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Closed unbounded subset problems

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CLOSED UNBOUNDED SUBSET PROBLEMS

A Thesis

Presented to

The Faculty of the Department of Mathematics and Computer Science

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Parisa Safa

December 2001

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ABSTRACT
CLOSED UNBOUNDED SUBSET PROBLEMS

by **Parisa Safa**

This thesis discusses questions about closed unbounded (club) subsets of infinite cardinals, specifically of ω_1 and ω_2 . It gives first order characterizations of (i) subsets of ω_1 that can have club subsets in an ω_1 preserving outer model, (ii) families of subsets of ω_1 such that, in some ω_1 preserving outer model, there exists a club subset of ω_1 that is eventually contained in each member of the family, (iii) subsets of ω_2 that can have club subsets in an ω_1 and ω_2 preserving outer model that adds no new bounded sets, (iv) bounded pattern width subsets of ω_2 that can have club subsets in an ω_1 and ω_2 preserving outer model. It also shows that if κ is large enough and the successor of a regular cardinal and $X \subseteq \kappa$, then in some outer model $\kappa = \omega_2$ and either X or $\kappa \setminus X$ has a club subset.

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1. Introduction

Closed unbounded sets play a fundamental role in modern set theory. It is therefore of interest to know which subsets of an uncountable cardinal can contain closed unbounded subsets in a cardinal perserving outer model.

Let κ be a regular uncountable cardinal. The set $A \subseteq \kappa$ is *closed* in κ if for every $\lambda < \kappa$, whenever $\sup(A \cap \lambda) = \lambda$, then $\lambda \in A$. The set $B \subseteq \kappa$ is *unbounded* in κ if $\sup(B) = \kappa$. We say that a set $C \subseteq \kappa$ is *closed unbounded* or *club* if it is both closed and unbounded in κ .

For any non-empty set A , a *filter* on A is a subset $\mathcal{F} \subset \mathcal{P}(A)$ such that

- (a) $A \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.
- (b) $\forall X, Y \in \mathcal{F} (X \cap Y \in \mathcal{F})$.
- (c) $\forall X \in \mathcal{F} \forall Y \subset A (X \subset Y \rightarrow Y \in \mathcal{F})$.

The filter generated by the club subsets of κ , where $\text{cf}(\kappa) > \omega$, consists of all $X \subseteq \kappa$ that contain a closed unbounded subset. This filter is called the *club filter* on κ . So, the club filter on κ is

$$\{ X \subseteq \kappa : \exists C \subseteq X (C \text{ is club in } \kappa) \}.$$

If κ is a regular uncountable cardinal and $\mathcal{F}$ is a filter on a non-empty set A , then $\mathcal{F}$ is called κ -*complete* if $\mathcal{F}$ is closed under intersections of less than κ sets, that is, if whenever $\{X_\alpha : \alpha < \gamma\}$ is a family of subsets of A , $\gamma < \kappa$, and $X_\alpha \in \mathcal{F}$

for every $\alpha < \gamma$, then

$$\bigcap_{\alpha < \gamma} X_\alpha \in \mathcal{F}.$$

The intersection of any family of less than $\text{cf}(\kappa)$ club subsets of κ is club, so the club filter on κ is $\text{cf}(\kappa)$ -complete. Therefore if κ is regular, the club filter on κ is κ -complete.

Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of subsets of κ . The *diagonal intersection* of X_α , for $\alpha < \kappa$, is defined as follows:

$$\Delta_{\alpha < \kappa} X_\alpha = \{ \delta < \kappa : \delta \in \bigcap_{\alpha < \delta} X_\alpha \}$$

We say that a filter $\mathcal{F}$ on κ is *normal* if it is closed under diagonal intersections:

$$\text{if } X_\alpha \in \mathcal{F}, \alpha < \kappa, \text{ then } \Delta_{\alpha < \kappa} X_\alpha \in \mathcal{F}$$

The diagonal intersection of a κ -sequence of club sets is club, so the club filter on κ is closed under diagonal intersections. Therefore the club filter on κ is *normal*.

We say that a set $S \subseteq \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every club subset C of κ . For example, if $\kappa > \omega_1$, $S = \{ \alpha < \kappa : \text{cf}(\alpha) = \omega_1 \}$ is stationary in κ because for any club subset of κ , C , the ω_1^{st} element of C has cofinality ω_1 , and is therefore in S . Note that the notion of being stationary is equivalent to being positive with respect to the club filter.

The following is a consequence of the fact that the diagonal intersection of a sequence of club subsets of κ being club.

Fodor's Lemma. *If $\kappa > \omega$ is regular, $S \subseteq \kappa$ is stationary, $\text{dom}(f) = S$, and f is regressive, then there exists an α such that*

$$\{ \delta \in S : f(\delta) = \alpha \} \text{ is stationary.}$$

Using Fodor's lemma, it can be shown that every regular uncountable κ is the union of κ disjoint stationary sets. In fact, it has also been shown that a stationary subset of a regular uncountable cardinal κ can be partitioned into κ many disjoint stationary sets.

If V is a standard transitive model of ZFC (the collection of axioms of Zermelo-Fraenkel together with the Axiom of Choice), we say that W is an *outer model* of V if $W \supseteq V$ is also a standard transitive model of ZFC and $V \cap \text{OR} = W \cap \text{OR}$. To study club subsets of κ , we need sentences that are absolute. For example, If κ is a regular cardinal in V and X is club in κ , then “ X is club in κ ” is absolute for an outer model W in which κ remains regular. This is to say that in W , X is club in κ . In general, however, “ X is stationary” is not absolute for such outer models.

We can use an alternate strong form of “stationary” that is trivially absolute, namely “ X has a club subset in some outer model in which κ remains regular”. One problem with this is that in general, we are interested in preserving more than the regularity of κ . For example, if $\kappa = \omega_2$, we want, in W , to preserve both ω_1 and ω_2 . The other problem is that quantifying over outer models isn't first-order.

Thus we need a criterion in V so that we can give a first-order characterization of sets that have club subsets in an outer model in which κ remains the same.

Section 2 is devoted to giving a first-order characterization of those subsets of ω_1 that have club subsets in an ω_1 preserving outer model. It shows that if S is a stationary subset of ω_1 , then there exists a ω_1 preserving outer model in which S has a club subset.

Section 3 gives a first-order characterization of those families of subsets of ω_1 that can have club subsets in an ω_1 preserving outer model simultaneously. The non-triviality of this question is discussed and it is shown that those families of subsets of ω_1 that generate a normal filter will have club subsets in an ω_1 preserving outer model.

Section 4 discusses the problem of characterizing those subsets of ω_2 that can have club subsets in an ω_1 and ω_2 preserving outer model. It is shown that being stationary is no longer sufficient to guarantee the existences of club subsets in a cardinal preserving outer model. We then proceed to give characterizations of different subsets of ω_2 that have club subsets in an ω_1 and ω_2 preserving outer model. First-order characterizations are given for (1) those subsets of ω_2 that have a club subset in an ω_1 and ω_2 preserving outer model that doesn't add bounded subsect of ω_2 and (2) those bounded pattern width subsets of ω_2 that have a club subset in an ω_1 and ω_2 preserving outer model. We then construct complicated stationary sets that are subsets of ω_2 and reduce the question of characterization

to the question of which partitions of ω_2 have large homogeneous sets.

In Section 5 we begin with noting that in general there is no first-order characterization of those subsets X of ω_2 such that either X or $\omega_2 \setminus X$ has a club subset in an ω_1 and ω_2 preserving outer model. It is then shown that if $X \subseteq \kappa$ where κ is sufficiently large and the successor of a regular cardinal, then there is an outer model in which $\kappa = \omega_2$ and either X or $\kappa \setminus X$ has a club subset.

2. Closed unbounded subsets of ω_1

Let κ be a regular uncountable cardinal. Recall that a set $C \subset \kappa$ is *closed unbounded* or *club* in κ if

- (i) $\sup(C) = \kappa$ (unbounded), and
- (ii) if $\lambda < \kappa$ and $\sup(C \cap \lambda) = \lambda$ then, $\lambda \in C$ (closed).

We say that $S \subset \kappa$ is *stationary* in κ if $S \cap C \neq \emptyset$ for every closed unbounded subset C of κ .

This section gives a characterization of those subsets of ω_1 that have closed unbounded subsets in an ω_1 preserving outer model. To do so, it first needs to be shown that every stationary subset of ω_1 has arbitrarily long countable subsets.

Proposition 2.1. [3] *Every stationary subset of ω_1 contains arbitrarily long countable closed subsets.*

Proof. Let S be a stationary subset of ω_1 . It will be shown by induction on $\alpha < \omega_1$ that S has a closed subset of length α . Let the induction hypothesis be that for all $\beta < \alpha$ and for each $\gamma < \omega_1$, there is a closed subset $c \subseteq S \setminus \gamma$ of length β .

Case 1. $\alpha = \lambda + 1$, where λ is a successor ordinal. Fix $\gamma < \omega_1$. By the induction hypothesis, there exists a closed $c' \subseteq S \setminus \gamma$ of length λ . Since S is unbounded, we can find $\delta \in S$ where $\delta > \sup(c')$. Let $c = c' \cup \{\delta\}$. Then c is closed and of length $\lambda + 1$, as required.

Case 2. α is a limit ordinal. Fix $\gamma < \omega_1$ and choose $\beta_0 < \beta_1 < \dots < \alpha$, where $\sup(\beta_n) = \alpha$. By the induction hypothesis, for each β_n and each $\delta < \omega_1$ there exists a closed set c of length β_n , where $c \subseteq S \setminus \delta$. Let $\langle c_n : n < \omega \rangle$ be a sequence of closed subsets of $S \setminus \gamma$ where length of c_0 is $\beta_0 + 1$, $c_{n+1} \subseteq S \setminus (\sup(c_n) + 1)$, and for each n , the length of c_{n+1} is $\beta_{n+1} + 1$. Then the set $c = \bigcup_{n \in \omega} c_n$ has order type at least α .

Case 3. $\alpha = \lambda + 1$, where λ is a limit ordinal. Fix $\gamma < \omega_1$. Define an increasing continuous sequence $\langle \tau_\xi : \xi < \omega_1 \rangle$ of ordinals less than ω_1 and a sequence of closed sets $\langle c_\xi : \xi < \omega_1 \rangle$ of order type λ as follows. Let $c_0 \subseteq S \setminus \gamma$ have order type λ . Set $\tau_\xi = \sup(c_\xi)$, unless ξ is a limit ordinal. In this case, set $\tau_\xi = \sup_{\zeta < \xi} \tau_\zeta$. Let $c_{\xi+1} \subseteq S \setminus \tau_\xi$ be closed with order type λ . Then $\{\tau_\xi : \xi < \omega_1\}$ is club in ω_1 , so it meets S and therefore for some $\delta < \omega_1$, there exists a $\tau_\delta \in S$. If δ is a successor ordinal, then $c_\delta \cup \{\tau_\delta\}$ is as required. If δ is a limit ordinal, let $\langle \delta_n : n \in \omega \rangle$ be an increasing and cofinal sequence in δ , and $\langle \eta_n : n \in \omega \rangle$ be increasing and cofinal in λ . Let d_n be the first $\eta_n + 1$ many elements of c_{δ_n+1} . Then $\sup(d_n) \in d_n$ and if $m < n$, then $\sup(d_m) < \min(d_n)$. Set $c = (\bigcup_{n \in \omega} d_n) \cup \{\tau_\delta\}$. Then $c \subseteq S \setminus \gamma$ and the length of c is at least $\sup_{n \in \omega} (\text{ot}(d_n) + 1) + 1 = (\sup_{n \in \omega} (\eta_n + 1)) + 1 = \lambda + 1$.

