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Analysis on vector product spaces

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Analysis on Vector Product Spaces

A Thesis

Presented to

the Faculty of

the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

Claude Michael Cassano

May 2004

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ABSTRACT

ANALYSIS ON VECTOR PRODUCT SPACES

by Claude Michael Cassano

The general purpose of this paper is to define and develop algebraic properties and analysis on vector product spaces, and to generalize concepts and develop material cutting across and unifying subject areas; most specifically, to investigate "smoothness" criteria and consequences on such "smooth" functions in the space. Associative vector product spaces are examined in the most detail, including generalizations of the Cauchy-Riemann equations and the Cauchy Integral Theorem. Non-associative vector product spaces are treated as well. To establish the existence of non-associative vector product spaces - a generalization of ordinary matrix multiplication - is defined and developed.

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Preface

This thesis is a study of certain properties of a class of algebras and analysis performed thereon. This study is an initial culmination of my efforts toward a unified field theory. In these efforts, eventually, I settled on three points that became the groundwork for this foundation of it.

First, closure, so that "interactions" modeled as binary operations did not cause members of the space to exit the space "vanishing without trace."

Second, a "smoothness" criteria on functions within the space would determine the "equations of motion" in the space - suggested by the Cauchy-Riemann equations of an analytic function within the complex plane, and the faint similarity they have to Maxwell's equations of an electromagnetic field reduced to two dimensions.

Third, the space in which these functions "reside" has, to first approximation, a four-dimensional special relativity spacetime, with ordinary invariants of dot and cross products in the embedded three-space.

Within the associative R -algebras considered, I came up with a generalization of the Cauchy-Riemann equations for "smooth" functions, and later, a generalization of the Cauchy Integral Theorem, as well.

However, after spending quite a bit of time on associative R -algebras, being unable to construct a four-dimensional space meeting the above criteria, within which Maxwell's equations followed from the "smoothness" criteria, I became convinced that a non-associative

binary product was necessary, and fashioned one to fit the bill satisfying all needs.

Not wanting this to be a mere “theoretical abstraction” and already

knowing that the matrix pair $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is isomorphic to $(1, i)$ (of the complex field); I searched for a generalization of ordinary matrix multiplication, and eventually hit on what I called the “weighted matrix product” matching weight matrix entries with the operands of the product described herein, which is, generally, non-associative. This serves as the construction point for R -algebras of desired characteristics, and that is why I begin with it.

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Chapter I

Certain Features on a Class of Algebras

This thesis is concerned with certain properties of and analysis of algebras. The vector properties of algebras will figure significantly in the examination that will be done, herein, which is why artistic license has been taken with the usage "vector product spaces." First, then a standard definition of the term algebra is presented, followed by a handful of common and well-known examples, along with a spoonful of their properties relative to matrices (A1-26) (which will also make extended appearance).

An algebra is a vector space (A1-27), V , over a field (A1-23), $\mathcal{F}$, with products, $\otimes, \circ$, such that $\forall \lambda, \mu \in \mathcal{F}, \forall u, v, w, x \in V$:

$$(\lambda \otimes u + \mu \otimes v) \circ w = \lambda \otimes (u \circ w) + \mu \otimes (v \circ w)$$

$$u \circ (\lambda \otimes v + \mu \otimes w) = \lambda \otimes (u \circ v) + \mu \otimes (u \circ w) .$$

When a field, $\mathcal{F}$ is explicitly stated, a common term used is $\mathcal{F}$ -algebra.

See [1], and [30] for further depth into algebras and $\mathcal{F}$ -algebras..

In this thesis, the vector space, figuring prominently, must be specified, so, the notation $(\mathcal{F}, \otimes, V)$ shall be used to designate a vector space, and $((\mathcal{F}, \otimes, V), \circ)$ an $\mathcal{F}$ -algebra throughout this thesis.

Some well-known algebras figuring prominently herein, are introduced in the examples that follow.

