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On deciding whether a surface is parabolic or hyperbolic

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ON DECIDING WHETHER A SURFACE IS
PARABOLIC OR HYPERBOLIC

Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements

for the Degree of

Masters of Science

by

Dean Kenneth Leonardi

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Abstract

On Deciding Whether a Surface is Parabolic or Hyperbolic

by Dean Kenneth Leonardi

San Jose State University, 2002

A simply-connected open Riemann surface is conformally equivalent either to the non-compact unit disk or to the entire complex plane. In the first case the surface is said to be hyperbolic; in the second case it is said to be parabolic. Equivalently, a hyperbolic surface is one on which Brownian motion is transient, and a parabolic surface is one on which Brownian motion is recurrent. The type problem for Riemann surfaces is the problem of determining whether a given non-compact Riemann surface is hyperbolic or parabolic.

Milnor gave a Gaussian curvature criterion for the conformal type of a surface. He investigated the case of a simply connected, non-compact, symmetric C^∞ Riemannian 2-manifold, with C^∞ Riemannian metric. In this thesis, Milnor's criterion for the non-symmetric case will be investigated.

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Chapter 1

Manifolds

The type problem was the main theme in the early stages of the classification theory of Riemann surfaces. The classical form of the type problem was born from the Riemann mapping theorem, which asserts that there are only two conformal types of simply connected plane regions. In the proof Riemann used the then unproved Dirichlet principle. This was criticized by Weierstrass who exhibited a functional similar to the Dirichlet integral for which the problem of minimizing the functional has no solution. Subsequently Schwartz and Neumann put Riemann's result on a sound basis and later Hilbert gave a rigorous proof of the Dirichlet principle, thus justifying Riemann's original approach.

All these methods of the early pioneers were of potential theoretic nature and required certain smoothness of the boundary. It was Osgood who, using a purely function theoretic proof, established the Riemann mapping theorem for arbitrary simply connected plane regions.

Weyl introduced the concept of abstract Riemann surface as a two manifold with conformal structure: his additional requirement of a countable basis was later

proved to be redundant by Rado. Koebe extended the mapping problem to planar abstract Riemann surfaces and showed by his uniformization theory that they all are conformally equivalent to the extended plane with parallel slits. In particular, the most general form of the Riemann mapping theorem is that every simply connected abstract open Riemann surface is conformally equivalent either to the unit disk or the plane. The surface is called, accordingly, hyperbolic or parabolic. The type problem is to determine which alternative occurs for a simply connected open Riemann surface.

Although the type problem goes back, implicitly, to the original Riemann mapping theorem, its concrete and explicit study seems to have been initiated by Speiser in [11]. In subsequent attacks on the type problem, Nevanlinna used the harmonic measure in connection with his value distribution theory, Ahlfors employed conformal metrics, Kobayashi [30] introduced a net now bearing his name, Teichmüller worked with Euclidean and Lobatschevsky metrics, and P. Myrberg utilized symmetries.

This thesis investigates an aspect of the type problem due to Milnor [38], Doyle [12] and Grigor'yan [22] by considering geodesically complete Riemannian manifolds with and without a pole, and proving a parabolicity and non-parabolicity test. In particular, geodesically complete, C^∞ Riemannian 2-manifold, M , having a global geodesic polar coordinate system (r, θ) about some point p will be considered. In this coordinate system the metric takes the form

$$ds^2 = dr^2 + g(r, \theta) d\theta^2.$$

thus

$$G(r) = \int_0^{2\pi} g(r, \theta) d\theta$$

is the length of a geodesic circle a distance r from p and the Gaussian curvature $K(r, \theta)$ satisfies

$$K(r, \theta)g + \frac{\partial^2 g}{\partial r^2} = 0.$$

Milnor wanted to relate the conformal type of the surface to the Gaussian curvature. Specifically, he proposed the following two conjectures:

P: If $K \geq -1/(r^2 \log r)$ for large r then the surface is conformally *parabolic*, i.e. can be mapped conformally onto the complex plane.

H: If $K \leq -(1 + \epsilon)/(r^2 \log r)$ for large r , and if $G(r)$ is unbounded, then the surface is conformally *hyperbolic*, i.e. can be mapped conformally onto the unit disk.

Milnor established these criteria for the case of a surface symmetric about p (that is, for a surface where G depends only on r) by constructing explicitly a conformal mapping of the surface onto a disk or the whole complex plane. He then asked for a proof in the more general case. Evidently, Robert Osserman pointed out to him that by applying a method of Ahlfors it can be concluded that

$$\int_a^\infty \frac{dr}{G(r)} = \infty \Rightarrow \text{conformally parabolic.}$$

The criterion **P** is a simple consequence of this parabolicity criterion. Hence what was missing was a proof of the criterion **H**.

Doyle states, to someone familiar with Rayleigh's method, the expression appearing in Ahlfors's criterion has a particularly concrete significance. Viewing the Riemann surface as being made of an isotropic resistive material of constant thickness, then the type problem can be interpreted as the problem of determining whether the resistance of the surface out to infinity is infinite (parabolic type) or

finite (hyperbolic type). The expression

$$\int_{\alpha}^{\infty} \frac{dr}{G(r)}$$

represents the resistance out to infinity of the electrical system obtained from the Riemann surface by "shorting the points of the surface together along the circles $r = \text{constant}$. As the resistance of this new system is less than or equal to the resistance of the old system,

$$\int_{\alpha}^{\infty} \frac{dr}{G(r)} = \infty \Rightarrow \text{conformally parabolic.}$$

Doyle's next step is to get an upper bound for the resistance of the surface by making some cuts. The most natural thing to do is to cut along the rays $\theta = \text{constant}$. The conductance (i.e. inverse resistance) of the resulting system out to infinity is

$$\frac{\int_0^{2\pi} d\theta}{\int_{\alpha}^{\infty} \frac{dr}{g(r, \theta)}}.$$

This is less than or equal to the conductance of the original Riemann surface out to infinity, hence

$$\frac{\int_0^{2\pi} d\theta}{\int_{\alpha}^{\infty} \frac{dr}{g(r, \theta)}} > 0 \Rightarrow \text{conformally hyperbolic.}$$

This hyperbolicity criterion obviously resembles the parabolicity criterion just obtained, and just as **P** followed from the earlier criterion, so **H** follows from this one.

In this chapter the following will be investigated [38]: Let M be a simply connected open surface (section 1.1), differentiable say of class C^{∞} , with C^{∞} Riemannian metric, defined in section 1.2. It follows from complex function theory

that M is conformally diffeomorphic either to the complex plane, or to the unit disc. In order to prove this result use will be made of the Uniformization Theorem as well as the fact that the conformal structure and the Riemannian structure of a surface are equivalent. That is, [42] a Riemannian manifold R is a connected, orientable, C^∞ manifold of dimension n with smooth metric tensor g_{ij} corresponding to a positive symmetric form $g_{ij}\eta^i\eta^j$. Since R is a metric space equipped with the natural distance

$$d(x_1, x_2) = \inf_{\gamma} \int_{\gamma} ds, \quad ds^2 = g_{ij} dx^i dx^j,$$

where the infimum is taken with respect to all arcs γ joining x_1 and x_2 . Moreover, R satisfies the first axiom of countability and thus admits an *exhaustion* $\{R_m\}_{m=1}^\infty$, i.e., a sequence of subregions R_m of R with

$$\bar{R}_m \subset R_{m+1}, \quad R = \cup_1^\infty R_m.$$

In case $n = 2$, an isothermal parametric system can be found. That is $g_{11} = g_{22}$, and $g_{12} = g_{21} = 0$ over the entire parametric ball. For a complete exposition on this construction see [8] and [26]. Therefore R is a Riemann surface. Conversely, for a Riemann surface R there always exists a positive density ρ such that $\rho |dz|$ is invariant. This gives isothermal coordinates with $g_{ij} = \rho^2 \delta_{ij}$, for the parameter $z = x^1 + ix^2$, and R becomes a Riemannian manifold. δ_{ij} is understood to be the Kronecker delta.

1.1 Topology

Definition 1.1.1. By a curve in a topological space X we mean a continuous mapping $u : I \rightarrow X$, where $I = [0, 1] \subset \mathbb{R}$ is the unit interval. The point $a = u(0)$

is called the initial point and $b = u(1)$ the end point of u .

Definition 1.1.2. A space is *arcwise connected* if any two points are the end points of a curve. A space is *locally arcwise connected* if it has a basis consisting of arcwise connected sets.

Definition 1.1.3. Let X be a topological space and $a, b \in X$. Two curves $u, v : I \rightarrow X$ from a to b are called homotopic, denoted $u \sim v$, if there exists a continuous mapping $A : I \times I \rightarrow X$ with the following properties:

$$A(t, 0) = u(t) \quad \text{for all } t \in I$$

$$A(t, 1) = v(t) \quad \text{for all } t \in I$$

$$A(0, s) = a \text{ and } A(1, s) = b \quad \text{for all } s \in I$$

Theorem 1.1.1. Suppose X is a topological space and $a, b \in X$. Then the notion of homotopy is an equivalence relation on the set of all curves from a to b .

Definition 1.1.4. Suppose a, b and c are three points in a topological space X , $u : I \rightarrow X$ is a curve from a to b and $v : I \rightarrow X$ is a curve from b to c .

1. The product curve $u \cdot v : I \rightarrow X$ from a to c is defined by

$$(u \cdot v)t = \begin{cases} u(2t), & t \in [0, \frac{1}{2}]; \\ v(2t - 1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

2. The inverse curve $u^- : I \rightarrow X$ from a to b is defined by

$$u^-(t) = u(1 - t) \quad \text{for all } t \in I$$

Definition 1.1.5. Suppose X is a topological space and $a \in X$. A constant curve at a is the constant mapping $u_0 : I \rightarrow X$, i.e., $u_0(t) = a \quad \forall t \in I$.

Theorem 1.1.2. *Suppose that X is a topological space and $a, b, c, d \in X$. Suppose $u, v, w : I \rightarrow X$ are curves such that*

$$u(0), \quad u(1) = b = v(0), \quad v(1) = c = w(0), \quad w(1) = d.$$

Further let u_0 be a constant curve at a , v_0 the constant curve at b . The following homotopies exist:

$$u_0 \cdot u \sim u \cdot v_0$$

$$u \cdot u^- \sim u_0$$

$$(u \cdot v) \cdot w \sim u \cdot (v \cdot w)$$

Definition 1.1.6. A curve $u : I \rightarrow X$ in a topological space X is called closed if $u(0) = u(1)$. A closed curve $u : I \rightarrow X$ with initial and end point a is said to be null-homotopic if it is homotopic to the constant curve at a .

Theorem 1.1.3. *Suppose X is a topological space and $a \in X$ is a point. The set $\pi_1(X, a)$ of homotopy classes of closed curves in X with initial and endpoint a forms a group under the operation induced by the product of curves. This group is called the fundamental group of X with base point a .*

Theorem 1.1.4. *For an arcwise connected space X , the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic for any $x, y \in X$. In this case denote the fundamental group by $\pi(X)$.*

Definition 1.1.7. An arcwise connected topological space X is called simply connected if $\pi(X) = 0$.

1.2 Riemannian manifolds

This section contains excerpts from [6] Chapters 2, 5, and 10 which cover some results from the theory of manifolds, differential forms, and surface theory in $\mathbb{R}^3$.

Definition 1.2.1. Let X be a set, and $p \geq 1$ an integer. A d -dimensional atlas of class C^p on X is a set of pairs $(U_i, \phi_i)_{i \in I}$ satisfying the following axioms:

1. Each U_i is a subset of X and the U_i cover X .
2. Each ϕ_i is a bijection from U_i into a subset of $\phi_i(U_i)$ of $\mathbb{R}^n$,
and $\phi_i(U_i \cap U_j) \subset \mathbb{R}^n$ is open for every i and j .
3. For every pair (i, j) the map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ is a diffeomorphism.

Definition 1.2.2. The pairs (U_i, ϕ_i) are called charts of the atlas (U_i, ϕ_i) . A chart at or about $x \in X$ is one whose domain contains x , and a chart centered at x is one mapping x to the origin in $\mathbb{R}^d$. The local coordinates associated with the chart (U_i, ϕ_i) are the functions $\phi_{i,k} : U_i \rightarrow \mathbb{R} (1 \leq k \leq d)$ such that $\phi_i(x) = (\phi_{i,1}, \dots, \phi_{i,d}(x))$.

Definition 1.2.3. Let $(U_i, \phi_i)_{i \in I}$ be an atlas on X , let U be a subset of X and $\phi : U \rightarrow \mathbb{R}^d$ be a bijection onto an open subset of $\mathbb{R}^d$. The pair (U, ϕ) is said to be a chart compatible with the atlas $(U_i, \phi_i)_{i \in I}$ if the union $(U, \phi) \cup (U_i, \phi_i)_{i \in I}$ is an atlas. Two atlases (of same dimension and differentiability class) are compatible if their union is still an atlas.

Definition 1.2.4. Let X and Y be manifolds, of dimension d and e and class C^p and C^r , respectively. Let $p \leq \inf(q, r)$. A continuous map $f : X \rightarrow Y$ is of class

C^p , or C^p differentiable, if for every chart (U, ϕ) at $x \in X$ and every chart (V, ψ) at $f(x) \in Y$, the map $\psi \circ f \circ \phi^{-1} : \phi(U \cap f^{-1}(V)) \rightarrow \mathbb{R}^e$ is of class C^p . Denote by $C^p(X; Y)$ the set of C^p differentiable maps from X into Y .

Definition 1.2.5. A C^p differentiable structure ($p \geq 1$) on a set X is an equivalence class of d -dimensional atlases of class C^p on X . A d -dimensional manifold of class C^p is a set X endowed with a C^p differentiable structure. A chart on X is any chart belonging to any atlas in the differential structure of X .

Proposition 1.2.1. Let X be a d -dimensional manifold of class C^p and $x \in X$ a point. Consider all triples of the form (U, ϕ, u) , where (U, ϕ) is a chart at x and u is a vector in $\mathbb{R}^d$. The relation $\sim$ defined by " $(U, \phi, u) \sim (V, \psi, v)$ if and only if $(\psi \circ \phi^{-1})'(\phi(x))(u) = v$ " is an equivalence relation.

Definition 1.2.6. Let X be a manifold and $x \in X$ a point. A tangent vector to X at x is an equivalence class of the triples (U, ϕ, u) . The set of tangent vectors to X at x will be denoted $T_x X$.

Definition 1.2.7. Let X be a manifold and consider the disjoint union

$$TX = \bigcup_{x \in X} T_x X$$

of the tangent spaces to X at all of its points. We call TX the tangent bundle of X .

Definition 1.2.8. Let X be a d -dimensional manifold and Y a subset of X . We say Y is an e -dimensional submanifold of X if for every $y \in Y$ there exists a chart (U, ϕ) of X at y such that $\phi(U \cap Y) = \phi(U) \cap \mathbb{R}^e$.

Definition 1.2.9. Let X be a manifold of class C^p . A vector field on X is a C^{p-1} map $\xi : X \rightarrow TX$ such that $\xi(x) \in T_x X$ for every $x \in X$.

Notation 1.2.2. Let E be a vector space and $E^* = L(E; \mathbb{R})$ its dual. We denote by $\Lambda^r E^*$ the vector space of alternating r -linear forms on E . Let X be an n -dimensional C^q manifold, where $q \geq 2$. For $x \in X$, let

$$T_x^* X = L(T_x X; \mathbb{R}) = (T_x X)^*$$

be the dual of $T_x X$. For every $0 \leq r \leq n$, let

$$\Lambda^r T^* X = \bigcup_{x \in X} \Lambda(T_x^* X)$$

be the disjoint union of spaces $\Lambda^r(T_x^* X)$, and let $p : \Lambda^r T^* X \rightarrow X$ be the map taking $\alpha \in \Lambda(T_x^* X)$ to $x = p(\alpha)$.

Definition 1.2.10. Let X be an n -dimensional $C^{q'}$ manifold and q, r integers such that $0 \leq q \leq q' - 1$ and $0 \leq r \leq n$. A C^q differential form of degree r , or r -form, on X is a map $\omega \in C^q(X; \Lambda^r T^* X)$ such that $p \circ \omega = Id_X$. The space of r -forms of class C^q on X will be denoted by $\Omega_q^r(X)$.

Definition 1.2.11. Let X and Y be manifolds, $f : X \rightarrow Y$ a map of class $C^{q'}$ and $\omega \in \Omega_q^r(Y)$ ($q \leq q' - 1$) an r -form on Y . Define $f^* \omega : X \rightarrow \Lambda^r T^* X$ by

$$f^* \omega = (Tf)^*(\omega \circ f),$$

This map is called the pullback of ω by f .

Definition 1.2.12. Let V be an oriented, codimension-one, C^p submanifold of a Euclidean vector space E . The canonical normal vector field to V is the unique vector field $\nu \in C^{p-1}(V; E)$ defined by the following conditions:

1. $\nu(x) \in \theta_x \left((T_x V)^\perp \right)$ for all $x \in V$,

2. $\|\nu(x)\| = 1,$

3. if $\{e_1, \dots, e_d\}$ is a positively oriented basis for $T_x V$, then

$$\nu(x), \theta_x(e_1), \dots, \theta_x(e_d)$$

is a positively oriented basis for E .

If (U, g) is a positively oriented parameterization of a hypersurface V in a d -dimensional Euclidean space E , we have

$$\nu = \frac{\frac{\partial g}{\partial x_1} \wedge \dots \wedge \frac{\partial g}{\partial x_{d-1}}}{\left\| \frac{\partial g}{\partial x_1} \wedge \dots \wedge \frac{\partial g}{\partial x_{d-1}} \right\|}$$

Here $\wedge$ denotes the exterior product in $\bigwedge E$ and $\|\cdot\|$ the associated norm. In dimension three $\wedge$ is just the cross product $\times$.

