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The geometry of third degree curves

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THE GEOMETRY OF THIRD DEGREE CURVES

A Thesis

Presented to

The Faculty of the Department of Mathematics

at San Jose State University

in Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Richard Jones

December 2001

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A handwritten signature in cursive script, appearing to read "Roger Alperin", written over a horizontal line.

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Dr. Marilyn Blockus

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ABSTRACT

THE GEOMETRY OF THIRD DEGREE CURVES

by Richard Jones

The collection of third-degree curves is described three times using three different definitions of geometric equivalence: equivalence under euclidean transformations (or rigid motions), equivalence under affine transformations and equivalence under projective transformations.

Even under projective equivalence, there are uncountably many different third-degree curves; the collection of such curves is defined as a circle in a convenient coefficient space.

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CHAPTER ONE

BASIC IDEAS

A cubic curve is a curve defined by some equation of form

$$\text{3rd-degree polynomial}(x, y) = 0.$$

Such curves typically lack any obvious symmetry, such as reflection or rotational symmetry. A cubic curve may, among other things, look like the curve of a nose or other body part or like a stroke of a cursive letter.

This paper will attempt to classify the cubic curves by their appearance. Two equations, such as $x^3 + 2y^2 + 5 = 0$ and $2x^3 + 4y^2 + 10 = 0$, which yield the same points will be treated as equivalent and even casually referred to as the same equation. Since the plane people visualize is the real plane, real variables and polynomials with real coefficients will be used.

In order to describe the different appearances cubic curves may have, one needs to define “same appearance”. Thus I will choose a group of geometric transformations and say that two curves have the “same appearance” if and only if a geometric transformation from the group changes one of the curves to the other.

I will discuss the cubic curves three times, each time using a different (and more general) group of transformations to define “same appearance”. In chapter 2, I will choose the group of euclidean transformations (the rigid motions); the list of “euclidean different” curves will be a collection of six vector spaces of up to six dimensions each. In chapter 3, I will choose the group of affine transformations (dilations, projections from infinity); the list of “affine different” curves will be a collection of eight vector spaces of up to three dimensions each, and three isolated curves. In chapter 4, I will choose the group of projective transformations (projections from an arbitrary point); the list of “projective different” curves will be a circle and eleven isolated curves.

More formally, I will make a group action from each of these three groups of transformations of the set of cubic curves. Each of these group actions will generate a collection of cosets of equivalent curves. The greater part of the paper will be a description of these cosets.

In the familiar “affine plane”, each point is defined by a pair of coordinates (x, y) ; (x_1, y_1) and (x_2, y_2) are equivalent if and only if $x_1 = x_2$ and $y_1 = y_2$.

In the “projective plane”, each point is defined by *three* coordinates, (X, Y, W) ; (X_1, Y_1, W_1) and (X_2, Y_2, W_2) are equivalent if and only if $X_1 = mX_2$, $Y_1 = mY_2$ and $W_1 = mW_2$ for some nonzero m . Thus different sets of coordinates may define the same point.

The conventional mapping from the projective plane to the affine plane (and the one used in this paper) is division by W :

$$(X, Y, W) \rightarrow (x, y) = \left(\frac{X}{W}, \frac{Y}{W} \right).$$

Points for which $W = 0$ map to the “line at infinity”, a sort of horizon around the affine plane.

Since (X, Y, W) and (mX, mY, mW) are equivalent (as long as $m \neq 0$), the only meaningful polynomials on projective coordinates are those in which every term has the same degree, called “homogenous” polynomials. Thus in a polynomial for a cubic curve, every term would be a third degree term.

To rewrite a projective polynomial for the affine plane, simply remove the W s:

$$AX^3 + GXYW + KYW^2 + MW^3 \rightarrow Ax^3 + Gxy + Ky + M.$$

To convert an affine polynomial for the projective plane, pad each term with W s until it is third degree: $Bx^2y + Fx^2 + Jx \rightarrow BX^2Y + FX^2W + JXW^2$.

Projective coordinates may seem clumsy, but for algebraic geometry they have a special advantage: any projective transformation is a linear transformation of projective coordinates. Thus one may represent any projective transformation simply as a product of a 3x3 matrix (transformation) with a 3x1 matrix (coordinates of the point). This device will represent geometric transformations in chapters 2, 3 and 4.

The “order” of a curve at a point (a, b) is the order of the smallest non-zero derivative of the curve at that point.

Thus if $F(a, b) \neq 0$, point a is said to have order zero, since there the first non-zero derivative of F is F itself. Point (a, b) is not even on the curve $F(x, y) = 0$.

If $F(a, b) = 0$, but at least one of its first derivatives is non-zero, (a, b) is said to have order one. The curve $F(z) = 0$ passes through (a, b) in a well-defined, unique direction (the behavior that one normally expects of points on curves).

If $F(a, b) = 0$, the first derivatives of F are zero at (a, b) , but at least one of its second derivatives is not, then (a, b) is said to be a second order point. At such a point the curve $F(x, y) = 0$ may intersect itself or have a cusp or an isolated point.

If $F(a, b) = 0$ and its first and second derivatives are all zero at (a, b) , but at least one of its third derivatives is non-zero, (a, b) is said to be a third order point. The curve $F(x, y) = 0$ might pass through (a, b) three times or “bounce off” itself there or it might simultaneously be both an ordinary (1st order) point and an isolated (2nd order) point on the curve.

In general, the higher the order of a point on a curve, the more peculiar and subtle is the behavior of the curve there. A point of at least second order is called a “singular” point.

One may also define the order of a point on a curve $F(X, Y, W) = 0$, on the projective plane. It can be shown that if the point and curve are each defined in both planes, then the point’s “projective order” and its “affine order” must be the same.

The idea of “multiple intersections” is not defined precisely in this paper, but loosely one may say that an n -tuple intersection is one for which the distance between the two curves is proportional to the n th power of distance from their intersection. Thus if the distance between the two curves is directly proportional to distance from the intersection, the intersection is called a “single intersection”; if the distance between curves goes as the third power of that distance from the intersection, that intersection would be called a “triple intersection”.

Usually, when a line intersects a curve, the order of the intersection matches the order of the point (a 1st-order intersection at a 1st-order point, an n th order intersection at an n th order point). A special line whose intersection with a curve has “too high” an order (such as a line that makes a 2nd-order intersection with the curve at a 1st-order point) is said to be a “tangent” of that curve.

Every ordinary (1st-order) point on the curve has exactly one tangent.

At most tangents, the difference in the orders is only one. (The most common kind of tangent makes a 2nd-order intersection at a 1st-order point.) A point with a “supertangent” at which the orders differ by two or more is called a “point of inflection”. The only points of inflection appearing in this paper have 3rd-order tangent intersections at 1st-order points.

A line that is tangent to a curve at infinity is said to be an “asymptote” of that curve.

A cubic curve may have a pair of complex conjugate asymptotes; then I will call their intersection (which is still a real point) an “asymptotic center” of the curve.

Sometimes a curve may have the line at infinity as a tangent.

One could then say that the line at infinity is a double asymptote of the curve.

Those readers who find finite constructions easier to visualize may prefer an “asymptotic parabola” which is tangent to the line at infinity at the same place and with the same curvature.

Occasionally, a curve may be tangent to the line at infinity and also have zero curvature there. In that case, no asymptotic parabola can be constructed (the directrix of an asymptotic parabola is inversely proportional to the curve’s curvature at the tangent) and the line at infinity acts as a triple asymptote.

I refer to such curves as having “no asymptotes” (really no finite asymptotes) and call them “Case 2” in chapters 2 and 3.

CHAPTER TWO

CUBIC CURVES UNDER RIGID MOTIONS

Transformations which preserve distance are called “euclidean transformations” or “rigid motions”. In homogenous coordinates each “direct” (angle-preserving) rigid motion can be written in the form

$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} \cos \theta & -\sin \theta & a \\ \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix}$$

while each “retrograde” (angle-reversing) rigid motion can be written in the form

$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} \cos \eta & \sin \eta & a \\ \sin \eta & -\cos \eta & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix}.$$

Curves related through rigid motions are so obviously equivalent that people are likely to refer to them as the “same curve”; however, trying to describe the connection algebraically can be a little daunting. Conveniently, a rotation around any center may be factored into (rotation around origin)•(translation) , while any reflection or glide reflection may be factored into (reflection across line through origin)•(translation)*; the effect of each of these “factors” on a cubic polynomial is relatively simple (and is described in the appendix).

* A rotation by θ around some point (a, b) is equivalent to a rotation by θ around the origin, followed by the translation (a, b) – rotation θ around origin (a, b) .

A reflection across the line $Ax + By + C = 0$ is equivalent to a reflection across the line $Ax + By = 0$ (or $By = -Ax$),

followed by the translation $\left(\frac{-2AC}{A^2 + B^2}, \frac{-2BC}{A^2 + B^2} \right)$.

If one curve is equivalent to a rigid motion of a second curve, I will call the two curves “euclidean equivalent”. It is easy to see that this relation is an equivalence relation and I will call the partitions that follow “euclidean partitions”. This kind of relationship is so obvious to people that they routinely refer to curves equivalent in this way as “the same curve”, or, more precisely, as “curves of the same size and shape”. The object of this chapter will be a map of the euclidean partitions of the cubic curves.

I will show that any cubic equation has a sequence of rigid motions that transforms it into one of five ‘standard forms’; then there must be a one-to-one map from some subset of the equations in these five forms to the euclidean partitions of all cubic equations. I will treat this subset as a ‘picture’ of the collection of euclidean partitions of cubic curves.

Consider a cubic curve, that is the curve of an equation of the form

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0.$$

A rotation by an angle θ around the origin transforms it into

$$A_2x^3 + B_2x^2y + C_2xy^2 + D_2y^3 + F_2x^2 + G_2xy + H_2y^2 + J_2x + K_2y + M = 0,$$

where the new coefficients depend on θ ; in particular,

$$\begin{aligned} A_2 &= A \cos^3 \theta - B \cos^2 \theta \sin \theta + C \cos \theta \sin^2 \theta - D \sin^3 \theta \\ &= \cos^3 \theta (A - B \tan \theta + C \tan^2 \theta - D \tan^3 \theta) \end{aligned} .$$

If $D \neq 0$, the expression in parenthesis is a cubic on $\tan \theta$. Since a real cubic always has a real solution, there is a real $\tan \theta$ (and therefore a θ , $-90^\circ < \theta < 90^\circ$) for which $A_2 = 0$. If $D = 0$, a 90° rotation yields $A_2 = 0$. Either way, there is some rotation that transforms the cubic equation to the form

$$B_2 x^2 y + C_2 xy^2 + D_2 y^3 + F_2 x^2 + G_2 xy + H_2 y^2 + J_2 x + K_2 y + M = 0.$$

Geometrically, the curve was rotated until one asymptote (if it had any) was horizontal.

The next step depends on the value of B_2 and the other coefficients.

