$$

Since α is a root of the Fibonacci quadratic equation $\alpha^2 = \alpha + 1$, we have $\frac{1}{\alpha} = \alpha - 1$.

Thus, $2 + \frac{1}{\alpha} = 2 + (\alpha - 1) = 1 + \alpha = \alpha^2$, since $\alpha^2 = \alpha + 1$.

Substituting this value of $\left(2 + \frac{1}{\alpha} \right)$ in (4.6) we have

$$(4.7) \quad s = 2\sqrt{\alpha}(\alpha^2) = 2\alpha^{\frac{5}{2}}.$$

Therefore the minimal perimeter is $s = 2\alpha^{\frac{5}{2}}$. DeTemple [15] gives the dimensions of such a triangle ABC with perimeter s . They are:

$$s \cong 6.66, \quad \theta = \arccos\left(\frac{1}{\alpha}\right) = 51.8^\circ, \quad AE = BD = \sqrt{\alpha}, \quad BE = \frac{1}{\sqrt{\alpha}}, \quad AC = \alpha^{\frac{3}{2}},$$

$AD = \alpha$. Thus α plays a role in this minimization problem.

2. Properties Of The Golden Ellipse

We begin by explaining the term Golden Ellipse. Huntley[16] defines it as follows:

Golden Ellipse: It is an ellipse in which the ratio of the major axis to the minor axis is α , the golden section. If $2a$ is the length of the major axis and $2b$ is the length of the minor axis then, $\frac{a}{b} = \alpha$.

Figure 3: The Golden Ellipse

In Figure 3 of the golden ellipse let $OA = a = \alpha$ units and $OB = b = 1$ unit. The eccentricity e of an ellipse is given by $b^2 = a^2(1 - e^2)$. With $a = \alpha, b = 1$, e for the golden ellipse is computed as follows:

$$1 = \alpha^2(1 - e^2) \text{ implies } \alpha^2 e^2 = \alpha^2 - 1 \text{ which gives, } e^2 = 1 - \frac{1}{\alpha^2} = \frac{\alpha^2 - 1}{\alpha^2}.$$

$$\text{Substituting } \alpha^2 - 1 = \alpha \text{ we get } e^2 = \frac{1}{\alpha}.$$

$$(4.8) \quad e^2 = \frac{1}{\alpha} \text{ implies } e = \frac{1}{\sqrt{\alpha}} = \sqrt{-\beta}, \text{ since } \alpha\beta = -1.$$

We use Figure 3 and known properties of ellipse to write the following equations. Let S and S' be the foci of the golden ellipse. Then

$$(4.9) \quad OS = ae = \alpha \frac{1}{\sqrt{\alpha}} = \sqrt{\alpha}.$$

$$(4.10) \quad BS = \sqrt{b^2 + a^2 e^2} = \sqrt{1 + \alpha} = \sqrt{\alpha^2} = \alpha.$$

If $\angle OBS = \theta$. We can write

$$(4.11) \quad \frac{1}{\cos \theta} = \sec \theta = \frac{a}{b} = \alpha.$$

If ON is perpendicular to the directrix ND , then

$$(4.12) \quad ON = \frac{a}{e} = \frac{\alpha}{\frac{1}{\sqrt{\alpha}}} = \alpha\sqrt{\alpha} = \alpha^{\frac{3}{2}}.$$

$$(4.13) \quad SN = ON - OS = \frac{a}{e} - ae = \alpha\sqrt{\alpha} - \frac{\alpha}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}}.$$

If L is the length of semi-latus rectum PS , then L is given by

$$(4.14) \quad L = \frac{b^2}{a} = \frac{1}{\alpha} = -\beta.$$

We see from (4.14) that $b^2 = aL$ and $a:b:c = \alpha:1:-\beta = \alpha^2:\alpha:1$. Using (4.9), (4.12), and (4.13), we write

$$(4.15) \quad \frac{ON}{OS} = \frac{\frac{a}{e}}{ae} = \frac{1}{e^2} = \frac{1}{\frac{1}{\alpha}} = \alpha.$$

Using (4.9) and (4.12), we have

$$(4.16) \quad \frac{OS}{SN} = \alpha = -\beta.$$

(4.15) and (4.16) imply that the focus S divides ON both externally and internally in the golden section. If PP' is the latus rectum, then

$$(OP)^2 = (OS)^2 + (SP)^2 = \alpha + \frac{b^4}{a^2} = \alpha + \frac{1}{\alpha^2} \text{ which can be written as } \alpha + \frac{1}{\alpha^2} = \alpha + \beta^2.$$

