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# $C_4$ -face-magic toroidal labelings on $C_m \times C_n$

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## Abstract

For a graph  $G = (V, E)$  naturally embedded in the torus, let  $\mathcal{F}(G)$  denote the set of faces of  $G$ . Then,  $G$  is called a  $C_n$ -face-magic toroidal graph if there exists a bijection  $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$  such that for every  $F \in \mathcal{F}(G)$  with  $F \cong C_n$ , the sum of all the vertex labels along  $C_n$  is a constant  $S$ . Let  $x_v = f(v)$  for all  $v \in V(G)$ . We call  $\{x_v : v \in V(G)\}$  a  $C_n$ -face-magic toroidal labeling on  $G$ . We show that, for all  $m, n \geq 2$ ,  $C_m \times C_n$  admits a  $C_4$ -face-magic toroidal labeling if and only if either  $m = 2$ , or  $n = 2$ , or both  $m$  and  $n$  are even. We say that a  $C_4$ -face-magic toroidal labeling  $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$  on  $C_{2m} \times C_{2n}$  is antipodal balanced if  $x_{i,j} + x_{i+m,j+n} = \frac{1}{2}S$ , for all  $(i, j) \in V(C_{2m} \times C_{2n})$ . We show that there exists an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$  if and only if the parity of  $m$  and  $n$  are the same. Furthermore, when both  $m$  and  $n$  are even, an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$  is both row-sum balanced and column-sum balanced. In addition, when  $m = n$  is even, an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2n} \times C_{2n}$  is diagonal-sum balanced.

*Keywords:*  $C_4$ -face-magic graphs, polyomino, toroidal graphs, Cartesian products of cycles.

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## 1 Introduction

Kotzig and Rosa [11] formally introduced graph labelings in the 1970s. There are applications of graph labelings to graph decomposition problems, radar pulse code designs, X-ray crystallography and communication network models. The interested reader should read J.A. Gallian's comprehensive dynamic survey on graph labelings [8] for further investigation.

We refer the reader to Bondy and Murty [5] for concepts and notation not explicitly defined in this paper. All graphs in this paper are simple and connected. For a planar graph  $G = (V, E)$  embedded in  $\mathbb{R}^2$ , let  $\mathcal{F}(G)$  denote the set of faces of  $G$ . Then,  $G$  is called a  $C_n$ -face-magic graph if there exists a bijection  $f : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$  such that for every  $F \in \mathcal{F}(G)$  with  $F \cong C_n$ , the sum of all the vertex labels along  $C_n$  is a constant  $S$ . Here, the constant  $S$  is called a  $C_n$ -face-magic value of  $G$ . A  $C_n$ -face-magic toroidal (or cylindrical) graph  $G$  is defined similarly, where  $G$  is embedded in the torus (or cylinder), respectively.  $C_n$ -face-magic graph labelings are a special case of the more general  $(a, b, c)$ -magic labeling introduced by Lih [12]. For assorted values of  $a, b$  and  $c$ , Baca and others [1, 2, 3, 4, 9, 10, 12] have analyzed the problem for various classes of graphs. Wang [13] showed that the toroidal grid graphs  $C_m \times C_n$  are antimagic for all integers  $m, n \geq 3$ . Butt et al. [6] investigated face antimagic labelings on toroidal and Klein bottle grid graphs.

In this paper, we investigate  $C_4$ -face-magic toroidal labelings on  $C_m \times C_n$  with its natural embedding in the torus. We show that for all  $m, n \geq 2$ , there exists a  $C_4$ -face-magic toroidal labeling on  $C_m \times C_n$  if and only if either  $m = 2$ , or  $n = 2$ , or both  $m$  and  $n$  are even. In the case when  $m = n$ , we say that a  $C_4$ -face-magic toroidal labeling on  $C_{2n} \times C_{2n}$  is *torus symmetric* if the labeling is row-sum balanced, column-sum balanced and diagonal-sum balanced. Curran and Low [7] show that, up to symmetries on the torus, there are only three torus symmetric  $C_4$ -face-magic toroidal labelings on  $C_4 \times C_4$ . See Theorem 3.5 in Section 3 for details. In this paper, we search for  $C_4$ -face-magic toroidal labelings on  $C_{2m} \times C_{2n}$  that are row-sum balanced and column-sum balanced. This investigation leads naturally to the concept of an antipodal balanced labeling. We say that a  $C_4$ -face-magic toroidal labeling  $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$  on  $C_{2m} \times C_{2n}$  is *antipodal balanced* if  $x_{i,j} + x_{i+m,j+n} = \frac{1}{2}S$ , for all  $(i, j) \in V(C_{2m} \times C_{2n})$ . We show that there exists an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$  if and only if the parity of  $m$  and  $n$  are the same. Furthermore, when  $m = n$  is even, we show that any antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2n} \times C_{2n}$  is torus symmetric.

## 2 Preliminaries

**Theorem 2.1.** *Let  $m, n \geq 2$ . Then,  $P_m \times P_n$  is  $C_4$ -face-magic.*

*Proof.* Label the vertex set of  $P_m \times P_n$  as

$$V(P_m \times P_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and its edge set as

$$E(P_m \times P_n) = \{(i, j), (i+1, j)\} : 1 \leq i < m, 1 \leq j \leq n\} \\ \cup \{(i, j), (i, j+1)\} : 1 \leq i \leq m, 1 \leq j < n\}.$$

We will determine a label  $x_{i,j}$  for each vertex  $(i, j) \in V(P_m \times P_n)$  and check that this provides a  $C_4$ -face-magic labeling.

**Case 1.** Assume  $m - n$  is even. Color the vertex  $(i, j)$  white if  $i + j$  is even and black if  $i + j$  is odd. Note that the vertices  $(1, 1)$  and  $(m, n)$  are white. Let  $x_{i,j} = i + m(j - 1)$  for each white vertex  $(i, j)$ , and  $x_{i,j} = (m - i + 1) + m(n - j)$  for each black vertex  $(i, j)$ . An equivalent definition for  $\{x_{i,j} : (i, j) \in V(P_m \times P_n)\}$  would be to write the number  $i + m(j - 1)$  in each cell  $(i, j)$ , and then rotate the black vertices 180 degrees about the center of the board.

Let  $C_{i,j} = \{(i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1)\}$ . If  $(i, j)$  is white, then the two 4-cycles  $C_{i,j}$  and  $C_{i+1,j}$  have the same face sum, since  $x_{i,j} = x_{i+2,j} - 2$  and  $x_{i,j+1} = x_{i+2,j+1} + 2$ . If  $(i, j)$  is black, then the two 4-cycles  $C_{i,j}$  and  $C_{i+1,j}$  have the same face sum, since  $x_{i,j} = x_{i+2,j} + 2$  and  $x_{i,j+1} = x_{i+2,j+1} - 2$ . A similar proof for  $C_{i,j}$  and  $C_{i,j+1}$  shows that  $\{x_{i,j}\}$  is  $C_4$ -face-magic. The sum on each face must be the sum on  $C_{1,1}$ , which is  $1 + (mn - 1) + (m + 2) + m(n - 1) = 2mn + 2$ . This completes Case 1.

**Case 2.** Without loss of generality, we may assume that  $m$  is even and  $n$  is odd. Let  $m = 2m_1$  and  $n = 2n_1 - 1$  for some positive integers  $m_1$  and  $n_1$ . Again, color vertex  $(i, j)$  white if  $i + j$  is even and black if  $i + j$  is odd.

We first label the white vertices. Let  $x_{i,j} = m(j - 1) + i$  if both  $i$  and  $j$  are odd, and  $x_{i,j} = m(j - 1) + i - 1$  if both  $i$  and  $j$  are even. We observe that  $x_{2k-1,2\ell-1} = m(2\ell - 2) + 2k - 1$  for all  $1 \leq k \leq m_1$  and  $1 \leq \ell \leq n_1$ , and  $x_{2k,2\ell} = m(2\ell - 1) + 2k - 1$  for all  $1 \leq k \leq m_1$  and  $1 \leq \ell < n_1$ . Thus each odd label  $1, 3, 5, \dots, mn - 1$  is used exactly once on a white vertex.

Next, we label the black vertices. Let  $x_{i,j} = m(n - j + 1) - i + 1$  if  $i$  is odd and  $j$  is even, and  $x_{i,j} = m(n - j + 1) - i + 2$  if  $i$  is even and  $j$  is odd. We observe that whenever vertex  $(i, j)$  is white, then vertex  $(m - i + 1, n - j + 1)$  is black and  $x_{m-i+1, n-j+1} = x_{i,j} + 1$ . Thus each even label  $2, 4, 6, \dots, mn$  is used exactly once on a black vertex.

Let  $C_{i,j} = \{(i, j), (i + 1, j), (i + 1, j + 1), (i, j + 1)\}$ . If  $(i, j)$  is white, then  $x_{i+2,j} = x_{i,j} + 2$  and  $x_{i,j+2} = x_{i,j} + 2m$ . If  $(i, j)$  is black, then  $x_{i+2,j} = x_{i,j} - 2$  and  $x_{i,j+2} = x_{i,j} - 2m$ . An argument similar to that in Case (i) shows that each 4-cycle  $C_{i,j}$  has the same face sum as  $C_{1,1}$ , which is  $1 + mn + (m + 1) + m(n - 1) = 2mn + 2$ . This completes Case 2.  $\square$

**Lemma 2.2.** *Let  $m$  and  $n$  be positive integers. A  $C_4$ -face-magic labeling on  $P_{2m} \times P_{2n}$  always yields a  $C_4$ -face-magic labeling on  $C_{2m} \times C_{2n}$  with its natural embedding in the torus. Furthermore, the  $C_4$ -face-magic value is  $S = 2(4mn + 1)$ .*

*Proof.* Let  $x_{i,j}$  be the  $C_4$ -face-magic labeling on vertex  $(i, j)$ , for  $i = 1, 2, \dots, 2m$  and  $j = 1, 2, \dots, 2n$ . Since  $S$  is the  $C_4$ -face-magic value on  $P_{2m} \times P_{2n}$ , we have  $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = S$ , for all  $i = 1, 2, \dots, 2m - 1$  and  $j = 1, 2, \dots, 2n - 1$ . We observe that

$$mnS = \sum_{i=1}^m \sum_{j=1}^n (x_{2i-1,2j-1} + x_{2i,2j-1} + x_{2i-1,2j} + x_{2i,2j}) = \sum_{i=1}^{4mn} i = \frac{1}{2}(4mn)(4mn + 1).$$

Thus,  $S = 2(4mn + 1)$ . Since  $x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i+1,j} + x_{i+2,j} + x_{i+1,j+1} + x_{i+2,j+1}$ , we have  $x_{i,j} + x_{i,j+1} = x_{i+2,j} + x_{i+2,j+1}$ . An induction argument shows that  $x_{i,j} + x_{i,j+1} = x_{i+2k,j} + x_{i+2k,j+1}$ . Since  $x_{i+2k,j} + x_{i+2k,j+1} + x_{i+2k+1,j} + x_{i+2k+1,j+1} = S$ , we have  $x_{i,j} + x_{i,j+1} + x_{i+2k+1,j} + x_{i+2k+1,j+1} = S$ .