Case 1: $B_2 \neq 0$

A translation, $\left(\frac{C_2 F_2}{(B_2)^2} + \frac{G_2}{2B_2}, \frac{F_2}{B_2} \right)$, followed by division by B_2 yields an equation of form $x^2 y + C_3 xy^2 + D_3 y^3 + H_3 y^2 + J_3 x + K_3 y + M_3 = 0$ with only six arbitrary coefficients. Geometrically, the translation moves the curve until one asymptote lies on the x-axis (a Case 1 curve always has at least one asymptote).

I will think of such an equation as a point in a six-dimensional "CDHJKM-space". Every Case 1 cubic is equivalent by some rigid motion to at least one of the cubics in this CDHJKM-space (so the inclusion map from the CDFJKM-space to the space of euclidean partitions of Case 1 cubics is an onto map). One should then expect that the CDFJKM-space would have some subset in which each euclidean partition is represented exactly once. Such a subset could be used as a list of the euclidean partitions of Case 1.

To find such a subset of CDHJKM-space, one should find out which curves of this space are connected by a rigid motion and therefore in the same euclidean partition. Knowing this, one could set up a subset of CDHJKM-space in which each Case-1 euclidean partition is represented exactly once.

Each point in CDHJKM-space stands for a curve which uses the x-axis as an asymptote (if $B_2 \neq 0$, the x-axis is an asymptote if and only if $A_2 = 0$ and $F_2 = 0$). I start by assuming that a rigid motion that stays inside CDHJKM-space must map the x-axis to itself.

This rigid motion can be factored as translation (α, β) followed by a rotation around the origin or as translation (α, β) followed by a reflection across the origin. $\beta \neq 0$ implies that the translation of the x-axis does not go through the origin; the following rotation/reflection will not repair this. $\beta = 0, \alpha \neq 0$ implies that the new value of G is not zero; the following rotation/reflection will leave either a nonzero G or a nonzero F ; but in CDHJKM-space $F = G = 0$. Thus a rigid motion that stays within CDHJKM-space and maps the x-axis to itself must have a “translation component” of $(0, 0)$; the rigid motion is itself either a rotation around the origin or a reflection across a line through the origin.

It follows that a rigid motion that stays inside CDHJKM-space and maps the x-axis to itself must be the identity, the 180° rotation, the reflection across the x-axis or the reflection across the y-axis (the 180° rotation and the reflection across the x-axis must be combined with multiplication of the entire polynomial by -1).

The effects of these transformations on C, D, H, J, K and M are especially simple: the rotation multiplies H and M by -1 , the reflection across the x-axis multiplies C and J by -1 , and the reflection across the y-axis multiplies C,H,J and M by -1 .

One thereby obtains a “clique” of one, two or four “standard” curves connected by these rigid motions. All curves in a clique belong to the same euclidean partition; if any of them is in CDHJKM-space, they all must be in CDHJKM-space.

It is easy to construct some subset of the CDFJKM-space such as .
 $\{(C,D,H,J,K,M) \text{ where } C \geq 0, H > 0 \text{ or } C \geq 0, H = 0, M \geq 0\}$

in which each clique is “represented” by exactly one curve.

The flaw in this argument is that some Case-1 curves have two or three asymptotes; one might rotate or reflect another asymptote of such a curve onto the x-axis and thereby obtain a new clique, linked to the old clique by a rigid motion and therefore part of the same euclidean partition. A single euclidean partition might intersect CDHJKM-space at up to twelve curves from three different cliques.

The slopes of the asymptotes of a Case-1 curve are solutions to

$$\tan \theta + C \tan^2 \theta + D \tan^3 \theta = 0, \text{ so}$$

$$\tan \theta = 0 \quad \text{or} \quad \tan \theta = \frac{-C_3 - \sqrt{C_3^2 - 4D_3}}{2D_3} \quad \text{or} \quad \tan \theta = \frac{-C_3 + \sqrt{C_3^2 - 4D_3}}{2D_3}.$$

When $4D_3 > C_3^2$, the curve has one real asymptote, the x-axis.

When $4D_3 = C_3^2$, the curve may have one, two or three real asymptotes; only the one on the x-axis will be horizontal (two horizontal asymptotes implies “Case 1.5”).

When $4D_3 < C_3^2$, the curve always has three different real asymptotes in three different directions; only these curves will form cliques that map different asymptotes to the x-axis.

The three asymptotes define three angles which I label, going counter-clockwise from the x-axis, “p”, “q” and “r”. A rigid motion that permutes the asymptotes must also permute these angles (so that angle “p” on the asymptotes of one curve might be angle “q” on the asymptotes of an equivalent curve); since the value of each angle is unaffected by a rigid motion, permuting them also permutes their relative values (one curve might have $p > q > r$, while an equivalent curve has $q > r > p$).

If the $4D_3 \prec C_3^2$ region is divided into six subregions according to the relative values of p, q and r , a rigid motion that permutes asymptotes will appear as a jump from one subregion to another. Each euclidean partition is represented by a clique of twelve curves (the two in each subregion are separated by a 180° rotation). The boundaries between the subregions represent curves with some sort of symmetry; the euclidean partitions here have six, four, three, two or one curves apiece (depending one the type of symmetry).

Using such ideas, one may construct a set

$$\left\{ \begin{array}{l} x^2y + Cxy^2 + Dy^3 + Hy^2 + Jx + Ky + M = 0, \text{ where} \\ -\frac{3D+1}{\sqrt{2D+1}} \prec C \prec 0, \quad (H \succ 0 \text{ or } H = 0, M \geq 0) \\ \text{or } -\frac{3D+1}{\sqrt{2D+1}} = C \prec 0, \quad J + \frac{K}{\sqrt{2D+1}} + \frac{\sqrt{2D+1}}{(D+1)^2} H^2 \leq 0, \quad (H \succ 0 \text{ or } H = 0, M \geq 0) \\ \text{or } -\frac{1}{3} \prec D, \quad C = 0, \quad 0 \leq J, \quad (H \succ 0, \text{ or } H = 0, M \geq 0) \\ \text{or } D = -\frac{1}{3}, \quad C = 0, \quad J + \sqrt{3}K + 0.75\sqrt{3}H^2 \leq 0, \quad 0 \leq J, \quad (H \succ 0 \text{ or } H = 0, M \geq 0) \end{array} \right\}$$

which intersects each of the euclidean partitions in the CDHJKM-space exactly once.

Since each Case 1 euclidean partition appears in the CDHJKM-space, it follows that the above set includes exactly one curve from each euclidean partition of Case 1.

HERE ARE 2 OF THE 6 DIMENSIONS OF (DHJKM-SPACE OF CUBIC CURVES.

THE SHADED AREA REPRESENTS THE SET DESCRIBED ON THE PREVIOUS PAGE (OMITTING RESTRICTIONS ON H, M, J, K).

Case 1.5: $B_2 = 0; \quad C_2 \neq 0.$

A rotation of $\arctan \frac{C_2}{D_2}$ (or 90° if $D_2 = 0$) converts

$$C_2 xy^2 + D_2 y^3 + F_2 x^2 + G_2 xy + H_2 y^2 + J_2 x + K_2 y + M_2 = 0 \quad \text{into a Case 1 curve.}$$

Thus any Case 1.5 curve already belongs to one of the euclidean partitions for Case 1;
no new partitions are needed.

Case 2: $B_2 = C_2 = 0; \quad D_2 \neq 0; \quad F_2 \neq 0.$

A translation, $\left(\frac{-J_2}{2F_2}, \frac{-H_2}{3D_2} \right)$, followed by division by D_2 , yields a curve of form

$$y^3 + F_3 x^2 + G_3 xy + K_3 y + M_3 = 0, \quad F_3 \neq 0.$$

I think of these curves as points in a four-dimensional “FGKM-space”. Since every partition appears in this space, there must be a one-to-one onto function between the set of euclidean partitions of the Case 2 curves and some subset of it. To find the subset, one wants to find out which of these curves are connected by rigid motions.

As before, a rigid motion that connects two of these curves may be factored into a rotation or reflection about the origin followed by some translation, (α, β) .

One can quickly show that for a rigid motion that connects two curves from FGKM-space the rotation/reflection must be one of the identity, the 180° rotation, the reflection across the x-axis or the reflection across the y-axis.

Since the rigid motion connects two curves for which $H = 0$, one can show that $\beta = 0$. Then since the two curves both have $J = 0$, one can also show that $\alpha = 0$. Thus the translation component of the rigid motion is $(0, 0)$; the rigid motion itself must be one of the identity, the 180° rotation, the reflection across the x-axis or the reflection across the y-axis.

These rigid motions generate a “clique” which is the intersection between FGKM-space and some euclidean partition of Case 2 curves; it has two or four curves, exactly one of which lies in

$$\{y^3 + Fx^2 + Gxy + Ky + M = 0, \quad \text{where} \quad F > 0, \quad G \geq 0\}.$$

This set is a possible map of the Case 2 euclidean partitions.

This argument did not need to refer to asymptotes, which is fortunate since Case 2 curves do not have asymptotes.

Case 3: $B_2 = C_2 = 0$; $D_2 \neq 0$; $F_2 = 0$; $G_2 \neq 0$.

A translation, $\left(\frac{-K_2}{G_2}, \frac{-H_2}{3D_2} \right)$, followed by division by D_2 , yields a curve of the form $y^3 + G_3xy + J_3x + M_3 = 0$, $G_3 \neq 0$. Such a curve has one horizontal asymptote and one “asymptotic parabola” with a horizontal axis; the translation moved the vertex of the asymptotic parabola to the origin.

A Case 3 curve is called a “trident”; such curves were studied by Descartes and Newton. If the asymptote is horizontal or vertical, one variable must be a rational function of the other (in this case $x = \frac{-y^3 - M_3}{G_3y + J_3}$), so these curves are easy to graph.

Curves of the form $y^3 + G_3xy + J_3x + M_3 = 0$ may be thought of as points in a three-dimensional “GJM-space”. Since each Case 3 curve is connected to something in GJM-space by a rigid motion, the subset lemma guarantees a one-to-one onto map between the euclidean partitions from Case 3 and some subset of GJM-space. By an argument almost identical to that used for Case 2, one can show that two curves that are both in GJM-space and in the same euclidean partition must be connected by the identity, the 180° rotation, the reflection across the x-axis or the reflection across the y-axis. The set

$$\{y^3 + Gxy + Jx + M = 0, \quad \text{where} \quad G > 0, \quad J \geq 0\}$$

contains exactly one curve from each Case 3 euclidean partition.

Case 4: $B_2 = C_2 = 0$; $D_2 \neq 0$; $F_2 = G_2 = 0$; $J_2 \neq 0$.

A translation, $\left(\frac{-M_2}{J_2}, \frac{-H_2}{3D_2}\right)$, followed by division by D_2 , transforms the equation into the form $y^3 + J_3x + K_3y = 0$, $J_3 \neq 0$. Every Case 4 curve is symmetrical under some 180° rotation; the translation moved the center of this rotation to the origin.

