1-1-2021

# On the edge irregular reflexive labeling of corona product of graphs with path

Kooi Kuan Yoong
*Universiti Malaysia Terengganu*

Roslan Hasni
*Universiti Malaysia Terengganu*

Muhammad Irfan
*University of Okara*

Ibrahim Taraweh
*Khalid Ibn Al-Walid School*

Ali Ahmad
*College of Computer Science and Information Technology*

*See next page for additional authors*


AKCE International Journal of Graphs and Combinatorics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20

# On the edge irregular reflexive labeling of corona product of graphs with path

**Kooi-Kuan Yoong, Roslan Hasni, Muhammad Irfan, Ibrahim Taraweh, Ali Ahmad & Sin-Min Lee**

**To cite this article:** Kooi-Kuan Yoong, Roslan Hasni, Muhammad Irfan, Ibrahim Taraweh, Ali Ahmad & Sin-Min Lee (2021) On the edge irregular reflexive labeling of corona product of graphs with path, AKCE International Journal of Graphs and Combinatorics, 18:1, 53-59, DOI: 10.1080/09728600.2021.1931555

**To link to this article:** https://doi.org/10.1080/09728600.2021.1931555



Published online: 02 Jun 2021.

AKCE INTERNATIONAL JOURNAL OF GRAPHS AND COMBINATORICS
2021, VOL. 18, NO. 1, 53–59
https://doi.org/10.1080/09728600.2021.1931555



# On the edge irregular reflexive labeling of corona product of graphs with path

Kooi-Kuan Yoong[a], Roslan Hasni[a], Muhammad Irfan[b], Ibrahim Taraweh[c], Ali Ahmad[d], and Sin-Min Lee[e]

[a]Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Kuala Nerus, Malaysia; [b]Department of Mathematics, University of Okara, Okara, Pakistan; [c]Mathematics, Khalid Ibn Al-Walid School, Karak, Jordan; [d]College of Computer Science and Information Technology, Jazan, Saudi Arabia; [e]Department of Computer Science, San Jose State University, San Jose, California, USA

**ABSTRACT**

We define a total $k$-labeling $\varphi$ of a graph $G$ as a combination of an edge labeling $\varphi_e : E(G) \rightarrow \{1, 2, ..., k_e\}$ and a vertex labeling $\varphi_v : V(G) \rightarrow \{0, 2, ..., 2k_v\}$, such that $\varphi(x) = \varphi_v(x)$ if $x \in V(G)$ and $\varphi(x) = \varphi_e(x)$ if $x \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The total $k$-labeling $\varphi$ is called an *edge irregular reflexive k-labeling* of G if every two different edges has distinct edge weights, where the edge weight is defined as the summation of the edge label itself and its two vertex labels. Thus, the smallest value of $k$ for which the graph $G$ has the edge irregular reflexive $k$-labeling is called the *reflexive edge strength* of G. In this paper, we study the edge irregular reflexive labeling of corona product of two paths and corona product of a path with isolated vertices. We determine the reflexive edge strength for these graphs.

**KEYWORDS**
Edge irregular reflexive labeling; reflexive edge strength; corona product; path; complete graph

**2010 MATHEMATICAL SUBJECT CLASSIFICATION**
05C78; 05C38

## 1. Introduction

An edge irregular reflexive labeling is introduced by Ryan et al. [25] and is inspired by the problems of an irregular assignment and an edge irregular total labeling. Let us start with a brief review of the origins and some background information of these labelings.

Chartrand et al. [13] proposed a labeling problem in 1988, that is, determine the minimum value of parallel edges between every two vertices to ensure that a loopless multigraph has vertex irregularity. This problem is created as a consequence of Handshaking Lemma, i.e., no simple graph can have each distinct vertex degree, however, it is possible in multigraphs.

They defined this labeling problem as an edge $k$-labeling $\delta : E(G) \rightarrow \{1, 2, ..., k\}$ of a graph G such that the vertex weight is $w_\delta(x) \neq w_\delta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$, where $w_\delta(x) = \sum \delta(xy)$ the summation is over all vertices $y$ adjacent to $x$. Such labeling is called *irregular assignment* and the *irregularity strength* of G, $s(G)$ is known as the minimum $k$ for which $G$ has an irregular assignment using labels not greater than $k$. In other words, irregularity strength is interpreted as the minimum number of parallel edges, such that every vertex has a distinct degree in multigraph. For further results, see papers [6, 14, 17, 23, 24]. For comprehensive survey of graph labelings, please refer [15].

Bača et al. [10] introduced a total $k$-labeling $\rho : V(G) \cup E(G) \rightarrow \{1, 2, ..., k\}$ to be an edge irregular total $k$-labeling of a graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ one has $wt(xy) = \rho(x) + \rho(xy) + \rho(y) \neq wt(x'y') = \rho(x') + \rho(x'y') + \rho(y')$. The *total edge irregularity strength*, denoted by $tes(G)$ is defined as the minimum $k$ for which $G$ has an edge irregular total $k$-labeling. Some other results on the total edge irregularity strength can be referred to [2–5, 7, 11, 12, 21, 22, 26].

Therefore, Ryan et al. [25] were subsequently combined these two labeling problems by allowing for the vertex labels representing as loops. They noticed that: (a) the vertex labels are even non-negative integers, which also representing the fact that each loop added 2 to the vertex degree; and (b) vertex label 0 is permissible as representing a loopless vertex.

Thus, they defined the edge irregular reflexive $k$-labeling as a combination of an edge labeling $\varphi_e : E(G) \rightarrow \{1, 2, ..., k_e\}$ and a vertex labeling $\varphi_v : V(G) \rightarrow \{0, 2, ..., 2k_v\}$, in which labeling $\varphi$ is a total $k$-labeling of the graph $G$ such that $\varphi(x) = \varphi_v(x)$ if $x \in V(G)$ and $\varphi(x) = \varphi_e(x)$ if $x \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The total $k$-labeling $\varphi$ is called an edge irregular reflexive $k$-labeling of $G$ if for every two different edges $xy$, $x'y'$ of $G$ one has $wt(xy) = \varphi_v(x) + \varphi_e(xy) + \varphi_v(y) \neq wt(x'y') = \varphi_v(x') + \varphi_e(x'y') + \varphi_v(y')$. The smallest value of $k$ for which such labeling exists is called the *reflexive edge strength* of G and is denoted by $res(G)$. For more results of reflexive edge strength of graphs, see [1, 8, 9, 16, 18–20, 27, 28].