A similar argument shows that  $x_{i,j} + x_{i+1,j} + x_{i,j+2\ell+1} + x_{i+1,j+2\ell+1} = S$ . This yields  $x_{i,j} + x_{i+2k+1,j} + x_{i,j+2\ell+1} + x_{i+2k+1,j+2\ell+1} = S$ . Hence, we have  $x_{1,j} + x_{1,j+1} + x_{2m,j} + x_{2m,j+1} = S$ , for all  $j = 1, 2, \dots, 2n-1$ . Similarly, we have  $x_{i,1} + x_{i+1,1} + x_{i,2n} + x_{i+1,2n} = S$ , for all  $i = 1, 2, \dots, 2m-1$ . Lastly, we have  $x_{1,1} + x_{2m,1} + x_{1,2n} + x_{2m,2n} = S$ . Therefore, the  $C_4$ -face-magic labeling on  $P_{2m} \times P_{2n}$  yields a  $C_4$ -face-magic labeling on  $C_{2m} \times C_{2n}$  with its natural embedding in the torus.  $\square$

**Lemma 2.3.** *Let  $m$  and  $n$  be integers such that  $m \geq 3$  and  $n \geq 2$ . Suppose  $P_m \times C_n$  is a  $C_4$ -face-magic cylindrical graph with the natural embedding of  $P_m \times C_n$  on the cylinder. Then,  $n$  is even.*

*Proof.* For the purpose of contradiction, suppose  $n$  is odd, and let  $n = 2n_1 + 1$  for some positive integer  $n_1$ . Label the vertex set of  $P_m \times C_n$  as

$$V(P_m \times C_n) = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

and its edge set as

$$E(P_m \times C_n) = \{(i, j), (i + 1, j) : 1 \leq i < m, 1 \leq j \leq n\} \cup \{(i, j), (i, j + 1)\}, \{(i, n), (i, 1)\} : 1 \leq i \leq m, 1 \leq j < n\}.$$

Let  $\{x_{i,j} : (i, j) \in V(P_m \times C_n)\}$  be a  $C_4$ -face-magic labeling on  $P_m \times C_n$  with  $C_4$ -face-magic value  $S$ . Let  $S_i = x_{i,1} + x_{i+1,1}$ , for  $i = 1, 2$ . Equating the following  $C_4$ -face sums to each other:

$$x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = S = x_{i,j+1} + x_{i+1,j+1} + x_{i,j+2} + x_{i+1,j+2},$$

we obtain  $x_{i,j} + x_{i+1,j} = x_{i,j+2} + x_{i+1,j+2}$ , where the index  $j$  is taken modulo  $n$ . Thus,

$$x_{i,1} + x_{i+1,1} = x_{i,2j+1} + x_{i+1,2j+1} \quad \text{and} \\ x_{i,2} + x_{i+1,2} = x_{i,2j} + x_{i+1,2j}, \quad \text{for } j = 1, 2, \dots, n_1.$$

Also, we have  $x_{i,n-1} + x_{i+1,n-1} = x_{i,n+1} + x_{i+1,n+1} = x_{i,1} + x_{i+1,1}$ . Hence,  $S_i = x_{i,j} + x_{i+1,j}$ , for all  $i = 1, 2$  and  $j = 1, 2, \dots, n$ . From the  $C_4$ -face sum

$$S = (x_{i,j} + x_{i+1,j}) + (x_{i,j+1} + x_{i+1,j+1}) = S_i + S_i = 2S_i,$$

we have  $S_i = \frac{1}{2}S$ . Hence,  $x_{1,1} + x_{2,1} = \frac{1}{2}S = x_{2,1} + x_{3,1}$ , which in turn, implies that  $x_{1,1} = x_{3,1}$ . This is a contradiction. Therefore,  $n$  is even.  $\square$

**Proposition 2.4.** *Let  $m$  be an integer where  $m \geq 2$ . Then, there is a  $C_4$ -face-magic toroidal labeling on  $C_m \times C_2$ .*

*Proof.* Let  $x_{i,1} = i$  and  $x_{i,2} = 2m+1-i$ , for  $i = 1, 2, \dots, m$ . Then,  $x_{i,1} + x_{i,2} = 2m+1$ , for  $i = 1, 2, \dots, m$ . Thus,  $x_{i,1} + x_{i,2} + x_{i+1,1} + x_{i+1,2} = 2(2m+1)$ , for  $i = 1, 2, \dots, m$ . Hence,  $\{x_{i,j} : (i, j) \in V(C_m \times C_2)\}$  is a  $C_4$ -face-magic toroidal labeling on  $C_m \times C_2$ .  $\square$

**Proposition 2.5.** *Let  $m$  and  $n$  be integers where  $m, n \geq 2$ . Then,  $C_m \times C_n$  has a  $C_4$ -face-magic toroidal labeling if and only if either  $m = 2$ , or  $n = 2$ , or both  $m$  and  $n$  are even.*

*Proof.* ( $\Rightarrow$ ) Suppose  $C_m \times C_n$  has a  $C_4$ -face-magic toroidal labeling. If either  $m = 2$  or  $n = 2$ , we are done. So assume that  $m, n \geq 3$ . The  $C_4$ -face-magic toroidal labeling on  $C_m \times C_n$  is simultaneously a  $C_4$ -face-magic cylindrical labeling on both  $C_m \times P_n$  and  $P_m \times C_n$ . By Lemma 2.3, both  $m$  and  $n$  are even.

( $\Leftarrow$ ) Suppose either  $m = 2$ , or  $n = 2$ , or both  $m$  and  $n$  are even. On the one hand, if  $m = 2$  or  $n = 2$ , by Proposition 2.4,  $C_m \times C_n$  has a  $C_4$ -face-magic toroidal labeling. On the other hand, if both  $m$  and  $n$  are even, by Theorem 2.1,  $P_m \times P_n$  has a  $C_4$ -face-magic labeling. By Lemma 2.2, the  $C_4$ -face-magic labeling on  $P_m \times P_n$  yields a  $C_4$ -face-magic toroidal labeling on  $C_m \times C_n$ .  $\square$

Throughout this paper, if  $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$  is a labeling on  $C_{2m} \times C_{2n}$ , then for convenience we consider the index  $i$  modulo  $2m$  and the index  $j$  modulo  $2n$ .

**Definition 2.6.** We say that the  $C_4$ -face-magic torus labeling  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$  on  $C_{2m} \times C_{2n}$  is *antipodal balanced* if  $x_{i,j} + x_{i+m,j+n} = 4mn + 1$ , for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ .

**Remark 2.7.** We give a brief explanation for the term *antipodal balanced*. On the  $n$ -sphere  $S^n \subseteq \mathbb{R}^{n+1}$ , the antipodal map  $p : S^n \rightarrow S^n$  is given by  $p(\mathbf{x}) = -\mathbf{x}$ . Similarly, on the torus  $T^2 = S^1 \times S^1 \subseteq \mathbb{C}^2 \cong \mathbb{R}^4$ , we define the antipodal map  $p : T^2 \rightarrow T^2$  by  $p(e^{i\theta_1}, e^{i\theta_2}) = -(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\theta_1+\pi)}, e^{i(\theta_2+\pi)})$ . Thus an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$  is one in which the sum of the labels at a vertex and its antipodal vertex is constant for all vertices in  $C_{2m} \times C_{2n}$ .

**Lemma 2.8.** Let  $m$  and  $n$  be positive integers. Let  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . For all integers  $i$  where  $1 \leq i \leq m$ , we define

$$d_i = x_{i(m-1)+1, in+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

Then for all integers  $i$  and  $j$  where  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1} d_i.$$

*Proof.* By the definition of  $d_i$ , we have

$$x_{i(m-1)+1, in+1} = x_{(i-1)(m-1)+1, (i-1)n+1} + d_i.$$

We apply an induction argument on  $j$ . Thus we assume that

$$x_{i(m-1)+1, in+j-1} = x_{(i-1)(m-1)+1, (i-1)n+j-1} + (-1)^j d_i. \quad (2.1)$$

Since the labeling is antipodal balanced, we have

$$x_{i(m-1)+2, in+j-1} = \frac{1}{2}S - x_{(i-1)(m-1)+1, (i-1)n+j-1}, \text{ and} \quad (2.2)$$

$$x_{i(m-1)+2, in+j} = \frac{1}{2}S - x_{(i-1)(m-1)+1, (i-1)n+j}. \quad (2.3)$$

Since  $\{x_{i,j}\}$  is a  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ , we have

$$x_{i(m-1)+1, in+j-1} + x_{i(m-1)+1, in+j} + x_{i(m-1)+2, in+j-1} + x_{i(m-1)+2, in+j} = S. \quad (2.4)$$

When we substitute the expressions from equations (2.1), (2.2) and (2.3) into equation (2.4), we obtain

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i.$$

This completes the proof. □

We next show that if  $C_{2m} \times C_{2n}$  has an antipodal balanced  $C_4$ -face-magic toroidal labeling, then the parity of  $m$  and  $n$  are the same.

**Lemma 2.9.** *Let  $m$  and  $n$  be positive integers. Let  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . Then, the parity of  $m$  and  $n$  are the same.*

*Proof.* We may assume that  $m$  is even and  $n$  is odd. Let  $\{x_{i,j}\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . We will show that this leads to a contradiction. For all integers  $i$  such that  $1 \leq i \leq m$ , we define

$$d_i = x_{i(m-1)+1, in+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

By Lemma 2.8, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i.$$

Fix  $j$  such that  $1 \leq j \leq 2n$ . The equations

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i \text{ for } i = 1, 2, \dots, m,$$

yield

$$x_{m+1, j} = x_{m(m-1)+1, mn+j} = x_{1, j} + (-1)^{j+1}(d_1 + d_2 + \dots + d_m). \tag{2.5}$$

Setting  $j = 1$ , equation (2.5) becomes

$$x_{m+1, 1} = x_{1, 1} + (d_1 + d_2 + \dots + d_m). \tag{2.6}$$

Setting  $j = n + 1$ , equation (2.5) becomes

$$x_{m+1, n+1} = x_{1, n+1} - (d_1 + d_2 + \dots + d_m). \tag{2.7}$$

Because  $\{x_{i,j}\}$  is antipodal balanced, we have

$$x_{1, 1} + x_{m+1, n+1} = \frac{1}{2}S = x_{m+1, 1} + x_{1, n+1}. \tag{2.8}$$

Substituting equations (2.6) and (2.7) into equation (2.8) and simplifying yields

$$d_1 + d_2 + \dots + d_m = 0. \tag{2.9}$$

This implies that  $x_{m+1, j} = x_{1, j}$ , for all  $j = 1, 2, \dots, 2n$ . This is a contradiction. □

We next investigate how the conditions of Lemma 2.8 apply to the case  $C_{2m} \times C_{2n}$  where both  $m$  and  $n$  are even.

**Lemma 2.10.** *Let  $m$  and  $n$  be positive even integers. Let  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . For all integers  $i$  such that  $1 \leq i \leq m$ , we define*

$$d_i = x_{i(m-1)+1, in+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

*Then by Lemma 2.8, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have*

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1} d_i.$$

*Also, for all integers  $i$  and  $j$  such that  $m+1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ , we have*

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^j d_{i-m}.$$

*For all integers  $j$  such that  $1 \leq j \leq n$ , we define*

$$d'_j = x_{jm+1, j(n-1)+1} - x_{(j-1)m+1, (j-1)(n-1)+1}.$$

*Then for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq n$ , we have*

$$x_{jm+i, j(n-1)+1} = x_{(j-1)m+i, (j-1)(n-1)+1} + (-1)^{i+1} d'_j.$$

*Also, for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $n+1 \leq j \leq 2n$ , we have*

$$x_{jm+i, j(n-1)+1} = x_{(j-1)m+i, (j-1)(n-1)+1} + (-1)^i d'_{j-n}.$$

*Proof.* By Lemma 2.8, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1} d_i. \quad (2.10)$$

Since the indices  $in+j$  and  $(i-1)n+j$  are reduced modulo  $2n$ , equation (2.10) holds for all integers  $j$ . Let  $i$  be an integer such that  $m+1 \leq i \leq 2m$ . We replace  $i$  with  $i-m$  and  $j$  with  $n+j$  in equation (2.10) to obtain

$$x_{(i-m)(m-1)+1, (i-m)n+n+j} = x_{(i-m-1)(m-1)+1, (i-m-1)n+n+j} + (-1)^{n+j+1} d_{i-m}.$$

This reduces to

$$x_{i(m-1)+m+1, (i+1)n+j} = x_{(i-1)(m-1)+m+1, in+j} + (-1)^{j+1} d_{i-m}. \quad (2.11)$$

Since  $\{x_{i,j}\}$  is antipodal balanced, we have

$$x_{i(m-1)+m+1, (i+1)n+j} = \frac{1}{2}S - x_{i(m-1)+1, in+j} \quad \text{and} \quad (2.12)$$

$$x_{(i-1)(m-1)+m+1, in+j} = \frac{1}{2}S - x_{(i-1)(m-1)+1, (i-1)n+j}. \quad (2.13)$$

When we substitute the expressions in equations (2.12) and (2.13) into equation (2.11), we have, for all integers  $i$  and  $j$  such that  $m+1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ ,

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^j d_{i-m}. \quad (2.14)$$

When we interchange the roles of  $i$  and  $j$  in the previous argument, when  $1 \leq i \leq 2m$  and  $1 \leq j \leq n$ , we have

$$x_{jm+i, j(n-1)+1} = x_{(j-1)m+i, (j-1)(n-1)+1} + (-1)^{i+1} d'_j;$$

and when  $1 \leq i \leq 2m$  and  $n+1 \leq j \leq 2n$ , we have

$$x_{jm+i, j(n-1)+1} = x_{(j-1)m+i, (j-1)(n-1)+1} + (-1)^i d'_{j-n}.$$

□



We next investigate how the conditions of Lemma 2.8 apply to the case  $C_{2m} \times C_{2n}$  where both  $m$  and  $n$  are odd.