□

Assume the generalized continuum hypothesis (GCH), that is, $2^\kappa = \kappa^+$. The following theorem, which has been proved in [1], and independently in [2], shows that only stationary subsets of ω_1 can have club subsets in an ω_1 preserving outer

model.

Theorem 2.2. (GCH) *The following three properties are equivalent:*

- (1) *S is stationary in ω_1 .*
- (2) *There exists a cardinal and GCH preserving set generic extension $V[G]$ in which S has a club subset.*
- (3) *S has a club subset in some ω_1 -preserving outer model W of V .*

Proof. For (1) $\Rightarrow$ (2), let S be a stationary subset of ω_1 and define $\mathbb{P}$ to be a notion of forcing such that $p \in \mathbb{P}$ iff $p \subseteq S$ is a bounded, closed set of ordinals. Say that $p \leq p'$ iff p end-extends p' , and let G be a V -generic filter over $\mathbb{P}$.

By Proposition 2.1, S has arbitrarily long countable subsets, so every condition in $\mathbb{P}$ has a proper extension. $\mathbb{P}$ satisfies the $< \omega_2$ -c.c. because $|\mathbb{P}| = \omega_1^{<\omega_1}$ by the GCH, and therefore forcing with $\mathbb{P}$ preserves ω_2 and larger cardinals.

Suppose that G is $\mathbb{P}$ generic over V . Note that $C = \bigcup \{p : p \in G\}$ is a club subset of S in $V[G]$. It is clear that C is closed. To see that C is unbounded, pick $\alpha < \omega_1$ and define the set $D_\alpha = \{c : \sup(c) \geq \alpha\}$. Now, D_α is dense because for any $p \in \mathbb{P}$, we can find $q \in D_\alpha$ such that $q \leq p$ as follows: let δ be the least element in S that is greater than α and set $q = p \cup \{\delta\}$. So, D_α meets G , which means that for any $\alpha < \omega_1$, there exists a condition $r \in G$ such that $\sup(r) \geq \alpha$ and therefore C is unbounded.

To see that cardinals smaller than ω_2 are preserved in $V[G]$, it suffices to show that $\mathbb{P}$ is countably distributive. Choose $\bar{c} \in \mathbb{P}$ and let $\langle D_n : n \in \omega_1 \rangle$ be a

sequence of dense subsets in $\mathbf{P}$. It needs to be shown that $\bar{c}$ has an extension that meets all the D_n 's for $n < \omega_1$.

Recall that H_{ω_2} is the collection of sets of hereditary cardinality less than ω_2 . All $p \in \mathbf{P}$ are countable subsets of ω_1 , so $\mathbf{P}$ has size $\omega_1^{<\omega_1} < \omega_2$ and $\mathbf{P} \in H_{\omega_2}$. Hence each D_n belongs to H_{ω_2} ; Indeed $\langle D_n : n \in \omega \rangle \in H_{\omega_2}$. Define a tower of elementary substructures of H_{ω_2} , $\langle M_i : i < \omega_1 \rangle$, by

$$M_i = \text{the least } M \prec H_{\omega_2} \text{ s.t. } \{\bar{c}, \langle D_n : n < \omega_1 \rangle\} \cup \{M_j : j < i\} \subseteq M,$$

and set $\alpha_i = (M_i \cap \omega_1)$ for each $i < \omega_1$. Consider the set $K = \{\alpha_j : j < \omega_1\}$.

When i is a limit ordinal,

$$M_i = \bigcup_{j < i} M_j, \text{ so } \alpha_i = \bigcup_{j < i} (M_j \cap \omega_1) = \bigcup_{j < i} \alpha_j = \sup_{j < i}(\alpha_j).$$

Therefore K is closed. Now, $\omega_1 \in M_{j+1}$ and $M_j \in M_{j+1}$, so $\alpha_j \in M_{j+1}$. But $M_{j+1} \prec H_{\omega_2}$, so $M_{j+1} \models \text{"}\alpha_j \text{ is countable"}$. Also

$$\alpha_{j+1} = \sup(\text{countable ordinals in } M_{j+1})$$

It follows that $\alpha_j < \alpha_{j+1}$. Thus K is unbounded.

So the set $B = \{\alpha_j : j < \omega_1 \text{ and } j \text{ is a limit ordinal}\}$ is also a club subset of ω_1 and therefore B meets S .

Let $\alpha_j \in (B \cap S)$ and let $\langle i_n : n < \omega \rangle$ be a monotonically increasing sequence of ordinals cofinal in j . Consider the sequence $\langle M_{i_n} : n < \omega \rangle$ of elementary substructures of H_{ω_2} and let $c_0 = \bar{c}$. D_0 is dense in M_{i_0} , so there exists a condition $c'_0 \leq c_0$ such that $c'_0 \in D_0$ and $c'_0 \in M_{i_0}$. Then $\sup(c'_0) < \alpha_{i_0}$.

In M_{i_1} , let $c_1 \leq c'_0$ such that $\sup(c_1) > \alpha_{i_0}$. Continue in this way to define $\langle c_n : n < \omega \rangle$. Then $\sup(\bigcup_{n \in \omega} c_n) = \alpha_j$, so $c = \bigcup_{n < \omega} c_n \cup \{\alpha_j\}$ is a condition in $\mathbb{P}$ and meets D_n for all $n < \omega_1$. Hence $\mathbb{P}$ is countably distributive.

We now know that V and $V[G]$ have the same cardinals. It remains to be shown that for any infinite cardinal μ , $[2^\mu]^{V[G]} = \mu^+$ (i.e. the GCH is preserved.)

To do so, first note that in V ,

- (a) $\mathbb{P}$ is countably distributive
- (b) $\mathbb{P}$ has antichains of cardinality at most ω_1 , and
- (c) $|\mathbb{P}| \leq 2^{\omega_1} = \omega_2$.

Let μ be an infinite cardinal. Recall that V and $V[G]$ have the same cardinals.

If $\mu = \omega$, then $[2^\mu]^{V[G]} = [2^\mu]^V = \mu^+$

If $\mu \geq \omega$, then

$$\begin{aligned}
 [2^\mu]^{V[G]} &\leq [(|\mathbb{P}|^{\omega_1})^\mu]^V \\
 &\leq [((\omega_1^{<\omega_1})^{\omega_1})^\mu]^V \\
 &= [((\omega_2)^{\omega_1})^\mu]^V \\
 &= [(\omega_2)^\mu]^V \\
 &= [\omega_2 \cdot 2^\mu]^V \\
 &= [2^\mu]^V \\
 &= \mu^+
 \end{aligned}$$

(2) $\Rightarrow$ (3) is obvious.

For (3) $\Rightarrow$ (1), let W be an ω_1 -preserving outer model of V and $K \in W$ be a club subset of S . Let $C \in V$ be club in ω_1 . Then $C \cap K \subseteq C \cap S$ and $C \cap K \neq \emptyset$. Then, in V , S is stationary. $\square$

Recall that a partial ordering $\mathbb{P}$ is λ -closed if for every $\alpha \leq \lambda$ and $\xi < \alpha$, if

$$p_0 \geq p_1 \geq \dots \geq p_\xi \geq \dots$$

is a descending sequence of conditions, then there is a $p \in \mathbb{P}$ stronger than all p_ξ , $\xi < \alpha$.

If $S \subseteq \omega_1$ is stationary, then the forcing used to add a club subset to it in the previous theorem satisfies the ω_2 -c.c. (because it has cardinality ω_1) and is countably distributive, but not countably closed. The following two propositions show that if S does not contain a club subset in the ground model (so $\omega_1 \setminus S$ is stationary), then there is no ω_1 -c.c. forcing, nor any countably closed forcing that can add a club to S .

Proposition 2.3. *Let $\mathbb{P}$ be a notion of forcing and κ a regular uncountable cardinal. If $\mathbb{P}$ is λ -closed for every $\lambda < \kappa$ then every stationary set S in the ground model remains stationary in any $\mathbb{P}$ -generic extension.*

Proof. Let $\kappa > \omega_1$ be a regular cardinal and let S be stationary in κ . Fix $p \in \mathbb{P}$ such that $p \Vdash \text{“}\dot{c} \text{ is club in } \check{\kappa}\text{”}$. To show that S remains stationary in a generic extension, it suffices to show that there exists $q \leq p$ such that $q \Vdash \text{“}S \cap \dot{c} \neq \emptyset\text{”}$.

Construct an increasing sequence of ordinals, $\langle \delta_\alpha : \alpha < \kappa \rangle$, and a decreasing sequence of conditions, $\langle p_\alpha : \alpha < \kappa \rangle$, as follows. Let $p_0 \leq p$ and δ_0 be such that $p_0 \Vdash "\delta_0 \in \dot{c}"$. Given p_α and δ_α , let $p_{\alpha+1} \leq p_\alpha$ and $\delta_{\alpha+1} > \delta_\alpha$ be such that $p_{\alpha+1} \Vdash "\delta_{\alpha+1} \in \dot{c}"$. When α is a limit point, let p_α be a lower bound of $\langle p_\beta : \beta < \alpha \rangle$ and $\delta_\alpha = \sup(\{\delta_\beta : \beta < \alpha\})$, then $p_\alpha \Vdash "\delta_\alpha \in \dot{c}"$, because $p \Vdash "\dot{c}$ is closed".

Since $\langle \delta_\alpha : \alpha < \kappa \rangle$ is increasing and continuous and S is stationary, there is some α such that $\delta_\alpha \in S$. Then $p_\alpha \Vdash "\delta_\alpha \in \dot{c} \cap S"$ as required. $\square$

Recall that in a partial ordering $\mathbb{P}$, an antichain is a family of pairwise incompatible conditions. We say that $\mathbb{P}$ satisfies the κ -c.c. (κ chain condition) if all antichains in $\mathbb{P}$ have cardinality less than κ .

Proposition 2.4. *Let $\mathbb{P}$ be a notion of forcing and κ a regular uncountable cardinal. If $\mathbb{P}$ satisfies the $< \kappa$ -c.c., then every stationary subset of κ in the ground model remains stationary in any $\mathbb{P}$ -generic extension.*

Proof. Suppose $S \subseteq \kappa$ is stationary in the ground model, V . To see that S remains stationary in any $\mathbb{P}$ -generic extension, it needs to be shown that S has non-empty intersection with an arbitrary club set in the generic extension.

Fix $\bar{p} \in \mathbb{P}$ such that $\bar{p} \Vdash "\dot{c} \text{ is club in } \dot{\kappa}"$. Choose a regular cardinal $\lambda > \kappa$, large enough that $\mathbb{P}, \dot{c}, \bar{p} \in H_\lambda$. Define a tower of elementary substructures, $\langle M_i : i < \kappa \rangle$, such that $S, \bar{p}, \dot{c}, \mathbb{P} \in M_0$, and $M_i = \text{least } M \prec H_\lambda$ such that

$\{\mathbb{P}, S, \dot{c}, \bar{p}, \kappa\} \cup \{M_j : j < i\} \subseteq M$ and $(M \cap \kappa) \in \kappa$.

First, note that $K = \{M_i \cap \kappa : i < \kappa\}$ is club in κ . Let $\alpha_i = M_i \cap \kappa$; then $\alpha_i \in \kappa$. To see that K is closed, note that when i is a limit ordinal, $M_i = \bigcup_{j < i} M_j$ and $\alpha_i = \bigcup_{j < i} M_j \cap \kappa = \bigcup_{j < i} \alpha_j = \sup_{j < i} (\alpha_j)$. So, $\{\alpha_i : i < \kappa\}$ is closed. To see that $\{\alpha_i : i < \kappa\}$ is unbounded, note that $\kappa \in M_{j+1}$ and that $\alpha_{j+1} > \alpha_j$ because $M_j \in M_{j+1}$. This makes $M_j \cap \kappa \in M_{j+1}$ and $\alpha_j \in M_{j+1}$. M_{j+1} is an elementary substructure of H_λ , so $M_{j+1} \models \alpha_j < \kappa$ and hence

$$\alpha_{j+1} = \sup(\text{ordinals less than } \kappa \text{ in } M_{j+1}) > \alpha_j.$$

Consequently, $\{\alpha_j : j < \kappa\}$ is unbounded.

