2003

Special boundary value problems

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DOI: <https://doi.org/10.31979/etd.nvse-7htf>
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SPECIAL BOUNDARY VALUE PROBLEMS

A Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

Graciela Pousa Cochran

August 2003

UMI Number: 1417472

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ABSTRACT
SPECIAL BOUNDARY VALUE PROBLEMS

by Graciela Pousa Cochran

In this thesis we derive the solutions of the torsion problem of a cylinder with different cross sections bounded by simple closed curves. The torsion problem can be reduced to solving the two dimensional Dirichlet problem. Four different methods are shown.

The first method, convenient for rectangular curves, involves separation of variables and Fourier analysis. The second method, used to solve cross sections such as those bounded by an ellipse, triangle, or limaçon, involves the use of complex functions. The third method, used to solve the problem for cross sections bounded by curves with equation $r = \cos^n(\theta/n)$ or $r = \sin^n(\theta/n)$, involves the use of combinatorial trigonometric identities. And the fourth method, used to solve the torsion problem for cross sections bounded by curvilinear polygons, involves the use of conformal mappings to arrive to solutions.

ACKNOWLEDGEMENTS

I would like to express my thanks to my committee members, Dr. Leslie Foster and Dr. Mohammad Saleem, for their helpful suggestions and comments. I also wish to thank Dr. Richard Kubelka for his patience and invaluable guidance with many administrative issues. A personal thanks to Fernanda Karp for running around with all my papers and having them signed on time to meet registration deadlines, at a time when I needed that due to the fact I was about nine months pregnant. A special thanks to my husband, Stephen, who drove me every Tuesday for five months, to see my advisor and waited for me in the car with my three wonderful and active children. I am also grateful to my daughter Victoria—who was born in the midst of my thesis writing preparations—for keeping me company during my many hours of study during the first eight months of her life.

But all over and above I am most thankful to Dr. Samih Obaid, my advisor, who contributed grandly to the writing of this thesis by working tirelessly and patiently, with an incredible attention to detail and professionalism. In addition, I am thankful to Dr. Obaid for teaching me so much about the teaching of mathematics and for modeling what caring about your students' learning really means: being kind and patient, and demanding high quality work, all at the same time—the perfect formula. I am extremely lucky for having found such an outstanding mathematician and exceptional human being to guide me with my thesis. I will not forget to thank Dr. Evelyn Obaid, Dr. S. Obaid's wife, whose writings in math are models of beautiful prose and, last but not least, for letting her husband work so many extra hours with my thesis.

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INTRODUCTION

A very important problem in mathematical physics is the two-dimensional Dirichlet's problem for a simple closed curve. One of the situations from which it arises is the classical Saint-Venant's torsion problem of a homogeneous right cylinder of length l with lateral surfaces free of external load. When a twisting motion is applied to the bases of the beam, the beam is subjected to stresses. The calculation of these stresses is called the torsion problem. The stresses can cause deformations and potential failure of the beam and therefore the solution to the torsion problem is important in practice.

There are two common approaches to doing these calculations—numerical techniques that calculate approximate solutions and analytic techniques that lead to exact solutions. Analytical solutions, when they can be determined, are often preferable because they are exact and can usually be calculated quickly. In this thesis we will show that analytic solutions to the torsion problem are possible for a variety of cylindrical shaped beams with a uniform cross section bounded by a simple closed curve.

Several authors used various methods to solve the problem to obtain exact solutions, including Stevenson (1938) [19], Morris (1940) [10], Higgins (1942) [7], Muskhelishvili (1954) [11], Sokolnikoff (1956) [17], Bassali (1960) [1], Bassali with Elsirafy (1970) [2], Bassali with S. Obaid (1981) [5], and S. Obaid with Rung (1985) [14]. In addition to deriving exact solutions, Higgins (1943) [8] also derived approximate solutions.

We assume that one of the bases of the cylinder lies in the xy plane and the other in the plane $Z=l$. We will consider different uniform, simply connected, cross sections S of the cylinder, bounded by the curves Γ . Twisting the cylinder by two couples (parallel forces) of the same magnitude and opposite directions, applied at its two bases, results in displacements and shearing stresses, in which engineers are interested. So, besides deriving the torsion function, we will calculate the shearing stresses for some cross

sections. Another measure of interest to engineers is the torsional rigidity. The torsional rigidity, a measure of the strength of the bar, will be calculated for each cross section. Now, it has been shown that the torsion problem can be reduced to finding a function $\psi(x, y)$, harmonic conjugate of the torsion function $\phi(x, y)$, which is harmonic in the cross section S of the cylinder, and that satisfies the boundary condition $\psi(x, y) = (x^2 + y^2)/2$ on the boundary of S , Γ . We will introduce four different methods to solve the torsion problem, each one for different bounding curves, and we will either start with the derivation of the torsion function or its harmonic conjugate.

In chapter one we show two methods. The first method, which will be used to solve the torsion problem for squares or rectangular bounding curves, involves separation of variables and Fourier series [15]. The second method, which was derived by Bassali and Obaid [4], will be used to solve the problem for cross sections bounded by an ellipse, triangle, limacon, lemniscate of Booth, cardioid, as well as others. This method involves the use of complex function theory and it is much more efficient than the use of Fourier series. In chapter two we will show the third method. The third method, originally developed by Bassali and Obaid [5], will be used to solve the torsion problem for cross sections bounded by curves with polar equation $r = \sin^n(\theta/n)$. Later, this method was used by Obaid and Rung [14] to solve the torsion problem for cross sections bounded by curves with polar equation $r = \cos^n(\theta/n)$, which we will derive as well. It involves the use of combinatorial trigonometric identities. And finally, we introduce the fourth method in chapter three. The fourth method, which will be used to solve the torsion problem for cross sections bounded by curvilinear polygons, involves the use of conformal mappings to arrive to solutions [3]. Using the fourth method, the author independently solved the torsion problem for additional curves.

We would like to mention that MAPLE and a TI 89 automatic grapher have been used to compute some lengthy and complicated integrals and residues, as well as to generate graphs. Whenever we considered appropriate we included the MAPLE commands used.

CHAPTER 1

THE TORSION PROBLEM AND ITS SOLUTION USING THE FOURIER SERIES AND BASSALI-OBAID'S METHOD

1.1 The Problem

Let S be a uniform cross section of the cylinder described in the introduction bounded by a simple closed curve Γ in the complex z -plane which we will choose to be $Z=0$, perpendicular to the axis of the cylinder. (We work with axes x, y , and Z and $z = x + iy$.) Twisting the cylinder at its two bases by two couples of the same magnitude and opposite directions, with moments directed along the axis of the bar, we get the well known displacements (u, v, w) ($u = -Tzy$, $v = Tzx$) and the non-vanishing stresses Z_x and Z_y (Bassali and Obaid [4], Sokolnikoff [17]), which are given by

$$(1.1.1) \quad u + iv = iTzZ, \quad w = T \phi(x, y) = \frac{1}{2} T [\Omega(z) + \bar{\Omega}(\bar{z})],$$

$$(1.1.2) \quad Z_x = \mu T \left(\frac{\partial \phi}{\partial x} - y \right) = \mu T \left(\frac{\partial \psi}{\partial y} - y \right),$$

$$(1.1.3) \quad Z_y = \mu T \left(\frac{\partial \phi}{\partial y} + x \right) = \mu T \left(-\frac{\partial \psi}{\partial x} + x \right),$$

$$(1.1.4) \quad Z_x - iZ_y = \mu T [\Omega'(z) - i\bar{z}],$$

where $z = x + iy$, T is the constant twist per unit length, μ is the rigidity of the material of the cylinder, ϕ is the torsion function, ψ is the conjugate harmonic function, related to $\phi(x, y)$ by the Cauchy-Riemann equations,

$$(1.1.5) \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \quad \text{and}$$

$$(1.1.6) \quad \Omega(z) = \phi(x, y) + i\psi(x, y)$$

is the complex torsion function which is analytic in S . It is also known that

$\psi(x, y)$ satisfies the boundary condition

$$(1.1.7) \quad \psi(x, y) = \frac{1}{2}(x^2 + y^2) \quad \text{on } \Gamma$$

(For a proof see [14], p. 114), and

$$(1.1.8) \quad \nabla^2 \psi = 0$$

in the interior of S (Dirichlet's problem). In order to solve the torsion problem for a particular cylinder, we must find $\psi(x, y)$ or, equivalently, the stress function

$$(1.1.9) \quad \Psi(x, y) = \psi(x, y) - \frac{1}{2}(x^2 + y^2),$$

which satisfies Poisson's equation

$$(1.1.10) \quad \nabla^2 \Psi = -2$$

in the interior of S ; and

$$(1.1.11) \quad \Psi = 0 \quad \text{on } \Gamma.$$

The stresses, in terms of Ψ , are obtained by partial differentiation of (1.1.9) with respect to y and x , and inserting in (1.1.2) and (1.1.3) respectively. They are given by:

$$(1.1.12) \quad Z_x = \mu T \frac{\partial \Psi}{\partial y} \quad \text{and} \quad Z_y = -\mu T \frac{\partial \Psi}{\partial x}.$$

The torsional rigidity, which represents the strength of the beam and is of interest to engineers, is given by

$$(1.1.13) \quad D = 2\mu \iint_S \Psi(x, y) dx dy.$$

In polar coordinates (r, θ) the stresses are determined by

$$(1.1.14) \quad Z_r = \frac{\mu T}{r} \frac{\partial \Psi}{\partial \theta} \quad \text{and} \quad Z_\theta = -\mu T \frac{\partial \Psi}{\partial r},$$

and the torsional rigidity is given by

$$(1.1.15) \quad D = 2\mu \iint_S \Psi(r, \theta) r dr d\theta.$$

The shearing stress, or the amount of force per unit area the cross section of the bar is subjected to when twisting it, is given by

$$(1.1.16) \quad \sigma(r, \theta) = \sqrt{Z_r^2 + Z_\theta^2}.$$

It is well known that the maximum shearing stress occurs on the boundary of the cross section Γ . The maximum and/or minimum shearing stress on Γ can be found by solving the equation

$$(1.1.17) \quad \frac{d\sigma(\theta)}{d\theta} = 0.$$

Notice that both functions, ψ and Ψ , must be continuous and finite in S .

1.2 Method 1: Fourier Series Method for the Square Cross Section

In this case we will solve the torsion problem when Γ is a square of side a with vertices $(0,0)$, $(a,0)$, (a,a) , and $(0,a)$ (**Fig. 1**). (The results in this section, 1.2, were obtained independently by the author). The equation of Γ is given by $xy(x-a)(y-a) = 0$

$$\begin{array}{c} \psi = (x^2 + a^2)/2 \\ \begin{array}{c} (a,a) \\ \square \\ \nabla^2 \psi = 0 \\ S \\ (0,0) \end{array} \\ \psi = y^2/2 \quad \psi = (a^2 + y^2)/2 \\ (a,0) \end{array}$$

Figure 1. Boundary Conditions on the Function ψ on the Square

So we must solve

$$(1.2.1) \quad \nabla^2 \psi = 0, \quad 0 < x < a, \quad 0 < y < a,$$

$$(1.2.2) \quad \psi(x, 0) = \frac{1}{2}(x^2 + 0^2), \quad 0 < x < a,$$

$$(1.2.3) \quad \psi(x, a) = \frac{1}{2}(x^2 + a^2), \quad 0 < x < a,$$

$$(1.2.4) \quad \psi(0, y) = \frac{1}{2}(0^2 + y^2), \quad 0 < y < a,$$

$$(1.2.5) \quad \psi(a, y) = \frac{1}{2}(a^2 + y^2), \quad 0 < y < a.$$

We will use the method of separation of variables. For simplicity, we write the function ψ as the sum of four conveniently chosen functions, then we solve the corresponding boundary value problem for each one, and finally we add the solutions obtained, which is the sought function ψ . (This can be done because of the superposition principle.)

Let

$$(1.2.6) \quad \psi(x, y) = \psi_1(x, y) + \psi_2(x, y) + \psi_3(x, y) + \psi_4(x, y),$$

where

$$(1.2.7) \quad \nabla^2 \psi_1(x, y) = 0, \text{ or } \frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0 \quad \text{in } S$$

$$(1.2.8) \quad \psi_1(0, y) = 0,$$

$$(1.2.9) \quad \psi_1(a, y) = 0,$$

$$(1.2.10) \quad \psi_1(x, 0) = \frac{1}{2}x^2, \text{ and}$$

$$(1.2.11) \quad \psi_1(x, a) = 0.$$

Separating variables, differentiating with respect to x and y (“’” means differentiation with respect to the variable of each function) and inserting in (1.2.7) we obtain

$$(1.2.12) \quad \psi_1(x, y) = X(x)Y(y),$$

$$(1.2.13) \quad \psi_{1xx}(x, y) = X''(x)Y(y), \quad \psi_{1yy}(x, y) = X(x)Y''(y),$$

and

$$(1.2.14) \quad X''Y + XY'' = 0.$$

Since we have two zeros on the “ x ’s” (at $x=0$ and $x=a$) we want sine and cosine solutions with X . Dividing both sides of (1.2.14) by XY and subtracting Y''/Y we obtain

$$(1.2.15) \quad X''/X = -Y''/Y.$$

Now, X contains x only, and Y contains y only, then (1.2.15) is possible only if each side is constant, which we will name $-\lambda^2$. So

$$(1.2.16) \quad X''/X = Y''/Y = -\lambda^2,$$

$$(1.2.17) \quad X'' + \lambda^2 X = 0,$$

and solving this ordinary differential equation we obtain:

$$(1.2.18) \quad X(x) = A \cos(\lambda x) + B \sin(\lambda x)$$

where A and B are real constants. Using the boundary conditions (1.2.8) and (1.2.9) we obtain:

$$(1.2.19) \quad X(0) = A \cos 0 + B \sin 0 = 0,$$

which implies that $A = 0$, and

$$(1.2.20) \quad X(a) = B \sin(\lambda a) = 0.$$

$B \neq 0$, otherwise $\psi_1 \equiv 0$ for all points in its domain, which would contradict, for

example, $\psi_1(1, 0) = \frac{1}{2}$. Therefore,

$$(1.2.21) \quad \sin(\lambda a) = 0,$$

and thus

$$(1.2.22) \quad \lambda_n = n\pi / a, \quad n = 1, 2, 3, \dots$$

(λ can't be zero, since in that case we would have, substituting in (1.2.17), $X'' = 0$,

leading to $X = cx + d$ with c and d being real numbers, and using the boundary

conditions, we would obtain $X = 0$, which, as already shown, is not a solution).