Now $(OP)^2 = \alpha + \beta^2 = \alpha^2 - 1 + \beta^2$, since $\alpha = \alpha^2 - 1$. So

$$(OP)^2 = \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta - 1 = (\alpha + \beta)^2 - 2\alpha\beta - 1, \text{ since } \alpha + \beta = 1 \text{ and } \alpha\beta = -1.$$

We get $(OP)^2 = 1 + 2 - 1 = 2$. Therefore

$$(4.17) \quad (OP)^2 = 2.$$

We find the angle, γ , the latus rectum subtends at the center O . Using the *cosine* rule

for $\triangle POP'$ and the fact that $(OP')^2 = (OP)^2 = 2$, we have

$$\cos \gamma = \frac{(OP)^2 + (OP')^2 - (PP')^2}{2(OP)(OP')} = \frac{2+2-\frac{4}{\alpha^2}}{2(\sqrt{2})(\sqrt{2})} = \frac{4-\frac{4}{\alpha^2}}{4}. \text{ Therefore,}$$

$$\cos \gamma = 1 - \frac{1}{\alpha^2} = \frac{\alpha^2 - 1}{\alpha^2} = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}. \text{ So } \cos \gamma = \frac{1}{\alpha}. \text{ From (4.5) we can write}$$

$$(4.18) \quad \gamma = \theta$$

A known property of the ellipse is that a tangent at P passes through N and $\cot \angle SPN = e$.

For the golden ellipse, since $\cot \theta = \frac{b}{ae}$ it follows that

$$\cot \angle SPN = \frac{SP}{SN} = \frac{\frac{1}{\alpha}}{\frac{1}{\sqrt{\alpha}}} = \frac{\sqrt{\alpha}}{\alpha^2} = \frac{1}{\sqrt{\alpha}}. \text{ We know that } \cot \theta = \frac{1}{\sqrt{\alpha}}. \text{ So we get}$$

$$(4.19) \quad \angle SPN = \theta.$$

Now from Figure 3, MB and BS are parallel because the corresponding angles are equal. Thus $MPSB$ is a parallelogram. This implies that

$$(4.20) \quad MP = BS = a = \alpha.$$

Since $\frac{MP}{PN} = \frac{OS}{SN} = \alpha$, we have

$$(4.21) \quad PN = 1 \text{ and } MN = \alpha + 1 = \alpha^2.$$

Thus P divides MN in the golden section. We demonstrated some of the places that α appears in this special ellipse. We move on to look for α in another place.

3. The Golden Section: Yet Another Place

Huntley [17] finds α when trying to find centroid of plane figures under some conditions.

Consider an area A of any outline, regular or irregular symmetric about an axis.

Consider a cut-area B in A having the same outline under the following conditions:

- (1) The peripheries of A and B have one point O in common.
- (2) B is oriented so that O and the centroids C_a and C_b of A and B are collinear.

In such a case the centroid C of remnant $(A - B)$ also lie on the straight line OC_aC_b as shown in Figure 4 below.

Figure 4.

Let the ratio of linear dimensions of A and B be $a : b$ and their areas be λa^2

and λb^2 respectively. $\frac{OC_a}{OC_b} = \frac{a}{b}$. Taking moments about O , we get

$$(4.22) \quad \lambda b^2 OC_b + \lambda(a^2 - b^2)OC = \lambda a^2 OC_a.$$

Multiplying throughout by $\frac{1}{\lambda b^2 OC_b}$ and using the fact that $\frac{OC_a}{OC_b} = \frac{a}{b}$ we get

$$(4.23) \quad 1 + \left(\frac{a^2}{b^2} - 1 \right) \frac{OC}{OC_b} = \frac{a^3}{b^3}.$$

We simplify the above equation as follows:

$$1 + \left(\frac{a}{b} - 1 \right) \left(\frac{a}{b} + 1 \right) \frac{OC}{OC_b} = \frac{a^3}{b^3}, \quad \text{or} \quad \frac{1}{\left(\frac{a}{b} - 1 \right)} + \left(\frac{a}{b} + 1 \right) \frac{OC}{OC_b} = \frac{\frac{a^3}{b^3}}{\left(\frac{a}{b} - 1 \right)}. \quad \text{Thus we have}$$

$$\left(\frac{a}{b} + 1 \right) \frac{OC}{OC_b} = \frac{a^3}{b^2(a-b)} - \frac{b}{(a-b)} = \frac{a^3 - b^3}{b^2(a-b)} = \frac{a^2 + ab + b^2}{b^2}. \quad \text{Therefore, we get}$$

$$(4.24) \quad \left(\frac{a}{b} + 1 \right) \frac{OC}{OC_b} = \frac{a^2}{b^2} + \frac{a}{b} + 1.$$