Since this is all taking place in the ground model, where S is stationary, we can fix $i < \kappa$ such that $\alpha_i = M_i \cap \kappa \in S$. Let $A \in M_i$ be a maximal antichain such that $|A| = \delta$. Because $\mathbb{P}$ is $< \kappa$ -c.c., we know that $\delta < \kappa$. Now, $M_i \prec H_\lambda$, so $M_i \models \delta < \kappa$. Thus $\delta < \alpha_i$. Say $M_i \models \exists f(f : A \rightarrow \delta)$, where f is a bijection. Since $\delta \subseteq M_i$, we see that $A \subseteq M_i$.

Let G be a $\mathbb{P}$ -generic filter. We shall see that $G \cap M_i$ is a $(\mathbb{P} \cap M_i)$ -generic filter over M_i . Recall that for $G \cap M_i$ to be $(\mathbb{P} \cap M_i)$ -generic over M_i , $G \cap M_i$ has to meet all dense sets in the ground model M_i (which is equivalent to meeting all maximal antichains), and be a filter on $\mathbb{P} \cap M_i$.

Let $A \in M_i$ be a maximal antichain. Since G is $\mathbb{P}$ -generic, it has a nonempty intersection with A , but $A \subseteq M_i$, so $(G \cap M_i)$ meets A . $(G \cap M_i)$ is a filter on $(\mathbb{P} \cap M_i)$ if the following three conditions hold.

(1) $G \cap M_i \subseteq \mathbb{P} \cap M_i$

(2) If $p \in G \cap M_i, q \in M_i, q \geq p$, then $q \in G \cap M_i$.

(3) If $p, q \in G \cap M_i$, then there exists an r in $G \cap M_i$ such that $r \leq p, q$.

(1) follows from $G \subseteq \mathbb{P}$ and (2) follows from the fact that G is a filter. To see that

(3) holds, for $p, q \in (G \cap M_i)$ let

$$D = \{ r : r \leq p, q \text{ or } r|p \text{ or } r|q \},$$

where $p|q$ iff $\exists r \in \mathbb{P}(r \leq p \wedge r \leq q)$.

It is clear that D is dense, so $G \cap M_i$ meets D and therefore we can find an r in $D \cap (G \cap M_i)$ such that $r \leq p, q$.

We want to see that M_i correctly calculates c up to its version of κ namely, $\alpha_i = M_i \cap \kappa$. For this, choose $\gamma < (M_i \cap \kappa)$, then $\gamma \in \dot{c}^{M_i \cap G}$ iff there exists $p \in A \cap (G \cap M_i)$ such that $p \Vdash \dot{\gamma} \in \dot{c}$ iff there exists $p \in (A \cap G)$ such that $p \Vdash \dot{\gamma} \in \dot{c}$ iff $\gamma \in \dot{c}^G$. So, $\dot{c}^{M_i \cap G} = \dot{c}^G \cap (M_i \cap \kappa)$.

Now, $M_i[G] \models \text{"}\dot{c}^{G \cap M_i} \text{ is club in } \kappa\text{"}$, so $\dot{c}^{G \cap M_i}$ is unbounded in $(M_i \cap \kappa)$ and therefore $(M_i \cap \kappa) \in \dot{c}^{G \cap M_i} = \dot{c}^G$. But $(M_i \cap \kappa) \in S$, so S remains stationary in the generic extension. $\square$

3. Normal filters

In Section 2 it was shown that only stationary subsets of ω_1 can have club subsets in an ω_1 preserving outer model. We want to find an analog of Theorem 2.2 for subsets of ω_2 , but first we must consider which families of subsets of ω_1 can have club subsets in an ω_1 preserving outer model simultaneously. This section discusses these families of subsets of ω_1 .

For each ordinal $\alpha < \omega_2$ having cofinality ω_1 , let $f_\alpha : \omega_1 \rightarrow \alpha$ be a continuous, cofinal, monotonically increasing function. (So the range of f_α is a club subset of α having order type ω_1 .) If S is a subset of ω_2 , define the *pattern of S at α* by

$$\text{ptn}(S, \alpha) = \{ \delta < \omega_1 : f_\alpha(\delta) \in S \}.$$

Note that if S has a club subset C , then the set of limit points of C of cofinality ω_1 is stationary. (Suppose that K is club subset of ω_2 . $K \cap \lim(C)$ is in fact club itself. Let α be the ω_1^{st} element of $K \cap \lim(C)$. Then α is in the set of limit points of C of cofinality ω_1 , and hence this set is stationary.)

Furthermore, if α is a limit point of C having cofinality ω_1 , then $\text{ptn}(S, \alpha)$ contains a club, namely, the inverse image of $C \cap \alpha$ under f_α .

Thus if S is a stationary subset of ω_2 that can contain a club set in an ω_2 preserving outer model, then in general it may be necessary to add club subsets to many subsets of ω_1 to obtain such an outer model, namely to many of the sets $\text{ptn}(S, \alpha)$. So it's natural to consider which families of subsets of ω_1 can get club subsets in a forcing extension.

Example 1. Partition ω_1 into ω_1 -many stationary cells. Then it is impossible to force a club subset in any two cells at the same time, as the intersection of the two club sets will be in both cells.

Example 2. Partition ω_1 into ω -many stationary cells, $\langle S_n : n < \omega \rangle$. Set $P_n = \bigcup_{m \geq n} S_m$. Then it is possible to force a club subset into finitely many P_n 's simultaneously by forcing a club in the biggest P_n . However, ω_1^V is collapsed in any outer model in which infinitely many P_n 's have club subsets, as $\bigcap_{n \in d} P_n = \emptyset$, where $|d| = \omega$.

Recall that for any non-empty set A , a *filter* on A is a subset $\mathcal{F} \subset \mathcal{P}(A)$ such that:

- (i) $A \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$
- (ii) $\forall X, Y \in \mathcal{F} (X \cap Y \in \mathcal{F})$
- (iii) $\forall X \in \mathcal{F} \forall Y \subset A (X \subset Y \rightarrow Y \in \mathcal{F})$

In the definition of *filter*, $\emptyset \notin \mathcal{F}$ is sometimes omitted. In that terminology, *filter* is used to mean *proper filter*.

Suppose that $\langle X_\alpha : \alpha < \kappa \rangle$ is a sequence of subsets of $\kappa > \omega$. Recall that the *diagonal intersection* of the X_α 's is defined by

$$\Delta_{\alpha < \kappa} X_\alpha = \{ \delta < \kappa : \delta \in \bigcap_{\alpha < \delta} X_\alpha \}$$

Recall that a *normal filter* is one that is closed under diagonal intersections.

To proceed, we need a generalization of the notions closed, unbounded, and stationary.

If I is a set of cardinality at least κ and κ is an infinite cardinal, let $[I]^\kappa$ denote the set of all subsets of I having cardinality κ . That is,

$$[I]^\kappa = \left\{ A \subseteq I : |A| = \kappa \right\}.$$

Say that a subset U of $[I]^\kappa$ is *unbounded* if given any $x \in [I]^\kappa$, there exists a $y \in U$ such that $x \subseteq y$.

Say that a subset C of $[I]^\kappa$ is *closed* if given any increasing (under the subset relation) sequence $\langle x_i : i < \theta \rangle$ where $\theta < \kappa^+$, $\bigcup_{i < \theta} x_i \in C$.

Say that a subset S of $[I]^\kappa$ is *stationary* if it has non-empty intersection with every club subset of $[I]^\kappa$.

Note that the intersection of two club subsets A and B of $[I]^\kappa$ is club. (Let $\theta < \kappa^+$. To see that the intersection is closed, let $\langle x_i : i < \theta \rangle$ be an increasing sequence in $A \cap B$. Then $\bigcup_{i < \theta} x_i \in A$ and $\bigcup_{i < \theta} x_i \in B$. So, $\bigcup_{i < \theta} x_i \in A \cap B$. To see that $A \cap B$ is unbounded, let $x \in [I]^\kappa$ be arbitrary. Define an increasing sequence $\langle x_i : i < \omega \rangle$ from $[I]^\kappa$ by

$$x_0 = x$$

$$x_{2i+1} \supseteq x_{2i} \text{ and } x_{2i+1} \in A$$

$$x_{2i+2} \supseteq x_{2i+1} \text{ and } x_{2i+2} \in B.$$

Set $y = \bigcup_{i < \omega} x_i$. Then $y = \bigcup_{i < \omega} x_{2i} \in A$ and $y = \bigcup_{i < \omega} x_{2i+2} \in B$. Hence $y \supseteq x$ and $y \in A \cap B$.

A stronger statement is true: The intersection of κ many club subsets of $[I]^\kappa$ is club. To see this, proceed by induction on β such that $0 < \beta \leq \kappa$ to see that if C_α is club in $[I]^\kappa$, for $\alpha < \beta$, then $\bigcap_{\alpha < \beta} C_\alpha$ is club in $[I]^\kappa$. As before, it is clear that $\bigcap_{\alpha < \beta} C_\alpha$ is closed. To see that it is unbounded, we consider two cases. If $\beta = \bar{\beta} + 1$, then $\bigcap_{\alpha < \bar{\beta}} C_\alpha$ is club by induction (or by hypothesis, if $\bar{\beta} = 0$). So $\bigcap_{\alpha < \bar{\beta}+1} C_\alpha = (\bigcap_{\alpha < \bar{\beta}} C_\alpha) \cap C_{\bar{\beta}}$ is the intersection of two club sets and is therefore itself club. If β is a limit ordinal, let $x \in [I]^\kappa$ be arbitrary. Define an increasing sequence $\langle x_\alpha : \alpha \leq \beta \rangle$ from $[I]^\kappa$ by

$$x_0 = x$$

$$x_{\alpha+1} \supseteq x_\alpha \text{ and } x_{\alpha+1} \in \bigcap_{\delta < \alpha} C_\delta$$

$$x_\alpha = \bigcup_{\delta < \alpha} x_\delta, \text{ if } \alpha \leq \beta \text{ is a limit ordinal.}$$

Note that $x_\beta \in [I]^\kappa$ since $\beta \leq \kappa$. Furthermore, $x_\alpha \in C_\delta$ whenever $\alpha > \delta$. Hence $x_\beta \in \bigcap_{\delta < \beta} C_\delta$ and $x_\beta \supseteq x$.

The following propositions and theorem show that it is possible to give a nice characterization of ground model families of stationary subsets of ω_1 that contain club sets in an ω_1 preserving outer model.

Proposition 3.1. *Assume that $\mathcal{F}$ is a filter on ω_1 . The following two assertions are equivalent.*

- (1) *If $\langle S_i : i < \omega_1 \rangle$ is a sequence of sets from $\mathcal{F}$, then $\Delta_{i < \omega_1} S_i$, the diagonal intersection of the S_i 's, is stationary.*

(2) Suppose $\lambda \geq \omega_2$ is a regular cardinal. Then

$$(*) \quad \{ M \in [H_\lambda]^\omega : \sup(M \cap \omega_1) \text{ lies in every } S \in (M \cap \mathcal{F}) \}$$

is stationary in $[H_\lambda]^\omega$.

