2001

On Turan's pure power sum problem

Andrew H. Ledoan
San Jose State University

ON TURÁN'S PURE POWER SUM PROBLEM

A Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree of

Master of Science

by

Andrew H. Ledoan

May 2001

UMI Number: 1403980

UMI®

UMI Microform 1403980

Bell & Howell Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Dr. Daniel A. Goldston

Dr. Brian Peterson

Dr. Samih Obaid

APPROVED FOR THE UNIVERSITY

Abstract

On Turán's Pure Power Sum Problem

by Andrew H. Ledoan

Let $z_1, \ldots, z_n$ be complex numbers, and write $s_j = \sum_{k=1}^{n} z_k^j$ $(j \geq 1)$ for their pure power sums. P. Turán [1] started the investigation of the sequence

$$R_n = \min_{z_1,\ldots,z_n} \max_{j=1,\ldots,n} |s_j|$$

subject to the normalization

$$\max_{k=1,\ldots,n} |z_k| = 1.$$

In 1942, Turán conjectured that $R_n \geq A$ for some constant $A > 0$ independent of n and F. V. Atkinson [2] provided the proof in 1961. Atkinson showed that $R_n > \frac{1}{6}$. In 1994, A. Biró [5] showed that $R_n > \frac{1}{2}$. This thesis offers detailed proofs of Atkinson's and Biró's results, refinements for Atkinson's method, and computations on the Atkinson numbers and Biró numbers. It discovers surprisingly that Atkinson's method produces some lower bounds that exceed $\frac{1}{2}$ for small n.

Acknowledgment

This thesis was conducted under the direction of Professor Daniel Goldston, and was approved by Professors Brian Peterson and Samih Obaid as readers. My advisor has made a significant contribution to the work described in this thesis and to my mathematical education. I am grateful to him for suggesting Turán's pure power sum problem for the topic of this thesis, for teaching me how to develop intuition about mathematical problems, for getting me started in LaTeX, for reading my work, and for making helpful suggestions. I am also grateful to other professors at San Jose State University for providing encouragement and for helping me to develop the academic self-confidence that sustained me during difficult times. I would like to dedicate my efforts on this thesis to Professors Daniel Goldston, Brian Peterson, Samih Obaid, Jane Day, Maurice Stanley, Howard Swann, Eloise Hamann, and Martin Billik, for whom I have the greatest esteem, and think their ethic, creativity, and enthusiasms have given me a totally different view of the purpose of my graduate study.

Contents

Abstract iv

Acknowledgment v

1 Introduction 1

1.1 Formulating the problem 1

1.2 Proof of Turán's theorem 3

2 Atkinson's Method 6

2.1 Proof of Atkinson's theorem 6

2.2 Refinements 12

2.3 Atkinson numbers 15

2.4 Notes and comments 17

3 Biró's Method 19

3.1 Proof of Biró's theorem 19

3.2 Biró numbers 25

3.3 Conclusion 30

References 33

Chapter 1

Introduction

1.1 Formulating the problem

Let $z_1, \ldots, z_n$ be complex numbers, and let, for $j \geq 1$,

$$s_j = \sum_{k=1}^{n} z_k^j$$

be the jth pure power sum of these n complex numbers. (This means that no order is specified for the z_k.) In connection with the distribution of zeros of $\zeta(s)$, P. Turán started the investigation of the sequence

$$R_n = \min_{z_1, \ldots, z_n} \max_{j=1, \ldots, n} |s_j| ,$$

where the minimum is taken over all the sets of complex numbers $z_1, \ldots, z_n$ with

$$\max_{k=1, \ldots, n} |z_k| = 1. \tag{1.1}$$

The minimum R_n exists by Weierstrass's theorem.

Suppose $|z_1| = 1$. By replacing z_k by $\frac{z_k}{z_1}$ we do not change the value of $|s_j|$ and therefore R_n, and so we may suppose that $z_1 = 1$, thus replacing the condition (1.1) by $z_1 = 1$.

In this thesis, we study the problem of determining how small $|s_j|$ can be for all $1 \le j \le m$ and for any configuration of complex numbers satisfying (1.1). There are several results on the determination of R_n, now for arbitrary n. For instance, $R_1 = 1$, one can compute $R_2 = \frac{\sqrt{5}-1}{\sqrt{2}} = 0.8740320488\cdots$ with equality for the pair $\left(1, \frac{\sqrt{5}-1}{2}e^{\pm 2\pi i/3}\right)$, and $R_3 = 0.8247830309\cdots$. The only known case where R_n is computed exactly is the following, Turán [1, p. 31].

Theorem 1 (Turán). *Let $n = 2k+1$, and let $z_1 = \max_{j=1,\dots,n}|z_j| = 1$. Suppose further that $s_1 = s_2 = \cdots = s_k = 0$. Then*

$$\min_{z_1,\dots,z_n} \max_{j=1,\dots,2k+1} |s_j| \le \left(\sum_{j=k+1}^{2k+1} \frac{1}{j} \right)^{-1}.$$

with equality if and only if the z_j's are the roots of

$$z^{2k+1}\left(\sum_{j=k+1}^{2k+1} \frac{1}{j} \right) - \sum_{j=0}^{k} \frac{z_j}{2k+1-j} = 0.$$

There is a corresponding result for n even.

The first and simplest lower bound is given by the following theorem.

Theorem 2 (Turán). *The inequality*

$$R_n > \frac{\log 2}{\sum_{k=1}^{n} \frac{1}{k}}$$

holds.

This bound tends to zero with n.

In 1942, Turán [1] conjectured

Conjecture 1 (Turán). *$R_n > A$ for some constant $A > 0$ independent of n.*

In 1961, F. V. Atkinson [2] gave an involved proof using analytical methods. He showed that $R_n > \frac{1}{6}$. In 1963, he replaced $\frac{1}{6}$ by $\frac{1}{3}$. In his paper [3] Atkinson showed that $R_n > \frac{3}{8}$ for $n < 1600$, and, for sufficiently large n, $R_n > s_0$, where $0 < s_0 < \frac{1}{2}$ and s_0 is the only zero of the transcendental equation

$$\frac{s_n^2}{2\pi} \int_0^{\infty} \left[\exp\left\{2s_0 T(\varphi)\right\} - 1\right]^2 \varphi^{-2}\, d\varphi = 1,$$

where

$$T(x) = \int_0^x y^{-1} |\sin y|\, dy.$$

Atkinson did not give the exact value of s_0, only that $s_0 < \frac{3}{8}$.

H. L. Montgomery [4] showed that $R_n > \frac{1}{5}$. We will show that $R_n > \frac{10}{47}$. The best lower bound was obtained by A. Biró [5] by a new method. He showed that $R_n > \frac{1}{2}$.

1.2 Proof of Turán's theorem

Let us introduce the Newton-Girard formulas by way of proving Theorem 2.

Proof of Theorem 2: We will make use of

$$P(z) = \prod_{k=1}^{n} (z - z_k) = \sum_{k=0}^{n} a_k z^{n-k} \quad (z_1 = 1, a_0 = 1). \tag{1.2}$$

where the a_k are complex numbers and $z_1, \ldots, z_n$ satisfy (1.1). Let

$$s = \max_{j=1,\ldots,n} |s_j|. \tag{1.3}$$

The Newton-Girard formulas give the connection between the coefficients of the equation $P(z) = 0, a_0 = 1$, and the pure power sums

$$s_1 + a_1 = 0,$$

$$s_2 + a_1 s_1 + 2a_1 = 0,$$

$$s_3 + a_1 s_2 + a_2 s_1 + 3a_3 = 0,$$

$$s_4 + a_1 s_3 + a_2 s_2 + a_3 s_1 + 4a_4 = 0,$$

$$\vdots$$

$$s_{n-1} + a_1 s_{n-2} + a_2 s_{n-3} + \cdots + a_{n-2} s_1 + (n-1) a_{n-1} = 0.$$

The proof follows by comparing the expressions

$$P'(z) = \sum_{k=1}^{n} \frac{P(z)}{z - z_k} = \sum_{k=1}^{n} \frac{P(z) - P(z_k)}{z - z_k}$$

$$= \sum_{k=1}^{n} \Big[z^{n-1} + (z_k + a_1) z^{n-2} + \left(z_k^2 + a_1 z_k + a_2\right) z^{n-3} + \cdots$$

$$+ \left(z_k^{n-1} + a_1 z_k^{n-2} + \cdots + a_{n-1}\right)\Big]$$

$$= nz^{n-1} + (s_1 + na_1) z^{n-2} + (s_2 + a_1 s_1 + na_2) z^{n-3} + \cdots$$

$$+ (s_{n-1} + a_1 s_{n-2} + \cdots + a_{n-2} s_1 + na_{n-1})$$

and

$$P'(z) = nz^{n-1} + (n-1) a_1 z^{n-2} + (n-2) a_2 z^{n-3} + \cdots + a_{n-1}.$$

(See Ciarlet and Lions [7, p. 635].) We shall obtain an upper bound on the a_k. By the binomial coefficients formula

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{1\cdot 2\cdot 3\cdots k}$$

$$(m \geq k, m \geq 0, k \geq 0).$$

we have

$$|a_1| = |-s_1| \leq s,$$

$$|2a_2| = |-s_2 - a_1 s_1| \leq |s_2| + |a_1|\,|s_1| \leq s + s^2,$$

$$|3a_3| \leq |-s_3 - a_1 s_2 - s_2 s_1| \leq |s_3| + |a_1|\,|s_2| + |a_2|\,|s_1|$$

$$\leq s\left(1 + \binom{s}{1} + \binom{s+1}{2}\right) = s\binom{s+2}{2},$$

$$|4a_4| \leq |-s_4 - a_1 s_3 - a_2 s_2 - a_3 s_1| \leq |s_4| + |a_1|\,|s_3| + |a_2|\,|s_2| + |a_3|\,|s_1|$$

$$\leq s\left(1 + \binom{s}{1} + \binom{s+1}{2} + \binom{s+2}{3}\right) = s\binom{s+3}{3},$$

or, more elegantly,

$$|a_1| \leq \binom{s}{1}, |a_2| \leq \binom{s+1}{2}, |a_3| \leq \binom{s+2}{3}, |a_4| \leq \binom{s+3}{4}.$$