Each curve of form $y^3 + Jx + Ky$ may be thought of as a point in some JK-plane. Since every Case 4 curve is connected to one of these by a rigid motion, some subset of the JK-plane is a map of the Case 4 euclidean partitions. By analogy with Case 2, one finds that two curves in the JK-plane that are also in the same euclidean partition must be connected by the identity or the reflection across the x-axis (since all these curves have 180° symmetry, the 180° rotation is equivalent to the identity and the reflection across the y-axis is equivalent to the reflection across the x-axis). Thus $\{y^3 + Jx + Ky = 0, \text{ where } J > 0\}$ contains exactly one curve from each Case 4 euclidean partition.

Case 5: $B_2 = C_2 = 0$; $D_2 \neq 0$; $F_2 = G_2 = J_2 = 0$.

The equation has only one variable; the “curve” is merely a set of one, two or three parallel lines. The translation, $\left(0, \frac{-H_2}{3D_2}\right)$, followed by division by D_2 , transforms the equation into the form $y^3 + K_3y + M_3 = 0$.

Since every Case 5 curve is connected by a rigid motion to at least one of these equations, some subset of them forms a map of all euclidean partitions of Case 5 curves.

If two of these equations belong to the same euclidean partition, they are connected by a rigid motion; it may be factored into a rotation/reflection across the origin and a translation (α, β) . Since both curves have $H = 0$, one can show that $\beta = 0$. Since neither equation mentions x , α is irrelevant (different values of α give the same curve). The rotation/reflection may be the identity (reflection across the y -axis is equivalent to the identity) or a 180° rotation (reflection across the x -axis is equivalent to this rotation). The only possible change in the equation is reversal of the sign of M . Thus the set $\{y^3 + Kx + M = 0, \text{ where } M \geq 0\}$ includes exactly one curve from each Case 5 euclidean partition.

I have found one-to-one onto maps of the Case 1, Case 2, Case 3, Case 4 and Case 5 euclidean partitions. I will claim that their union is a one-to-one onto map of all the euclidean partitions, but first I should check that there is no overlap among the five maps, that is that no euclidean partition can contain curves from two different cases (this was the problem that eliminated “Case 1.5”).

First one can show that no euclidean partition includes both a Case 1 curve and a curve from Case 2, 3, 4 or 5.

Such a partition would include a curve of form

$$B'x^2y + C'xy^2 + D'y^3 + H'y^2 + J'x + K'y + M' = 0, \quad B' \neq 0$$

(since every Case 1 curve is connected to one of these “standard” curves by

a rigid motion) and a curve of the form ,

$$Dy^3 + Fx^2 + Gxy + Jx + Ky + M = 0, \quad D \neq 0,$$

(since every curve of Case 2, 3, 4 or 5 is connected to such a “standard” curve by

a rigid motion). Since the two “standard” curves would belong to the same

euclidean partition, there would also have to be a rigid motion from the curve,

$$Dy^3 + Fx^2 + Gxy + Jx + Ky + M = 0,$$

 to the curve
$$B'x^2y + C'xy^2 + D'y^3 + H'y^2 + J'x + K'y + M' = 0.$$

This rigid motion could be factored into a rotation or reflection about the origin followed by a translation. The translation has no effect on A, B, C or D ;

the transformation from the terms $Dy^3, D \neq 0$ to the terms

$B'x^2y + C'xy^2 + D'y^3, \quad B' \neq 0$ would come from the rotation/reflection.

A standard equation from Case 2, 3, 4 or 5 has $A = B = C = 0$, $D \neq 0$.

A rotation θ would transform it into an equation with

$$\begin{aligned} A' &= -D \sin^3 \theta \\ B' &= 3D \sin^2 \theta \cos \theta \\ C' &= -3D \sin \theta \cos^2 \theta, \\ D' &= D \cos^3 \theta \end{aligned}$$

so that $A' = 0$, implying $\sin \theta = 0$, which implies $B' = 0$.

A reflection across $y = x \tan \eta$ would transform it into

an equation with

$$\begin{aligned} A' &= D \sin^3 2\eta \\ B' &= -3D \sin^2 2\eta \cos 2\eta \\ C' &= 3D \sin 2\eta \cos^2 2\eta, \\ D' &= -D \cos^3 2\eta \end{aligned}$$

so that $A' = 0$, implying $\sin 2\eta = 0$, which implies $B' = 0$.

Neither a rotation around the origin nor a reflection across the origin can transform an equation with $A = B = C = 0$, $D \neq 0$ into an equation with $A' = 0$, $B' \neq 0$.

Since a translation has no effect on A, B, C or D , this implies that no rigid motion can transform an equation with $A = B = C = 0$, $D \neq 0$ into an equation with $A' = 0$, $B' \neq 0$; so no euclidean partition can contain both an equation with $A = B = C = 0$, $D \neq 0$ and an equation with $A' = 0$, $B' \neq 0$; thus no euclidean partition can contain both a Case 1 curve and a curve of Case 2, 3, 4 or 5.

Next, one can argue that no euclidean partition can contain curves from any two of Cases 2, 3, 4 and 5.

Such a partition would also include the two corresponding standard curves,

$$Dy^3 + Fx^2 + Gxy + Jx + Ky + M = 0 \quad \text{and} \quad D'y^3 + F'x^2 + G'xy + J'x + K'y + M' = 0;$$

which, since they belong to the same euclidean partition, would also have to be connected by a rigid motion.

This rigid motion could be expressed either as a rotation θ , followed by a translation, (α, β) , or as a reflection across the line $y = x \tan \eta$, followed by a translation, (α, β) .

$$\begin{aligned} \text{A rotation, } \theta, \text{ followed by translation } (\alpha, \beta), \text{ would effect } & A' = -D \sin^3 \theta \\ & B' = 3D \sin^2 \theta \cos \theta \\ & C' = -3D \sin \theta \cos^2 \theta \\ & D' = D \cos^3 \theta \end{aligned}$$

$$\begin{aligned} F' &= 3D\alpha \sin^3 \theta - 3D\beta \sin^2 \theta \cos \theta + F \cos^2 \theta - G \sin \theta \cos \theta \\ G' &= -6D\alpha \sin^2 \theta \cos \theta + 6D\beta \sin \theta \cos^2 \theta + 2F \sin \theta \cos \theta + G(\cos^2 \theta - \sin^2 \theta) \\ H' &= 3D\alpha \sin \theta \cos^2 \theta - 3D\beta \cos^3 \theta + F \sin^2 \theta + G \sin \theta \cos \theta \\ J' &= -3D\alpha^2 \sin^3 \theta + 6D\alpha\beta \sin^2 \theta \cos \theta - 3D\beta^2 \sin \theta \cos^2 \theta \\ &\quad - 2F\alpha^2 \sin^2 \theta + 2G\alpha \sin \theta \cos \theta - 2F\beta \sin \theta \cos \theta - G\beta(\cos^2 \theta - \sin^2 \theta) \\ &\quad + J \cos \theta - K \sin \theta \end{aligned}$$

$$A' = 0, \text{ so } \sin \theta = 0, \text{ and therefore } \cos^2 \theta = 1 \text{ and } H' = -3D\beta \cos \theta.$$

$$\text{However } H' = 0, \text{ so } \beta = 0.$$

$$\text{Thus, } F' = F, \quad G' = G, \quad J' = J \cos \theta - 2F\alpha.$$

A reflection across $y = x \tan \eta$, followed by a translation (α, β) , would effect

$$A' = D \sin^3 2\eta$$

$$B' = -3D \sin^2 2\eta \cos 2\eta$$

$$C' = 3D \sin 2\eta \cos^2 2\eta$$

$$D' = -D \cos^3 2\eta$$

$$F' = -3D\alpha \sin^3 2\eta + 3D\beta \sin^2 2\eta \cos 2\eta + F \cos^2 2\eta + G \sin 2\eta \cos 2\eta$$

$$G' = 6D\alpha \sin^2 2\eta \cos 2\eta - 6D\beta \sin 2\eta \cos^2 2\eta + 2F \sin 2\eta \cos 2\eta - G[\cos^2 2\eta - \sin^2 2\eta]$$

$$H' = -3D\alpha \sin 2\eta \cos^2 2\eta + 3D\beta \cos^3 2\eta + F \sin^2 2\eta - G \sin 2\eta \cos 2\eta$$

$$J' = 3D\alpha^2 \sin^3 2\eta - 6D\alpha\beta \sin^2 2\eta \cos 2\eta + 3D\beta^2 \sin 2\eta \cos^2 2\eta - 2F\alpha \cos^2 2\eta - 2G\alpha \sin 2\eta \cos 2\eta - 2F\beta \sin 2\eta \cos 2\eta + G\beta(\cos^2 2\eta - \sin^2 2\eta) + J \cos 2\eta + K \sin 2\eta$$

$$A' = 0 \text{ so } \sin 2\eta = 0, \text{ therefore } \cos^2 2\eta = 1, \quad H' = -3D\beta \cos 2\eta.$$

$$H' = 0, \text{ so } \beta = 0.$$

$$\text{Thus, } F' = F, \quad G' = -G, \quad J' = J \cos 2\eta - 2F\alpha.$$

Whether the rigid motion is a rotation with a translation or a reflection with a translation, one finds that

$$G' = G \text{ or } -G$$

$$J' = J - 2F\alpha \text{ or } -J - 2F\alpha$$

Evidently, a rigid motion cannot map a standard Case 2 curve ($F \neq 0$) to a standard curve of Case 3, 4 or 5 ($F = 0$); it follows that a rigid motion cannot map any Case 2 curve to any curve of Case 3, 4 or 5 and that a euclidean partition that includes a Case 2 curve cannot also include a Case 3, 4 or 5 curve.

In the same way, a rigid motion cannot map a standard Case 3 curve ($G \neq 0$) to a standard curve of Case 4 or 5 ($G = 0$); a euclidean partition that contains any Case 3 curve cannot also contain a curve of Case 4 or 5.

Curves of Case 4 or 5 have $F = 0$, so that the effect on J of a rigid motion is merely $J' = J$ or $-J$. A standard Case 4 curve ($J \neq 0$) could not be mapped to a standard Case 5 curve ($J = 0$), so no euclidean partition that contains any Case 4 curve could also contain a Case 5 curve.

One might also try to prove the separateness of the five cases by a geometric argument. Curves from each case show geometric properties that are invariant under rigid motions and (with one exception) appear only with that case:

Case 1: Curve has two or three asymptotes, not all in the same direction
or one asymptote and one “asymptotic center”,
or one asymptote and one “asymptotic parabola”
whose axis intercepts the asymptote,
or one asymptote.

Case 2: Curve has no asymptotes;
at large distances, it approaches a semicubic parabola.

Case 3: Curve has one asymptote and one “asymptotic parabola” whose axis
is in the same direction as the asymptote.

Case 4: Curve has no asymptotes;
at large distances it approaches a cubic parabola.

Case 5: Curve consists of lines all in the same direction.
(These may be thought of as asymptotes.)