Definition 1.2.13. Consider an arbitrary abstract differential manifold X . To each tangent space $T_x X$ associate the tensor product

$$\left(\bigoplus^p T_x X \right) \otimes \left(\bigoplus^q T_x^* X \right)$$

whose elements are the tensors of type (p, q) at x . The union of these vector spaces for all x forms a vector bundle over X , denoted by $T^{(p,q)}$. A tensor of type (p, q) on X is a C^∞ section of this bundle. A q -form is a skew-symmetric tensor of type $(0, q)$. A *Riemannian structure* on X is a tensor of type $(0, 2)$ that is symmetric and everywhere positive definite. A vector field on X is a tensor of type $(1, 0)$. All tensors of type $(0, q)$, including q -forms and Riemannian structures, are contravariant. That is, given a differentiable map $f : X \rightarrow Y$ and a tensor ω of type $(0, q)$ on Y , we have a pullback $f^* \omega$ on X . On the other hand, if $p > 0$, a

tensor of type (p, q) cannot be pulled back any more than a vector field can (except under diffeomorphisms).

An immersed submanifold of $\mathbb{R}^n$ inherits a natural Riemannian structure, called its *first fundamental form*.

Definition 1.2.14. This notation applies to local, global, and immersed surfaces. For simplicity, let's consider the case $V \subset \mathbb{R}^3$. We also assume that V is oriented and we denote by ν the canonical normal to V . Thus we have a differentiable map $\nu : V \rightarrow S^2$, called the Gauss map. Since by construction, $T_v V = T_{\nu(v)} S^2$, the tangent map $T_v \nu$ is a linear map from $T_v V$ into itself, called the *Weingarten endomorphism*. This map is self-adjoint with respect to the first fundamental form g , and thus comes from a symmetric bilinear form on $T_v V$, called the *second fundamental form*, and denoted by

$$\mathbf{II}_v(x, y) = \left(T_v \nu(x) | y \right).$$

Assume that V is locally the image of an immersion $(u, v) \mapsto f(u, v)$, defined on an open set $U \subset \mathbb{R}^2$. The pullback f^*g is a Riemannian structure on U , which can be written in coordinates; we say that g has been expressed in coordinates (u, v) . The standard notation is

$$f^*(ds^2) = Edu^2 + 2Fdudv + Gdv^2,$$

where $du^2, dudv$ and dv^2 are tensor products of the one-forms du and dv . The functions E, F and G are given by

$$E = \left\| \frac{\partial f}{\partial u} \right\|^2, \quad F = \left(\frac{\partial f}{\partial u} \mid \frac{\partial f}{\partial v} \right), \quad G = \left\| \frac{\partial f}{\partial v} \right\|^2.$$

These three functions completely determine the Riemannian structure, but they are not determined by it.

1.3 Laplace operator of the Riemannian metric

Let g_{ij} be the Riemannian metric tensor on M . This means that, in any coordinate chart $(x^1, x^2, \dots, x^d)$ on M , the length element can be computed by

$$ds^2 = g_{ij} dx^i dx^j,$$

assuming the summation on the repeated indices. Denote by g^{ij} the elements of the inverse matrix $\|g_{ij}\|^{-1}$ and let $g = \det \|g_{ij}\|$. Then the Laplace operator Δ associated with the metric g_{ij} is defined by

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right).$$

Sometimes it is useful to represent the Laplacian in the form

$$\Delta = \operatorname{div} \nabla$$

where the gradient ∇ acts on a function f by

$$(\nabla f)^i = g^{ij} \frac{\partial f}{\partial x^j}$$

and the divergence div acts on a vector field $F = F^i \frac{\partial}{\partial x^i}$ by

$$\operatorname{div} F = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} F^i).$$

Definition 1.3.1. A uniform topological space X is said to be precompact if its Hausdorff completion is compact. A subset A of a uniform space X is said to be a precompact subset if the uniform subspace A of X is precompact.

Green's formula states that for any precompact region U , and any functions $u, v \in C_0^2(U)$,

$$\int_U v \Delta u \, d\mu = - \int_U \nabla v \nabla u \, d\mu,$$

where $\nabla v \nabla u$ is the inner Riemannian product of the vectors

$$\nabla v \nabla u = g_{ij} (\nabla u)^i (\nabla v)^j = g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j},$$

and $d\mu$ is the Riemannian volume element, defined by

$$d\mu = \sqrt{g} \, dx^1 dx^2 \dots dx^d.$$

If the boundary ∂U is C^1 and $u, v \in C^2(U) \cap C^1(\bar{U})$, then

$$\int_U v \Delta u \, d\mu = \int_{\partial U} v \frac{\partial u}{\partial \nu} \, d\mu' - \int_U \nabla v \nabla u \, d\mu, \quad (1.3.1)$$

where $d\mu'$ in the middle integral is the Riemannian volume element on the submanifold U , and ν is the outward unit normal vector field on ∂U , see [22].

1.4 Riemann surfaces

This section covers basic definitions from the theory of Riemann surfaces, see [5], [18], and [26].

Definition 1.4.1. Let X be a two-dimensional manifold (surface). A complex chart on X is a homomorphism $\varphi : U \rightarrow V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_i : U_i \rightarrow V_i, i = 1, 2$ are said to be homomorphically compatible if the map, called a transition map,

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

is biholomorphic.

Definition 1.4.2. A complex atlas on X is a system

$$\mathfrak{A} = \varphi_i : U_i \rightarrow V_i \mid i \in I$$

of charts which are holomorphically compatible and which cover X i.e.,

$$\bigcup_{i \in I} U_i = X.$$

A complex atlas is called *conformal* the transitions maps are holomorphic. Two complex atlases $\mathfrak{A}$ and $\mathfrak{A}'$ on X are called *analytically equivalent* if every chart of $\mathfrak{A}$ is homomorphically compatible with every chart of $\mathfrak{A}'$.

Definition 1.4.3. A complex structure on a two-dimensional manifold X is an equivalence class of analytically equivalent atlases on X .

Definition 1.4.4. A Riemann surface is a pair (X, Σ) , where X is a connected two-dimensional manifold, and Σ is a complex structure on X .

Example 1.4.1. *The simplest example of an open Riemann surface is the complex plane $\mathbb{C}$. The single coordinate chart $(\mathbb{C}, id)$ defines the Riemann structure on $\mathbb{C}$.*

Given any Riemann surface M , the a domain D (connected open subset) on M is also a Riemann surface. The coordinate charts on D are obtained by restricting coordinate charts of M to D . Thus every domain in $\mathbb{C}$ is a Riemann surface.

Remark 1.4.1. Coordinate charts are also called *local coordinates*, and *uniformizing variables*. Furthermore, the local coordinate $\{U, z\}$ can be identified with the mapping z (when its domain is clear or not material). U , and $f(U)$ can always be chosen to be simply connected, and a bounded domain in $\mathbb{C}$ respectively. In this case U will be called a *parametric disc*.

1.5 Metrics on Riemann surfaces

A conformal Riemann metric on a Riemann surface R is given in local coordinates by

$$ds^2 = \lambda^2(z) dz d\bar{z} \quad \lambda(z) > 0,$$

assuming λ is C^∞ . If $w \mapsto z(w)$ is a transformation of local coordinates, then the metric is transformed to

$$\lambda^2(z) \frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial \bar{w}} dw d\bar{w},$$

with $w = u + iv$, $\frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$, $\frac{\partial}{\partial \bar{w}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$. The length of a rectifiable curve γ on R is given by

$$L(\gamma) = \int_{\gamma} \lambda(z) |dz|.$$

For two points z_1 and z_2 on R we set

$$s(z_1, z_2) = \inf_{\gamma} \int_{\gamma} ds,$$

where γ runs over all rectifiable curves joining z_1 and z_2 . Then s is a distance function on R . Here we assume that

$$\lim_n s(z, z_n) = \infty,$$

for any z in R and for any sequence $\{z_n\}$ of points in R which does not cluster in R . This may be expressed by saying that the distance from z to the ideal boundary of R is infinite.

Fix a point z_0 in R . Set $\Gamma(\rho) = \{z : z \in R, s(z, z_0) = \rho\}$ for each $\rho > 0$. If ρ is sufficiently small, then $\Gamma(\rho)$ consists of a single curve. As ρ increases, $\Gamma(\rho)$ generally breaks up at a certain ρ_1 into two or more curves. As ρ continues to

increase, new decompositions may take place. Let

$$\rho_1 < \cdots < \rho_n < \cdots$$

be those values of ρ for which there occurs a decomposition of the components of $\Gamma(\rho)$. Assume these values do not accumulate anywhere, and if this sequence is finite then add to the ρ 's an infinite sequence such that the sequence obtained tends to infinity.

1.6 Metrics and conformal structures

Let M be a two-dimensional differential manifold, a Riemannian metric on M is given in local co-ordinates $z = (x, y)$ by

$$ds^2 = E dx^2 + 2F dx dy + G dy^2$$

with $E > 0$, $EG - F^2 > 0$. Assume the coefficients E, F and G of the metric are C^∞ . It will be shown that every Riemannian metric determines a conformal structure on M , see [26], with respect to which the metric has conformal form $\lambda^2(z) dz d\bar{z}$. Assume the validity of this result for the moment.

Lemma 1.6.1. *By introducing complex coordinates $z = x + iy$, $\bar{z} = x - iy$, the metric can be brought into the form*

$$\sigma(z) |dz + \mu(z) d\bar{z}|^2 = \sigma(z) (dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz) \quad (1.6.1)$$

where $\sigma > 0$, and μ is complex-valued, $|\mu| < 1$.

Proof. By hypothesis, $dz = dx + idy$ and $d\bar{z} = dx - idy$. Let $\mu = \alpha + i\beta$. Then after a simple calculation and comparison of coefficients of dx^2 , $dx dy$, dy^2 the metric can

be brought into the desired form if the following equations can be solved:

$$E = 2\sigma\alpha + \sigma(1 + \alpha^2 + \beta^2),$$

$$F = 2\sigma\beta,$$

$$G = -2\sigma\alpha + \sigma(1 + \alpha^2 + \beta^2)$$

for α, β and σ . Then

$$\beta = \frac{F}{2\sigma}, \quad \alpha = \frac{E - G}{4\sigma}$$

and

$$\sigma = \frac{E + G}{4} \pm \frac{1}{2}\sqrt{EG - F^2}.$$

Since $EG - F^2 > 0$ by assumption, we can take the positive square root in the above expression for σ . Then

$$|\mu|^2 = \alpha^2 + \beta^2 \leq \frac{4F^2 + E^2 - 2EG + G^2}{(E + G)^2} < 1,$$

again since $F^2 < EG$. □

Theorem 1.6.2. *Let M be an oriented surface with C^∞ Riemannian metric. Then M admits a conformal structure, i.e., can be made into a Riemann surface. Local holomorphic co-ordinates are given by diffeomorphic solutions of the equation*

$$u_{\bar{z}} = \mu u_z \tag{1.6.2}$$

where μ is as in lemma 1.6.1. In such coordinates, the metric has the conformal form

$$\frac{\sigma(z)}{u_z u_{\bar{z}}} du d\bar{u}.$$

The equation, [24] page 21, $u_{\bar{z}} = \mu u_z$ is called the *Beltrami equation*. Moreover, for oriented 2-dimensional Riemannian manifolds (M, ds^2) and (N, ds_1^2) , an

orientation preserving diffeomorphism $f : M \rightarrow N$ is a *conformal mapping* if the pull back of ds_1^2 by f is equal to $\exp(\phi)ds^2$ on M , where ϕ is a real-valued smooth function on M . Geometrically, it means that the angle, measured by ds^2 between any smooth curves C_1 and C_2 on M equals the angle, measured by ds_1^2 , between $f(C_1)$ and $f(C_2)$ on N . (M, ds^2) and (N, ds_1^2) are *conformally equivalent* or have the *same conformal structure* if there exists a conformal mapping between them.

It is easy to see that the uniqueness of the representation expressed in lemma 1.6.1 leads to the following:

Theorem 1.6.3. *Let R and S be Riemann surfaces induced by oriented 2-dimensional Riemannian manifolds (M, ds^2) and (N, ds_1^2) , respectively. Then $f : (M, ds^2) \rightarrow (N, ds_1^2)$ is conformal if and only if $f : R \rightarrow S$ is biholomorphic.*

This theorem shows that, in two-dimensional, the concepts of complex structure and of conformal structure are equivalent.

Proof. Write the differential equation (1.6.2) in real form with $z = x + iy$ and as in lemma 1.6.1,

$$\begin{aligned} Edx^2 + 2Fdx dy + Gdy^2 &= \sigma(z)|dz + \mu(z)d\bar{z}|^2 \\ &= \sigma(z)(dz + \mu d\bar{z})(d\bar{z} + \bar{\mu} dz). \end{aligned}$$

With $g = \sqrt{EG - F^2}$, and $u = v + iw$ equation (1.6.2) becomes

$$\begin{aligned} v_x &= -\frac{F}{g}w_x + \frac{E}{g} \\ v_y &= -\frac{G}{g}w_x + \frac{F}{g}. \end{aligned} \tag{1.6.3}$$

By differentiation and equality of mixed partials, $v_{xy} = v_{yx}$

$$a_{11}w_{xx} - 2a_{12}w_{xy} + a_{22}w_{yy} + b_1w_x + b_2w_y = 0 \tag{1.6.4}$$

where

$$a_{11} = \frac{G}{g}, \quad a_{12} = \frac{F}{g}, \quad a_{22} = \frac{E}{g},$$

$$b_1 = \frac{\partial}{\partial x} a_{11} - \frac{\partial}{\partial y} a_{12}, \quad b_2 = \frac{\partial}{\partial y} a_{22} - \frac{\partial}{\partial x} a_{12}$$

In some neighborhood of an arbitrary point z_0 , look for a solution w of equation (1.6.4) whose differential at z_0 does not vanish, thereby determining v by (1.3).

Then

$$v_x w_y - v_y w_x = \frac{1}{g}(E w_y^2 - 2F w_x w_y + G w_x^2) > 0 \quad (1.6.5)$$

since the matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is positive definite. A conformal structure can be defined on Σ by declaring diffeomorphic solutions u of (1.6.2) as local co-ordinates. If u and t are solutions of (1.6.2) and u satisfies (1.6.5), then

$$t_{\bar{u}} = t_z z_{\bar{u}} + t_{\bar{z}} \bar{z}_{\bar{u}} = \frac{1}{u_z \bar{u}_{\bar{z}} - \bar{u}_z u_{\bar{z}}} (-t_z \bar{u}_z + t_{\bar{z}} u_z) = 0.$$

Thus the transition maps between charts are holomorphic as required.

Moreover, since

$$du = u_z dz + u_{\bar{z}} d\bar{z} = u_z (dz + \mu d\bar{z}),$$

$$d\bar{u} = \bar{u}_z dz + \bar{u}_{\bar{z}} d\bar{z} = \bar{u}_{\bar{z}} (\bar{\mu} dz + d\bar{z}),$$

the metric has conformal form with respect to these charts as asserted.

It remains to show that equation (1.6.4) has local solutions with non-vanishing gradient. This can be accomplished by letting z_0 be an element of Σ . By a suitable linear change in coordinates, it can be assumed that

$$a_{11}(z_0) = a_{22}(z_0) = 1, \quad a_{12}(z_0) = 0.$$

Thus at z_0 , the principal part of the operator

$$L = a_{11} \frac{\partial^2}{\partial^2 x} - 2a_{12} \frac{\partial^2}{\partial x \partial y} + a_{22} \frac{\partial^2}{\partial^2 y} + b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} \quad (1.6.6)$$

is just the Laplacian Δ . Now for any function ϕ , set

$$\psi = (\Delta - L)\phi = \psi_1 + \psi_2, \quad (1.6.7)$$

where

$$\begin{aligned} \psi_1 &= (1 - a_{11}) \frac{\partial^2 \phi}{\partial^2 x} + 2a_{12} \frac{\partial^2 \phi}{\partial x \partial y} + (1 - a_{22}) \frac{\partial^2 \phi}{\partial^2 y} \\ \psi_2 &= -b_1 \frac{\partial \phi}{\partial x} - b_2 \frac{\partial \phi}{\partial y}. \end{aligned}$$

Notation 1.6.4. Let Ω be a bounded set in $\mathbb{R}^d$. Then

$$L^2(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable}, \|u\|_L^2(\Omega) = \int_{\Omega} u^2 < \infty\}$$

is a Hilbert space with the scalar product

$$(u, v) = \int_{\Omega} uv,$$

Let Ω be as above, we consider spaces

$$\|f\|_{C^k(\Omega)} = \sum_{j=1}^k \sup_{x \in \Omega} |D^j f(x)|,$$

D^j standing for all derivatives of f of order j .

Definition 1.6.1. Let $f : \Omega \rightarrow \mathbb{R}$, $x_0 \in \Omega$ and $\alpha \in (0, 1)$. Then f is said to be Hölder continuous at x_0 with exponent α if

$$\sup_{x \in \Omega \setminus \{x_0\}} \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|^\alpha} < \infty.$$

Moreover, $C^{k,\alpha}(\Omega)$ is the space of $f \in C^k(\Omega)$ whose k -th derivatives are Hölder continuous with exponent α .

For $R > 0$, consider on $C^{3,\alpha}(B(z_0, R))$ the norm

$$\|v\|_R = \left(\|D^3 v\|_{C^\alpha} + \frac{1}{R} \|D^2 v\|_{C^\alpha} + \frac{1}{R^2} \|Dv\|_{C^\alpha} + \frac{1}{R^3} \|v\|_{L^2} \right) \quad (0 < \alpha < 1).$$

Since the coefficients $1 - a_{11}$, $2a_{12}$ and $1 - a_{22}$ are Lipschitz continuous, in fact C^∞ and vanish at z_0 ,

$$\left(\|D\psi_1\| + \frac{1}{R} \|\psi_1\| \right)_{C^\alpha(B(z_0, R))} \leq c_1 R \left(\|D^3 \phi\| + \frac{1}{R} \|D^2 \phi\| \right)_{C^\alpha(B(z_0, R))} \quad (1.6.8)$$

with constant c_1 independent of R . Also, if ϕ has compact support

$$\begin{aligned} \left(\|D\psi_2\| + \frac{1}{R} \|D\psi_2\| \right)_{C^\alpha(B(z_0, R))} &\leq c_2 \left(\|D^2 \phi\| + \frac{1}{R} \|\phi\| \right)_{C^\alpha(B(z_0, R))} \\ &\leq c_3 \left(\|D^3 \phi\| + \frac{1}{R} \|D^2 \phi\| \right)_{C^\alpha(B(z_0, R))} \end{aligned} \quad (1.6.9)$$

Now assume ϕ , hence also ψ , has compact support in $B(z_0, R)$. Thus ψ can be defined on all of $\mathbb{C}$, and vanishes outside $B(z_0, R)$. Let

$$G(z - \xi) = \frac{1}{2\pi} \log |z - \xi|$$

be the Green function on a set

$$X(z) = \int G(z - \xi) \psi(\xi) d\xi.$$

Use will be made of the following theorem.