Therefore,

$$(1.2.23) \quad X_n(x) = \sin(\lambda_n x), \quad \lambda_n = n\pi / a, \quad n = 1, 2, 3, \dots$$

The functions Y 's that accompany the X 's are the solutions to:

$$(1.2.24) \quad Y'' - Y\lambda^2 = 0,$$

which are

$$(1.2.25) \quad Y_n(y) = a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y).$$

Now $X_n Y_n$ is a solution to (1.2.7) that satisfies (1.2.8) and (1.2.9). By the superposition

principle, the sum of these functions will satisfy the same conditions, so:

$$(1.2.26) \quad \psi_1(x, y) = \sum_{n=1}^{\infty} ([a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)] \sin(\lambda_n x)),$$

From the boundary condition (1.2.10) we get

$$(1.2.27) \quad \psi_1(x, 0) = \sum_{n=1}^{\infty} ([a_n \cosh(\lambda_n 0) + b_n \sinh(\lambda_n 0)] \sin(\lambda_n x)) = \sum_{n=1}^{\infty} (a_n \sin(\lambda_n x)) = \frac{1}{2} x^2$$

which is the half-range Fourier Series expansion. Thus, a_n must be the Fourier sine

coefficients of $\frac{1}{2} x^2$

$$a_n = \frac{2}{a} \int_0^a \frac{1}{2} x^2 \sin(n\pi x / a) dx.$$

Computing the a_n using MAPLE we obtain

`a[n]:=Int((x^2*sin(n*Pi*x/a))/a, x=0...(a)): a[n]:=value(a[n]);`

$$(1.2.28) \quad a_n := - \frac{a^2 (-2 \cos(n \pi) + n^2 \pi^2 \cos(n \pi) - 2 n \pi \sin(n \pi) + 2)}{n^3 \pi^3}$$

or, simplifying we have

$$(1.2.29) \quad a_n = \frac{a^2}{n^3 \pi^3} [2(-1)^n - n^2 \pi^2 (-1)^n - 2]$$

ψ_1 must also satisfy the second boundary condition (1.2.11), then

$$(1.2.30) \quad \psi_1(x, a) = \sum_{n=1}^{\infty} ([a_n \cosh(\pi n a / a) + b_n \sinh(\pi n a / a)] \sin(\lambda_n x)) = 0.$$

Again we recognize a problem in Fourier series where

$$(1.2.31) \quad a_n \cosh(n\pi) + b_n \sinh(n\pi) = c_n = \frac{2}{a} \int_0^a 0 \cdot \sin(\lambda_n x) dx = 0$$

and so we obtain:

$$(1.2.32) \quad b_n = \frac{-a_n \cosh(\pi n)}{\sinh(\pi n)} = -a_n \coth(\pi n).$$

Thus the harmonic function solution ψ_1 is determined by

(1.2.33)

$$\psi_1(x, y) = \sum_{n=1}^{\infty} \frac{a^2}{n^3 \pi^3 \sinh(n\pi)} [2(-1)^n + n^2 \pi^2 (-1)^{(n+1)} - 2] \sinh\left[\frac{n\pi}{a}(a-y)\right] \sin\left(\frac{n\pi}{a}x\right)$$

Similarly, we set the conditions on ψ_2 , as follows:

$$(1.2.34) \quad \nabla^2 \psi_2(x, y) = 0, \text{ or } \quad \frac{\partial^2 \psi_2}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial y^2} = 0,$$

$$(1.2.35) \quad \psi_2(0, y) = 0,$$

$$(1.2.36) \quad \psi_2(a, y) = 0,$$

$$(1.2.37) \quad \psi_2(x, 0) = 0,$$

and

$$(1.2.38) \quad \psi_2(x, a) = \frac{1}{2}(x^2 + a^2)$$

obtaining, after separation of variables and solving the eigenvalue problem as before

$$\psi_2(x, y) = \sum_{n=1}^{\infty} ([a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)] \sin(\lambda_n x)), \quad \lambda_n = \pi n / a, \quad n = 1, 2, 3,$$

...

To find the coefficients a_n and b_n we use the boundary conditions (1.2.37) and (1.2.38)

as follows:

$$(1.2.39) \quad \psi_2(x, 0) = \sum_{n=1}^{\infty} ([a_n \cosh(\lambda_n 0) + b_n \sinh(\lambda_n 0)] \sin(\lambda_n x)) = \sum_{n=1}^{\infty} (a_n \sin(\lambda_n x)) = 0.$$

This implies that $a_n = 0$. Using (1.2.38) we obtain b_n .

$$(1.2.40) \quad \psi_2(x, a) = \sum_{n=1}^{\infty} b_n (\sinh n\pi a / a) \sin(\lambda_n x) = \frac{1}{2}(x^2 + a^2)$$

$$(1.2.41) \quad b_n \sinh(n\pi) = c_n = \frac{2}{a} \int_0^a \frac{x^2 + a^2}{2} * \sin(\lambda_n x) dx$$

This integral's value, using MAPLE, is:

$$c[n] := \text{Int}(((x^2 + a^2) * \sin(n * \text{Pi} * x / a)) / a, x = 0 \dots (a)): c[n] := \text{value}(c[n]);$$

$$c_n := - \frac{a^2 (-2 \cos(n \pi) + 2 n^2 \pi^2 \cos(n \pi) - 2 n \pi \sin(n \pi) - n^2 \pi^2 + 2)}{n^3 \pi^3}$$

$$c_n = \frac{a^2 [2(-1)^n - 2n^2 \pi^2 (-1)^n + n^2 \pi^2 - 2]}{n^3 \pi^3}.$$

Therefore, b_n is given by

$$b_n = \frac{c_n}{\sinh(n\pi)}, n = 1, 2, 3, \dots$$

simplifying we obtain

$$(1.2.42) \quad b_n = \frac{a^2 [2(-1)^n - 2n^2 \pi^2 (-1)^n + n^2 \pi^2 - 2]}{n^3 \pi^3 \sinh(n\pi)}.$$

Thus, the harmonic function ψ_2 is given by

$$(1.2.43) \quad \psi_2(x, y) = \sum_{n=1}^{\infty} \frac{a^2 [2(-1)^n - 2n^2 \pi^2 (-1)^n + n^2 \pi^2 - 2]}{n^3 \pi^3 \sinh(n\pi)} \sinh\left(\frac{\pi n y}{a}\right) \sin\left(\frac{\pi n x}{a}\right).$$

In the same manner we set the conditions on ψ_3 , as follows:

$$(1.2.44) \quad \nabla^2 \psi_3(x, y) = 0, \text{ or } \quad \frac{\partial^2 \psi_3}{\partial x^2} + \frac{\partial^2 \psi_3}{\partial y^2} = 0,$$

$$(1.2.45) \quad \psi_3(0, y) = \frac{1}{2} y^2,$$

$$(1.2.46) \quad \psi_3(a, y) = 0,$$

$$(1.2.47) \quad \psi_3(x, 0) = 0,$$

and

$$(1.2.48) \quad \psi_3(x, a) = 0.$$

The steps to obtain ψ_3 are the same as those to obtain ψ_1 but interchanging x and y .

$$(1.2.49)$$

$$\psi_3(x, y) = \sum_{n=1}^{\infty} \frac{a^2}{n^3 \pi^3 \sinh(n\pi)} [2(-1)^n + n^2 \pi^2 (-1)^{(n+1)} - 2] \sinh\left[\frac{n\pi}{a}(a-x)\right] \sin\left(\frac{n\pi}{a}y\right).$$

The conditions for ψ_4 , symmetric to those for ψ_2 , are:

$$(1.2.50) \quad \nabla^2 \psi_4(x, y) = 0, \text{ or } \frac{\partial^2 \psi_4}{\partial x^2} + \frac{\partial^2 \psi_4}{\partial y^2} = 0$$

$$(1.2.51) \quad \psi_4(0, y) = 0,$$

$$(1.2.52) \quad \psi_4(a, y) = \frac{1}{2}(y^2 + a^2),$$

$$(1.2.53) \quad \psi_4(x, 0) = 0,$$

and

$$(1.2.54) \quad \psi_4(x, a) = 0.$$

As expected, the steps to obtain ψ_4 are similar to those to obtain ψ_2 . Thus we get

$$(1.2.55) \quad \psi_4(x, y) = \sum_{n=1}^{\infty} \frac{a^2 [2(-1)^n - 2n^2 \pi^2 (-1)^n + n^2 \pi^2 - 2]}{n^3 \pi^3 \sinh(n\pi)} \sinh\left(\frac{\pi n x}{a}\right) \sin\left(\frac{\pi n y}{a}\right).$$

Therefore, the solution of the torsion problem in the square is given by

$$(1.2.56) \quad \psi(x, y) = \sum_{i=1}^4 \psi_i(x, y).$$

1.3 Method 2: Bassali-Obaid's Method (BOM)

Bassali and Obaid developed an efficient method [4] to obtain the torsion function in the interior and boundary of a simply connected region, even for more complicated bounding curves than a square. The idea of their method, which we will call BOM, is to start with a complex analytic function, write the boundary Γ in terms of the analytic function—as we'll show in the next paragraph—and recognize for which region S would be solved.

Assume that the closed boundary Γ of the region S has the form:

$$(1.3.1) \quad \operatorname{Re} F(z) - (\alpha x^2 + \beta y^2 + \gamma) = 0$$

$$(\text{or } \operatorname{Re} F(z) = \alpha x^2 + \beta y^2 + \gamma)$$

where $F(z)$ is an arbitrary function of z , analytic in S , $\alpha, \beta, \gamma \in \mathbf{R}$, $\alpha + \beta \neq 0$.

We recall that on the boundary Γ , ψ satisfies the condition

$$(1.3.2) \quad \psi(x, y) = \frac{1}{2}(x^2 + y^2)$$

or in terms of the stress function

$$(1.3.3) \quad \Psi(x, y) = \psi(x, y) - \frac{1}{2}(x^2 + y^2) = 0 \text{ on } \Gamma.$$

We assume that the solution of the torsion problem corresponding to the region S bounded by the curve Γ (1.3.1) is given by:

$$(1.3.4) \quad \psi(x, y) = A \operatorname{Re} F(z) + B(x^2 - y^2) + C$$

where A, B, and C are constants to be determined by the boundary condition (1.3.2).

Substituting (1.3.1) and (1.3.2) in (1.3.4), we obtain on Γ :

$$(1.3.5) \quad A(\alpha x^2 + \beta y^2 + \gamma) + B(x^2 - y^2) + C = \frac{1}{2}(x^2 + y^2).$$

Rearranging terms we get

$$(A\alpha + B)x^2 + (A\beta - B)y^2 + (A\gamma + C) = \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

which implies

$$A\alpha + B = \frac{1}{2}, \quad (A\beta - B) = \frac{1}{2}, \quad A\gamma + C = 0.$$

Solving these equations for A, B, and C we obtain

$$(1.3.6) \quad A = \frac{1}{\alpha + \beta}, \quad B = \frac{\beta - \alpha}{2(\alpha + \beta)}, \quad C = \frac{-\gamma}{\alpha + \beta}.$$

Substituting (1.3.6) into (1.3.4) we get

$$(1.3.7) \quad \psi(x, y) = \frac{1}{\alpha + \beta} [\operatorname{Re} F(z) + \frac{1}{2}(\beta - \alpha)(x^2 - y^2) - \gamma]$$

Substituting (1.3.7) into (1.3.3) and simplifying the stress function becomes

$$(1.3.8) \quad \Psi(x, y) = \frac{1}{\alpha + \beta} [\operatorname{Re} F(z) - (\alpha x^2 + \beta y^2 + \gamma)].$$

We notice that $\nabla^2 \Psi = -2$ in the interior of S and $\Psi = 0$ on its boundary Γ , and also that

(1.3.8) is proportional to the expression appearing in (1.3.1).

Now we proceed to solve the torsion problem for several curves. We will use Bassali and Obaid's method to arrive to solutions in the rest of this chapter.

1.4 Solution for the Elliptical Cross Section

We start with $F(z) = z^2$ and we write the equation of Γ in the form given by (1.3.1)

$$(1.4.1) \quad F(x + iy) = x^2 - y^2 + 2ixy$$

$$(1.4.2) \quad \operatorname{Re} F(z) = x^2 - y^2.$$

Substituting (1.4.2) in (1.3.1) we obtain:

$$(1.4.3) \quad (x^2 - y^2) - (\alpha x^2 + \beta y^2 + \gamma) = 0,$$

or

$$(1.4.4) \quad (1 - \alpha)x^2 + (-1 - \beta)y^2 + \gamma = 0,$$

dividing by $-\gamma$ (and assuming $\gamma \neq 0$) we get:

$$(1.4.5) \quad \frac{(1 - \alpha)}{-\gamma}x^2 + \frac{(-1 - \beta)}{-\gamma}y^2 + \frac{\gamma}{-\gamma} = 0$$

or

$$(1.4.6) \quad \frac{\frac{x^2}{\gamma}}{\alpha - 1} + \frac{\frac{y^2}{\gamma}}{\beta + 1} - 1 = 0.$$

We immediately recognize the equation of an ellipse centered at the origin, so our bounding curve is

$$(1.4.7) \quad \Gamma: \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

with $a^2 = \frac{\gamma}{\alpha - 1}$ and $b^2 = \frac{\gamma}{\beta + 1}$, solving for α and β we obtain:

$$(1.4.8) \quad \alpha = \frac{-\gamma}{a^2} + 1 \text{ and } \beta = \frac{-\gamma}{b^2} - 1,$$

and substituting these values in (1.3.8) we get the expression for the stress function:

$$(1.4.9) \quad \Psi(x, y) = \frac{-a^2 b^2}{a^2 + b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right).$$

The torsional rigidity D (1.1.13) is:

$$(1.4.10) \quad D = 8\mu \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{-a^2 b^2}{a^2 + b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) dy dx = \frac{a^3 b^3 \pi \mu}{a^2 + b^2}$$

1.5 Solution for the Equilateral Triangular Cross Section

We start with $F(z) = z^3$ and we write the equation of Γ in the form given by (1.3.1)

$$(1.5.1) \quad F(x + iy) = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3),$$

$$(1.5.2) \quad \operatorname{Re} F(z) = x^3 - 3xy^2.$$

Substituting (1.5.2) in (1.3.1) we obtain:

$$(1.5.3) \quad (x^3 - 3xy^2) - (\alpha x^2 + \beta y^2 + \gamma) = 0.$$

Taking $\alpha = \beta = -3a$ and $\gamma = 4a^3$, the relation in (1.5.3) reduces to

$$(1.5.4) \quad x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3 = 0.$$

Equation (1.5.4) can be written in the form $(x-a)[(y - (\frac{x+2a}{\sqrt{3}}))][(y + (\frac{x+2a}{\sqrt{3}}))] = 0$,

which is clearly the equation of an equilateral triangle of side $2\sqrt{3}a$ See **Fig. 2**.

Figure 2. The equilateral triangle section ($a=1$)

Thus, the stress function takes the form

$$(1.5.5) \quad \Psi(x, y) = \frac{-1}{6a}(x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3)$$

And the torsional rigidity for this cross section is

$$(1.5.6) \quad D = 4\mu \int_{-2a}^a \int_0^{(x+2a)/\sqrt{3}} \frac{-1}{6a}(x^3 - 3xy^2 + 3ax^2 + 3ay^2 - 4a^3) dy dx = \frac{a\sqrt{3}}{5} \mu a^4.$$

1.6 Solution for the Cross Section Bounded by $r^m = a^m \cos(n\theta)$

We start with $F(z) = z^n$, where $z = r(\cos \theta + i \sin \theta)$ and $z^n = r^n(\cos n\theta + i \sin n\theta)$. The polar form of equation (1.3.1) is

$$(1.6.1) \quad \Gamma_n : r^n \cos n\theta - (\alpha r^2 \cos^2 \theta + \beta r^2 \sin^2 \theta + \gamma) = 0$$

Taking $\alpha = \beta, \gamma = 0$, we get

$$(1.6.2) \quad \Gamma_n : r^n \cos n\theta - \alpha r^2 = 0$$

Putting $m=2-n$, $\frac{1}{\alpha} = a^m$ (or $\alpha = a^{-m} = a^{n-2}$) we obtain

$$(1.6.3) \quad \Gamma_n : r^m = a^m \cos n\theta .$$

Using the polar form of (1.3.8) we arrive at the stress function $\Psi(r, \theta)$, where

$$(1.6.4) \quad \Psi(r, \theta) = \frac{1}{2\alpha} (r^n \cos n\theta - \alpha r^2),$$

or

$$(1.6.5) \quad \Psi(r, \theta) = \frac{1}{2a^{n-2}} \left(\frac{a^{n-2}}{a^{n-2}} r^n \cos n\theta - a^{n-2} r^2 \right).$$

Simplifying we have

$$(1.6.6) \quad \Psi(r, \theta) = \frac{1}{2} (a^m r^n \cos n\theta - r^2).$$

To ensure continuity at $r = 0$, we put the condition $n \geq 0$ and to obtain closed curves we put $n < 2$. Notice that if $n = 2$ $\Gamma : \theta = \pi k$ a line through the origin. To evaluate the torsional rigidity we consider two cases.