An easy induction shows

$$|a_j| \le \binom{s+j-1}{j} \quad (j = 1, 2, \ldots, n).$$

Since $P(1) = 0$ implies $1 = -(a_1 + \cdots + a_n)$, taking absolute values and applying the triangle inequality yield

$$1 = |a_1 + \cdots + a_n| \le |a_1| + \cdots + |a_n| \le \binom{s}{1} + \cdots + \binom{s+n-1}{n} = \binom{s+n}{n} - 1,$$

we have

$$\begin{aligned} 2 &\le \frac{(s+n)(s+n-1)(s+n-2)\cdots(s+1)}{1\cdot 2\cdot 3\cdots n} \\ &\le \left(1+\frac{s}{n}\right)\left(1+\frac{s}{n-1}\right)\left(1+\frac{s}{n-2}\right)\cdots\left(1+\frac{s}{1}\right) \\ &< \exp\left\{s\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\}. \end{aligned}$$

The last inequality holds because $e^x > 1 + x$ if $x > 0$. Hence

$$\log 2 < s\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right),$$

and solving for s completes the proof. QED

Concerning upper bounds, it is trivial that $R_n \le 1$, but it is not known whether one can replace the constant $\frac{1}{2}$ by a function tending to 1 or some other limiting value as $n \to \infty$. This is Problem 12 in Turán's monograph [1, p. 190], which is an enlargement of his "Eine neue Methode in der Analysis und deren Anwendungen." In 1995, A. Y. Cheer and D. A. Goldston [6] used Lawrynowicz's variational method to compute R_n and showed that an upper bound for R_n decreases for $n \le 55$, which suggested that R_n decreases to a limiting value less than 0.7 as $n \to \infty$. Very recently, Biró (Acta Math. Sci. Hungar., to appear) showed that $R_n < \frac{5}{6}$ for large n, which is the sharpest known upper bound.

Chapter 2

Atkinson's Method

This chapter offers a summary of Atkinson's method, refinements for the method, and computations on the Atkinson numbers.

2.1 Proof of Atkinson's theorem

Atkinson's analytic attack upon Turán's pure power sum problem relies heavily on the Cauchy-Schwarz inequality and Parseval's identity. Without any suggestion that this is a precise value, Atkinson showed

Theorem 3 (Atkinson). *The inequality $R_n > \frac{1}{6}$ holds.*

Proof: As in the proof of Theorem 2, there is no loss in generality to assume that $z_1 = 1$. By the power series

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{m=1}^{\infty} \frac{x^m}{m} \quad (|x| < 1),$$

we have

$$\log \prod_{k=1}^{n} \frac{1}{1 - z_k z} = \sum_{k=1}^{n} \log \frac{1}{1 - z_k z} = \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{(z_k z)^j}{j} = \sum_{j=1}^{\infty} \frac{z^j}{j} \sum_{k=1}^{n} z_k^j = \sum_{j=1}^{\infty} \frac{s_j}{j} z^j$$

for $|z_k z| < 1$, so

$$\exp\left\{-\sum_{j=1}^{\infty} \frac{s_j}{j} z^j\right\} = \prod_{k=1}^{n} (1 - z_k z)$$

for small z. By decomposing the sum into two parts, $\sum_{j=1}^{\infty} = \sum_{j=1}^{n} + \sum_{j\geq n+1}$, and applying the Taylor series of e^z, we obtain

$$\begin{aligned}
\exp\left\{-\sum_{j=1}^{n}\frac{s_j}{j}z^j\right\} &= \exp\left\{\sum_{j\geq n+1}\frac{s_j}{j}z^j\right\}\prod_{k=1}^{n}(1-z_k z) \\
&= \left\{1+\sum_{j\geq n+1}c_j z^j\right\}\prod_{k=1}^{n}(1-z_k z) \\
&= \prod_{k=1}^{n}(1-z_k z)+\sum_{j\geq n+1}c'_j z^j,
\end{aligned}$$

the coefficients being $c_j = c_j(z_1, \ldots, z_n)$ and $c'_j = c'_j(z_1, \ldots, z_n)$. Here we note that the left-hand side and the first term on the right are entire functions, and so the power series $\sum_{j\geq n+1} c'_j z^j$ is also entire. Hence this identity holds for all complex numbers z.

Let $z = e^{i\theta}$, and, for the sake of brevity, let

$$g(\theta) = -\sum_{j=1}^{n}\frac{s_j}{j}e^{ij\theta}.$$

Then

$$e^{g(\theta)} = \prod_{k=1}^{n}\left(1-z_k e^{i\theta}\right)+\sum_{j\geq n+1}c'_j e^{ij\theta}.$$

The special case $\theta = 0$ is of particular interest, for then

$$e^{g(0)} = \sum_{j\geq n+1} c'_j, \tag{2.1}$$

where the c'_j can be readily calculated by integration by parts as the Fourier coefficients of $e^{g(\theta)}$:

$$\begin{aligned}
c'_j &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta}\left\{e^{g(\theta)}-\prod_{k=1}^{n}\left(1-z_k e^{i\theta}\right)\right\}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta}e^{g(\theta)}\,d\theta \qquad \text{(because } j\geq n+1\text{)} \\
&= \frac{1}{2\pi i j}\left\{-\,e^{-ij\theta+g(\theta)}\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi} g'(\theta)\,e^{-ij\theta+g(\theta)}\,d\theta\right\} \\
&= \frac{1}{2\pi i j}\int_{-\pi}^{\pi} g'(\theta)\,e^{-ij\theta+g(\theta)}\,d\theta.
\end{aligned} \tag{2.2}$$

Then

$$\epsilon^{g(0)} = \frac{1}{2\pi i} \sum_{j \geq n+1} \frac{1}{j} \int_{-\pi}^{\pi} g'(\theta)\, \epsilon^{-ij\theta + g(\theta)}\, d\theta.$$

By the theory of mean-square convergence, we have a fortuitous opportunity to invert the order of summation and integration.

Now let

$$h(\theta) = \sum_{j \geq n+1} \frac{\epsilon^{-ij\theta}}{j} \quad (-\pi \leq \theta \leq \pi, \theta \neq 0).$$

and let

$$\psi(\theta) = g(\theta) - g(0) = -\sum_{j=1}^{n} \frac{s_j}{j} \left(\epsilon^{ij\theta} - 1\right).$$

Then

$$1 = \frac{1}{2\pi i} \int_{-\pi}^{\pi} g'(\theta)\, \epsilon^{\psi(\theta)} h(\theta)\, d\theta. \tag{2.3}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned} 1 &\leq \left| \frac{1}{2\pi i} \int_{-\pi}^{\pi} g'(\theta)\, \epsilon^{\psi(\theta)} h(\theta)\, d\theta \right|^2 \\ &\leq \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} |g'(\theta)|^2\, d\theta \int_{-\pi}^{\pi} \left|\epsilon^{\psi(\theta)} h(\theta)\right|^2 d\theta \\ &\leq \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} |g'(\theta)|^2\, d\theta \int_{-\pi}^{\pi} \left|\epsilon^{\psi(\theta)}\right|^2 |h(\theta)|^2\, d\theta. \end{aligned}$$

By Parseval's identity and (1.3),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |g'(\theta)|^2\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| -i \sum_{j=1}^{n} s_j \epsilon^{ij\theta} \right|^2 d\theta = \sum_{j=1}^{n} |s_j|^2 \leq ns^2.$$

Thus

$$1 \leq \frac{ns^2}{2\pi} \int_{-\pi}^{\pi} \left|\epsilon^{\psi(\theta)}\right|^2 |h(\theta)|^2\, d\theta \leq \frac{ns^2}{2\pi} \int_{-\pi}^{\pi} \epsilon^{2|\psi(\theta)|} |h(\theta)|^2\, d\theta. \tag{2.4}$$

Atkinson's argument consists of showing that if s is too small, then the coefficients c_j' are so small — because $g(\theta)$ and $g'(\theta)$ are both small — that (2.1) cannot hold. Here we note that s is bounded below from zero. To make the remark more precise, we shall estimate the integrand in (2.4). Let us divide the integral over $|\theta| \leq \pi$ into ranges. Symbolically we may write $\int_{-\pi}^{\pi} = \int_{-\pi}^{-\pi/n} + \int_{-\pi/n}^{\pi/n} + \int_{\pi/n}^{\pi}$. We consider first the case $|\theta| \leq \frac{\pi}{n}$ and give a pointwise bound for $w(\theta)$ that is valid for all s_j and satisfies (1.3). By the use of $|\sin \alpha| \leq \sin |\alpha| \leq |\alpha|$ and $|\theta| \leq \frac{\pi}{n}$, we estimate that

$$
\begin{aligned}
|w(\theta)| &= \left| -\sum_{j=1}^{n} \frac{s_j}{j} \left(e^{ij\theta} - 1\right) \right| \\
&\leq \sum_{j=1}^{n} \frac{|s_j|}{j} \left| e^{ij\theta} - 1 \right| \\
&= \sum_{j=1}^{n} \frac{|s_j|}{j} \left| 2ie^{ij\theta/2} \left(\frac{e^{ij\theta/2} - e^{-ij\theta/2}}{2i} \right) \right| \\
&= \sum_{j=1}^{n} \frac{|s_j|}{j} \left| 2 \sin \frac{j\theta}{2} \right| \\
&\leq \sum_{j=1}^{n} 2 \frac{|s_j|}{j} \sin \left| \frac{j\theta}{2} \right| \\
&\leq \sum_{j=1}^{n} 2 \frac{|s_j|}{j} \left| \frac{j\theta}{2} \right| \\
&= \sum_{j=1}^{n} |s_j| \, |\theta| \\
&\leq ns \, |\theta| \\
&\leq \pi s.
\end{aligned}
$$

By Parseval's identity and the Integral Test,

$$
\begin{aligned}
\int_{-\pi/n}^{\pi/n} e^{2|w(\theta)|} |h(\theta)|^2 \, d\theta &\leq \int_{-\pi/n}^{\pi/n} e^{2\pi s} |h(\theta)|^2 \, d\theta \\
&\leq e^{2\pi s} \int_{-\pi}^{\pi} |h(\theta)|^2 \, d\theta \\
&= 2\pi e^{2\pi s} \sum_{j \geq n+1} \frac{1}{j^2} \\
&< 2\pi e^{2\pi s} \int_{n}^{\infty} x^{-2} \, dx \\
&= \frac{2\pi e^{2\pi s}}{n}. \qquad (2.5)
\end{aligned}
$$

We now obtain upper estimations for the quantities $\psi(\theta)$ and $h(\theta)$ for $\frac{\pi}{n} \leq \theta \leq \pi$.