A. Example 1-1

The Complex Plane

The complex plane is defined as the set of complex numbers, written as ordered pairs [4],[7] : (an isomorphism [34],[46] (A1-14) with $\mathcal{R}^2$) (Generally, within this chapter, whenever a set of sets and relations is given in different forms, these forms will be referred to as isomorphic [34],[46]).

Throughout, the symbol $\equiv$ defines the left operand to be what the right operand is. So, using ordinary multiplication as product, $u_0 \equiv 1$, and $u_1 \equiv i$, then

$$u_0 \circ u_0 = 1 \cdot 1 = 1 = u_0 \quad u_1 \circ u_0 = i \cdot 1 = i = u_1$$

$$u_0 \circ u_1 = 1 \cdot i = i = u_1 \quad u_1 \circ u_1 = i \cdot i = -1 = -u_0$$

Using matrix multiplication (A1-41) as product, and basis [40] (A1-39)

$$\mathbf{u}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then

$$\begin{aligned} \mathbf{u}_0 \circ \mathbf{u}_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{u}_0 \\ \mathbf{u}_0 \circ \mathbf{u}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{u}_1 \\ \mathbf{u}_1 \circ \mathbf{u}_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{u}_1 \\ \mathbf{u}_1 \circ \mathbf{u}_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{u}_0. \end{aligned}$$

These may be expressed compactly in Cayley Tables [22], as follows.

	1	i
1	1	i
i	i	-1

	$\mathbf{u}_0$	$\mathbf{u}_1$
$\mathbf{u}_0$	$\mathbf{u}_0$	$\mathbf{u}_1$
$\mathbf{u}_1$	$\mathbf{u}_1$	$-\mathbf{u}_0$

Any two dimensional vector spaces are isomorphic to one another, but here, they are also isomorphic with respect to the product relation (ordinary multiplication for $(1, i)$, matrix multiplication for $(\mathbf{u}_0, \mathbf{u}_1)$).

B. Example 1-2

The Cross Product

For the following discussion, vectors in $\mathcal{R}^3$ may be expressed using the matrices:

$$\mathbf{u}_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u}_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{u}_3 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If $c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 = \mathbf{0}$, then

$$\begin{pmatrix} (c_1) & (c_2 + c_3) \\ (c_2 - c_3) & (-c_1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent; (A1-38) and, the basis: $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans (A1-37) the set of matrices of the form:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix};$$

i.e., spans the set:

$$V = \left\{ W \mid W = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, \forall x, y, z \in \mathcal{R} \right\},$$

which is a vector space, as the reader may verify.

Given any $(x, y, z) \in \mathcal{R}$, $\exists$ unique $(p, q, r) \in \mathcal{R} \ni$

$$p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} x & y \\ z & -x \end{pmatrix},$$

then

$$\begin{aligned} p &= x, q + r = y \\ q - r &= z \\ 2q &= y + z \Rightarrow q = \frac{1}{2}(y + z) \\ 2r &= y - z \Rightarrow r = \frac{1}{2}(y - z). \end{aligned}$$

$$\text{So, for: } \begin{pmatrix} (x_1) & (x_2 + x_3) \\ (x_2 - x_3) & (-x_1) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix},$$

$$x_1 = u_{11}, \quad x_2 + x_3 = u_{12}, \quad x_2 - x_3 = u_{21}$$

$\therefore$

$$x_1 = u_{11}, \quad x_2 = \frac{u_{12} + u_{21}}{2}, \quad x_3 = \frac{u_{12} - u_{21}}{2}.$$

This mapping (A1-7):

$$m : (x_1, x_2, x_3) \in \mathcal{R}^3 \rightleftharpoons (u_{11}, u_{12}, u_{21}) \in \mathcal{R}^3,$$

is bijective (A1-10) (one-to-one (A1-9) and onto (A1-8)).

Using this set of matrices, observe that $\mathcal{R}^3$ with the usual cross product is isomorphic to the following construction.