Proof. For $(1) \Rightarrow (2)$, let $\mathcal{C}$ be a club subset of $[H_\lambda]^\omega$. It needs to be shown that $\mathcal{C}$ has nonempty intersection with the set $(*)$. Let $\langle M_i : i < \omega_1 \rangle$ be an increasing sequence from $\mathcal{C}$ where for each $i < \omega_1$, M_i is the least $M \prec H_{\lambda^+}$ such that $\{\mathcal{F}, \mathcal{C}\} \cup \{M_j : j < i\} \subseteq M$.

Let $\alpha_i = (M_i \cap \omega_1)$ for each $i < \omega_1$, and $\langle S_\delta : \delta < \omega_1 \rangle$ be a sequence such that $\langle S_\delta : \delta < \alpha_i \rangle \in M_{i+1}$ and $\langle S_\delta : \delta < \alpha_i \rangle$ enumerates $(M_i \cap \mathcal{F})$. $\{\alpha_i : i < \omega_1\}$ is closed and unbounded in ω_1 (a similar case has been demonstrated in Proposition 2.2), and $\Delta_{\delta < \omega_1} S_\delta$ is stationary, so for some $i < \omega_1$, $\alpha_i \in \Delta_{\delta < \omega_1} S_\delta$, which means that $\alpha_i \in S_\delta$ for $\delta < \alpha_i$, hence $\alpha_i \in \bigcap (M_i \cap \mathcal{F})$, and so M_i lies in the set $(*)$ and in $\mathcal{C}$. Because $\mathcal{C}$ is arbitrary, the set $(*)$ is stationary in $[H_\lambda]^\omega$.

For $(2) \Rightarrow (1)$, suppose that S_i lies in $\mathcal{F}$, for each $i < \omega_1$. Fix a club subset K of ω_1 . Note that the following set $\mathcal{C}$ is club in $[H_\lambda]^\omega$.

$$\mathcal{C} = \left\{ M \in [H_\lambda]^\omega : M \prec H_\lambda \text{ and } K, \langle S_i : i < \omega_1 \rangle \in M \right\}$$

To see that $\mathcal{C}$ is closed, suppose that the sequence $\langle M_i : i < \omega \rangle$ is increasing under $\subseteq$ and each $M_i \in \mathcal{C}$, and let $M = \bigcup_{i < \omega} M_i$. Then, $K \in M$, $\langle S_i : i < \omega_1 \rangle \in M$, $|M| < \omega_1$, and by Chang & Keisler [4], $M \prec H_\lambda$, so $M \in \mathcal{C}$ and therefore $\mathcal{C}$ is

closed. To see that $\mathcal{C}$ is unbounded, pick $X \in [H_\lambda]^\omega$ and let

$$M = \text{Skolem Hull}\left(X \cup \{K, \langle S_i : i < \omega_1 \rangle\}\right).$$

Then $X \in [H_\lambda]^\omega$ and $M \prec H_\lambda$, so $M \in \mathcal{C}$ and $X \subseteq M$ and therefore $\mathcal{C}$ is unbounded.

Since $\mathcal{C}$ is club and the set $(*)$ is stationary, we can find $M \in \mathcal{C}$ such that $\sup(M \cap \omega_1)$ lies in every $S \in (M \cap \mathcal{F})$. Let $\alpha = (M \cap \omega_1)$, then $\alpha \in K$ because $K \in M \prec H_\lambda$, and α lies in each S_i for $i < \alpha$ because $\langle S_i : i < \omega_1 \rangle \in M$, so $\alpha \in \Delta_{i < \omega_1} S_i$ which means that $(\Delta_{i < \omega_1} S_i) \cap K \neq \emptyset$ and therefore $\Delta_{i < \omega_1} S_i$ is stationary. $\square$

Proposition 3.2. (CH) *If $\mathcal{F}$ is a normal filter on ω_1 , then there exists a countably distributive, GCH preserving, $< \omega_2$ -c.c. forcing property that adds a club subset of ω_1 that is eventually contained in each element of $\mathcal{F}$.*

Proof. Let $\mathbb{P}$ be a notion of forcing such that

$$\mathbb{P} = \left\{ (c, S) : c \text{ is a closed bounded subset of } \omega_1, c \subseteq S, S \in \mathcal{F} \right\}.$$

Say that $(c', S') \leq (c, S)$ if c' end-extends c and $S' \subseteq S$. Let G be $\mathbb{P}$ -generic filter over V and define

$$C = \bigcup \left\{ c : (c, S) \in G \text{ for some } S \right\}.$$

It needs to be shown that $\mathbb{P}$ is $< \omega_2$ -c.c., countably distributive, GCH preserving, and that C is club in ω_1 and eventually contained in each element of $\mathcal{F}$.

To see that $\mathbb{P}$ satisfies the $< \omega_2$ -c.c., note that for any two conditions (c, S) and (c, S') in $\mathbb{P}$, $(c, S \cap S') \leq (c, S), (c, S')$. There are $\omega_1^{<\omega_1}$ (which using CH, is ω_1) possibilities for c , so it is impossible to have more than ω_1 pairwise incompatible conditions and therefore $\mathbb{P}$ is $< \omega_2$ -c.c.

It is clear that the set C is closed in ω_1 . To see that C is unbounded, let $D_\alpha = \{(c, S) : \sup(c) > \alpha\}$, for $\alpha < \omega_1$. To see that D_α is dense, choose $(c, S) \in \mathbb{P}$. If $\sup(c) > \alpha$, then $(c, S) \in D_\alpha$, otherwise let $\bar{c} = c \cup \{\beta\}$ where $\beta \in S$ and $\sup(c) > \beta > \alpha$. Then $(\bar{c}, S) \in D_\alpha$ and $(\bar{c}, S) \leq (c, S)$. Because D_α is dense, it meets G . This means that for any $\alpha < \omega_1$ there exists $(c, S) \in G$ such that $\sup(c) > \alpha$, and therefore C is an unbounded subset of ω_1 .

Let $D_S = \{(c, S') : S' \subseteq S\}$, for $S \in \mathcal{F}$. D_S meets G because it is dense, so for some c , $(c, S) \in G$. Let $\alpha = \sup(c) + 1$, then $C \setminus \alpha \subseteq S$ and hence C is eventually contained in each element of $\mathcal{F}$.

To see that $\mathbb{P}$ is countably distributive, fix a condition $(\bar{c}, \bar{S}) \in \mathbb{P}$ and a sequence $\langle D_n : n < \omega \rangle$ of dense subsets of $\mathbb{P}$. For $\mathbb{P}$ to be countably distributive, it needs to be shown that $\bigcap_{n < \omega} D_n$ is dense in $\mathbb{P}$.

Let λ be a sufficiently large regular cardinal. Note that the following set $\mathcal{B}$ is club in $[H_\lambda]^\omega$.

$$\mathcal{B} = \left\{ M \in [H_\lambda]^\omega : M \prec H_\lambda \text{ and } \{\langle D_n : n < \omega \rangle, (\bar{c}, \bar{S}), \mathcal{F}, \mathbb{P}\} \subseteq M \right\}$$

To see that $\mathcal{B}$ is closed, suppose that the sequence $\langle M_i : i < \omega \rangle$ is increasing under $\subseteq$ and that each $M_i \in \mathcal{B}$. Let $M = \bigcup_{i < \omega} M_i$.

Then $\{(\bar{c}, \bar{S}), \langle D_n : n < \omega \rangle, \mathcal{F}, \mathbb{P}\} \subseteq M$, $|M| < \omega_1$, and by the Elementary Chain Theorem [CK], $M \prec H_\lambda$. So $M \in \mathcal{B}$ and therefore $\mathcal{B}$ is closed. To see that $\mathcal{B}$ is unbounded, pick $X \in [H_\lambda]^\omega$ and let

$$M = \text{Skolem Hull}(X \cup \{\langle D_n : n < \omega \rangle, (\bar{c}, \bar{S}), \mathcal{F}, \mathbb{P}\}).$$

Then $X \in [H_\lambda]^\omega$ and $M \prec H_\lambda$, so $M \in \mathcal{B}$, $X \subseteq M$ and therefore M is unbounded.

Using Proposition 3.1, we can find an M in $\mathcal{B}$ such that $\alpha = M \cap \omega_1$ lies in the intersection of all the elements of $\mathcal{F}$ that lie in M . Let $\langle \alpha_n : n < \omega \rangle$ be monotonically increasing and cofinal in α . Let $\langle (c_n, S_n) : n < \omega \rangle$ be a descending sequence of conditions in M such that $(c_0, S_0) = (\bar{c}, \bar{S})$ and (c_{n+1}, S_{n+1}) is in $(\mathbb{P} \cap M)$, meeting D_n and extending (c_n, S_n) such that $\sup(c_{n+1}) \geq \alpha_n$. Set $c = \bigcup_{n < \omega} c_n \cup \{\alpha\}$ and $S = \bigcap (\mathcal{F} \cap M)$. S is in $\mathcal{F}$ because $\mathcal{F}$ is countably closed. It can also be seen that $S \subseteq S_n$, because S_n is in $(\mathcal{F} \cap M)$, for each $n < \omega$. Now, $\alpha \in \bigcap (\mathcal{F} \cap M)$, so $c \subseteq S$. Thus we have found a condition (c, S) in $\mathbb{P}$ extending every (c_n, S_n) and meeting all of the D_n 's, for $n < \omega$. Hence $\bigcap_{n < \omega} D_n$ is dense in $\mathbb{P}$ and $\mathbb{P}$ is countably distributive.

It remains to be shown that for any infinite cardinal μ , $[2^\mu]^{V[G]} = \mu^+$ (i.e. the GCH is preserved.) To do so, first note that in V ,

- (a) $\mathbb{P}$ is countably distributive
- (b) $\mathbb{P}$ has antichains of cardinality at most ω_1 , and
- (c) $|\mathbb{P}| \leq 2^{\omega_1} = \omega_2$.

Let μ be an infinite cardinal. Recall that V and $V[G]$ have the same cardinals.

If $\mu = \omega$, then $[2^\mu]^{V[G]} = [2^\mu]^V = \mu^+$

If $\mu \geq \omega$, then

$$\begin{aligned}
 [2^\mu]^{V[G]} &\leq [(|\mathbb{P}|^{\omega_1})^\mu]^V \\
 &\leq [((\omega_1^{<\omega_1})^{\omega_1})^\mu]^V \\
 &= [((\omega_2)^{\omega_1})^\mu]^V \\
 &= [(\omega_2)^\mu]^V \\
 &= [\omega_2 \cdot 2^\mu]^V \\
 &= [2^\mu]^V \\
 &= \mu^+
 \end{aligned}$$

□

Theorem 3.3. (*GCH*) *Let B be a family of subsets of ω_1 . The following statements are equivalent.*

- (1) $\Delta_{i < \omega_1} S_i$ is stationary whenever $\langle S_i : i < \omega_1 \rangle$ is a sequence of elements of B .
- (2) There exists a cardinal and GCH preserving set generic extension $V[G]$ in which every $S \in B$ has a club subset.
- (3) There exists an ω_1 -preserving outer model W of V in which every element $S \in B$ has a club subset.

Proof. For (1) $\Rightarrow$ (2), let B be a family of subsets of ω_1 with the property that

$\Delta_{i < \omega_1} S_i$ is stationary, whenever $\langle S_i : i < \omega_1 \rangle$ is a sequence of elements of B . We may assume that $B \neq \emptyset$, or else (2) is trivial. Define a family $\mathcal{F} \subseteq B$ by

$X \in \mathcal{F}$ iff $\omega_1 \supseteq X \supseteq C \cap \Delta_{i < \omega_1} S_i$, for some club $C \subseteq \omega_1$ and some sequence $\langle S_i : i < \omega_1 \rangle$ of elements of B .

Claim. $\mathcal{F}$ is a normal filter.