Claim 1. *For $1 \leq a < b$, the inequality*

$$\sum_{a<n\leq b} \frac{1}{n} \leq \log\frac{b}{a} + 1$$

holds.

Proof: Since $a \geq 1$, we have

$$\begin{aligned}\sum_{a<n\leq b} \frac{1}{n} &= \sum_{[a]+1\leq n\leq [b]} \frac{1}{n} \\ &\leq \int_{[a]}^{[b]} x^{-1}\,dx \\ &\leq \int_a^b x^{-1}\,dx + [a]^{-1} \\ &\leq \log\frac{b}{a} + 1,\end{aligned}$$

which proves the claim. QED

Since

$$\begin{aligned}|\psi(\theta)| &\leq \sum_{j=1}^{n} \frac{|s_j|}{j}\left|e^{ij\theta} - 1\right| \\ &= \sum_{j=1}^{n} \frac{|s_j|}{j}\left|2\sin\frac{j\theta}{2}\right| \\ &\leq \sum_{j=1}^{n} 2\frac{|s_j|}{j}\min\left(\left|\frac{j\theta}{2}\right|, 2\right) \\ &\leq \sum_{j\leq\pi/\theta} 2\frac{|s_j|}{j}\left|\frac{j\theta}{2}\right| + \sum_{\pi/\theta<j\leq n} 2\frac{|s_j|}{j} \\ &\leq s\left(\pi + 2\sum_{\pi/\theta<j\leq n}\frac{1}{j}\right) \\ &\leq s\pi + 2s\left(\log\frac{n}{\frac{\pi}{\theta}} + 1\right) \qquad \text{(by Claim 1)} \\ &\leq s(\pi+2) + s\log\left(\frac{n\theta}{\pi}\right)^2.\end{aligned}$$

we have

$$\epsilon^{|\psi(\theta)|} \le \epsilon^{s(\pi+2)} \left(\frac{n\theta}{\pi}\right)^{2s}.$$

This bound grows exponentially in s.

The function $h(\theta)$ will control the growth in n in the quantity $\epsilon^{\psi(\theta)}$, since

$$\left|\int h(\theta)\, d\theta\right| = \left|\sum_{j\ge n+1} \frac{\epsilon^{ij\theta}}{ij^2}\right| \le \sum_{j\ge n+1} \frac{1}{j^2} < \int_n^\infty x^{-2}\, dx = \frac{1}{n}.$$

Summation by parts yields

$$\begin{aligned}
\left(1-\epsilon^{-i\theta}\right) h(\theta) &= \sum_{j\ge n+1} \frac{\epsilon^{-ij\theta}}{j} - \sum_{j\ge n+1} \frac{\epsilon^{-i(j+1)\theta}}{j} \\
&= \sum_{j\ge n+1} \frac{\epsilon^{-ij\theta}}{j} - \sum_{j\ge n+2} \frac{\epsilon^{-ij\theta}}{j-1} \\
&= \left(\frac{\epsilon^{-i(n+1)\theta}}{n+1} + \sum_{j\ge n+2} \frac{\epsilon^{-ij\theta}}{j}\right) - \sum_{j\ge n+2} \frac{\epsilon^{-ij\theta}}{j-1} \\
&= \frac{\epsilon^{-i(n+1)\theta}}{n+1} + \sum_{j\ge n+2} \epsilon^{-ij\theta}\left(\frac{1}{j} - \frac{1}{j-1}\right).
\end{aligned}$$

so

$$\begin{aligned}
\left|\left(1-\epsilon^{-i\theta}\right) h(\theta)\right| &= \left|1-\epsilon^{-i\theta}\right| |h(\theta)| \\
&= \left|2\sin\frac{\theta}{2}\right| |h(\theta)| \\
&\le \frac{1}{n+1} + \sum_{j\ge n+2} \left(\frac{1}{j-1} - \frac{1}{j}\right) \\
&= \frac{1}{n+1} + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) \\
&\quad + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \left(\frac{1}{n+4} - \frac{1}{n+5}\right) + \cdots \\
&= \frac{1}{n+1} + \frac{1}{n+1} \\
&= \frac{2}{n+1}.
\end{aligned}$$

By the use of the fact that $|\sin\alpha| \ge \frac{2|\alpha|}{\pi}$ for $0 \le |\alpha| \le \frac{\pi}{2}$, it is recognized that

$|\csc \alpha| \le \frac{\pi}{2|\alpha|}$, and hence

$$|h(\theta)| \le (n+1)^{-1} \left|\csc \frac{\theta}{2}\right| \le (n+1)^{-1} \frac{\pi}{|\theta|} \le \frac{\pi}{n|\theta|}$$

for $\left|\frac{\theta}{2}\right| \le \frac{\pi}{2}$, or $|\theta| \le \pi$. Thus collecting all estimates we have

$$\begin{aligned}
\int_{\pi/n}^{\pi} \epsilon^{2|v(\theta)|} |h(\theta)|^2 \, d\theta &\le \epsilon^{2s(\pi+2)} \int_{\pi/n}^{\pi} \left(\frac{n\theta}{\pi}\right)^{4s} \left(\frac{\pi}{n\theta}\right)^2 d\theta \\
&= \epsilon^{2s(\pi+2)} \int_{\pi/n}^{\pi} \left(\frac{n\theta}{\pi}\right)^{4s-2} d\theta \\
&= \epsilon^{2s(\pi+2)} \int_{1}^{n} u^{4s-2} \left(\frac{\pi}{n}\right) du \\
&< \frac{\pi\epsilon^{2s(\pi+2)}}{n} \int_{1}^{\infty} u^{4s-2} \, du \\
&= \left.\frac{\pi\epsilon^{2s(\pi+2)} u^{4s-1}}{n(4s-1)}\right|_{1}^{\infty} \\
&= \frac{\pi\epsilon^{2s(\pi+2)}}{n(1-4s)}
\end{aligned}$$

if $s < \frac{1}{4}$. (Otherwise, we are done.) We have a similar result to the integral over $-\pi \le \theta \le -\frac{\pi}{n}$. Substituting the last estimate and (2.5) into (2.4) we obtain

$$1 < \frac{ns^2}{2\pi} \left(\frac{2\pi\epsilon^{2s\pi}}{n} + \frac{2\pi\epsilon^{2s(\pi+2)}}{n(1-4s)}\right) = s^2\epsilon^{2s\pi} \left(1 + \frac{\epsilon^{4s}}{1-4s}\right). \tag{2.6}$$

Since the right-hand side is independent of n and monotonic increasing in $0 \le s < \frac{1}{4}$, and its value at $s = \frac{1}{6}$ is $0.5416877131\cdots$, we conclude that $s > \frac{1}{6}$. Thus (2.6) implies the validity of Theorem 3. QED

2.2 Refinements

A further improvement in the bound $R_n > \frac{1}{6}$ may be developed by using an estimate for $v(\theta)$ due to Montgomery. Let $[\alpha]$ be the integer part of α and $\{\alpha\}$ be the fractional part, so that $\alpha = [\alpha] + \{\alpha\}$. Montgomery showed that the quantity ϵ^{4s} in (2.6) can be deleted using the estimate

$$
\begin{aligned}
|\psi(\theta)| &\le \sum_{j=1}^{n} \frac{|s_j|}{j} \left|e^{ij\theta} - 1\right| \\
&= \sum_{j=1}^{n} \frac{|s_j|}{j} \left|2\sin\frac{j\theta}{2}\right| \\
&\le \sum_{j=1}^{n} 2\frac{|s_j|}{j} \min\left(\left|\frac{j\theta}{2}\right|, 2\right) \\
&\le \sum_{j\le\pi/\theta} |s_j|\,|\theta| + \sum_{\pi/\theta<j\le n} 2\frac{|s_j|}{j} \\
&\le s\left(\sum_{j\le\pi/\theta} \theta + 2\sum_{\pi/\theta<j\le n} \frac{1}{j}\right) \\
&\le s\left(\theta\left[\frac{\pi}{\theta}\right] + 2\log\frac{n}{\left[\frac{\pi}{\theta}\right]}\right) \\
&\le s\left(\theta\left(\frac{\pi}{\theta} - \left\{\frac{\pi}{\theta}\right\}\right) + 2\log\left(\frac{n}{\frac{\pi}{\theta} - \left\{\frac{\pi}{\theta}\right\}}\right)\right) \\
&= s\left(\pi - \theta\left\{\frac{\pi}{\theta}\right\} + 2\log\left(\frac{\frac{n\theta}{\pi}}{1 - \left\{\frac{\pi}{\theta}\right\}/\left(\frac{\pi}{\theta}\right)}\right)\right) \\
&= s\left(\pi + 2\log\frac{n\theta}{\pi} + 2\log\left(1 - \frac{\left\{\frac{\pi}{\theta}\right\}}{\frac{\pi}{\theta}}\right)^{-1} - \pi\frac{\left\{\frac{\pi}{\theta}\right\}}{\frac{\pi}{\theta}}\right) \\
&\le s\left(\pi + 2\log\frac{n\theta}{\pi}\right).
\end{aligned}
$$