This paper focuses on the edge irregular reflexive labeling of two classes of corona product of graphs, that is, corona product of two paths and corona product of a path with isolated vertices. All graphs considered here are simple, finite and undirected. At the end of this paper, we are able to

**CONTACT** Roslan Hasni hroslan@umt.edu.my Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, Kuala Nerus, Malaysia.

determine the reflexive edge strength of these graphs with condition that they admit such labeling.

## 2. Significant lemma and conjecture

It is known that Lemma 1 is proved in [25].

**Lemma 1.** *For every graph G,*

$$\operatorname{res}(G) \geq \begin{cases} \lceil \frac{|E(G)|}{3} \rceil & \text{if } |E(G)| \not\equiv 2,3 \pmod 6, \\ \lceil \frac{|E(G)|}{3} \rceil + 1 & \text{if } |E(G)| \equiv 2,3 \pmod 6. \end{cases}$$

The following conjecture is proved by Bača et al. [9].

**Conjecture 1.** *Any graph G with maximum degree* $\Delta(G)$ *satisfies:*

$$res(G) = \max\left\{ \left\lfloor \frac{\Delta+2}{2} \right\rfloor, \left\lfloor \frac{|E(G)|}{3} \right\rfloor + r \right\}$$

*where* $r = 1$ *for* $|E(G)| \equiv 2,3 \pmod 6$, *and zero otherwise.*

## 3. Corona product of two paths

Suppose $P_n$ is a path of order $n$ and $P_m$ is another path of order $m$. The corona product of two paths, denoted by $P_n \odot P_m$ is defined as a graph obtained by taking one copy of $P_n$ (with $n$ vertices) and $n$ copies of $P_m$, and then joining the $i$-th vertex of $P_n$ to every vertex of the $i$-th copy of $P_m$.

Therefore, the vertex set and edge set of $P_n \odot P_m$ are defined as $V(P_n \odot P_m) = \{x_i, y_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P_n \odot P_m) = \{x_i y_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i^j y_i^{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{x_i x_{i+1} : 1 \leq i \leq n-1\}$, respectively.

The following theorem shows the edge irregular reflexive labeling on $P_n \odot P_m$ and its reflexive edge strength, $res(P_n \odot P_m)$.

**Theorem 1.** *For* $n \geq 2$ *and* $m \geq 2$,

$$res(P_n \odot P_m) = \begin{cases} \lceil \frac{2nm-1}{3} \rceil, & \text{if } nm \not\equiv 2 \pmod 3, \\ \lceil \frac{2nm-1}{3} \rceil + 1, & \text{if } nm \equiv 2 \pmod 3. \end{cases}$$

*Proof.* Note that the graph $P_n \odot P_m$ has $2nm - 1$ edges. By using Lemma 1, we obtain the following lower bound:

$$res(P_n \odot P_m) \geq k = \begin{cases} \lceil \frac{2nm-1}{3} \rceil, & \text{if } nm \not\equiv 2 \pmod 3, \\ \lceil \frac{2nm-1}{3} \rceil + 1, & \text{if } nm \equiv 2 \pmod 3. \end{cases}$$

It clearly shows that $k$ is odd only when $n, m \equiv 1 \pmod 3$ or $n, m \equiv 2 \pmod 3$, otherwise, $k$ is even.

Now, we prove that $k$ is the upper bound for $res(P_n \odot P_m)$, where $n, m \geq 2$. First, we define a total $k$-labeling $\varphi$ of $P_n \odot P_m$ by labeling the vertex set and edge set.

(a) All vertices $x_i$ and $y_i^j$ are labeled with the even integers in the following ways.
  (i) $\varphi(x_1) = 0, \varphi(x_2) = 2\lceil \frac{m-1}{3} \rceil$, otherwise, $\varphi(x_i) = 2\lceil \frac{im-1}{3} \rceil$ if $i \geq 3$.
  (ii) For $1 \leq j \leq m, \varphi(y_1^j) = 2\lceil \frac{j-2}{2} \rceil$, otherwise, $\varphi(y_i^j) = 2\lceil \frac{im-1}{3} \rceil$ if $i \geq 2$.

(b) The edges $x_i y_i^j, y_i^j y_i^{j+1}$ and $x_i x_{i+1}$ are labeled as follows.
  (a) $\varphi(x_1 y_1^j) = 1$ if $j$ is odd, whereas $\varphi(x_1 y_1^j) = 2$ if $j$ is even. For $1 \leq j \leq m, \varphi(x_2 y_2^j) = j$, otherwise, $\varphi(x_i y_i^j) = 2m(i-1) - 4\lceil \frac{im-1}{3} \rceil + j$ if $i \geq 3$.
  (b) For $1 \leq j \leq m-1, \varphi(y_1^j y_1^{j+1}) = m + 2 - j$, otherwise, $\varphi(y_i^j y_i^{j+1}) = m(2i-1) - 4\lceil \frac{im-1}{3} \rceil + j$ if $i \geq 2$.
  (c) The edges $x_i x_{i+1}$ are labeled as follows:

$$\varphi(x_i x_{i+1}) = \begin{cases} 2m - 2\lceil \frac{m-1}{3} \rceil, & \text{if } i = 1,2, \\ 2im - 2\lceil \frac{im-1}{3} \rceil - 2\lceil \frac{(i+1)m-1}{3} \rceil, & \text{if } i \geq 3. \end{cases}$$

Evidently, all vertex labels and edge labels are at most $k$ under the labeling $\varphi$, thus, labeling $\varphi$ is a total $k$-labeling of $P_n \odot P_m$. Next, we show the edge weights of all edges in $P_n \odot P_m$ are distinct under the total $k$-labeling $\varphi$.

(a) $wt_\varphi(x_i y_i^j) = \varphi(x_i) + \varphi(x_i y_i^j) + \varphi(y_i^j)$.
  (i) For $j$ odd, $wt_\varphi(x_1 y_1^j) = 0 + 1 + 2\lceil \frac{j-2}{2} \rceil = 1 + j - 1 = j$, whereas for $j$ even, $wt_\varphi(x_1 y_1^j) = 0 + 2 + 2\lceil \frac{j-2}{2} \rceil = 2 + j - 2 = j$.
  (ii) For $1 \leq j \leq m, wt_\varphi(x_2 y_2^j) = 2\lceil \frac{m-1}{3} \rceil + j + 2\lceil \frac{im-1}{3} \rceil = 2(\lceil \frac{m-1}{3} \rceil + \lceil \frac{2m-1}{3} \rceil) + j = 2m + j$.
  (iii) For $i \geq 3$ and $1 \leq j \leq m, wt_\varphi(x_i y_i^j) = 2\lceil \frac{im-1}{3} \rceil + 2m(i-1) - 4\lceil \frac{im-1}{3} \rceil + j + 2\lceil \frac{im-1}{3} \rceil = 2m(i-1) + j$.