First case: $\frac{1}{2} \leq n < 2$. We show in **Fig. 3** the cross sections for $n = \frac{1}{2}, 1, \frac{3}{2}$ generated by

MAPLE. The MAPLE command is as follows:

```
with(plots):polarplot([[cos((1/2)*t),t,t=-Pi ...Pi],[cos(t),t,t=-Pi/2
...Pi/2],[cos(3/2*t)^2,t,t=-
Pi/3 ...Pi/3]],color=[black,black,black]);
```

Fig. 3. Cross sections bounded by $r^m = a^m \cos(n\theta)$, $n=1/2, 1, 3/2$ (outer to inner)

The torsional rigidity is given by

$$(1.6.7) \quad D_n = 2\mu \int_{-\pi/2n}^{\pi/2n} \int_0^{a[\cos(n\theta)]^{1/(2-n)}} \frac{1}{2} (\alpha^{2-n} r^n \cos(n\theta) - r^2) r dr d\theta.$$

Now we calculate the torsional rigidities D_n for several values of n using MAPLE

(MAPLE uses the lower case d_n rather than D_n). Here is the MAPLE command:

```
for n from 1/2 by 1/12 to 7/4 do d[n]:=int (mu*a^4*(cos(n*theta)^(4/(2-n))*(2-
n)/(2*(n+2))),theta=0 ...Pi/(2*n))end do
```

$$d_{1/2} := \frac{9}{32} \frac{\mu a^4 \Gamma\left(\frac{5}{6}\right) \sqrt{3} \Gamma\left(\frac{2}{3}\right)}{\sqrt{\pi}} \quad d_{7/12} := \frac{289}{784} \frac{\mu a^4 \Gamma\left(\frac{31}{34}\right) \sqrt{\pi}}{\Gamma\left(\frac{7}{17}\right)} \quad d_{2/3} := \frac{1}{4} \mu a^4$$

$$d_{3/4} := \frac{5}{288} \frac{\mu a^4 \pi^{(3/2)} \csc\left(\frac{1}{10} \pi\right)}{\Gamma\left(\frac{9}{10}\right) \Gamma\left(\frac{3}{5}\right)} \quad d_{5/6} := \frac{21}{800} \frac{\mu a^4 \Gamma\left(\frac{3}{14}\right) \sqrt{\pi}}{\Gamma\left(\frac{5}{7}\right)}$$

$$\begin{aligned}
d_{\frac{11}{12}} &:= \frac{117}{3872} \frac{\mu a^4 \Gamma\left(\frac{9}{26}\right) \sqrt{\pi}}{\Gamma\left(\frac{11}{13}\right)} & d_1 &:= \frac{1}{32} \pi \mu a^4 & d_{\frac{13}{12}} &:= \frac{1815}{10816} \frac{\mu a^4 \Gamma\left(\frac{15}{22}\right) \sqrt{\pi}}{\Gamma\left(\frac{2}{11}\right)} \\
d_{\frac{7}{6}} &:= \frac{225}{3136} \frac{\mu a^4 \Gamma\left(\frac{9}{10}\right) \sin\left(\frac{2}{5} \pi\right) \Gamma\left(\frac{3}{5}\right)}{\sqrt{\pi}} & d_{\frac{5}{4}} &:= \frac{21}{1600} \frac{\pi^{(3/2)} \mu a^4}{\Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{2}{3}\right)} \\
d_{\frac{4}{3}} &:= \frac{3}{256} \pi \mu a^4 & d_{\frac{17}{12}} &:= \frac{5733}{184960} \frac{\mu a^4 \Gamma\left(\frac{13}{14}\right) \sqrt{\pi}}{\Gamma\left(\frac{3}{7}\right)} & d_{\frac{3}{2}} &:= \frac{5}{768} \pi \mu a^4 \\
d_{\frac{19}{12}} &:= \frac{16445}{5175296} \frac{\pi^{(3/2)} \mu a^4 \csc\left(\frac{3}{10} \pi\right)}{\Gamma\left(\frac{7}{10}\right) \Gamma\left(\frac{4}{5}\right)} & d_{\frac{5}{3}} &:= \frac{63}{20480} \pi \mu a^4 & d_{\frac{7}{4}} &:= \frac{429}{229376} \pi \mu a^4
\end{aligned}$$

Decimal approximations can be generated by MAPLE with the following command:

```

for n from .5 by .1 to 1.9 do d[n]:=int (mu*a^4*(cos(n*theta)^(4/(2-n))*(2-n) /
(2*(n+2))), theta=0 ...Pi/(2*n)) end do

```

Some of the results are

$$\begin{aligned}
d_{.5} &:= .4200942377 \mu a^4 \\
d_{.8} &:= .1707644986 \mu a^4 \\
d_{1.0} &:= .09817477044 \mu a^4 \\
d_{1.2} &:= .05555555556 \mu a^4 \\
d_{1.5} &:= .02045307718 \mu a^4 \\
d_{1.8} &:= .004046340383 \mu a^4
\end{aligned}$$

Other interesting results, obtained using a similar command, are

$$d_{1.25} := .04781405963 \mu a^4 \qquad d_{1.75} := .005875694271 \mu a^4$$

Second case: $0 \leq n < 1/2$. We show in **Fig. 4** the cross sections for $n = 0, 1/4, 1/3$. The

MAPLE command is

```
with(plots):polarplot([[cos((0)*t),t,t=-Pi...Pi],[cos(t/4)^(4/7),t,t=-Pi...Pi],[cos(t/3)^(3/5),t,t=-Pi...Pi]], color=black);
```

Fig. 4. Cross sections bounded by $r^m = a^m \cos(n\theta)$, $n=0, 1/4, 1/3$ (from outer to inner)

The torsional rigidities, calculated with the TI 89 are shown after the grapher's command:

```
seq(∫ (cos(n * x) ^ (4/(2 - n)) * (2 - n) / (4 * n + 8), x, -π, π), n, 0, .45, .05)
```

n	D_n
0	$1.5708 \mu a^2$
.05	$1.4816 \mu a^2$
.10	$1.3730 \mu a^2$
.15	$1.2487 \mu a^2$
.20	$1.1138 \mu a^2$
.25	$.9744 \mu a^2$
.30	$.8373 \mu a^2$
.35	$.7091 \mu a^2$
.40	$.5950 \mu a^2$
.45	$.4986 \mu a^2$

1.7 Solution for the Cross Section Bounded by the Cardioid

In sections 1.3 to 1.6 we started with a complex analytic function $F(z)$, found its real part and inserted it in relation (1.3.1). From there, we found the curve that bounds the cross section of the cylinder. In this section we start with the curve and try to determine the $F(z)$ that will lead to the solution of the torsion problem corresponding to the curve.

Let the equation of the cardioid be

$$(1.7.1) \quad r = a(1 + \cos \theta)$$

where a is a real constant and $-\pi \leq \theta \leq \pi$. Then

$$(1.7.2) \quad r^2 = a(r + r \cos \theta) = a(\sqrt{r}\sqrt{r} + r \cos \theta).$$

Now $\sqrt{r}$ can be written as follows:

$$(1.7.3) \quad \sqrt{r} = \sqrt{a(1 + \cos \theta)} = \sqrt{a2 \cos^2\left(\frac{\theta}{2}\right)} = \sqrt{2a} \cos\left(\frac{\theta}{2}\right),$$

and inserting this result in (1.7.2) we obtain

$$(1.7.4) \quad r^2 = a(\sqrt{r}\sqrt{2a} \cos\left(\frac{\theta}{2}\right) + r \cos \theta).$$

From elementary complex variable theory we know that if $z = re^{i\theta}$, then $\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}}$ and

$$(1.7.5) \quad \operatorname{Re}(\sqrt{z}) = \operatorname{Re}[\sqrt{r}(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})] = \sqrt{r} \cos \frac{\theta}{2}.$$

Substituting this result in (1.7.4) and simplifying we have

$$(1.7.6) \quad r^2 = a[\sqrt{2a} \operatorname{Re}(\sqrt{z}) + \operatorname{Re}(z)] = a \operatorname{Re}(\sqrt{2a}\sqrt{z} + z) = a \operatorname{Re}(\sqrt{2a}z + z).$$

So we take

$$(1.7.7) \quad F(z) = \sqrt{2az} + z, \quad \alpha = \beta = \frac{1}{a} \text{ and } \gamma = 0.$$

From (1.7.6) we can easily verify that $\operatorname{Re}(\sqrt{2az} + z) = \frac{r^2}{a} = \alpha(x^2 + y^2) = \alpha x^2 + \beta y^2 + \gamma$.

Therefore the stress function is given by

$$(1.7.8) \quad \Psi(r, \theta) = \frac{a}{2} \left[\sqrt{2ar} \cos\left(\frac{\theta}{2}\right) + r \cos \theta - \frac{1}{a} r^2 \right] = \frac{1}{2} \left[a\sqrt{2ar} \cos\left(\frac{\theta}{2}\right) + ar \cos \theta - r^2 \right].$$

And the torsional rigidity is

$$(1.7.9) \quad D = 2 * 2\mu \int_0^\pi \int_0^{a(1+\cos\theta)} \frac{1}{2} \left[a\sqrt{2ar} \cos\left(\frac{\theta}{2}\right) + ar \cos \theta - r^2 \right] r dr d\theta = \frac{17}{16} \mu a^4$$

The peripheral shearing stress can be obtained using formulas (1.1.14) and (1.1.16) as we show next. First we find the stresses

$$Z_r = \frac{\mu T}{r} \frac{\partial \Psi}{\partial \theta} = -\frac{\mu T}{4} \left(a \sin \frac{\theta}{2} \frac{\sqrt{2ar}}{r} + 2 \sin \theta \right)$$

and

$$Z_\theta = -\mu T \frac{\partial \Psi}{\partial r} = -\frac{\mu T}{4} \left(2 \cos \theta - 4r + \frac{\sqrt{2a^2} \cos(\theta/2)}{\sqrt{ar}} \right),$$

then we insert the two values in (1.1.16) obtaining

$$\sigma(r, \theta) = \sqrt{Z_r^2 + Z_\theta^2},$$

and finally we substitute r by $a(1 + \cos \theta)$ obtaining

$$(1.7.10) \quad \sigma(\theta) = \mu T a \left(\cos \frac{\theta}{2} + \frac{1}{4} \sec \frac{\theta}{2} \right)$$

1.8 Solution for the Cross Section Bounded by the Elliptic Limacon

Here we will also start with the equation of the elliptic limacon and find later the corresponding $F(z)$ and the numbers α , β , and γ . Before applying BOM to find the stress function Ψ for the cross section bounded by the elliptic limacon (**Fig. 5**):

$$(1.8.1) \quad \Gamma: \quad r = a + b \cos \theta \quad (a > b)$$

we will show some results from geometry regarding this curve.

In **Fig. 5** we show all the curves we will use to derive Ψ : the elliptic limacon Γ , with polar coordinates (r, θ) , circle with center at the origin and radius f , and ellipse Γ' , with polar coordinates (r', θ) , which is the inverse of the elliptic limacon with respect to the circle. The MAPLE command for **Fig. 5** follows

```
with(plots):polarplot([[3/5+2/5*cos(t),t,t=0...2*Pi],[1,t,t=0...2*Pi],[(5/3)/(1+(2/3)*cos(t)),t,t=0...2*Pi]],linestyle=[1,2,3],color=black);
```

Figure 5. Elliptic Limacon (solid) Γ : $a=3/5$, $b=2/3$. Ellipse (dash) Γ' : $e=2/3$, $a'=3$, center $(-2,0)$, focal distance $=2a'$, $e=4$. Foci: $(0,0)$ and $(-4,0)$. $c=5/4$. $H(-5/4,0)$. Circle with center at O , radius $f (=1)$ (dot). Γ and Γ' inverses of each other w/r circle

Let the equation of an ellipse referred to a focus O as pole be

$$(1.8.2) \quad \Gamma': \quad r' = \frac{L}{(1 + e \cos \theta)},$$

where it is known that $L = a'(1 - e^2)$, $2a'$ is the length of the major axis of the ellipse, and e is the eccentricity of the ellipse. (**Fig. 6**).

Let $P'(r', \theta)$ be a point on Γ' . If $P(r, \theta)$ is the inverse of $P'(r', \theta)$ with respect to the circle with center at O and radius f we have, since $rr' = f^2$,

$$(1.8.3) \quad r = \frac{f^2}{r'} = f^2 \left(\frac{1 + e \cos \theta}{L} \right) = a + b \cos \theta$$

where $\frac{f^2}{L} = a$ and $\frac{f^2 e}{L} = b$, and therefore $ae = b$ or $e = \frac{b}{a}$. Let H be the inverse, with

respect to the circle, of O' the other focus of the ellipse. Then,

$$(1.8.4) \quad OH \cdot OO' = f^2$$

But since $OO' = 2a'e$ and letting $OH = c$, we obtain

$$(1.8.5) \quad c = \frac{f^2}{2a'e} = \frac{La}{2a'e} = \frac{a'(1 - e^2)a}{2a'e} = \frac{(1 - e^2)a}{2e} = \frac{(1 - (\frac{b}{a})^2)a}{2\frac{b}{a}} = \frac{a^2 - b^2}{2b}.$$

Using the fact that $a > b$, we have

$$(1.8.6) \quad \frac{a^2 - b^2}{2b} > \frac{(a + b)(a - b)}{a + b} = a - b.$$

Therefore H lies outside the limaçon (on its axis, at a distance c from O).

Let P be a point on the elliptic limaçon Γ , and (R, Θ) the coordinates of P referred to H as a pole (**Fig. 6**),

Fig. 6. Elliptic limaçon Γ and ellipse Γ'

then

$$(1.8.7) \quad R = \sqrt{(c + r \cos \theta)^2 + (r \sin \theta)^2} = \sqrt{c^2 + r^2 + 2cr \cos \theta} = \frac{a^2 + b^2}{2b} + a \cos \theta,$$

where we used (1.8.5), or

$$(1.8.8) \quad R = \frac{a^2 - b^2 + 2b^2}{2b} + a \cos \theta = \frac{a^2 - b^2}{2b} + b + a \cos \theta = c + b + a \cos \theta.$$

Adding c to both sides of (1.8.8) and then simplifying its RHS we obtain

$$(1.8.9) \quad R + c = b + 2c + a \cos \theta = b + 2 \frac{a^2 - b^2}{2b} + a \cos \theta = \frac{a}{b} [a + b \cos \theta] = \frac{a}{b} r$$

or

$$(1.8.10) \quad R + c = \frac{a}{b} \left[\frac{b^2}{a} + \frac{2b}{a} \left(\frac{a^2 - b^2}{2b} \right) + b \cos \theta \right].$$

Using the Law of Cosines on triangle HOP (**Fig. 7**), we obtain

$$(1.8.11) \quad r = \sqrt{c^2 + R^2 - 2Rc \cos \Theta}$$

Figure 7. Triangle HOP

Substituting (1.8.11) in (1.8.9), squaring both sides and simplifying we arrive to

$$(1.8.12) \quad R^2 - \frac{R}{b}(b^2 + a^2 \cos \Theta) + c^2 = 0$$

which is a polar equation of Γ referred to H as a pole.