The last inequality holds because $2\log(1-x)^{-1} < \pi x$ for $0 \le x \le \frac{1}{2}$ and $0 \le \left\{\frac{\pi}{\theta}\right\}/\left(\frac{\pi}{\theta}\right) < \frac{1}{2}$ for $0 < \theta \le \pi$ trivially. Thus

$$
e^{|\psi(\theta)|} \le e^{s\pi}\left(\frac{n\theta}{\pi}\right)^{2s}.
$$

A similar argument applies to $-\pi \le \theta \le -\frac{\pi}{n}$. Proceeding analogously as in the proof of Theorem 3, Montgomery obtained

$$
\int_{\pi/n}^{\pi} e^{2|\psi(\theta)|}\,|h(\theta)|^2\,d\theta \le e^{2s\pi}\int_{\pi/n}^{\pi}\left(\frac{n\theta}{\pi}\right)^{4s-2} d\theta < \frac{\pi e^{2s\pi}}{n}\int_1^{\infty} u^{4s-2}\,du = \frac{\pi e^{2s\pi}}{n(1-4s)}
$$

if $s < \frac{1}{4}$. Consequently,

$$1 < s^2 e^{2s\pi} \left(1 + \frac{1}{1 - 4s} \right).$$

The right-hand side is monotonic increasing in $0 \le s < \frac{1}{4}$, and its value at $s = \frac{1}{5}$ is $0.8432605498\cdots$, so $s > \frac{1}{5}$. Summing up we have:

Theorem 4 (Montgomery). *The inequality $R_n > \frac{1}{5}$ holds.*

We can replace $\frac{1}{5}$ by $\frac{10}{47}$. By orthogonality

$$\int_{-\pi}^{\pi} g'(\theta)\, h(\theta)\, d\theta = 0,$$

so that

$$1 = \frac{1}{2\pi i} \int_{-\pi}^{\pi} g'(\theta) \left(e^{\psi(\theta)} - 1 \right) h(\theta)\, d\theta.$$

The Cauchy-Schwarz inequality and Parseval's identity apply and show that

$$1 \le \frac{ns^2}{2\pi} \int_{-\pi}^{\pi} \left| e^{\psi(\theta)} - 1 \right|^2 |h(\theta)|^2\, d\theta \le \frac{ns^2}{2\pi} \int_{-\pi}^{\pi} \left(e^{|\psi(\theta)|} - 1 \right)^2 |h(\theta)|^2\, d\theta. \qquad (2.7)$$

The last inequality holds because $\left| e^{\psi_n(\theta)} - 1 \right| \le e^{|\psi_n(\theta)|} - 1$. By a computation,

$$\int_{-\pi/n}^{\pi/n} \left(e^{|\psi(\theta)|} - 1 \right)^2 |h(\theta)|^2\, d\theta < \frac{2\pi}{n} \left(e^{s\pi} - 1 \right)^2.$$

Applying Montgomery's estimate for $\psi(\theta)$, we obtain

$$\int_{\pi/n}^{\pi} \left(e^{|\psi(\theta)|} - 1 \right)^2 |h(\theta)|^2\, d\theta < \frac{\pi}{n} \left(\frac{e^{2s\pi}}{1 - 4s} - \frac{2e^{s\pi}}{1 - 2s} + 1 \right)$$

if $s < \frac{1}{4}$. We have a similar result to the integral over $-\pi \le \theta \le -\frac{\pi}{n}$. Following the argument of proof of (2.6), we obtain

$$1 < s^2 \left(\left(e^{s\pi} - 1 \right)^2 + \frac{e^{2s\pi}}{1 - 4s} - \frac{2e^{s\pi}}{1 - 2s} + 1 \right).$$

The right-hand side is increasing in s, and its value at $s = \frac{10}{47}$ is $0.9358636980\cdots$, so $s > \frac{10}{47}$. As a result we have:

Theorem 5. *The inequality $R_n > \frac{10}{47}$ holds.*

2.3 Atkinson numbers

We now report on some computations on the Atkinson numbers which verify our knowledge of R_n. All of our computations make use of *Mathematica* running on an IBM ThinkPad 600X.

We use

$$|\psi_n(\theta)| \leq s_n \chi_n(\theta) \quad \text{and} \quad \chi_n(\theta) = \sum_{j=1}^{n} \frac{2}{j} \left|\sin \frac{j\theta}{2}\right|$$

to compute numerically $\psi_n(\theta)$, $s_n\chi_n(\theta)$ being a finite sum. The quantity $h_n(\theta)$ can be computed numerically (and exactly) using the power series

$$-\ln(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j} \quad (|z| < 1).$$

where the logarithm has its principal value. We can evaluate the right-hand side at $z = e^{-i\theta} \neq 1$ even though the point lies on the circle of convergence $|z| = 1$. Hence by a theorem of Abel (see Whittaker and Watson [8, p. 57]), we have

$$\lim_{z \to e^{-i\theta}} \{-\ln(1-z)\} = \sum_{j=1}^{\infty} \frac{e^{-ij\theta}}{j}.$$

On the other hand, since $\ln(1-z)$ makes sense for $\mathbb{C} - [1, \infty)$, we can evaluate the left-hand side of the power series at $z = e^{-i\theta}$, decompose the sum into two parts as in the proof of Theorem 3, and apply the definition of $h_n(\theta)$ on the right-hand side to obtain the closed form

$$-\ln\left(1 - e^{-i\theta}\right) = \sum_{j=1}^{n} \frac{e^{-ij\theta}}{j} + h_n(\theta).$$

We now write Atkinson's basic inequality (2.4) as

$$1 \leq \frac{n s_n^2}{2\pi} \int_{-\pi}^{\pi} \exp\{2 s_n \chi_n(\theta)\} |h_n(\theta)|^2 \, d\theta. \tag{2.8}$$

and define the Atkinson numbers $A_n^{(1)} = 0.a_{-1}a_{-2}\cdots a_{-i}\cdots$ to be the numbers s_n that satisfy the identity in (2.8). We compute these numbers by finding the largest

TABLE 1. Values of $A_n^{(1)}$ and $A_n^{(2)}$.

n	$A_n^{(1)}$	$A_n^{(2)}$	n	$A_n^{(1)}$	$A_n^{(2)}$
1	0.7385342869	0.9077423015	21	0.4485347564	0.4934819162
2	0.6119802868	0.7247445753	22	0.4470586905	0.4914153119
3	0.5637748126	0.6564875608	23	0.4456825121	0.4894896157
4	0.5364876686	0.6175338028	24	0.4443870066	0.4876613561
5	0.5190871119	0.5930151348	25	0.4431618339	0.4859584999
6	0.5062957578	0.5748063786	26	0.4420182757	0.4843594553
7	0.4968117930	0.5614572908	27	0.4409363871	0.4828460645
8	0.4891682498	0.5506444835	28	0.4399066024	0.4814053086
9	0.4829518040	0.5418743128	29	0.4389379073	0.4800519426
10	0.4777323312	0.5341536889	30	0.4380071825	0.4787224663
11	0.4733176711	0.5283101903	31	0.4370436499	0.4775262308
12	0.4694174226	0.5228017336	32	0.4362904195	0.4763523344
13	0.4660543914	0.5180802100	33	0.4354839775	0.4752264704
14	0.4630517847	0.5138602284	34	0.4347231629	0.4741635155
15	0.4603532618	0.5100651866	35	0.4339870202	0.4731361616
16	0.4579170545	0.5066415259	36	0.4332787875	0.4721465462
17	0.4557207905	0.5035625033	37	0.4326022493	0.4712019362
18	0.4536938312	0.5007155552	38	0.4319519959	0.4702961237
19	0.4518467654	0.4981276460	39	0.4313264805	0.4694237415
20	0.4501254675	0.4957118384	40	0.4307220739	0.4685380356

value for a_{-1} that does not contradict the identity, and proceed in this fashion to find a_{-2}, a_{-3}, a_{-4}, and so on. The results of our computations are indicated in Table 1. The values are truncated at 10 digits. It will be noted at once that Atkinson's method gives a better lower bound than Biró's method for $n \leq 6$. For larger values of n, we obtain $A_{100}^{(1)} = 0.4127148745\cdots$ and $A_{1000}^{(1)} = 0.3870053970\cdots$. The values of $A_n^{(1)}$ can be raised by replacing the quantity $e^{\psi_n(\theta)}$ by $e^{\psi_n(\theta)} - 1$ and applying the inequality $|e^{\psi_n(\theta)} - 1| \leq e^{|\psi_n(\theta)|} - 1$ in (2.4), so that

$$1 \leq \frac{ns_n^2}{2\pi} \int_{-\pi}^{\pi} [\exp\{s_n \chi_n(\theta)\} - 1]^2 \, |h_n(\theta)|^2 \, d\theta.$$

the constant 1 being best possible. We define the $A_n^{(2)}$ similarly and place the results in Table 1. There are two consequences: The slightly higher values for $A_n^{(2)}$ and the tripling of the configuration of points that exceed $\frac{1}{2}$. Thus the second approach gives a better lower bound than Biró's method for $n \leq 18$. For larger values of n, we obtain

$A_{100}^{(2)} = 0.4435535502\cdots$ and $A_{1000}^{(2)} = 0.4082201908\cdots$.

2.4 Notes and comments

By an argument in [3] (see Atkinson's proof of Lemma 9, p. 202), we have

$$v_n(\theta) = 2s_n\Re\sum_{j=1}^{n}\frac{1}{j}\left(1-e^{ij\theta}\right) = 2s_n\Re\sum_{j=1}^{n}\frac{1}{j}e^{ij\theta/2}\cos\frac{j\theta}{2} = 2s_n\sum_{j=1}^{n}\frac{1}{j}\cos^2\frac{j\theta}{2}.$$

Applying this quantity in (2.4), we obtain

$$1 \le \frac{ns_n^2}{2\pi}\int_{-\pi}^{\pi}\exp\left\{4s_n\sum_{j=1}^{n}\frac{1}{j}\cos^2\frac{j\theta}{2}\right\}|h(\theta)|^2\,d\theta.$$

Letting $A_n^{(3)}$ be the numbers s_n in the above equality, we then find that the $A_n^{(3)}$ to be substantially lower than both $A_n^{(1)}$ and $A_n^{(2)}$. The values in Table 2 indicate that the third approach yields weaker results.