(b) $wt_\varphi(y_i^j y_i^{j+1}) = \varphi(y_i^j) + \varphi(y_i^j y_i^{j+1}) + \varphi(y_i^{j+1})$.
  (i) For $1 \leq j \leq m-1, wt_\varphi(y_1^j y_1^{j+1}) = 2\lceil \frac{j-2}{2} \rceil + m + 2 - j + 2\lceil \frac{(j+1)-2}{2} \rceil = 2(j-1) + m + 2 - j = m + j$.
  (ii) For $i \geq 2$ and $1 \leq j \leq m-1, wt_\varphi(y_i^j y_i^{j+1}) = 2\lceil \frac{im-1}{3} \rceil + m(2i-1) - 4\lceil \frac{im-1}{3} \rceil + j + 2\lceil \frac{im-1}{3} \rceil = m(2i-1) + j$.

(c) $wt_\varphi(x_i x_{i+1}) = \varphi(x_i) + \varphi(x_i x_{i+1}) + \varphi(x_{i+1})$.
  (i) $wt_\varphi(x_1 x_2) = 0 + 2m - 2\lceil \frac{m-1}{3} \rceil + 2\lceil \frac{m-1}{3} \rceil = 2m$.
  (ii) $wt_\varphi(x_2 x_3) = 2\lceil \frac{m-1}{3} \rceil + 2m - 2\lceil \frac{m-1}{3} \rceil + 2\lceil \frac{(i+1)m-1}{3} \rceil = 2m + 2\lceil \frac{3m-1}{3} \rceil = 2m + 2m = 4m$.
  (iii) For $i \geq 3, wt_\varphi(x_i x_{i+1}) = 2\lceil \frac{im-1}{3} \rceil + 2im - 2\lceil \frac{im-1}{3} \rceil - 2\lceil \frac{(i+1)m-1}{3} \rceil + 2\lceil \frac{(i+1)m-1}{3} \rceil = 2im$.

We can see that the edge weights of all edges in $P_n \odot P_m$ are distinct integers from the set $\{1, 2, \ldots, 2nm-1\}$, in other words, every edge has a distinct weight. Thus, the total $k$-labeling $\varphi$ is an edge irregular reflexive $k$-labeling of $P_n \odot P_m$ and $k$ is the reflexive edge strength of $P_n \odot P_m$. This completes the proof. □

Figures 1 and 2 show the corresponding edge irregular reflexive $k$-labelings of $P_4 \odot P_4$ and $P_4 \odot P_5$.

## 4. Corona product of a path with isolated vertices

Assume $P_n$ is a path of order $n$ and $mK_1$ is a disjoint union of $m$ copies of isolated vertex. The corona product of a path with $m$ copies of isolated vertex, denoted by $P_n \odot mK_1$ is defined as a graph obtained by taking one copy of $P_n$ (with $n$ vertices) and $n$ copies of $mK_1$ by joining the $i$-th vertex of

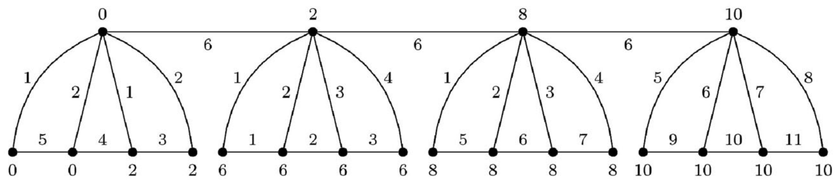


**Figure 1.** The edge irregular reflexive 11-labeling of $P_4 \odot P_4$.

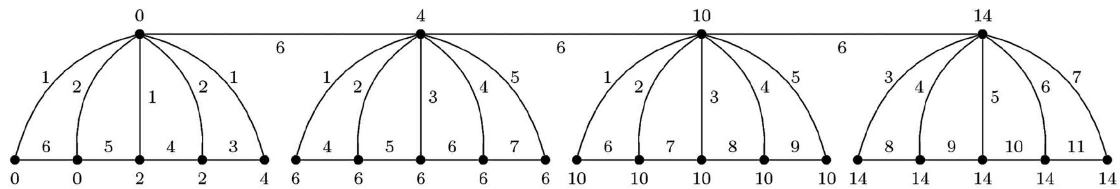


**Figure 2.** The edge irregular reflexive 14-labeling of $P_4 \odot P_5$.

$P_n$ to every vertex of the $i$-th copy of $mK_1$. Note that $P_n \odot mK_1$ is also known as a subclass of caterpillars.

Therefore, the vertex set and edge set of $P_n \odot mK_1$ are $V(P_n \odot mK_1) = \{x_i, y_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(P_n \odot mK_1) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$, respectively. The number of edges of $P_n \odot mK_1$, denoted by $|E(P_n \odot mK_1)|$ is $n(m+1)-1$. Thus, according to Lemma 1,

$$res(P_n \odot mK_1) \geq k = \begin{cases} \lceil \frac{n(m+1)-1}{3} \rceil, & \text{if } n(m+1)-1 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{n(m+1)-1}{3} \rceil + 1, & \text{if } n(m+1)-1 \equiv 2,3 \pmod 6. \end{cases} \quad (1)$$

We notice that $k$ is odd when $n \equiv 1 \pmod 3, m \equiv 1 \pmod 6$ or $n \equiv 2 \pmod 3, m \equiv 3 \pmod 6$ or $n \equiv 2 \pmod 6, m \equiv 0 \pmod 6$ or $n$, $m \equiv 4 \pmod 6$. Otherwise, $k$ is even.

The following lemmas show the reflexive edge strength of $P_n \odot mK_1$ by distinguishing $m$ into odd and even cases. First, we deal with $P_n \odot mK_1$ when $m$ is odd.