A line through H makes an angle Θ with $\overline{OH}$ meets the upper half of Γ at (R_1, Θ_1) and (R_2, Θ_2) . To find R_1 and R_2 we solve the quadratic equation (1.8.12) using the quadratic formula. After lengthy algebraic manipulation we obtain

$$(1.8.13) \quad R_{1,2} = \frac{1}{2b} \left(a \cos \frac{\Theta}{2} \pm \sqrt{b^2 - a^2 \sin^2 \frac{\Theta}{2}} \right)^2 \text{ for } 0 \leq \Theta \leq \Theta_0$$

where Θ_0 is the angle obtained when $R_1 = R_2$ (line is tangent to Γ).

Θ_0 is obtained setting the discriminant equal to zero and solving for Θ_0 as follows:

$$b^2 - a^2 \sin^2 \frac{\Theta}{2} = 0$$

and therefore

$$(1.8.14) \quad \Theta_0 = 2 \sin^{-1} \frac{b}{a}.$$

From (1.8.12) and using the property of quadratic equations that the product of its roots equals the constant term we have

$$(1.8.15) \quad R_1 R_2 = c^2 = \text{square of the length of the tangent from } H \text{ to } \Gamma$$

This is a property that circles have with respect to any point outside them, but for the limaçon it holds only for H .

Now we proceed to determine the analytic function $F(z)$, α , β , and γ corresponding to (1.8.1) which can be written as follows:

$$(1.8.16) \quad \frac{r^2}{b} = \frac{ar}{b} + r \cos \theta$$

By the Law of Cosines applied to triangle HOP angle (see **Fig. 7**) and the trigonometry formula of a half angle we obtain

$$r^2 = R^2 + c^2 - 2cR \cos \Theta$$

or

$$\frac{1 + \cos \Theta}{2} = \frac{R^2 + 2cR + c^2 - r^2}{4cR}$$

and therefore

$$\cos^2 \frac{\Theta}{2} = \frac{(R+c)^2 - r^2}{4cR}$$

resulting in

$$(1.8.17) \quad \cos \frac{\Theta}{2} = \sqrt{\frac{(R+c)^2 - r^2}{4cR}}.$$

Substituting $R+c = \frac{ar}{b}$ and $c = \frac{a^2 - b^2}{2b}$ in the equation above we have

$$(1.8.18) \quad r = \sqrt{2bR} \cos\left(\frac{\Theta}{2}\right).$$

Now we substitute (1.8.18) in the RHS of (1.8.16) and obtain

$$(1.8.19) \quad \frac{r^2}{b} = \frac{a\sqrt{2bR} \cos\left(\frac{\Theta}{2}\right)}{b} + r \cos \theta.$$

Now, if we let $Z = Re^{i\Theta}$ and $z = re^{i\theta}$ we have (see **Fig.7**) $Z = z + c$ and

$$(1.8.20) \quad z + c = Re^{i\Theta}$$

or

$$(1.8.21) \quad \sqrt{z + c} = \sqrt{Re^{i\frac{\Theta}{2}}} = \sqrt{R}\left(\cos \frac{\Theta}{2} + i \sin \frac{\Theta}{2}\right).$$

Therefore $\operatorname{Re}(\sqrt{z + c}) = \sqrt{R} \cos \frac{\Theta}{2}$. Substituting this last expression in (1.8.19) we obtain

$$(1.8.22) \quad \frac{r^2}{b} = \operatorname{Re}\left[z + a\sqrt{\frac{2}{b}}(z + c)\right] \text{ on } \Gamma.$$

This leads to

$$(1.8.23) \quad F(z) = z + a\sqrt{\frac{2}{b}}(z + c), \quad \alpha = \beta = \frac{1}{b}, \quad \gamma = 0.$$

Hence the stress function (1.3.8) takes the form

$$(1.8.24) \quad \Psi = \frac{b}{2}\left[r \cos \theta + \frac{a}{b}\sqrt{2bR} \cos \frac{\Theta}{2} - \frac{r^2}{b}\right] = \frac{b}{2}\left[r \cos \theta - \frac{r^2}{b} + a\sqrt{\frac{2R}{b}} \cos \frac{\Theta}{2}\right].$$

If u and v are real numbers, we will show that

$$(1.8.25) \quad \operatorname{Re} \sqrt{u+iv} = \frac{1}{\sqrt{2}} \sqrt{u + \sqrt{u^2 + v^2}}$$

Proof:

If we let $\sqrt{u+iv} = x+iy$

then $u+iv = (x+iy)^2 = x^2 + y^2 - 2xyi$

and $u^2 + v^2 = (x^2 + y^2)^2$.

So $\sqrt{u^2 + v^2} = x^2 + y^2$

and $u + \sqrt{u^2 + v^2} = x^2 + y^2 + x^2 - y^2 = 2x^2$.

Therefore $\sqrt{u + \sqrt{u^2 + v^2}} = \sqrt{2x^2}$

and $\frac{1}{\sqrt{2}} \sqrt{u + \sqrt{u^2 + v^2}} = x = \operatorname{Re} \sqrt{u+iv}$

which proves (1.8.25).

Applying (1.8.25) to $\sqrt{z+c}$, the stress function can be written in terms of r and θ as follows:

$$(1.8.26) \quad \Psi = \frac{b}{2} \left[r \cos \theta - \frac{r^2}{b} + \frac{a}{\sqrt{b}} \sqrt{r \cos \theta + c + \sqrt{r^2 + 2cr \cos \theta + c^2}} \right]$$

The torsional rigidity is given by

$$(1.8.27) \quad D = 4\mu \int_0^\pi \int_0^{a+b \cos \theta} \Psi(r, \theta) r dr d\theta = \frac{1}{16} \mu \pi (8a^4 + 8a^2 b^2 + b^4)^4,$$

The shearing stresses are

$$(1.8.28) \quad Z_r = -\frac{1}{2}\mu T b \sin \theta \left(1 + \frac{a^2}{a^2 + b^2 + 2ab \cos \theta}\right),$$

$$(1.8.29) \quad Z_\theta = \frac{1}{2}\mu T b r \left(1 + \frac{a^2}{a^2 + b^2 + 2ab \cos \theta}\right),$$

and

$$(1.8.30) \quad \sigma = \frac{\mu T (2a^2 + b^2 + 2ab \cos \theta)}{2\sqrt{a^2 + b^2 + 2ab \cos \theta}} = \frac{\mu T (2ar + b^2)}{2\sqrt{b^2 - a^2 + 2ar}}$$

(For more details concerning the calculations of the torsional rigidity and stresses see [4]).

We notice that if we put $a=b$ we obtain the formula of the cardioid and the formulas in section 1.8 will reduce to the ones in 1.7.

1.9 Solution for Cross Sections Bounded by a Set of Special Curves

To solve the torsion problem for some curves we mention below we start with $F(z)$ and specific values of α , β , and γ , and then we write the analytic function in terms of the curve Γ_n , as we did in sections 1.4, 1.5, and 1.6.

Let $F(z) = \sqrt{a^4 - z^4}$, $\alpha = \beta = \sqrt{\frac{n}{2}}$, $\gamma = 0$, and $z = re^{i\theta}$ in polar coordinates. Writing

$F(z)$ in terms of r and θ we obtain

$$(1.9.1) \quad F(re^{i\theta}) = \sqrt{a^4 - r^4 e^{4i\theta}}$$

expanding and then associating real and imaginary parts in the RHS of (1.9.1) we get

$$(1.9.2) \quad F(re^{i\theta}) = \sqrt{(a^4 - r^4 \cos 4\theta) + i(-r^4 \sin 4\theta)}.$$

Applying formula (1.8.25) that states $\operatorname{Re}\sqrt{u+iv} = \frac{1}{\sqrt{2}}\sqrt{u + \sqrt{u^2 + v^2}}$ to (1.9.2) we have

$$(1.9.3) \quad \begin{aligned} \operatorname{Re}F(re^{i\theta}) &= \frac{1}{\sqrt{2}}\sqrt{(a^4 - r^4 \cos 4\theta) + \sqrt{(a^4 - r^4 \cos 4\theta)^2 + (r^8 \sin^2 4\theta)}} \\ &= \frac{1}{\sqrt{2}}\sqrt{(a^4 - r^4 \cos 4\theta) + \sqrt{(a^8 - 2a^4 r^4 \cos 4\theta + r^8)}} \end{aligned}$$

Simplifying (1.9.3) and setting it equal to the equation of Γ_n , as the method dictates, we arrive to

$$(1.9.4) \quad \frac{1}{\sqrt{2}}\sqrt{(a^4 - r^4 \cos 4\theta) + \sqrt{(a^4 - r^4 \cos 4\theta)^2 + (r^8 \sin^2 4\theta)}} = \sqrt{\frac{n}{2}}r^2,$$

Squaring both sides, simplifying, isolating the root that is left and performing several algebraic steps we arrive to

$$(1.9.5) \quad r^4 = \frac{2na^4}{n^2 + 2n \cos 4\theta + \cos^2 4\theta - 1},$$

where we take $n > 2$ to obtain closed curves. See **Fig. 8** for the graphs for $n=100, 3$, and

2.1. We observe that the larger the n , the closer the curves are to circles.

```
with(plots): polarplot([[(6/((2+cos(4*t))*(4+cos(4*t))))^(1/4),t,t=-
Pi...Pi],[[4.2/((1.1+cos(4*t))*(3.1+cos(4*t))))^(1/4),t,t=-
Pi...Pi],[[200/((101+cos(4*t))*(99+cos(4*t))))^(1/4),t,t=-Pi...Pi]],color=black);
```

Figure 8. Cross sections bounded by Γ_n , for $n=2.1, 3, 100$ (outer to inner)

Now we write the stress function $\Psi(r, \theta)$

$$(1.9.6) \quad \Psi(r, \theta) = \frac{1}{2\sqrt{\frac{n}{2}}} (\operatorname{Re} \sqrt{a^4 - z^4} - r^2 \sqrt{\frac{n}{2}})$$

or

$$(1.9.7) \quad \Psi(r, \theta) = \frac{1}{2\sqrt{n}} (\sqrt{(a^4 - r^4 \cos 4\theta) + \sqrt{a^8 - 2a^4 r^4 \cos 4\theta + r^8}} - \sqrt{n} r^2).$$

And the torsional rigidity (for details see [12]) is given by

$$(1.9.8) \quad D_n = \frac{\mu a^4 \pi}{2\sqrt{n}} (\sqrt{n+2} - \sqrt{n-2}).$$

CHAPTER 2

METHOD 3: SOLUTION OF THE TORSION PROBLEM USING SPECIAL TRIGONOMETRIC AND COMBINATORIAL IDENTITIES

In this chapter we will derive solutions to the torsion problem for cross sections bounded by members of the families

$$C_n: \quad r = a \cos^n \frac{\theta}{n} \quad \text{with } -\pi < \theta \leq \pi, a > 0 \text{ and } n=1, 2, 3, \dots$$

and

$$\Gamma_n: \quad r = a \left| \sin \frac{\theta}{n} \right|^n \quad \text{with } -\pi < \theta \leq \pi, a > 0, \text{ and } n=2, 3, \dots$$

We show the solutions for the first family in section 2.1 and for the second in section 2.2. We will use several not common trigonometric and/or combinatorial identities to arrive to the solutions, whose proofs we will include at the end of sections 2.1. and 2.2.

2.1 Solutions for Cross Sections Bounded by $r = a \cos^n \frac{\theta}{n}$

In this section we derive solutions to the torsion problem for cross sections bounded by members of the family

$$(2.1.1) \quad C_n: \quad r = a \cos^n \frac{\theta}{n} \quad \text{with } -\pi < \theta \leq \pi, a > 0 \text{ and } n = 1, 2, 3, \dots$$

These curves are symmetric with respect to the polar axis. See **Fig. 9**.

Next follows the MAPLE command for the graphs of C_n for $n=1, 2, 3$, and 4.

```
with(plots): polarplot([cos(t),t,t=-Pi...Pi],[cos(t/2))^2,t,t=-
Pi...Pi],[cos(t/3))^3,t,t=-Pi...Pi],[cos(t/4))^4,t,t=-Pi...Pi], color=black);
```

Fig. 9. C_1 , C_2 , C_3 , and C_4 (inner to outer) ($\alpha=1$)

Before starting the derivation of the solutions we will prove the following identity [5]:

$$(2.1.2) \quad 2^{2n} \cos^{2n} \theta = \sum_{k=1}^n 2^{k+1} \binom{2n-k-1}{n-1} \cos^k \theta \cos k\theta$$

Proof:

Let

$$(2.1.3) \quad z = e^{i\theta}, \quad \frac{1}{z} = e^{-i\theta}.$$

From the definition of $\cos \theta$ we have

$$(2.1.4) \quad \cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right), \quad \cos^k \theta = \frac{1}{2^k} \left(z + \frac{1}{z} \right)^k, \text{ and } \cos k\theta = \frac{1}{2} \left(z^k + \frac{1}{z^k} \right).$$

Using (2.1.4), the RHS of (2.1.2) can be written as

$$(2.1.5) \quad I = \sum_{k=1}^n p_k^n \left(z + \frac{1}{z} \right)^k \left(z^k + \frac{1}{z^k} \right),$$

where $p_k^n = \binom{2n-k-1}{n-1}$, or

$$(2.1.6) \quad I = \sum_{k=1}^n p_k^n (z^2 + 1)^k (z^k + \frac{1}{z^k}) \frac{1}{z^k} = \sum_{k=1}^n p_k^n (z^2 + 1)^k (1 + z^{-2k}).$$

Distributing and simplifying (2.1.6) we obtain

$$(2.1.7) \quad I = \sum_{k=1}^n p_k^n (z^2 + 1)^k + \sum_{k=1}^n p_k^n (1 + z^{-2})^k.$$

Now, the first summation in (2.1.7) can be expanded as follows

$$(2.1.8) \quad \sum_{k=1}^n p_k^n \left[\binom{k}{0} + \binom{k}{1} z^2 + \binom{k}{2} z^4 + \dots + \binom{k}{n-m} z^{2(n-m)} + \dots + \binom{k}{k} z^{2k} \right],$$

and the second one can be expanded in a similar manner, with powers of z^{2m-2n} instead

of powers of z^{2n-2m} . The coefficient of z^{2m-2n} or z^{2n-2m} is given by $\sum_{k=1}^n p_k^n \binom{k}{n-m}$ which

is equal to $\binom{2n}{m}$ (a proof is included at the end of section 2.1).

The independent term of (2.1.8) is given by $\sum_k p_k^n \binom{k}{0} + \sum_k p_k^n \binom{k}{0} = 2 \sum_k p_k^n$ which is

equal to $\binom{2n}{n}$ (a proof at the end of section 2.1).