Theorem 1.6.5. *Let $\Omega \in \mathbb{R}^d$ be open and bounded and*

$$u(x) = \int_{\Omega} G(x - y) f(y) dy$$

with

$$G(x - y) = \begin{cases} \frac{1}{2\pi} \log |x - y|, & d = 2 \\ \frac{1}{d(2-d)\omega_d} |x - y|^{2-d}, & d > 2 \end{cases}$$

Then if $f \in C_0^\alpha$ $u \in C^{2,\alpha}(\Omega)$, and

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq k \|f\|_C^\alpha(\Omega) \quad (0 < \alpha < 1).$$

The constant, k , depends only on α, d and $|\Omega|$

By this theorem and the above

$$\|D^2 X\|_{C^\alpha(\mathbb{C})} \leq k \|\psi\|_{C^\alpha(B(z_0, R))} \quad (1.6.10)$$

and similarly

$$\|D^3 X\|_{C^\alpha(\mathbb{C})} \leq k \|D\psi\|_{C^\alpha(B(z_0, R))}. \quad (1.6.11)$$

Further

$$\begin{aligned} \|DX\|_{L^2} &\leq c_4 R \|\psi\|_{L^2}, \\ \|X\|_{L^2} &\leq c_5 R^2 \log R \|\psi\|_{L^2}. \end{aligned} \quad (1.6.12)$$

Now from the theory of harmonic functions

Corollary 1.6.6. *Let $\psi \in C_0^2(\Omega)$, then*

$$\psi = \int_{\Omega} G(x-y) \Delta \psi(x) dx = \Delta_y \int_{\Omega} G(x-y) \psi(x) dx$$

for all $y \in \Omega$. The subscript y on the Laplacian operator signifies that it operates on the variable y , and Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary.

By this corollary and (1.7),

$$\Delta X = \psi = (\Delta - L)\phi, \quad (1.6.13)$$

and (1.8 - 1.12) imply

$$\|X\|_R \leq c_6 R \log R \|\phi\|_R. \quad (1.6.14)$$

Now choose a cut-off function $\eta \in C_0^2(B(z_0, R))$ with $\eta \equiv 1$ on $B(z_0, \frac{R}{2})$

$$|D\eta| \leq c_7 \frac{1}{R} |D^2\eta| \leq c_8 \frac{1}{R^2}; \quad (1.6.15)$$

set $\phi = \eta w$ for $w \in C^{3,\alpha}(B(z_0, R))$, and $X = Tw$ (X being the function constructed above for ϕ). Then by equations (1.6.14) and (1.6.15),

$$\|Tw\|_R \leq c_9 R \log R \|w\|_R. \quad (1.6.16)$$

Thus T is a bounded linear operator of $C^{3,\alpha}(B(z_0, R))$ into itself. If R is chosen so small that

$$c_9 R \log R = q < 1, \quad (1.6.17)$$

then the series $1 + T + T^2 + \dots$ converges in the norm $\|\cdot\|_R$ by comparison with $\sum q^n$. Thus

$$\sum_{n=0}^{\infty} T^n = (1 - T)^{-1}.$$

For example, take $\xi(z) = x + y$ and $w(z) = (1 - T)^{-1}\xi(z)$. Then

$$\begin{aligned} 0 &= \Delta(\xi) = \Delta(w - Tw) = \Delta w - (\Delta - L)\eta\xi \\ &= \Delta((1 - \xi)w) + L(\eta\xi) \end{aligned}$$

and since $\eta \equiv 1$ on $B(z_0, \frac{R}{2})$, $Lw = 0$. Also,

$$\|w - \xi\|_r = \|((1 - T)^{-1} - 1)\xi\| \leq \frac{q}{1 - q} \|\xi\|_R \quad (1.6.18)$$

By choosing R small enough, the right hand side becomes small. Then since the $C^{3,\alpha}$ -norm controls the C^1 -norm,

$$|D(z_0) - D\xi(z_0)|$$

will also be arbitrarily small. In particular, $Dw(\xi_0) \neq 0$. Thus finishing the proof for the existence of a solution of differential equation (1.6.4) in a neighborhood of z_0 with non-vanishing differential at z_0 , and hence of theorem 1.6.2. $\square$

1.7 Uniformization theorem

The uniformization theorem will be stated without proof. For a detailed discussion on this deep result, see [1] and [18].

Theorem 1.7.1. *Every simply connected Riemann surface is biholomorphically equivalent to either $\mathbb{C}$, $\mathbb{C} \cup \{\infty\}$ or the unit disc.*

The background covered up to this point clarifies the first result in Milnor's paper [38].

Theorem 1.7.2. *Let M be a simply connected open surface, differentiable of class C^∞ , with C^∞ Riemannian metric. Then M is conformally diffeomorphic either to the complex plane or the open unit disc.*

Proof. The conformal structure and the Riemannian structure are equivalent. That is, M can be given a conformal structure making M into a Riemann surface. Since the metric induces a topology on M and simple connectivity is invariant under diffeomorphisms, M as Riemann surface, is simply connected and open. Thus by the uniformization theorem, M with its conformal structure is biholomorphically equivalent to either $\mathbb{C}$ or the unit disc. M cannot be biholomorphically equivalent to the Riemann sphere since the sphere is compact. $\square$

Chapter 2

Spherically Symmetric Manifolds

The problem of deciding whether a manifold M is parabolic, conformally diffeomorphic to the complex plane, or M is hyperbolic, conformally diffeomorphic to the open unit disc, under the special circumstance that M is complete and rotationally symmetric about a point p will now be investigated.

2.1 Local surface theory

The account of surface theory, as discussed in the sequel, was taken from Berger [6], and Struik [45].

Definition 2.1.1. Let V be a surface in $\mathbb{R}^3$, considered as an abstract Riemannian manifold. Let $([a, b], f)$ be a curve on V . The length of which is given by

$$\text{length}(f) = \int_a^b \|f'(t)\| dt.$$

Note that this number depends only on the first fundamental form. The distance between two points $v, w \in V$ is defined as the infimum of the length of curves with

endpoints v and w . That is,

$$d(u, w) = \inf\{\text{leng}(f) : f \text{ is a curve in } V \text{ with } f(a) = v, f(b) = w\}.$$

This defines an intrinsic metric of V , called the metric of a local surface.

Definition 2.1.2. Consider the problem of finding a curve $C = ([a, b], f)$ on V that has the shortest length among all curves with the same endpoints. The calculus of variations shows a necessary condition, which is also sufficient if the length has zero derivative, namely $f''(t) \in (T_{f(t)})^\perp$ for all t in the interval $[a, b]$. This implies $\|f'(t)\|$ is constant. Without loss of generality this constant is equal to one. Such curves are called *geodesics*.

Definition 2.1.3. Let $C = ([a, b], f)$ be a curve on V . A vector field Z along C , such the $Z(t)$ is everywhere tangent to V , or parallel to V , if its derivative is everywhere perpendicular to V . That is, $Z'(t) \in (T_{f(t)})^\perp$ for all $t \in [a, b]$.

Theorem 2.1.1. *Given a curve C on V and a vector $z \in T_{f(a)}V$, there exists a unique parallel vector field Z along C satisfying $Z(a) = z$. In addition, Z only depends on the first fundamental form.*

The vectors $Z(t)$ are obtained from z by *parallel transport*. Geodesics are then just self-parallel curves. In other words their velocity vector field is parallel.

2.1.1 Covariant derivative and geodesic curvature

Definition 2.1.4. Let Z be a vector field along C . The covariant derivative of Z along C is the orthogonal projection of $Z'(t)$ on $T_{f(t)}(V)$ and is denoted by $D_{f'}Z$. The geodesic curvature of a curve in V is defined as $\|D_{f'}f'\|$.

The covariant derivative depends only on the first fundamental form. The reason is that, given any Riemannian manifold (X, g) , associate to each pair of vector fields (ξ, η) on X a new vector field, called the *covariant derivative* of η with respect to ξ and is denoted by $D_\xi \eta$. It is characterized by the following conditions, D_ξ preserves the scalar product g for every ξ , and

$$D_\xi \eta - D_\eta \xi = [\xi, \eta]$$

for every ξ and η . The brackets are defined as follows:

Definition 2.1.5. If X is a C^∞ manifold, denote by $C^\infty(X)$ the $\mathbb{R}$ -algebra of C^∞ real-valued functions on X , and by $V(X)$ the $\mathbb{R}$ -vector space of C^∞ vector fields on X , that is, C^∞ maps $\xi : X \rightarrow TX$ such that $p \circ \xi = Id_X$. A derivation on $C^\infty X$ is an $\mathbb{R}$ -linear map $D : C^\infty(X) \rightarrow C^\infty(X)$ such that $D(fg) = D(f)g + fD(g)$ for any $f, g \in C^\infty(X)$. The set of derivations on $C^\infty(X)$ is denoted by $der(C^\infty(X))$.

Take $\xi \in V(X)$, $f \in C^\infty(X)$, and define $df(\xi) = \xi(f) \in C^\infty(X)$ by

$$(df(\xi))(x) = df(x)(\xi(x))$$

for every $x \in X$. Now if $\xi, \eta \in V(X)$ then it can be shown that the map

$$f \mapsto \xi(\eta(f)) - \eta(\xi(f))$$

is a derivation on $C^\infty(X)$. Denote by $[\xi, \eta]$ the vector field associated to this map by the above and call it the bracket of ξ and η

In order to write the covariant derivative in coordinates, one introduces the *Christoffel symbols*. These are numbers which can be calculated using just the first order derivatives of E, F , and G . In particular, let $\mathbf{x} = \mathbf{x}(u, v)$ be a parametric

representation of a surface in $\mathbb{R}^3$. Define a *moving trihedron* at a point P of the surface as the three linearly independent vectors $\mathbf{x}_u, \mathbf{x}_v$, and $\mathbf{N}$ of which $\mathbf{x}_u$ and $\mathbf{x}_v$ lie in the tangent plane normal to $\mathbf{N}$. Every vector can be expressed as a linear combination of the moving trihedron vectors. Doing this for $\mathbf{x}_{uu}, \mathbf{x}_{uv}, \mathbf{x}_{vv}$ gives

$$\mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{11}^2 \mathbf{x}_v + e \mathbf{N}$$

$$\mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + f \mathbf{N}$$

$$\mathbf{x}_{vv} = \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + g \mathbf{N},$$

where the $\Gamma_{jk}^i (i, j, k = 1, 2)$ are defined as

$$\begin{aligned} \Gamma_{11}^1 &= \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, & \Gamma_{11}^2 &= \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)} \\ \Gamma_{12}^1 &= \frac{GE_v - FG_u}{2(EG - F^2)}, & \Gamma_{12}^2 &= \frac{EG_u - FE_v}{2(EG - F^2)} \\ \Gamma_{22}^1 &= \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, & \Gamma_{22}^2 &= \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)} \\ \Gamma_{jk}^i &= \Gamma_{kj}^i. \end{aligned}$$

and

$$e = -\mathbf{x}_u \cdot \mathbf{N}_u, \quad 2f = -(\mathbf{x}_u \cdot \mathbf{N}_v + \mathbf{x}_v \cdot \mathbf{N}_u), \quad g = -\mathbf{x}_v \cdot \mathbf{N}_v$$

If the coordinates are assumed to be orthogonal, $F = 0$), the formula for the geodesic curvature of a curve $t \mapsto (u(t), v(t))$ on a surface is given by

$$\frac{1}{\sqrt{Eu'^2 + Gv'^2}} \left(\frac{d\phi}{dt} + \frac{1}{2\sqrt{EG}} \left(\frac{\partial G}{\partial u} v' - \frac{\partial E}{\partial v} u' \right) \right),$$

where ϕ is defined by $\tan \phi = \sqrt{\frac{G}{E}} \frac{dv}{du}$.

If the surface is in geodesic polar coordinates, that is,

$$ds^2 = du^2 + J^2(u, v)dv^2,$$

and v is given as a function of u ,

$$\frac{d\phi}{dt} = \frac{1}{1 + J^2 v'^2} \frac{d}{du} (J(u, v(u))v'),$$

and the geodesic curvature simplifies to

$$\frac{1}{(1 + J^2 v'^2)^{3/2}} \left(\frac{\partial J}{\partial u} (2v' + J^2 v'^3) + \frac{\partial J}{\partial v} v'^2 + Jv'' \right).$$

The equations above show that $D_{f'} f' = 0$, the equation characterizing a geodesic, is a second-order quadratic differential equation.

Theorem 2.1.2. *Given $v \in V$ there exists $\epsilon > 0$ such that, for every unit vector $\xi \in T_v V$, there exists a geodesic of V with initial speed ξ and defined at least on the interval $(-\epsilon, \epsilon)$. As ξ runs over the set of unit vectors in $T_v V$, the associated geodesic γ_ξ make up exactly the open ball*

$$B(v, \epsilon) = \{x \in V : d(x, v) < \epsilon\}.$$

Furthermore, for $\epsilon' < \epsilon$ the restriction of γ_ξ to $[0, \epsilon']$ is the shortest path on V joining its endpoints. This shows that the topology of M as a surface coincides with its topology as a metric space.

Definition 2.1.6. If ξ is a unit tangent vector and γ_ξ its geodesic, then by theorem 2.1.2 $d(\gamma(0), \gamma(t)) = t$ for t small enough. This cannot be true if t gets too large: consider the diameter of a compact manifold. It make sense to look at the (possibly infinite) infimum of the values of t for which $d(\gamma(0), \gamma(t)) < t$, this is called the *cut value* of ξ , and is denoted $cutval(\xi)$.

Theorem 2.1.3 (6). *If $t = cutval(\xi)$, at least one of the two conditions must hold:*

1. *there exists another segment δ from $\gamma(0)$ to $\gamma(t)$, distinct from $\gamma|_{[0,t]}$; or*

2. the exponential map $\exp_{\gamma(0)} : T_{\gamma(0)}M \rightarrow X$ has rank < 2 at $t\xi$.

In addition, if $\exp_{\gamma(0)}$ has rank < 2 at η , we have $\text{cutval}(\eta/\|\eta\|) \leq \|\eta\|$.

Definition 2.1.7. When $\exp_{\gamma(0)}$ has rank < 2 as in the theorem, we say that the points $m = \gamma(0)$ and $n = \gamma(t)$ are conjugate along the geodesic γ_ξ . The cut locus $\text{cutloc}(m)$ of $m \in (M, g)$ is the set of points of the form $\exp_m(\text{cutval}(\xi)\xi)$.

Definition 2.1.8. The injectivity radius of (M, g) at m is defined by

$$\text{inj}(m) = \text{inj}(M, m) = \inf\{\text{cutval}(\xi) : \xi \in U_m M\},$$

where $U_m M$ is the unit tangent bundle at m . This number is always positive because the cut value is a continuous function on UM . The injectivity radius of (M, g) is then

$$\text{inj}(M, g) = \text{inj}(M) = \inf\{\text{inj}(m) : m \in M\};$$

this number can be zero in general but is positive if M is compact.

One reason the injectivity radius is important is that every disc $B(m, r)$ with $r \leq \text{inj}(m)$ is diffeomorphic to $\mathbb{R}^2$. Similarly, for every $m \in M$, the complement $M \setminus \text{cutloc}(m)$ is diffeomorphic to $\mathbb{R}^2$.

For a generalization of these notions of a cut locus and injectivity radius to Riemannian manifolds of higher dimensions see [7].

2.1.2 Polar coordinates

Fix a point $o \in M$ and denote by $\text{cutloc}(o)$ the cut locus of o . Away from $\text{cutloc}^*(o) = \text{cutloc}(o) \cup \{o\}$, define the polar coordinates with the pole o as follows: For any point $x \in M - \text{cutloc}^*(o)$ there corresponds a polar radius $\rho = \text{dist}(x, o)$

and a polar "angle" $\theta \in \mathbb{S}^{d-1}$, ($\mathbb{S}^d$ is the d -dimensional unit sphere in $\mathbb{R}^{d+1}$) such that the shortest geodesics from o to x start at o to the direction θ in $T_o M$. Identify $T_o M$ with $\mathbb{R}^d$ so that θ can be regarded as a point on $\mathbb{S}^{d-1}$. In particular, $M - \text{cutloc}^*(o)$ is diffeomorphic to a star-like region on $\mathbb{R}_+ \times \mathbb{S}^{d-1}$, see [31] and [19] for proofs of the facts mentioned.

The Riemannian metric in $M - \text{cutloc}^*(o)$ has in the polar coordinates the form

$$ds^2 = d\rho^2 + A_{ij}(\rho, \theta) d\theta^i d\theta^j, \quad (2.1.1)$$

where $(\theta^1, \theta^2, \dots, \theta^{d-1})$ are coordinates on $\mathbb{S}^{d-1}$ and $\|A_{ij}(\rho, \theta)\|$ is a positive definite matrix. In fact, $A_{ij}(\rho, \cdot)$ is the Riemannian metric tensor on the geodesic sphere $S_\rho = \partial B(o, \rho) - \text{cutloc}(o)$. Denote $A = \det \|A_{ij}\|$. Then the area element on S_ρ is given by

$$d\mu'|_{S_\rho} = \sqrt{A} d\theta^1 d\theta^2 \dots d\theta^{d-1}.$$

In particular,

$$\mu'(S_\rho) = \int_{\mathbb{S}^{d-1}} \sqrt{A} d\theta^1 d\theta^2 \dots d\theta^{d-1},$$

where $\theta^1, \theta^2, \dots, \theta^{d-1}$ are defined almost everywhere on $\mathbb{S}^{d-1}$. It follows that the Laplace operator has in the polar coordinates the form

$$\Delta = \frac{1}{\sqrt{A}} \frac{\partial}{\partial \rho} \left(\sqrt{A} \frac{\partial}{\partial \rho} \right) + \Delta_{S_\rho} = \frac{\partial^2}{\partial \rho^2} + (\log \sqrt{A})' \frac{\partial}{\partial \rho} + \Delta_{S_\rho} \quad (2.1.2)$$

where $(\cdot)'$ denotes $\frac{\partial}{\partial \rho}$ and Δ_{S_ρ} is the Laplace operator on the submanifold S_ρ .