The one difficulty with this argument, oddly enough, involves lines. Either a Case 1 equation or a Case 5 equation may describe something that looks like one ordinary straight line. This seems strange, since I have just shown that no rigid motion can connect a Case 1 equation to a Case 5 equation.

The paradox disappears when you look at these curves in the complex plane. The Case 1 equation describes not only a line in the real plane, but also a conic or pair of intersecting lines outside the real plane. The Case 5 equation can only describe lines in one direction, even outside the real plane; one only finds (if anything) more lines in that same direction.

The geometric properties I have chosen are invariant not only under rigid motions, but under affine transformations as well. This same argument could be used to argue that an affine transformation cannot map a cubic curve to a cubic curve from a different case, and therefore that the affine partitions are also separated by Cases 1, 2, 3, 4 and 5.

CHAPTER THREE

CUBIC CURVES UNDER AFFINE TRANSFORMATIONS

Transformations which map parallel lines to parallel lines are called “affine transformations”; in homogenous coordinates such a transformation would take the form

$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } ae - bd \neq 0 .$$

In affine coordinates it would take the form

$$(x, y) \rightarrow (ax + by + c, dx + ey + f) .$$

The affine transformations include all the rigid motions (both direct and retrograde) as well as more exotic transformations such as the “dilation”, (e.g. $(x, y) \rightarrow (\alpha x, \beta y)$) and the “slant”, (e.g. $(x, y) \rightarrow (x + \mu y, y)$). “Affine equivalence” is not quite so obvious as “euclidean equivalence”, but it is still readily visible to people; traditional terms such as “hyperbola”, “parabola” and “pair of parallel lines” refer to collections of affine equivalent curves (“affine partitions”).

Any affine transformation may be factored into a rotation about the origin,

a horizontal slant, a translation and an xy dilation:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{ae-bd}{\sqrt{d^2+e^2}} & 0 & 0 \\ 0 & \sqrt{d^2+e^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1 & 0 & c(d^2+e^2) \\ 0 & 1 & f(ae-bd) \\ 0 & 0 & 1 \end{bmatrix} \bullet$$

(continued...)

$$\begin{matrix} (xy \text{ dilation}) & (translation) \\ \begin{bmatrix} 1 & \frac{ad+be}{ae-bd} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \bullet \begin{bmatrix} \frac{e}{\sqrt{d^2+e^2}} & \frac{-d}{\sqrt{d^2+e^2}} & 0 \\ \frac{d}{\sqrt{d^2+e^2}} & \frac{e}{\sqrt{d^2+e^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ (horizontal \text{ slant}) & (rotation \text{ around origin}) \end{matrix}$$

The appendix describes the effect of each of these four simpler transformations on the coefficients of a cubic equation.

Consider a curve of the form

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0.$$

A rotation, θ , about the origin, where $-D \tan^3 \theta + C \tan^2 \theta - B \tan \theta + A = 0$,

transforms it to the form

$$B_2x^2y + C_2xy^2 + D_2y^3 + F_2x^2 + G_2xy + H_2y^2 + J_2x + K_2y + M_2 = 0.$$

What happens next depends on the value of B_2 .

Case 1: $A_2 = 0, \quad B_2 \neq 0$

A translation, $\left(\frac{B_2 G_2 - 2C_2 F_2}{2B_2^2}, \frac{F_2}{B_2} \right)$, followed by division by B_2 , transforms the equation to the form $x^2 y + C_3 x y^2 + D_3 y^3 + H_3 y^2 + J_3 x + K_3 y + M_3 = 0$.

Then the horizontal slant, $x \rightarrow x - C_3 y$, eliminates the xy^2 term:

$$x^2 y + D_4 y^3 + H_4 y^2 + J_4 x + K_4 y + M_4 = 0.$$

A dilation can be used to set D to 1, -1 or 0 and H to 1 or 0, thereby generating six “subcases”:

Subcase 1A: $A_4 = C_4 = F_4 = G_4 = 0, \quad B_4 = 1, \quad \text{and } D_4 > 0, \quad H_4 \neq 0$.

The dilation $(x, y) \rightarrow \left(\frac{\sqrt{D_4}}{H_4} x, \frac{D_4}{H_4} y \right)$, followed by multiplication by $\frac{D_4^2}{H_4^3}$, changes the equation to the form $x^2 y + y^3 + y^2 + J_5 x + K_5 y + M_5 = 0$.

The curves of the form $x^2 y + y^3 + y^2 + J_5 x + K_5 + M_5 = 0$ together form a “JKM-space” which includes at least one curve from each affine partition of the subcase 1A curves. If two of these curves belong to the same affine partition, they must be connected by some affine transformation which stays inside JKM-space.

In this case, the only such transformations are the dilations of the form

$$(x, y) \rightarrow (\alpha x, \beta y) \quad \text{where } \alpha \neq 0, \quad \beta \neq 0.$$

Any affine transformation can be factored into

(xy dilation)(translation)(horizontal slant)(rotation about origin).

The only products of form *(translation)(horizontal slant)(rotation about origin)*

that would not transform a curve out of the JKM-space are the identity and

the 180° rotation. Since each of these can be expressed as an xy-dilation

($\alpha = \beta = 1$ or $\alpha = \beta = -1$), it follows that an affine transformation that

“stays inside” the JKM-space must be a product of two xy-dilations,

which of course is also an xy-dilation.

Thus any affine transformation that stays inside this JKM-space is an “xy dilation”,

a dilation of the form $(x, y) \rightarrow (\alpha x, \beta y)$ where $\alpha \neq 0$, $\beta \neq 0$.

Such a dilation resets five of the coefficients:

$$\begin{aligned} D' &= \frac{\alpha^2 \beta}{\beta^3} D = \frac{\alpha^2}{\beta^2} D & H' &= \frac{\alpha^2 \beta}{\beta^2} H = \frac{\alpha^2}{\beta} H \\ J' &= \frac{\alpha^2 \beta}{\alpha} J = \alpha \beta J & K' &= \frac{\alpha^2 \beta}{\beta} K = \alpha^2 K & M' &= \frac{\alpha^2 \beta}{1} M = \alpha^2 \beta M \end{aligned}$$

(So far this argument applies to any of the six subcases of Case 1.)

However (in subcase 1A) $D' = D = 1$ and $H' = H = 1$, so $\beta = 1$, $\alpha = 1$ or -1 ,

so that $J' = J$ or $-J$ $K' = K$ $M' = M$. An affine partition would

meet this JKM-space (i) at two points on opposite sides of the KM-plane

(with opposite values of J) or (ii) at one point on the KM-plane (with $J = 0$).

Thus $\{x^2y + y^3 + y^2 + Jx + Ky + M = 0, \text{ where } J \geq 0\}$ is a possible map of the affine partitions of the Case 1A curves; including exactly one curve from each partition.

Each subcase 1A curve has one asymptote and one asymptotic point not on the asymptote.

Each “standard” curve from this JKM-space uses the x-axis for an asymptote and has an asymptotic point at $(0, -1/2)$.

Subcase 1B: $A_4 = C_4 = F_4 = G_4 = 0, \quad B_4 = 1, \quad \text{and } D_4 > 0, \quad H_4 = 0$

The dilation, $(x, y) \rightarrow (x, \sqrt{D_4}y)$, followed by multiplication by $\sqrt{D_4}$, changes the equation to the form $x^2y + y^3 + J_5x + K_5y + M_5 = 0$.

If two of the curves in this JKM-space are in the same affine partition, they are connected by a dilation of the form $(x, y) \rightarrow (\alpha x, \beta y)$.

The dilation alters the coefficients: $D' = \frac{\alpha^2}{\beta^2} D, \quad J' = \alpha\beta J, \quad K' = \alpha^2 K, \quad M' = \alpha^2 \beta M$.

In this JKM-space, $D' = D = 1$, so $\beta = \alpha \text{ or } -\alpha$, so that $J' = \beta^2 J \text{ or } -\beta^2 J, \quad K' = \beta^2 K, \quad M' = \beta^3 M$.

An affine partition meets this JKM-space at (i) a pair of semicubic parabolas on either side of the KM-plane or at (ii) a single semicubic parabola in the KM-plane or at (iii) a ray on the M-axis or at (iv) a pair of rays in the JK-plane, one on each side of the J-axis or at (v) a ray on the J-axis or at (vi) the origin of JKM-space.

The subset

$$\{x^2y + y^3 + Jx + Ky + M = 0 \text{ where } J^2 + K^2 + M^2 = 1, J \geq 0, M \geq 0\}$$

$$\cup \{x^2y + y^3 = 0\}$$

would intersect each of these affine partitions exactly once, and therefore serve as a map of them.

Each subcase 1B curve has one asymptote and one asymptotic point lying on that asymptote.

Each of the “standard” curves in this JKM-space (in which all curves are Subcase 1B curves) has an asymptote at the x-axis and an asymptotic point at the origin.

Subcase 1C: $A_4 = C_4 = F_4 = G_4 = 0, B_4 = 1,$ and $D_4 < 0, H_4 \neq 0$

The dilation, $(x, y) \rightarrow \left(\frac{\sqrt{-D_4}}{H_4} x, \frac{-D_4}{H_4} y \right)$, followed by multiplication by $\frac{D_4^2}{H_4^3}$,

transforms the equation to the form $x^2y - y^3 + y^2 + J_5x + K_5y + M_5 = 0$.

If two of these curves are in the same affine partition, they might be connected by a dilation of the form, $(x, y) \rightarrow (\alpha x, \beta y)$, in which case

$$D' = \frac{\alpha^2}{\beta^2} D, \quad H' = \frac{\alpha^2}{\beta} H, \quad J' = \alpha\beta J, \quad K' = \alpha^2 K, \quad M' = \alpha^2 \beta M.$$

Then within this JKM-space, $D' = D = -1, H' = H = 1$, so $\alpha = 1 \text{ or } -1, \beta = 1$;

therefore $J' = J \text{ or } -J, K' = K, M' = M$.

However each of these curves has three asymptotes; so there is another way that an affine transformation could “stay inside” the JKM-space: a combination slant and rotation might permute the asymptotes. One such transformation is

$$\begin{bmatrix} -.5 & -1.5 & 0.25 \\ +.5 & -.5 & 0.25 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad (x, y) \rightarrow (0.25 - .5x - 1.5y, \quad 0.25 + .5x - .5y).$$

It transforms a curve of form $x^2y - y^3 + y^2 + Jx + Ky + M = 0$ to one of form $x^2y - y^3 + y^2 + J'x + K'y + M' = 0$, where

$$J' = -0.125 - 0.5J - 0.5K$$

$$K' = -0.375 + 1.5J - 0.5K$$

$$M' = +0.0625 - 0.25J + 0.25K + M.$$

The square of this transformation is of course another permutation of asymptotes.

Combining these transformations with reflection across the y-axis,

$((x, y) \rightarrow (-x, y), \quad J' = -J)$, generates the remaining permutations of the asymptotes.