Thus,

$$(2.1.9) \quad I = 2 \sum_{k=1}^n p_k^n + \sum_{m=0}^{n-1} \binom{2n}{m} [z^{2n-2m} + z^{2m-2n}] = \binom{2n}{n} + \sum_{m=0}^{n-1} \binom{2n}{m} z^{2n-2m} + \sum_{m=0}^{n-1} \binom{2n}{m} z^{2m-2n}$$

which can also be written as

$$(2.1.10) \quad I = \frac{1}{z^{2n}} \sum_{m=0}^{n-1} \binom{2n}{m} (z^2)^m + \binom{2n}{n} + \sum_{m=0}^{n-1} \binom{2n}{m} (z^2)^{n-m} \frac{(z^2)^n}{z^{2n}}.$$

We now combine the first sum and $\binom{2n}{n}$ in (2.1.10) and make the change of variables

$j = 2n - m$ in the second sum, obtaining:

$$(2.1.11) \quad I = \frac{1}{z^{2n}} \sum_{m=0}^{n-1} \binom{2n}{m} (z^2)^m + \binom{2n}{n} + \frac{1}{z^{2n}} \sum_{j=n+1}^{2n} \binom{2n}{2n-j} (z^2)^j,$$

which is equal to

$$(2.1.12) \quad I = \frac{1}{z^{2n}} \sum_{m=0}^{2n} \binom{2n}{m} (z^2)^m = \frac{1}{z^{2n}} (1 + z^2)^{2n} = \left(\frac{1}{z} + z\right)^{2n} = 2^{2n} \cos^{2n} \theta$$

which was what we wanted to prove.

Now we derive the stress function for C_n : $r = a \cos^n \frac{\theta}{n}$ with the conditions specified at

the beginning of this section. C_n can also be written as:

$$(2.1.13) \quad r^{\frac{1}{n}} = a \cos \frac{\theta}{n}.$$

Now we replace θ by $\frac{\theta}{n}$ in (2.1.2)

$$(2.1.14) \quad 2^{2n} \cos^{2n} \frac{\theta}{n} = \sum_{k=1}^n 2^{k+1} \binom{2n-k-1}{n-1} \cos^k \frac{\theta}{n} \cos k \frac{\theta}{n},$$

and next we substitute $\cos \frac{\theta}{n}$ by $\frac{r^{\frac{1}{n}}}{a}$ in (2.1.14) and arrive to

$$(2.1.15) \quad \frac{2^{2n}}{a^{2n}} (r^{\frac{1}{n}})^{2n} = \sum_{k=1}^n 2^{k+1} \binom{2n-k-1}{n-1} \frac{r^{\frac{k}{n}}}{a^k} \cos \frac{k}{n} \theta$$

or

$$(2.1.16) \quad \frac{r^2}{2} = \frac{a^{2n}}{2^{2n}} \sum_{k=1}^n 2^k \binom{2n-k-1}{n-1} \frac{r^{\frac{k}{n}}}{a^k} \cos \frac{k}{n} \theta$$

Therefore, the stress function is given by

$$(2.1.17) \quad \Psi_n(r, \theta) = -\frac{r^2}{2} + \sum_{k=1}^n 2^{k-2n} a^{2n-k} \binom{2n-k-1}{n-1} r^{\frac{k}{n}} \cos \frac{k}{n} \theta.$$

We now show expressions for the stress function and torsional rigidities for $n=2, 3$, and 4, and $a=1$. The TI 89 graphing calculator is used to evaluate the torsional rigidities.

$$(2.1.18) \quad \Psi_2(r, \theta) = -\frac{r^2}{2} + \frac{r^{1/2}}{4} \cos \frac{\theta}{2} + \frac{r}{4} \cos \theta$$

$$(2.1.19) \quad \Psi_3(r, \theta) = -\frac{r^2}{2} + \frac{3r^{1/2}}{16} \cos \frac{\theta}{3} + \frac{3r^{2/3}}{16} \cos \frac{2\theta}{3} + \frac{r}{8} \cos \theta$$

$$(2.1.20) \quad \Psi_4(r, \theta) = -\frac{r^2}{2} + \frac{5r^{1/4}}{32} \cos \frac{\theta}{4} + \frac{5r^{1/2}}{32} \cos \frac{\theta}{2} + \frac{r^{3/4}}{8} \cos \frac{3\theta}{4} + \frac{r}{128} \cos \theta$$

$$(2.1.21) \quad D_2 = 4\mu \int_0^\pi \int_0^{\cos^2 \frac{\theta}{2}} \left(-\frac{r^2}{2} + \frac{r^{1/2}}{4} \cos \frac{\theta}{2} + \frac{r}{4} \cos \theta \right) r dr d\theta \approx \mu \cdot 0.212904$$

$$(2.1.22) \quad D_3 = 4\mu \int_0^\pi \int_0^{\cos^3 \frac{\theta}{3}} \left(-\frac{r^2}{2} + \frac{3r^{1/2}}{16} \cos \frac{\theta}{3} + \frac{3r^{2/3}}{16} \cos \frac{2\theta}{3} + \frac{r}{8} \cos \theta \right) r dr d\theta \approx \mu \cdot 0.83289$$

$$(2.1.23) \quad D_4 = 4\mu \int_0^\pi \int_0^{\cos^4 \frac{\theta}{4}} \left(-\frac{r^2}{2} + \frac{5r^{1/4}}{32} \cos \frac{\theta}{4} + \frac{5r^{1/2}}{32} \cos \frac{\theta}{2} + \frac{r^{3/4}}{8} \cos \frac{3\theta}{4} + \frac{r}{128} \cos \theta \right) r dr d\theta$$

$$\approx \mu \cdot 2.28112$$

These D 's were not known. D_2 is the torsional rigidity for a cardioid.

Proof of the combinatorial identities used in (2.1.8) and (2.1.9)

We now give proofs for: $2 \sum_{v=1}^n p_v^n = \binom{2n}{n}$ and $\sum_{v=1}^n p_v^n \binom{v}{n-m} = \binom{2n}{m}$

Let m and n be positive integers, $n > m$. It has been shown (Bassali and Obaid [5], Riordan [16]) that

$$(2.1.24) \quad \sum_{k=0}^{n-m} \binom{n-1-k}{m-1} = \binom{n}{m}$$

and

$$(2.1.25) \quad \sum_{k=0}^{n-m} \binom{n-j-k}{m-k} \binom{j+k-1}{j-1} = \binom{n}{m}.$$

Replacing n by $2n$ and m by n in (2.1.23) we obtain

$$(2.1.26) \quad \sum_{k=0}^{n-m} \binom{2n-1-k}{n-1} = \binom{2n}{n};$$

thus

$$(2.1.27) \quad \binom{2n-1}{n-1} + \sum_{k=1}^{n-m} \binom{2n-1-k}{n-1} = \binom{2n}{n}.$$

Taking into account that

$$(2.1.28) \quad \binom{2n-1}{n-1} = \frac{1}{2} \binom{2n}{n}$$

we conclude that

$$(2.1.29) \quad 2 \sum_{k=1}^{n-m} \binom{2n-1-k}{n-1} = \binom{2n}{n}$$

which is what we wanted to prove. To prove the second identity we notice (2.1.25) can be written as follows:

$$(2.1.30) \quad \sum_{k=0}^m \binom{n-j-k}{n-j-m} \binom{j+k-1}{j-1} = \binom{n}{m}.$$

Replacing n by $2n$, $j+k$ by $v+1$, and $j+m$ by $n+1$ we arrive to the identity we wanted to prove:

$$(2.1.31) \quad \sum_{v=n-m}^n \binom{2n-v-1}{n-1} \binom{v}{n-m} = \binom{2n}{m}$$

2.2 Solution for Cross Sections Bounded by $r = a \left| \sin \frac{\theta}{n} \right|^n$

In this section our boundary curves are given by

$$(2.2.1) \quad \Gamma_n: r = a \left| \sin \frac{\theta}{n} \right|^n$$

with $-\pi < \theta \leq \pi$, $a > 0$, and $n=2, 3, \dots$

These curves are symmetric with respect to the polar axis and have a cusp at the pole O.

See **Fig. 10**.

Next follows the MAPLE command for the graphs of Γ_n for $n=2, 3$, and 4.

```
with(plots): polarplot([abs(sin(t/2))^2,t=-Pi...Pi],[abs(sin(t/3))^3,t=-Pi...Pi],
```

`[abs(sin(t/4))^4,t=-Pi..Pi]], linestyle=[3,2,1], color=black);`

Figure 10. Γ_2 , Γ_3 , and Γ_4 (dash, dot, solid) ($\alpha=1$)

We want to find $\psi(r, \theta)$ such that on the boundary the condition $\psi(r, \theta) = \frac{1}{2}r^2$ is

satisfied on Γ_n . For the curve Γ_n we need

$$\frac{1}{2}r^2 = \frac{1}{2}a^2 \left(\sin \frac{\theta}{n}\right)^{2n}$$

or

$$(2.2.2) \quad \frac{r^2}{a^2} = \left(\sin \frac{\theta}{n}\right)^{2n}.$$

So we proceed to find a special expression for $\left(\sin \frac{\theta}{n}\right)^{2n}$.

From (2.1.2) we know that

$$(2.2.3) \quad (2 \cos x)^{2n} = \sum_{v=1}^n \binom{2n-v-1}{n-1} 2^{v+1} (\cos x)^v \cos(vx).$$

Setting $\frac{\pi}{2} - x$ instead of x in (2.2.3) we have

$$(2.2.4) \quad (2 \cos(\frac{\pi}{2} - x))^{2n} = \sum_{v=1}^n \binom{2n-v-1}{n-1} (2 \cos(\frac{\pi}{2} - x))^v 2 \cos v(\frac{\pi}{2} - x),$$

but from elementary trigonometry (and adopting the notation $s_v = 2 \sin vx$) we have

$$(2.2.5) \quad 2 \cos(x - \frac{\pi}{2}) = 2 \sin x = s_1.$$

Then (2.2.4) becomes

$$(2.2.6) \quad (2 \sin x)^{2n} = \sum_{v=1}^n \binom{2n-v-1}{n-1} (2 \sin x)^v [2 \cos \frac{v\pi}{2} \cos(vx) + 2 \sin \frac{v\pi}{2} \sin(vx)],$$

thus we have

$$(2.2.7) \quad s_1^{2n} = \sum_{v=1}^n \binom{2n-v-1}{n-1} s_1^v (2 \cos(vx)) \cos \frac{v\pi}{2} + \sum_{v=1}^n \binom{2n-v-1}{n-1} s_1^v (2 \sin(vx)) \sin \frac{v\pi}{2}$$

or

$$(2.2.8) \quad s_1^{2n} = \sum_{v=1}^n \binom{2n-v-1}{n-1} s_1^v c_v \cos \frac{v\pi}{2} + \sum_{v=1}^n \binom{2n-v-1}{n-1} s_1^v s_v \sin \frac{v\pi}{2},$$

where $c_v = 2 \cos vx$.

We can rewrite (2.2.8) so that we skip all the zero terms. To do it, we will consider the cases when n is odd and when n is even.

Case n odd. When n is odd we have a total of $(n-1)/2$ nonzero terms in the first summation and $(n+1)/2$ in the second summation of (2.2.8). The zero terms are due to the fact that $\cos v \frac{\pi}{2}$ is zero for v odd (and ± 1 for v even) and $\sin v \frac{\pi}{2}$ is zero for v even (and ± 1 for v odd).

Case n even. When n is even, and for the same reasons as with n odd, we have a total of $n/2$ nonzero in each summation of (2.1.16).

So, changing the subindices $v \rightarrow n - 2v + 1$ (or $v \rightarrow n - 2v$) to reflect the simplifications mentioned above we have:

$$(2.2.9) \quad s_1^{2n} = (-1)^{(n-1)/2} \left[\sum_{v=1}^{(n-1)/2} \binom{2n - (n - 2v + 1) - 1}{n-1} s_1^{n-2v+1} c_{n-2v+1} (-1)^{v-1} + \right. \\ \left. + \sum_{v=0}^{(n-1)/2} \binom{2n - (n - 2v) - 1}{n-1} s_1^{n-2v} s_{n-2v} (-1)^v \right]$$

for n odd, and

$$(2.2.10) \quad s_1^{2n} = (-1)^{n/2} \left[\sum_{v=0}^{(n/2)-1} \binom{n+2v-1}{n-1} s_1^{n-2v} c_{n-2v} (-1)^v + \right. \\ \left. + \sum_{v=1}^{n/2} \binom{2n+2v-2}{n-1} s_1^{n-2v+1} s_{n-2v+1} (-1)^v \right]$$

for n even.

If $P(r, \theta)$ is any point in the upper half of Γ_n ($0 \leq \theta \leq \pi$) then we have, from (2.2.1),

$$(2.2.11) \quad \sin \frac{\theta}{n} = \left(\frac{r}{a} \right)^{\frac{1}{n}}.$$

Replacing x by $\frac{\theta}{n}$ in $s_1^{2n} = (2 \sin x)^{2n}$, (2.2.11) yields

$$(2.2.12) \quad s_1^{2n} = \left(2 \sin \frac{\theta}{n}\right)^{2n} = 2^{2n} \left[\left(\frac{r}{a}\right)^{\frac{1}{n}}\right]^{2n} = 2^{2n} \frac{r^2}{a^2},$$

and then substituting (2.2.12) in (2.2.9) and (2.2.10) we have for n odd

$$(2.2.13) \quad \frac{r^2}{2a^2} = \frac{(-1)^{(n-1)/2}}{2^n} \left\{ \sum_{v=1}^{(n-1)/2} \frac{(-1)^{v-1}}{2^{2v-1}} \binom{n+2v-2}{n-1} \left(\frac{r}{a}\right)^{1-(2v-1)/n} \cos\left[\left(1 - \frac{2v-1}{n}\right)\theta\right] + \sum_{v=0}^{(n-1)/2} \frac{(-1)^v}{2^{2v}} \binom{n+2v-1}{n-1} \left(\frac{r}{a}\right)^{1-(2v)/n} \sin\left[\left(1 - \frac{2v}{n}\right)\theta\right] \right\},$$

and for n even we have

$$(2.2.14) \quad \frac{r^2}{2a^2} = \frac{(-1)^{n/2}}{2^n} \left\{ \sum_{v=0}^{(n/2)-1} \frac{(-1)^v}{2^{2v}} \binom{n+2v-1}{n-1} \left(\frac{r}{a}\right)^{1-(2v)/n} \cos\left[\left(1 - \frac{2v}{n}\right)\theta\right] + \sum_{v=0}^{n/2} \frac{(-1)^v}{2^{2v-1}} \binom{n+2v-2}{n-1} \left(\frac{r}{a}\right)^{1-(2v-1)/n} \sin\left[\left(1 - \frac{2v-1}{n}\right)\theta\right] \right\}$$

For any point $P(z = re^{i\theta})$ inside Γ_n the stress function is as follows:

For n odd

$$(2.2.15) \quad \Psi_n(r, \theta) = -\frac{r^2}{2} + \frac{a^2 (-1)^{(n-1)/2}}{2^n} \left\{ \sum_{v=1}^{(n-1)/2} \frac{(-1)^{v-1}}{2^{2v-1}} \binom{n+2v-2}{n-1} \left(\frac{r}{a}\right)^{1-(2v-1)/n} \cos\left[\left(1 - \frac{2v-1}{n}\right)\theta\right] + \sum_{v=0}^{(n-1)/2} \binom{n+2v-1}{n-1} \left(\frac{r}{a}\right)^{1-(2v)/n} \sin\left[\left(1 - \frac{2v}{n}\right)\theta\right] \frac{(-1)^v}{2^{2v}} \right\}$$

and for n even we have

$$(2.2.16) \quad \Psi_n(r, \theta) = -\frac{r^2}{2} + \frac{a^2 (-1)^{n/2}}{2^n} \left\{ \sum_{v=0}^{(n/2)-1} \binom{n+2v-1}{n-1} \left(\frac{r}{a}\right)^{1-(2v)/n} \cos\left[\left(1 - \frac{2v}{n}\right)\theta\right] \frac{(-1)^v}{2^{2v}} + \sum_{v=0}^{n/2} \frac{(-1)^v}{2^{2v-1}} \binom{n+2v-2}{n-1} \left(\frac{r}{a}\right)^{1-(2v-1)/n} \sin\left[\left(1 - \frac{2v-1}{n}\right)\theta\right] \right\}$$

$$\sum_{v=0}^{n/2} \binom{n+2v-2}{n-1} \left(\frac{r}{a}\right)^{1-(2v-1)/n} \sin\left[\left(1-\frac{2v-1}{n}\right)\theta\right] \frac{(-1)^v}{2^{2v-1}} \}.$$

The value of the stress function on the lower half of our cross section, i.e. at $P(\bar{z} = re^{-i\theta})$ is defined as equal to its value at $P(z = re^{i\theta})$; which is possible by application of Schawarz' reflection principle and the fact that the cross sections are symmetric.