M is said to be a *manifold with a pole* if M possesses a point o with an empty cut locus $\text{cutloc}(o)$. The point o is called the pole of M , and the polar coordinates are defined on $M - \{o\}$. If, in addition, M is geodesically complete, then M is diffeomorphic to $\mathbb{R}^d$.

2.1.3 Gaussian curvature

A local surface is written in *normal coordinates* if all geodesics going through the origin of the coordinate plane are straight lines in that plane, parameterized by arc length. In normal coordinates the expansion of ds^2 at the origin is

$$ds^2 = dx^2 + dy^2 + K(xdy + ydx)^2 + o(x^2 + y^2)$$

where K is a real number, depending only on the metric and on the point taken as the origin. If ds^2 is expressed in geodesic polar coordinates, comparing with

$$ds^2 = \dot{u}r^2 + J^2(r, \theta)d\theta^2$$

and taking the limit we get

$$K = -\frac{1}{J(0,0)} \frac{\partial^2 J}{\partial r^2}(0,0).$$

In fact, see [6], the formula for the Gaussian curvature in some sense measures how far the surface is from being Euclidean. The inhabitants of a surface V can calculate K at a point p by drawing circles of small radius ϵ around p and measuring their circumference. The radius and length of a circle are related by the formula

$$\text{length}(C(p, \epsilon)) = 2\pi\epsilon \left(1 - \frac{K(p)}{6}\epsilon^3 + o(\epsilon^3) \right).$$

A similar formula relating the area of the disc $B(p, \epsilon)$ and its radius

$$\text{Area}(B(p, \epsilon)) = \pi\epsilon^2 \left(1 - \frac{K(p)}{12}\epsilon^4 + o(\epsilon^4) \right)$$

exists. These results follow from the relations of the circumference C of a curve $u = u_0 = \text{const.}$ and the area A it encloses for a system of geodesic polar coordinates:

$$\begin{aligned} C &= \int_0^{2\pi} J dv, & J &= J(u_0, v), \\ A &= \int_0^u \int_0^{2\pi} J du dv, & J &= J(u, v). \end{aligned}$$

We call K the Gaussian curvature of the surface of the point in question. Gauss in his "Theorema egregium", expressed K explicitly as a function of E, G and F . His formula reads

$$4K(EG - F^2)^2 = \begin{vmatrix} 4F_{uv} - 2E_{vv} - 2G_{uu} & E_u & 2F_u - E_v \\ 2F_v - G_u & E & F \\ G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & E_v & G_u \\ E_v & E & F \\ G_u & F & G \end{vmatrix}$$

Thus if the surface is in the form $ds^2 = du^2 + J^2(u, v)dv^2$,

$$K = -\frac{1}{J} \frac{\partial^2 J}{\partial u^2}, \quad (2.1.3)$$

see [45] page 139.

Thus, if M is rotationally symmetric about a point p then the Gaussian curvature is expressed as a function of the distance r from p . Let $2\pi g(r)$ denote the circumference of a circle of radius r centered at p . Introducing geodesic polar coordinates r, θ then by the above result, the function g satisfies the linear differential equation $\ddot{g} + Kg = 0$ with the initial conditions $g(0) = 0, \dot{g}(0) = 1$.

We now give an alternate calculation of the Gaussian curvature based on Milnor's suggestion using [45] pages 136-138. Consider at an arbitrary point O on the surface the geodesics starting from this point in all directions $v = \text{const.}$, as well at their orthogonal directions $u = \text{const.}$ then ds^2 takes the form

$$ds^2 = du^2 + G(u, v)dv^2,$$

in which u is the arc length along the geodesics, measured from O . We call these coordinates geodesic polar coordinates. And the Gaussian curvature becomes

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.$$

2.2 Model manifold

The forgoing results were taken from Milnor [38].

Lemma 2.2.1. *Let M be a simply connected, open, complete surface with rotational symmetry about a point p . Then M is hyperbolic if and only if the integral*

$$\int_a^\infty \frac{dr}{g(r)}$$

is finite.

Proof. Introducing a new coordinate ρ in place of r by setting $\rho = \int_a^r ds/g(s)$. Then by the fundamental theorem of calculus of a single real variable, the metric takes the form

$$dr^2 + g^2 d\theta^2 = g^2 (d\rho^2 + d\theta^2)$$

Specifically, let

$$\rho(r) = \int_a^r \frac{ds}{g(s)}$$

then

$$\frac{d\rho}{dr}(r) = \frac{1}{g(r)}$$

$$dr^2 = g^2 d\rho^2$$

$$ds^2 = g^2 (d\rho^2 + d\theta^2).$$

Now map M conformally into the complex numbers, $f : \mathbb{R}^+ \times S^1 \rightarrow \mathbb{C}$, defined by $f(\rho, \theta) = \exp(\rho + i\theta)$. f is conformal since it is defined, holomorphic with non-vanishing derivative at all points except possibly at the base point p , since $g(0) = 0$. However, f is bounded, smooth, and conformal throughout a neighborhood of p ,

except at the point p itself. So p is a removable singularity and we obtain a smooth conformal mapping which is defined and one-to-one throughout M .

Evidently, the image of this mapping is either the disk of radius

$$\exp \int_a^\infty dr/g(r)$$

or the entire complex plane according as this integral is finite or infinite. $\square$

Theorem 2.2.2. *If $K(r) \geq -1/(r^2 \log r)$ for large r , then the surface is conformally parabolic. But if $K(r) \leq -(1 + \epsilon)/(r^2 \log r)$ for large r with $\epsilon > 0$ and the function $g(r)$ is unbounded, then the surface is hyperbolic.*

Proof. Let $g(r) > 0$ be any smooth function whose associated curvature $K(r) = -\ddot{g}/g$ satisfies inequality $K(r) \geq K_0(r)$ for $r \geq a$, and $K_0(r) = -1/(r^2 \log r)$. Multiplying g by a small positive quantity if necessary, assume the initial inequalities

$$g(a) < g_0(a), \tag{2.2.1}$$

$$\dot{g}(a) < \dot{g}_0(a) \tag{2.2.2}$$

are satisfied. It follows that $g(r) < g_0(r)$ for all $r \geq a$. Otherwise, if b were the smallest number greater than a with $g(b) = g_0(b)$, then integrating the inequality

$$\ddot{g} \leq -K(r)g \leq -K_0(r)g_0 = \ddot{g}_0$$

from a to r and adding $\dot{g}(a) < \dot{g}_0(a)$ we would obtain $\dot{g}(r) < \dot{g}_0(r)$ for $a \leq r \leq b$. Integrating this new inequality from a to b and adding $g(a) < g_0(a)$ we would obtain $g(b) < g_0(b)$, which contradicts the choice of b . Thus $g(r) < g_0(r)$ for all $r \geq a$. Hence the integral $\int_a^\infty dr/g(r)$ is also divergent and the surface is parabolic.

For a proof in the other direction, we use the comparison function of the form

$$g_\epsilon(r) = r(\log r)^{1+\epsilon} \quad r \geq 2.$$

This yields a hyperbolic surface since the integral

$$\int_2^\infty dr/r(\log r)^{1+\epsilon} = [-1/\epsilon(\log r)^\epsilon]_2^\infty$$

is finite. Computation shows that the corresponding Gauss curvature is given by

$$K_\epsilon(r) = -(1+\epsilon)(1+\epsilon/\log r)/(r^2 \log r) \geq -(1+2\epsilon)/(r^2 \log r)$$

for large r .

Now consider a smooth function $g(r) > 0$ whose associated Gaussian curvature $K = -\ddot{g}/g$ satisfies

$$K(r) \leq -(1+2\epsilon)/(r^2 \log r) \leq K_\epsilon(r)$$

for $r \leq a$. If $\dot{g}(a) > 0$ then, after multiplying the function $g(r)$ by a large constant if necessary, a completely analogous argument shows that $g(r) > g_\epsilon(r)$ for all $r \geq a$. Hence the integral $\int_a^\infty dr/g(r)$ is convergent, and the surface is hyperbolic by the lemma.

The above argument will work whenever $\dot{g}(r) > 0$ for some sufficiently large r . In particular, it will certainly work whenever the function $g(r)$ is unbounded. On the other hand, If $\dot{g}(r) \leq 0$ for all large r , then the function $g(r)$ is bounded, and hence by the lemma is parabolic. $\square$

Chapter 3

Type-problem for Riemann Surfaces

According to Milnor, Robert Osserman suggested that by applying a method of Ahlfors [5], it can be concluded that

$$\int_{\alpha}^{\infty} \frac{dr}{G(r)} = \infty \Rightarrow \text{conformally parabolic.}$$

Results from Sario and Nakai [42] will be used to prove this fact.

3.1 Royden's compactification

Given a Riemann surface R , consider the *Royden algebra* $\mathfrak{M}(R)$, i.e. the normed ring which is the completion of $\{f \in C^{\infty}(R), \|f\| < \infty\}$ under the norm $\|f\| = \sup_R |f| + (D_r(f))^{1/2}$. Let $\mathfrak{M}_0(R)$ be the class of $\phi \in \mathfrak{M}(R)$ with compact supports on R , and $\mathfrak{R}_{\Delta}(R)$ the subalgebra of $\mathfrak{M}(R)$ consisting of $f \in \mathfrak{M}(R)$ for which there exists a sequence $\{\phi_n\} \subset \mathfrak{M}_0(R)$ with the following properties: $\sup_n(\sup_R |\phi_n|) < \infty$, $D_r(\phi_n - f) \rightarrow 0$, and $\phi_n \rightarrow f$ uniformly on each compact subset of R . Let R^* be

the *Royden compactification* of R , i.e. the maximal ideal space of $\mathfrak{M}(R)$; it is seen to contain R as its open and dense subspace. The part $\partial_\Delta R = \{p \in R^*; M_\Delta(R)(p) = 0\}$ of the Royden boundary $\partial R = \{p \in R^*; M_0(R)(p) = 0\} = R^* - R$ is called the Royden *harmonic boundary* of R . Royden showed that $R \in O_G$ if and only if $\partial_\Delta R = \emptyset$. If T is a quasiconformal mapping, of R onto another Riemann surface R' , then Royden, in essence, proved that T extends to a homomorphism T^* of R^* onto R'^* with $T^*(\partial_\Delta R) = \partial_\Delta R'$. In particular, $\partial_\Delta R = \emptyset$ if and only if $\partial_\Delta R' = \emptyset$, or what amounts to the same, $R \in O_G$ if and only if $R' \in O_G$. This generalizes the Katutani theorem of quasiconformal invariance of types. It was only recently proved that the converse of the above is true: If T^* is a homeomorphism of R^* onto R'^* , then $T = T^*|_R$, deformed in a compact subset of R , is automatically a quasiconformal mapping of R onto R' . We can thus say that R^* and R'^* are homeomorphic if and only if R and R' are quasiconformally equivalent. In other words, the Royden compactification (as a topological structure) characterizes the quasiconformal structure of a Riemann surface. Thus the quasiconformal invariance of types unveiled the phenomenon that parabolicity is determined by quasiconformal structures on surfaces.

Following the development in Sario and Nakai [42], let U (resp. $\bar{U}$) be open (resp. closed) unit disk and $\mathfrak{D}^0(\bar{U})$ the space of all (not necessarily finitely) continuous functions on $\bar{U}$ with finite Dirichlet integrals over U . Consider the problem of minimizing $D(f)$ for $f \in \mathfrak{D}^0(\bar{U})$ such that $f|_{\partial U} = \phi$, a fixed function. The *Dirichlet principle* states that there exists a unique minimizing function $u \in \mathfrak{D}^0(\bar{U})$ with $u|_{\partial U} = \phi$ and u in the class $H\mathfrak{D}^0(\bar{U})$ of those functions in $\mathfrak{D}^0(\bar{U})$ which are harmonic on U , as defined in Jost [26]. This can be restated following Royden in

[41]:

$$\mathfrak{D}^0(\bar{U}) = H\mathfrak{D}^0(\bar{U}) + \mathfrak{D}_1^0(\bar{U})$$

where the sum stands for the orthogonal decomposition with respect to the inner product $D(\phi, \psi)$, and $\mathfrak{D}_1^0(\bar{U})$ is the space of functions in $\mathfrak{D}^0(\bar{U})$ which vanish on ∂U . Hence the space $H\mathfrak{D}^0(\bar{U})$ is isomorphic to the space $\mathfrak{D}^0(\bar{U})|_{\partial U}$.

The global study of the class $H\mathfrak{D}^0(\bar{U})$ is thus equivalent to that of $\mathfrak{D}^0(\bar{U})|_{\partial U}$. Since $H\mathfrak{D}^0(\bar{U})$ is dense in the $HD(U)$ with respect to the Dirichlet norm and $\mathfrak{D}^0(\bar{U})|_{\partial U}$ is dense in the space $\tilde{C}(\partial U)$ of integrable continuous functions on ∂U with respect to the L^1 norm, information about $HD(U)$ should be obtainable by studying $\tilde{C}(\partial U)$. The advantage of this idea is the fact that all analytic properties in $HD(U)$ are eliminated in $\tilde{C}(\partial U)$, and the treatment of $\tilde{C}(\partial U)$ is easier than that of $HD(U)$.

However an application of this approach to an arbitrary surface R is a priori confronted with the following difficulty: R has in general no natural boundary such as ∂U in the case U . We thus need some abstract compactification of R . As a substitute for $\mathfrak{D}^0(\bar{U})$, we take the class $\tilde{\mathfrak{M}}(R)$ of continuous functions on R with finite Dirichlet integrals; we wish to make each function in $\tilde{\mathfrak{M}}(R)$ continuous at the ideal boundary of R . We reverse the process and take the orthogonal complement $\tilde{\mathfrak{M}}_{\Delta}(R)$ of $HD(R)$ in $\tilde{\mathfrak{M}}(R)$:

$$\tilde{\mathfrak{M}}(R) = HD(R) + \tilde{\mathfrak{M}}_{\Delta}(R).$$

To avoid the ambiguity of additive constants we require that each $f \in \tilde{\mathfrak{M}}_{\Delta}(R)$ be approximated by functions with compact supports with respect to the Dirichlet norm but also uniform convergence on compact sets.

This choice of $\tilde{\mathfrak{M}}_{\Delta}(R)$ will have the effect on $\tilde{\mathfrak{M}}(R)$ that there exists a smallest compact Hausdorff space R^* , called Royden's compactification, which contains R as an open dense subspace such that $\tilde{\mathfrak{M}}(R) \subset C(R^*)$. The boundary of R is $\Gamma = R^* - R$ and we take the set $\Delta \subset \Gamma$ of points in Γ on which every function in $\tilde{\mathfrak{M}}_{\Delta}$ vanishes. Then Δ corresponds to ∂U and $\Gamma - \Delta$ corresponds to the set of "irregular" points; this set is empty in case $R = U$. The space $HD(R)$ is isomorphic to the "dense" subspace $\tilde{\mathfrak{M}}(R)|_{\Delta}$ of the space $\tilde{C}(\Delta)$ of "integrable" continuous functions on Δ . The study of $HD(R)$ is thus reduced to that of $\tilde{C}(\Delta)$.

At this point there will be an investigation of the totality $\mathfrak{M}(R)$ of bounded members of $\tilde{\mathfrak{M}}(R)$, called *Royden's algebra* associated with R , and a discussion of the topology of R^* . Not only O_{HD} , the class of harmonic functions with finite Dirichlet integral, but the class O_G of parabolic surfaces can be considered from this viewpoint.

Definition 3.1.1 (Tonelli Functions). A real-valued function $f(x, y)$ defined on $(a, b) \times (c, d)$ is called a *Tonelli function* on a plane region R if it satisfies the following conditions for every $(a, b) \times (c, d) \subset R$.

1. f is continuous on $(a, b) \times (c, d)$,
2. $x \mapsto f(x, y)$ is absolutely continuous on (a, b) for almost every fixed $y \in (c, d)$, and the same holds if x and y are interchanged.
3. $(\frac{\partial f}{\partial x})(x, y)$ and $(\frac{\partial f}{\partial y})(x, y)$ are square integrable on any compact subset of $(a, b) \times (c, d)$.

A complex-valued function f on a plane region R is a Tonelli function by definition if $Re f$ and $Im f$ are Tonelli functions on R .

Definition 3.1.2. A function f on a Riemann surface R is a Tonelli function if f is one on every parametric disc of R .

Theorem 3.1.1. If f and g are Tonelli functions on a Riemann surface R then so are $f \cup g = \max(f, g)$ and $f \cap g = \min(f, g)$.

Definition 3.1.3 (Royden's Algebra). The real (or complex) Royden algebra $\mathfrak{M}(R)$ associated with a Riemann surface R is the totality of real (or complex) functions f on R satisfying the following conditions:

1. f is bounded on R ,
2. f is a Tonelli function on R ,
3. the Dirichlet integral

$$D_R(f) = \int_R \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] dx dy$$

of f on R is finite.

The Royden's algebra $\mathfrak{M}(R)$ forms an algebra over the real (or complex) number field with the usual addition and multiplication of functions and scalar multiplication defined pointwise. We remark for division that, if $f \in \mathfrak{M}(R)$ then the function $1/f \in \mathfrak{M}(R)$ if and only if $\inf_R |f| > 0$.

Theorem 3.1.2. Functions $f \cup g$ and $f \cap g$ belong to $\mathfrak{M}(R)$ along with f and g .

Let f_n be a sequence of functions on R and f a function on R and consider topologies on $\mathfrak{M}(R)$.