Thus the KM-plane ($J = 0$) of JKM-space is transformed to two other planes ($-J + K + 0.25 = 0$ and $J + K + 0.25 = 0$); these meet at a single line ($J = 0, K = -0.25$) and divide up the rest of JKM-space into six “wedges”.

An affine partition of subcase 1C may consist of (i) six points, one in each of the “wedges” of JKM-space, or (ii) three points, one on each of the three planes, or (iii) one point on the line ($J = 0, K = -0.25$) where the three planes meet.

The subset $\{x^2y - y^3 + y^2 + Jx + Ky + M = 0, \quad \text{where} \quad 0 \leq J \leq K + 0.25\}$

will intersect each of these affine partitions exactly once.

Each curve from subcase 1C has three asymptotes which define a triangle.

Each curve from this JKM-space has three asymptotes ($y = 0$, $x + y - 0.5 = 0$, $x - y + 0.5 = 0$); these define a right isosceles triangle.

Subcase 1D: $A_4 = C_4 = F_4 = G_4 = 0$, $B_4 = 1$, and $D_4 < 0$, $H_4 = 0$

The dilation $(x, y) \rightarrow (x, \sqrt{-D_4}y)$, followed by multiplication by $\sqrt{-D_4}$, transforms the equation to the form $x^2y - y^3 + J_5x + K_5y + M_5 = 0$.

Two of these curves in the same affine partition might be connected by a dilation of form $(x, y) \rightarrow (\alpha x, \beta y)$, in which case $D' = \frac{\alpha^2}{\beta^2}D$, $J' = \alpha\beta J$, $K' = \alpha^2 K$, $M' = \alpha^2 \beta M$. If they are both “standard” curves, $D' = D = -1$, so $\alpha = \beta$ or $-\beta$, therefore $J' = \beta^2 J$ or $-\beta^2 J$, $K' = \beta^2 K$, $M' = \beta^3 M$.

However each of these curves has three asymptotes; a combination rotation and slant that permuted these asymptotes would also stay inside this JKM-space.

One such transformation is

$$\begin{bmatrix} -0.5 & -1.5 & 0 \\ +0.5 & -0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad (x, y) \rightarrow (-.5x - 1.5y, +.5x - .5y) .$$

It transforms a curve of form $x^2y - y^3 + Jx + Ky + M = 0$ to a curve of form $x^2y - y^3 + J'x + K'y + M' = 0$, where $J' = -0.5J - 0.5K$

$$K' = +1.5J - 0.5K$$

$$M' = M.$$

The square of this transformation, the reflection across the y-axis and their various products generate the other permutations of the three asymptotes.

These permutations transform the KM-plane ($J = 0$) into two other planes ($-J + K = 0$ and $J + K = 0$); the three planes meet the M-axis and divide up the rest of JKM-space into six “wedges”. An affine partitions of subcase 1D may consist of (i) six semicubic parabolas, one in each of the wedges, or (ii) three semicubic parabolas, one on each of the planes between the wedges, or (iii) the M-axis (except the origin), or (iv) six rays on the JK-plane, one in each wedge (v) the three rays at the intersections of the JK-plane with the other three planes, or (vi) the origin of JKM-space.

The subset

$$\left\{ x^2y - y^3 + Jx + Ky + M = 0, \text{ where } J^2 + K^2 + M^2 = 1, 0 \leq J \leq K, 0 \leq M \right\} \\ \cup \left\{ x^2y - y^3 = 0 \right\}$$

intersects each of the affine partitions exactly once.

Each curve for subcase 1D has three asymptotes that meet at a single point.

Each “standard” curve from this JKM-space has three asymptotes,

($y = 0$, $-x + y = 0$, $x + y = 0$), that meet at the origin.

Subcase 1E: $A_4 = C_4 = F_4 = G_4 = 0$, $B_4 = 1$, and $D_4 = 0$, $H_4 \neq 0$.

The dilation, $(x, y) \rightarrow (x, H_4 y)$, followed by multiplication by H_4 , transforms the equation to the form $x^2 y + y^2 + J_5 x + K_5 y + M_5 = 0$.

If two of these curves are in the same affine partition, they must be connected by a dilation of form $(x, y) \rightarrow (\alpha x, \beta y)$. Such a dilation would make $H' = \frac{\alpha^2}{\beta} H$, $J' = \alpha \beta J$, $K' = \alpha^2 K$, $M' = \alpha^2 \beta M$. If the dilation stays inside this JKM-space, $H' = H = 1$, so $\beta = \alpha^2$, therefore $J' = \alpha^3 J$, $K' = \alpha^2 K$, $M' = \alpha^4 M$.

An affine partition of subcase 1E where $K \neq 0$ may consist of (i) a curve of the form $J^2 = c_1 K^3$ and $M = c_2 K^2$, or (ii) half of a parabola in the KM-plane, or (iii) a semicubic parabola in the JK-plane, or (iv) half of the K-axis.

An affine partition of subcase 1E where $K = 0$ may consist of

(v) a curve of form $M^3 = k_1 J^4$ in the JM-plane, or (vi) the J-axis, or (vii) half of the M-axis, or (viii) the origin of JKM-space.

The subset $\{x^2 y + y^2 + Jx + Ky + M = 0, \text{ where } J^2 + K^2 + M^2 = 1, 0 \leq J\}$
 $\cup \{x^2 y + y^2 = 0\}$
 would intersect each of these partitions exactly once.

Each curve of subcase 1E has an asymptote and an asymptotic parabola (see definition on page 4) whose axis intersects the asymptote.

Each curve of this JKM-space uses the x-axis for an asymptote and has an asymptotic parabola whose axis is the y-axis.

Subcase 1F: $A_4 = C_4 = F_4 = G_4 = 0$, $B_4 = 1$, and $D_4 = H_4 = 0$.

No initial dilation is needed; the equation already has the form

$$x^2 y + J_4 x + K_4 y + M_4 = 0 .$$

If two of these curves are in the same affine partition, they must be connected by a dilation of form $(x, y) \rightarrow (\alpha x, \beta y)$.

(A Subcase 1F curve may have a single asymptote and a double asymptote, but no affine transformation will map a single asymptote to a double asymptote or vice versa. The curve may have three asymptotes, two of them parallel, but no affine transformation will map the transversal to one of the pair of parallel lines. In Subcase 1F, the x-axis is the single asymptote or the transversal; it is not mapped to any other asymptote.)

Thus $J' = \alpha\beta J$, $K' = \alpha^2 K$, $M' = \alpha^2 \beta M$.

The subset

$$\begin{aligned} & \left\{ x^2 y + Jx + y + M = 0, \text{ where } J^2 + M^2 = 1, J \geq 0, M \geq 0 \right\} \\ & \cup \left\{ x^2 y + y = 0 \right\} \\ & \cup \left\{ x^2 y + Jx - y + M = 0, \text{ where } J^2 + M^2 = 1, J \geq 0, M \geq 0 \right\} \\ & \cup \left\{ x^2 y - y = 0 \right\} \\ & \cup \left\{ x^2 y + x + 1 = 0, x^2 y + x = 0, x^2 y + 1 = 0, x^2 y = 0 \right\} \end{aligned}$$

will intersect each of these partitions exactly once.

A subcase 1F curve may have (i) one asymptote, or (ii) an asymptote and a double asymptote (which intersect), or (iii) three asymptotes, two of them parallel.

A subcase 1F curve from this JKM-space has (i) an asymptote along the x-axis, or (ii) an asymptote along the x-axis and a double asymptote along the y-axis, or (iii) an asymptote along the x-axis and two asymptotes parallel to the y-axis.

Case 1.5: $B_2 = 0, C_2 \neq 0$.

The slant $(x, y) \rightarrow \left(x + \frac{D_2}{C_2} y, y \right)$ transforms the curve to the form

$$C_3 xy^2 + F_3 x^2 + G_3 xy + H_3 y^2 + J_3 x + K_3 y + M_3 = 0.$$

A 90° rotation changes it to $C_3 x^2 y + H_3 x^2 - G_3 xy + F_3 y^2 - K_3 x + J_3 y + M_3 = 0$ and the translation $\left(\frac{H_3}{C_3}, \frac{-G_3}{2C_3} \right)$, followed by division by C_3 , puts it in the form,

$$x^2 y + H_4 y^2 + J_4 x + K_4 y + M_4 = 0, \text{ which is equivalent to subcase 1E or 1F of Case 1.}$$

Case 2: $A_2 = B_2 = C_2 = 0$, $D_2 \neq 0$, $F_2 \neq 0$.

A horizontal slant, $(x, y) \rightarrow \left(x + \frac{G_2}{2F_2}y, y\right)$, followed by division by D_2 , transforms the curve to $y^3 + F_3x^2 + H_3y^2 + J_3x + K_3y + M_3 = 0$;
then the translation $\left(\frac{J_3}{2F_3}, \frac{H_3}{3}\right)$ transforms it to $y^3 + F_4x^2 + K_4y + M_4 = 0$.
(In this case, $F_4 = F_3 = F_2 \neq 0$.)

A dilation, $(x, y) \rightarrow \left(x, \frac{y}{\sqrt[3]{F_3}}\right)$, followed by division by F_3 , transforms the curve to the form $y^3 + x^2 + K_5y + M_5 = 0$. Each such curve may be regarded as a point on a "KM-plane".

If two points on the KM-plane stand for curves in the same affine partition, the curves must be connected by some affine transformation. It can be shown that no non-trivial rotation, translation or slant stays inside this KM-plane; the affine transformation must be a dilation such as $(x, y) \rightarrow (\alpha x, \beta y)$.
(The 180° rotation stays inside the KM-plane, but the 180° rotation may be regarded as the dilation $(x, y) \rightarrow (-x, -y)$.)

The dilation alters the coefficients: $F' = \frac{\beta^3}{\alpha^2}$, $K' = \frac{\beta^3}{\beta}K = \beta^2K$, $M' = \beta^3M$.
 $F' = F = 1$, so $\beta^3 = \alpha^2$ and $\beta \geq 0$.

In the KM-plane, one of these affine partitions may appear as

(i) half of a “semicubic parabola” (see below), or (ii) half of the K-axis,
or (iii) half of the M-axis, or (iv) the origin of the KM-plane.

The subset $\{y^3 + x^2 + Ky + M = 0, \text{ where } K^2 + M^2 = 1\} \cup \{y^3 + x^2 = 0\}$

would intersect each of these partitions exactly once. It appears (along with examples of the affine partitions) in the illustration at the end of this paper.

Each Case 2 curve approaches a curve $k_1 y^3 + k_2 x^2 = 0$, where $k_1 \neq 0, k_2 \neq 0$;
which is called a “semicubic parabola”.

Each curve in this KM-plane approaches the semicubic parabola $y^3 + x^2 = 0$;
moreover each of these curves has bilateral symmetry (across the y-axis),
since x is always to an even power.