**Lemma 2.11.** *Let  $m$  and  $n$  be positive odd integers. Let  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . For all integers  $i$  such that  $1 \leq i \leq m$ , we define*

$$d_i = x_{i(m-1)+1, in+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

Then, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$\begin{aligned} x_{i(m-1)+1, in+j} &= x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i \text{ and} \\ x_{i(m-1)+m+1, in+j} &= x_{(i-1)(m-1)+m+1, (i-1)n+j} + (-1)^{j+1}d_i. \end{aligned}$$

For all integers  $i$  such that  $1 \leq j \leq n$ , we define

$$d'_j = x_{jm+1, j(n-1)+1} - x_{(j-1)m+1, (j-1)(n-1)+1}.$$

Then, for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq n$ , we have

$$\begin{aligned} x_{jm+i, j(n-1)+1} &= x_{(j-1)m+i, (j-1)(n-1)+1} + (-1)^{i+1}d'_j \text{ and} \\ x_{jm+i, j(n-1)+n+1} &= x_{(j-1)m+i, (j-1)(n-1)+n+1} + (-1)^i d'_j. \end{aligned}$$

*Proof.* By Lemma 2.8, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i. \tag{2.15}$$

Because  $in + j$  and  $(i - 1)n + j$  are reduced modulo  $2n$ , equation (2.15) holds for all integers  $j$ . Because  $\{x_{i,j}\}$  is antipodal balanced, we have

$$x_{i(m-1)+1, in+j} = \frac{1}{2}S - x_{i(m-1)+m+1, (i+1)n+j} \text{ and} \tag{2.16}$$

$$x_{(i-1)(m-1)+1, (i-1)n+j} = \frac{1}{2}S - x_{(i-1)(m-1)+m+1, in+j}. \tag{2.17}$$

When we substitute the expressions from equations (2.16) and (2.17) into equation (2.15), we obtain

$$x_{i(m-1)+m+1, (i+1)n+j} = x_{(i-1)(m-1)+m+1, in+j} + (-1)^j d_i. \tag{2.18}$$

When we replace  $j$  with  $j + n$  in equation (2.18), we have, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ ,

$$x_{i(m-1)+m+1, in+j} = x_{(i-1)(m-1)+m+1, (i-1)n+j} + (-1)^{j+1}d_i.$$

When we interchange the roles of  $i$  and  $j$  in the above argument, we have for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq n$ ,

$$\begin{aligned} x_{jm+i, j(n-1)+1} &= x_{(j-1)m+i, (j-1)(n-1)+1} + (-1)^{i+1}d'_j \text{ and} \\ x_{jm+i, j(n-1)+n+1} &= x_{(j-1)m+i, (j-1)(n-1)+n+1} + (-1)^i d'_j. \end{aligned}$$

□

### 3 Results on $C_4 \times C_4$

Curran and Low [7] determine all antipodal balanced  $C_4$ -face-magic toroidal labelings on  $C_4 \times C_4$  (up to symmetries on a torus). In order to state this result precisely, we must introduce some definitions. This result, as stated in Theorem 3.5, is the basis for the investigation of antipodal balanced  $C_4$ -face-magic toroidal labelings on  $C_{2m} \times C_{2n}$  in this paper.

**Definition 3.1.** Let  $n$  be a positive integer. Let  $\{x_{i,j} : i, j = 1, 2, \dots, 2n\}$  be a  $C_4$ -face-magic torus labeling on  $C_{2n} \times C_{2n}$ . We say that the  $C_4$ -face-magic labeling  $\{x_{i,j}\}$  on  $C_{2n} \times C_{2n}$  is *torus symmetric* if all row sums, column sums, and diagonal sums have a constant value  $S'$ . In other words, the sums

$$R_i = \sum_{j=1}^{2n} x_{i,j} = S' \quad \text{for all } i = 1, 2, \dots, 2n,$$

$$C_j = \sum_{i=1}^{2n} x_{i,j} = S' \quad \text{for all } j = 1, 2, \dots, 2n,$$

$$D_j = \sum_{i=1}^{2n} x_{i,i+j} = S' \quad \text{for all } j = 1, 2, \dots, 2n \text{ and}$$

$$D'_j = \sum_{i=1}^{2n} x_{i,j-i} = S' \quad \text{for all } j = 1, 2, \dots, 2n,$$

are constant.

According to Lemma 3.2, a torus symmetric  $C_4$ -face-magic toroidal labeling on  $C_4 \times C_4$  is equivalent to a  $C_4$ -face-magic toroidal labeling on  $C_4 \times C_4$  in which the four  $2 \times 2$  block sums given by  $B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2}$ , for all  $i, j = 1, 2$ , also add up to the  $C_4$ -face-magic value 34.

**Lemma 3.2** ([7], Lemma 8). *Consider the system of linear equations  $x_{i,j} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} = 34$  (S1), for all  $i = 1, 2, 3$  and  $j = 1, 2, 3$  for a  $C_4$ -face-magic labeling on  $P_4 \times P_4$ . Let (S2) be the system S1 together with the equations  $B_{i,j} = x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = 34$ , for all  $i, j = 1, 2$ . If the labeling  $\{x_{i,j}\}$  satisfies system (S2), then  $\{x_{i,j}\}$  is torus symmetric. Also, let (S3) be the system (S1) together with the equations  $R_1 = x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 34$ ,  $C_1 = x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} = 34$ , and  $D_4 = x_{1,1} + x_{2,2} + x_{3,3} + x_{4,4} = 34$ . Then, (S2) is equivalent to (S3).*

**Definition 3.3.** Consider the natural embedding of  $C_{2n} \times C_{2n}$  in the torus. We say that two torus symmetric  $C_4$ -face-magic toroidal labelings on  $C_{2n} \times C_{2n}$  are *torus equivalent* if there is a homeomorphism of the torus that maps  $C_{2n} \times C_{2n}$  onto itself such that the first  $C_4$ -face-magic toroidal labeling on  $C_{2n} \times C_{2n}$  is mapped to the second  $C_4$ -face-magic toroidal labeling on  $C_{2n} \times C_{2n}$ .

By Lemma 3.4, a torus symmetric  $C_4$ -face-magic toroidal labeling on  $C_4 \times C_4$  is antipodal balanced.

**Lemma 3.4** ([7], Lemma 13). *Let  $\{x_{i,j}\}$  be a torus symmetric  $C_4$ -face-magic toroidal labeling on  $C_4 \times C_4$ . Then, for all  $i$  and  $j$ , we have  $x_{i,j} + x_{i+2,j+2} = 17$  where the indices are taken modulo four. In other words, the labeling  $\{x_{i,j}\}$  is antipodal balanced.*

All torus symmetric  $C_4$ -face-magic labelings on  $C_4 \times C_4$  (up to torus equivalence) are given in the following theorem.

**Theorem 3.5** ([7], Theorem 10). *There are three distinct torus nonequivalent torus symmetric  $C_4$ -face-magic toroidal labelings on  $C_4 \times C_4$ . These three distinct torus nonequivalent torus symmetric  $C_4$ -face-magic toroidal labelings on  $C_4 \times C_4$  are given below:*

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Table 1: Torus symmetric  $C_4$ -face-magic toroidal labeling A on  $C_4 \times C_4$ .

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

Table 2: Torus symmetric  $C_4$ -face-magic toroidal labeling B on  $C_4 \times C_4$ .

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

Table 3: Torus symmetric  $C_4$ -face-magic toroidal labeling C on  $C_4 \times C_4$ .

An interesting observation about these three labelings is that the row sums and column sums of any two labelings are the 16  $C_4$ -face sums in the third labeling. In the remark below, we indicate how the three torus nonequivalent torus symmetric  $C_4$ -face-magic toroidal labelings in Theorem 3.5 can be regarded as coming from one particular labeling on  $C_4 \times C_4$ .

**Remark 3.6** ([7], Remark 24). Label the vertices of  $C_4 \times C_4$  with the elements from the set  $\{0, 1\}^4$  so that the labelings on each  $C_4$  face adds up to  $(2, 2, 2, 2)$ . This labeling is given in Table 4. Then the corresponding  $C_4$ -face-magic torus labelings on  $C_4 \times C_4$  are given by  $x_{i,j} = x_{1,1} + a_1d_1 + a_2d_2 + a_3d_3 + a_4d_4$  where  $x_{1,1} = 1$ ,  $(a_1, a_2, a_3, a_4)$  is the labeling on vertex  $(i, j)$  in  $C_4 \times C_4$  given in Table 4, and  $(d_1, d_2, d_3, d_4)$  is one of the three choices of either  $(1, 2, 4, 8)$ ,  $(1, 4, 2, 8)$  or  $(1, 8, 2, 4)$ . The choices of  $(1, 2, 4, 8)$ ,  $(1, 4, 2, 8)$  or  $(1, 8, 2, 4)$  for  $(d_1, d_2, d_3, d_4)$  result in the labelings A, B and C, respectively, in Theorem 3.5.

(0, 0, 0, 0)	(1, 1, 1, 0)	(0, 0, 1, 1)	(1, 1, 0, 1)
(1, 0, 1, 1)	(0, 1, 0, 1)	(1, 0, 0, 0)	(0, 1, 1, 0)
(1, 1, 0, 0)	(0, 0, 1, 0)	(1, 1, 1, 1)	(0, 0, 0, 1)
(0, 1, 1, 1)	(1, 0, 0, 1)	(0, 1, 0, 0)	(1, 0, 1, 0)

Table 4:  $C_4$ -face-magic toroidal labeling with elements from  $\{0, 1\}^4$  on  $C_4 \times C_4$ .

## 4 Results on $C_{2m} \times C_{2n}$

We first consider antipodal balanced  $C_4$ -face-magic toroidal labelings on  $C_6 \times C_6$ .

**Lemma 4.1.** *Let  $\{x_{i,j}\}$  be an antipodal balanced toroidal  $C_4$ -face-magic labeling on  $C_6 \times C_6$ . Let  $d_i = x_{2i+1,3i+1} - x_{2i-1,3i-2}$ , for  $i = 1, 2, 3$ , and let  $d_{j+3} = d'_j = x_{3j+1,2j+1} - x_{3j-2,2j-2}$ , for  $j = 1, 2, 3$ . Then the value of  $x_{i,j}$  is determined by the values of  $x_{1,1}$  and  $d_i$ , for  $i = 1, 2, 3, 4, 5, 6$  as displayed in Table 5.*

*Proof.* By Lemma 2.11, for integers  $i$  and  $j$ , we have

$$x_{2i+1,3i+j} = x_{2i-1,3i-3+j} + (-1)^{j+1}d_i,$$

when  $1 \leq i \leq 3$  and  $1 \leq j \leq 6$  and

$$x_{2i+4,3i+j} = x_{2i+2,3i-3+j} + (-1)^{j+1}d_i,$$

when  $1 \leq i \leq 3$  and  $1 \leq j \leq 6$ . Thus  $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$ . Furthermore, it suffices to establish the result for the expressions  $x_{1,j}$  and  $x_{4,j}$ , for  $j = 1, 2, 3$ . Also, by Lemma 4.1, for integers  $i$  and  $j$ , we have

$$x_{3j+i,2j+1} = x_{3j-3+i,2j-1} + (-1)^{i+1}d_{j+3},$$

when  $1 \leq i \leq 6$  and  $1 \leq j \leq 3$  and

$$x_{3j+i,2j+4} = x_{3j-3+i,2j+2} + (-1)^i d_{j+3},$$

when  $1 \leq i \leq 6$  and  $1 \leq j \leq 3$ . Thus  $x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$  and  $x_{4,3} = x_{1,1} + d_4$ . From  $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$  and  $x_{1,4} = x_{4,2} - d_6$ , we have  $x_{4,2} = x_{1,1} + d_1 + d_2 + d_3 + d_6$ . From  $x_{4,1} = x_{1,1} + d_4 + d_5 + d_6$  and  $x_{1,3} = x_{4,1} - d_4$ , we have  $x_{1,1} = x_{1,1} + d_5 + d_6$ . Lastly, from  $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3$ ,  $x_{6,3} = x_{3,1} + d_4$ ,  $x_{3,5} = x_{6,3} + d_5$  and  $x_{1,2} = x_{3,5} - d_1$ , we have  $x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$ .  $\square$

We next consider antipodal balanced  $C_4$ -face-magic toroidal labelings on  $C_8 \times C_8$ .