1. f_n converges to f on R in C -topology, $f = C\text{-}\lim_n f_n$, if $\lim_n \sup_R |f_n - f| = 0$ for each compact subset K of R ;

2. f_n converges to f on R in B -topology, $f = B\text{-}\lim_n f_n$, if f_n is uniformly bounded on R and $f = C\text{-}\lim_n f_n$ on R ;
3. f_n converges to f on R in D -topology, $f = D\text{-}\lim_n f_n$, if $\lim_n D_R(f_n - f) = 0$;
4. f_n converges to f on R in BD -topology, $f = BD\text{-}\lim_n f_n$, if $f = B\text{-}\lim_n f_n$ on R and $f = D\text{-}\lim_n f_n$ on R .

Definition 3.1.4. Denote by $\mathfrak{M}_0(R)$ the totality of functions in $\mathfrak{M}(R)$ with compact supports, and by $\mathfrak{M}_\Delta(R)$ the BD -closure of $\mathfrak{M}_0(R)$ in $\mathfrak{M}(R)$. Then $\mathfrak{M}_0(R) \subset \mathfrak{M}_\Delta(R) \subset \mathfrak{M}(R)$. Here $\mathfrak{M}_0$ is a subalgebra of $\mathfrak{M}_\Delta$ and also $\mathfrak{M}$, and $\mathfrak{M}_\Delta$ is a subalgebra of $\mathfrak{M}$.

Definition 3.1.5 (Royden Compactification). Let R be a Riemann surface and $\mathfrak{M}(R)$ Royden's algebra associated with R . Consider the following conditions on a topological space R^* :

1. R^* is a compact Hausdorff space,
2. R^* contains every R as an open and dense subspace,
3. every function in $\mathfrak{M}(R)$ can be continuously extended to R^* . Denote by $B(R^*)$ the set of (complex- or real-valued) bounded continuous functions on R^* . The continuous extension of any $f \in \mathfrak{M}(R)$ to R^* is unique by (2). Hence we may use the same notation f for the extension and write $\mathfrak{M}(R) \subset B(R^*)$.
With this convention consider one more condition:

4. $\mathfrak{M}(R)$ separates points of R^*

This means if p and q are distinct points in R^* then there exists a function $f \in \mathfrak{M}(R)$ such that $f(p) \neq f(q)$. R^* is called the *Royden compactification* of R if R^* satisfies conditions (1)-(4).

Definition 3.1.6 (Royden's Boundary). The set $\Gamma = R^* - R$ is a compact subset of R^* . Call Γ Royden's boundary of R .

$$\Gamma = \{ p : p \in R^*, f(p) = 0 \quad \forall f \in \mathfrak{M}_0(R) \}$$

and thus $\Gamma = \emptyset$ if and only if R is compact.

Definition 3.1.7 (Royden's Harmonic Boundary). Distinguish the following part of Royden's boundary Γ of R

$$\Delta = \{ p : p \in R^*, f(p) = 0 \quad \forall f \in \mathfrak{M}(R) \}.$$

This is a compact set in R^* , and since $\mathfrak{M}_0(R) \subset \mathfrak{M}(R)$, Δ is a compact subset of Γ , where Δ is called Royden's harmonic boundary of R

For an open Riemann surface R let $\{R_n\}_0^\infty$ be an exhaustion with connected $R - \bar{R}_0$. Let $\omega_n(z; R_n, R_0)$ be the continuous function on R such that $\omega_n|_{\bar{R}_0} = 0$, $\omega_n|(R - R_n) = 1$, and $\omega_n \in H(R_n - \bar{R}_0)$, (H is the class of harmonic functions). As found in [41] and [3] the function ω_n is called the *harmonic measure* of ∂R_n with respect to $R_n - \bar{R}_0$. By the maximum principle we see that

$$\omega_{n+p} \leq \omega_n.$$

Thus $\lim_{n \rightarrow \infty} \omega_n(z)$ exists on R , vanishes on $\bar{R}_0$, and is harmonic on $R_n - \bar{R}_0$. This function will be denoted by

$$\omega(z) = \omega(z; R, R_0)$$

and referred to as the *harmonic measure* of the ideal boundary of R with respect to $R_\pi - \bar{R}_0$.

A surface R which is either closed, or open with $\omega \equiv 0$, is called *parabolic*. Otherwise it is said to be *hyperbolic*. The class of parabolic surfaces will be denoted by O_G . It can be shown that $R \in O_G$ is characterized by the nonexistence of Green's function on R .

We state without proof, following [42], a characterization of the class O_G in terms of the harmonic boundary which will be used in the sequel.

Theorem 3.1.3. *The following conditions for Riemann surfaces R are equivalent*

1. $R \in O_G$,
2. $1 \in \mathfrak{M}_\Delta(R)$,
3. $\mathfrak{M}_\Delta(R) = \mathfrak{M}(R)$,
4. $\Delta = \emptyset$

3.2 Ahlfors parabolicity test

It was Ahlfors in [28], who explicitly pointed out that a sufficient condition for R to be parabolic is the existence of a conformal metric $f(w)|dw|$ such that $\int_0^\infty L(p)^{-1}dp = \infty$, with $L(p)$ the length of the circle of radius p in this metric with fixed center. The metric must also give an infinite distance to the boundary of R from any point of R . Also in [28], Kakutani points out that if one chooses the metric $f(w)|dw|$ in such a way that $\log f(w)$ is subharmonic, see chapter 4, then there exists one and only one geodesic $l(w, w_0)$ between two arbitrary given points w and w_0 . In this

case, fix a point w_0 in R and express an arbitrary point w in R by the polar coordinates (ρ, θ) , where $\rho = \rho(w)$ is the length of the geodesic $l(w, w_0)$ and $\theta = \theta(w)$ is the angle between $l(w, w_0)$ and the real axis measured at w_0 . Map R to the z -plane by

$$w \rightarrow z : z = re^{i\theta}, \text{ with } r = \rho (\rho \leq 1) \text{ and } r = 1 + \int_1^\rho L(p)^{-1} dp (\rho \geq 1).$$

Kakutani showed that if this mapping is *quasiconformal*, then Ahlfors condition $\int_0^\infty L(p)^{-1} dp = \infty$ is not only sufficient but also necessary for R to be parabolic. However, no concrete condition for the above mapping to be quasiconformal has been found.

3.2.1 Modular test

Sario [43] develops the following *modular test*. On a Riemann surface R consider a relatively compact region E whose relative boundary ∂E consists of a finite number (≥ 2) of Jordan curves, divided into two disjoint classes α and β . Let x be a continuous function on $\bar{E}$ which is harmonic on E with

$$x|_\alpha = 0, \quad x|_\beta = \log \mu$$

where $\mu > 1$ is a constant such that

$$\int_\alpha * dx = 2\pi.$$

Here the integral is to be understood as $\lim_{\lambda > 0, \lambda \rightarrow 0} \int_{x=\lambda} * dx$ with $\int_\alpha * dx = \int_\alpha (\partial u / \partial n) ds$ is the flux. The number μ , a conformal invariant of the configuration (E, α, β) , is called its *modulus* $\text{mod}(E, \alpha, \beta)$, in analogy, $\log \mu$ shall be referred to as the *logarithmic modulus*.

Let x be the *modulus function* for (E, α, β) , and $w = w(\zeta, E, \beta)$ be the *harmonic measure* of β with respect to E i.e., the continuous function on $\bar{E}$ which is harmonic on E with

$$w|_{\alpha} = 0, \quad w|_{\beta} = 1.$$

By Stokes' formula

$$\int_{\alpha} *dw = \int_E dw \wedge *dw = D_E(w)$$

and therefore

$$\log \mu = \frac{2\pi}{D_E(w)}$$

Now suppose (E_j, α_j, β_j) , $j = 1, \dots, n$, are configurations as above. If the E_j are disjoint by pairs and $\alpha \cap \beta_i = \emptyset$ ($i \neq j$) then we set $E = \bigcup_{j=1}^n E_j$, $\alpha = \bigcup_{j=1}^n \alpha_j$ and $\beta = \bigcup_{j=1}^n \beta_j$. For (E, α, β) we can also define the modulus function x , the harmonic measure w , and the modulus $mod(E, \alpha, \beta)$ in the same fashion as before. In particular the expression for $\log \mu$ remains valid.

Definition 3.2.1. Let R be an open Riemann surface and $\{R_n\}_0^\infty$ be an exhaustion of R . Denoted by

$$\nu_n = mod(R_n - \bar{R}_0, \partial R_0, \partial R_n)$$

the modulus of the configuration $(R_n - \bar{R}_0, \partial R_0, \partial R_n)$.

Theorem 3.2.1. A necessary and sufficient condition for R to belong to O_G is that

$$\lim_{n \rightarrow \infty} \nu_n = \infty \tag{3.2.1}$$

Proof. Let $\omega_n(z) = \omega_n(z; R_n, R_0)$ be the harmonic measure of ∂R_n with respect to $R_n - \bar{R}_0$, i.e., $\omega_n \in \mathfrak{A}(R)$, $\omega_n|_{R_n - \bar{R}_0} \in H(R_n - \bar{R}_0)$, $\omega_n|_{\bar{R}_0} = 0$, and $\omega_n|_{R - R_n} = 1$. Then

$$\log \nu_n = \frac{2\pi}{D_R(\omega_n)}. \tag{3.2.2}$$

Thus (3.1) is equivalent to $\lim_n D_R(\omega_n) = 0$ and this in turn to $\lim_n \omega_n = 0$, i.e., $R \in O_G$. $\square$

3.2.2 Divergent modular product

Denote by μ_n the modulus of the configuration $(R_n - \bar{R}_{n-1}, \partial R_{n-1}, \partial R_n)$:

$$\mu_n = \text{mod}(R_n - \bar{R}_{n-1}, \partial R_{n-1}, \partial R_n).$$

Theorem 3.2.2. *In order that $R \in O_G$ it is necessary and sufficient that there exist an exhaustion $\{R_n\}_0^\infty$ of R with*

$$\prod_{n=1}^{\infty} \mu_n = \infty \quad (3.2.3)$$

Proof. Let $w_n(z)$ be the harmonic measure of ∂R_n with respect to $R_n - \bar{R}_{n-1}$. Then

$$\log \mu_n = \frac{2\pi}{D_R(w_n)}. \quad (3.2.4)$$

Set $u_n = \omega_n / D_R(\omega_n)$ and $v_n = w_n / D_R(w_n)$. Clearly

$$\int_{\partial R_n} *du_n = \int_{\partial R_n} \omega_n * d\omega_n / D_R(\omega_n) = 1$$

and similarly $\int_{\partial R_n} *dv_n = 1$. By (3.2) and (3.4)

$$\log \nu_n = 2\pi D_R(u_n), \quad \log \mu_n = 2\pi D_R(v_n). \quad (3.2.5)$$

Schwartz's inequality gives for $1 \leq i \leq n$

$$\begin{aligned} D_{R_i - \bar{R}_{i-1}}(u_n) &\geq \frac{D_{R_i - \bar{R}_{i-1}}^2(v_i, u_n)}{D_{R_i - \bar{R}_{i-1}}(v_i)} = \frac{(\int_{\partial R_i - \partial R_{i-1}} v_i * du_n)^2}{\int_{\partial R_i - \partial R_{i-1}} v_i * dv_i} \\ &= \frac{(1/(D_R(w_i)))^2}{1/(D_R(w_i))} = \frac{1}{D_R(w_i)} = \frac{1}{2\pi} \log \mu_i. \end{aligned}$$

Therefore $D_R(u_n) = \sum_{i=1}^n D_{R_i - \bar{R}_{i-1}}(u_n) \geq (2\pi)^{-1} \sum_{i=1}^n \log \mu_i$. Since $D_R(u_n) = (2\pi)^{-1} \log \nu_n$ we obtain

$$\nu_n \geq \prod_{i=1}^n \mu_i.$$

By theorem 3.2.1, (3.2.3) implies $R \in O_G$.

Conversely assume $R \in O_G$. Take regular regions R_0 and R_1 with $R_1 \supset \bar{R}_0$ and a regular region $R_2 \supset \bar{R}_1$ with $\mu_2 \geq 2$. This is possible by (3.2.1). Here R_2 can be arbitrarily large. By repeating this procedure an exhaustion $\{R_n\}_0^\infty$ of R with $\mu_i \geq i$ is obtained and thus with (3.2.3). $\square$

3.2.3 Conformal metric test

Recall from section 3.1 that a Riemann surface R *parabolic* if it is closed or if the harmonic measure of its ideal boundary vanishes; the class of parabolic surfaces is denoted by O_G . In contrast an $R \notin O_G$ is called *hyperbolic*.

Theorem 3.2.3. *If the length $L(\rho)$ grows so slowly that*

$$\int_\epsilon^\infty \frac{d\rho}{L(\rho)} = \infty \quad \epsilon > 0 \quad (3.2.6)$$

then R belongs to O_G .

Proof. Let w_n be the harmonic measure of ∂R_n with respect to $R_n - \bar{R}_{n-1}$ and w_n^* a conjugate harmonic function of w_n on $R_n - \bar{R}_{n-1}$. Then the function

$$W_n = w_n + iw_n^*$$

maps $R_n - \bar{R}_{n-1}$, less suitable slits γ_j on which w_n^* is constant, conformally onto the horizontal sliced rectangle $T : 0 \leq w_n \leq 1, 0 \leq w_n^* \leq 2\pi / \log \mu_n$ in the W plane.

The Euclidean length of $\Gamma(\rho)$ in T is at least $2\pi/\log \mu_n$:

$$\frac{2\pi}{\log \mu_n} \leq \int_{\Gamma(\rho)} \left| \frac{dW_n}{dz} \right| |dz|.$$

By Schwartz's inequality

$$\left(\frac{2\pi}{\log \mu_n} \right)^2 \leq \int_{\Gamma(\rho)} \left| \frac{dW_n}{dz} \right|^2 \frac{|dz|}{\gamma} \int_{\Gamma(\rho)} \gamma |dz|.$$

Denote by $A(\rho)$ the area of that part of T which lies to the left of the image of $\Gamma(\rho)$. Since on $\Gamma(\rho)$

$$\frac{|dz|}{\gamma} = |dz| \frac{|dz|}{ds} = |dz| \frac{|dz|}{d\rho},$$

it can be inferred that

$$\int_{\Gamma(\rho)} \left| \frac{dW_n}{dz} \right|^2 \frac{|dz|}{\gamma} = \frac{d}{d\rho} \int_{\rho_{n-1}}^{\rho} \int_{\Gamma(\rho)} \left| \frac{dW_n}{dz} \right|^2 |dz| \frac{|dz|}{d\rho} d\rho = \frac{d}{d\rho} A(\rho).$$

Therefore

$$\left(\frac{2\pi}{\log \mu_n} \right)^2 \leq \frac{A(\rho)}{d\rho} L(\rho)$$

and

$$\int_{\rho_{n-1}}^{\rho} \frac{d\rho}{L(\rho)} \leq \left(\frac{\log \mu_n}{2\pi} \right)^2 \int_{\rho_{n-1}}^{\rho} dA = \frac{1}{2\pi} \log \mu_n.$$

Therefore

$$\frac{1}{2\pi} \log \prod_1^{\infty} \mu_n \geq \int_{\rho_0}^{\infty} \frac{d\rho}{L(\rho)} = \infty.$$

So (3.2.3) is satisfied and $R \in O_G$. □

3.3 Quasiconformal mappings

One of the most important contributions to the type problem was given in the paper by Shizou Kakutani, see [27]. This paper was the pioneering work for some of the results that will be considered in the sequel.

It was Kakutani's remarkable idea to deform a given surface by a quasiconformal mapping T to a surface suitable for the determination of the type, often called the *Kakutani surface*. Consider a point p of a surface. If T transforms an infinitesimal circle to an infinitesimal ellipse. The ratio $g_T(p) \geq 1$ of the major and minor axes of this ellipse describes the deviation of T from the conformal mapping at p . If $\sup_p g_T(p) < +\infty$, then T is called *quasiconformal*. Also, define a *pseudoregular* function as one of the form $f \circ T$ where f is regular and T is quasiconformal.

Kakutani's paper [28] consists of three parts. In the first part differentiable pseudoregular functions are considered and the great Picard theorem, see Ahlfors [3], is generalized to a certain class of such functions. In the second part, the *quasiconformal invariance* of types is established i.e., if there exists a quasiconformal mapping of a simply connected open Riemann surface R onto another R' , then R is parabolic if and only if R' is parabolic. This important result is then applied to the third part of the paper. In particular, consider a simply connected covering surface R of the w -plane which has algebraic or logarithmic branch points over a finite number of points $w_1, w_2, \dots, w_n$ of the w -plane. Let π be the projection of R onto the w -plane, and f the conformal mapping of R onto the unit or the infinite disk of the z -plane. Take a Jordan curve Γ passing through $w_1, w_2, \dots, w_n$ in this order, a point denoted by $\circ$ in the interior of the region encircled by Γ , a point $\times$ in the exterior of that region, and n simple arcs $l_1, l_2, \dots, l_n$ which each join $\circ$ and $\times$ and each intersect one of the subarcs $w_1w_2, w_2w_3, \dots, w_nw_1$ of R , respectively. Utilizing π , configuration topologically such that the images of $l_1, l_2, \dots, l_n$ all become straight line segments. The resulting complex consisting of countable points marked by $\circ$ or $\times$ and line segments joining them referred to as the *topological tree* (line complex) of R in the Speiser sense, see [39]. The information the topological

tree supplies consists of the ramification numbers of the nodes of the tree from one generation to the next. There are several ways to describe these numbers; Kakutani employed the *tree function* $\mu(\theta)$ of Kobayashi, see [30].