It is amusing that while each Case 2 curve approaches a semi-cubic parabola
(in the xy-plane), most affine partitions of these Case 2 curves appear as
semi-cubic parabolas (in the KM-plane).

Case 3: $A_2 = B_2 = C_2 = 0, \quad D_2 \neq 0, \quad F_2 = 0, \quad G_2 \neq 0.$

A horizontal slant, $(x, y) \rightarrow \left(x + \left[\frac{H_2}{G_2} - \frac{3D_2 J_2}{G_2^2} \right] y, y \right),$

followed by the translation $\left(\frac{3D_2 J_2^2}{G_2^3} - \frac{2H_2 J_2}{G_2^2} + \frac{K_2}{G_2}, \frac{J_2}{G_2} \right),$ and division by $D_2,$

transforms the equation to the form $y^3 + G_4 xy + M_4 = 0.$

If $M_4 \neq 0$, the dilation $(x, y) \rightarrow \left(\frac{G_4}{M_4^{2/3}} x, \frac{1}{M_4^{1/3}} y \right)$,

followed by division by M_4 , transforms the curve to $y^3 + xy + 1 = 0$.

All trident curves are affine-equivalent to this curve.

Each such curve has one asymptote and one asymptotic parabola, whose axis goes in the same direction as the asymptote.

If $M_4 = 0$, the dilation $(x, y) \rightarrow (G_4 x, y)$ transforms the curve to

$y^3 + xy = 0$. Every degenerate Case 3 curve is affine-equivalent to this curve, the union of a parabola and a line in the direction of its axis.

Case 4: $A_2 = B_2 = C_2 = 0$, $D_2 \neq 0$, $F_2 = G_2 = 0$, $J_2 \neq 0$.

A horizontal slant, $(x, y) \rightarrow \left(x + \left[\frac{K_2}{J_2} - \frac{H_2^2}{3D_2 J_2} \right] y, y \right)$,

followed by the translation $\left(\frac{M_2}{J_2} - \frac{H_2^3}{27D_2^2 J_2}, \frac{H_2}{3D_2} \right)$ and division by D_2 ,

transforms the equation to $y^3 + \frac{J_2}{D_2} x = 0$.

Then the dilation $(x, y) \rightarrow \left(\frac{J_2}{D_2} x, y \right)$ transforms the equation to $y^3 + x = 0$.

Each Case 4 curve is affine equivalent to the curve $y^3 + x = 0$, which is known as a “cubic parabola”; thus all Case 4 curves are in the same affine partition.

Notice that while Case 1 and Case 2 each have uncountably many affine partitions, Case 3 has only two affine partitions and Case 4 has only one.

Case 5: $A_2 = B_2 = C_2 = 0$, $D_2 \neq 0$, $F_2 = G_2 = J_2 = 0$.

Since the equation has no terms with x , its curve is merely a set of horizontal lines , so the horizontal translation, the horizontal dilation and the horizontal slant have no effect.

A translation, $\left(0, \frac{H_2}{3D_2}\right)$, followed by division by D_2 , transforms the equation to the form $y^3 + K_3y + M_3 = 0$. Each such equation may be thought of as point on a “KM-plane”.

Two points on this plane represent curves in the same affine partition if and only if the curves are connected by an affine transformation. The only affine transformations that “stay inside” this KM-plane are dilations of the form $(x, y) \rightarrow (\alpha x, \beta y)$. Such a dilation changes the coefficients to $K' = \frac{\beta^3}{\beta} K = \beta^2 K$ $M' = \beta^3 M$.

An affine partition of the KM-plane would appear as (i) a semicubic parabola, or (ii) the K-axis, or (iii) half of the M-axis or (iv) the origin of the KM-plane.

The subset $\{y^3 + Ky + M = 0, \text{ where } K^2 + M^2 = 1, M \geq 0\} \cup \{y^3 = 0\}$ would intersect each of these (infinitely many) affine partitions exactly once.

“Three-line” curves belong to the same affine partition if and only the three distances between their three component lines show matching ratios.

The union of the ten subsets devised for subcases 1A, 1B, 1C, 1D, 1E, 1F and Cases 2, 3, 4 and 5 includes curves from every affine partition of the cubic curves. This union would be a list of these affine partitions (or more vaguely, a list of the kinds of shapes that a third-degree curve can take). I have already shown that each affine partition appears only once in each case/subcase; now I will show that no partition appears in two different cases or subcases.

An affine transformation always maps asymptotes to asymptotes and asymptotic centers to asymptotic centers. It maps asymptotic parabolas to asymptotic parabolas (but the asymptotic parabola of a curve is not unique; it may be replaced by any other parabola with the same axis and directrix). An affine transformation maps intersecting lines to intersecting lines and parallel lines to parallel lines (and it maps parabolas to parabolas). These properties of affine transformations can serve to segregate most of the cases and subcases in this chapter.

Thus each subcase 1A curve has an asymptote and an asymptotic center off the asymptote, a configuration that does not appear in any other case or subcase. Likewise each subcase 1B curve has an asymptote and an asymptotic center lying on the asymptote; each subcase 1C curve has three asymptotes that form a triangle; each subcase 1D curve has three asymptotes through a single point; each subcase 1E curve has an asymptote and an asymptotic parabola whose axis goes in a different direction; each Case 3 curve has an asymptote and an asymptotic parabola whose axis goes in the same direction. Curves of subcase 1F and Case 5 have one, two or three asymptotes of which no more than two intersect. Curves of Case 2 and Case 4 have no asymptotes at all.

To segregate Case 2 from Case 4, note that each Case 2 curve approaches a semicubic parabola; while each Case 4 curve approaches a cubic parabola. An affine transformation cannot map a semicubic parabola to a cubic parabola, nor can it map a curve that approaches a semicubic parabola to a curve that approaches a cubic parabola.

To segregate subcase 1F from Case 5, note that an affine transformation that mapped a subcase 1F curve to a Case 5 curve could be factored into the form

$(rotation)(horizontal\ slant)(translation)(xy\ dilation)$.

The $(xy\ dilation)$ cannot change a subcase 1F curve to a Case 5 curve; it does not change the zero/nonzero status of any of the equation's coefficients.

The product $(rotation)(horizontal\ slant)$ cannot change a subcase 1F curve to a Case 5 curve; an angle/slant combination that yielded $A' = B' = 0$

would not yield $C' = 0$. Neither can $(translation)$ change a subcase 1F curve into a Case 5 curve; it has no effect on the cubic coefficients A, B, C, D .

Since each individual transformation transforms subcase 1F to subcase 1F, one must expect that their product will also transform subcase 1F to subcase 1F.

If any affine partition included curves from two different cases or subcases, there would be an affine transformation connecting each case/subcase with the other. Since no such affine transformation seems to exist, one must conclude that no affine partition includes curves from two different cases or subcases.

Thus the ten lists of affine partitions in this chapter are completely separate; together they form a list of all affine partitions of cubic curves.

CHAPTER FOUR

CUBIC CURVES UNDER PROJECTIVE TRANSFORMATIONS

Transformations which always map straight lines to straight lines are called “projective transformations”; in homogenous coordinates they are equivalent to multiplication by an arbitrary invertible 3x3 matrix:

$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ \pi & \rho & \sigma \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \quad (3 \times 3 \text{ matrix has determinant } \neq 0) .$$

In affine coordinates such a transformation looks like a rational transformation of form

$$(x, y) \rightarrow \left(\frac{ax + by + c}{\pi x + \rho y + \sigma}, \frac{dx + ey + f}{\pi x + \rho y + \sigma} \right) .$$

Geometrically, a “homology” of a curve can be seen as its “shadow” under projection from an arbitrary point to an arbitrary plane; any projective transformation may be interpreted as a composition of several such homologies.

Affine transformations (and therefore euclidean transformations) are projective transformations. Thus, one may be sure that the product of any list of euclidean, affine and projective transformations is itself a projective transformation.

A projective transformation may transform the “line at infinity” to an ordinary line or transform or vice versa. This means that a projective transformation need not map asymptotes to asymptotes (an asymptote of a curve may be defined as its tangent at the point where it intersects the line at infinity). The classification that appears in chapters two and three is largely based on mapping asymptotes to asymptotes; a different scheme must be used in this chapter.

A point of at least second order on a curve (see chapter one) is called a “singularity”. Most polynomial curves of any degree do not have any singularities; a curve that has at least one singularity is therefore called a “singular” curve.

It will be convenient to divide the cubic curves into a “Generic Case” (curves are nondegenerate and have no singular points), a “Singular Case” (curves are nondegenerate but each has at least one singular point), and a “Degenerate Case” (curves are degenerate and therefore must have singular points, at least in the complex plane).

The “Hessian” of a polynomial $F(X, Y, W)$ (of any degree) is defined as

$$\text{Hessian}(F)(X, Y, W) = \det \begin{bmatrix} \frac{\delta^2 F}{\delta X^2} & \frac{\delta^2 F}{\delta X \delta Y} & \frac{\delta^2 F}{\delta X \delta W} \\ \frac{\delta^2 F}{\delta Y \delta X} & \frac{\delta^2 F}{\delta Y^2} & \frac{\delta^2 F}{\delta Y \delta W} \\ \frac{\delta^2 F}{\delta W \delta X} & \frac{\delta^2 F}{\delta W \delta Y} & \frac{\delta^2 F}{\delta W^2} \end{bmatrix} .$$

(The “Hessian curve” of a polynomial is the curve $\text{Hessian}(\text{polynomial}) = 0$.)

Thus the Hessian of the cubic

$$AX^3 + BX^2Y + CXY^2 + DY^3 + FX^2W + GXYW + HY^2W + JXW^2 + KYW^2 + MW^3 = 0$$

is

$$\det \begin{bmatrix} 6AX + 2BY + 2FW & 2BX + 2CY + GW & 2FX + GY + 2JW \\ 2BX + 2CY + GW & 2CX + 6DY + 2HW & GX + 2HY + 2KW \\ 2FX + GY + 2JW & GX + 2HY + 2KW & 2JX + 2KY + 6MW \end{bmatrix} ;$$

the Hessian of a cubic in homogenous coordinates is itself another cubic.

The Hessian of the corresponding affine cubic

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

is

$$\det \begin{bmatrix} 6Ax + 2By + 2F & 2Bx + 2Cy + G & 2Fx + Gy + 2J \\ 2Bx + 2Cy + G & 2Cx + 6Dy + 2H & Gx + 2Hy + 2K \\ 2Fx + Gy + 2J & Gx + 2Hy + 2K & 2Jx + 2Ky + 6M \end{bmatrix} .$$

The Hessian of a curve goes to zero at a singularity of that curve.

(Move the singularity to the origin; then the polynomial for the curve will have

$M = J = K = 0$; the Hessian of such a curve must go to zero at the origin.)

The Hessian of a curve goes to zero at a point of inflection of the curve.