Atkinson in his paper [3] obtained a more precise estimate for $h(\theta)$ for $0 \le |\theta| \le \pi$. Using

$$\left|\sin\frac{(\theta-it)}{2}\right|^2 = \sin^2\frac{\theta}{2} + \sinh^2\frac{\theta}{2} \ge \sin^2\frac{\theta}{2},$$

he showed

$$\begin{aligned}|h(\theta)| &= \left|\int_0^\infty e^{-(n+1/2)t}\left\{2\sin\frac{(\theta-it)}{2}\right\}^{-1}dt\right| \\ &\le \left(2\sin\frac{\theta}{2}\right)^{-1}\int_0^\infty e^{-(n+1/2)t}\,dt \\ &= (2n+1)^{-1}\left(\sin\frac{\theta}{2}\right)^{-1}. \end{aligned} \tag{2.9}$$

He then proved

Lemma 1. *For α, such that $0 \le \alpha \le \frac{\pi}{2}$, we have*

$$|h(\theta)|^2 \le \left(n+\frac{1}{2}\right)^{-2}\left(\theta^{-2} + 4\csc^2\frac{\alpha}{2}\right) \quad (0 \le \theta \le \alpha) \tag{2.10}$$

$$|h(\theta)|^2 \le (2n+1)^{-1}\csc^2\frac{\alpha}{2} \quad (\alpha \le \theta \le \pi). \tag{2.11}$$

TABLE 2. Values of $A_n^{(3)}$.

n	$A_n^{(3)}$	n	$A_n^{(3)}$
1	0.4964674666	11	0.2461145156
2	0.3832913737	12	0.2418943979
3	0.3375131378	13	0.2381568545
4	0.3113013527	14	0.2348134521
5	0.2937770015	15	0.2317970776
6	0.2809790257	20	0.2201317919
7	0.2710841091	25	0.2119811005
8	0.2631228354	30	0.2058264319
9	0.2565264822	35	0.2009399630
10	0.2509365879	40	0.1969218683

Proof: In view of a Mittag-Leffler partial fraction expansion.

$$\csc^2\frac{\theta}{2} - 4\theta^{-2} = 4\sum_{q=1}^{\infty}\left\{(2q\pi-\theta)^{-2} + (2q\pi+\theta)^{-2}\right\}.$$

The right-hand side increases as θ increases in $(0, \pi/2)$. so

$$\csc^2\frac{\theta}{2} - 4\theta^{-2} \le \csc^2\frac{\alpha}{2} - 4\alpha^{-2} \le \csc^2\frac{\alpha}{2}$$

for $0 \le \theta \le \alpha$. Hence (2.10) follows from (2.9). Similarly. we get (2.11) from (2.9) on noting that $\csc\frac{\theta}{2}$ decreases as θ increases in $\alpha \le \theta \le \pi$. QED

There are similar results to (2.9). (2.10). and (2.11) for negative θ. We apply these bounds for $h(\theta)$ in (2.9). for some $\alpha. 0 \le \alpha \le \frac{\pi}{2}$. we apply (2.10) over $(0, \alpha)$. and similarly over $(-\alpha, 0)$. We apply (2.11) over (α, π). and likewise over $(-\pi, -\alpha)$. Combining these results we get another form of the basic inequality

$$1 \le \frac{ns^2}{2\pi\left(n+\frac{1}{2}\right)^2}\int_{-\alpha}^{\alpha}\left(e^{|\psi(\theta)|}-1\right)^2\theta^{-2}\,d\theta + \frac{ns^2}{2\pi\left(n+\frac{1}{2}\right)^2 4\sin^2\left(\frac{\alpha}{2}\right)}\int_{-\pi}^{\pi}\left(e^{|\psi(\theta)|}-1\right)^2 d\theta.$$

the first integral being dominated by

$$\frac{ns^2}{\pi\left(n+\frac{1}{2}\right)^2}\int_0^{\alpha}\left(e^{s\chi(\theta)}-1\right)^2\theta^{-2}\,d\theta.$$

Here we have used the fact that $|\psi(\theta)| \le s\chi(\theta)\,.\,e^{|\psi(\theta)|} - 1 \le e^{s\chi(\theta)} - 1$. and $\chi(\theta)$ is an even function of θ. For further details we refer to [3]. Our computations reveal that the last approach also fails to yield sharper Atkinson numbers.

Chapter 3

Biró's Method

3.1 Proof of Biró's theorem

Now let us turn to the proof of Biró's theorem. Biró, at the time a third year student at the University of Budapest, showed

Theorem 6 (Biró). *The inequality $R_n > \frac{1}{2}$ holds.*

Proof: Let us replace k by j in (1.2), and derive the Newton-Girard formulas for $Q(z)$ by obtaining the generating function for s_k. The logarithmic derivative of $P(z)$ for $\left|\frac{z_j}{z}\right| < 1$, given by $\frac{P'(z)}{P(z)}$, is the quantity

$$\sum_{j=1}^{n} \frac{1}{z - z_j} = \sum_{j=1}^{n} \frac{1}{z}\left(\frac{1}{1 - \frac{z_j}{z}}\right) = \sum_{j=1}^{n} \frac{1}{z} \sum_{k=1}^{\infty} \left(\frac{z_j}{z}\right)^k = \sum_{k=1}^{\infty} \frac{1}{z^{k+1}} \sum_{j=1}^{n} z_j^k = \sum_{k=1}^{\infty} \frac{s_k}{z^{k+1}}.$$

Moreover,

$$P'(z) = P(z) \sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}} \quad (s_0 = n).$$

since

$$\frac{P'(z)}{P(z)} = \frac{n}{z} + \frac{s_1}{z^2} + \frac{s_2}{z^3} + \cdots = \sum_{k=0}^{\infty} \frac{s_k}{z^{k+1}}$$

with $s_0 = n$. By (1.2), we have

$$\sum_{j=0}^{n} (n - j)\, a_j z^{n-j-1} = \sum_{j=0}^{n} a_j z^{n-j-1} \sum_{k=0}^{\infty} s_k z^{-k} = \sum_{j=0}^{n} \sum_{k=0}^{\infty} a_j s_k z^{n-j-k-1}.$$

Let $m = j + k$. Then $j = m - k$, $j \leq m$, and $k \leq m$. If $0 \leq j \leq n$, then $m - n \leq k$. Since $0 \leq k$ also, the lower limit for the inner sum is then $\max(0, m - n) \leq k$. Applying these conditions on the right-hand side and replacing j by m on the left-hand side, we obtain

$$\sum_{m=0}^{n} (n-m)\, a_m z^{n-m-1} = \sum_{m=0}^{\infty} \left\{ \sum_{k \geq \max(0, m-n)} a_{m-k} s_k \right\} z^{n-m-1}.$$

Hence

$$\sum_{\max(0,m-n) \leq k \leq m} a_{m-k} s_k = \begin{cases} (n-m)\, a_m, & m = 0, 1, 2, \ldots, n; \\ 0, & m = n+1, n+2, \ldots. \end{cases}$$

We now write $P(z) = (z-1)\, Q(z)$, where

$$Q(z) = (z - z_2)(z - z_3) \cdots (z - z_n) = z^{n-1} + b_1 z^{n-2} + \cdots + b_{n-1},$$

and derive the Newton-Girard formulas for $Q(z)$ by computing for $m = 0, 1, 2, \ldots, n, n+1, n+2, \ldots$ the following:

- $\sum_{k \geq \max(0,-n)}^{0} a_{0-k} s_k = n a_0,\ a_0 s_0 = n a_0,\ s_0 = n;$
  $\sum_{k=0}^{1} a_{1-k} s_k = (n-1)\, a_1,\ a_1 s_0 + a_0 s_1 = (n-1)\, a_1,\ s_1 + a_1 = 0;$
  $\sum_{k=0}^{2} a_{2-k} s_k = (n-2)\, a_2,\ a_2 s_0 + a_1 s_1 + a_0 s_2 = (n-2)\, a_2,\ s_2 + a_1 s_1 + 2a_2 = 0;$
  $\vdots$
  $\sum_{k=0}^{m-1} a_k s_{m-k} + m a_m = 0 \quad (m = 1, 2, \ldots, n-1).$

- $\sum_{k=0}^{n} a_n s_k = 0,\ a_n s_0 + a_{n-1} s_1 + \cdots + a_0 s_n = 0,\ s_n + a_1 s_{n-1} + \cdots + n a_n = 0;$
  $\sum_{k=1}^{n+1} a_{n+1} s_k = 0,\ a_n s_1 + a_{n-1} s_2 + \cdots + a_1 s_n + s_{n+1} = 0;$
  $\sum_{k=1}^{n+2} a_{n+2} s_k = 0,\ a_n s_2 + a_{n-1} s_3 + \cdots + a_1 s_{n+1} + s_{n+2} = 0;$
  $\vdots$
  $a_n s_{m-n} + a_{n-1} s_{m-n+1} + \cdots + a_1 s_{m-1} + s_m = 0 \quad (m = n+1, n+2, \ldots).$

In general,

$$s_k + a_1 s_{k-1} + \cdots + a_{k-1} s_1 + k a_k = 0 \quad (k = 1, 2, \ldots, n). \tag{3.1}$$

$$s_{n+1} + a_1 s_n + a_2 s_{n-1} + \cdots + a_n s_1 = 0. \tag{3.2}$$

Since the jth pure power sum of $P(z)$ is

$$s_j = z_1^j + z_2^j + \cdots + z_n^j = 1 + z_2^j + \cdots + z_n^j = 1 + \widehat{s}_j,$$

where $\widehat{s}_j = \sum_{k=2}^{n} z_k^j$ is the jth pure power sum of $Q(z)$, it follows that by shifting the index n to $n-1$ and applying $\widehat{s}_j$ to (3.1) and (3.2) we have, respectively,

$$\widehat{s}_k + b_1\widehat{s}_{k-1} + \cdots + b_{k-1}\widehat{s} + kb_k = 0 \quad (k = 1, 2, \ldots, n-1),$$

$$\widehat{s}_n + b_1\widehat{s}_{n-1} + \cdots + b_{n-1}\widehat{s}_1 = 0.$$

Taking into consideration $\widehat{s}_j = s_j - 1$ we obtain the formulas

$$s_k + b_1 s_{k-1} + \cdots + b_{k-1}s_1 = 1 + b_1 + \cdots + b_{k-1} - kb_k \tag{3.3}$$

$$(k = 1, 2, \ldots, n-1),$$

$$s_n + b_1 s_{n-1} + \cdots + b_{n-1}s_1 = 1 + b_1 + \cdots + b_{n-1}. \tag{3.4}$$

We will make use of the following lemmas.