Proof. Because $B \neq \emptyset$, also $\mathcal{F} \neq \emptyset$; in particular, $\omega_1 \in \mathcal{F}$. Furthermore, $\emptyset \notin \mathcal{F}$ because $\Delta_{i < \omega_1} S_i$ is stationary when $S_i \in B$, for all $i < \omega_1$. Clearly, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq \omega_1$, then $Y \in \mathcal{F}$. It is enough to see that $\mathcal{F}$ is closed under diagonal intersections. Suppose that $X_i \in \mathcal{F}$, for $i < \omega_1$. Say $X_i \supseteq C_i \cap \Delta_{j < \omega} S_j^i$, where $S_j^i \in B$, for $i, j < \omega_1$, and C_i is club in ω_1 . Recall the Gödel pairing function, $(i, j) \mapsto \langle \langle i, j \rangle \rangle$, a one-to-one function from $\text{OR} \times \text{OR}$ onto OR . In particular, recall that ω_1 is closed under this function. Indeed

$$K = \{ \delta < \omega_1 : i, j < \delta \iff \langle \langle i, j \rangle \rangle \in \delta \}$$

is club in ω_1 . Set

$$A_{\langle \langle i, j \rangle \rangle} = S_j^i, \text{ for } i, j < \omega_1.$$

Then

$$\begin{aligned} \Delta_{i < \omega_1} X_i &\supseteq \Delta_{i < \omega_1} (C_i \cap \Delta_{j < \omega_1} S_j^i) \\ &\supseteq K \cap \Delta_{i < \omega_1} (C_i \cap \Delta_{j < \omega_1} S_j^i) \\ &= K \cap \Delta_{i < \omega_1} C_i \cap \Delta_{i < \omega_1} \Delta_{j < \omega_1} S_j^i \\ &= K \cap \Delta_{i < \omega_1} C_i \cap \Delta_{\alpha < \omega_1} A_\alpha \end{aligned}$$

Because $\Delta_{\alpha < \omega_1} A_\alpha$ is a diagonal intersection of elements of B and because

$K \cap \Delta_{i < \omega_1} C_i$ is club, it follows that $\Delta_{i < \omega_1} X_i \in \mathcal{F}$. $\square$

Now, by Proposition 3.2, there exists a countably distribute, GCH preserving, ω_2 -c.c. forcing property that adds a club subset of ω_1 that is eventually contained in each element of $\mathcal{F}$.

(2) $\Rightarrow$ (3) is obvious.

For (3) $\Rightarrow$ (1), assume that $\langle S_i : i < \omega_1 \rangle$ is a sequence of elements of B . Let $K \in V$ be club in ω_1 . In W , let $C_i \subseteq S_i$ be club ω_1 . Then $\Delta_{i < \omega_1} C_i$ is club, hence $K \cap \Delta_{i < \omega_1} C_i$ is club. And $K \cap \Delta_{i < \omega_1} C_i \subseteq K \cap \Delta_{i < \omega_1} S_i$. It follows that $\Delta_{i < \omega_1} S_i$ is stationary (in V). $\square$

4. Closed unbounded subsets of ω_2

As the following example shows, for a subset of ω_2 to have a club subset in an ω_1 and ω_2 preserving outer model, unlike a subset of ω_1 , it is not sufficient for the subset to be stationary.

Example 1. Let $S = \{ \alpha < \omega_2 : \text{cf}(\alpha) = \omega \}$. Then S is stationary. (Let $C \subseteq \omega_2$ be club. The ω^{th} element of C has cofinality ω and is therefore in S . Hence, $S \cap C \neq \emptyset$.) Suppose that S has a club subset in an ω_1 and ω_2 preserving outer model, W . That is, $C \in W$ and $C \subseteq S \in V$. Let α be the ω_1^{st} element of C . Then α has cofinality ω_1 in W and therefore in V . But if α is in C , then $\alpha \in S$ and therefore $\text{cf}(\alpha) = \omega$ in V .

We need several definitions in order to state the propositions and theorems in this section.

Let $S \subset \text{OR}$. For each $\alpha < \sup(S)$ of uncountable cofinality, fix a monotonically increasing continuous sequence

$$\langle \rho_i^\alpha : i < \text{cf}(\alpha) \rangle$$

cofinal in α .

Define $\text{ptn}(S, \alpha)$, the *pattern of S at α* by

$$\text{ptn}(S, \alpha) = \{ i < \text{cf}(\alpha) : \rho_i^\alpha \in S \}$$

Say that $S \in \kappa$ is *fat stationary* if for each uncountable regular cardinal $\mu < \kappa$

$$\{ \alpha \in S : \text{cf}(\alpha) = \mu \text{ and } \text{ptn}(S, \alpha) \text{ has a club subset} \}$$

is stationary in κ .

Lemma 4.1. *The following are equivalent.*

- (1) S is a fat stationary subset of κ
- (2) If $\omega_1 \leq \mu < \kappa$ is a regular cardinal, then

$$\{ \alpha \in S : \text{cf}(\alpha) = \mu \text{ and } S \cap \alpha \text{ contains a club} \}$$

is stationary.

- (3) If $S \subseteq \kappa$ and C is club in κ , then $S \cap C$ contains closed subsets of arbitrarily great order type less than κ .

Proof. To see that (1) and (2) are equivalent, fix a regular μ such that $\omega_1 \leq \mu < \kappa$ and fix $\alpha \in S$ such that $\text{cf}(\alpha) = \mu$. On the one hand, if C is club in μ , then $\{ \rho_i^\alpha : i \in C \}$ is club in α . On the other, if C is club in α , then $\{ i < \mu : \rho_i^\alpha \in C \}$ is club in μ . Thus $\text{ptn}(S, \alpha)$ has a club subset iff $S \cap \alpha$ has a club subset.

For (2) $\Rightarrow$ (3), suppose that (2) fails. Let δ be least such that $C \cap S$ does not have a closed subset of order type $\delta + 1$ and set $\mu = \max(\omega_1, \text{cf}(\delta))$ for some C club in κ . Because S is stationary, δ is a limit ordinal. Fix such a club set C . Note that there exists a club set $K \subseteq \kappa$ such that if $\beta < \beta'$ are in K and $\gamma < \delta$, then $C \cap S \cap [\beta, \beta')$ has a closed subset of order type $\gamma + 1$. [Let K consist of fixed

points of the function f_γ defined by $f_\gamma(\beta) = \sup(C_{\beta,\gamma})$, where $C_{\beta,\gamma}$ is chosen to be a closed subset of $S \cap (C \setminus \beta)$ having order type $\gamma + 1$.]

Using (2), let $\alpha \in S \cap \lim(C) \cap \lim(K)$ be such that $\text{cf}(\alpha) = \mu$ and $S \cap \alpha$ contains a club subset, say D . Let $\langle \beta_i : i \leq \text{cf}(\delta) \rangle$ enumerate the first $\text{cf}(\delta) + 1$ many elements of $D \cap C \cap K$ monotonically and let $\langle \gamma_i : i < \text{cf}(\delta) \rangle$ be cofinal in δ . Let $c_i \subseteq C \cap S \cap [\beta_i, \beta_{i+1})$ be closed and have type γ_{i+1} . Set

$$c = \bigcup_{i < \text{cf}(\delta)} c_i \cup \{\beta_i : i \leq \text{cf}(\delta)\}.$$

Then, c is closed and $c \subseteq C \cap S$. Also $\text{ot}(c \cap \beta_{\text{cf}(\delta)}) \geq \sup_{i < \text{cf}(\delta)} \gamma_{i+1} = \delta$, so $\text{ot}(c) \geq \delta + 1$.

To see that (3) $\Rightarrow$ (2), let C be club in κ and fix a regular cardinal μ such that $\omega_1 \leq \mu < \kappa$. Let c be a closed subset of $S \cap C$ with order type $\mu + 1$ and set $\alpha = \sup(c)$. Then $\alpha \in c$ and therefore in $S \cap C$. Now, $\alpha = \sup(c \cap \alpha)$ and $\text{ot}(c \cap \alpha) = \mu$, so $\text{cf}(\alpha) = \mu$, and $S \cap \alpha$ contains a club subset, namely $c \cap \alpha$. $\square$

The following theorem gives a complete characterization of all subsets of ω_2 that can have club subsets in an ω_1 and ω_2 preserving outer model that doesn't add new subsets of ω_1 (i.e. bounded subsets of ω_2).

Theorem 4.2. *Let κ be either a strongly inaccessible cardinal or the successor of a regular cardinal μ such that $\mu = \mu^{<\mu}$. Let $S \subseteq \kappa$ be fat stationary. Then there*

exists a poset $\mathbb{P}$ such that the following hold.

- (1) Forcing with $\mathbb{P}$ adds a club $C \subseteq S$.
- (2) Forcing with $\mathbb{P}$ does not add new sets of size $< \kappa$. (Hence cardinals and cofinalities $< \kappa$ remain unchanged.)
- (3) $|\mathbb{P}| = 2^{<\kappa}$, so if $2^{<\kappa} = \kappa$, cardinals above κ aren't collapsed.

Proof. Let S be a fat stationary subset of κ and define $\mathbb{P}$ to be a notion of forcing such that $p \in \mathbb{P}$ iff $p \subseteq S$ is a bounded and closed set of ordinals. Say that $p \leq p'$ iff p end-extends p' and let G be a V -generic filter over $\mathbb{P}$.

S is fat stationary, so it contains closed subsets of arbitrarily large order type less than κ , and for every $p \in \mathbb{P}$, $|p| < \kappa$. Therefore $|\mathbb{P}| = 2^{<\kappa}$, so if $2^{<\kappa} = \kappa$, then $\mathbb{P}$ satisfies the κ^+ -c.c. and forcing with $\mathbb{P}$ preserves cardinals above κ .

Note that $C = \bigcup \{p : p \in G\}$ is a club subset of S in $V[G]$. Because each $p \in G$ is closed, it suffices to see that C is unbounded in κ . To see that C is unbounded, pick $\alpha < \kappa$ and define the set $D_\alpha = \{c \in \mathbb{P} : \sup(c) \geq \alpha\}$. D_α is dense because for any $p \in \mathbb{P}$, we can find a $q \in D_\alpha$ such that $q \leq p$ as follows: let δ be the least element of S that is greater than α and set $q = p \cup \{\delta\}$. So D_α meets G , therefore for any $\alpha < \kappa$ there exists $r \in G$ such that $\sup(r) \geq \alpha$.

To see that no new bounded subsets of κ are added, hence κ and smaller cardinals are not collapsed in $V[G]$, it suffices to show that $\mathbb{P}$ is $< \kappa$ -distributive. In other words, given $\bar{p} \in \mathbb{P}$, $\tau < \kappa$, and a sequence $\bar{D} = \langle D_i : i \in \tau \rangle$ of dense open subsets of $\mathbb{P}$, it has to be shown that $\bar{p}$ has an extension that meets $\bigcap_{i < \tau} D_i$.

Let λ be a large enough regular cardinal so that $\mathbb{P} \in H_\lambda$ and all subsets of $\mathbb{P}$ are contained in H_λ , specifically all dense open subsets of $\mathbb{P}$ are in H_λ . Because λ is regular, any sequence of subsets of $\mathbb{P}$ having length less than κ lies in H_λ . In particular $\bar{D} \in H_\lambda$. Define $\langle M_\alpha : \alpha < \kappa \rangle$, a tower of elementary substructures of H_λ , by

$$M_\alpha = \text{the least } M \prec H_\lambda \text{ such that } \{\bar{p}, \mathbb{P}, \bar{D}, \tau + 1\} \cup \{M_\beta : \beta < \alpha\} \subseteq M.$$

Set $\eta_\alpha = M_\alpha \cap \kappa$ for each $\alpha < \kappa$. Consider $E = \{\eta_\gamma : \gamma < \kappa\}$. When $\alpha < \kappa$ is a limit ordinal, $M_\alpha = \bigcup_{\gamma < \alpha} M_\gamma$, so $\eta_\alpha = \bigcup_{\gamma < \alpha} (M_\gamma \cap \kappa) = \bigcup_{\gamma < \alpha} \eta_\gamma$ and E is closed. Now, $\kappa \in M_{\gamma+1}$ and $M_\gamma \in M_{\gamma+1}$, so $\eta_\gamma \in M_{\gamma+1}$. But $M_{\gamma+1} \prec H_\lambda$, so $M_\gamma \models \text{"}\eta_\gamma < \kappa\text{"}$, so $\eta_{\gamma+1} = \sup(\text{ordinals smaller than } \kappa \text{ in } M_{\gamma+1})$ which means that E is unbounded and therefore club. Then $\langle \eta_\alpha : \alpha < \kappa \rangle$ is a continuous and increasing sequence cofinal in κ .