In the proof of his Theorem III.6.3, Walker uses the derivatives of F at some point of inflection a to define two polynomials,

$$L_a(X_1, X_2, X_3) = \sum \left(\frac{\partial F}{\partial X_i} \text{ at } a \right) X_i$$

and $Q_a(X_1, X_2, X_3) = \sum \left(\frac{\partial^2 F}{\partial X_i \partial X_j} \text{ at } a \right) X_i X_j$.

The curve $Q_a = 0$ has a triple intersection with the cubic curve at a (it is called the “osculating conic” at that point). The line $L_a = 0$ is the tangent to the cubic at a ; since this line is the tangent at a point of inflection, it too has a triple intersection with $F = 0$ at a . This means that $Q_a = 0$ and $L_a = 0$ also have a triple intersection with each other at a . But then $L_a = 0$ must be a component of $Q_a = 0$ (an independent line and conic can intersect no more than two times in the plane) and therefore $Q_a = 0$ must be degenerate.

The matrix $A_{ij} = \left(\frac{\partial^2 F}{\partial X_i \partial X_j} \text{ at } a \right)$ must also be degenerate and the Hessian at a (which is its determinant) must be zero.

Any point where the Hessian goes to zero must be a point of inflection or a singularity. Suppose the Hessian at point a is zero. Then matrix A_{ij} is degenerate and polynomial Q_a is also degenerate; thus Q_a is the product of two linear (but maybe complex) polynomials. As ever, $Q_a = 0$ has a triple intersection with $F = 0$ at a . If a is not a singularity, $F = 0$ has only one tangent at a ; this tangent must be one of the components of $Q_a = 0$ (otherwise Q_a could not have a triple intersection with F there); the other component does not go through a at all (if it did, a would be a singularity of Q_a and therefore a singularity of F), so our tangent has a triple intersection with $F = 0$ at a ; a is a point of inflection. A point that is not a singularity where the Hessian is zero must be a point of inflection.

Together these three arguments show that a point is a singularity or point of inflection of some cubic if and only if the cubic intersects its own Hessian there.

GENERIC CUBIC CURVES (nondegenerate cubic curves with no singularity)

A generic cubic has a Hessian which is an independent cubic and therefore intersects it exactly nine times in the complex projective plane (by Bezout's theorem). There may be multiple intersections at the same point.

It can be shown that if polynomial $\bar{F}$ is conjugate to polynomial F , then $Hessian(\bar{F})$ is conjugate to $Hessian(F)$ and each intersection between curves $\bar{F}$ and $Hessian(\bar{F})$ is conjugate to an intersection between curves F and $Hessian(F)$. If F is a real polynomial, $\bar{F} = F$ and each intersection between curves F and $Hessian(F)$ is conjugate to another such intersection (or to itself). If F is also a generic cubic, there are nine such intersections; since nine is odd, at least one of these intersections must be conjugate to itself, that is it must lie in the real projective plane. This intersection must be at a point of inflection or a singularity, but a generic cubic curve has no singularities. Thus a real generic cubic curve must have at least one point of inflection in the real projective plane.

Any real generic cubic is projectively equivalent (by a real projection) to a Case 2 curve. To show this, find a real point of inflection on such a cubic and project it to the infinite point, $(X, Y, W) = (1, 0, 0)$, then rotate it about that point until it is tangent to the line at infinity.

The result has form

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0,$$

where $A = 0$ (because the curve goes through $(1, 0, 0)$), $B = 0$ (because the curve is tangent to the line at infinity at $(1, 0, 0)$) and $C = 0$ (because $(1, 0, 0)$ is also a point of inflection). It then follows that $D \neq 0$ ($D = 0$ would mean that the polynomial was not cubic) and $F \neq 0$ ($F = 0$ would mean that it was degenerate).

Thus this result has form

$$Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0 \text{ with } D \neq 0, F \neq 0 ;$$

it is a Case 2 curve.

Since every Case 2 curve is connected by an affine transformation either to a curve of form $y^3 + x^2 + Ky + M = 0$, where $K^2 + M^2 = 1$ or to the curve $y^3 + x^2 = 0$, it follows that every generic cubic curve is connected by a projective transformation to such a curve. Thus every projective partition of the generic cubics has at least one representative in the set $\{y^3 + x^2 + Ky + M = 0, \text{ where } K^2 + M^2 = 1\} \cup \{y^3 + x^2 = 0\}$; some subset of this set has precisely one curve from each of these projective partitions.

This subset cannot contain any singular or degenerate curves (a projective transformation maps a singular curve to a singular curve and a degenerate curve to a degenerate curve). In the KM-plane, nonsingularity is equivalent to the condition $4K^3 + 27M^2 \neq 0$. All curves in the KM-plane are nondegenerate since Case 2 curves are never degenerate.

In The Arithmetic of Elliptic Functions, Joseph Silverman defines a “J invariant”

for non-singular Case 2 curves; then he shows that two such curves are

complex projective equivalent if and only if they have the same J invariant.

For curves of form $y^3 + x^2 + Ky + M = 0$, the J invariant is $\frac{6912K^3}{4K^3 + 27M^2}$

(since a non-singular curve must have $4K^3 + 27M^2 \neq 0$, it is always defined).

For a given K, M , the curves $y^3 + x^2 + Ky + M = 0$, $y^3 + x^2 + Ky - M = 0$

have the same J (and are connected by the dilation $(x, y) \rightarrow (ix, -y)$).

The curves $y^3 + x^2 + y = 0$ and $y^3 + x^2 - y = 0$ also have the same J

(and are connected by the dilation $(x, y) \rightarrow (\sqrt{i}x, -iy)$). One can show that there are

no other duplications of a value of J as long as K and M are real

(in the two quadrants of the “circle” $K^2 + M^2 = 1$ where $M \geq 0$,

J decreases monotonically from 1 to $-\infty$ and then from $+\infty$ to 1;

in the other two quadrants it does the reverse).

The “circle” of non-singular Case 2 curves,

$$\{y^3 + x^2 + Ky + M = 0, \text{ where } K^2 + M^2 = 1, 4K^3 + 27M^2 \neq 0\},$$

has two curves in each complex projective partition .

However this chapter is supposed to be about real projective partitions. Are these pairs of complex-projective equivalent curves also real-projective equivalent?

I think not. However I can offer no evidence for my conjecture other than the appearance of (the real parts of) their graphs. Several of these pairs of equations yield curves that look so different that it is hard to imagine any (real) projective transformation which would bring them together.

If my conjecture is correct, the set

$$\{y^3 + x^2 + Ky + M = 0, \text{ where } K^2 + M^2 = 1, 4K^3 + 27M^2 \neq 0\}$$

includes exactly one curve from each real projective partition of the generic cubic curves. If my conjecture fails, at least some of the real projective partitions are represented twice (and are equivalent to complex projective partitions); the above set would not be a one-to-one map of the complex real partitions, but some subset of it would be such a map.

SINGULAR (but nondegenerate) CUBIC CURVES

Use a translation (or a central projection followed by a translation) to move the singularity to the origin. The new curve has $M_2 = 0$ (because it passes through the origin) and $J_2 = K_2 = 0$ (because the origin is a singularity), so its equation has the form $A_2x^3 + B_2x^2y + C_2xy^2 + D_2y^3 + F_2x^2 + G_2xy + H_2y^2 = 0$.

Rotate the curve until $C_3F_3^2 - A_3F_3H_3 - B_3F_3G_3 + A_3G_3^2 = 0$.

(A 180° rotation reverses the sign of this polynomial, which is a continuous function of the angle; by the intermediate value theorem there must be some rotation which reduces its value to zero.) The singularity is still at the origin, so the new curve's equation still has the form

$$A_3x^3 + B_3x^2y + C_3xy^2 + D_3y^3 + F_3x^2 + G_3xy + H_3y^2 = 0.$$

subcase: $F_3 \neq 0$.

$$\text{The central projection, } \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{A}{F} & \frac{BF - AG}{F^2} & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix},$$

transforms the curve into a curve with $A_4 = B_4 = C_4 = 0$. On this new curve, $D_4 \neq 0$

($D_4 = 0$ implies that it is quadratic, and thus that the original curve was quadratic)

and $F_4 \neq 0$ ($F_4 = 0$ implies that it is degenerate, and thus that the original curve was degenerate). The new curve is therefore a Case 2 curve.

Note that when $C_3 F_3^2 - A_3 F_3 H_3 - B_3 F_3 G_3 + A_3 G_3^2 = 0$,

$F_3 = 0$ implies $A_3 G_3^2 = 0$, and therefore $A_3 = 0$ or $G_3 = 0$.

subcase: $F_3 = A_3 = 0$.

The curve is degenerate (one of its components is the x -axis), so the original curve must also have been degenerate.

subcase: $F_3 = G_3 = 0$, $A_3 \neq 0$, $H_3 \neq 0$.

The central projection,
$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{C_3}{H_3} - \frac{B_3^2}{3A_3H_3} & \frac{D_3}{H_3} - \frac{B_3^3}{27A_3^2H_3} & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix},$$

followed by the slant,
$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{B_3}{3A_3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
 transforms the curve to the form

$$A_4 x^3 + H_4 y^2 = 0, \quad \text{where } A_4 \neq 0, H_4 \neq 0,$$

which is a Case 2 curve (a 90° rotation of the “standard” Case 2 curve).

It can be shown that a curve of this subcase has a horizontal cusp at the origin.

The central projection and horizontal slant transform it into a semicubic parabola.

subcase: $F_3 = G_3 = 0$, $A_3 \neq 0$, $H_3 = 0$.

The curve is degenerate (each component is either a line through the origin or the point at the origin), so the original cubic must have been degenerate also.

In each of the subcases, the curve is projective-equivalent to either a Case 2 curve or a degenerate curve. A curve that is projective-equivalent to a degenerate curve must be degenerate itself; thus any singular nondegenerate cubic must be projective-equivalent to a Case 2 curve (which is also singular).

There are only three affine partitions of singular curves in Case 2.

(a curve in the “standard” form, $y^3 + x^2 + Ky + M = 0$, is singular if and only if

$4K^3 + 27M^2 = 0$. The three singular affine partitions correspond to the cases

$M > 0$, $M < 0$ and $M = 0$.) In one of these partitions (e.g. $y^3 + x^2 - 3y + 2 = 0$),

the singularity is always an isolated point; in another (e.g. $y^3 + x^2 - 3y - 2 = 0$),

it is always a “node” or self-intersection; in the third (e.g. $y^3 + x^2 = 0$),

the singularity is always a cusp. Since a real projective transformation cannot

change an isolated point to a node or change either to a cusp, it is evident that

the three affine partitions are also three separate projective partitions.

Since every singular, nondegenerate cubic curve is projective-equivalent

to one of these partitions, it is also evident that these are the only three

projective partitions of singular, nondegenerate cubics.

ONE OF THE REAL PROJECTIVE PARTITIONS
(THE "CUSP" PARTITION) APPEARS AT ONLY ONE POINT,
THE CENTER OF THE CIRCLE.