**Lemma 4.2.** *Let  $\{x_{i,j}\}$  be an antipodal balanced toroidal  $C_4$ -face-magic labeling on  $C_8 \times C_8$ . Let  $d_i = x_{3i+1,4i+1} - x_{3i-2,4i-3}$ , for  $i = 1, 2, 3, 4$ , and let  $d_{j+4} = d'_j = x_{4j+1,3j+1} - x_{4j-3,3j-2}$ , for  $j = 1, 2, 3, 4$ . Then the value of  $x_{i,j}$  is determined by the values of  $x_{1,1}$  and  $d_i$ , for  $i = 1, 2, 3, 4, 5, 6, 7, 8$  as displayed in Figure 1.*

$x_{1,1}$	$x_{1,2} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_5$	$x_{1,3} =$ $x_{1,1} + d_5$ $+d_6$	$x_{1,4} =$ $x_{1,1} + d_1$ $+d_2 + d_3$	$x_{1,5} =$ $x_{1,1} + d_4$ $+d_5$	$x_{1,6} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_5 + d_6$
$x_{2,1} =$ $x_{1,1} + d_1$ $+d_2 + d_4$ $+d_5 + d_6$	$x_{2,2} =$ $x_{1,1} + d_3$ $+d_6$	$x_{2,3} =$ $x_{1,1} + d_1$ $+d_2 + d_4$	$x_{2,4} =$ $x_{1,1} + d_3$ $+d_4 + d_5$ $+d_6$	$x_{2,5} =$ $x_{1,1} + d_1$ $+d_2 + d_6$	$x_{2,6} =$ $x_{1,1} + d_3$ $+d_4$
$x_{3,1} =$ $x_{1,1} + d_2$ $+d_3$	$x_{3,2} =$ $x_{1,1} + d_1$ $+d_4 + d_5$	$x_{3,3} =$ $x_{1,1} + d_2$ $+d_3 + d_5$ $+d_6$	$x_{3,4} =$ $x_{1,1} + d_1$	$x_{3,5} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_5$	$x_{3,6} =$ $x_{1,1} + d_1$ $+d_5 + d_6$
$x_{4,1} =$ $x_{1,1} + d_4$ $+d_5 + d_6$	$x_{4,2} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_6$	$x_{4,3} =$ $x_{1,1} + d_4$	$x_{4,4} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_5 + d_6$	$x_{4,5} =$ $x_{1,1} + d_6$	$x_{4,6} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4$
$x_{5,1} =$ $x_{1,1} + d_1$ $+d_2$	$x_{5,2} =$ $x_{1,1} + d_3$ $+d_4 + d_5$	$x_{5,3} =$ $x_{1,1} + d_1$ $+d_2 + d_5$ $+d_6$	$x_{5,4} =$ $x_{1,1} + d_3$	$x_{5,5} =$ $x_{1,1} + d_1$ $+d_2 + d_4$ $+d_5$	$x_{5,6} =$ $x_{1,1} + d_3$ $+d_5 + d_6$
$x_{6,1} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_5 + d_6$	$x_{6,2} =$ $x_{1,1} + d_1$ $+d_6$	$x_{6,3} =$ $x_{1,1} + d_2$ $+d_3 + d_4$	$x_{6,4} =$ $x_{1,1} + d_1$ $+d_4 + d_5$ $+d_6$	$x_{6,5} =$ $x_{1,1} + d_2$ $+d_3 + d_6$	$x_{6,6} =$ $x_{1,1} + d_1$ $+d_4$

Table 5:  $C_4$ -face-magic toroidal labeling involving the differences  $d_1, d_2, d_3, d_4, d_5$  and  $d_6$  on  $C_6 \times C_6$ .

*Proof.* By Lemma 2.10, for integers  $i$  and  $j$ , we have

$$x_{3i+1,j} = x_{3i-2,j+4} + (-1)^{j+1}d_i,$$

when  $1 \leq i \leq 4$  and  $1 \leq j \leq 8$  and

$$x_{3i+1,j} = x_{3i-2,j+4} + (-1)^j d_{i-4},$$

when  $5 \leq i \leq 8$  and  $1 \leq j \leq 8$ . Thus  $x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$ . Furthermore, we only need to verify the expressions for  $x_{1,j}$  for  $2 \leq j \leq 8$ . Also, by Lemma 2.10, for all integers  $i$  and  $j$  such that we have

$$x_{i,3j+1} = x_{i+4,3j-2} + (-1)^{i+1}d'_j,$$

when  $1 \leq i \leq 8$  and  $1 \leq j \leq 4$  and

$$x_{i,3j+1} = x_{i+4,3j-2} + (-1)^i d'_{j-4},$$

when  $1 \leq i \leq 8$  and  $5 \leq j \leq 8$ . Hence,  $x_{1,7} = x_{1,1} + d_5 + d_6, x_{1,5} = x_{1,7} + d_7 + d_8, x_{1,3} = x_{1,5} - d_5 - d_6$ . Thus  $x_{1,5} = x_{1,1} + d_5 + d_6 + d_7 + d_8$  and  $x_{1,3} = x_{1,1} + d_7 + d_8$ . Also,

$x_{1,4} = x_{5,1} + d_5$ ,  $x_{1,2} = x_{1,4} + d_6 + d_7$ ,  $x_{1,8} = x_{1,2} + d_8 - d_5$  and  $x_{1,6} = x_{1,8} - d_6 - d_7$ . Since  $x_{5,1} = x_{1,1} + d_1 + d_2 + d_3 + d_4$ , we have  $x_{1,4} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5$ ,  $x_{1,2} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7$ ,  $x_{1,8} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_6 + d_7 + d_8$  and  $x_{1,6} = x_{1,1} + d_1 + d_2 + d_3 + d_4 + d_8$ .  $\square$

**Lemma 4.3.** *Let  $\{x_{i,j}\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{4m} \times C_{4n}$ . Then, we have*

$$R_i = \sum_{j=1}^{4n} x_{i,j} = 2n(16mn + 1), \text{ for all } i = 1, 2, \dots, 4m$$

and

$$C_j = \sum_{i=1}^{4m} x_{i,j} = 2m(16mn + 1), \text{ for all } j = 1, 2, \dots, 4n.$$

Furthermore, if  $m = n$ , then we have

$$D_i = \sum_{j=1}^{4n} x_{j,i+j} = 2n(16n^2 + 1), \text{ for all } i = 1, 2, \dots, 4n$$

and

$$D'_i = \sum_{j=1}^{4n} x_{j,i-j} = 2n(16n^2 + 1), \text{ for all } i = 1, 2, \dots, 4n.$$

*Proof.* Let  $S = 2(16mn + 1)$  be the  $C_4$ -face-magic value of  $\{x_{i,j}\}$ . We have, for all  $i = 1, 2, \dots, 4m$ ,

$$R_i + R_{i+1} = \sum_{j=1}^{2n} (x_{i,2j-1} + x_{i,2j} + x_{i+1,2j-1} + x_{i+1,2j}) = 2nS = 4n(16mn + 1).$$

Thus it suffices to show that  $R_1 = 2n(16mn + 1)$ . We first observe that for all  $j = 1, 2, \dots, n$ ,

$$x_{2m+1,2j-1+2n} = x_{1,2j-1+2n} + (d_1 + d_2 + \dots + d_{2m}).$$

Thus,

$$\frac{1}{2}S = x_{1,2j-1} + x_{2m+1,2j-1+2n} = x_{1,2j-1} + x_{1,2j-1+2n} + (d_1 + d_2 + \dots + d_{2m}).$$

We next observe that for all  $j = 1, 2, \dots, n$ ,

$$x_{2m+1,2j+2n} = x_{1,2j+2n} - (d_1 + d_2 + \dots + d_{2m}).$$

Thus,

$$\frac{1}{2}S = x_{1,2j} + x_{2m+1,2j+2n} = x_{1,2j} + x_{1,2j+2n} - (d_1 + d_2 + \dots + d_{2m}).$$

$x_{1,1}$	$x_{1,2} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_5$ $+d_6 + d_7$	$x_{1,3} =$ $x_{1,1} + d_7$ $+d_8$	$x_{1,4} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_5$	$x_{1,5} =$ $x_{1,1} + d_5$ $+d_6 + d_7$ $+d_8$	$x_{1,6} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_8$	$x_{1,7} =$ $x_{1,1} + d_5$ $+d_6$	$x_{1,8} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_6$ $+d_7 + d_8$
$x_{2,1} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_5 + d_6$ $+d_7 + d_8$	$x_{2,2} =$ $x_{1,1} + d_4$ $+d_8$	$x_{2,3} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_5 + d_6$	$x_{2,4} =$ $x_{1,1} + d_4$ $+d_6 + d_7$ $+d_8$	$x_{2,5} =$ $x_{1,1} + d_1$ $+d_2 + d_3$	$x_{2,6} =$ $x_{1,1} + d_4$ $+d_5 + d_6$ $+d_7$	$x_{2,7} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_7 + d_8$	$x_{2,8} =$ $x_{1,1} + d_4$ $+d_5$
$x_{3,1} =$ $x_{1,1} + d_3$ $+d_4$	$x_{3,2} =$ $x_{1,1} + d_1$ $+d_2 + d_5$ $+d_6 + d_7$	$x_{3,3} =$ $x_{1,1} + d_3$ $+d_4 + d_7$ $+d_8$	$x_{3,4} =$ $x_{1,1} + d_1$ $+d_2 + d_5$	$x_{3,5} =$ $x_{1,1} + d_3$ $+d_4 + d_5$ $+d_6 + d_7$ $+d_8$	$x_{3,6} =$ $x_{1,1} + d_1$ $+d_2 + d_8$	$x_{3,7} =$ $x_{1,1} + d_3$ $+d_4 + d_5$ $+d_6$	$x_{3,8} =$ $x_{1,1} + d_1$ $+d_2 + d_6$ $+d_7 + d_8$
$x_{4,1} =$ $x_{1,1} + d_1$ $+d_5 + d_6$ $+d_7 + d_8$	$x_{4,2} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_8$	$x_{4,3} =$ $x_{1,1} + d_1$ $+d_5 + d_6$	$x_{4,4} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_6 + d_7$ $+d_8$	$x_{4,5} =$ $x_{1,1} + d_1$	$x_{4,6} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_5 + d_6$ $+d_7$	$x_{4,7} =$ $x_{1,1} + d_1$ $+d_7 + d_8$	$x_{4,8} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_5$
$x_{5,1} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4$	$x_{5,2} =$ $x_{1,1} + d_5$ $+d_6 + d_7$	$x_{5,3} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_7$ $+d_8$	$x_{5,4} =$ $x_{1,1} + d_5$	$x_{5,5} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_5$ $+d_6 + d_7 + d_8$	$x_{5,6} =$ $x_{1,1} + d_8$	$x_{5,7} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_4 + d_5$ $+d_6$	$x_{5,8} =$ $x_{1,1} + d_6$ $+d_7 + d_8$
$x_{6,1} =$ $x_{1,1} + d_4$ $+d_5 + d_6$ $+d_7 + d_8$	$x_{6,2} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_8$	$x_{6,3} =$ $x_{1,1} + d_4$ $+d_5 + d_6$	$x_{6,4} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_6 + d_7$ $+d_8$	$x_{6,5} =$ $x_{1,1} + d_4$	$x_{6,6} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_5 + d_6$ $+d_7$	$x_{6,7} =$ $x_{1,1} + d_4$ $+d_7 + d_8$	$x_{6,8} =$ $x_{1,1} + d_1$ $+d_2 + d_3$ $+d_5$
$x_{7,1} =$ $x_{1,1} + d_1$ $+d_2$	$x_{7,2} =$ $x_{1,1} + d_3$ $+d_4 + d_5$ $+d_6 + d_7$	$x_{7,3} =$ $x_{1,1} + d_1$ $+d_2 + d_7$ $+d_8$	$x_{7,4} =$ $x_{1,1} + d_3$ $+d_4 + d_5$	$x_{7,5} =$ $x_{1,1} + d_1$ $+d_2 + d_5$ $+d_6 + d_7$ $+d_8$	$x_{7,6} =$ $x_{1,1} + d_3$ $+d_4 + d_8$	$x_{7,7} =$ $x_{1,1} + d_1$ $+d_2 + d_5$ $+d_6$	$x_{7,8} =$ $x_{1,1} + d_3$ $+d_4 + d_6$ $+d_7 + d_8$
$x_{8,1} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_5 + d_6$ $+d_7 + d_8$	$x_{8,2} =$ $x_{1,1} + d_1$ $+d_8$	$x_{8,3} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_5 + d_6$	$x_{8,4} =$ $x_{1,1} + d_1$ $+d_6 + d_7$ $+d_8$	$x_{8,5} =$ $x_{1,1} + d_2$ $+d_3 + d_4$	$x_{8,6} =$ $x_{1,1} + d_1$ $+d_5 + d_6$ $+d_7$	$x_{8,7} =$ $x_{1,1} + d_2$ $+d_3 + d_4$ $+d_7 + d_8$	$x_{8,8} =$ $x_{1,1} + d_1$ $+d_5$

Figure 1:  $C_4$ -face-magic toroidal labeling involving the differences  $d_1, d_2, d_3, d_4, d_5, d_6, d_7$  and  $d_8$  on  $C_8 \times C_8$ .