**Lemma 2.** *For $n \geq 2$ and $m$ odd,*

$$res(P_n \odot mK_1) = \begin{cases} \lceil \frac{n(m+1)-1}{3} \rceil, & \text{if } n(m+1)-1 \not\equiv 2,3 \pmod 6, \\ \lceil \frac{n(m+1)-1}{3} \rceil + 1, & \text{if } n(m+1)-1 \equiv 2,3 \pmod 6. \end{cases}$$

*Proof.* As a fact that $P_n \odot mK_1$ has $n(m+1)-1$ edges. According to Lemma 1, the lower bound for $res(P_n \odot mK_1)$ is shown as (1). Now, we prove that $k$ is the upper bound for $res(P_n \odot mK_1)$ when $m$ is odd. We first define a total $k$-labeling $\varphi$ of $P_n \odot mK_1$.

(a) All vertices $x_i$ and $y_i^j$ are labeled with the even integers in the following ways.

(i) $\varphi(x_1) = 0$. For $m \equiv 1 \pmod 6$, $i \equiv 2 \pmod 3$ or $m \equiv 3 \pmod 6$, $i \equiv 4 \pmod 3$, $\varphi(x_i) = \lceil \frac{i(m+1)+2}{3} \rceil$. Otherwise, $\varphi(x_i) = \lceil \frac{i(m+1)-2}{3} \rceil$.

(ii)

$$\varphi(y_i^j) = \begin{cases} 0, & \text{if } i=1, m \equiv 1 \pmod 6, 1 \leq j \leq \lceil \frac{2(m+2)}{3} \rceil, \\ & \text{or } m \equiv 3,5 \pmod 6, 1 \leq j \leq \lceil \frac{2m}{3} \rceil, \\ & \text{if } i=2, m \equiv 1 \pmod 6, 1 \leq j \leq \lceil \frac{m+3}{3} \rceil, \\ & \text{or } m \equiv 3,5 \pmod 6, 1 \leq j \leq \lceil \frac{m}{3} \rceil, \\ \lceil \frac{2(m+2)}{3} \rceil, & \text{if } i=1, m \equiv 1 \pmod 6, \lceil \frac{2(m+2)}{3} \rceil + 1 \leq j \leq m, \\ & \text{or } i=2, m \equiv 1 \pmod 6, \lceil \frac{m+3}{3} \rceil + 1 \leq j \leq m, \\ \lceil \frac{2m}{3} \rceil, & \text{if } i=1, m \equiv 3,5 \pmod 6, \lceil \frac{2m}{3} \rceil + 1 \leq j \leq m, \\ & \text{or } i=2, m \equiv 3,5 \pmod 6, \lceil \frac{m}{3} \rceil + 1 \leq j \leq m, \\ \varphi(x_i), & i \geq 3, 1 \leq j \leq m. \end{cases}$$

(b) The edges $x_i y_i^j$ and $x_i x_{i+1}$ are labeled as follows.

(i) $\varphi(x_1 y_1^j) = j$ if $m \equiv 1 \pmod 6, 1 \leq j \leq \lceil \frac{2(m+2)}{3} \rceil$ or $m \equiv 3,5 \pmod 6, 1 \leq j \leq \lceil \frac{2m}{3} \rceil$, otherwise, $\varphi(x_1 y_1^j) = j - \varphi(x_2)$. Next, $\varphi(x_2 y_2^j) = m+1-\varphi(x_2)+j$ if $m \equiv 1 \pmod 6, 1 \leq j \leq \lceil \frac{m+3}{3} \rceil$ or $m \equiv 3,5 \pmod 6$, $1 \leq j \leq \lceil \frac{m}{3} \rceil$, otherwise, $\varphi(x_2 y_2^j) = m+1-2\varphi(x_2)+j$. For $i \geq 3$ and $1 \leq j \leq m, \varphi(x_i y_i^j) = (i-1)(m+1)-2\varphi(x_i)+j$.

(ii) $\varphi(x_1 x_2) = \lceil \frac{m-1}{3} \rceil$ if $m \equiv 1 \pmod 6$, whereas $\varphi(x_1 x_2) = \lceil \frac{m+1}{3} \rceil$ if $m \equiv 3,5 \pmod 6$. Next, for $m \equiv 1 \pmod 6$, $i \equiv 2,3 \pmod 3$ or $m \equiv 3 \pmod 6$, $i \equiv 4 \pmod 3$, $\varphi(x_i x_{i+1}) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m-3}{3} \rceil$. For $m \equiv 5 \pmod 6$, $i \geq 2$, $\varphi(x_i x_{i+1}) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m+1}{3} \rceil$. Otherwise, $\varphi(x_i x_{i+1}) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m+3}{3} \rceil$.

Evidently, all vertex labels and edge labels are at most $k$ under the labeling $\varphi$, thus, labeling $\varphi$ is a total $k$-labeling of $P_n \odot mK_1$. Next, we show the edge weights of all edges in $P_n \odot mK_1$ are distinct under the total $k$-labeling $\varphi$.

(a) $wt_\varphi(x_i y_i^j) = \varphi(x_i) + \varphi(x_i y_i^j) + \varphi(y_i^j)$.

(i) For $i = 1$,

(A) $wt_\varphi(x_1 y_1^j) = 0 + j + 0 = j$ if $m \equiv 1 \pmod 6, 1 \leq j \leq \lceil \frac{2(m+2)}{3} \rceil$ or $m \equiv 3,5 \pmod 6$ $1 \leq j \leq \lceil \frac{2m}{3} \rceil$.

(B) $wt_\varphi(x_1 y_1^j) = 0 + j - \varphi(x_2) + \lceil \frac{2(m+2)}{3} \rceil = j - \lceil \frac{2(m+2)}{3} \rceil + \lceil \frac{2(m+2)}{3} \rceil = j$ if $m \equiv 1 \pmod 6), \lceil \frac{2(m+2)}{3} \rceil + 1 \leq j \leq m$.

(C) $wt_\varphi(x_1y_1^j)=0+j-\varphi(x_2)+\lceil\frac{2m}{3}\rceil=j-\lceil\frac{2m}{3}\rceil+\lceil\frac{2m}{3}\rceil=j$ if $m\equiv 3,5 \pmod 6$, $\lceil\frac{2m}{3}\rceil+1\le j\le m$.