IF MY CONJECTURE IS CORRECT,
EACH OF THE OTHER REAL PROJECTIVE PARTITIONS
OCCUPIES ONE OF THE DOTTED RAYS
(EACH RAY IS HALF OF A SEMICUBIC PARABOLA)
AND INTERSECTS THE CIRCLE EXACTLY ONCE,

DEGENERATE CUBIC CURVES

One can hardly expect that a projective transformation would map one of the degenerate cubic curves to a Case 2 curve; it must map a degenerate curve to a degenerate curve and the Case 2 curves are never degenerate.

A degenerate cubic polynomial must have a factorization in one of the forms, a line * quadratic polynomial, a line * line * line, a line * double line, or a triple line.

If it is a triple line, simply map it to the line at infinity.

Otherwise, use a central projection to map a single-line factor to the line at infinity. The finite part of the curve is then some sort of quadratic curve. Next, use an affine transformation to map the quadratic curve to one of the nine “standard” quadratic curves:

$$x^2 + y^2 + 1 = 0$$

$$x^2 + y^2 - 1 = 0$$

$$x^2 + y = 0$$

$$x^2 - y^2 - 1 = 0$$

$$x^2 + y^2 = 0$$

$$x^2 + 1 = 0$$

$$x^2 - y^2 = 0$$

$$x^2 - 1 = 0$$

$$x^2 = 0 .$$

Since any affine transformation maps the line at infinity to itself, this algorithm
(a central projection with an affine transformation, and so a projective transformation)

maps any degenerate cubic to one of

$W(X^2 + Y^2 + W^2) = 0$	(line and imaginary conic)
$W(X^2 + Y^2 - W^2) = 0$	(line outside real conic)
$W(X^2 + YW) = 0$	(line tangent to real conic)
$W(X^2 - Y^2 - W^2) = 0$	(line cutting across real conic)
$W(X^2 + Y^2) = 0$	(line and isolated point)
$W(X^2 + W^2) = 0$	(line and one of its own points)
$W(X^2 - Y^2) = 0$	(three lines forming a triangle)
$W(X^2 - W^2) = 0$	(three lines through a common point)
$WY^2 = 0$	(line and double line)

(or, if the degenerate cubic is a triple line)

$$W^3 = 0 \quad . \quad (\text{triple line}) .$$

A projective transformation always maps a line to a line, and separate lines
to separate lines. It maps a double line to a double line and a triple line to a triple line.

It maps intersections to intersections and tangents to tangents.

It maps a conic curve to a conic curve.

In addition, a real projective transformation maps a real conic curve to a real
conic curve, an imaginary conic to an imaginary conic, and maps isolated points
to isolated points.

Thus each of the ten curves listed on the last page differs from the other nine in some property that is “conserved” by real projective transformations; so each must be in a separate real projective partition. Since it is already shown that every degenerate cubic is projective-equivalent to one of these ten, it follows that the degenerate cubics have exactly ten real-projective partitions, one for each of the ten curves on the list.

Here, then, is the list of real-projective partitions of cubic curves. The center of the circle in KM-space and the points on its circumference represent the partitions of the nondegenerate cubics; the ten curves from the last page represent the partitions of the degenerate cubics.

**APPENDIX: Six “Standard” Transformations and
their effects on Third-degree Curves**

Here are the geometric effects of six “standard” transformations on a generic cubic curve. The transformations are expressed as movement of the curve against fixed coordinates.

Since any projective transformation can be expressed as a product of these six transformations (see page 36), this table may be used to compute the effect of any projective transformation on any cubic curve.

The TRANSLATION, $(x, y) \rightarrow (x + c, y + f)$ (in affine coordinates)

$$\text{or } \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \quad (\text{in projective coordinates}),$$

changes the curve of

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

to the curve of

$$A'x^3 + B'x^2y + C'xy^2 + D'y^3 + F'x^2 + G'xy + H'y^2 + J'x + K'y + M' = 0,$$

where

$$A' = A$$

$$B' = B$$

$$C' = C$$

$$D' = D$$

$$F' = F - 3cA - fB$$

$$G' = G - 2cB - 2fC$$

$$H' = H - cC - 3fD$$

$$J' = J - 2cF - fG + 3c^2A + 2cfB + f^2C$$

$$K' = K - cG - 2fH + c^2B + 2cfC + 3f^2D$$

$$M' = M - cJ - fK + c^2F + cfG + f^2H - c^3A - c^2fB - cf^2C - f^3D.$$

The ROTATION about the origin,

$$\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \text{ (in projective coordinates),}$$

changes the curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

to the curve

$$A'x^3 + B'x^2y + C'xy^2 + D'y^3 + F'x^2 + G'xy + H'y^2 + J'x + K'y + M' = 0,$$

where

$$A' = A \cos^3 \theta - B \sin \theta \cos^2 \theta + C \sin^2 \theta \cos \theta - D \sin^3 \theta$$

$$B' = 3A \sin \theta \cos^2 \theta + B(\cos^3 \theta - 2 \sin^2 \theta \cos \theta) + C(\sin^3 \theta - 2 \sin \theta \cos^2 \theta) + 3D \sin^2 \theta \cos \theta$$

$$C' = 3A \sin^2 \theta \cos \theta - B(\sin^3 \theta - 2 \sin \theta \cos^2 \theta) + C(\cos^3 \theta - 2 \sin^2 \theta \cos \theta) - 3D \sin \theta \cos^2 \theta$$

$$D' = A \sin^3 \theta + B \sin^2 \theta \cos \theta + C \sin \theta \cos^2 \theta + D \cos^3 \theta$$

$$F' = F \cos^2 \theta - G \sin \theta \cos \theta + H \sin^2 \theta$$

$$G' = 2F \sin \theta \cos \theta + G(\cos^2 \theta - \sin^2 \theta) - 2H \sin \theta \cos \theta$$

$$H' = F \sin^2 \theta + G \sin \theta \cos \theta + H \cos^2 \theta$$

$$J' = J \cos \theta - K \sin \theta$$

$$K' = J \sin \theta + K \cos \theta$$

$$M' = M$$

REFLECTION across the line, $y(\cos \eta) = x(\sin \eta)$,

is equivalent to the projective transformation $\begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} \cos 2\eta & \sin 2\eta & 0 \\ \sin 2\eta & -\cos 2\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix}$

and changes the curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

to the curve

$$A'x^3 + B'x^2y + C'xy^2 + D'y^3 + F'x^2 + G'xy + H'y^2 + J'x + K'y + M' = 0,$$

where

$$A' = A \cos^3 2\eta + B \sin 2\eta \cos^2 2\eta + C \sin^2 2\eta \cos 2\eta + D \sin^3 2\eta$$

$$B' = 3A \sin 2\eta \cos^2 2\eta - B(\cos^3 2\eta - 2 \sin^2 2\eta \cos 2\eta) + C(\sin^3 2\eta - 2 \sin 2\eta \cos^2 2\eta) - 3D \sin^2 2\eta$$

$$C' = 3A \sin^2 2\eta \cos 2\eta + B(\sin^3 2\eta - 2 \sin 2\eta \cos^2 2\eta) + C(\cos^3 2\eta - 2 \sin^2 2\eta \cos 2\eta) + 3D \sin 2\eta$$

$$D' = A \sin^3 2\eta - B \sin^2 2\eta \cos 2\eta + C \sin 2\eta \cos^2 2\eta - D \cos^3 2\eta$$

$$F' = F \cos^2 2\eta + G \sin 2\eta \cos 2\eta + H \sin^2 2\eta$$

$$G' = 2F \sin 2\eta \cos 2\eta - G(\cos^2 2\eta - \sin^2 2\eta) - 2H \sin 2\eta \cos 2\eta$$

$$H' = F \sin^2 2\eta - G \sin 2\eta \cos 2\eta + H \cos^2 2\eta$$

$$J' = J \cos 2\eta + K \sin 2\eta$$

$$K' = J \sin 2\eta - K \cos 2\eta$$

$$M' = M.$$

The HORIZONTAL SLANT,

$$(x, y) \rightarrow (x + \mu y, y) \quad (\text{in affine coordinates})$$

$$\text{or} \quad \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \quad (\text{in projective coordinates}),$$

changes the cubic curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

to the cubic curve

$$A'x^3 + B'x^2y + C'xy^2 + D'y^3 + F'x^2 + G'xy + H'y^2 + J'x + K'y + M' = 0$$

where

$$A' = A$$

$$B' = -3A\mu + B$$

$$C' = +3A\mu^2 - 2B\mu + C$$

$$D' = -A\mu^3 + B\mu^2 - C\mu + D$$

$$F' = F$$

$$G' = -2F\mu + G$$

$$H' = F\mu^2 - G\mu + H$$

$$J' = J$$

$$K' = -J\mu + K$$

$$M' = M$$

The DILATION from the origin,

$$(x, y) \rightarrow (\alpha x, \beta y) \quad (\text{in affine coordinates})$$

$$\text{or } \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \quad (\text{in projective coordinates}),$$

changes the cubic curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

to the curve

$$A'x^3 + B'x^2y + C'xy^2 + D'y^3 + F'x^2 + G'xy + H'y^2 + J'x + K'y + M' = 0$$

where

$$\begin{aligned} A' &= \frac{A}{\alpha^3} & B' &= \frac{B}{\alpha^2\beta} & C' &= \frac{C}{\alpha\beta^2} & D' &= \frac{D}{\beta^3} \\ F' &= \frac{F}{\alpha^2} & G' &= \frac{G}{\alpha\beta} & H' &= \frac{H}{\beta^2} \\ J' &= \frac{J}{\alpha} & K' &= \frac{K}{\beta} \\ M' &= M \end{aligned}$$

The CENTRAL PROJECTION,

$$(x, y) \rightarrow \left(\frac{x}{\pi x + \rho y + 1}, \frac{y}{\pi x + \rho y + 1} \right) \quad (\text{affine coordinates})$$

$$\text{or } \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \pi & \rho & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ W \end{bmatrix} \quad (\text{projective coordinates}),$$

projects the line $\pi x + \rho y + 1 = 0$ to infinity

and changes the cubic curve

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Fx^2 + Gxy + Hy^2 + Jx + Ky + M = 0$$

to the cubic curve

$$A'x^3 + B'x^2y + C'xy^2 + D'y^3 + F'x^2 + G'xy + H'y^2 + J'x + K'y + M' = 0$$

where

$$A' = A - \pi F + \pi^2 J - \pi^3 M$$

$$B' = B - \rho F - \pi G + 2\pi\rho J + \pi^2 K - 3\pi^2 \rho M$$

$$C' = C - \rho G - \pi H + \rho^2 J + 2\pi\rho K - 3\pi\rho^2 M$$

$$D' = D - \rho H + \rho^2 K - \rho^3 M$$

$$F' = F - 2\pi J + 3\pi^2 M$$

$$G' = G - 2\rho J - 2\pi K + 6\pi\rho M$$

$$H' = H - 2\rho K + 3\rho^2 M$$

$$J' = J - 3\pi M$$

$$K' = K - 3\rho M$$

$$M' = M .$$

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