The expressions obtained for the stress function in (2.2.15) and (2.2.16) are sums of finite number of terms that are known harmonic functions of r and θ , then they are finite, continuous and satisfy Poisson's equation, $\nabla^2 \Psi = -2$ at any point in the interior of Γ_n .

In addition, from (2.2.13) and (2.2.14) is clear that $\Psi_n = 0$ on the boundary Γ_n .

We now show the expressions for Ψ_2 , Ψ_3 , and Ψ_4 and their corresponding torsional rigidities D_2 , D_3 , and D_4 . From (2.2.15) and (2.2.16) we have:

$$(2.2.17) \quad \Psi_2(r, \theta) = \frac{1}{4}(-2r^2 - ra \cos \theta + a^{\frac{3}{2}} r^{\frac{1}{2}} \sin \frac{1}{2} \theta),$$

$$(2.2.18) \quad \Psi_3(r, \theta) = \frac{1}{8}(-4r^2 - ra \sin \theta - \frac{3}{2} a^{\frac{4}{3}} r^{\frac{2}{3}} \cos \frac{2}{3} \theta + \frac{3}{2} a^{\frac{5}{3}} r^{\frac{1}{3}} \sin \frac{2}{3} \theta),$$

and

$$(2.2.19) \quad \Psi_4(r, \theta) = \frac{1}{16}(-8r^2 + ra \cos \theta - 2a^{\frac{5}{4}} r^{\frac{3}{4}} \sin \frac{3}{4} \theta - \frac{5}{2} a^{\frac{3}{2}} r^{\frac{1}{2}} \cos \frac{\theta}{2} + \frac{5}{2} a^{\frac{7}{4}} r^{\frac{1}{4}} \sin \frac{\theta}{4}).$$

The expressions for the torsional rigidities (1.1.15) are obtained computing the integrals below (a TI 89 graphing calculator has been used). For details on stresses see [5].

$$(2.2.20) \quad D_2 = 4\mu \int_0^\pi \int_0^{a \sin^2(\frac{\theta}{2})} \frac{1}{4} a \left(-\frac{2r^2}{a} - r \cos \theta + a^{\frac{1}{2}} r^{\frac{1}{2}} \sin \frac{1}{2} \theta \right) r dr d\theta = \frac{17}{256} \pi \mu a^4,$$

$$(2.2.21) \quad D_3 = 4\mu \int_0^\pi \int_0^{a \sin^3(\frac{\theta}{3})} \Psi_3(r, \theta) r dr d\theta = \frac{243}{2048} \left(\frac{131}{243} \pi - \frac{487}{560} \sqrt{3} \right) \mu a^4,$$

and

$$(2.2.22) \quad D_4 = 4\mu \int_0^\pi \int_0^{a \sin^4(\frac{\theta}{4})} \Psi_4(r, \theta) r dr d\theta = \frac{5}{8} \left(\frac{797}{8192} \pi - \frac{211}{693} \right) \mu a^4.$$

CHAPTER 3

SOLUTION OF THE TORSION PROBLEM USING CONFORMAL MAPPINGS

In this chapter we will solve the torsion problem for several cross sections bounded by different curvilinear polygons with n sides and n vertices using conformal mappings. By means of a conformal mapping (see Sokolnikoff [17], Bassali and Elsirafy [2], and Nehari [12]), we will map the boundary of the polygonal region on the unit circle centered at the origin. Then we solve the torsion problem on the circle—arriving to Schwarz and Poisson's formulas [17]. The solution inside the circle gives good information about the corresponding solution in the original problem since a conformal mapping is locally 1-1 (See Nehari [12]). The same method will be applied to cross-sections bounded by some irregular curvilinear polygons of n sides and n vertices. But before starting to work on the solution of the torsion problem we will state some facts from the theory of functions of complex variable, including some facts about conformal mappings, and the derivation of the solution of the torsion problem on the circle using Schwarz' formulas.

3.1 Facts about Functions of Complex Variable and Mappings

Definition: A single valued function

$$(3.1.1) \quad \Omega(z) = \phi(x, y) + i\psi(x, y)$$

of the complex variable $z = x + iy$ is called analytic in a given region S if it has a unique derivative at every point of S .

Cauchy-Riemann Equations: The necessary and sufficient conditions for a function $\Omega(z) = \phi(x, y) + i\psi(x, y)$ to be analytic in a region S are given by the Cauchy-Riemann equations

$$(3.1.2) \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

where it is assumed that the partial derivatives involved are continuous functions of x and y in S .

Harmonic Functions: It is also known that if $\Omega(z)$ is analytic in the region S , not only the first partial derivatives exist, but also all those of higher orders; so it follows that both the real and imaginary parts of an analytic function satisfy Laplace's equation, that is

$$(3.1.3) \quad \nabla^2 \phi = 0 \text{ and } \nabla^2 \psi = 0.$$

Functions such as $\phi(x, y)$ and $\psi(x, y)$ that satisfy Laplace's equation are called harmonic functions.

Cauchy's Integral Theorem: If $\Omega(z)$ is continuous in the closed region R bounded by a simple closed contour Γ and analytic in the interior of Γ , then

$$(3.1.4) \quad \int_{\Gamma} \Omega(z) dz = 0$$

Theorem (Generalization of Cauchy's Integral Theorem for multiply connected regions): If $\Omega(z)$ is continuous in the closed, multiply connected region R bounded by an exterior simple closed contour Γ_0 and by the interior simple contours $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ and analytic in the interior of R then

$$(3.1.5) \quad \int_{\Gamma_0} \Omega(z) dz = \int_{\Gamma_1} \Omega(z) dz + \int_{\Gamma_2} \Omega(z) dz + \dots + \int_{\Gamma_n} \Omega(z) dz,$$

where the integration over all the contours is performed in the same direction.

Cauchy's Integral Formula: If $\Omega(z)$ is continuous in the closed region S bounded by a simple closed contour Γ , analytic in the interior of Γ , and $z=a$ is an interior point of Γ , then

$$(3.1.6) \quad \Omega(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Omega(z)}{z-a} dz.$$

Cauchy's Integral Formula involving the derivatives of $\Omega(z)$. If the variable of integration in (3.1.6) is changed to ζ and if z is any interior point to S , then (3.1.6) implies

$$(3.1.7) \quad \Omega^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{\Omega(\zeta)}{(\zeta-z)^{n+1}} d\zeta$$

Cauchy's Integral Formulas can be used to establish the fact that an analytic function in a region S can be expanded in Taylor's series as follows:

$$(3.1.8) \quad \Omega(z) = \Omega(a) + \Omega'(a)(z-a) + \dots + \frac{\Omega^{(n)}(a)}{n!}(z-a)^n + \dots$$

If z is any interior point of a circle that lies in the region S and whose center is at a , then this series converges to $\Omega(z)$, and this representation is unique.

Now we consider the region S bounded by two concentric circles Γ_1 and Γ_2 and let $z=a$ be the center of those circles, if $\Omega(z)$ is continuous in the closed annulus and analytic in its interior then we can represent $\Omega(z)$ by the Laurent series

$$(3.1.9) \quad \Omega(z) = \sum_{k=-\infty}^{k=\infty} b_k (z-a)^k$$

where

$$(3.1.10) \quad b_k = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Omega(z)}{(z-a)^{k+1}} dz$$

with k an integer and Γ is an arbitrary path drawn in S that encloses Γ_1 . In the special

case $k = -1$ we have $b_{-1} = \frac{1}{2\pi i} \int_{\Gamma} \Omega(z) dz$ which can be obtained integrating (3.1.9) term

by term using special cases of (3.1.4) and (3.1.6). The quantity b_{-1} is called the residue of $\Omega(z)$ at $z=a$.

Calculation of residues. To obtain the residues of a function $\Omega(z)$ at $z=a$ when a is a pole of order n there is a simple formula, given by:

$$(3.1.11) \quad b_{-1} = \lim_{z \rightarrow a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n \Omega(z)\}.$$

If $n = 1$ (simple pole) we have

$$(3.1.12) \quad b_{-1} = \lim_{z \rightarrow a} (z-a)^n \Omega(z).$$

The Residue Theorem. Let $\Omega(z)$ be single valued and analytic inside and on a simple closed curve Γ except at the singularities $a, b, c, \dots$ inside Γ which have residues $a_{-1}, b_{-1}, \dots$. Then

$$(3.1.13) \quad \int_{\Gamma} \Omega(z) dz = 2\pi i (b_{-1} + a_{-1} + c_{-1} + \dots).$$

An alternative to (3.1.13) is

$$(3.1.14) \quad \int_{\Gamma} \Omega(z) dz = -2\pi i (\text{sum of the residues at the poles outside } \Gamma)$$

(3.1.14) is valid if $\Omega(z)$ is a rational function and if the degree of the denominator is larger than the degree of the numerator by at least 2.

Notation: To avoid misinterpretation with the formulas we adopt the following notation:

we will draw the unit circle in the ζ -plane where $\zeta = \xi + i\eta$ (being ξ and η are real).

The boundary of the unit circle $|\zeta| \leq 1$ will be denoted by γ and the points on its

boundary by $\sigma = e^{i\theta}$. All functions of θ will be assumed periodic, so that

$f(\theta + 2\pi) = f(\theta)$. Our curvilinear polygonal region S will be in the z -plane, $z = x + iy$, its

bounding curve will be denoted by Γ_n , and the points on the boundary will be $z = re^{i\theta}$

Theorem of Harnack. If $f(\theta)$ and $\varphi(\theta)$ are continuous real functions of θ (defined on the unit circle γ) and if

$$(3.1.15) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{f(\theta)}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i} \int_{\gamma} \frac{\varphi(\theta)}{\sigma - \zeta} d\sigma$$

for all values of ζ in the interior of γ , then $f(\theta) = \varphi(\theta)$. If the point ζ is exterior to γ , and (3.1.15) is true for every ζ , then

$$(3.1.16) \quad f(\theta) = \varphi(\theta) + \text{const.}$$

For a proof see [17], p.144.

Conformal mapping. Definition. Let S' be a region in the complex ζ -plane,

$\zeta = \xi + i\eta$. We say that $z = \omega(\zeta)$ is a conformal mapping in S' if $\omega(\zeta)$ is analytic in S'

and $\omega'(\zeta) \neq 0$ in S' .

Note. A conformal mapping preserves harmonicity and angle measures [18].

Riemann's Mapping Theorem. Let Γ be a simple closed curve forming the boundary of a region S , and let γ be a circle of radius one and center at the origin forming the boundary of the region S' . Then this theorem states that there exists a bijective conformal map $z = \omega(\zeta)$ analytic in S , that maps each point of S into a corresponding point in S' and each point of Γ into a corresponding point in γ .

3.2 Formulas of Schwarz and Poisson

In this section we solve the torsion problem when the boundary of the cross section of the cylinder is a circle. As we already mentioned, finding the torsion function $\phi(x, y)$ or its harmonic conjugate $\psi(x, y)$ can be reduced to solving the two-dimensional problem of Dirichlet.

Let S be the region bounded by the unit circle just described.

We want to find a harmonic function $\phi(\xi, \eta)$ such that

$$(3.2.1) \quad \phi|_{\gamma} = g(\theta),$$

where $g(\theta)$ is a real continuous function of θ . Let $\psi(\xi, \eta)$ be the conjugate harmonic function of $\phi(\xi, \eta)$ where $\psi(\xi, \eta)$ can be determined within an arbitrary constant from $\phi(\xi, \eta)$ using the Cauchy-Riemann equations.

Both functions ϕ and ψ are analytic, then the function

$$(3.2.2) \quad \Omega(\zeta) = \phi(\xi, \eta) + i \psi(\xi, \eta)$$

is an analytic function of the variable $\zeta = \xi + i\eta$ in the interior of $|\zeta| = 1$. Assuming that $\Omega(\zeta)$ is continuous on the closed unit disk $|\zeta| \leq 1$, we can write (3.2.2) as follows:

$$(3.2.3) \quad \Omega(\sigma) = g(\theta) + i \psi|_{\gamma}.$$

Adopting the notation $\bar{\Omega}(\bar{\sigma}) = \overline{\Omega(\sigma)}$ and $\bar{\Omega}(\sigma) = \overline{\Omega(\bar{\sigma})}$ (see Sokolnikoff [17]), we also have

$$(3.2.4) \quad \bar{\Omega}(\bar{\sigma}) = g(\theta) - i \psi|_{\gamma},$$

and if we add the two equations we get

$$(3.2.5) \quad \Omega(\sigma) + \bar{\Omega}(\bar{\sigma}) = 2g(\theta).$$

Multiplying both sides of (3.2.5) by $\frac{1}{2\pi i} \frac{d\sigma}{\sigma - \zeta}$, where ζ is any point in the interior of γ ,

and integrating over the circle we obtain:

$$(3.2.6) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{\Omega(\sigma)}{\sigma - \zeta} d\sigma + \frac{1}{2\pi i} \int_{\gamma} \frac{\bar{\Omega}(\bar{\sigma})}{\sigma - \zeta} d\sigma = \frac{1}{\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma - \zeta} d\sigma,$$

which is equivalent to (3.2.5) by the theorem of Harnack.

Now,

$$(3.2.7) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{\Omega(\sigma)}{\sigma - \zeta} d\sigma = \Omega(\zeta),$$

by Cauchy's Integral Formulas, and

$$(3.2.8) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{\bar{\Omega}(\bar{\sigma})}{\sigma - \zeta} d\sigma = \bar{\Omega}(0).$$

To prove (3.2.8) we write the Taylor expansion for $\Omega(\zeta)$ around 0, which exists and is unique since $\Omega(\zeta)$ is analytic,

$$(3.2.9) \quad \Omega(\zeta) = \Omega(0) + \Omega'(0)(\zeta - 0) + \dots + \frac{\Omega^{(n)}(0)}{n!}(\zeta - 0)^n + \dots$$

and on the circle $|\zeta| = 1$ we have $\zeta = e^{i\theta} = \sigma$ and $\bar{\zeta} = e^{-i\theta} = \frac{1}{\sigma}$, thus

$$(3.2.10) \quad \bar{\Omega}(\bar{\sigma}) = \bar{\Omega}(0) + \bar{\Omega}'(0)\frac{1}{\sigma} + \dots + \frac{\bar{\Omega}^{(n)}(0)}{n!}\frac{1}{\sigma^n} + \dots$$

and term by term integration gives the desired result, taking into account that

$$(3.2.11) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{d\sigma}{\sigma^n(\sigma - \zeta)} = 0 \quad \text{for } n \geq 1$$

and

$$(3.2.12) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{d\sigma}{\sigma^n(\sigma - \zeta)} = 1 \quad \text{for } n=0.$$

We notice that (3.2.12) is true by Cauchy's Integral Formula (3.1.6), and (3.2.11) can be calculated by adding the residues at the pole of order n , $\sigma = \text{zero}$, and the pole $\sigma = \zeta$ of order 1 by applying (3.1.11), (3.1.12), and (3.1.13).