Hence, we have

$$\begin{aligned} R_1 &= \sum_{j=1}^{4n} x_{1,j} = \sum_{j=1}^{2n} (x_{1,2j-1} + x_{1,2j}) \\ &= \sum_{j=1}^n (x_{1,2j-1} + x_{1,2j-1+2n}) + \sum_{j=1}^n (x_{1,2j} + x_{1,2j+2n}) \\ &= nS = 2n(16mn + 1). \end{aligned}$$

By interchanging the roles of  $i$  and  $j$ , we have

$$C_j = \sum_{i=1}^{4m} x_{i,j} = 2m(16mn + 1), \text{ for all } j = 1, 2, \dots, 4n.$$

Finally, we assume that  $m = n$ . Then,

$$D_i = \sum_{j=1}^{4n} x_{j,i+j} = \sum_{j=1}^{2n} (x_{j,i+j} + x_{j+2n,i+j+2n}) = (2n)\left(\frac{1}{2}S\right) = 2n(16n^2 + 1).$$

A similar argument shows that for all  $i = 1, 2, \dots, 4n$ ,

$$D'_i = \sum_{j=1}^{4n} x_{j,i-j} = 2n(16n^2 + 1).$$

□

**Proposition 4.4.** *Let  $m$  and  $n$  be integers where  $m, n \geq 3$ . Let  $\{x_{i,j}\}$  be a  $C_4$ -face-magic labeling on  $P_m \times P_n$  with face-magic value  $S$ . Suppose that for all integers  $i$  and  $j$  such that  $1 \leq i \leq m - 2$  and  $1 \leq j \leq n - 2$ , we have*

$$x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = S.$$

Then,  $m \leq 4$  and  $n \leq 4$ .

*Proof.* For the purpose of contradiction, we assume that  $m \geq 5$ . We first observe that for all integers  $i$  and  $j$  such that  $1 \leq i \leq m - 2$  and  $1 \leq j \leq n - 2$ , we have

$$x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1} = S = x_{i,j+1} + x_{i,j+2} + x_{i+1,j+1} + x_{i+1,j+2}.$$

Thus,

$$x_{i,j} + x_{i+1,j} = x_{i,j+2} + x_{i+1,j+2}. \tag{4.1}$$

Replacing  $i$  with  $i + 1$  in equation (4.1) yields

$$x_{i+1,j} + x_{i+2,j} = x_{i+1,j+2} + x_{i+2,j+2}. \tag{4.2}$$

When we subtract equation (4.2) from equation (4.1) and rearrange terms, we obtain

$$x_{i,j} + x_{i+2,j+2} = x_{i+2,j} + x_{i,j+2}. \tag{4.3}$$



Since

$$x_{i,j} + x_{i,j+2} + x_{i+2,j} + x_{i+2,j+2} = S,$$

we have

$$x_{i,j} + x_{i+2,j+2} = \frac{1}{2}S = x_{i+2,j} + x_{i,j+2}. \tag{4.4}$$

Let  $i = 1$  or  $3$ , and  $j = 1$ . Then equation (4.4) yields

$$x_{1,1} + x_{3,3} = \frac{1}{2}S = x_{5,1} + x_{3,3}.$$

Hence,  $x_{1,1} = x_{5,1}$ . This is a contradiction. Therefore,  $m \leq 4$ . A similar argument shows that  $n \leq 4$ . □

**Proposition 4.5.** *Let  $m$  and  $n$  be even positive integers. Let  $y_j$ , for  $j = 1, 2, \dots, 2n$ , be a positive integer, and let  $d_1, d_2, \dots, d_m$  be integers. We define a labeling  $\{x_{i,j}\}$  on  $C_{2m} \times C_{2n}$  by letting, for all integers  $i$  and  $j$  such that  $0 \leq i \leq m$  and  $1 \leq j \leq 2n$ ,*

$$x_{i(m-1)+1, in+j} = y_j + (-1)^{j+1} \sum_{k=1}^i d_k \text{ and}$$

$$x_{i(m-1)+m+1, in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^m d_k.$$

Let  $A = \{\sum_{k=1}^i d_k, \sum_{k=i}^m d_k : 1 \leq i \leq m\} \cup \{0\}$ . For all integers  $j$  such that  $1 \leq j \leq 2n$ , let

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\}.$$

Suppose that

1. for all integers  $j$  such that  $1 \leq j \leq n$ ,  $y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^m d_k) = 4mn + 1$  and
2. the set  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ .

Then,  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ .

*Proof.* We first show that the  $C_4$ -face sum is preserved for all of the relevant  $C_4$  faces on  $C_{2m} \times C_{2n}$ . We note that  $m - 1$  is relatively prime to  $2m$ . Thus,  $m - 1$  is a generator of  $\mathbb{Z}_{2m}$ . Hence,

$$\{x_{i(m-1)+1, in+j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$$

is the set  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\}$ . We observe that  $(i+m)(m-1) + m + 1 = i(m-1) + 1 \pmod{2m}$ . We have, for all integers  $i$  and  $j$  such that  $0 \leq i \leq m$  and  $1 \leq j \leq 2n$ ,

$$x_{i(m-1)+1, in+j} = y_j + (-1)^{j+1} \sum_{k=1}^i d_k \text{ and} \tag{4.5}$$

$$x_{i(m-1)+m+1, in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^m d_k. \tag{4.6}$$

We replace  $i$  with  $i - 1$  and  $j$  with  $j + n$  in equation (4.6) to obtain, for all integers  $i$  and  $j$  such  $1 \leq i \leq m + 1$  and  $1 \leq j \leq 2n$ ,

$$x_{(i-1)(m-1)+m+1, (i-1)n+j+n} = x_{i(m-1)+2, in+j} = y_{j+n} + (-1)^{j+1} \sum_{k=i}^m d_k.$$

Hence, for all integers  $i$  and  $j$  such  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$\begin{aligned} & x_{i(m-1)+1, in+j} + x_{i(m-1)+1, in+j+1} + x_{i(m-1)+2, in+j} + x_{i(m-1)+2, in+j+1} \\ &= y_j + (-1)^{j+1} \left( \sum_{k=1}^i d_k \right) + y_{j+1} + (-1)^{j+2} \left( \sum_{k=1}^i d_k \right) \\ & \quad + y_{j+n} + (-1)^{j+1} \left( \sum_{k=i}^m d_k \right) + y_{j+n+1} + (-1)^{j+2} \left( \sum_{k=i}^m d_k \right) \\ &= \left( y_j + y_{j+n} + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) \right) \\ & \quad + \left( y_{j+1} + y_{j+n+1} + (-1)^{j+2} \left( \sum_{k=1}^m d_k \right) \right) \\ &= \frac{1}{2}S + \frac{1}{2}S = S. \end{aligned}$$

Next, we replace  $i$  with  $i - 1$  and  $j$  with  $j + n$  in equation (4.5) to obtain, for all integers  $i$  and  $j$  such  $1 \leq i \leq m + 1$  and  $1 \leq j \leq 2n$ ,

$$x_{(i-1)(m-1)+1, (i-1)n+j+n} = x_{i(m-1)+m+2, in+j} = y_{j+n} + (-1)^{j+1} \sum_{k=1}^{i-1} d_k.$$

Hence, for all integers  $i$  and  $j$  such  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$\begin{aligned} & x_{i(m-1)+m+1, in+j} + x_{i(m-1)+m+1, in+j+1} + x_{i(m-1)+m+2, in+j} + x_{i(m-1)+m+2, in+j+1} \\ &= y_j + (-1)^{j+1} \left( \sum_{k=i+1}^m d_k \right) + y_{j+1} + (-1)^{j+2} \left( \sum_{k=i+1}^m d_k \right) \\ & \quad + y_{j+n} + (-1)^{j+1} \left( \sum_{k=1}^{i-1} d_k \right) + y_{j+n+1} + (-1)^{j+2} \left( \sum_{k=1}^{i-1} d_k \right) \\ &= \left( y_j + y_{j+n} + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) \right) \\ & \quad + \left( y_{j+1} + y_{j+n+1} + (-1)^{j+2} \left( \sum_{k=1}^m d_k \right) \right) \\ &= \frac{1}{2}S + \frac{1}{2}S = S. \end{aligned}$$

Hence,  $\{x_{i,j}\}$  is a  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ .

We show that each integer  $k$ , where  $1 \leq k \leq 4mn$ , is used exactly once in the labeling  $\{x_{i,j}\}$ . For each integer  $j$  such that  $1 \leq j \leq 2n$ , we show that  $\{x_{i(m-1)+1, in+j} : 1 \leq i \leq$

$2m\} = A_j$ . We have

$$\begin{aligned} \{x_{i(m-1)+1, in+j} : 1 \leq i \leq 2m\} &= \{x_{i(m-1)+1, in+j}, x_{i(m-1)+m+1, in+j} : 1 \leq i \leq m\} \\ &= \{y_j, y_j + (-1)^{j+1} \left(\sum_{k=1}^i d_k\right), y_j + (-1)^{j+1} \left(\sum_{k=i}^m d_k\right) : 1 \leq i \leq m\} \\ &= \{y_j + (-1)^{j+1} a : a \in A\} = A_j. \end{aligned}$$

Since  $\{A_j : 1 \leq j \leq 2n\}$  is a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ , each integer  $k$ , where  $1 \leq k \leq 4mn$ , is used exactly once in the labeling  $\{x_{i,j}\}$ .

Finally, we show that  $\{x_{i,j}\}$  is an antipodal balanced labeling on  $C_{2m} \times C_{2n}$ . When we replace  $j$  with  $j + n$  in equation (4.6), we have

$$x_{(i+m)(m-1)+1, (i+m)n+j+n} = x_{i(m-1)+1+m, in+j+n} = y_{j+n} + (-1)^{j+1} \left(\sum_{k=i+1}^m d_k\right). \tag{4.7}$$

When we add equations (4.5) and (4.7), we have

$$\begin{aligned} x_{i(m-1)+1, in+j} + x_{i(m-1)+1+m, in+j+n} &= y_j + (-1)^{j+1} \left(\sum_{k=1}^i d_k\right) + y_{j+n} + (-1)^{j+1} \left(\sum_{k=i+1}^m d_k\right) \\ &= y_j + y_{j+n} + (-1)^{j+1} \left(\sum_{k=1}^m d_k\right) = \frac{1}{2}S = 4mn + 1. \end{aligned}$$

When we replace  $i$  with  $i + m$  in this equation, we have

$$\begin{aligned} x_{(i+m)(m-1)+1+m, (i+m)n+j} + x_{(i+m)(m-1)+1, (i+m)n+j+n} &= x_{i(m-1)+1, in+j} + x_{i(m-1)+1+m, in+j+n} = 4mn + 1. \end{aligned}$$

This completes the proof. □

**Proposition 4.6.** We define a labeling  $\{x_{i,j}\}$  on  $C_{4m} \times C_{4n}$  in the following manner. For integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ , when  $j$  is odd, we let

$$\begin{aligned} x_{i(2m-1)+1, j+i(2n)} &= 4m(j-1) + 2i, \\ x_{i(2m-1)+1, j+(i+1)(2n)} &= 4m(4n-j) + 2i, \\ x_{i(2m-1)+2m+1, j+i(2n)} &= 4mj - 2i + 1 \text{ and} \\ x_{i(2m-1)+2m+1, j+(i+1)(2n)} &= 4m(4n-j+1) - 2i + 1; \end{aligned}$$

and when  $j$  is even, we let

$$\begin{aligned} x_{i(2m-1)+1, j+i(2n)} &= 4mj - 2i + 1, \\ x_{i(2m-1)+1, j+(i+1)(2n)} &= 4m(4n-j+1) - 2i + 1, \\ x_{i(2m-1)+2m+1, j+i(2n)} &= 4m(j-1) + 2i \text{ and} \\ x_{i(2m-1)+2m+1, j+(i+1)(2n)} &= 4m(4n-j) + 2i. \end{aligned}$$

Then,  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{4m} \times C_{4n}$ . Furthermore, by Lemma 4.3,  $\{x_{i,j}\}$  is row-sum balanced and column-sum balanced, and  $\{x_{i,j}\}$  is torus symmetric whenever  $m = n$ .