(ii) For $i=2$,

(A) when $m\equiv 1 \pmod 6$, $wt_\varphi(x_2y_2^j)=\lceil\frac{i(m+1)+2}{3}\rceil+m+1-\varphi(x_2)+j+0=\lceil\frac{i(m+1)+2}{3}\rceil+m+1-\lceil\frac{i(m+1)+2}{3}\rceil+j=m+1+j$ if $1\le j\le\lceil\frac{m+3}{3}\rceil$, otherwise, $wt_\varphi(x_2y_2^j)=\lceil\frac{i(m+1)+2}{3}\rceil+m+1-2\varphi(x_2)+j+\lceil\frac{2(m+2)}{3}\rceil=\lceil\frac{i(m+1)+2}{3}\rceil+m+1-2\lceil\frac{i(m+1)+2}{3}\rceil+j+\lceil\frac{i(m+1)+2}{3}\rceil=m+1+j$ if $\lceil\frac{m+3}{3}\rceil+1\le j\le m$.

(B) when $m\equiv 3,5 \pmod 6$, $wt_\varphi(x_2y_2^j)=\lceil\frac{i(m+1)-2}{3}\rceil+m+1-\varphi(x_2)+j+0=\lceil\frac{i(m+1)-2}{3}\rceil+m+1-\lceil\frac{i(m+1)-2}{3}\rceil+j=m+1+j$ if $1\le j\le\lceil\frac{m}{3}\rceil$, otherwise, $wt_\varphi(x_2y_2^j)=\lceil\frac{i(m+1)-2}{3}\rceil+m+1-2\varphi(x_2)+j+\lceil\frac{2m}{3}\rceil=\lceil\frac{i(m+1)-2}{3}\rceil+m+1-2\lceil\frac{i(m+1)-2}{3}\rceil+j+\lceil\frac{i(m+1)-2}{3}\rceil=m+1+j$ if $\lceil\frac{m}{3}\rceil+1\le j\le m$.

(iii) For $i\ge 3$ and $1\le j\le m$,

(A) $wt_\varphi(x_iy_i^j)=\lceil\frac{i(m+1)-2}{3}\rceil+(i-1)(m+1)-2\varphi(x_i)+j+\lceil\frac{i(m+1)-2}{3}\rceil=\lceil\frac{i(m+1)-2}{3}\rceil+(i-1)(m+1)-2\lceil\frac{i(m+1)-2}{3}\rceil+j+\lceil\frac{i(m+1)-2}{3}\rceil=(i-1)(m+1)+j$ if $m\equiv 1 \pmod 6$, $i\equiv 0,1 \pmod 3$ or $m\equiv 3 \pmod 6$, $i\equiv 0,2 \pmod 3$ or $m\equiv 5 \pmod 6$, $i\ge 3$.

(B) $wt_\varphi(x_iy_i^j)=\lceil\frac{i(m+1)+2}{3}\rceil+(i-1)(m+1)-2\varphi(x_i)+j+\lceil\frac{i(m+1)+2}{3}\rceil=\lceil\frac{i(m+1)+2}{3}\rceil+(i-1)(m+1)-2\lceil\frac{i(m+1)+2}{3}\rceil+j+\lceil\frac{i(m+1)+2}{3}\rceil=(i-1)(m+1)+j$ if $m\equiv 1 \pmod 6$, $i\equiv 2 \pmod 3$ or $m\equiv 3 \pmod 6$, $i\equiv 1 \pmod 3$. Take note that $i\ne 1,2$ in (iii)(A) and (iii)(B).

(b) $wt_\varphi(x_ix_{i+1})=\varphi(x_i)+\varphi(x_ix_{i+1})+\varphi(x_{i+1})$.

(i) For $i=1$,

(A) $wt_\varphi(x_1x_2)=0+\lceil\frac{m-1}{3}\rceil+\lceil\frac{(i+1)(m+1)+2}{3}\rceil=\lceil\frac{m+1}{3}\rceil+\lceil\frac{2m+4}{3}\rceil=\frac{1}{3}[(m-1)+2(m+2)]=m+1$ if $m\equiv 1 \pmod 6$.

(B) $wt_\varphi(x_1x_2)=0+\lceil\frac{m+1}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=\lceil\frac{m+1}{3}\rceil+\lceil\frac{2m}{3}\rceil=\frac{1}{3}[(m+3)+2m]=m+1$ if $m\equiv 3 \pmod 6$.

(C) $wt_\varphi(x_1x_2)=0+\lceil\frac{m+1}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=\lceil\frac{m+1}{3}\rceil+\lceil\frac{2m}{3}\rceil=\frac{1}{3}[(m+1)+(m+2)]=m+1$ if $m\equiv 5 \pmod 6$.

(ii) For $i\ge 2$,

(A) when $i\equiv 0 \pmod 3$, $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)-2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m-3}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=\lceil\frac{i(m+1)-2}{3}\rceil+i(m+1)-2\lceil\frac{i(m+1)-2}{3}\rceil-\lceil\frac{m-3}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=i(m+1)+\frac{1}{3}[-i(m+1)-(m-1)+i(m+1)+(m-1)]=i(m+1)$ if $m\equiv 1 \pmod 6$, otherwise, $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)-2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m+3}{3}\rceil+\lceil\frac{(i+1)(m+1)+2}{3}\rceil=i(m+1)+\frac{1}{3}[-i(m+1)-(m+3)+i(m+1)+(m+3)]=i(m+1)$ if $m\equiv 3 \pmod 6$.

(B) when $i\equiv 1 \pmod 3$, $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)-2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m+3}{3}\rceil+\lceil\frac{(i+1)(m+1)+2}{3}\rceil=i(m+1)+\frac{1}{3}\{-[i(m+1)-2]-(m+5)+[i(m+1)-2]+(m+5)\}=i(m+1)$ if $m\equiv 1 \pmod 6$, otherwise, $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)+2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m-3}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=i(m+1)+\frac{1}{3}\{-[i(m+1)+2]-(m-3)+[i(m+1)+2]+(m-3)\}=i(m+1)$ if $m\equiv 3 \pmod 6$.

(C) when $i\equiv 2 \pmod 3$, $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)+2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m-3}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=i(m+1)+\frac{1}{3}\{-[i(m+1)+2]-(m-1)+[i(m+1)+2]+(m-1)\}=i(m+1)$ if $m\equiv 1 \pmod 6$, otherwise, $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)-2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m+3}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=i(m+1)+\frac{1}{3}\{-[i(m+1)-2]-(m+3)+[i(m+1)-2]+(m+3)\}=i(m+1)$ if $m\equiv 3 \pmod 6$.