So, inserting (3.2.7) and (3.2.8) in (3.2.6) we obtain

$$(3.2.13) \quad \Omega(\zeta) + \bar{\Omega}(0) = \frac{1}{\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma - \zeta} d\sigma.$$

Letting $\Omega(0) = a_0 + ib_0$ and therefore $\bar{\Omega}(0) = a_0 - ib_0$ in (3.2.13) we get

$$(3.2.14) \quad \Omega(\zeta) = \frac{1}{\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma - \zeta} d\sigma - a_0 + ib_0$$

for all ζ in the interior and boundary of γ . In particular, if we put $\zeta = 0$, formula (3.2.14)

becomes:

$$(3.2.15) \quad \Omega(0) = \frac{1}{\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma} d\sigma - a_0 + ib_0,$$

or,

$$(3.2.16) \quad a_0 + ib_0 = \frac{1}{\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma} d\sigma - a_0 + ib_0,$$

and therefore,

$$(3.2.17) \quad a_0 = \frac{1}{2\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma} d\sigma.$$

Substituting (3.2.17) in (3.2.14) we obtain:

$$(3.2.18) \quad \Omega(\zeta) = \frac{1}{\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma - \zeta} d\sigma - \frac{1}{2\pi i} \int_{\gamma} \frac{g(\theta)}{\sigma} d\sigma + ib_0,$$

or, combining both integrals:

$$(3.2.19) \quad \Omega(\zeta) = \frac{1}{2\pi i} \int_{\gamma} g(\theta) \frac{(\sigma + \zeta)}{(\sigma - \zeta)\sigma} d\sigma + ib_0,$$

which is the formula of Schwarz. The solution to our problem on the circle is given by

the real part of $\Omega(\zeta)$, which is possible to find by substituting $\zeta = \rho e^{i\Theta}$ and $\sigma = e^{i\theta}$.

Doing so, we obtain the integral of Poisson:

$$(3.2.20) \quad \operatorname{Re} \Omega(\zeta) = \phi(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - \rho^2)g(\theta)}{1 - 2\rho \cos(\theta - \Theta) + \rho^2} d\theta,$$

Which is the solution to the problem of Dirichlet on the circle. It is possible to generalize (3.2.18), (3.2.19) and (3.2.20) so that they will apply to any simply connected region, which is done by means of a mapping function.

3.3 Method 4: The Conformal Mappings Method

Let

$$(3.3.1) \quad \Omega(z) = \phi(x, y) + i \psi(x, y)$$

the complex torsion function, where $z = x + iy$, $\phi(x, y)$ is the torsion function, and $\psi(x, y)$ is its conjugate harmonic function which can be determined within an arbitrary constant from $\phi(x, y)$ using the Cauchy-Riemann equations. $\Omega(z)$ is analytic in the region S —bounded by Γ —representing the cross section of the cylinder. Since we assume S is simply connected, we can map it on the unit circle in the ζ -plane. Let $z = \omega(\zeta)$ the function that maps the region S to the unit disk $|\zeta| \leq 1$. The function $\Omega(z)$ can be written in terms of $\zeta = \xi + i\eta$ as follows:

$$(3.3.2) \quad \phi(\xi, \eta) + i \psi(\xi, \eta) = \Omega[\omega(\zeta)] = f(\zeta),$$

where $f(\zeta)$ is an analytic function of the variable $\zeta = \xi + i\eta$ in the interior of the circle $|\zeta| = 1$.

As we stated in section 1.2 the torsion function ψ satisfies, the boundary condition

$$(3.3.3) \quad \psi = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}(z\bar{z}) \quad \text{on } \Gamma.$$

Therefore, the imaginary part of (3.3.2) must satisfy, on γ ,

$$(3.3.4) \quad \psi = \frac{1}{2} \omega(\zeta) \bar{\omega}(\bar{\zeta}).$$

Solving the torsion problem is equivalent to finding the real part of

$$(3.3.5) \quad \frac{1}{i} f(\zeta) = \psi - i\phi,$$

which on the boundary γ of the unit circle $|\zeta| = 1$ takes the values $\psi = \frac{1}{2} \omega(\sigma) \bar{\omega}(\bar{\sigma})$.

Now this is a special case of the problem solved in section 3.2. Putting

$g(\theta) = \frac{1}{2} \omega(\sigma) \bar{\omega}(\bar{\sigma})$ and $\Omega(\zeta) = \frac{1}{i} f(\zeta)$ in (3.2.14) we obtain

$$(3.3.6) \quad \frac{1}{i} f(\zeta) = \frac{1}{\pi i} \int_{\gamma} \frac{\frac{1}{2} \omega(\sigma) \bar{\omega}(\bar{\sigma})}{\sigma - \zeta} d\sigma - a_0 + i b_0$$

or

$$(3.3.7) \quad f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{\omega(\sigma) \bar{\omega}(\bar{\sigma})}{\sigma - \zeta} d\sigma + \text{constant}.$$

Noting that on the boundary of $|\zeta| = 1$ $\sigma = e^{i\theta}$ and $\bar{\sigma} = e^{-i\theta} = \frac{1}{\sigma}$, (3.3.7) reduces to

$$(3.3.8) \quad f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{\omega(\sigma) \bar{\omega}(\frac{1}{\sigma})}{\sigma - \zeta} d\sigma + \text{constant}.$$

Torsional Rigidity. We can express the torsional rigidity D (See Sokolnikoff [17]) in terms of $\Omega(\zeta)$ and $\omega(\zeta)$ as follows:

$$(3.3.9) \quad D = \mu I_0 + \mu D_0,$$

where

$$(3.3.10) \quad I_0 = -\frac{i}{4} \int_{\gamma} [\bar{\omega}(\bar{\sigma})]^2 \omega(\sigma) d\omega(\sigma)$$

and

$$(3.3.11) \quad D_0 = -\frac{1}{4} \int_{\gamma} [\bar{\Omega}(\bar{\sigma}) + \Omega(\sigma)] d[\omega(\sigma) \bar{\omega}(\bar{\sigma})].$$

Now we have all the elements needed to solve the torsion problem for cross sections bounded by regular curvilinear curves.

3.4 Solution for a Cross Section Bounded by a Regular Curvilinear Polygon

Stevenson [20] considered briefly the torsion problem of a beam for a simply connected cross section. He applied this method to a grooved column whose cross section was bounded by a regular curvilinear polygon of n sides and n vertices—polygon whose sides are arcs rather than line segments (if the sides are line segments we call them rectilinear polygons), for more details see [20], [1]. The boundary curve of the cross section in the z -plane is mapped into the unit circle γ in the ζ -plane by a conformal mapping

$z = \omega(\zeta) = c\zeta(1 + m\zeta^n)$, where both c and m are real positive constants and m and n

satisfy $0 \leq m(n+1) \leq 1$. Some special cases solved by Stevenson include cross sections

bounded by the elliptic limacon and the cardioid. In this paper we apply the methods

described by Muskhelishvili [11] to solve the torsion problem corresponding to the

mapping function $z = \omega(\zeta) = \frac{c\zeta}{(1 + m\zeta^n)}$ when n is a natural number, $n \geq 2$ (see **Fig. 10**

for a two-sided polygon), and $-1 \leq m(n-1) \leq 1$. In the next page we explain why we

impose such conditions on m and n . Notice that this method works for curvilinear

polygons, for rectilinear polygons the work involves the use of the Schwarz-Christoffel transformation .

The mapping function. We consider cross sections bounded by curves Γ_n , regular curvilinear polygons of n sides and n vertices. We map these cross sections on the unit circle γ by

$$(3.4.1) \quad z = \omega(\zeta) = \frac{c\zeta}{(1 + m\zeta^n)}.$$

For (3.4.1) to be conformal everywhere inside γ , $\omega'(\zeta)$ must not be zero or infinite in the interior of γ . Thus, taking into account that $n \geq 2$, we find the conditions on m as follows:

$$\frac{dz}{d\zeta} = \frac{c(1 + m\zeta^n) - c\zeta mn\zeta^{n-1}}{(1 + m\zeta^n)^2} = 0 \text{ if and only if } \zeta^n = \frac{1}{m(n-1)}.$$

So we impose the condition $|\zeta|^n = \left| \frac{1}{m(n-1)} \right| \geq 1$ so that ζ is on the boundary or outside

the unit circle, which is equivalent to

$$(3.4.2) \quad |m(n-1)| \leq 1.$$

It is sufficient to consider either positive or negative values of m , since $m\zeta^n (= me^{i\theta n})$ and $-m\zeta^n$ take exactly the same values when θ takes its values from 0 to 2π . For the special case $n=2$ we must take $|m| < 1$ to ensure a finite cross section.

If $z = x + iy = re^{i\theta}$ is the point on Γ_n that corresponds to $\zeta = \xi + i\eta = e^{i\theta}$ on γ , then we have

$$z = \omega(\zeta)$$

$$(3.4.3) \quad x + iy = \frac{c(\xi + i\eta)}{1 + m(\xi + i\eta)^n} = \frac{ce^{i\theta}}{1 + me^{in\theta}} = \frac{c(\cos \theta + i \sin \theta)}{1 + m(\cos n\theta + i \sin n\theta)}.$$

Dividing both sides of (3.4.3) by c we have

$$(3.4.4) \quad \frac{x}{c} + i \frac{y}{c} = \frac{\cos \theta + i \sin \theta}{1 + m \cos n\theta + im \sin n\theta},$$

and then, separating real parts from imaginary parts and using trigonometric identities we obtain the parametric equations of the curves Γ_n

$$(3.4.5) \quad \frac{x}{c} = \frac{\cos \theta + m \cos(n-1)\theta}{1 + 2m \cos n\theta + m^2}$$

and

$$(3.4.6) \quad \frac{y}{c} = \frac{\sin \theta - m \sin(n-1)\theta}{1 + 2m \cos n\theta + m^2}.$$

The polar equation of Γ_n , obtained squaring and adding (3.4.5) and (3.4.6)), is given by

$$(3.4.7) \quad \frac{x^2}{c^2} + \frac{y^2}{c^2} = \frac{r^2}{c^2} = \frac{1}{1 + 2m \cos n\theta + m^2},$$

where

$$(3.4.8) \quad \frac{y}{x} = \tan \Theta = \frac{\sin \theta - m \sin(n-1)\theta}{\cos \theta + m \cos(n-1)\theta}$$

which is the formula of a regular curvilinear polygon of n sides and n rounded vertices. In

Figures 10 to 13 we show the graphs of Γ_n for $n=2, 3, 4$, and 5 and $m = -0.25$. The

MAPLE command for such graphs follows:

```
for n from 2 to 5 do plot( [Re(exp(I*t)/(1-.25*exp(I*t*(n))))],Im(exp(I*t)/(1-
.25*exp(I*t*(n))))],t=0..2*Pi,color=black)end do;
```

Figure 10. Γ_2 $m=-0.25$

Figure 11. Γ_3 $m=-0.25$

Figure 12. Γ_4 $m=-0.25$

Figure 13. Γ_5 , $m=-0.25$

We notice that if $n=2$, we must take $|m|<1$, otherwise we don't obtain a finite/closed cross section, as we show in **Fig. 14**.

```
for n from 2 to 2 do plot( [Re(exp(I*t)/(1-exp(I*t*(n))))],Im(exp(I*t)/(1-
exp(I*t*(n))))],t=0..2*Pi,color=black)end do;
```

Figure 14. Γ_2 , $m=1$

The Complex Torsion Function. As we have seen, the complex torsion function

$\Omega(z) = \phi(x, y) + i\psi(x, y)$ can be expressed in terms of ζ as $\Omega(z) = f(\zeta)$, where $f(\zeta)$ is

the function that solves the torsion problem on the unit circle, which can be obtained

evaluating the integral (3.3.8). Substituting $\omega(\sigma) = \frac{c\sigma}{(1+m\sigma^n)}$ and

$$\overline{\omega(\sigma)} = \overline{\omega}(1/\sigma) = \frac{c/\sigma}{(1+m(1/\sigma)^n)} \text{ in}$$

$$f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{\omega(\sigma) \overline{\omega}(\frac{1}{\sigma})}{\sigma - \zeta} d\sigma + \text{constant}$$

we obtain:

$$(3.4.9) \quad f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{(c\sigma)(c\frac{1}{\sigma})}{(1+m\sigma^n)(1+m\frac{1}{\sigma^n})(\sigma - \zeta)} d\sigma + \text{constant}$$

or

$$(3.4.10) \quad f(\zeta) = \frac{c^2}{2\pi} \int_{\gamma} \frac{\sigma^n}{(m + \sigma^n)(\sigma^n + m^{-1})(\sigma - \zeta)} d\sigma + \text{constant}$$

being ζ a point inside γ . To find $f(\zeta)$ we use the theorem of residues. From (3.4.2) we

have that $\left| \frac{1}{m} \right| \geq n - 1 \geq 1$, therefore, the integrand of (3.4.10), which we will call $h(\zeta)$,

has n simple poles that lie outside the unit circle, the n th roots of $-1/m$. Since $|m| \leq 1$

$h(\zeta)$ has n simple poles that lie inside the unit circle, the n th roots of $-m$ which we

will call m_s , $s=1, 2, \dots, n$. $h(\zeta)$ also has a simple pole at $\sigma = \zeta$. In order to calculate the

required residues (abbreviated “Res”) we will need the following facts:

$$(3.4.11) \quad m_s^n = -m,$$

$$(3.4.12) \quad \lim_{\sigma \rightarrow m_s} \frac{\sigma - m_s}{\sigma^n + m} = \frac{1}{nm_s^{n-1}},$$

$$(3.4.13) \quad \frac{n\sigma^{n-1}}{(\sigma^n + m)} = \sum_{s=1}^n \frac{1}{\sigma - m_s}.$$

To prove (3.4.13) we start with

$$\sigma^n + m = (\sigma - m_1)(\sigma - m_2) \dots (\sigma - m_n),$$

we differentiate both sides and obtain:

$$n\sigma^{n-1} = \frac{\sigma^n + m}{\sigma - m_1} + \frac{\sigma^n + m}{\sigma - m_2} + \dots + \frac{\sigma^n + m}{\sigma - m_n},$$

and therefore

$$\frac{n\sigma^{n-1}}{(\sigma^n + m)} = \sum_{s=1}^n \frac{1}{\sigma - m_s}$$

as desired. To prove (3.4.12) it suffices to apply L'Hospital's rule and the result will follow.