The surfaces considered in [28] page 6, which deals with the type problem are taken from the class of simply connected open Riemann surfaces represented as covering surfaces of the plane with logarithmic branch points over the three points 1 , $e^{2\pi i/3}$, and $e^{4\pi i/3}$. Such a class of Riemann surfaces had earlier been considered by Nevanlinna, Speiser, Ullrich, and Kobayashi, and several conditions for a surface R of this class to be parabolic has been obtained in terms of the rate of growth of the tree function $\mu(\theta)$ of the topological tree of R . If the rate is sufficiently small, then R is parabolic; the greater this rate becomes, the more the surface R is likely to be hyperbolic. The Speiser version of the type problem, stated in a concrete fashion, is to obtain necessary and sufficient condition on μ for R to be parabolic (or hyperbolic). Kakutani gave a surprising answer that deeply influenced the evolution of the problem: it is *impossible* to determine the type of R by considering only the rate of growth of its tree function μ . Explicitly, there exists two surfaces R_1 and R_2 of this class, with R_1 parabolic and R_2 hyperbolic and yet $\mu_1 \geq \mu_2$. This result is an application of the main result in the third part of Kakutani's paper, a sufficient condition for hyperbolicity. To establish this condition, the given surface R is deformed quasiconformally to a more manageable one, the Kakutani surface, and essential use is made of the quasiconformal invariance obtained in the second part of the paper.

3.4 Hyperbolic surfaces

Given a surface of the form $z = f(x, y)$ over the whole plane, it is called *hyperbolic* if it can be mapped conformally onto the interior of the unit circle. The question whether any such surfaces exist was raised by Loewner and has been answered by the construction of a number of examples. In particular, Osserman in [40] constructed a polyhedral hyperbolic surface.

By a conformal map of a polyhedral surface onto a plane region, is a topological map of the whole surface which is conformal in the interior of each plane face. The behavior of the map along the edges at the vertices is then uniquely determined by the classical theorems in function theory.

Define a surface S over the strip $x \geq 0, 0 \leq y \leq 1$. The chief property of S is that Z oscillates rapidly between 0 and 1 as y goes from 0 to 1, and the rate of oscillation increasing exponentially with x . More specifically, on each ray: $x \geq n, y = k/3^n$, we have $z = 0$ or $z = 1$ according as k is even or odd.

Denote by S_0 the part of the surface over $0 \leq x \leq 1, 0 \leq y \leq 1$. It may be chosen as an arbitrary polyhedral surface (the union of a finite number of plane triangles) of the form $z = f(x, y)$, satisfying

1. the boundary of S_0 consists of the line segments joining successively the points $(0, 0, 0), (1, 0, 0), (1, \frac{1}{3}, 1), (1, \frac{2}{3}, 0), (1, 1, 1), (0, 1, 1), (0, 0, 0)$;
2. none of the triangles of S_0 lies in a plane parallel to the y -axis.

Denote by S_n the part of the surface over the rectangle $n \leq x \leq n + 1$,

$0 \leq y \leq \frac{1}{3^n}$, defined in terms of S_0 by setting

$$f(x, y) = f(x - n, 3^n y)$$

then extend the surface over the square $n \leq x \leq n + 1$, $0 \leq y \leq 1$, by successive reflections in the vertical plane $y = k 3^n$, $k = 1, 2, \dots, 3^n - 1$. By virtue of property 1 of S_0 a polyhedral surface S of the form $z = f(x, y)$ over the whole strip $x \geq 0$, $0 \leq y \leq 1$ is obtained.

Now define the map ϕ of S onto the rectangle R : $0 \leq \xi \leq \frac{3}{2}$, $0 \leq \eta \leq 1$ by setting, for $n \leq x \leq n + 1$,

$$\xi = \frac{3}{2} \left(1 - \frac{1}{3^n} \right) + \frac{1}{3^n} (x - n)$$

$$\eta = y.$$

The map ϕ is thus a topological map of S onto R , and it is furthermore linear in each triangle of S . Now it will be shown that the dilatations of all these linear maps are uniformly bounded as above.

Let T_n be an arbitrary triangle of S_n , and let T_0 be the corresponding triangle in S_0 . If the plane of T_0 is $z = ax + by + c$ then T_n lies in the plane $z = a(x - n) + 3^n by + c$. In terms of the variables ξ, η we obtain

$$ds^2 = dx^2 + dy^2 + dz^2 = 3^{2n} (E d\xi^2 + 2F d\xi d\eta + G d\eta^2)$$

where

$$E = a^2 + 1, \quad F = ab, \quad G = b^2 + 3^{-2n}$$

for the points on T_n . Hence

$$\frac{E + G}{\sqrt{EG - F^2}} = \frac{a^2 + b^2 + 1 + 3^{-2n}}{\sqrt{b^2 + (a^2 + 1)3^{-2n}}} < \frac{2 + a^2 + b^2}{b}$$

where we note that $b \neq 0$ by property 2 of S_0 .

Thus an upper bound for the dilatation (see [6] for a development of dilatations in the context of affine geometry, and [2] in the context of C^1 homeomorphisms) of T_n is obtained. This upper bound does not depend on n , but depends only on the corresponding triangle T_0 of S_0 . Since S_0 consists of a finite number of triangles a uniform bound for all the triangles of S is obtained. Along the common edge of two triangles the map ϕ will not be differentiable but it is known [2], the maximal dilatation remains unchanged if we delete analytic arcs. Thus ϕ is indeed quasiconformal.

To obtain the example cited above, extend S by successive reflections to a surface W over the whole plane, and then extend the map ϕ again by reflection, to the map of W onto the infinite strip $T : -\frac{3}{2} < \xi < \frac{3}{2}$. This extended map will be quasiconformal (with the same maximal dilatation as ϕ) and composing it with an elementary conformal map of T onto the unit disk D , this gives an explicit quasiconformal map of W onto D .

3.5 Potential theory and harmonic functions

The following development for necessity and sufficiency for a Riemann surface to be hyperbolic was taken from Farkas and Kra [16], see also Abikoff [1], and Ahlfors [4] for a more complete treatment of the Dirichlet principle, Green's function, harmonic measures and hyperbolic Riemann surface classification. The point here is to give highlights in relation to Milnor's conjecture as well as a historical context that gave rise to the notion of a Riemann surface being made of an isotropic resistive material of constant thickness and the surface type being determined by its resistance out

to infinity as discussed by Doyle in [12].

Let F be a closed subset of M . Denote by $e_F(x)$ the $\mathbb{P}_x$ -probability that the process X_t hits F ever, that is, $X_t \in F$ for some $t \geq 0$. Obviously, $e_F(x) = 1$ on F . It turns out that $e_F(x)$ is a harmonic function outside F and superharmonic on M .

Another function $h_F(x)$ to be considered here is the $\mathbb{P}_x$ -probability of the event that X_t visits F at arbitrarily large moments of time. This function turns out to be harmonic on all of M . Both of the above functions are called hitting probabilities.

Definition 3.5.1. A function u defined in a region of M is harmonic if

$$\Delta u = 0. \tag{3.5.1}$$

The harmonic equation is understood either in the sense of distribution or pointwise. In the latter case, the function u is initially C^2 smooth. In both cases, u will be actually C^∞ . Indeed, u satisfies locally an elliptic equation of the second order with smooth coefficients. Therefore, smoothness of u follows from the general theory of elliptic PDE. Other consequences are the maximum principle, the local Harnack inequality and the convergence principles.

The standard way of constructing harmonic functions is by solving the Dirichlet problem. If B is a precompact open set on M with smooth boundary, then for any continuous function f on ∂B , there exists a unique function $u \in C(\bar{B}) \cap C^2(B)$ such that

$$\begin{cases} \Delta u = 0, \\ u|_{\partial B} = f. \end{cases} \tag{3.5.2}$$

The proof is similar that on $\mathbb{R}^d$, see [15].

Alternatively, the solution of (4.5.2) is given by the formula of Kakutani,

$$u(x) = \mathbb{E}_x(f(X_\tau)),$$

where τ is the first hitting time of the boundary ∂B by the process X_t .

The Green formula (1.3.1) implies that, for a harmonic function u and for any precompact open set Ω in the domain of u , the flux of u through the boundary $\partial\Omega$ is zero, that is

$$\text{flux}_{\partial\Omega} u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\mu' = 0, \quad (3.5.3)$$

where ν is the outward normal unit vector field on $\partial\Omega$ (assuming is smooth enough). Moreover, (4.5.3) is equivalent to harmonicity of u . Indeed, it implies by (1.3.1)

$$\int_{\Omega} \Delta u d\mu = 0,$$

for any Ω in the domain of u , whence $\Delta u = 0$.

A function s defined in the region $\Omega \subset M$ is called superharmonic if s is continuous and if, for any precompact region $U \subset\subset \Omega$ and any harmonic function $u \in C^2(U) \cap C(\bar{U})$, $s \geq u$ on ∂U implies $s \geq u$ on U . If $s \in C^2(\Omega)$ then the *superharmonicity* of s is equivalent to

$$s \leq 0, \quad (3.5.4)$$

which follows from the maximum principle. Conversely, if $s \in C(\Omega)$ and (4.4) holds in the sense of distributions, then s is superharmonic.

The following properties of superharmonic functions will be needed in the sequel.

1. If $\{\Omega_\alpha\}$ is a family of open sets and the function s is superharmonic in each Ω_α , then s is superharmonic in their union $\cup_\alpha \Omega_\alpha$.
2. The minimum of two superharmonic functions is also superharmonic.

A function u is *subharmonic* if $-u$ is superharmonic. Note, on a Riemann surface subharmonic functions assume the role of convex functions. A subharmonic function is a continuous real-valued function whose value at a point $\zeta_0 \in M$ is less than or equal to the average of the values of the function on the boundary of any small symmetric neighborhood of ζ_0 .

Theorem 3.5.1. *Let $u : M \rightarrow \mathbb{R}$ be continuous. The u is subharmonic if and only if*

$$u(0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta$$

for every conformal disc on M , [16].

Definition 3.5.2. A family $\mathfrak{F}$ of subharmonic functions on a Riemann surface M is called a *Perron class* (on M) provided:

- $\mathfrak{F}$ is non-empty,
- For all conformal discs $\Delta \subset M$, and for all $u \in \mathfrak{F}$ there exists $v \in \mathfrak{F}$ such that v restricted to Δ is harmonic and $v \geq u$,
- $\forall u_1, u_2 \in \mathfrak{F} \exists v \in \mathfrak{F}$ such that $v \geq \max\{u_1, u_2\}$.

Alternately, a *Perron class* $\mathfrak{F}$ is a non void set of subharmonic functions $s : S \rightarrow \mathbb{R}$ which is closed under the following operations:

- taking the maximum of two functions

- local harmonic majorization.

Definition 3.5.3. Let M be a Riemann surface and $P \in M$. A function g is called a *Green's function*, or potential of a point charge, for M with singularity at P provided g is harmonic in $M - \{P\}$, $g > 0$ in $M - \{P\}$, if z is a local parameter vanishing at P , then $g(z) + \log |z|$ is harmonic in a neighborhood of P and if $\hat{g}$ is another function satisfying the above, then $\hat{g} \geq g$. In other words, [1] the potential should be the smallest potential which is positive and grows as $-\log |z|$ near the point charge.

Theorem 3.5.2. Let M be a Riemann surface. There exists a Green's function on M (with singularity at some point $P \in M$) if and only if M is hyperbolic.

Proof. Let g be a Green's function on M with singularity at P . Let $m > 0$. Set $f = \min\{m, g\}$. Then f is superharmonic in $M - \{P\}$. Since $g(Q) \rightarrow \infty$ as $Q \rightarrow P$, $f \equiv m$ in a neighborhood of P . Thus f is superharmonic on M and $f > 0$. If f were constant, then $g \geq m$, and thus $g - m$ would be another candidate for the Green's function with $g - m < g$, a contradiction. Hence f cannot be constant and thus M is hyperbolic.

Conversely, assume that M is hyperbolic. Let $P \in M$ and Δ be a conformal disc about P with local coordinate z . Set

$$\begin{aligned} \mathfrak{F} = \{ & v : v \text{ is subharmonic in } M - \{P\} \wedge \\ & v \geq 0 \wedge v \text{ has compact support} \wedge \\ & v(z) + \log |z| \text{ is subharmonic in } |z| < 1 \}. \end{aligned}$$

$\mathfrak{F}$ is non-empty since

$$v_0(z) = \begin{cases} -\log |z|, & 0 < |z| < 1 \\ 0, & |z| \geq 1. \end{cases}$$

and observe that $v_0 \in \mathfrak{F}$. Then $\mathfrak{F}$ is a Perron family on $M - \{P\}$.

Lemma 3.5.3. *Outside every neighborhood of P , $\mathfrak{F}$ is uniformly bounded.*

Proof. Choose $r \in (0, 1)$. Let ω_r be the harmonic measure of $\{|z| \leq r\}$, it is only here we use the fact that M is hyperbolic. Thus ω_r is harmonic on $|z| > r$, $\omega_r = 1$ on $|z| = r$, and $0 < \omega_r < 1$ for $|z| > r$. Let $\lambda_r = \max\{\omega_r(z) : |z| = 1\}$. Thus $\lambda \in (0, 1)$. For $u \in \mathfrak{F}$, let $u_r = \max\{u(z) : |z| = r\}$. We claim

$$u_r z_r - u \geq 0 \text{ for } |z| \geq r. \quad (3.5.5)$$

Clearly, $u_r z_r - u$ is superharmonic on $|z| > 1$, and $u_r z_r - u \geq 0$ on $|z| = r$. Let $\mathfrak{K}$ be the support of u . Then $u = 0$ on $\partial\mathfrak{K}$, and thus (3.5.5) holds on $\partial\mathfrak{K}$. Hence (3.5.5) holds on $\mathfrak{K} - \{|z| \leq r\}$ by the minimum principle for subharmonic functions. Inequality (3.5.5) holds on $M - \mathfrak{K}$.

Finally, $u(z) + \log |z|$ is subharmonic in $|z| < 1$ and continuous on $|z| \leq 1$.

Thus

$$u_r + \log r = \max_{|z|=r} u + \log r \leq \max_{|z|=1} u + \log 1 = \max_{|z|=1} u \leq u_r \lambda_r$$

the last inequality is a consequence of (3.5.5). Thus

$$u_r \leq \frac{\log r}{\lambda_r - 1}$$

Since $u = 0$ off a compact set $\mathfrak{K}$, we conclude that

$$\max\{u(z) : |z| \geq r\} \leq \frac{\log r}{\lambda_r - 1} \quad (3.5.6)$$

This completes the proof of the lemma. $\square$

Continuing the proof of theorem 3.5.2, set

$$g(Q) = \sup_{u \in \mathfrak{F}} u(Q), \quad Q \in M - \{P\}.$$

Inequality (3.5.6) implies that $g(Q) < \infty$ for $Q \neq P$. Thus by Perron's principle, see [4], g is harmonic in $M - \{P\}$. The fact that $g > 0$ on $M - \{P\}$ follows from the corresponding fact that $u < 0$ on $M - \{P\}$ for all $u \in \mathfrak{F}$. Since $u > 0$ implies $g \geq 0$ on $M - \{P\}$. If $g(Q) = 0$ for some $Q \in M - \{P\}$, then g is constant. Thus $g > 0$ on $M - \{P\}$. Note that for $|z| < r$, $u \in \mathfrak{F}$,

$$u(z) + \log |z| \leq u_r + \log r \leq \frac{\log r}{\lambda_r - 1} + \log r = \frac{\lambda_r \log r}{\lambda_r - 1}.$$

Thus also

$$g(z) + \log |z| \leq \frac{\lambda_r \log r}{\lambda_r - 1} \text{ for } |z| \leq r.$$

Hence $z = 0$ is a removable singularity for the bounded harmonic functions $g(z) + \log |z|$. Note this is precisely condition two above for a function to be a Green's function.

To finish the proof that g is the Green's function, let $\hat{g}$ be a Green's function. Let $u \in \mathfrak{F}$. Then $\hat{g} - u$ is superharmonic on M . Since u has compact support, say K' , $\hat{g} - u \geq 0$ on $M - K'$ and by the minimum principle for superharmonic functions also on K' . Thus $\hat{g} \geq u$ on $M - \{P\}$. Thus $\hat{g} \geq g$. In fact, either $\hat{g} > g$ or $\hat{g} = g$ on $M - \{P\}$ where $\hat{g}$ is the harmonic measure. $\square$

Remark 3.5.1. A little more than claimed in the theorem has been shown, namely:

Existence of Green's function at one point

$\Downarrow$

Hyperbolic

$\Downarrow$

Existence of harmonic measures

$\Downarrow$

Existence of Green's function at every point.

Define $g(Q, P)$ as the value at Q of the Green's function, on the hyperbolic surface M , with singularity at P .

3.6 Remarks on type problem

If a simply connected open Riemann surface R is conformally mapped onto the unit disk $|z| < 1$ by $z = f(w)$ with $f(w_0) = 0$ then $g(w, w_0) = -\log |f(w)|$ is the harmonic Green's function on R with pole w_0 . Conversely, if there exists the Green's function $g(w, w_0)$ on R with pole w_0 then

$$z = f(w) = \exp(-g(w, w_0) - i * g(w, w_0)),$$

where $*g(w, w_0)$ is the harmonic conjugate of $g(w, w_0)$ on $R - w_0$, and maps conformally onto the unit disk. Therefore, the parabolicity of R is characterized by the nonexistence of the Green's function on R . Since this property has no relation to the surface being simply connected, parabolicity and hyperbolicity can be generalized: an *arbitrary* open Riemann surface R is, by definition, *parabolic* (or *hyperbolic*) if there does not (or does) exist a harmonic *Green's function* on R , see [28] page 382.

Despite striking partial results so obtained, for an account see [39], the classical type problem in the original sense of Speiser remains unsolved; several conditions are known that are either necessary or sufficient, but none that are both in

the concrete sense Speiser proposed.

As happens in many branches of mathematics, the type problem, without being solved completely, has been absorbed in a more comprehensive area of inquiry, the *harmonic classification theory*, which started around 1950. Characterization of O_G by the nonexistence of various potentials have been given by Ohtsuka, Ahlfors, Myrberg, Kuramochi, Sario, and many others, see [42]. In particular, six phases can be distinguished in the historical development of the classification theory: Riemann's mapping theorem, the classical type problem, Harmonic and analytic classification of Riemann surfaces, Harmonic classification of Riemannian manifolds, Biharmonic classification of Riemannian manifolds and the Biharmonic type problem [43].