*Proof.* In the notation of Proposition 4.5, we have  $d_1 = 1$  and  $d_k = 2$ , for  $2 \leq k \leq 2m$ . Also, for integers  $j$  such that  $1 \leq j \leq 2n$ , we have  $y_j = 4m(j - 1) + 1$  and  $y_{j+2n} = 4m(4n - j) + 1$  when  $j$  is odd, and  $y_j = 4mj$  and  $y_{j+2n} = 4m(4n - j + 1)$  when  $j$  is even. Then the labeling  $\{x_{i,j}\}$  of  $C_{4m} \times C_{4n}$  satisfies, for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ , when  $j$  is odd, we have

$$\begin{aligned} x_{i(2m-1)+1,i(2n)+j} &= y_j + (-1)^{j+1} \left( \sum_{k=1}^i d_k \right) \\ &= (4m(j - 1) + 1) + (2i - 1) = 4m(j - 1) + 2i \quad \text{and} \\ x_{i(2m-1)+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1} \left( \sum_{k=1}^i d_k \right) \\ &= (4m(4n - j) + 1) + (2i - 1) = 4m(4n - j) + 2i, \end{aligned}$$

and when  $j$  is even, we have

$$\begin{aligned} x_{i(2m-1)+1,i(2n)+j} &= y_j + (-1)^{j+1} \left( \sum_{k=1}^i d_k \right) \\ &= (4mj) - (2i - 1) = 4mj - 2i + 1 \quad \text{and} \\ x_{i(2m-1)+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1} \left( \sum_{k=1}^i d_k \right) \\ &= (4m(4n - j + 1)) - (2i - 1) = 4m(4n - j + 1) - 2i + 1. \end{aligned}$$

We let  $i = 2m$  in each of the previous four equations. Thus, for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ , when  $j$  is odd, we have

$$x_{2m+1,j} = 4mj \quad \text{and} \quad x_{2m+1,j+2n} = 4m(4n - j + 1),$$

and when  $j$  is even, we have

$$x_{2m+1,j} = 4m(j - 1) + 1 \quad \text{and} \quad x_{2m+1,j+2n} = 4m(4n - j) + 1.$$

Next, we observe that the labeling  $\{x_{i,j}\}$  on  $C_{4m} \times C_{4n}$  satisfies, for all integers  $i$  and  $j$  such that  $1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ , when  $j$  is odd, we have

$$\begin{aligned} x_{i(2m-1)+2m+1,i(2n)+j} &= y_j + (-1)^{j+1} \left( \sum_{k=i+1}^{2m} d_k \right) \\ &= (4m(j - 1) + 1) + 2(2m - i) = 4mj - 2i \quad \text{and} \\ x_{i(2m-1)+2m+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1} \left( \sum_{k=i+1}^{2m} d_k \right) \\ &= (4m(4n - j) + 1) + 2(2m - i) = 4m(4n - j + 1) - 2i, \end{aligned}$$

and when  $j$  is even, we have

$$\begin{aligned} x_{i(2m-1)+2m+1,i(2n)+j} &= y_j + (-1)^{j+1} \left( \sum_{k=i+1}^{2m} d_k \right) \\ &= (4mj) - 2(2m - i) = 4m(j - 1) + 2i \quad \text{and} \\ x_{i(2m-1)+2m+1,i(2n)+j+2n} &= y_{j+2n} + (-1)^{j+1} \left( \sum_{k=i+1}^{2m} d_k \right) \\ &= (4m(4n - j + 1)) - 2(2m - i) = 4m(4n - j) + 2i. \end{aligned}$$

We show that condition (1) of Proposition 4.5 is satisfied. Let  $j$  be an integer such that  $1 \leq j \leq 2n$ . When  $j$  is odd, we have

$$y_j + y_{j+2n} + (-1)^{j+1} \left( \sum_{k=1}^{2m} d_k \right) = (4m(j-1)+1) + (4m(4n-j)+1) + (4m-1) = 16mn+1.$$

When  $j$  is even, we have

$$y_j + y_{j+2n} + (-1)^{j+1} \left( \sum_{k=1}^{2m} d_k \right) = (4mj) + (4m(4n - j + 1)) - (4m - 1) = 16mn + 1.$$

We show that condition (2) of Proposition 4.5 is satisfied. We first observe that

$$\begin{aligned} A &= \left\{ \sum_{k=1}^i d_k, \sum_{k=i}^{2m} d_k : 1 \leq k \leq 2m \right\} \cup \{0\} \\ &= \{2i - 1 : 1 \leq i \leq 2m\} \cup \{4m - 2i + 2 : 2 \leq i \leq 2m\} \cup \{0\} \\ &= \{k \in \mathbb{Z} : 0 \leq k \leq 4m - 1\}. \end{aligned}$$

Let  $j$  be an integer such that  $1 \leq j \leq 2n$ . When  $j$  is odd, we have

$$\begin{aligned} A_j &= \{y_j + (-1)^{j+1}a : a \in A\} = \{4m(j - 1) + 1 + i : i \in \mathbb{Z}, 0 \leq i \leq 4m - 1\} \\ &= \{k \in \mathbb{Z} : 4m(j - 1) + 1 \leq k \leq 4mj\} \end{aligned}$$

and

$$\begin{aligned} A_{j+2n} &= \{y_{j+2n} + (-1)^{j+1}a : a \in A\} \\ &= \{4m(4n - j) + 1 + i : i \in \mathbb{Z}, 0 \leq i \leq 4m - 1\} \\ &= \{k \in \mathbb{Z} : 4m(4n - j) + 1 \leq k \leq 4m(4n - j + 1)\}. \end{aligned}$$

When  $j$  is even, we have

$$\begin{aligned} A_j &= \{y_j + (-1)^{j+1}a : a \in A\} = \{4mj - i : i \in \mathbb{Z}, 0 \leq i \leq 4m - 1\} \\ &= \{k \in \mathbb{Z} : 4m(j - 1) + 1 \leq k \leq 4mj\} \end{aligned}$$

and

$$\begin{aligned} A_{j+2n} &= \{y_{j+2n} + (-1)^{j+1}a : a \in A\} \\ &= \{4m(4n - j + 1) - i : i \in \mathbb{Z}, 0 \leq i \leq 4m - 1\} \\ &= \{k \in \mathbb{Z} : 4m(4n - j) + 1 \leq j \leq 4m(4n - j + 1)\}. \end{aligned}$$

Hence, for all integers  $j$ , when  $1 \leq j \leq 2n$ , we have

$$A_j = \{k \in \mathbb{Z} : 4m(j - 1) + 1 \leq j \leq 4mj\}$$

and when  $2n + 1 \leq j \leq 4n$ , we have

$$A_j = \{k \in \mathbb{Z} : 4m(4n - j) + 1 \leq j \leq 4m(4n - j + 1)\}.$$

Therefore,  $\{A_j : 1 \leq j \leq 4n\}$  is a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 16mn\}$ . This completes the proof.  $\square$

**Example 4.7.** We consider the example of Proposition 4.6 where  $m = n = 2$ . This example is given in Table 6.

1	16	17	32	57	56	41	40
62	51	46	35	6	11	22	27
5	12	21	28	61	52	45	36
58	55	42	39	2	15	18	31
8	9	24	25	64	49	48	33
59	54	43	38	3	14	19	30
4	13	20	29	60	53	44	37
63	50	47	34	7	10	23	26

Table 6: An antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_8 \times C_8$ .

We next prove the converse to Proposition 4.5.

**Proposition 4.8.** *Let  $m$  and  $n$  be even positive integers. Let  $\{x_{i,j} : (i, j) \in V(C_{2m} \times C_{2n})\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . For all integers  $j$  such that  $1 \leq j \leq 2n$ , let  $y_j = x_{1,j}$ . For all integers  $i$  such that  $1 \leq i \leq m$ , let*

$$d_i = x_{i(m-1)+1, in+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

Then, for all integers  $i$  and  $j$  such that  $0 \leq i \leq m$  and  $1 \leq j \leq 2n$ ,

$$x_{i(m-1)+1, in+j} = y_j + (-1)^{j+1} \sum_{k=1}^i d_k \text{ and} \tag{4.8}$$

$$x_{i(m-1)+m+1, in+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^m d_k. \tag{4.9}$$

Let  $A = \{\sum_{k=1}^i d_k, \sum_{k=i}^m d_k : 1 \leq i \leq m\} \cup \{0\}$ . For all integers  $j$  such that  $1 \leq j \leq 2n$ , let  $A_j = \{y_j + (-1)^{j+1}a : a \in A\}$ . Then

1. for all integers  $j$  such that  $1 \leq j \leq n$ ,  $y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^m d_k) = 4mn + 1$  and
2. the set  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ .

*Proof.* By Lemma 2.10, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i. \tag{4.10}$$

For all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , repeated use of equation (4.10) yields

$$x_{i(m-1)+1, in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^i d_k. \tag{4.11}$$

Thus equation (4.8) holds. When we let  $i = m$  in equation (4.11), we have

$$x_{m+1,j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^m d_k. \tag{4.12}$$

By Lemma 2.10, for all integers  $i$  and  $j$  such that  $m + 1 \leq i \leq 2m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^j d_{i-m}. \tag{4.13}$$

Let  $1 \leq i \leq m$ . Replacing  $i$  with  $i + m$  in equation (4.13) yields

$$x_{i(m-1)+m+1, in+j} = x_{(i-1)(m-1)+m+1, (i-1)n+j} + (-1)^j d_i. \tag{4.14}$$

For all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , repeated use of equation (4.14) yields

$$x_{i(m-1)+m+1, in+j} = x_{m+1,j} + (-1)^j \sum_{k=1}^i d_k. \tag{4.15}$$

When we combine equations (4.12) and (4.15), we have

$$x_{i(m-1)+m+1, in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=i+1}^m d_k. \tag{4.16}$$

Thus equation (4.9) holds.

Since  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ , we have  $x_{1,n+j} + x_{m+1,j} = 4mn + 1$  for all integers  $j$  such that  $1 \leq j \leq n$ . By equation (4.12), we have  $x_{m+1,j} = y_j + (-1)^{j+1} \sum_{k=1}^m d_k$ . Since  $x_{1,n+j} = y_{j+n}$ , we have  $y_j + y_{j+n} + (-1)^{j+1}(\sum_{k=1}^m d_k) = x_{1,n+j} + x_{m+1,j} = 4mn + 1$ .

By equations (4.8) and (4.9), for integers  $j$  such that  $1 \leq j \leq 2n$ , we have

$$A_j = \{x_{i(m-1)+1, in+j}, x_{i(m-1)+m+1, in+j} : 0 \leq i \leq m - 1\}.$$

Thus the set  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of the set  $\{x_{i,j} : 1 \leq i \leq 2m \text{ and } 1 \leq j \leq 2n\} = \{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ . □

**Proposition 4.9.** *Let  $m$  and  $n$  be odd positive integers. Let  $y_j$  and  $z_j$ , for  $j = 1, 2, \dots, n$ , be a positive integers, and let  $d_1, d_2, \dots, d_m$  be integers. We define a labeling  $\{x_{i,j}\}$  on  $C_{2m} \times C_{2n}$  by letting, for all integers  $i$  and  $j$  such  $0 \leq i \leq m$  and  $1 \leq j \leq n$ ,*

$$x_{i(m-1)+1, in+j} = y_j + (-1)^{j+1} \sum_{k=1}^i d_k, \quad (4.17)$$

$$x_{i(m-1)+1, (i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^m d_k, \quad (4.18)$$

$$x_{i(m-1)+m+1, in+j} = z_j + (-1)^{j+1} \sum_{k=1}^i d_k \text{ and} \quad (4.19)$$

$$x_{i(m-1)+m+1, (i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=i+1}^m d_k. \quad (4.20)$$

Let  $A = \{\sum_{k=1}^i d_k, \sum_{k=i}^m d_k : 1 \leq i \leq m\} \cup \{0\}$ . For all integers  $j$  such that  $1 \leq j \leq n$ , let

$$A_j = \{y_j + (-1)^{j+1} a : a \in A\} \text{ and } A_{j+n} = \{z_j + (-1)^{j+1} a : a \in A\}.$$

Suppose

1. for all integers  $j$  such that  $1 \leq j \leq n$ ,  $y_j + z_j + (-1)^{j+1}(\sum_{k=1}^m d_k) = 4mn + 1$ , and
2. the set  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ .

Then,  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ .

*Proof.* We first show that the  $C_4$ -face sum is preserved for all relevant  $C_4$  faces on  $C_{2m} \times C_{2n}$ . Since  $\gcd(2m, m-1) = 2$ ,  $m-1$  generates the subgroup  $\langle 2 \rangle$  of  $\mathbb{Z}_{2m}$ . Hence,

$$\{x_{i(m-1)+1, in+j}, x_{(i+1)(m-1)+1, in+j}, x_{i(m-1)+m+1, in+j}, x_{(i+1)(m-1)+m+1, in+j} : i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n\}$$

is the set

$$\{x_{i,j} : i = 1, 2, \dots, 2, m \text{ and } j = 1, 2, \dots, 2n\}.$$

We replace  $i$  with  $i-1$  in equations (4.17), (4.18), (4.19) and (4.20) to obtain

$$x_{i(m-1)+m+2, (i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=1}^{i-1} d_k, \quad (4.21)$$

$$x_{i(m-1)+m+2, in+j} = y_j + (-1)^{j+1} \sum_{k=i}^m d_k, \quad (4.22)$$

$$x_{i(m-1)+2, (i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=1}^{i-1} d_k \text{ and} \quad (4.23)$$

$$x_{i(m-1)+2, in+j} = z_j + (-1)^{j+1} \sum_{k=i}^m d_k. \quad (4.24)$$



We replace  $j$  with  $j + 1$  in equations (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24) to obtain

$$x_{i(m-1)+1, in+j+1} = y_{j+1} + (-1)^{j+2} \sum_{k=1}^i d_k, \tag{4.25}$$

$$x_{i(m-1)+1, (i+1)n+j+1} = y_{j+1} + (-1)^{j+2} \sum_{k=i+1}^m d_k, \tag{4.26}$$

$$x_{i(m-1)+m+1, in+j+1} = z_{j+1} + (-1)^{j+2} \sum_{k=1}^i d_k, \tag{4.27}$$

$$x_{i(m-1)+m+1, (i+1)n+j+1} = z_{j+1} + (-1)^{j+2} \sum_{k=i+1}^m d_k, \tag{4.28}$$

$$x_{i(m-1)+m+2, (i+1)n+j+1} = y_{j+1} + (-1)^{j+2} \sum_{k=1}^{i-1} d_k, \tag{4.29}$$

$$x_{i(m-1)+m+2, in+j+1} = y_{j+1} + (-1)^{j+2} \sum_{k=i}^m d_k, \tag{4.30}$$

$$x_{i(m-1)+2, (i+1)n+j+1} = z_{j+1} + (-1)^{j+2} \sum_{k=1}^{i-1} d_k \text{ and} \tag{4.31}$$

$$x_{i(m-1)+2, in+j+1} = z_{j+1} + (-1)^{j+2} \sum_{k=i}^m d_k. \tag{4.32}$$

When we add equations (4.17), (4.25), (4.24) and (4.32) together, for  $1 \leq i \leq m$  and  $1 \leq j \leq n - 1$ , we obtain

$$\begin{aligned} &x_{i(m-1)+1, in+j} + x_{i(m-1)+1, in+j+1} + x_{i(m-1)+2, in+j} + x_{i(m-1)+2, in+j+1} \\ &= \left( y_j + (-1)^{j+1} \left( \sum_{k=1}^i d_k \right) \right) + \left( y_{j+1} + (-1)^{j+2} \left( \sum_{k=1}^i d_k \right) \right) \\ &\quad + \left( z_j + (-1)^{j+1} \left( \sum_{k=i}^m d_k \right) \right) + \left( z_{j+1} + (-1)^{j+2} \left( \sum_{k=i}^m d_k \right) \right) \\ &= \left( y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) \right) + \left( y_{j+1} + z_{j+1} + (-1)^{j+2} \left( \sum_{k=1}^m d_k \right) \right) \\ &= S. \end{aligned}$$

When we add equations (4.18), (4.26), (4.23) and (4.31) together, for  $1 \leq i \leq m$  and  $1 \leq j \leq n - 1$ , we obtain

$$\begin{aligned} &x_{i(m-1)+1, (i+1)n+j} + x_{i(m-1)+1, (i+1)n+j+1} \\ &\quad + x_{i(m-1)+2, (i+1)n+j} + x_{i(m-1)+2, (i+1)n+j+1} = S. \end{aligned}$$

Similar  $C_4$ -face-magic sums occur when we add equations (4.19), (4.49), (4.22), and (4.52) together; or when we add equations (4.20), (4.50), (4.21) and (4.51) together.

Suppose  $i$  is odd and let  $j = n$  in equations (4.18), (4.20), (4.21), and (4.23), we have

$$x_{i(m-1)+1,n} = y_n + (-1)^{n+1} \sum_{k=i+1}^m d_k, \quad (4.33)$$

$$x_{i(m-1)+m+1,n} = z_n + (-1)^{n+1} \sum_{k=i+1}^m d_k, \quad (4.34)$$

$$x_{i(m-1)+m+2,n} = y_n + (-1)^{n+1} \sum_{k=1}^{i-1} d_k \text{ and} \quad (4.35)$$

$$x_{i(m-1)+2,n} = z_n + (-1)^{n+1} \sum_{k=1}^{i-1} d_k. \quad (4.36)$$

Suppose  $i$  is odd and let  $j = 1$  in equations (4.17), (4.19), (4.22), and (4.24), we have

$$x_{i(m-1)+1,n+1} = y_1 + (-1)^2 \sum_{k=1}^i d_k, \quad (4.37)$$

$$x_{i(m-1)+m+1,n+1} = z_1 + (-1)^2 \sum_{k=1}^i d_k, \quad (4.38)$$

$$x_{i(m-1)+m+2,n+1} = y_1 + (-1)^2 \sum_{k=i}^m d_k \text{ and} \quad (4.39)$$

$$x_{i(m-1)+2,n+1} = z_1 + (-1)^2 \sum_{k=i}^m d_k. \quad (4.40)$$

When we add equations (4.33), (4.37), (4.36) and (4.40) together, we have

$$\begin{aligned} & x_{i(m-1)+1,n} + x_{i(m-1)+1,n+1} + x_{i(m-1)+2,n} + x_{i(m-1)+2,n+1} \\ &= \left( y_n + \left( \sum_{k=i+1}^m d_k \right) \right) + \left( y_1 + \left( \sum_{k=1}^i d_k \right) \right) \\ & \quad + \left( z_n + \left( \sum_{k=1}^{i-1} d_k \right) \right) + \left( z_1 + \left( \sum_{k=i}^m d_k \right) \right) \\ &= \left( y_1 + y_n + \left( \sum_{k=1}^m d_k \right) \right) + \left( z_1 + z_n + \left( \sum_{k=1}^m d_k \right) \right) = S. \end{aligned}$$

When we add equations (4.34), (4.38), (4.35) and (4.39) together, we have

$$x_{i(m-1)+m+1,n} + x_{i(m-1)+m+1,n+1} + x_{i(m-1)+m+2,n} + x_{i(m-1)+m+2,n+1} = S.$$

Suppose  $i$  is even and let  $j = n$  in equations (4.17), (4.19), (4.22) and (4.24), we have

$$x_{i(m-1)+1,n} = y_n + (-1)^{n+1} \sum_{k=1}^i d_k, \quad (4.41)$$

$$x_{i(m-1)+m+1,n} = z_n + (-1)^{n+1} \sum_{k=1}^i d_k, \quad (4.42)$$

$$x_{i(m-1)+m+2,n} = y_n + (-1)^{n+1} \sum_{k=i}^m d_k \text{ and} \quad (4.43)$$

$$x_{i(m-1)+2,n} = z_n + (-1)^{n+1} \sum_{k=i}^m d_k. \quad (4.44)$$

Suppose  $i$  is even and let  $j = 1$  in equations (4.18), (4.20), (4.21) and (4.23), we have

$$x_{i(m-1)+1,n+1} = y_1 + (-1)^2 \sum_{k=i+1}^m d_k, \quad (4.45)$$

$$x_{i(m-1)+m+1,n+1} = z_1 + (-1)^2 \sum_{k=i+1}^m d_k, \quad (4.46)$$

$$x_{i(m-1)+m+2,n+1} = y_1 + (-1)^2 \sum_{k=1}^{i-1} d_k \text{ and} \quad (4.47)$$

$$x_{i(m-1)+2,n+1} = z_1 + (-1)^2 \sum_{k=1}^{i-1} d_k. \quad (4.48)$$

When we add equations (4.41), (4.45), (4.44) and (4.48) together, we have

$$\begin{aligned} & x_{i(m-1)+1,n} + x_{i(m-1)+1,n+1} + x_{i(m-1)+2,n} + x_{i(m-1)+2,n+1} \\ &= \left( y_n + \left( \sum_{k=1}^i d_k \right) \right) + \left( y_1 + \left( \sum_{k=i+1}^m d_k \right) \right) \\ & \quad + \left( z_n + \left( \sum_{k=i}^m d_k \right) \right) + \left( z_1 + \left( \sum_{k=1}^{i-1} d_k \right) \right) \\ &= \left( y_1 + y_n + \left( \sum_{k=1}^m d_k \right) \right) + \left( z_1 + z_n + \left( \sum_{k=1}^m d_k \right) \right) = S. \end{aligned}$$

When we add equations (4.42), (4.46), (4.43) and (4.47) together, we have

$$x_{i(m-1)+m+1,n} + x_{i(m-1)+m+1,n+1} + x_{i(m-1)+m+2,n} + x_{i(m-1)+m+2,n+1} = S.$$

For  $1 \leq j \leq n$ , by equations (4.17) and (4.18), we have

$$\begin{aligned} & \{x_{i(m-1)+1, in+j} : i = 0, 1, \dots, 2m-1\} \\ &= \{x_{i(m-1)+1, in+j}, x_{(i+m)(m-1)+1, (i+m)n+j} : i = 0, 1, \dots, m-1\} \\ &= \{x_{i(m-1)+1, in+j}, x_{i(m-1)+1, (i+1)n+j} : i = 1, 2, \dots, m\} \\ &= \{y_j + (-1)^{j+1} \sum_{k=1}^i d_k, y_j + (-1)^{j+1} \sum_{k=i+1}^m d_k : i = 0, 1, \dots, m-1\} \\ &= \{y_j + (-1)^{j+1} a : a \in A\} = A_j. \end{aligned}$$

For  $1 \leq j \leq n$ , by equations (4.19) and (4.20), we have

$$\begin{aligned} & \{x_{i(m-1)+m+1, in+j} : i = 0, 1, \dots, 2m-1\} \\ &= \{x_{i(m-1)+m+1, in+j}, x_{(i+m)(m-1)+m+1, (i+m)n+j} : i = 0, 1, \dots, m-1\} \\ &= \{x_{i(m-1)+m+1, in+j}, x_{i(m-1)+m+1, (i+1)n+j} : i = 1, 2, \dots, m\} \\ &= \{z_j + (-1)^{j+1} \sum_{k=1}^i d_k, z_j + (-1)^{j+1} \sum_{k=i+1}^m d_k : i = 0, 1, \dots, m-1\} \\ &= \{z_j + (-1)^{j+1} a : a \in A\} = A_{j+n}. \end{aligned}$$

Since  $\{A_j, A_{j+n} : j = 1, 2, \dots, n\}$  is a partition of  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ ,  $\{x_{i,j} : i = 1, 2, \dots, 2m \text{ and } j = 1, 2, \dots, 2n\} = \{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ . By Lemma 2.2,  $\{x_{i,j}\}$  is a  $C_4$ -face-magic labeling on  $C_{2m} \times C_{2n}$ .

We need to show that  $\{x_{i,j}\}$  is antipodal balanced. Let  $i$  and  $j$  be integers such that  $0 \leq i \leq m$  and  $1 \leq j \leq n$ . We add equations (4.17) and (4.20) together to obtain

$$\begin{aligned} & x_{i(m-1)+1, in+j} + x_{i(m-1)+m+1, (i+1)n+j} = \\ &= y_j + (-1)^{j+1} \sum_{k=1}^i d_k + z_j + (-1)^{j+1} \sum_{k=i+1}^m d_k \\ &= y_j + z_j + (-1)^{j+1} \sum_{k=1}^m d_k = \frac{1}{2}S = 4mn + 1. \end{aligned}$$

We add equations (4.18) and (4.19) together to obtain

$$\begin{aligned} & x_{i(m-1)+1, (i+1)n+j} + x_{i(m-1)+m+1, in+j} = \\ &= y_j + (-1)^{j+1} \sum_{k=i+1}^i d_k + z_j + (-1)^{j+1} \sum_{k=1}^i d_k \\ &= y_j + z_j + (-1)^{j+1} \sum_{k=1}^m d_k = \frac{1}{2}S = 4mn + 1. \end{aligned}$$

Hence,  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ .  $\square$

**Proposition 4.10.** *Suppose  $m$  and  $n$  are odd positive integers. Suppose  $i$  and  $j$  are integers such that  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and  $j$  is odd. Let*

$$\begin{aligned} x_{i(m-1)+1, in+j} &= 2m(j-1) + 2i, \\ x_{i(m-1)+1, (i+1)n+j} &= 2mj - 2i + 1, \\ x_{i(m-1)+m+1, in+j} &= 2m(2n-j) + 2i \text{ and} \\ x_{i(m-1)+m+1, (i+1)n+j} &= 2m(2n-j+1) - 2i + 1. \end{aligned}$$

*Suppose  $i$  and  $j$  are integers such that  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and  $j$  is even. Let*

$$\begin{aligned} x_{i(m-1)+1, in+j} &= 2mj - 2i + 1, \\ x_{i(m-1)+1, (i+1)n+j} &= 2m(j-1) + 2i, \\ x_{i(m-1)+m+1, in+j} &= 2m(2n-j+1) - 2i + 1 \text{ and} \\ x_{i(m-1)+m+1, (i+1)n+j} &= 2m(2n-j) + 2i. \end{aligned}$$

*Then,  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ .*

*Proof.* This labeling  $\{x_{i,j}\}$  corresponds to the labeling in Proposition 4.9 where for integers  $j$  such that  $1 \leq j \leq n$  and  $j$  is odd,  $y_j = 2m(j-1) + 1$  and  $z_j = 2m(2n-j) + 1$ ; for integers  $j$  such that  $1 \leq j \leq n$  and  $j$  is even,  $y_j = 2mj$  and  $z_j = 2m(2n-j+1)$ ; and  $d_1 = 1$  and  $d_k = 2$  for integers  $k$  such that  $2 \leq k \leq m$ .