Now we proceed to find $f(\zeta)$ such that

$$f(\zeta) = \frac{c^2}{2\pi} 2\pi i (\text{sum of the residues at the poles inside the circle}) + \text{const. The residues,}$$

for which we have used (3.4.11) and (3.4.12) are:

$$\begin{aligned} \text{Res}(m_s) &= \lim_{\sigma \rightarrow m_s} \frac{\sigma^n (\sigma - m_s)}{(\sigma^n + m)(\sigma^n + \frac{1}{m})(\sigma - \zeta)} = \frac{m_s^n}{nm_s^{n-1}(-m + \frac{1}{m})(m_s - \zeta)} = \\ &= \frac{mm_s}{n(-m^2 + 1)} \frac{1}{m_s - \zeta} = \frac{m}{n(m^2 - 1)} \left(-1 + \frac{\zeta}{\zeta - m_s} \right). \end{aligned}$$

Taking the summation of the above result and using (3.4.13) we have,

$$(3.4.14) \quad \sum_{s=1}^n \operatorname{Res}(m_s) = \frac{m}{n(m^2-1)} \sum_{s=1}^n \left(-1 + \frac{\zeta}{\zeta - m_s} \right) = \frac{-m^2}{(m^2-1)(\zeta^n + m)}.$$

Now we calculate the residue at $\sigma = \zeta$

$$(3.4.15) \quad \operatorname{Res}(\sigma = \zeta) = \lim_{\sigma \rightarrow \zeta} (\sigma - \zeta) g(\sigma) = \frac{\zeta^n}{(\zeta^n + m)(\zeta^n + \frac{1}{m})} = \frac{m\zeta^n}{(\zeta^n + m)(m\zeta^n + 1)}.$$

The sum of all the residues at the poles in (3.4.14) and (3.4.15) yields

$$(3.4.16) \quad \sum_{s=1}^n \operatorname{Res}(m_s) + \operatorname{Res}(\sigma = \zeta) = \frac{-m}{(m^2-1)(m\zeta^n + 1)},$$

and therefore

$$f(\zeta) = \frac{-c^2 i}{(m^2-1)(m\zeta^n + 1)} + \text{constant}.$$

Choosing the constant $= \frac{-c^2 i}{2(1-m^2)}$ we have

$$(3.4.17) \quad f(\zeta) = \frac{c^2 i(1 - m\zeta^n)}{2(1-m^2)(m\zeta^n + 1)}$$

Putting $\zeta = \rho e^{i\theta}$ in (3.4.17) ($\rho \leq 1$) and taking its imaginary part we obtain the harmonic conjugate of the torsion function:

$$(3.4.18) \quad \operatorname{Im} f(\rho e^{i\theta}) = \psi = \frac{c^2(1 - m^2 \rho^{2n})}{2(1-m^2)(1 + 2m\rho^n \cos n\theta + m^2 \rho^{2n})}$$

On the boundary $\rho = 1$, so we have

$$(3.4.19) \quad \psi = \frac{c^2}{2(1 + 2m \cos n\theta + m^2)}.$$

It is easy to see that $\psi = \frac{1}{2} z \bar{z}$ on Γ when we use (3.4.7).

The Torsional Rigidity. As we mentioned before, the torsional rigidity is given by

$$(3.4.20) \quad D = \mu(I + D_0),$$

where

$$(3.4.21) \quad I = -\frac{1}{4}i \int_{\gamma} \bar{\omega}^2(\bar{\sigma}) d[\omega^2(\sigma)]$$

and

$$(3.4.22) \quad D_0 = -\frac{1}{4} \int_{\gamma} [f(\sigma) + \bar{f}(\bar{\sigma})] d[\omega(\sigma) \bar{\omega}(\bar{\sigma})]$$

Inserting the formula of $\omega(\sigma)$ and $f(\zeta)$ from (3.4.1) and (3.4.17) respectively in

(3.4.20) and using the Residues Theorem we arrive to (Bassali [3]):

$$(3.4.23) \quad D = \frac{1}{2} \mu \pi c^4 \frac{1 + (n-1)m^4}{1 - m^2}$$

Notice that setting $m=0$ in (3.4.23) we obtain the torsional rigidity for a circular cross section of radius c .

3.5 Solution for a Cross Section Bounded by Other Curvilinear Polygons

As we mentioned in the introduction of section 3.4, Stevenson [20], in 1943, used the conformal mapping $z = \omega(\zeta) = c\zeta(1 + m\zeta^n)$ to map an n -side regular curvilinear polygon in the unit circle.

In 1960, Bassali [3] used the conformal mapping $z = \omega(\zeta) = c\zeta/(1 + m\zeta^n)$ to map regular curvilinear polygons on the unit circle, as we just showed in section 3.4.

In another paper, Bassali [1] used the conformal mapping

$z = \omega(\zeta) = c\zeta / (1 + m\zeta^n + p\zeta^{2n})$ to map curvilinear polygons on the unit circle, obtaining

the complex torsion function

$$\begin{aligned} f(\zeta) &= \frac{c^2}{2\pi} \int_{\gamma} \frac{\sigma^{2n}}{(1 + m\sigma^n + p\sigma^{2n})(\sigma^{2n} + m\sigma^n + p)(\sigma - \zeta)} d\sigma + \text{const.} = \\ &= \frac{c^2 i(1 + p + mp\zeta^n)}{(1 - p)\{(1 + p)^2 - m^2\}(1 + m\zeta^n + p\zeta^{2n})} + \text{const.} \end{aligned}$$

Choosing the constant as

$$\frac{-c^2 i(1 + p)}{2(1 - p)\{(1 + p)^2 - m^2\}},$$

and calling $k = (1 + p^2) - m^2$, the torsional rigidity is given by

$$D = \frac{\mu\pi c^4}{2(1 - p)^4 k^4} [(1 + p)^4 (1 + p^4) - 4p(1 + p)(1 + p^3)m^2 - 2p^2 m^4]$$

Later in 1970, Bassali and Elsirafy [2] obtained the torsion function using the mapping

function $z = \omega(\zeta) = c\zeta / (1 + m\zeta^n + p\zeta^{2n})$ with $n > 2$, $m = -\frac{2}{n(n-1)}$ and $p = \frac{n-2}{n^2(2n-1)}$

by a method that involves solving algebraic equations. In this section we will use the

mapping function $z = \omega(\zeta) = c\zeta / (1 + m\zeta^n + p\zeta^2)$, with p a positive real number, to

obtain the torsion function. The author obtained all the results that follow independently.

We do many of the calculations of residues and graphs using MAPLE, some calculations are done with the TI 89.

The mapping function. As before, we consider cross sections bounded by curves Γ_n ,

and we map them on the unit circle γ by

$$(3.5.1) \quad z = \omega(\zeta) = \frac{c\zeta}{(1 + m\zeta^n + p\zeta)}.$$

For (3.5.1) be conformal everywhere on the cross section, $\omega'(\zeta)$ must not be zero or infinite in the interior of Γ_n . So, taking into account that $n \geq 2$, we find the conditions on m as we did in section 3.4:

$$\frac{dz}{d\zeta} = \frac{c(1 + m\zeta^n + p\zeta) - c\zeta(mn\zeta^{n-1} + p)}{(1 + m\zeta^n + p\zeta)^2} = 0 \text{ if and only if } \zeta^n = \frac{1}{m(n-1)}.$$

So we impose the condition $|\zeta|^n = \left| \frac{1}{m(n-1)} \right| \geq 1$ so that $\omega'(\zeta)$ is never zero inside the

unit circle, which is equivalent to

$$(3.5.2) \quad |m(n-1)| \leq 1.$$

We notice that p has no influence on conformality

It is sufficient to consider either positive or negative values of m , since $m\zeta^n (= me^{i\theta n})$ and $-m\zeta^n$ take exactly the same values when θ takes its values from 0 to 2π . For the special case $n=2$ we must take $|m| < 1$ to ensure a closed curve (In **Figure 19** we show the curve obtained for $n=2$ and $m=1$)

If $z = x + iy = re^{i\theta}$ is the point on Γ_n that corresponds to $\zeta = \xi + i\eta = e^{i\theta}$ on γ , then we have

$$z = \omega(\zeta),$$

$$\begin{aligned}
 (3.5.3) \quad x + iy &= \frac{c(\xi + i\eta)}{1 + m(\xi + i\eta)^n + p(\xi + i\eta)} = \frac{ce^{i\theta}}{1 + me^{in\theta} + pe^{i\theta}} = \\
 &= \frac{c(\cos \theta + i \sin \theta)}{1 + m(\cos n\theta + i \sin n\theta) + p(\cos \theta + i \sin \theta)},
 \end{aligned}$$

or dividing both sides of (3.5.3) by c and associating terms we have

$$(3.5.4) \quad \frac{x}{c} + i \frac{y}{c} = \frac{\cos \theta + i \sin \theta}{1 + m \cos n\theta + p \cos \theta + i(\sin n\theta + \sin \theta)},$$

and then, we obtain the parametric equations

$$\begin{aligned}
 (3.5.5) \quad \frac{x}{c} &= \operatorname{Re}\left(\frac{\cos \theta + i \sin \theta}{1 + m \cos n\theta + p \cos \theta + i(\sin n\theta + \sin \theta)}\right) = \\
 &= \frac{m \cos(n-1)\theta + \cos \theta + p}{2m(p \cos \theta + 1) \cos(n\theta) + 2mp \sin \theta \sin n\theta + 2p \cos \theta + m^2 + p^2 + 1}
 \end{aligned}$$

and

$$\begin{aligned}
 (3.5.6) \quad \frac{y}{c} &= \operatorname{Im}\left(\frac{\cos \theta + i \sin \theta}{1 + m \cos n\theta + p \cos \theta + i(\sin n\theta + \sin \theta)}\right) = \\
 &= \frac{-m \sin(n-1)\theta + \sin \theta + p}{2m(p \cos \theta + 1) \cos(n\theta) + 2mp \sin \theta \sin n\theta + 2p \cos \theta + m^2 + p^2 + 1},
 \end{aligned}$$

and the polar equations (squaring and adding (3.5.5) and (3.5.6)):

$$\begin{aligned}
 (3.5.7) \quad \frac{x^2}{c^2} + \frac{y^2}{c^2} &= \frac{r^2}{c^2} = \\
 &= \frac{1}{2m(p \cos \theta + 1) \cos(n\theta) + 2mp \sin \theta \sin n\theta + 2p \cos \theta + m^2 + p^2 + 1}
 \end{aligned}$$

and

$$(3.5.8) \quad \frac{y}{x} = \tan \Theta = \frac{-m \sin(n-1)\theta + \sin \theta + p}{m \cos(n-1)\theta + \cos \theta + p}$$

which is the formula of a curvilinear polygon of n sides and n vertices. It is clear that (3.5.5), (3.5.6), and (3.5.8) are generalizations of (3.4.5), (3.4.6), and (3.4.8), which were obtained by Bassali [3] in 1960. In **Figures 15 to 18** we show the graphs of Γ_n for $n=2$, 3, 4, and 5, $m=-0.25$, and $p=1$. The MAPLE command for such graphs follows:

```
for n from 2 to 5 do plot( [Re(exp(I*t)/(1-
.25*exp(I*t*(n))+exp(I*t))),Im(exp(I*t)/(1-
.25*exp(I*t*(n))+exp(I*t))),t=0..2*Pi], color=black)end do;
```

Figure 15. Γ_2

Figure 16. Γ_3

Figure 17. Γ_4 Figure 18. Γ_5

In the next figure we can see clearly the reason we must impose $|m| < 1$ when $n=2$.

The MAPLE command for $n=2$ and $m=1$ follows:

```
for n from 2 to 2 do plot( [Re(exp(I*t)/(1-
1*exp(I*t*(n))+exp(I*t))),Im(exp(I*t)/(1-
exp(I*t*(n))+exp(I*t))),t=0..2*Pi],color=black)end do;
```

Figure 19. $\Gamma_2, m=1$

The Complex Torsion Function. In a similar fashion as we did in section 3.4 we proceed to find complex torsion function $\Omega(z) = \phi(x, y) + i\psi(x, y)$, which can be expressed in terms of ζ as $\Omega(z) = f(\zeta)$, where $f(\zeta)$ is an analytic function in the

interior of the unit disk. Substituting $\omega(\sigma) = \frac{c\sigma}{(1+m\sigma^n+p\sigma)}$ and

$$\overline{\omega(\sigma)} = \overline{\omega}(1/\sigma) = \frac{c/\sigma}{(1+m(1/\sigma)^n + p/\sigma)} \text{ in}$$

$$f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{\omega(\sigma) \overline{\omega}(\frac{1}{\sigma})}{\sigma - \zeta} d\sigma + \text{constant}$$

we obtain:

$$(3.5.9) \quad f(\zeta) = \frac{1}{2\pi} \int_{\gamma} \frac{(c\sigma)(c\frac{1}{\sigma})}{(1+m\sigma^n+p\sigma)(1+m\frac{1}{\sigma^n}+p\frac{1}{\sigma})(\sigma-\zeta)} d\sigma + \text{constant}$$

or

$$(3.5.10) \quad f(\zeta) = \frac{c^2}{2\pi} \int_{\gamma} \frac{\sigma^n}{(1+m\sigma^n+p\sigma)(m+p\sigma^{n-1}+\sigma^n)(\sigma-\zeta)} d\sigma + \text{constant}$$

where ζ a point inside γ . To find $f(\zeta)$ we use the theorem of residues (3.1.14).

We will calculate $f(\zeta)$ for the special case $n=2$, $m = -\frac{1}{4}$, and $p = 1$. Substituting these

values in (3.5.10), factoring the denominator of the integrand, and taking common factor

$-\frac{1}{4}$ from the denominator we obtain:

$$(3.5.11)$$

$$f(\zeta) = -2 \frac{c^2}{\pi} \int_{\gamma} \frac{\sigma^2 d\sigma}{[\sigma - 2(\sqrt{2}+1)][\sigma + 2(\sqrt{2}-1)][\sigma + \frac{\sqrt{2}+1}{2}][\sigma - \frac{\sqrt{2}-1}{2}](\sigma - \zeta)} + \text{const.}$$

$$f(\zeta) = -2 \frac{c^2}{\pi} (-2\pi i) [\text{Res}(2(\sqrt{2}+1)) + \text{Res}(-\frac{\sqrt{2}+1}{2})]$$

$f(\zeta)$ has two poles outside of the circle and three inside. So we will calculate the

residues of the poles outside and apply the second form of the theorem of residues.

The MAPLE command for the calculation of the residues and $f(\zeta)$ follows

```
R1:=residue(sigma^(2)/((sigma-2*(sqrt(2)+1))*(sigma+2*(sqrt(2)-1))*(sigma+(sqrt(2)+1)/2)*(sigma-(sqrt(2)-1)/2)*(sigma-zeta)),
sigma=2*(sqrt(2)+1));R2:=residue(sigma^(2)/((sigma-2*(sqrt(2)+1))*(sigma+2*(sqrt(2)-1))*(sigma+(sqrt(2)+1)/2)*(sigma-(sqrt(2)-1)/2)*(sigma-zeta)),
sigma=-(sqrt(2)+1)/2);
f(zeta):=4*i*c^2*(R1+R2);
```

$$R1 := \frac{-32\sqrt{2} - 48}{-1520 + 320\zeta + 220\sqrt{2}\zeta - 1080\sqrt{2}}$$

$$R2 := \frac{-4\sqrt{2} - 6}{-10 - 40\zeta + 10\sqrt{2}\zeta - 15\sqrt{2}}$$

(3.5.12)

$$f(\zeta) := 4i c^2 \left(\frac{-32\sqrt{2} - 48}{-1520 + 320\zeta + 220\sqrt{2}\zeta - 1080\sqrt{2}} + \frac{-4\sqrt{2} - 6}{-10 - 40\zeta + 10\sqrt{2}\zeta - 15\sqrt{2}} \right)$$

As we did before, we put $\zeta = \rho e^{i\theta}$ in (3.5.12) ($\rho \leq 1$) and take the resulting imaginary part obtaining the harmonic conjugate of the torsion function. The calculations can be easily done with an automatic grapher like the TI 89, and are completely similar to those in section 3.4. The torsion function, obtained with the TI 89 follows:

(3.5.13)

$$\text{Im } f(\rho e^{i\theta}) = \psi \approx \frac{c^2 .5912}{(\rho^2 - 9.6569\rho \cos \theta + 23.3137)(\rho^2 + 2.4142\rho \cos \theta + 1.4571)} *$$

$$\{ \rho^3 (\cos \theta - 12.2038 \sin \theta) + 117.8501 \rho^2 [\sin \theta \cos \theta - .0205 \{ \sin^2 \theta + 1 \}]$$

$$-10.1997 \rho (\cos \theta + 27.8943 \sin \theta) - 7.0355 \}.$$

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