Chapter 4

Brownian Motion on Riemannian Manifolds

4.1 Introduction

In probabilistic terms, Milnor's result means that if the curvature is negative enough a particle diffusing around on a surface will eventually wander off and never come back, while if the curvature is just a little less negative the particle is bound to come back near where it started, no matter how far off it may have wandered. Let D be a relatively compact subregion of a Riemann surface R , with the boundary ∂D of D consisting of a finite number of disjoint Jordan curves. Take an elementary set E in ∂D , and let $P(\zeta, E, D)$ be the probability that the *Brownian motion* starting from a point ζ in D will enter E without first entering the other part ∂D of D . Katutani proved that $P(\zeta, E, D)$ is the unique bounded harmonic function of ζ on D which has boundary values 1 on the interior of $\partial D - E$. Thus it is the harmonic measure of E with respect to D measured at ζ , see [28] page 382.

Consider the exhaustion $\{D_n\}_0^\infty$ of a simply connected open Riemann surface R i.e., each D_n is a subregion of R of the type D above, $\bar{D}_n \subset D_{n+1}$ ($n = 0, 1, \dots$), $\cup_{n=0}^\infty D_n = R$, and D_0 is a disk on R . Set $u_n(\zeta) = P(\zeta, \partial D_0, D_n - \bar{D}_0)$ for $n = 1, 2, \dots$. Since $u_n \leq u_{n+1}$, the limit $u = \lim u_n$ exists and can be viewed as the probability that the Brownian motion on R starting from ζ will enter ∂D_0 . In current terminology, $u(\zeta)$ is thus the harmonic measure of ∂D_0 with respect to $R - \bar{D}_0$. The main result of [28] is that R is parabolic (or hyperbolic) if and only if $u \equiv 1$ (or $u < 1$) on $R - \bar{D}_0$.

4.2 Heat semigroup on Riemannian manifolds

The reference for the results of the rest of this chapter is Grigor'yan [22].

Definition 4.2.1. A standard Brownian motion means a family $\{X_t(\omega) : t \geq 0\}$ of real random variables defined on some probability space $(\Omega, \mathfrak{B}, P)$ that satisfies:

- $X_0 = 0$
- If $0 = t_0 < t_1 < \dots < t_n$, then the random variables $X_{t_k} - X_{t_{k-1}}$ for $k = 1, 2, \dots, n$ are independent.
- For each $s, t \geq 0$, the random variable $X_{t+s} - X_t$ is normally distributed with mean 0 and variance s .
- For almost all $\omega \in \Omega$, the function $X_t = X_t(\omega)$ is everywhere continuous in t .

Construct Brownian motion on a Riemannian manifold M by constructing first *the heat kernel* which will also serve as the density of the transition probability. The heat kernel will be denoted by $p(t, x, y)$ where $t > 0$ is a time and x, y are points

on M . Thus, the probability that the Brownian motion starting at the point x lies in a measurable set $\Omega \subset M$ at the time t is given by

$$\int_{\Omega} p(t, x, y) d\mu(y).$$

In $\mathbb{R}^d$, the heat kernel is given by the classical formula

$$p(t, x, y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x - y|^2}{2t}\right).$$

It is known to satisfy the heat equation

$$\frac{\partial p}{\partial t} - \frac{1}{2} \Delta p = 0 \tag{4.2.1}$$

in the variables (t, x) the point y is considered as fixed and the initial data

$$p(t, \cdot, y) \xrightarrow[t \rightarrow 0+]{} \delta_y \tag{4.2.2}$$

where δ_y is the delta function of Dirac.

The properties (4.2.1) and (4.2.2) can be used to define the heat kernel on an arbitrary Riemannian manifold M , which is done below.

4.3 Heat kernel and Brownian motion

Any function on $(0, \infty) \times M \times M$ satisfying (4.2.1) and (4.2.2) is called a fundamental solution of the heat equation (4.2.1) on M . The *heat kernel* is the *smallest positive* fundamental solution of the heat equation on M . The heat kernel always exists (regardless of geodesic completeness) and is smooth in (t, x, y) . Moreover, the heat kernel possesses the following properties.

1. Symmetry in x, y , that is $p(t, x, y) = p(t, y, x)$.

2. The semigroup identity: for any $s \in (0, t)$

$$p(t, x, y) = \int_M p(s, x, z)p(t-s, z, y) d\mu(y). \quad (4.3.1)$$

3. For all $t > 0$ and $x \in M$,

$$\int_M p(t, x, y) d\mu(y) \leq 1. \quad (4.3.2)$$

As soon as one has (4.3.1) and (4.3.2), a (sub)Markov process X_t on M can be constructed with the transition density p by using standard probabilistic tools. The process X_t turns out to be a diffusion and is referred to as the *Brownian motion* or the *Wiener process* on M . The corresponding measure in the space of paths emanating from a point x will be denoted by $\mathbb{P}_x$.

Given an open set $\Omega \subset M$, one can treat Ω as a manifold itself. Let us denote by p_Ω the heat kernel of Ω . Minimality of the heat kernel implies that p_Ω vanishes on the boundary $\partial\Omega$, at least if $\partial\Omega$ is smooth. This implies, in turn, that p_Ω increases on enlarging of Ω .

The global heat kernel p can be constructed as follows: one first defines p_Ω for precompact sets Ω (which can be done by using the eigenfunction expansion) letting

$$p = \lim_{k \rightarrow \infty} p_{\Omega_k}$$

where $\{\Omega_k\}$ is an increasing sequence of precompact open sets with smooth boundaries, which exhaust M . Due to (4.3.1) and (4.3.2), the heat kernel $p(x, y, t)$ can be considered as a kernel of the *submarkovian* operator semigroup P_t which acts on functions on M by

$$P_t f = \int_M p(\cdot, y, t) f(y) d\mu(y).$$

The semigroup corresponding to p_Ω will be denoted by P_t^Ω . If f is a continuous bounded function on M , then the function $u(x, t) = P_t f(x)$ solves the Cauchy problem in $M \times (0, \infty)$:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2} \Delta u, \\ u(\cdot, 0) = f. \end{cases}$$

Moreover, if $f \geq 0$ then $P_t f$ is the smallest non-negative solution to this problem.

Definition 4.3.1. Brownian motion X_t on a manifold M is recurrent if, for any non-empty open set Ω and for any point $x \in M$,

$$\mathbb{P}_x \{ \text{there is a sequence } t_k \rightarrow \infty \text{ such that } X_{t_k} \in \Omega \} = 1.$$

Otherwise X_t is transient.

Definition 4.3.2. Brownian motion X_t is stochastically complete i.e., possesses the conservation property or the non-explosion property if, for all $x \in M$ and $t > 0$,

$$\int_M p(t, x, y) d\mu(y) = 1.$$

In other words, $\mathbb{P}_x$ is a probability measure in the sense that its total mass is equal to 1.

It is convenient to say that a manifold is stochastically complete (recurrent, transient) when the Brownian motion on it has this property.

4.4 Green's function revisited

The easiest way to introduce the Green's function $G(x, y)$ is to set

$$G(x, y) = \frac{1}{2} \int_0^\infty p(t, x, y) dt. \quad (4.4.1)$$

The factor $\frac{1}{2}$ appears because the heat kernel is generated by $\frac{1}{2}\Delta$ rather than by Δ .

An independent definition is as follows: $G(x, y)$ is the smallest positive fundamental solution of the Laplace equation on M and $G \equiv +\infty$ if there is no positive fundamental solution, which matches the case when the integral in (4.6.1) diverges. If $G \neq \infty$ then, for any fixed y ,

$$G(\cdot, y) = -\delta_y.$$

Definition 4.4.1. Let Ω be an open set on M and K be a compact set in Ω . Call the pair (K, Ω) a *capacitor* and define the capacity $\text{cap}(K, \Omega)$ by

$$\text{cap}(K, \Omega) = \inf_{\phi \in \mathcal{L}(K, \Omega)} \int_{\Omega} |\nabla \phi|^2 d\mu, \quad (4.4.2)$$

where $\mathcal{L}(K, \Omega)$ is a set of locally Lipschitz functions ϕ on M with a compact support in $\bar{\Omega}$ such that $0 \leq \phi \leq 1$ and $\phi|_K = 1$.

For an open precompact set $K \subset \Omega$, define its capacity by

$$\text{cap}(K, \Omega) = \text{cap}(\bar{K}, \Omega).$$

Therefore, definition (4.6.2) can be applied in this case too, since $\phi|_K = 1$ is equivalent to $\phi|_{\bar{K}} = 1$. Since $\nabla \phi = 0$ on $\bar{K}$, the capacity $\text{cap}(K, \Omega)$ is determined by the intrinsic properties of $\Omega - \bar{K}$. If $\Omega = M$ then write $\text{cap}(K)$ for $\text{cap}(K, \Omega)$. It is obvious from the definition that the set $\mathcal{L}(K, \Omega)$ increases on expansion of Ω (and on shrinking of K). Therefore, the capacity $\text{cap}(K, \Omega)$ decreases on expanding and shrinking of K . In particular, it can be proved that, for any exhaustion sequence $\{\mathcal{E}_k\}$,

$$\text{cap}(K) = \lim_{k \rightarrow \infty} \text{cap}(K, \mathcal{E}_k).$$

It is well known that the Dirichlet integral in (4.6.2) is minimized by a harmonic function. Therefore, the infimum in (4.6.2) is attained at the function $\phi = u$ which is the Perron solution to the following Dirichlet problem in $\Omega - K$:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = 0 \\ u|_{\partial K} = 1. \end{cases} \quad (4.4.3)$$

The function u is called the *equilibrium potential* or the *capacity potential* of the capacitor (K, Ω) . It is obvious that if the boundaries of Ω and K are smooth enough, then $u \in \mathcal{L}(K, \Omega)$. Then by the Green formula and (4.6.3),

$$\begin{aligned} \text{cap}(K, \Omega) &= \int_{\Omega} |\nabla u|^2 d\mu = \int_{\Omega-K} |\nabla u|^2 d\mu \\ &= - \int_{\Omega-K} u \Delta u d\mu + \int_{\partial K \cup \partial\Omega} \frac{\partial u}{\partial \nu} u d\mu' \\ &= -\text{flux}_{\partial K} u \end{aligned} \quad (4.4.4)$$

where ν is the outward unit normal vector field on $\partial(\Omega - K)$, the negative sign in (4.6.4) appears because ν points inward to K . On the other hand, the harmonicity of u and (4.5.3) imply

$$0 = \text{flux}_{\partial\Omega-K} u = \text{flux}_{\partial\Omega} u - \text{flux}_{\partial K} u \quad (4.4.5)$$

In general, despite the fact that u may not be in $\mathcal{L}(K, \Omega)$, the Dirichlet integral of u is still equal to the capacity.

If M is a manifold with boundary, then all the above remain true, with the additional property that the capacity potential u of (K, Ω) should satisfy the Neumann boundary condition on $\partial M \cap (\Omega - K)$ should the latter be non-empty. If M is made of a conducting material, then a physical meaning of $\text{cap}(K, \Omega)$ is

a conductivity of the piece of M between ∂K and $\partial\Omega$. Put differently, the flux of u through ∂K and $\partial\Omega$ is equal to the current through M provided the potential difference between ∂K and $\partial\Omega$ is equal to one.

Example 4.4.1. On a the model manifold M_σ with pole o and $0 < r < R$. The function

$$u(\rho, \theta) = u(\rho) = a \int_r^R \frac{d\xi}{S(\xi)} \quad (4.4.6)$$

is the capacity potential of the capacitor $(B(o, r), B(o, R))$ where the constant a is chosen to ensure $u(r) = 1$, and thus

$$\text{cap}(B(o, r), B(o, R)) = \left(\int_r^\infty \frac{d\xi}{S(\xi)} \right)^{-1} \quad (4.4.7)$$

Definition 4.4.2. Given an open set $\Omega \subset M$, we say that a function $v \geq 0$ is an *admissible subharmonic function* for Ω if it is a bounded subharmonic function on M such that $v = 0$ in $M - \Omega$ and $\sup_\Omega v > 0$. An open set Ω is called *massive* if there is at least one admissible subharmonic function for Ω . Alternatively, Ω is massive if there exists an *admissible superharmonic function* u for Ω , i.e. a bounded superharmonic function $u \geq 0$ on M such that $u \equiv 1$ outside Ω and $\inf_\Omega u = 0$.

We say that an open set is *D-massive* if there is an admissible subharmonic function v for Ω (or an admissible superharmonic function u) which has a finite Dirichlet integral:

$$|\nabla v|^2 d\mu < \infty.$$

Massiveness is an intrinsic property of Ω , despite the function v being formally defined on the whole M . The manifold itself is always massive because the constant function is an admissible subharmonic function. The empty set is always non-massive.

An open set Ω is *proper* if $\bar{\Omega} \neq M$. By the maximum principle, a proper open precompact set is never massive.

4.5 Equivalent definitions of recurrence

Manifold M is said to be *non-parabolic* if it admits a non-constant positive superharmonic function, and parabolic otherwise. The parabolicity of M turns out to be equivalent to the recurrence of the Brownian motion X_t on M .

Theorem 4.5.1. *Let M be a Riemannian manifold. The following properties are equivalent.*

1. *Brownian motion on M is transient; i.e. for some open set U and for some point $x \in M$, the process X_t eventually leaves U with a positive probability:*

$$\mathbb{P}_x\{\exists T : \forall t > T X_t \notin U\} > 0.$$

2. *For any precompact set $U \subset M$ and any point $x \in M$, the process X_t eventually leaves U with the probability 1, i.e.*

$$\mathbb{P}_x\{\exists T : \forall t > T X_t \notin U\} = 1.$$

3. *There exists a proper massive set Ω on M .*
4. *The exterior of any compact set on M is D -massive.*
5. *There exists a non-constant positive superharmonic function on M (there exists a non-constant bounded subharmonic function on M).*
6. *The Green function $G(x, y)$ on M is finite for all $x \neq y$.*

7. For all $x \in M$,

$$\int_1^\infty p(t, x, x) dt < \infty. \quad (4.5.1)$$

8. The capacity of some compact/any precompact open set is positive.

9. There exists a non-zero bounded solution on M to the equation

$$\Delta u - q(x)u = 0, \quad (4.5.2)$$

for any function $q(x) \in C_0^\infty(M)$, non-negative and not identically 0.

10. If M is a simply connected Riemann surface, then all the above are equivalent to M being of hyperbolic type; i.e. M is conformally equivalent to $\mathbb{H}^2$, two-dimensional hyperbolic space.

4.6 Sufficient conditions for parabolicity

Theorem 4.6.1. *Let M be geodesically complete. If, for some point $x \in M$,*

$$\int_1^\infty \frac{\rho d\rho}{V(x, \rho)} \quad (4.6.1)$$

where $V(x, \rho) = \mu V(B(x, \rho))$ is the volume growth function then M is parabolic, as discussed by Grigor'yan in [22].

Denote

$$S(x, \rho) = \mu'(\partial B(x, \rho)),$$

that is, $S(x, \rho)$ is the boundary area of the geodesic sphere $\partial B(x, \rho)$. Observe that $S(x, \rho) = V'(x, \rho)$, for almost all ρ . By using the capacity estimate in the proof of the previous theorem, the following result follows:

Theorem 4.6.2. *Let M be geodesically complete and, for some $x \in M$*

$$\int^{\infty} \frac{d\rho}{S(x, \rho)} = \infty. \quad (4.6.2)$$

Then M is parabolic.

This theorem was proved by Ahlfors and Nevanlinna, see Chapter 3, for Riemann surfaces. For Riemannian manifolds, it was proved by Lyons and Sullivan [36] and by Grigor'yan [20] and [21].

4.7 Transience on manifolds with a pole

In this section, we assume that M is a geodesically complete manifold with a pole o and prove a non-parabolicity test for such a manifold. As was mentioned in Section 4.1, the domain of the polar coordinates (ρ, θ) is $M - \{o\}$, and the Riemannian metric of M has the form

$$ds^2 = d\rho^2 + A_{ij}(\rho, \theta) d\theta^i d\theta^j. \quad (4.7.1)$$

Assume $\theta^1, \theta^2, \dots, \theta^{d-1}$ are the normal coordinates on $\mathbb{S}^{d-1}$. Denote $B_\rho = B(o, \rho)$. The area element on the geodesic sphere ∂B_ρ is $\sqrt{\det \|A_{ij}\|} |d\theta|$ where $|\cdot|$ is the standard volume on $\mathbb{S}^{d-1}$ and set

$$D(\rho, \theta) = \sqrt{\det \|A_{ij}\|}, \quad (4.7.2)$$

see [22] page 200.

The following result is essentially due to Doyle [12], but Grigor'yan in [22] gives the rigorous proof for any dimension.

Theorem 4.7.1. *If*

$$\int_{\mathbb{S}^{d-1}} \frac{|d\theta|}{\int_1^\infty D^{-1}(\rho, \theta) d\rho} > 0, \quad (4.7.3)$$

then the manifold M is non-parabolic.

This condition can be rewritten as

$$\text{meas} \left(\theta \in \mathbb{S}^{d-1} : \int_1^\infty \frac{d\rho}{D(\rho, \theta)} < \infty \right) > 0. \quad (4.7.4)$$

For comparison, the sufficient condition for parabolicity (4.8.2) can also be expressed via D as follows

$$\int_1^\infty \frac{d\rho}{\int_{\mathbb{S}^{d-1}} D(\rho, \theta) |d\theta|} = \infty. \quad (4.7.5)$$

Let ω be a small region on $\mathbb{S}^{d-1}$. To satisfy (4.9.4), it suffices that $D(\rho, \theta)$ is growing in ρ fast enough in a cone

$$N^\omega = \{(\rho, \theta) : \theta \in \omega\},$$

regardless of the values of D outside the cone. This reflects the nature of the transience for which it suffices that the Brownian motion has some escape route to infinity, such as the cone N . In order to satisfy (4.9.5), $D(\rho, \theta)$ should be growing relatively slowly in ρ after being averaged in θ .

The function D has especially clear meaning if $d = 2$, in which case the metric (4.9.1) is expressed directly via D as follows

$$ds^2 = d\rho^2 + D^2(\rho, \theta) d\theta^2$$

It suffices to prove the following capacity estimate

$$\text{cap}(B_r, B_R) \geq \int_{\mathbb{S}^{d-1}} \frac{|d\theta|}{\int_r^R D^{-1}(\rho, \theta) d\rho} \quad (4.7.6)$$

and the rest will follow by theorem 4.7.1.