Let  $j$  be an integer such that  $1 \leq j \leq n$  and  $j$  is odd. Then,

$$\begin{aligned} y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) &= (2m(j-1) + 1) \\ &\quad + (2m(2n-j) + 1) + (2m-1) = 4mn + 1. \end{aligned}$$

Let  $j$  be an integer such that  $1 \leq j \leq n$  and  $j$  is even. Then,

$$y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) = 2mj + 2m(2n-j+1) - (2m-1) = 4mn + 1.$$

Thus condition (1) of Proposition 4.9 is satisfied.

We have

$$\begin{aligned} A &= \left\{ \sum_{k=1}^i d_k, \sum_{k=i+1}^m d_k : i = 1, 2, \dots, m \right\} \cup \{0\} \\ &= \{2i-1, 2(m-i) : i = 1, 2, \dots, m\} \cup \{0\} = \{k \in \mathbb{Z} : 0 \leq k \leq 2m-1\}. \end{aligned}$$

Let  $j$  be an integer such that  $1 \leq j \leq n$  and  $j$  is odd. Then,

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\} = \{2m(j-1) + 1 + a : a \in A\}$$

and

$$A_{j+n} = \{z_j + (-1)^{j+1}a : a \in A\} = \{2m(2n-j) + 1 + a : a \in A\}.$$

Let  $j$  be an integer such that  $1 \leq j \leq n$  and  $j$  is even. Then,

$$\begin{aligned} A_j &= \{y_j + (-1)^{j+1}a : a \in A\} = \{2mj - a : a \in A\} \\ &= \{2m(j-1) + 1 + a' : a' \in A\} \end{aligned}$$

and

$$\begin{aligned} A_{j+n} &= \{z_j + (-1)^{j+1}a : a \in A\} = \{2m(2n-j+1) - a : a \in A\} \\ &= \{2m(2n-j) + 1 + a' : a' \in A\}. \end{aligned}$$

Hence,  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ . Thus condition (2) of Proposition 4.9 is satisfied. Therefore by Proposition 4.9,  $\{x_{i,j}\}$  is an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ .  $\square$

**Example 4.11.** We consider the example of Proposition 4.10 where  $m = n = 3$ . This example is given in Table 7.

1	12	13	6	7	18
34	27	22	33	28	21
5	8	17	2	11	14
31	30	19	36	25	24
4	9	16	3	10	15
35	26	23	32	29	20

Table 7: An antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_6 \times C_6$ .

We need the following converse to Proposition 4.9 in order to prove our last result in this paper.

**Proposition 4.12.** *Let  $m$  and  $n$  be odd positive integers. Let  $\{x_{i,j} : (i,j) \in V(C_{2m} \times C_{2n})\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$ . For all integers  $j$  such that  $1 \leq j \leq n$ , let  $y_j = x_{1,j}$  and  $z_j = x_{m+1,j}$ . For all integers  $i$  such that  $1 \leq i \leq m$ , let*

$$d_i = x_{i(m-1)+1, in+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

*Then, for all integers  $i$  and  $j$  such that  $0 \leq i \leq m$  and  $1 \leq j \leq n$ , we have*

$$x_{i(m-1)+1, in+j} = y_j + (-1)^{j+1} \sum_{k=1}^i d_k, \tag{4.49}$$

$$x_{i(m-1)+1, (i+1)n+j} = y_j + (-1)^{j+1} \sum_{k=i+1}^m d_k, \tag{4.50}$$

$$x_{i(m-1)+m+1, in+j} = z_j + (-1)^{j+1} \sum_{k=1}^i d_k \text{ and} \tag{4.51}$$

$$x_{i(m-1)+m+1, (i+1)n+j} = z_j + (-1)^{j+1} \sum_{k=i+1}^m d_k. \tag{4.52}$$

Let  $A = \{\sum_{k=1}^i d_k, \sum_{k=i}^m d_k : 1 \leq i \leq m\} \cup \{0\}$ . For all integers  $j$  such that  $1 \leq j \leq n$ , let

$$A_j = \{y_j + (-1)^{j+1}a : a \in A\} \text{ and } A_{j+n} = \{z_j + (-1)^{j+1}a : a \in A\}.$$

Then

1. for all integers  $j$  such that  $1 \leq j \leq n$ ,  $y_j + z_j + (-1)^{j+1}(\sum_{k=1}^m d_k) = 4mn + 1$ , and
2. the set  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ .

*Proof.* By Lemma 2.11, for all integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq 2n$ , we have

$$x_{i(m-1)+1, in+j} = x_{(i-1)(m-1)+1, (i-1)n+j} + (-1)^{j+1}d_i \text{ and} \tag{4.53}$$

$$x_{i(m-1)+m+1, in+j} = x_{(i-1)(m-1)+m+1, (i-1)n+j} + (-1)^{j+1}d_i. \tag{4.54}$$

For integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , repeated use of equation (4.53) yields

$$x_{i(m-1)+1, in+j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^i d_k. \tag{4.55}$$

Thus equation (4.49) holds. When we let  $i = m$  in equation (4.55), we have

$$x_{1, n+j} = x_{1,j} + (-1)^{j+1} \sum_{k=1}^m d_k. \tag{4.56}$$

When we replace  $j$  with  $j + n$  in equation (4.53), for integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , we have

$$x_{i(m-1)+1, (i+1)n+j} = x_{(i-1)(m-1)+1, in+j} + (-1)^j d_i. \tag{4.57}$$

For integers  $i$  and  $j$  such that  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , repeated use of equation (4.57) yields

$$x_{i(m-1)+1, (i+1)n+j} = x_{1, n+j} + (-1)^j \sum_{k=1}^i d_k. \tag{4.58}$$

Combining equation (4.58) with equation (4.56) yields

$$x_{i(m-1)+1, (i+1)n+j} = x_{1,j} + (-1)^{j+1} \sum_{k=i+1}^m d_k.$$

Thus equation (4.50) holds. A similar proof using equation (4.54) shows that equations (4.51) and (4.52) hold.

A proof similar to that in Proposition 4.8 shows that, for all integers  $j$  such that  $1 \leq j \leq n$ , we have  $y_j + z_j + (-1)^{j+1}(\sum_{k=1}^m d_k) = 4mn + 1$ , and the set  $\{A_j : 1 \leq j \leq 2n\}$  forms a partition of the set  $\{k \in \mathbb{Z} : 1 \leq k \leq 4mn\}$ .  $\square$

**Proposition 4.13.** *Let  $m$  and  $n$  be positive odd integers. Then,  $C_{2m} \times C_{2n}$  has no antipodal balanced  $C_4$ -face-magic toroidal labeling that is both row-sum balanced and column-sum balanced.*

*Proof.* Let  $\{x_{i,j}\}$  be an antipodal balanced  $C_4$ -face-magic toroidal labeling on  $C_{2m} \times C_{2n}$  that is both row-sum balanced and column-sum balanced. For all integers  $j$  such that  $1 \leq j \leq n$ , let  $y_j = x_{1,j}$  and  $z_j = x_{m+1,j}$ . For all integers  $i$  such that  $1 \leq i \leq m$ , let

$$d_i = x_{i(m-1)+1, i(m-1)+1} - x_{(i-1)(m-1)+1, (i-1)n+1}.$$

By Proposition 4.12, for all integers  $i$  such that  $1 \leq i \leq m$ , equations (4.49), (4.50), (4.51), and (4.52) hold.

Let

$$T = \sum_{j=1}^n \left( y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) \right).$$

By Proposition 4.12, we have  $y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) = 4mn + 1$ . Then

$$\begin{aligned} T &= \sum_{j=1}^n \left( y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) \right) \\ &= \sum_{j=1}^n (4mn + 1) = n(4mn + 1) \equiv 1 \pmod{2}. \end{aligned}$$

For  $i = 1, 2, \dots, 2m$ , let  $R_i = \sum_{j=1}^{2n} x_{i,j}$  be the row sum of the labels on the vertices in row  $i$ . Also, for  $j = 1, 2, \dots, 2n$ , let  $C_j = \sum_{i=1}^{2m} x_{i,j}$  be the column sum of the labels on the vertices in column  $j$ . Let  $m_1$  be the integer such that  $m = 2m_1 + 1$ . Then for any integer  $j$  such that  $1 \leq j \leq n$ , we have

$$\begin{aligned} C_j &= \sum_{i=1}^m (x_{i(m-1)+1, j} + x_{i(m-1)+m+1, j}) \\ &= \sum_{i=1}^{m_1+1} (x_{(2i-1)(m-1)+1, j} + x_{(2i-1)(m-1)+m+1, j}) \\ &\quad + \sum_{i=1}^{m_1} (x_{(2i)(m-1)+1, j} + x_{(2i)(m-1)+m+1, j}) \\ &= \sum_{i=1}^{m_1+1} \left( y_j + (-1)^{j+1} \left( \sum_{k=2i}^m d_k \right) + z_j + (-1)^{j+1} \left( \sum_{k=2i}^m d_k \right) \right) \\ &\quad + \sum_{i=1}^{m_1} \left( y_j + (-1)^{j+1} \left( \sum_{k=1}^{2i} d_k \right) + z_j + (-1)^{j+1} \left( \sum_{k=1}^{2i} d_k \right) \right) \\ &= m(y_j + z_j) + (-1)^{j+1} \left( \sum_{k=1}^m (m + (-1)^k) d_k \right). \end{aligned}$$



We also have

$$\begin{aligned}
 C_{n+j} &= \sum_{i=1}^m (x_{i(m-1)+1,n+j} + x_{i(m-1)+m+1,n+j}) \\
 &= \sum_{i=1}^{m_1+1} (x_{(2i-1)(m-1)+1,n+j} + x_{(2i-1)(m-1)+m+1,n+j}) \\
 &\quad + \sum_{i=1}^{m_1} (x_{(2i)(m-1)+1,n+j} + x_{(2i)(m-1)+m+1,n+j}) \\
 &= \sum_{i=1}^{m_1+1} \left( y_j + (-1)^{j+1} \left( \sum_{k=1}^{2i-1} d_k \right) + z_j + (-1)^{j+1} \left( \sum_{k=1}^{2i-1} d_k \right) \right) \\
 &\quad + \sum_{i=1}^{m_1} \left( y_j + (-1)^{j+1} \left( \sum_{k=2i+1}^m d_k \right) + z_j + (-1)^{j+1} \left( \sum_{k=2i+1}^m d_k \right) \right) \\
 &= m(y_j + z_j) + (-1)^{j+1} \left( \sum_{k=1}^m (m + (-1)^{k+1}) d_k \right).
 \end{aligned}$$

Since  $C_j = C_{n+j}$ , we have  $\sum_{k=1}^m (-1)^k d_k = 0$ .

We have

$$R_1 = \sum_{j=1}^n (x_{1,j} + x_{1,n+j}) = \sum_{j=1}^n \left( y_j + y_j + (-1)^{j+1} \sum_{k=1}^m d_k \right) = 2 \sum_{j=1}^n y_j + \sum_{k=1}^m d_k.$$

We also have

$$\begin{aligned}
 R_{m+1} &= \sum_{j=1}^n (x_{m+1,j} + x_{m+1,n+j}) = \sum_{j=1}^n \left( z_j + z_j + (-1)^{j+1} \sum_{k=1}^m d_k \right) \\
 &= 2 \sum_{j=1}^n z_j + \sum_{k=1}^m d_k.
 \end{aligned}$$

Since  $R_1 = R_{m+1}$ , we have  $\sum_{j=1}^n y_j = \sum_{j=1}^n z_j$ . Thus

$$\begin{aligned}
 T &= \sum_{j=1}^n \left( y_j + z_j + (-1)^{j+1} \left( \sum_{k=1}^m d_k \right) \right) \\
 &= \sum_{j=1}^n y_j + \sum_{j=1}^n z_j + \sum_{k=1}^m d_k + \sum_{k=1}^m (-1)^k d_k \\
 &\equiv 2 \left( \sum_{j=1}^n y_j \right) + \sum_{i=1}^{m_1} 2d_{2i} \equiv 0 \pmod{2}.
 \end{aligned}$$

This contradicts  $T \equiv 1 \pmod{2}$ . □

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