Let $\beta < \kappa$ and $A = \{a : a \subseteq \beta \text{ and } |a| < |\beta|\}$. Then $|A| = \beta^{<\beta}$. If κ is strongly inaccessible, then $\beta^{<\beta} < \kappa$ by definition, and if $\kappa = \mu^+$ where μ is regular, then $\beta^{<\beta} \leq \mu^{<\mu} = \mu < \kappa$ by hypothesis. So, $|A| < \kappa$ and $H_\lambda \models |A| < \kappa$, therefore $H_\lambda \models (\exists f)(\exists \delta < \kappa)(f : \delta \xrightarrow{\text{onto}} A)$. Now $M_\alpha \prec H_\lambda$, so $M_\alpha \models (\exists f)(\exists \delta < \kappa)(f : \delta \xrightarrow{\text{onto}} A)$, where $\delta < \eta_\alpha$. Since δ is a subset of M_α , A is a subset of M_α and therefore M_α contains each subset of β of cardinality $< \beta$.

S is fat stationary, hence $S \cap E$ contains a closed subset of order type $\tau + 1$. Let $B \subseteq S \cap E$ be closed and of order type $\tau + 1$, and $\alpha = \sup(B)$, then $\alpha \in B$. For each $\xi < \alpha$, let β be such that $\xi < \beta < \eta_\alpha$. Then $(B \cap \xi) \subseteq \beta$ and $|B \cap \xi| < \kappa$.

Therefore $(B \cap \xi) \in M_\alpha$ for each $\xi < \alpha$.

Construct $\langle p_i : i < \tau \rangle$, an increasing sequence in $\mathbb{P}$, such that $p_{i+1} \in D_i \cap M_\alpha$. Begin with $p_0 = \bar{p}$. If p_i is defined, set p_{i+1} to be the first member of D_i extending p_i such that the ordinal interval $(\sup(p_i), \sup(p_{i+1}))$ has nonempty intersection with A . When $\delta < \tau$ is a limit ordinal, set p_δ to be $\bigcup_{i < \delta} p_i \cup \left\{ \sup\left(\bigcup_{i < \delta} p_i\right) \right\}$.

Because only proper initial segments of A were used in the definition of p_δ , it can be concluded that $p_\delta \in M_\alpha$. Also, because $A \subseteq S$ is closed, $p_\delta \subseteq S$. Finally, set $p = \bigcup_{i < \tau} p_i$. Then $p \in \bigcap_{i < \tau} D_i$ and therefore $\bigcap_{i < \tau} D_i$ is dense in $\mathbb{P}$ and $\mathbb{P}$ is $< \kappa$ -distributive. $\square$

Theorem 4.3. (GCH) *Let S be a subset of ω_2 . The following are equivalent.*

- (1) *S is fat stationary*
- (2) *There exists a cardinal and GCH preserving set generic extension $V[G]$ in which S has a club subset and $\mathcal{P}^V(\omega_1) = \mathcal{P}^{V[G]}(\omega_1)$.*
- (3) *There exists an ω_1 and ω_2 preserving outer model W of V in which S has a club subset and $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$.*

Proof. Using Theorem 4.2, it is clear that (1) $\Rightarrow$ (2). It is also clear that (2) $\Rightarrow$ (3). To see that (3) $\Rightarrow$ (1), let W be an outer model of V , as in (3). Let $C \in W$ be a club subset of ω_2 such that $C \subseteq S$. Let $K \in V$ be club in ω_2 . By Lemma 4.1, it suffices to see that there exists $\alpha \in S$ such that in V , $\text{cf}(\alpha) = \omega_1$, and $S \cap \alpha$ has a club subset. In W , let α be the ω_1^{st} element of $C \cap K$. Then $\text{cf}^V(\alpha) = \text{cf}^W(\alpha) = \omega_1$, since W is an ω_1 and ω_2

preserving outer model, and $\alpha \in S$, since $C \subseteq S$. Finally, $C \cap \alpha \in V$, since $\text{ot}(C \cap \alpha) = \omega_1$ and $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$. $\square$

Let X be a set of ordinals and κ a regular cardinal. Let

$$\lim(X) = \{ \alpha \in X : \sup(X \cap \alpha) = \alpha \} \text{ and}$$

$$X^{(\kappa)} = \{ \alpha \in \lim(X) : \text{cf}(\alpha) = \kappa \}$$

Proposition 4.4. (GCH) *If $S \subseteq \omega_2$ and $S' \subseteq S^{(\omega_1)}$ is stationary such that*

$$\left\{ \text{ptn}(S, \alpha) : \alpha \in S' \right\}$$

generates a normal filter of stationary sets on ω_1 , then S has a club subset in a cardinal and GCH preserving set generic extension.

Proof. By Proposition 3.2, there exists a countably distributive, ω_1 -c.c., GCH preserving forcing property that adds a club subset of ω_1 that is eventually contained in each $\left\{ \text{ptn}(S, \alpha) : \alpha \in S' \right\}$, so $S \cap \alpha$ eventually contains a club for $\alpha \in S'$. It follows that in this extension S is fat stationary, so, by Theorem 4.2, S has a club subset in a cardinal and GCH preserving set generic extension adding no more new bounded sets. $\square$

Suppose that κ is a regular uncountable cardinal. If X and Y are subsets of κ , then $X \subseteq_{NS} Y$ indicates that $X \cap C \subseteq Y$ for some club $C \subseteq \kappa$. Similarly, $X =_{NS} Y$ indicates that $X \subseteq_{NS} Y$ and $Y \subseteq_{NS} X$.

Say that $S \subseteq \omega_2$ has *pattern width* κ if κ is least such that there exists a family P of κ many subsets of ω_1 such that if $\alpha \in \lim(S)$ has cofinality ω_1 , then $\text{ptn}(S, \alpha) =_{NS} X$, for some $X \in P$. Say that S has *bounded pattern width*, if it has pattern width less than the maximum possible value, namely, less than ω_2 , assuming the GCH.

As the following theorem shows, it is also possible to characterize subsets of ω_2 with bounded pattern width that have club subsets in an ω_1 and ω_2 preserving outer model.

Theorem 4.5. *(GCH) Assume that $S \subseteq \omega_2$ has bounded pattern width. The following statements are equivalent.*

- (1) $B = \left\{ \alpha \in S^{(\omega_1)} : \text{ptn}(S, \alpha) \text{ is stationary in } \omega_1 \right\}$ is stationary in ω_2 .
- (2) There exists a cardinal and GCH preserving set generic extension $V[G]$ in which S has a club subset.
- (3) There exists an ω_1 and ω_2 preserving outer model W of V in which S has a club subset

Proof. To see that (1) $\Rightarrow$ (2), let $\langle X_\delta : \delta < \kappa \rangle$ be a one to one enumeration of a family of subsets of ω_1 witnessing that S has pattern width κ . Set

$$S' = \{ \alpha \in S^{(\omega_1)} : \text{ptn}(S, \alpha) \text{ is stationary in } \omega_1 \}.$$

So, for each $\alpha \in S'$, there exists a unique $\delta < \kappa$ such that $\text{ptn}(S, \alpha) =_{NS} X_\delta$. By hypothesis, S' is stationary. It follows by Fodor's Lemma that there exists a

$\delta < \kappa$ such that $S'_\delta = \{\alpha \in S' : \text{ptn}(S, \alpha) =_{NS} X_\delta\}$ is stationary in ω_2 . Then, trivially, $\{X_\delta\}$ generates a normal filter of stationary sets. By the Proposition 4.4, $S_\delta \cup \{\alpha \in S : \text{cf}(\alpha) < \omega_1\}$ has a club subset in a cardinal and GCH preserving set generic extension. But this set is a subset of S .

It is clear that (2) $\Rightarrow$ (3). To see that (3) $\Rightarrow$ (1), let $C \subseteq \omega_2$ be club in V and K be a club subset of S in W . Let α be such that $\text{cf}(\alpha) = \omega_1$ and $\alpha \in (\lim(K) \cap C)$. The $\text{ptn}(S, \alpha)$ contains a club in W , namely $K \cap \alpha$, and is therefore stationary in V . □

At this point we have the following:

- (1) A first-order characterization of those subsets of ω_1 that have a club subset in a ω_1 -preserving outer model, namely, “ S is stationary”
- (2) A first-order characterization of those subsets of ω_2 that have a club subset in an ω_1 and ω_2 preserving outer model with no new bounded subsets of ω_2 , namely, “ S is fat stationary”
- (3) A first-order characterization of those bounded pattern width subsets of ω_2 that have a club subset in an ω_1 and ω_2 preserving outer model, namely,

$$\left\{ \alpha \in S^{(\omega_1)} : \text{ptn}(S, \alpha) \text{ is stationary in } \omega_1 \right\} \text{ is stationary in } \omega_2$$

It is easy to see from the proof of Proposition 4.4 that, given an arbitrary $S \subseteq \omega_2$, the following condition is *sufficient* for S to have a club subset in an ω_1 and ω_2 preserving outer model: There exists a normal filter $\mathcal{F}$ of stationary

subsets of ω_1 such that $\{\alpha \in S^{(\omega_1)} : \text{ptn}(S, \alpha) \in \mathcal{F}\}$ is stationary in ω_2 .

This condition is not necessary in general. In [S1] it is shown that in general there is no first-order characterization of those subsets of ω_2 that have a club subset in an ω_1 and ω_2 preserving outer model.

The Box Principle ($\square_\kappa$): There exists a sequence $\langle C_\alpha : \alpha \in \text{lim}(\kappa^+) \rangle$ such that

- (1) C_α is a club subset of α ;
- (2) $\text{ot}(C_\alpha) \leq \kappa$;
- (3) if $\beta \in \text{lim}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.

The following theorem shows that the characterization of those subsets of ω_2 that have club subsets in an ω_1 and ω_2 preserving outer model, is at least as hard as the question of which partitions of ω_2 have large homogeneous sets.

Theorem 4.6 (Stationary Set Theorem). [5] *Assume that κ is regular, $\square_\kappa$ and the GCH. Let $\kappa \geq \omega_1$ and suppose that*

$$F : [\{\alpha < \kappa^+ : \text{cf}(\alpha) = \kappa\}]^2 \rightarrow 2.$$

There exists a partial ordering $\mathbf{S}$ such that

- (1) $\mathbf{S}$ has cardinality κ^+ , is $< \kappa$ -closed, and satisfies the $< \kappa^+$ -c.c.
- (2) *Forcing with $\mathbf{S}$ adds a stationary set $S^G \subseteq \kappa^+$ such that*
 - (a) *if $\alpha < \kappa^+$ and $\text{cf}(\alpha) = \kappa$, then $\alpha \in S^G$;*

- (b) if $F(\alpha, \beta) = 0$, then $|\text{ptn}(S^G, \alpha) \cap \text{ptn}(S^G, \beta)| < \kappa$; and
- (c) if H is F -homogeneous to 1, then $\{\text{ptn}(S^G, \alpha) : \alpha \in H\}$ generates a normal filter of stationary subsets of κ .