The term $\left(\frac{a}{b} - 1 \right) \neq 0$ since, if $\left(\frac{a}{b} - 1 \right) = 0$ then $\frac{a-b}{b} = 0$, which implies $a = b$. But,

we assumed $a \neq b$. If we let the ratio $\frac{OC}{OC_b} = 2$ and substitute in (4.24) then

$$\left(\frac{a}{b} + 1 \right) 2 = \frac{a^2}{b^2} + \frac{a}{b} + 1 \quad \text{which gives}$$

$$(4.25) \quad \frac{a^2}{b^2} - \frac{a}{b} - 1 = 0.$$

If we let $\frac{a}{b} = x$, we see that (4.25) is the Fibonacci quadratic equation. We thus

uncovered α here, in this case, $\frac{a}{b} = \alpha$.

4. Geometry Of An Identity

In this section, Horadam[18] looks at the geometry of a well-known identity, the Simson's formula in two dimensions. We discuss only some special cases. We again deal with the generalized sequence w_n defined in chapter I as

$$(4.26) \quad w_{n+2} = pw_{n+1} - qw_n, \quad w_0 = a, w_1 = b$$

where a, b, p, q are arbitrary numbers. The Simson's formula for the Fibonacci sequence is given by $F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$. Its generalized version is

$$(4.27) \quad w_n w_{n+2} - w_{n+1}^2 = eq^n \quad \text{where} \quad e = pab - qa^2 - b^2$$

With $w_n = w_n\{(0,1;1,-1)\}$ we get the Fibonacci version. (4.27) can be proved using induction. We use the recurrence relation (4.26) to write (4.27) as

$$w_n(pw_{n+1} - qw_n) - w_{n+1}^2 - eq^n = 0. \quad \text{Therefore, we get}$$

$$(4.28) \quad qw_n^2 + w_{n+1}^2 - pw_n w_{n+1} + eq^n = 0.$$

Putting $w_n = x$, $w_{n+1} = y$, (4.28) becomes

$$(4.29) \quad qx^2 + y^2 - pxy + eq^n = 0.$$

The above equation represents a conic in (x, y) . To remove the xy term from the equation (4.29), we rotate the co-ordinate axes about the origin through the acute angle

$$\theta = \tan^{-1} m, \quad \text{where } m \text{ is the positive root of } -\frac{p}{2}m^2 + (q-1)m + \frac{p}{2} = 0 \quad \text{or}$$

$$m^2 - \frac{2(q-1)}{p}m - 1 = 0. \quad \text{Therefore, } m \text{ is given by}$$

$$(4.30) \quad m = \frac{q-1}{p} + \sqrt{\left(\frac{q-1}{p}\right)^2 + 1}.$$

Let $k = \frac{q-1}{p}$. Then m can be written as $m = k + \sqrt{k^2 + 1}$ which implies

$\tan \theta = k + \sqrt{k^2 + 1}$. So $\tan^2 \theta = 2k^2 + 1 + 2k\sqrt{k^2 + 1}$. Now $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$ will be

$$\tan 2\theta = \frac{2(k + \sqrt{k^2 + 1})}{1 - (2k^2 + 1 + 2k\sqrt{k^2 + 1})}. \text{ Simplifying we get}$$

$$\tan 2\theta = \frac{2(k + \sqrt{k^2 + 1})}{-2k(k + \sqrt{k^2 + 1})} = -\frac{1}{k} = \frac{-1}{\left(\frac{q-1}{p}\right)} \text{ which implies}$$

$$(4.31) \quad 2\theta = \tan^{-1}\left(\frac{-p}{q-1}\right) \quad \text{or} \quad \theta = \frac{1}{2} \tan^{-1}\left(\frac{-p}{q-1}\right).$$

Thus, if we rotate the co-ordinate axes through an angle θ given by (4.31), it eliminates

the xy term in (4.29) and gives the canonical form of an ellipse if $p^2 < 4q$ and a

hyperbola if $p^2 > 4q$. As an example, we consider a special case, where

$p = 2, q = -1, a = 0, b = 1, e = -1$ which represents the case $p^2 > 4q$. When we substitute

these values in (4.29) we obtain

$$(4.32) \quad x^2 - y^2 + 2xy = (-1)^{n+1}.$$

The equation (4.32) gives rise to two cases, one when n is even and another when n is

odd. Below, we discuss the case when n is odd, which implies $(n+1)$ is even.