(D) $wt_\varphi(x_ix_{i+1})=\lceil\frac{i(m+1)-2}{3}\rceil+i(m+1)-2\varphi(x_i)-\lceil\frac{m+1}{3}\rceil+\lceil\frac{(i+1)(m+1)-2}{3}\rceil=i(m+1)+\frac{1}{3}[-i(m+1)-(m+1)+i(m+1)+(m+1)]=i(m+1)$ if $m\equiv 5 \pmod 6$.

It clearly shows that the edge weights of all edges in $P_n\odot mK_1$ are distinct integers from the set $\{1,2,\ldots,n(m+1)-1\}$, which means that all edges have distinct weights. Thus, the total $k$-labeling $\varphi$ is an edge irregular reflexive $k$-labeling of $P_n\odot mK_1$ and $k$ is the reflexive edge strength of $P_n\odot mK_1$, where $m$ is odd. This completes the proof. □

Figures 3 and 4 show the corresponding edge irregular reflexive 7-labeling of $P_5\odot 3K_1$ and edge irregular reflexive 8-labeling of $P_4\odot 5K_1$, respectively.

In the next lemma, we deal with $P_n\odot mK_1$ when $m$ is even.

**Lemma 3.** *For $n\ge 2$ and $m$ even,*

$$res(P_n\odot mK_1)=\begin{cases}\lceil\frac{n(m+1)-1}{3}\rceil, & \text{if } n(m+1)-1\not\equiv 2,3 \pmod 6,\\ \lceil\frac{n(m+1)-1}{3}\rceil+1, & \text{if } n(m+1)-1\equiv 2,3 \pmod 6.\end{cases}$$

*Proof.* Since the number of edges of $P_n\odot mK_1$ is $n(m+1)-1$, by Lemma 1, we obtain the lower bound as shown in (1). Now, we prove that $k$ is the upper bound for $res(P_n\odot mK_1)$ when $m$ is even. We first define a total $k$-labeling $\varphi$ of $P_n\odot mK_1$.

(a) All vertices $x_i$ and $y_i^j$ are labeled as follows.

(i) For $i\ne 3$,

(A) $\varphi(x_1)=0$. Then, $\varphi(x_i)=\lceil\frac{i(m+1)+2}{3}\rceil$ if $m\equiv 0 \pmod 6$, $i\equiv 3, 4 \pmod 6$ or

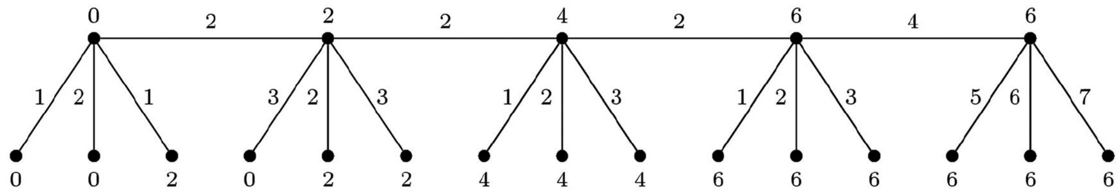


**Figure 3.** An edge irregular reflexive 7-labeling for $P_5 \odot 3K_1$.

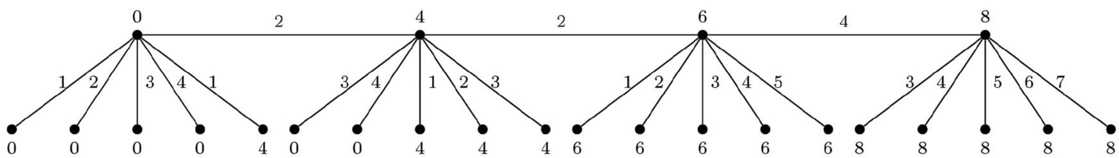


**Figure 4.** An edge irregular reflexive 8-labeling for $P_4 \odot 5K_1$.

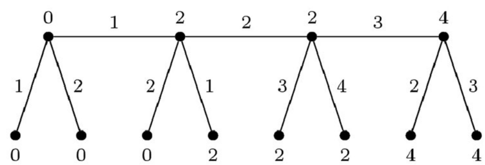


**Figure 5.** An edge irregular reflexive 4-labeling for $P_4 \odot 2K_1$.

$m \equiv 2 \pmod 6$, $i \equiv 1 \pmod 6$ or $m \equiv 4 \pmod 6$, $i \equiv 2,3 \pmod 6$. Next, $\varphi(x_i) = \lceil \frac{i(m+1)}{3} \rceil$ if $m \equiv 0 \pmod 6$, $i \equiv 5 \pmod 6$ or $m \equiv 4 \pmod 6$, $i \equiv 1 \pmod 6$.

Otherwise, $\varphi(x_i) = \lceil \frac{i(m+1)-2}{3} \rceil$.

(B) The vertices $\varphi(y_i^j)$ are labeled as follows.

$$\varphi(y_i^j) = \begin{cases} 0, & \text{if } i=1, m \equiv 0,2 \pmod 6, 1 \le j \le \lceil \frac{2m}{3} \rceil, \\ & \text{or } m \equiv 4 \pmod 6, 1 \le j \le \lceil \frac{2(m+2)}{3} \rceil, \\ & \text{if } i=2, m \equiv 0 \pmod 6, 1 \le j \le \lceil \frac{m-3}{3} \rceil, \\ & \text{or } m \equiv 2 \pmod 6, 1 \le j \le \lceil \frac{m}{3} \rceil, \\ & \text{or } m \equiv 4 \pmod 6, 1 \le j \le \lceil \frac{m+3}{3} \rceil, \\ \lceil \frac{2m}{3} \rceil, & \text{if } i=1, m \equiv 0,2 \pmod 6, \lceil \frac{2m}{3} \rceil + 1 \le j \le m, \\ & \text{or } i=2, m \equiv 0 \pmod 6, \lceil \frac{m-3}{3} \rceil + 1 \le j \le m, \\ & \text{or } m \equiv 2 \pmod 6, \lceil \frac{m}{3} \rceil + 1 \le j \le m, \\ \lceil \frac{2(m+2)}{3} \rceil, & \text{if } i=1, m \equiv 4 \pmod 6, \lceil \frac{2(m+2)}{3} \rceil + 1 \le j \le m, \\ & \text{or } i=2, m \equiv 4 \pmod 6, \lceil \frac{m+3}{3} \rceil + 1 \le j \le m, \\ \varphi(x_i), & \text{otherwise.} \end{cases}$$

(ii) For $i = 3$ and $1 \le j \le m$, $\varphi(x_3) = \varphi(y_3^j) = m$.