Proof. Let $\langle C_\alpha : \alpha \in \lim(\kappa^+) \rangle$ be a $\square_\kappa$ -sequence and let $\langle \rho_i^\alpha : i < \kappa \rangle$ enumerate the limit points of C_α in increasing order when $\text{cf}(\alpha) = \kappa$.

Note that if $\alpha, \beta \in \kappa^+$ have cofinality κ and $\rho_i^\alpha = \rho_j^\beta$, then $i = j$. Indeed, let $\delta = \rho_i^\alpha = \rho_j^\beta$, then $C_\alpha \cap \delta = C_\delta$ and $C_\beta \cap \delta = C_\delta$, so below δ , C_α and C_β are identical and therefore $i = j$.

Note also that if d is a set of fewer than κ many cofinality κ ordinals below κ^+ , then there exists $i < \kappa$ such that if α and β are distinct elements of d , then $C_\alpha \setminus \rho_i^\alpha$ and $C_\beta \setminus \rho_i^\beta$ are disjoint. Indeed, given $\alpha < \beta$, let $i(\alpha, \beta)$ be the least i such that $\rho_i^\beta \geq \alpha$. Then set $i = \sup_{\{\alpha, \beta\} \in [d]^2} i(\alpha, \beta)$.

Let $\mathbf{S}$ consist of all triples $s = (d_s, \beta_s, f_s)$ such that

- (i) $d_s \subseteq \{\alpha < \kappa^+ : \text{cf}(\alpha) = \kappa\}$ and $|d_s| < \kappa$;
- (ii) $\beta_s < \kappa$ and $(C_\alpha \setminus \rho_{\beta_s}^\alpha) \cap (C_{\alpha'} \setminus \rho_{\beta_s}^{\alpha'}) = \emptyset$, for all $\{\alpha, \alpha'\} \in [d_s]^2$; and
- (iii) $f_s : \bigcup_{\alpha \in d_s} \{\rho_i^\alpha : i < \beta_s\} \rightarrow \{0, 1\}$,

and say that $\bar{s} \geq s$ when

- (a) $d_{\bar{s}} \subseteq d_s$ and $\beta_{\bar{s}} \leq \beta_s$ and $f_{\bar{s}} \subseteq f_s$; and
- (b) if $\{\alpha, \alpha'\} \in [d_{\bar{s}}]^2$ and $F(\alpha, \alpha') = 0$, and if $f_s(\rho_i^\alpha) = 1$, for some i such that $\beta_{\bar{s}} \leq i \leq \beta_s$, then $f_{\bar{s}}(\rho_i^{\alpha'}) = 0$.

$|\mathbf{S}| = \kappa^+ \cdot \kappa \cdot \kappa = \kappa^+$ by the GCH. Also, because $|d_s| < \kappa$, $|\bigcup_{i < \theta} d_s| < \kappa$, and

$|\bigcup_{i<\theta} f_{s_i}| < \kappa$. Let $\langle s_i : i < \theta \rangle$ be a descending sequence of conditions in $\mathbf{S}$, where $\theta < \kappa$. Then

$$\inf_{i<\theta} s_i = \left(\bigcup_{i<\theta} d_{s_i}, \sup_{i<\theta} \beta_{s_i}, \bigcup_{i<\theta} f_{s_i} \right)$$

extends every s_i , and therefore $\mathbf{S}$ is $< \kappa$ -closed.

Let G be $\mathbf{S}$ generic over V and set

$$S^G = \left\{ \rho < \kappa^+ : \text{cf}(\rho) = \kappa \text{ or } f_s(\rho) = 1, \text{ for some } s \in G \right\}.$$

Extension Lemma. *Suppose that $\bar{s} \in \mathbf{S}$.*

- (a) *If $\delta < \kappa$, then $\bar{s}$ has an extension s such that $\beta_s > \delta$.*
- (b) *If $\alpha < \kappa^+$ has cofinality κ , then $\bar{s}$ has an extension s such that $\alpha \in d_s$.*

Proof. Parts (a) and (b) will be proven simultaneously. Suppose that $\delta < \kappa$ and that $\alpha < \kappa^+$ has cofinality κ . Assume that $\alpha \notin d_{\bar{s}}$ and set $d = d_{\bar{s}} \cup \{\alpha\}$. Choose β such that $\beta_{\bar{s}}, \delta < \beta < \kappa$ and $(C_\alpha \setminus \rho_\beta^\alpha) \cap (C_\gamma \setminus \rho_\beta^\gamma) = \emptyset$, for every $\gamma \in d$. Extend $f_{\bar{s}}$ to $f : \{\rho_i^\gamma : i < \beta \text{ and } \gamma \in d\} \rightarrow \{0, 1\}$ by setting $f(\rho) = 0$ when ρ is not in the domain of $f_{\bar{s}}$. Then (d, β, f) is a condition in $\mathbf{S}$.

To see that (d, β, f) extends $\bar{s}$, it is enough to show that $f(\rho_i^\gamma) = 0$ for $\gamma \in d_{\bar{s}}$ and $\beta_{\bar{s}} \leq i < \beta$. Suppose that $\rho_i^\gamma \in \text{dom}(f_{\bar{s}})$, then $\rho_i^\gamma = \rho_j^{\gamma'}$, which makes $i = j$. But $i \geq \beta_{\bar{s}}$, so $\rho_j^{\gamma'} \notin \text{dom}(f_{\bar{s}})$, and $f(\rho_i^\gamma) = 0$. $\square$

Antichain Lemma. *Every subset of $\mathbf{S}$ of size κ^+ has a subset of size κ^+ in which every two conditions are compatible.*

Proof. Suppose that $A \subseteq \mathbb{S}$ has cardinality κ^+ . Assume without loss of generality that

- (1) if $s, s' \in A$, then $\beta_s = \beta_{s'}$;
- (2) there exists $r \subseteq \kappa^+$ such that $\text{dom}(f_s) \cap \text{dom}(f_{s'}) = r$, if $s, s' \in A$ and $s \neq s'$; and
- (3) if $s, s' \in A$ and $\rho \in r$, then $f_s(\rho) = f_{s'}(\rho)$.

Assumptions (1) and (3) are possible by cardinality considerations. Assumption (2) is possible by the Δ -system Lemma [5].

Now, suppose that $s, s' \in A$. Set $d = d_s \cup d_{s'}$, and choose $\beta \geq \beta_s (= \beta_{s'})$ such that $(C_\alpha \setminus \rho_\beta^\alpha) \cap (C_{\alpha'} \setminus \rho_\beta^{\alpha'}) = \emptyset$, for all $\alpha \neq \alpha'$ that lie in d .

Let $f : \{\rho_i^\alpha : \alpha \in d \text{ and } i < \beta\} \rightarrow \{0, 1\}$ by setting

$$f(\rho_i^\alpha) = \begin{cases} f_s(\rho_i^\alpha), & \text{if } \alpha \in d_s \text{ and } i < \beta_s, \\ f_{s'}(\rho_i^\alpha), & \text{if } \alpha \in d_{s'} \text{ and } i < \beta_{s'}, \\ 0, & \text{otherwise.} \end{cases}$$

Using (2) and (3), the first two lines in the definition of $f(\rho_i^\alpha)$ don't conflict. It follows that $(d, \beta, f) \in \mathbb{S}$. To see that $(d, \beta, f) \leq s$, suppose that $\alpha \in d_s$ and $\beta_s \leq i < \beta$. Note that $\rho_i^\alpha \notin \text{dom}(f_s) \cup \text{dom}(f_{s'})$ because if $\rho_i^\alpha = \rho_j^\alpha$ then $j = i$. But $i \geq \beta_s$ and $\beta_s = \beta_{s'}$, so $f(\rho_i^\alpha) = 0$. It follows that $(d, \beta, f) \leq s$. Similarly, $(d, \beta, f) \leq s'$. $\square$

Filter Lemma. Suppose that G is $\mathbb{S}$ -generic.

- (a) If $F(\alpha, \alpha') = 0$, then $|\text{ptn}(S^G, \alpha) \cap \text{ptn}(S^G, \alpha')| < \kappa$.
- (b) If H lies in the generic extension by G and is F -homogenous to 1, then

$\{ \text{ptn}(S^G, \alpha) : \alpha \in H \}$ is contained in a normal filter of stationary subsets of κ .

Proof. For (a), let $\bar{s} \in G$ be such that $\alpha, \alpha' \in d_{\bar{s}}$. If s extends $\bar{s}$ and $\beta_{\bar{s}} \leq i \leq \beta_s$, then it is not the case that both $f_s(\rho_i^\alpha) = 1$ and $f_s(\rho_i^{\alpha'}) = 1$. Hence $\bar{s} \Vdash \text{ptn}(\dot{S}^G, \check{\alpha}) \cap \text{ptn}(\dot{S}^G, \check{\alpha}') \subseteq \check{\beta}_{\bar{s}}$.

For (b), it suffices to see that if $\langle \alpha_i : i < \kappa \rangle$ enumerates a subset of H (in the generic extension), then $\Delta_{i < \kappa} \text{ptn}(S^G, \alpha_i)$ is stationary in κ .

Work in the ground model. Suppose that $\mathbf{S} \Vdash \text{"}\dot{H} \text{ is } \hat{F}\text{-homogenous to 1"}$ and that $\mathbf{S} \Vdash \check{\alpha}_i \in \dot{H}_i$, for $i < \kappa$. Fix a condition $\bar{s}$ and a term $\dot{C}$ for a club subset of κ . Define a descending sequence $\langle s_n : n < \omega \rangle$ of conditions by recursion. Begin by setting $s_0 = \bar{s}$. Then choose s_{n+1} to be a condition $s \leq s_n$ such that

- (i) $s \Vdash \dot{C} \cap [\beta_{s_n}, \beta_s) \neq \emptyset$, and
- (ii) there exists $\langle \gamma_i : i < \beta_{s_n} \rangle$ such that for each $i < \beta_{s_n}$, $s \Vdash \check{\alpha}_i = \check{\gamma}_i$ and $\gamma_i \in d_s$.

Note that there is such a condition s , using the Extension Lemma and that $\mathbf{S}$ is $< \kappa$ -closed.

Set $s = \inf_{n < \omega} (s_n)$. Then $s \Vdash \check{\beta}_s \in \dot{C}$ and there exists $\langle \gamma_i : i < \beta_s \rangle$ such that $\gamma_i \in d_s$ and $s \Vdash \check{\alpha}_i = \check{\gamma}_i$, for all $i < \beta_s$.

Next define (d, β, f) . Set $\beta = \beta_s + 1$ and $d = d_s$. Let

$$f : \{ \rho_i^\alpha : \alpha \in d \text{ and } i \leq \beta_s \} \rightarrow \{0, 1\}$$

by

$$f(\rho_i^\alpha) = \begin{cases} f_s(\rho_i^\alpha), & \text{if } i < \beta_s \\ 1, & \text{if } i = \beta_s, \text{ and } \alpha \in \{\gamma_j : j < \beta_s\} \\ 0, & \text{if } i = \beta_s \text{ and } \alpha \in d_s \setminus \{\gamma_j : j < \beta_s\}. \end{cases}$$

Note that f is well-defined because $\rho_{\beta_s}^\alpha \neq \rho_{\beta_s}^{\alpha'}$ for $i < \beta_s$, and because $C_\alpha \setminus \rho_{\beta_s}^\alpha$ and $C_{\alpha'} \setminus \rho_{\beta_s}^{\alpha'}$ are disjoint, if $\alpha \neq \alpha'$ lie in d_s . It follows that (d, β, f) is a condition in $\mathbb{S}$.