Lemma 2. *Let $z \neq 0$ be a complex number, let $0 < \alpha < \frac{\pi}{2}$, and let $A > 0$. Then either* (3.5) *or* (3.6) *holds:*

$$|1 - Az|^2 \geq \sin^2\alpha \left(1 + \frac{\cos^2\alpha}{A + \sin^2\alpha}\right). \tag{3.5}$$

$$|1 + z| \geq 1 + (\cos\alpha)|z|. \tag{3.6}$$

Proof: Let us verify (3.5). Let $z = r(\cos\phi + i\sin\phi)$ be the trigonometric form of z, and suppose that (3.6) is invalid. Then $|1 + z| < 1 + (\cos\alpha)|z|$. Since

$$|1 + z|^2 = |1 + r\cos\phi + ir\sin\phi|^2 = (1 + r\cos\phi)^2 + (r\sin\phi)^2 = 1 + 2r\cos\phi + r^2$$

and

$$(1 + (\cos\alpha)|z|)^2 = (1 + r\cos\alpha)^2 = 1 + 2r\cos\alpha + r^2\cos^2\alpha,$$

we have

$$1 + 2r\cos\phi + r^2 < 1 + 2r\cos\alpha + r^2\cos^2\alpha.$$

Simplifying, we obtain

$$2\cos\phi < 2\cos\alpha + r\left(\cos^2\alpha - 1\right) = 2\cos\alpha - r\sin^2\alpha.$$

Applying this and completing the square in Ar, we obtain

$$\begin{aligned}
|1 - Az|^2 &= |1 - Ar\cos\phi - iAr\sin\phi|^2 \\
&= 1 - 2Ar\cos\phi + A^2r^2\cos^2\phi + A^2r^2\sin^2\phi \\
&= 1 - 2Ar\cos\phi + A^2r^2 \\
&> 1 - Ar\left(2\cos\alpha - r\sin^2\alpha\right) + A^2r^2 \\
&= 1 - 2Ar\cos\alpha + Ar^2\sin^2\alpha + A^2r^2 \\
&= 1 - 2Ar\cos\alpha + A^2r^2\left(1 + \frac{1}{A}\sin^2\alpha\right) \\
&= \left(1 + \frac{1}{A}\sin^2\alpha\right)\left(A^2r^2 - \frac{2Ar\cos\alpha}{1 + \frac{1}{A}\sin^2\alpha}\right) + 1 \\
&= \left(1 + \frac{1}{A}\sin^2\alpha\right)\left(Ar - \frac{\cos\alpha}{1 + \frac{1}{A}\sin^2\alpha}\right)^2 - \frac{\cos^2\alpha}{1 + \frac{1}{A}\sin^2\alpha} + 1. \\
&\geq \left(1 + \frac{1}{A}\sin^2\alpha\right)\left(0 - \frac{\cos^2\alpha}{\left(1 + \frac{1}{A}\sin^2\alpha\right)^2}\right) + 1 \\
&\geq 1 - \frac{\cos^2\alpha}{1 + \frac{1}{A}\sin^2\alpha} \\
&= \frac{1 + \frac{1}{A}\sin^2\alpha - \cos^2\alpha}{1 + \frac{1}{A}\sin^2\alpha} \\
&= \frac{A + \sin^2\alpha - A\cos^2\alpha}{A + \sin^2\alpha} \\
&= \frac{A\left(1 - \cos^2\alpha\right) + \sin^2\alpha}{A + \sin^2\alpha} \\
&= \frac{\sin^2\alpha\,(1 + A)}{A + \sin^2\alpha} \\
&= \sin^2\alpha\left(\frac{A + \sin^2\alpha - \sin^2\alpha + 1}{A + \sin^2\alpha}\right) \\
&= \sin^2\alpha\left(1 + \frac{\cos^2\alpha}{A + \sin^2\alpha}\right).
\end{aligned}$$

so in this case (3.5) is valid, which proves the lemma. QED

Lemma 3. *Let $0 < \alpha < \frac{\pi}{2}$. If $1 \le k \le n-1$, then either* (3.7) *or* (3.8) *holds:*

$$|1 + b_1 + \cdots + b_{k-1} - kb_k| \ge (\sin\alpha)\,|1 + b_1 + \cdots + b_{k-1}|\,. \tag{3.7}$$

$$|1 + b_1 + \cdots + b_{k-1} + b_k| \ge |1 + b_1 + \cdots + b_{k-1}| + (\cos\alpha)\,|b_k|\,. \tag{3.8}$$

If (3.6) *is valid for $k = 1, 2, \ldots, s$ $(s \le n-1)$, then*

$$|1 + b_1 + \cdots + b_s| > (\cos\alpha)\,(1 + |b_1| + \cdots + |b_s|)\,. \tag{3.9}$$

Proof: The proof is by induction on k. If $1 + b_1 + \cdots + b_{k-1} = 0$ or $b_k = 0$, then both (3.7) and (3.8) are true. Hence assume that $1 + b_1 + \cdots + b_{k-1} \ne 0$ and $b_k \ne 0$. Let $A = k$, put $z = \frac{b_k}{(1+b_1+\cdots+b_{k-1})}$, and delete $\frac{\cos^2\alpha}{(A+\sin^2\alpha)}$ in (3.5). We then have

$$\left|1 - \frac{kb_k}{1 + b_1 + b_2 + \cdots + b_{k-1}}\right|^2 = \left|\frac{1 + b_1 + b_2 + \cdots + b_{k-1} - kb_k}{1 + b_1 + b_2 + \cdots + b_{k-1}}\right|^2 \ge \sin^2\alpha.$$

$$|1 + b_1 + b_2 + \cdots + b_{k-1} - kb_k|^2 \ge \left(\sin^2\alpha\right)|1 + b_1 + b_2 + \cdots + b_{k-1}|^2\,.$$

$$|1 + b_1 + b_2 + \cdots + b_{k-1} - kb_k| \ge (\sin\alpha)\,|1 + b_1 + b_2 + \cdots + b_{k-1}|\,.$$

Similarly, for (3.6) we have

$$\left|1 + \frac{b_k}{1 + b_1 + b_2 + \cdots + b_{k-1}}\right| = \left|\frac{1 + b_1 + b_2 + \cdots + b_{k-1} + b_k}{1 + b_1 + b_2 + \cdots + b_{k-1}}\right|$$
$$\ge 1 + (\cos\alpha)\left|\frac{b_k}{1 + b_1 + b_2 + \cdots + b_{k-1}}\right|.$$

$$|1 + b_1 + b_2 + \cdots + b_{k-1} + b_k| \ge |1 + b_1 + b_2 + \cdots + b_{k-1}| + (\cos\alpha)\,|b_k|\,.$$

Suppose the last inequality is valid for $k = 1, 2, \ldots, s$ $(s \le n-1)$. Then Lemma 2 applies and shows that for $k = 1, 2, 3$, that

$$|1 + b_1| \ge 1 + (\cos\alpha)\,|b_1| > (\cos\alpha)\,(1 + |b_1|)\,.$$

$$|1 + b_1 + b_2| \ge |1 + b_1| + (\cos\alpha)\,|b_2| \ge 1 + (\cos\alpha)\,|b_1| + (\cos\alpha)\,|b_2|$$
$$= 1 + (\cos\alpha)\,(|b_1| + |b_2|) > (\cos\alpha)\,(1 + |b_1| + |b_2|)\,.$$

$$|1 + b_1 + b_2 + b_3| \ge |1 + b_1 + b_2| + (\cos\alpha)\,|b_3| \ge 1 + (\cos\alpha)\,(|b_1| + |b_2|) + (\cos\alpha)\,|b_3|$$
$$= 1 + (\cos\alpha)\,(|b_1| + |b_2| + |b_3|) > (\cos\alpha)\,(1 + |b_1| + |b_2| + |b_3|)\,.$$

An easy induction on k shows

$$|1 + b_1 + b_2 + \cdots + b_s| > (\cos\alpha)\,(1 + |b_1| + |b_2| + \cdots + |b_s|). \quad \text{QED}$$

Returning to the original problem we distinguish between two cases. Vaguely, the idea of Biró's proof is that if the s's are small, then the b_k must be near 1. But we cannot have both $1 + b_1 + b_2 + \cdots + b_{k-1} + b_k$ and $1 + b_1 + b_2 + \cdots + b_{k-1} - kb_k$ small compared with $1 + b_1 + b_2 + \cdots + b_{k-1}$. Theorem 6 then follows from Lemma 3.

Case 1. *Suppose* (3.8) *holds for* $k = 1, 2, \ldots, n-1$. *Then* $\max_{j=1,\ldots,n} |s_j| > \cos\alpha$.

Proof: By Lemma 3 and (3.4),

$$\begin{aligned}
&\left(\max_{j=1,\ldots,n} |s_j|\right)(1 + |b_1| + |b_2| + \cdots + |b_{n-1}|) \\
&\quad \geq |s_n| + |b_1|\,|s_{n-1}| + |b_2|\,|s_{n-2}| + \cdots + |b_{n-1}|\,|s_1| \\
&\quad \geq |s_n + b_1 s_{n-1} + b_2 s_{n-2} + \cdots + b_{n-1} s_1| \\
&\quad = |1 + b_1 + b_2 + \cdots + b_{n-1}| \\
&\quad \geq (\cos\alpha)\,(1 + |b_1| + |b_2| + \cdots + |b_{n-1}|). \qquad \text{(by (3.9))}
\end{aligned}$$

which implies $\max_{j=1,\ldots,n} |s_j| > \cos\alpha$. QED

Case 2. *If Case 1 is not satisfied, then* $\max_{j=1,\ldots,n} |s_j| > \sin\alpha\cos\alpha$.