(b) The edges $x_i y_i^j$ and $x_i x_{i+1}$ are labeled as follows. .

(i) For $i \ne 3$,

(A) $\varphi(x_1 y_1^j) = j$ if $m \equiv 0,2 \pmod 6$, $1 \le j \le \lceil \frac{2m}{3} \rceil$ or $m \equiv 4 \pmod 6$, $1 \le j \le \frac{2(m+2)}{3}$, otherwise, $\varphi(x_1 y_1^j) = j - \varphi(x_2)$. Next, $\varphi(x_2 y_2^j) = m+1-\varphi(x_2)+j$ if $m \equiv 0 \pmod 6$, $1 \le j \le \lceil \frac{m-3}{3} \rceil$ or $m \equiv 2 \pmod 6$, $1 \le j \le \lceil \frac{m}{3} \rceil$ or $m \equiv 4 \pmod 6$, $1 \le j \le \lceil \frac{m+3}{3} \rceil$, otherwise, $\varphi(x_2 y_2^j) = m+1-2\varphi(x_2)+j$. For $i \ge 4$, $1 \le j \le m$, $\varphi(x_i y_i^j) = (i-1)(m+1) - 2\varphi(x_i) + j$.

(B) $\varphi(x_1 x_2) = \lceil \frac{m+1}{3} \rceil$ if $m \equiv 0,2 \pmod 6$, otherwise, $\varphi(x_1 x_2) = \lceil \frac{m-1}{3} \rceil$. Next, $\varphi(x_2 x_3) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m-2}{3} \rceil$ if $m \equiv 0,2 \pmod 6$, otherwise, $\varphi(x_2 x_3) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m-4}{3} \rceil$. For $i \ge 4$,

$$\varphi(x_i x_{i+1}) = \begin{cases} i(m+1) - 2\varphi(x_i) - \lceil \frac{m-2}{3} \rceil, & \text{if } m \equiv 0 (\text{mod } 6), i \not\equiv 2 (\text{mod } 6), \\ & \text{or } m \equiv 2 (\text{mod } 6), i \equiv 1 (\text{mod } 2), \\ i(m+1) - 2\varphi(x_i) - \lceil \frac{m-4}{3} \rceil, & \text{if } m \equiv 4 (\text{mod } 6), i \equiv 3 (\text{mod } 6), \\ i(m+1) - 2\varphi(x_i) - \lceil \frac{m}{3} \rceil, & \text{if } m \equiv 4 (\text{mod } 6), i \not\equiv 3 (\text{mod } 6), \\ i(m+1) - 2\varphi(x_i) - \lceil \frac{m+4}{3} \rceil, & \text{otherwise.} \end{cases}$$

(ii) For $i = 3$,

(A) $\varphi(x_3 y_3^j) = 2 + j$ if $1 \le j \le m$.

(B) $\varphi(x_3 x_4) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m+4}{3} \rceil$ if $m \equiv 0,2 (\text{mod } 6)$, otherwise, $\varphi(x_3 x_4) = i(m+1) - 2\varphi(x_i) - \lceil \frac{m}{3} \rceil$ if $m \equiv 4 (\text{mod } 6)$.

Clearly, all vertex labels and edge labels are at most $k$ under the labeling $\varphi$, thus, labeling $\varphi$ is a total $k$-labeling of $P_n \odot mK_1$. Using similar approach as in the proof of Lemma 2, we are able to find the edge weights of all edges in $P_n \odot mK_1$.

$$wt_\varphi(x_i x_{i+1}) = \varphi(x_i) + \varphi(x_i x_{i+1}) + \varphi(x_{i+1})$$

and

$$wt_\varphi(x_i y_i^j) = \varphi(x_i) + \varphi(x_i y_i^j) + \varphi(y_i^j).$$

Therefore, the results of edge weights are: (a) $wt_\varphi(x_i x_{i+1}) = m+1$ if $i = 1$, otherwise, $wt_\varphi(x_i x_{i+1}) = i(m+1)$ if $i \ge 2$; and (b) $wt_\varphi(x_i y_i^j) = j$ if $i = 1$, $wt_\varphi(x_i y_i^j) = m+1+j$ if $i = 2$, $wt_\varphi(x_i y_i^j) = 2(m+1)+j$ if $i = 3$, otherwise, $wt_\varphi(x_i y_i^j) = (i-1)(m+1)+j$ if $i \ge 4$.

We can see that the edge weights of all edges in $P_n \odot mK_1$ are distinct integers from the set $\{1, 2, ..., n(m+1) - 1\}$, in other words, every edge has a distinct weight. Thus, the total $k$-labeling $\varphi$ is an edge irregular reflexive $k$-labeling of $P_n \odot mK_1$ and $k$ is the reflexive edge strength of $P_n \odot mK_1$, where $m$ is even. This completes the proof. □

Figures 5 and 6 show the corresponding edge irregular reflexive 4-labeling of $P_4 \odot 2K_1$ and edge irregular reflexive 6-labeling of $P_3 \odot 4K_1$, respectively.

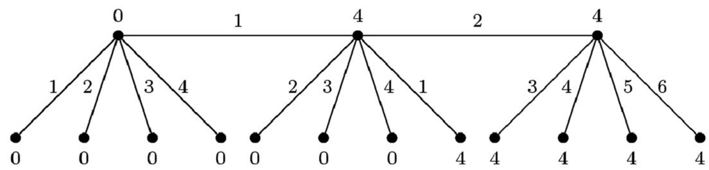


**Figure 6.** An edge irregular reflexive 6-labeling for $P_3 \odot 4K_1$.

Combining Lemmas 2 and 3, we obtain the concluding result for the reflexive edge strength of corona product of path with $mK_1$ as follows.