Now, $(d, \beta, f) \Vdash \check{\beta}_s \in \text{ptn}(\dot{S}^G, \check{\gamma}_i)$, for each $i < \beta_s$, so it suffices to see that $(d, \beta, f) \leq s$. Suppose that $\{\alpha, \alpha'\} \in [d_s]^2$, $F(\alpha, \alpha') = 0$, and $f(\rho_{\beta_s}^\alpha) = 1$. Then $\alpha = \gamma_i$, for some $i < \beta_s$. Because s forces that $\{\gamma_i : i < \beta_s\} \subseteq \dot{H}^G$ and that $\dot{H}^G$ is F -homogeneous to 1, it follows that $\alpha' \in d_s \setminus \{\gamma_i : i < \beta_s\}$. hence $f(\rho_{\beta_s}^{\alpha'}) = 0$, as required. $\square$

This completes the proof of the Stationary Set Theorem. $\square$

5. Deciding sets

Assuming the GCH, given a subset X of ω_1 , at least one of X and $\omega_1 \setminus X$ is stationary, hence at least one has a club subset in some ω_1 -preserving outer model. In contrast, if $X = \{ \alpha < \omega_2 : \text{cf}(\alpha) = \omega_1 \}$, then neither X nor $\omega_2 \setminus X$ has a club subset in any ω_1 and ω_2 preserving outer model. Furthermore, the “anticharacterization” theorems of [6] show that in general there is no first order characterization of those subsets $X \subseteq \omega_2$ such that either X or $\omega_2 \setminus X$ has a club subset in some ω_1 and ω_2 preserving outer model. The negative results generalize to successors of regular cardinals.

Nevertheless, it is possible to prove a positive theorem: If $X \subseteq \kappa$ where κ is sufficiently large and the successor of a regular cardinal, then there exists an outer model in which $\kappa = \omega_2$ and either X or $\kappa \setminus X$ has a club subset.

Let X be a set of ordinals. Say that C *decides* X if $\sup(C) = \sup(X)$ and either $C \subseteq X$ or $C \cap X = \emptyset$.

Let κ be a regular cardinal and let $\lambda > \kappa$ be an inaccessible cardinal. define the notion of forcing $(\mathbb{L}_\lambda^\kappa, <)$ (called the *Levy Collapse*) as follows:

- $p \in \mathbb{L}_\lambda^\kappa$ iff p is a function
- $\text{dom}(p) \subseteq \kappa \times \lambda$
- $p(\alpha, \beta) < \alpha$
- $|p| < \lambda$
- For any $p, q \in \mathbb{L}_\lambda^\kappa$, $p \leq q$ iff $q \subseteq p$.

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is stationary in ν .

Proof. (By induction on $\nu \in R$ that are greater than μ .) If ν is least in R above μ , then the claim is immediate by our choice of S_μ . Otherwise, let λ be the predecessor of ν in R . Then $\mu < \lambda < \nu$. Let C be club in ν . Choose $\beta \in \lim(C)$ such that $\text{ptn}(S_\nu, \beta) =_{NS} S_\lambda$. (We can find such a β because of our choice of S_λ .) By induction, $\{\alpha \in S_\lambda : \text{ptn}(S_\lambda, \alpha) =_{NS} S_\mu\}$ is stationary in λ . So, there exists $\gamma \in C \cap \beta$ such that $(S_\nu, \gamma) =_{NS} S_\mu$. $\square$

Let $\bar{S} = \sup(S) \setminus S$. For $\mu < \nu$ in R , let

$P(\mu, \nu)$ be the assertion “ $\{\alpha \in S_\nu : \text{ptn}(S_\nu, \alpha) =_{NS} S_\mu\}$ is stationary in ν .”

$\bar{P}(\mu, \nu)$ be the assertion “ $\{\alpha \in \bar{S}_\nu : \text{ptn}(\bar{S}_\nu, \alpha) =_{NS} \bar{S}_\mu\}$ is stationary in ν ”

Note that $\text{ptn}(\bar{S}_\nu, \alpha) =_{NS} \bar{S}_\mu$ iff $\text{ptn}(S_\nu, \alpha) =_{NS} S_\mu$.

Claim 2. If $\mu < \lambda < \nu$ in R , then $P(\mu, \lambda) \Rightarrow P(\mu, \nu)$ and $\bar{P}(\mu, \lambda) \Rightarrow \bar{P}(\mu, \nu)$.

Proof. Assume $P(\mu, \lambda)$. Fix a club set $C \subseteq \nu$ and choose $\alpha \in \lim(C)$ such that $\text{ptn}(S_\nu, \alpha) =_{NS} S_\lambda$. Then there exists a club set $K \in \lambda$ such that $\text{ptn}(S_\nu, \alpha) \cap K = S_\lambda \cap K$. Recall that $\text{ptn}(S_\nu, \alpha) = \{i < \lambda : \rho_i^\alpha \in S_\nu\}$, where $\langle \rho_i^\alpha : i < \lambda \rangle$ is a fixed continuous monotonically increasing sequence cofinal in α . Because $\alpha \in \lim(C)$, we have that $\bar{C} = \{i < \lambda : \rho_i^\alpha \in C\}$ is club in λ . Using $P(\mu, \lambda)$, let $\beta \in \lim(\bar{C}) \cap K \cap S_\lambda$ be such that $\text{ptn}(S_\lambda, \beta) =_{NS} S_\mu$. Then $\rho_\beta^\alpha \in C \cap S_\nu$ and $\text{ptn}(S_\nu, \rho_\beta^\alpha) =_{NS} \text{ptn}(S_\nu, \beta) =_{NS} S_\mu$. Thus $P(\mu, \nu)$ holds.

Similarly, $\bar{P}(\mu, \lambda) \Rightarrow \bar{P}(\mu, \nu)$. $\square$

Claim 3. *There exists $H \subseteq R$ such that $\kappa \in H$, $|H| > \hat{n}$ and either $P(\mu, \nu)$ or $\bar{P}(\mu, \nu)$, for all $\mu < \nu$ in H .*

Proof. If $\nu \in R \cap \kappa$, let $\nu^* = \min(R \setminus (\nu + 1))$. Set

$$A_0 = \{ \nu \in R \cap \kappa : P(\nu, \nu^*) \}$$

$$A_1 = \{ \nu \in R \cap \kappa : \bar{P}(\nu, \nu^*) \}$$

Now, $R \cap \kappa = A_0 \cup A_1$. If $|A_0| < \hat{n}$ and $|A_1| < \hat{n}$, then $|R \cap \kappa| < 2\hat{n}$ which will make $|R| < 2\hat{n} + 1$. But, by the hypothesis, $|R| \geq 2\hat{n} + 1$. Now if $|A_0| \geq \hat{n}$, set $H = A_0 \cup \{\kappa\}$, otherwise set $H = A_1 \cup \{\kappa\}$. Then by Claim 2 H is as required. $\square$

Now assume that there exists H as in Claim 3 such that $P(\mu, \nu)$ holds for all $\mu < \nu$ in H . (The alternate case in which $\bar{P}(\mu, \nu)$ holds is similar.)

Choose $\kappa_0 < \dots < \kappa_{\hat{n}}$ from H with $\kappa_{\hat{n}} = \kappa$. Note that by $P(\kappa_0, \kappa_1)$,

$$\{ \alpha \in S_{\kappa_1} : \text{ptn}(S_{\kappa_1}, \alpha) =_{NS} S_{\kappa_0} \}$$

is stationary in κ_1 , so S_{κ_1} is stationary in κ_1 .

We shall construct a generic extension in which $\kappa = \aleph_{\hat{n}}$ and X contains a club set through an $\hat{n}$ -stage iteration. At stage n of the iteration, we shall force over $V[G_1 * \dots * G_{n-1}]$ (over V , if $n = 1$) to obtain G_n . At the beginning of stage n the following five recursion hypotheses will hold in $V[G_1 * \dots * G_{n-1}]$.

(A) $\aleph_i = \kappa_i$, for $1 \leq i \leq n - 1$.

- (B) V -cardinals greater than or equal to κ_{n-1} are cardinals.
- (C) The GCH holds.
- (D) There exists club sets $C_{\kappa_i} \subseteq \kappa_i$ such that $C_{\kappa_i} \subseteq S_{\kappa_i}$, for $1 \leq i \leq n-1$.
- (E) If $n-1 \leq i \leq \hat{n}$ and $S \in V$ is stationary in κ_i (in V), then S remains stationary in κ_i in $V[G_1 * \dots * G_{n-1}]$.

Let κ be a regular cardinal and $S \subseteq \kappa$. Define $\mathbb{C}(S)$ as follows:

$c \in \mathbb{C}(S)$ iff $c \subseteq S$ is closed and $|c| < \kappa$

For $p, p' \in \mathbb{C}(S)$, $p' \leq p$ iff p' end-extends p .

Note. For $\kappa = \omega_1$ and S stationary, $\mathbb{C}(S)$ is $< \omega_1$ -distributive and therefore forcing with $\mathbb{C}(S)$ preserves all cardinals $\leq \omega_1$. For $\kappa > \omega_1$ and S fat stationary, $\mathbb{C}(S)$ is $< \kappa$ -distributive and therefore forcing with $\mathbb{C}(S)$ preserves all cardinals $\leq \kappa$.

Stage 1. Let G_1 be $\mathbb{L}_{\omega_1}^{\kappa_1} * \mathbb{C}(S_{\kappa_1})$ generic over V . Note that S_{κ_1} remains stationary (in $\kappa_1 = \omega_1$) in an $\mathbb{L}_{\omega_1}^{\kappa_1}$ -generic extension by Proposition 2.4. Hence the forcing $\mathbb{C}(S_{\kappa_1})$ defined in such an extension is countably distributive. $\mathbb{L}_{\omega_1}^{\kappa_1}$ forces κ_1 to become ω_1 in the extension, preserves all cardinals greater than or equal to κ_1 , and preserves the GCH. Therefore (A)-(C) hold. (D) and (E) hold by Theorem 4.3 and Proposition 2.3, respectively.

Stage n . ($1 < n \leq \hat{n}$).

Let G_n be $\mathbb{L}_{\kappa_{n-1}}^{\kappa_n} * \mathbb{C}(S_{\kappa_n})$ generic over $V[G_1 * \dots * G_{n-1}]$ where $\kappa_{n-1} = \omega_{n-1}$ in the current model, namely $V[G_1 * \dots * G_{n-1}]$.

Now, in V

$$\{ \alpha \in S_{\kappa_n} : \text{ptn}(S_{\kappa_n}, \alpha) =_{NS} S_{\kappa_{n-1}} \}$$

is stationary in κ_n , so, by (E), it remains stationary in $V[G_1 * \dots * G_{n-1}]$. Using (D) and that forcing with $\mathbb{L}_{\kappa_{n-1}}^{\kappa_n}$ preserves stationary subsets of κ_n , we see that S_{κ_n} is fat stationary in an $\mathbb{L}_{\kappa_{n-1}}^{\kappa_n}$ generic extension of $V[G_1 * \dots * G_{n-1}]$. We know that forcing with $\mathbb{L}_{\kappa_n}^{\kappa_{n-1}}$ collapses κ_n to be ω_n , and preserves the GCH (i.e. (A) and (C) hold) and Theorem 4.3 verifies that (B), (D), and (E) hold.

Finally, in $V[G_1 * \dots * G_{n-1}]$ we have that $\aleph_i = \kappa_i$, for $1 \leq i \leq \hat{n}$, hence $\aleph_{\hat{n}} = \kappa_{\hat{n}} = \kappa$ and there exists a club $C \subseteq \kappa$ such that $C \subseteq S_{\kappa_{\hat{n}}} = X$. $\square$

Open Question. Does there exist κ such that if $X \subseteq \kappa$, then there exists a forcing extension in which X is decided and $\kappa > \aleph_\omega$?

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