Proof: Let $1 \leq k_0 \leq n-1$ be the least positive integer for which (3.8) is invalid. Then (3.8) is valid only for $1 \leq k \leq k_0 - 1$. By Lemma 3 and (3.3),

$$\begin{aligned}
&\left(\max_{j=1,\ldots,k_0} |s_j|\right)(1 + |b_1| + |b_2| + \cdots + |b_{k_0-1}|) \\
&\quad \geq |s_{k_0}| + |b_1|\,|s_{k_0-1}| + |b_2|\,|s_{k_0-2}| + \cdots + |b_{k_0-1}|\,|s_1| \\
&\quad \geq |s_{k_0} + b_1 s_{k_0-1} + b_2 s_{k_0-2} + \cdots + b_{k_0-1} s_1| \\
&\quad = |1 + b_1 + b_2 + \cdots + b_{k_0-1} - k_0 b_{k_0}| \\
&\quad \geq (\sin\alpha)\,|1 + b_1 + b_2 + \cdots + b_{k_0-1}| \qquad \text{(by (3.7))} \\
&\quad > (\sin\alpha\cos\alpha)\,(1 + |b_1| + |b_2| + \cdots + |b_{k_0-1}|). \qquad \text{(by (3.9))}
\end{aligned}$$

hence the assertion

$$\max_{j=1,\dots,n} |s_j| \geq \max_{j,\dots,k_0} |s_j| > \sin\alpha\cos\alpha. \quad \text{QED}$$

So in both cases we have the estimate

$$\max_{j=1,\dots,n} |s_j| > \cos\alpha > \sin\alpha\cos\alpha = \frac{1}{2}\sin 2\alpha$$

since $0 \leq \sin\alpha \leq 1$. If $\alpha = \frac{\pi}{4}$, we have

$$\max_{j=1,\dots,n} |s_j| > \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}. \quad \text{QED}$$

3.2 Biró numbers

In this section we report on some computations on the Biró numbers. We will use the computational scheme in Cheer-Goldston [6], where the computing of R_n is formulated as a Lagrange multiplier problem via a result of Lawrynowicz which guarantees that there is always an extremal configuration of points z_k which takes on the value R_n having the property $|s_1| = |s_2| = \cdots = |s_n|$.

Let $z_1 = 1$, let $z_k = x_k + iy_k$ for $2 \leq k \leq n$, and let

$$F(x_2, x_3, \dots, x_n, y_2, \dots, y_n, \lambda_1, \lambda_2, \dots, \lambda_{n-1}) = |s_1|^2 - \sum_{k=1}^{n-1} \lambda_k \left(|s_1|^2 - |s_{k+2}|\right). \quad (3.10)$$

Then the equation $R_n = |s_1|$ is satisfied at a critical point of F, which requires the solution of the system

$$\frac{\partial F}{\partial x_{j+1}} = 0, \frac{\partial F}{\partial y_{j+1}} = 0, \frac{\partial F}{\partial \lambda_j} = 0 \quad (j = 1, 2, \dots, n-1). \quad (3.11)$$

For $n = 2$, Cheer-Goldston obtained

$$z_1 = 1,$$

$$z_2 = x_2 + iy_2,$$

$$|s_1|^2 = (1 + x_2)^2 + y_2^2,$$

$$|s_2|^2 = \left(1 + x_2^2 - y_x^2\right)^2 + 4x_2^2y_2^2.$$

and the system

$$\frac{\partial F}{\partial x_2} = 2(1+x_2) - \lambda_1 \left(2(1+x_2) - 4x_2\left(1+x_2^2-y_2^2\right) - 8x_2y_x^2\right) = 0.$$
$$\frac{\partial F}{\partial y_2} = 2y_2 - \lambda_1 \left(2y_2 + 4y_2\left(1+x_2^2-y_x^2\right) - 8x_2^2y_2\right) = 0.$$
$$\frac{\partial F}{\partial \lambda_2} = (1+x_2)^2 + y_2^2 - \left(1+x_2^2-y_2^2\right) - 4x_2^2y_2^2 = 0.$$

Let

$$R_n = \max\{|s_1|,|s_2|,\ldots,|s_n|\},$$

where the s_k are determined by the Newton-Girard formulas for $Q(z)$. From (3.3) we derive

$$s_1 = 1 - b_1,$$
$$s_2 + b_1 s_1 = 1 + b_1 - 2b_2,$$
$$s_3 + b_1 s_2 + b_2 s_1 = 1 + b_1 + b_2 - 3b_3,$$

and hence

$$R_n \geq |s_1| = |1 - b_1|,$$
$$R_n(1+|b_1|) \geq |s_2| + |b_1||s_1|$$
$$\geq |s_2 + b_1 s_1| = |1 + b_1 - 2b_2|,$$
$$R_n(1+|b_1|+|b_2|) \geq |s_3| + |b_1||s_2| + |b_2||s_1|$$
$$\geq |s_3 + b_1 s_2 + b_2 s_1| = |1 + b_1 + b_2 - 3b_3|.$$

In general,

$$R_n \geq \frac{|1 + b_1 + b_2 + \cdots + b_{k-2} - (k-1)b_{k-1}|}{1 + |b_1| + |b_2| + \cdots + |b_{k-2}|}$$

for $2 \leq k \leq n$.

From (3.4) we derive

$$R_n \geq \frac{|1 + b_1 + b_2 + \cdots + b_{n-1}|}{1 + |b_1| + |b_2| + \cdots + |b_{n-1}|}.$$

Letting

$$B_n = \max\left(|1-b_1|, \frac{|1+b_1-2b_2|}{1+|b_1|}, \frac{|1+b_1+b_2-3b_3|}{1+|b_1|+|b_2|}, \ldots, \frac{|1+b_1+b_2+\cdots+b_{n-2}-(n-1)b_{n-1}|}{1+|b_1|+|b_2|+\cdots+|b_{n-2}|}, \frac{|1+b_1+b_2+\cdots+b_{n-1}|}{1+|b_1|+|b_2|+\cdots+|b_{n-1}|}\right),$$

then $R_n \geq B_n$, and we choose the Biró numbers b_k $(k = 1, \ldots, n)$ that minimize B_n. (Recall that Biró proved that $B_n > \frac{1}{2}$.)

For $n = 2$, we have

$$s_1 = 1 - b_1, s_2 + b_1 s_1 = 1 + b_1.$$

$$s \geq |s_1| = |1 + b_1|, s(1 + |b_1|) \geq |s_2| + |b_1||s_1| \geq |1 + b_1|.$$

Thus

$$s \geq \max\left(|1-b_1|, \frac{|1+b_1|}{1+|b_1|}\right) = \widetilde{B}_2.$$

We do not actually solve this maximum problem, but instead the simpler problem of finding $b_1 = u + iv$ such that the maximum occurs when $|1 - b_1| = \frac{|1+b_1|}{1+|b_1|}$ is minimal, that is, we minimize $|1 - b_1|$ subject to the constraint $|1 - b_1|(1 + |b_1|) = |1 + b_1|$ over all complex numbers b_1. Letting $b_1 = w_1^2$, we then minimize $|1 - w_1^2|^2$ subject to the constraint $|1 - w_1^2|^2 \left(1 + |w_1|^2\right)^2 = |1 + w_1^2|^2$. This is a Lagrange multiplier problem, so we minimize

$$F(u_1, v_1, \lambda) = f(u_1, v_1) - i\lambda g(u_1, v_1),$$

where $f(u_1, v_1) = 1 - u_1^2 + v_1^2$ and $g(u_1, v_1) = u_1 v_1$, subject to the constraint

$$\left|1 - (u_1 + iv_1)^2\right|^2 \left(1 + u_1^2 + v_1^2\right)^2 - \left|1 + (u_1 + iv_1)^2\right|^2 = 0.$$

For $n = 3$,

$$s_1 = 1 - b_1,$$

$$s_2 + b_1 s_1 = 1 + b_1 - 2b_2,$$

$$s_3 + b_1 s_2 + b_2 s_1 = 1 + b_1 + b_2.$$

Thus

$$s \geq \max\left(|1-b_1|, \frac{|1+b_1-2b_2|}{1+|b_1|}, \frac{|1+b_1+b_2|}{1+|b_1|+|b_2|}\right) = \breve{B}_3.$$

and we minimize $|1-w_1^2|$ subject to the constraints

$$|1-w_1^2| = \frac{|1+w_1^2-2w_2^2|}{1+|w_1|^2},$$
$$|1-w_1^2| = \frac{|1+w_1^2+w_2^2|}{1+|w_1|^2+|w_2|^2}.$$

that is, we minimize $|1-w_1^2|^2$ subject to

$$|1-w_1^2|^2\left(1+|w_1|^2\right)^2 = |1+w_1^2-2w_2^2|^2,$$
$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2\right)^2 = |1+w_1^2+w_2^2|^2.$$

For $n = 4$,

$$s_1 = 1 - b_1,$$
$$s_2 + b_1 s_1 = 1 + b_1 - 2b_2,$$
$$s_3 + b_1 s_2 + b_2 s_1 = 1 + b_1 + b_2 - 3b_3,$$
$$s_4 + b_1 s_3 + b_2 s_2 + b_3 s_1 = 1 + b_1 + b_2 + b_3.$$

Thus

$$s \geq \max\left(|1-b_1|, \frac{|1+b_1-2b_2|}{1+|b_1|}, \frac{|1+b_1+b_2-3b_3|}{1+|b_1|+|b_2|}, \frac{|1+b_1+b_2+b_3|}{1+|b_1|+|b_2|+|b_3|}\right) = \breve{B}_4.$$

and we minimize $|1-w_1^2|^2$ subject to

$$|1-w_1^2|^2\left(1+|w_1|^2\right)^2 = |1+w_1^2-2w_2^2|^2,$$
$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2\right)^2 = |1+z_1^2+w_2^2-3w_3^2|^2,$$
$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2+|w_3|^2\right)^2 = |1+w_1^2+w_2^2+w_3^2|.$$