**Theorem 2.** *For $n \geq 2$ and all positive integers m,*

$$res(P_n \odot mK_1) = \begin{cases} \lceil \frac{n(m+1)-1}{3} \rceil, & \text{if } n(m+1)-1 \not\equiv 2,3 \pmod{6}, \\ \lceil \frac{n(m+1)-1}{3} \rceil + 1, & \text{if } n(m+1)-1 \equiv 2,3 \pmod{6}. \end{cases}$$

## 5. Conclusion

This paper has successfully determined the reflexive edge strength of corona product of graphs, that is, corona product of two paths and corona product of a path with isolated vertices, where these graphs have also proven to admit the edge irregular reflexive labeling. Moreover, these generalized results are not only strengthened the Conjecture 1, but also thoroughly replaced the weak and restricted results of the previous paper [19]. Last but not least, this interesting study found a problem that worths for further investigation, that is:

**Problem 1.** *Determine the reflexive edge strength of corona product of a path $P_n$ with m copies of complete graphs, i.e., $res(P_n \odot mK_t)$, where $n, m, t \geq 2$ and all positive integers m.*

## Acknowledgement

The authors would like to thank the referees for their valuable comments that improved the paper.

## Disclosure statement

No potential competing interest was reported by the authors.

## Funding

This research is supported by the Fundamental Research Grant Scheme (FRGS), Phase 1/2020, Ministry of Higher Education, Malaysia with Reference Number FRGS/1/2020/STG06/UMT/02/1 (Grant Vot. 59609).

## References

[1] Agustin, I. H., Utoyo, I, Venkatachalam, M. (2020). Edge irregular reflexive labeling of some tree graphs. *IOP Conf. Ser. J. Phy. Conf. Ser.* 1543: 012008.

[2] Ahmad, A., Bača, M., Bashir, Y, Siddiqui, M. K. (2012). Total edge irregularity strength of strong product of two paths. *Ars. Comb.* 106: 449–459.

[3] Ahmad, A., Bača, M, Siddiqui, M. K. (2014). On edge irregular total labeling of categorical product of two cycles. *Theory Comput. Syst.* 54(1): 1–12.

[4] Ahmad, A., Siddiqui, M. K, Afzal, D. (2012). On the total edge irregular strength of zigzag graphs. *Australasian J. Combin.* 54: 141–149.

[5] Al-Mushayt, O., Ahmad, A, Siddiqui, M. K. (2012). On the total edge irregularity strength of hexagonal grid graphs. *Australasian J. Combin.* 53: 263–271.

[6] Amar, D, Togni, O. (1998). Irregularity strength of trees. *Discrete Math.* 190(1-3): 15–38.

[7] Anholcer, M, Palmer, C. (2012). Irregular labelings of circulant graphs. *Discrete Math.* 312(23): 3461–3466.

[8] Bača, M., Irfan, M., Ryan, J., Semaničová-Feňovčíková, A, Tanna, D. (2019). Note on edge irregular reflexive labelings of graphs. *AKCE Int. J. Graphs Comb.* 16(2): 145–157.

[9] Bača, M., Irfan, M., Ryan, J., Semaničová-Feňovčíková, A, Tanna, D. (2017). On the edge irregular reflexive labelings for the generalized friendship graphs. *Mathematics* 5(4): 67.

[10] Bača, M., Jendrol', S., Miller, M, Ryan, J. (2007). On irregular total labelings. *Discrete Math.* 307(11-12): 1378–1388.

[11] Bača, M, Siddiqui, M. K. (2014). Total edge irregularity strength of generalized prism. *Appl. Math. Comp.* 235: 168–173.

[12] Brandt, S., Miškuf, J, Rautenbach, D. (2008). On a conjecture about edge irregular total labelings. *J. Graph Theory* 57(4): 333–343.

[13] Chartrand, G., Jacobson, M. S., Lehel, J., Oellermann, O. R., Ruiz, S, Saba, F. (1988). Irregular networks. *Congr. Numer.* 64: 187–192.

[14] Dinitz, J. H., Garnick, D. K, Gyárfás, A. (1992). On the irregularity strength of the $m \times n$ grid. *J. Graph Theory* 16(4): 355–374.

[15] Gallian, J. A. (2019). A dynamic survey of graph labeling. *Electron. J. Comb.* #DS6: 1–553.

[16] Guirao, J. L. G., Ahmad, S., Siddiqui, M. K, Ibrahim, M. (2018). Edge irregular reflexive labeling for the disjoint union of generalized Petersen graph. *Mathematics* 6(12): 304.

[17] Gyárfás, A. (1998). The irregularity strength of $K_{m,m}$ is 4 for odd *m*. *Discrete Math.* 71: 273–274.

[18] Ibrahim, M., Majeed, S, Siddiqui, M. K. (2020). Edge irregular reflexive labeling for star, double star and caterpillar graphs. *TWMS J. App. Eng. Math.* 10(3): 718–726.

[19] D. Indriati, Widodo, I. Rosyida, (2020). Edge irregular reflexive labeling on corona of path and other graphs. *IOP Conf. Series: Journal of Physics: Conf. Series* 1498: 012004.

[20] Irfan, M., Bača, M, Semaničová-Feňovčíková, A. On reflexive edge strength of generalized prism graphs. *Electron. J. Graph Theory Appl.* doi:10.3390/math5040067

[21] Ivančo, J, Jendrol', S. (2006). Total edge irregularity strength of trees. *Discuss. Math. Graph Theory* 26(3): 449–456.

[22] Jendrol', S., Miškuf, J, Soták, R. (2010). Total edge irregularity strength of complete graphs and complete bipartite graphs. *Discrete Math.* 310(3): 400–407.

[23] Lehel, J. (1991). Facts and quests on degree irregular assignment. In: *Graph Theory, Combinatorics and Applications*; New York, USA: Wiley, pp. 765–782.

[24] Nierhoff, T. (2000). A tight bound on the irregularity strength of graphs. *SIAM J. Discrete Math.* 13(3): 313–323.
[25] Ryan, J., Munasinghe, B, Tanna, D. (2017) Reflexive irregular labelings, preprint.
[26] Siddiqui, M. K. (2012). On the edge irregularity strength of a categorical product of a cycle and a path. *AKCE Int. J. Graphs Comb.* 9(1): 43–52.
[27] Tanna, D., Ryan, J, Semaničová-Feňovčíková, A. (2017). Edge irregular reflexive labeling of prisms and wheels. *Australasian J. Combin.* 69(3): 394–401.
[28] Zhang, X., Ibrahim, M., Bokhary, S. A. H, Siddiqui, M. K. (2018). Edge irregular reflexive labeling for the disjoint union of gear graphs and prism graphs. *Mathematics* 6(9): 142.