Therefore equation (4.32) can now be written as

$$(4.33) \quad x^2 - y^2 + 2xy = 1.$$

We rotate the co-ordinate axes through an angle θ given by (4.31). With $p = 2, q = -1$

angle θ becomes $\theta = \frac{1}{2} \tan^{-1}(1)$ which implies

$$(4.34) \quad 2\theta = \frac{\pi}{4}.$$

We substitute for x and y in the equation (4.33) in terms of (x', y') , the co-ordinates of the new system.

$$(4.35) \quad \begin{aligned} x &= x' \cos \theta - y' \sin \theta \\ y &= x' \sin \theta + y' \cos \theta \end{aligned}$$

Putting the values of x and y as given in (4.35) in equation (4.33) gives

$$(x' \cos \theta - y' \sin \theta)^2 - (x' \sin \theta + y' \cos \theta)^2 + 2(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) = 1.$$

Simplifying the above equation yields

$$x'^2(\cos 2\theta + \sin 2\theta) - y'^2(\cos 2\theta + \sin 2\theta) + 2x'y'(-2\sin 2\theta + 2\cos 2\theta) = 1.$$

Since, $2\theta = \frac{\pi}{4}$, $\cos 2\theta = \sin 2\theta = \frac{1}{\sqrt{2}}$, the above relation reduces to

$$(4.36) \quad x'^2(\sqrt{2}) - y'^2(\sqrt{2}) = 1 \text{ or } x'^2 - y'^2 = \left(\frac{1}{\sqrt{2}}\right).$$

The above equation is the equation of a hyperbola whose asymptotes are given by

$$(4.37) \quad y' = x' \text{ and } y' = -x'.$$

Multiplying the first equation of (4.35) by $\cos \theta$, and the second equation of (4.35) by $\sin \theta$ and subtracting yields

$$x \cos \theta = x' \cos^2 \theta - \frac{1}{2} y' \sin 2\theta$$

$$y \sin \theta = x' \sin^2 \theta - \frac{1}{2} y' \sin 2\theta$$

Adding the two equations we get $x' = x \cos \theta + y \sin \theta$. Now we multiply the first equation of (4.35) by $\sin \theta$, and the second equation of (4.35) by $\cos \theta$ and subtracting yields another system of equations similar to the one above again subtracting them gives

$$y' = y \cos \theta - x \sin \theta. \quad y' = x' \text{ implies } y \cos \theta - x \sin \theta = x \cos \theta + y \sin \theta \text{ which gives}$$

$$y = x \left(\frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \right). \text{ Simplifying the parentheses yields } y = x \left(\frac{1 + \sin 2\theta}{\cos 2\theta} \right) \text{ or}$$

$$y = x \left(\frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \right) \quad \text{or} \quad y = x(1 + \sqrt{2}). \text{ Similarly } y' = -x' \text{ implies } y = -\left(\frac{1}{1 + \sqrt{2}} \right) x.$$

Therefore the asymptotes are given by

$$(4.38) \quad y = x(1 + \sqrt{2}) \quad \text{and} \quad y = -\left(\frac{1}{1 + \sqrt{2}} \right) x.$$

These asymptotes are perpendicular. Such a hyperbola is called a rectangular hyperbola.

When n is even, we get yet another hyperbola which may be transformed into the corresponding hyperbola (hyperbola in the example) by a reflection in $y = x$ followed by a reflection in the x axis.

One another special case which can be proved similar to the one above is when

$p = 1, q = -1$. This again yields two rectangular hyperbolas one corresponding to n :odd

and other corresponding to n :even. The asymptotes are given by $y = \alpha x$ and $y = -\frac{1}{\alpha}x$

in which $\alpha = \frac{1+\sqrt{5}}{2}$, the positive root of $t^2 - t - 1 = 0$.

There can be more discussion about the geometry of Simson's formula but we stop here

as our purpose of looking into this topic was really to find $\alpha = \frac{1+\sqrt{5}}{2}$, the golden section

in geometry. In doing so, we have been able to provide different examples concerning various geometrical shapes. Again, what we have discussed here are just a few of the

many places where $\alpha = \frac{1+\sqrt{5}}{2}$ the golden section appears in geometry.

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