For $n = 5$,

$$s_1 = 1 - b_1,$$

$$s_2 + b_1 s_1 = 1 + b_1 - 2b_2,$$

$$s_3 + b_1 s_2 + b_2 s_1 = 1 + b_1 + b_2 - 3b_3,$$

$$s_4 + b_1 s_3 + b_2 s_2 + b_3 s_1 = 1 + b_1 + b_2 + b_3 - 4b_4,$$

$$s_5 + b_1 s_4 + b_2 s_3 + b_3 s_2 + b_4 s_1 = 1 + b_1 + b_2 + b_3 + b_4.$$

Thus

$$s \geq \max\left(|1 - b_1|, \frac{|1 + b_1 - 2b_2|}{1 + |b_1|}, \frac{|1 + b_1 + b_2 - 3b_3|}{1 + |b_1| + |b_2|}, \frac{|1 + b_1 + b_2 + b_3 - 4b_4|}{1 + |b_1| + |b_2| + |b_3|}, \frac{|1 + b_1 + b_2 + b_3 + b_4|}{1 + |b_1| + |b_2| + |b_3| + |b_4|} \right) = \widehat{B}_5,$$

and we minimize $|1 - w_1^2|^2$ subject to

$$|1 - w_1^2|^2 \left(1 + |w_1|^2\right)^2 = \left|1 + w_1^2 - 2w_2^2\right|^2,$$

$$|1 - w_1^2|^2 \left(1 + |w_1|^2 + |w_2|^2\right)^2 = \left|1 + w_1^2 + w_2^2 - 3w_3^2\right|^2,$$

$$|1 - w_1^2|^2 \left(1 + |w_1|^2 + |w_2|^2 + |w_3|^2\right)^2 = \left|1 + w_1^2 + w_2^2 + w_3^2 - 4w_4^2\right|,$$

$$|1 - w_1^2|^2 \left(1 + |w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2\right)^2 = \left|1 + w_1^2 + w_2^2 + w_3^2 + w_4^2\right|.$$

For $n = 6$,

$$s_1 = 1 - b_1,$$

$$s_2 + b_1 s_1 = 1 + b_1 - 2b_2,$$

$$s_3 + b_1 s_2 + b_2 s_1 = 1 + b_1 + b_2 - 3b_3,$$

$$s_4 + b_1 s_3 + b_2 s_2 + b_3 s_1 = 1 + b_1 + b_2 + b_3 - 4b_4,$$

$$s_5 + b_1 s_4 + b_2 s_3 + b_3 s_2 + b_4 s_1 = 1 + b_1 + b_2 + b_3 + b_4 - 5b_5,$$

$$s_6 + b_1 s_5 + b_2 s_4 + b_3 s_3 + b_4 s_2 + b_5 s_1 = 1 + b_1 + b_2 + b_3 + b_4 + b_5.$$

Thus

$$s \geq \max\left(|1-b_1|, \frac{|1+b_1-2b_2|}{1+|b_1|}, \frac{|1+b_1+b_2-3b_3|}{1+|b_1|+|b_2|}, \frac{|1+b_1+b_2+b_3-4b_4|}{1+|b_1|+|b_2|+|b_3|}, \frac{|1+b_1+b_2+b_3+b_4-5b_5|}{1+|b_1|+|b_2|+|b_3|+|b_4|}, \frac{|1+b_1+b_2+b_3+b_4+b_5|}{1+|b_1|+|b_2|+|b_3|+|b_4|+|b_5|}\right) = \breve{B}_6.$$

and we minimize $|1-w_1^2|^2$ subject to

$$|1-w_1^2|^2\left(1+|w_1|^2\right)^2 = \left|1+w_1^2-2w_2^2\right|^2,$$

$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2\right)^2 = \left|1+w_1^2+w_2^2-3w_3^2\right|^2,$$

$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2+|w_3|^2\right)^2 = \left|1+w_1^2+w_2^2+w_3^2-4w_4^2\right|,$$

$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2+|w_3|^2+|w_4|^2\right)^2 = \left|1+w_1^2+w_2^2+w_3^2+w_4^2-5z_5^2\right|,$$

$$|1-w_1^2|^2\left(1+|w_1|^2+|w_2|^2+|w_3|^2+|w_4|^2+|w_5|^2\right)^2 = \left|1+w_1^2+w_2^2+w_3^2+w_4^2+w_5^2\right|.$$

We compute $R_n = \min_{b_1,\ldots,b_{n-1}} \breve{B}_n$ for any complex b_k $(k = 1,\ldots,n)$ by finding suitable values for b_k (using w_k) that make R_n small, thus limiting Biró's method. As in [6], the equations (3.11) are solved numerically using Newton's algorithm. Our computations on the critical points for $n = 3,\ldots,7$, are indicated in Table 3. R_n being the smallest value in each column. Let us remark that $R_2 = 0.8740320488\cdots$, exactly. With several hundred runs only three critical points are found for $n = 3$, and with several thousand runs 26 additional critical points are found for $n = 6$ and 45 additional critical points are found for $n = 7$. Our computations on the numbers $b_k = w_k^2$ for $n = 2,\ldots,7$, are indicated in Table 4.

3.3 Conclusion

The extremal configuration of the b_k for different values of n has not developed sufficiently to predict the ultimate limiting configuration. The numbers found for $n \geq 3$ are smaller than the numbers found for the original Turán pure power sum problem. This shows that the min-max problem Biró used is genuinely different than the original problem.

TABLE 3. Values of $R_n < 1$ at critical points.

$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
0.8186568591	0.7859855693	0.7638412078	0.7475593330	0.7349293346
0.8965754721	0.8304250763	0.7924254402	0.7677638068	0.7501492347
0.9227628085	0.8440019164	0.8015139266	0.7732045626	0.7536202496
	0.8635229626	0.8231447035	0.7806223878	0.7583524918
	0.8634338223	0.8267526303	0.7947228759	0.7646380648
	0.8964146283	0.8279136892	0.7969036336	0.7736447266
	0.9137005034	0.8383913285	0.7995350437	0.7761696005
	0.9487367178	0.8428340816	0.8011889584	0.7777043244
		0.8643855247	0.8028117631	0.7800727779
		0.8644710194	0.8063579729	0.7815281877
		0.8647181326	0.8084359616	0.7821208574
		0.8734193522	0.8128290334	0.7838018044
		0.8767816135	0.8210285338	0.7839227317
		0.8854639303	0.8233161463	0.7840417913
		0.8911338831	0.8316330400	0.7879696743
		0.8942568861	0.8322039165	0.7914204358
		0.8956578813	0.8325903804	0.7934288389
		0.8965754721	0.8328541424	0.7940382539
		0.9148037307	0.8333520341	0.7988818639
			0.8335960363	0.8020283613
			0.8341487847	0.8030442650
			0.8401948448	0.8049364803
			0.8510403854	0.8069161213
			...	...
			0.9295896999	0.8436644606
			0.9313046760	0.8462544303
			0.9356218266	0.8466030849
			0.9486555735	0.8482627962
				0.8564317970
				0.8577031411
				0.8581072463
				0.8582187121
				...
				0.9052982306
				0.9066293679
				0.9087252946
				0.9243567762

TABLE 4. Values of w_k and b_k at critical points.

n	w_k	b_k
2	(0.6808270643, 0.3930756888)	(0.3090169943, 0.5352331346)
3	(0.7248611131, 0.3734659824)	(0.3859467933, 0.5414219356)
	(0.6458239277, 0.2486821420)	(0.3552457378, 0.3212097554)
4	(0.7486806765, 0.3613731962)	(0.4299321684, 0.5411062581)
	(0.6797882388, 0.2342658959)	(0.4072315397, 0.3185024016)
	(0.5133466149, 0.3738877782)	(0.1237326763, 0.3838680506)
5	(0.7639935032, 0.3529691650)	(0.4590988415, 0.5393322979)
	(0.7011041405, 0.2249666197)	(0.4409370359, 0.3154500571)
	(0.5847937350, 0.3012554577)	(0.2512288617, 0.3523446087)
	(0.4884724911, 0.3403791858)	(0.1227473845, 0.3325317377)
6	(0.7748454768, 0.3466871915)	(0.4801935042, 0.5372580044)
	(0.7159936074, 0.2183452864)	(0.4649721817, 0.3126676586)
	(0.6057307154, 0.2952785131)	(0.2797202993, 0.3577185301)
	(0.5145158300, 0.3360811881)	(0.1517759743, 0.3458381830)
	(0.4379339233, 0.3576416567)	(0.0638785666, 0.3132468277)
7	(0.7830340553, 0.3417559139)	(0.4963452270, 0.5352130385)
	(0.7271191258, 0.2133254328)	(0.4831944829, 0.3102260044)
	(0.5845094727, 0.3589549538)	(0.2128026648, 0.4196251416)
	(0.4928043782, 0.3907831750)	(0.0901446653, 0.3851593191)
	(0.4172623841, 0.4054417854)	(0.0097248557, 0.3383512120)
	(0.3536564884, 0.4102388916)	(0.0432230363, 0.2901672916)

References

1. P. Turán. *On a New Method of Analysis and its Application*. John Wiley and Sons. New York. 1984.

2. F. V. Atkinson. *On sums of power of complex numbers*. Acta Math. Acad. Sci. Hungar.. **12** (1961). 185-188.

3. F. V. Atkinson. *Some further estimates concerning sums of power of complex numbers*. Acta Math. Acad. Sci. Hungar.. **20** (1969). 209-216.

4. H. L. Montgomery. *Turán's method*. Institut Mittag-Leffler. Report No. **15**. 1978.

5. A. Biró. *On a problem of Turán concerning sums of power of complex numbers*. Acta Math. Hungar.. **65** (1994). No. **3**. 209-216.

6. A. Y. Cheer and D. A. Goldston. *Turán's pure power sum problem*. Mathematics of Computation. **65** (1995). No. **215**. 1349-1358.

7. P. G. Ciarlet and J. L. Lions. *Handbook of Numerical Analysis. Volume 3*. North-Holland. The Netherlands. 1994.

8. E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press. London. 1935.