Let ω be a region with a smooth boundary on $\mathbb{S}^{d-1}$. Consider the cone

$$N^\omega = \{(\rho, \theta) \in \mathbb{R}^d : \rho > 0 \text{ and } \theta \in \bar{\omega}\}$$

which is diffeomorphic to $\mathbb{R}_+ \times \bar{\omega}$ and can be regarded as a manifold with the boundary $\partial N^\omega = \mathbb{R}_+ \times \partial\omega$. Endow N^ω with the Riemannian metric (4.9.1) so that N^ω can be regarded as a geodesic cone in M with the vertex o .

Denote by B_r^ω "ball" $\{(\rho, \theta) \in N^\omega : \rho < r\}$.

Lemma 4.7.2. *If, for $\rho \in [r, R]$, the function $D(\rho, \theta)$ does not depend on θ , then, on the manifold N^ω ,*

$$\text{cap}(B_r^\omega, B_R^\omega) = |\omega| \left(\int_r^R \frac{d\rho}{D(\rho)} \right)^{-1}. \quad (4.7.7)$$

Proof. In [22], Grigor'yan computed the capacity $\text{cap}(B_r^\omega, B_R^\omega)$

$$\text{cap}(B(o, r), B(o, R)) = \left(\int_r^R \frac{d\xi}{S(\xi)} \right)^{-1},$$

similarly to the case of a model manifold, if take

$$S(\rho) = |\omega| D(\rho), \quad (4.7.8)$$

which is the boundary area of ∂B_ρ^ω . Indeed, by (4.6.7), the radial part of the Laplace operator has in the region $r < \rho < R$ the form

$$\Delta_{\text{radial}} = \frac{\partial^2}{\partial \rho^2} + \frac{D'}{D} \frac{\partial}{\partial \rho} = \frac{\partial^2}{\partial \rho^2} + \frac{S'}{S} \frac{\partial}{\partial \rho}.$$

Therefore, the function $u(\rho)$ defined by (4.6.6), that is,

$$u(\rho) = a \int_\rho^R \frac{d\eta}{S(\eta)}, \quad (4.7.9)$$

is harmonic in $B_R^\omega \setminus B_r^\omega$. In addition, u satisfies the Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\partial N^\omega} = 0$ because u is a constant on the sphere ∂B_ρ^ω that is orthogonal to ∂N^ω .

Furthermore, if

$$a = \left(\int_r^R \frac{d\eta}{S(\eta)} \right)^{-1},$$

then we have also $u(r) = 1$, $u(R) = 0$.

Hence, u is the capacity potential of the capacitor (B_r^ω, B_R^ω) , whence

$$\text{cap}(B_r^\omega, B_R^\omega) = -\text{flux}_{\partial B_r^\omega} u = \int_{\partial B_r^\omega} \frac{a}{S(r)} d\mu' = a = |\omega| \left(\int_r^R \frac{d\eta}{D(\eta)} \right)^{-1},$$

which was to be proved. $\square$

Lemma 4.7.3. *Let $\hat{D}(\rho)$ be a smooth positive function on $[r, R]$ such that*

$$D(\rho, \theta) \geq \hat{D}(\rho), \text{ for all } \rho \in [r, R], \theta \in \omega. \quad (4.7.10)$$

Then

$$\text{cap}(B_r^\omega, B_R^\omega) \geq |\omega| \left(\int_r^R \frac{d\rho}{\hat{D}(\rho)} \right)^{-1} \quad (4.7.11)$$

Proof. Denote by $L^\omega(r, R)$ the set of test functions for the capacitor (B_r^ω, B_R^ω) on N^ω , that is all locally Lipschitz functions ϕ on N^ω such that $0 \leq \phi \leq 1$, $\phi|_{B_r^\omega} = 1$ and $\phi = 0$ outside B_R^ω . We have

$$|\nabla \phi| \geq \left| \frac{\partial \phi}{\partial \rho} \right|^2$$

and

$$d\mu = D(\rho, \theta) |d\theta| d\rho,$$

whence

$$\begin{aligned}
\text{cap}(B_r^\omega, B_R^\omega) &= \inf_{\phi \in \mathcal{L}^\omega(r, R)} \int_{B_R^\omega} |\nabla \phi|^2 d\mu, \\
&\geq \inf_{\phi \in \mathcal{L}^\omega(r, R)} \int_{B_R^\omega} \left| \frac{\partial \phi}{\partial \rho} \right|^2 d\mu, \\
&= \inf_{\phi \in \mathcal{L}^\omega(r, R)} \int_r^R \int_\omega \left| \frac{\partial \phi}{\partial \rho} \right|^2 D(\rho, \theta) |d\theta| d\rho, \\
&= \inf_{\phi \in \mathcal{L}^\omega(r, R)} \int_r^R \int_\omega \left| \frac{\partial \phi}{\partial \rho} \right|^2 \hat{D}(\rho, \theta) |d\theta| d\rho.
\end{aligned} \tag{4.7.12}$$

Consider another metric on N^ω , which is given by

$$ds^2 = d\rho^2 + \sigma^2(\rho) d\theta^2, \tag{4.7.13}$$

where σ is a smooth positive function on $(0, \infty)$ such that $\sigma^{d-1}(\rho) = \hat{D}(\rho)$ for $\rho \in [r, R]$. Denote by N_σ^ω the cone N^ω with the metric (4.9.13). Then $\hat{D}(\rho)$ is the area density function in the region $r < \rho < R$ on N_σ^ω . It easy to see that the Euler-Lagrange equation for the functional (4.9.12) is

$$\frac{\partial}{\partial \rho} \left(\hat{D}(\rho) \frac{\partial \phi}{\partial \rho} \right) = 0$$

Its solution ϕ with the boundary conditions $\phi(r, \theta) = 1$ and $\phi(R, \theta) = 0$ is clearly independent of θ and is equal to the capacity potential of the capacitor (B_r^ω, B_R^ω) on the manifold N_σ^ω (cf. (12.9.9)). Hence, the infimum in (4.9.12) is equal to the capacity of (B_r^ω, B_R^ω) on the manifold N_σ^ω . This capacity was computed in lemma 4.9.2, whence (4.9.11) has been proved. $\square$

In order to prove (4.9.1), consider a finite family of small disjoint regions

$\omega_1, \omega_2, \dots$ on $\mathbb{S}^{d-1}$. Then

$$\begin{aligned}
\text{cap}(B_r, B_R) &= \inf_{\phi \in \mathcal{L}(B_r, B_R)} \int_{B_R} |\nabla \phi|^2 d\mu \\
&\geq \inf_{\phi \in \mathcal{L}(B_r, B_R)} \sum_i \int_{B_R \cap N^{\omega_i}} |\nabla \phi|^2 d\mu \\
&\geq \sum_i \inf_{\phi \in \mathcal{L}(B_r, B_R)} \int_{B_{\omega_i}^R} |\nabla \phi|^2 d\mu \\
&= \sum_i \text{cap}_{N^{\omega_i}}(B_r^{\omega_i}, B_R^{\omega_i}).
\end{aligned}$$

Assuming that, for any ω_i , chose a function $\hat{D}_i(\rho)$ as in (4.9.10) and applying (4.9.11) to all ω_i , obtaining

$$\text{cap}(B_r, B_R) \geq \sum_i |\omega_i| \left(\int_r^R \frac{d\rho}{\hat{D}_i(\rho)} \right)^{-1}. \quad (4.7.14)$$

If ω_i is small enough, then $\hat{D}_i(\rho)$ can be arbitrarily close to $D(\rho, \theta)$, uniformly in $\theta \in \omega_i$ and $\rho \in [r, R]$. Also, the sum on the right-hand side of (4.9.14) can be replaced by integration by refining the partition of $\mathbb{S}^{d-1}$ into ω_i . Therefore, the sum in (12.14) can be made arbitrarily close to

$$\int_{\mathbb{S}^{d-1}} \left(\int_r^R \frac{d\rho}{D(\rho, \theta)} \right)^{-1} |d\theta|,$$

whence (4.9.6) follows.

4.8 Conclusion

Theorem 4.8.1. *Let M be a two-dimensional geodesically complete manifold and $o \in M$. Let $\phi(\rho)$ be a smooth positive increasing function on $(1, \infty)$.*

(a) *Assume that, for all $x \in M$ with $\rho = \text{dist}(x, o)$ being large enough,*

$$K(x) \geq -\frac{\psi''(\rho)}{\psi(\rho)}, \quad (4.8.1)$$

and

$$\int^{\infty} \frac{dr}{\psi(r)} = \infty. \quad (4.8.2)$$

Then M is parabolic. (b) Let o be a pole. Assume that, for all ρ large enough and all $\theta \in \mathbb{S}^1$,

$$K(\rho, \theta) \leq -\frac{\psi''(\rho)}{\psi(\rho)}, \quad (4.8.3)$$

and

$$\int^{\infty} \frac{dr}{\psi(r)} < \infty. \quad (4.8.4)$$

Assume in addition that

$$\limsup_{r \rightarrow \infty} L(r) > 0, \quad (4.8.5)$$

where $L(r) = \mu'(\partial B(o, r))$. Then M is non-parabolic.

Example 4.8.2. Set $\psi(r) = r \log r$. Then the parabolicity test (4.10.1) becomes

$$K(\rho, \theta) \geq -\frac{1}{\rho^2 \log \rho}. \quad (4.8.6)$$

Set $\psi(r) = r \log^{1+\epsilon}$, $\epsilon > 0$, and assume that M is a manifold with a pole, satisfying (4.10.5). Then the non-parabolicity test (4.10.3) becomes

$$K(\rho, \theta) \leq -\frac{1+\epsilon}{\rho^2 \log \rho}. \quad (4.8.7)$$

Proof. (a) In the domain of the polar coordinates, the Riemannian metric of M has the form

$$ds^2 = d\rho^2 + \sigma^2(\rho, \theta)d\theta^2,$$

where $\sigma(\rho, \theta)$ is a smooth positive function on $\mathbb{R}_+ \times \mathbb{S}^1$. A straightforward computation of the Gauss curvature yields

$$K(\rho, \theta) = -\frac{\sigma''(\rho, \theta)}{\sigma(\rho, \theta)}, \quad (4.8.8)$$

where σ'' means $\partial^2 \sigma / \partial \rho^2$. The hypothesis (4.10.1) implies

$$\frac{\sigma''(\rho, \theta)}{\sigma(\rho, \theta)} \leq \frac{\psi''(\rho, \theta)}{\psi(\rho, \theta)}, \quad (4.8.9)$$

in the domain of the polar coordinates and for ρ large enough. Let us denote

$$L(r) = \int_{S^1} S\sigma(r, \theta) d\theta,$$

that is $L(r)$ is the length of the geodesic circle $\partial B(o, r)$ outside the cut locus $cutloc(o)$. Since the cut locus has measure zero, we have

$$L(r) = \frac{d}{dr} V(o, r)$$

The parabolicity of M will follow by Theorem 7.5 if we prove that

$$\int_{-\infty}^{\infty} \frac{dr}{L(r)} = \infty. \quad (4.8.10)$$

Integrating (15.39) in θ , pre-multiplied by $\sigma\psi$, we obtain

$$L''\psi - L\psi'' \leq 0,$$

whence

$$(L'\psi - L\psi')' \leq 0,$$

and the function

$$L'(r)\psi(r) - L(r)\psi'(r)$$

is decreasing, for large r . Therefore, it is bounded by a constant C , which yields

$$\left(\frac{L}{\psi}\right)' = \frac{L'\psi - L\psi'}{\psi^2} \leq \frac{C}{\psi^2}. \quad (4.8.11)$$

Integrating (4.10.9) from r_0 (being a large enough number) to $r > r_0$,

$$\frac{L}{\psi}(r) \leq \frac{L}{\psi}(r_0) + C \int_{r_0}^r \frac{d\xi}{\psi^2(\xi)}. \quad (4.8.12)$$

Assume first ψ satisfies

$$\int^{\infty} \frac{dr}{\psi^2(r)} < \infty. \quad (4.8.13)$$

Then the right-hand side of (4.10.12) is bounded by a constant, and we obtain $L(r) \leq \text{const } \psi(r)$. Therefore, (4.10.10) follows from the hypothesis (4.10.2). If it does not satisfy (4.10.13), that is, if

$$\int^{\infty} \frac{dr}{\psi^2(r)} = \infty. \quad (4.8.14)$$

then define the function ϕ by

$$\phi(r) = \psi(r) \int_{r_0}^r \frac{d\xi}{\psi^2(\xi)} \quad (4.8.15)$$

and rewrite (4.10.12) as

$$L(r) \leq \frac{L(r_0)}{\psi(r_0)} \psi(r) + C \phi(r). \quad (4.8.16)$$

Since (4.10.14) and (4.10.15) imply $\psi(r) = o(\phi(r))$ as $r \rightarrow \infty$, (4.10.16) yields, for r large enough, $L(r) \leq \text{const } \phi(r)$.

It is left to verify that

$$\int^{\infty} \frac{dr}{\phi(r)} = \infty. \quad (4.8.17)$$

Since ψ is increasing, assume that $c = \inf_{(r_0, \infty)} \psi > 0$, whence $\psi(r) \leq c^{-1} \psi^2(r)$. By setting $f = \frac{\phi}{\psi} = \int_r^{\infty} \frac{d\xi}{\psi(\xi)}$, obtaining

$$\int^{\infty} \frac{dr}{\psi(r)} = \int^{\infty} \frac{dr}{f(r)\psi(r)} \geq \int^{\infty} \frac{cdr}{f(r)\psi^2(r)}, \quad (4.8.18)$$

$$= c \int^{\infty} \frac{f'(r)dr}{f(r)} = c \log f(\infty) + \text{const} = \infty. \quad (4.8.19)$$

(b) In the previous notation,

$$\frac{\sigma''(\rho, \theta)}{\sigma(\rho, \theta)} \geq \frac{\psi''(\rho, \theta)}{\psi(\rho, \theta)}, \quad (4.8.20)$$

for all ρ large enough. By Theorem 4.9.1, the non-parabolicity of a manifold with a pole follows from

$$\int_{\mathbb{S}^1} \frac{d\theta}{\int_1^\infty \sigma^{-1}(r, \theta) dr} > 0, \quad (4.8.21)$$

which is equivalent to

$$\text{meas} \left\{ \theta \in \mathbb{S}^1 : \int_1^\infty \frac{dr}{\sigma(r, \theta)} < \infty \right\} > 0 \quad (4.8.22)$$

As in the previous proof, the function $\sigma'(r, \theta)\psi(r) - \psi'(r)\sigma(r, \theta)$ is increasing in r and therefore bounded below by a constant $-C$, uniformly for all $\theta \in \mathbb{S}^1$ and $r \geq R$, where R is a large enough number. This implies

$$\left(\frac{\sigma}{\psi} \right)' = \frac{\sigma'\psi - \psi'\sigma}{\psi^2} \geq -\frac{C}{\psi^2}. \quad (4.8.23)$$

The hypothesis (4.10.3) and the monotonicity of imply (4.10.13). Therefore, from (4.10.23), for all $r \geq r_0 \geq R$,

$$\frac{\sigma(r, \theta)}{\psi(r)} \geq \frac{\sigma(r_0, \theta)}{\psi(r_0)} - C \int_r^\infty \frac{d\xi}{\psi^2(\xi)}. \quad (4.8.24)$$

Defining the function ϕ by

$$\phi(r) = \psi(r) \int_r^\infty \frac{d\xi}{\psi^2(\xi)} \quad (4.8.25)$$

rewrite (4.10.24) as

$$\frac{\sigma(r, \theta)}{\psi(r)} \psi(r_0) \geq \sigma(r_0, \theta) - C\phi(r_0). \quad (4.8.26)$$

Split the set of all θ into two parts:

$$\Theta_0 = \{ \theta \in \mathbb{S}^1 : \sigma(r, \theta) < 2C\phi(r), \text{ for all } r \geq R \} \quad (4.8.27)$$

and $\Theta_1 = \mathbb{S}^1 - \Theta_0$.

If $\theta \in \Theta_1$ then, for some $r_0 \geq R$,

$$\sigma(r_0, \theta) \geq 2C\phi(r_0).$$

By (4.10.26), for any $r \geq r_0$,

$$\frac{\sigma(r, \theta)}{\psi(r)} \psi(r_0) \geq C\phi(r_0). \quad (4.8.28)$$

whence $\sigma(r, \theta) \geq \text{const}\psi(r)$ and, by (4.10.4),

$$\int_1^\infty \frac{dr}{\sigma(r, \theta)} < \infty. \quad (4.8.29)$$

It must be shown that $\text{meas}(\Theta_1) > 0$, which will imply (4.10.22).

If $\theta \in \Theta_0$ then, by (4.10.27), $\sigma(r, \theta) \leq 2C\phi(r)$, for r large enough, whence

$$\int_{\Theta_0} \sigma(r, \theta) d\theta \leq \text{const}\phi(r). \quad (4.8.30)$$

Since ψ is increasing, by (4.10.25) and (4.10.4),

$$\phi(r) \leq \int_r^\infty \frac{d\xi}{\psi(\xi)} \longrightarrow 0, \quad r \rightarrow \infty.$$

Thus, (4.10.30) and (4.10.5) imply

$$\limsup_{r \rightarrow \infty} \int_{\Theta_1} \sigma(r, \theta) d\theta > 0, \quad (4.8.31)$$

whence Θ_1 has positive measure. $\square$

Remark 4.8.1. As is clear from the proof, the hypothesis (4.10.5) can be relaxed as follows:

$$\limsup_{r \rightarrow \infty} \frac{L(r)}{\phi(r)}$$

where ϕ is defined by (4.10.25). For example, if $\psi(r) = r \log^\alpha r$, $\alpha > 1$, then $\phi(r) \asymp \log^{-\alpha} r$. Therefore, (4.10.5) can be replaced in this case by

$$\limsup_{r \rightarrow \infty} L(r) \log^\alpha r = \infty.$$

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