$\times :$

$$\mathbf{x} \times \mathbf{y} = \frac{1}{2}(\mathbf{x}^T \mathbf{y}^T - \mathbf{y}^T \mathbf{x}^T).$$

Therefore:

$$\mathbf{u}_1 \times \mathbf{u}_1 = \frac{1}{2}(\mathbf{u}_1^T \mathbf{u}_1^T - \mathbf{u}_1^T \mathbf{u}_1^T) = \mathbf{0}$$

$$\mathbf{u}_1 \times \mathbf{u}_2 = \frac{1}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \mathbf{u}_3$$

$$\begin{aligned}
\mathbf{u}_1 \times \mathbf{u}_3 &= \frac{1}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = -\mathbf{u}_2 \\
\mathbf{u}_2 \times \mathbf{u}_1 &= \frac{1}{2} (\mathbf{u}_2^T \mathbf{u}_1^T - \mathbf{u}_1^T \mathbf{u}_2^T) = -\frac{1}{2} (\mathbf{u}_1^T \mathbf{u}_2^T - \mathbf{u}_2^T \mathbf{u}_1^T) = -(\mathbf{u}_1 \circ \mathbf{u}_2) = -\mathbf{u}_3 \\
\mathbf{u}_2 \times \mathbf{u}_2 &= \frac{1}{2} (\mathbf{u}_2^T \mathbf{u}_2^T - \mathbf{u}_2^T \mathbf{u}_2^T) = \mathbf{0} \\
\mathbf{u}_2 \times \mathbf{u}_3 &= \frac{1}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \mathbf{u}_1 \\
\mathbf{u}_3 \times \mathbf{u}_1 &= \frac{1}{2} \frac{1}{2} (\mathbf{u}_3^T \mathbf{u}_1^T - \mathbf{u}_1^T \mathbf{u}_3^T) = -\frac{1}{2} (\mathbf{u}_1^T \mathbf{u}_3^T - \mathbf{u}_3^T \mathbf{u}_1^T) = -(\mathbf{u}_1 \circ \mathbf{u}_3) = \mathbf{u}_2 \\
\mathbf{u}_3 \times \mathbf{u}_2 &= \frac{1}{2} \frac{1}{2} (\mathbf{u}_3^T \mathbf{u}_2^T - \mathbf{u}_2^T \mathbf{u}_3^T) = -\frac{1}{2} (\mathbf{u}_2^T \mathbf{u}_3^T - \mathbf{u}_3^T \mathbf{u}_2^T) = -(\mathbf{u}_2 \circ \mathbf{u}_3) = -\mathbf{u}_1 \\
\mathbf{u}_3 \times \mathbf{u}_3 &= \frac{1}{2} (\mathbf{u}_3^T \mathbf{u}_3^T - \mathbf{u}_3^T \mathbf{u}_3^T) = \mathbf{0}
\end{aligned}$$

Notice that this is isomorphic to $\mathcal{R}^3$ with cross product (here represented with matrix product through the use of Cayley tables).

Cayley Tables:

	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$
$\hat{\mathbf{i}}$	$\mathbf{0}$	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$
$\hat{\mathbf{j}}$	$-\hat{\mathbf{k}}$	$\mathbf{0}$	$\hat{\mathbf{i}}$
$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	$\mathbf{0}$

	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$
$\mathbf{u}_1$	$\mathbf{0}$	$\mathbf{u}_3$	$-\mathbf{u}_2$
$\mathbf{u}_2$	$-\mathbf{u}_3$	$\mathbf{0}$	$\mathbf{u}_1$
$\mathbf{u}_3$	$\mathbf{u}_2$	$-\mathbf{u}_1$	$\mathbf{0}$

Note:

For all matrices: $(\alpha \mathbf{x})^T = \alpha \mathbf{x}^T$, and: $(\mathbf{x} + \mathbf{y})^T = (\mathbf{x}^T + \mathbf{y}^T)$, so:

$$\begin{aligned}
\mathbf{x} \times (\alpha \mathbf{y} + \beta \mathbf{z}) &= \frac{1}{2} [\mathbf{x}^T (\alpha \mathbf{y} + \beta \mathbf{z})^T - (\alpha \mathbf{y} + \beta \mathbf{z})^T \mathbf{x}^T] \\
&= \frac{1}{2} [\mathbf{x}^T (\alpha \mathbf{y}^T + \beta \mathbf{z}^T) - (\alpha \mathbf{y}^T + \beta \mathbf{z}^T) \mathbf{x}^T] \\
&= \frac{1}{2} [\alpha \mathbf{x}^T \mathbf{y}^T + \beta \mathbf{x}^T \mathbf{z}^T - \alpha \mathbf{y}^T \mathbf{x}^T + \beta \mathbf{z}^T \mathbf{x}^T] \\
&= \frac{1}{2} [\alpha (\mathbf{x}^T \mathbf{y}^T - \mathbf{y}^T \mathbf{x}^T) + \beta (\mathbf{x}^T \mathbf{z}^T - \mathbf{z}^T \mathbf{x}^T)] \\
&= \alpha (\mathbf{x} \times \mathbf{y}) + \beta (\mathbf{x} \times \mathbf{z}) \\
(\alpha \mathbf{x} + \beta \mathbf{y}) \times \mathbf{z} &= \frac{1}{2} [(\alpha \mathbf{x} + \beta \mathbf{y})^T \mathbf{z}^T - \mathbf{z}^T (\alpha \mathbf{x} + \beta \mathbf{y})^T] \\
&= \frac{1}{2} [(\alpha \mathbf{x}^T + \beta \mathbf{y}^T) \mathbf{z}^T - \mathbf{z}^T (\alpha \mathbf{x}^T + \beta \mathbf{y}^T)] \\
&= \frac{1}{2} [\alpha \mathbf{x}^T \mathbf{z}^T + \beta \mathbf{y}^T \mathbf{z}^T - \alpha \mathbf{z}^T \mathbf{x}^T - \beta \mathbf{z}^T \mathbf{y}^T] \\
&= \frac{1}{2} [\alpha (\mathbf{x}^T \mathbf{z}^T - \mathbf{z}^T \mathbf{x}^T) + \beta (\mathbf{y}^T \mathbf{z}^T - \mathbf{z}^T \mathbf{y}^T)] \\
&= \alpha (\mathbf{x} \times \mathbf{z}) + \beta (\mathbf{y} \times \mathbf{z}) .
\end{aligned}$$

Now, let:

$$\mathbf{A} \equiv \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \quad \mathbf{C} \equiv \begin{pmatrix} t & u \\ v & -t \end{pmatrix},$$

then

$$\begin{aligned}(\mathbf{A}^T \mathbf{B}^T) &= \begin{pmatrix} a & c \\ b & -a \end{pmatrix} \begin{pmatrix} p & r \\ q & -p \end{pmatrix} = \begin{pmatrix} (ap + cq) & (ar - cp) \\ (bp - aq) & (br + ap) \end{pmatrix} \\(\mathbf{B}^T \mathbf{A}^T) &= \begin{pmatrix} p & r \\ q & -p \end{pmatrix} \begin{pmatrix} a & c \\ b & -a \end{pmatrix} = \begin{pmatrix} (pa + rb) & (pc - ra) \\ (qa - pb) & (qc + pa) \end{pmatrix}.\end{aligned}$$

∴

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &\equiv \frac{1}{2}(\mathbf{A}^T \mathbf{B}^T - \mathbf{B}^T \mathbf{A}^T) = \begin{pmatrix} \frac{1}{2}(cq - rb) & (ar - cp) \\ (bp - aq) & \frac{1}{2}(br - qc) \end{pmatrix} \\&= \begin{pmatrix} \frac{1}{2}(cq - rb) & (ar - cp) \\ (bp - aq) & -\frac{1}{2}(cq - rb) \end{pmatrix}.\end{aligned}$$

Let: $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$, then:

$$\begin{aligned}u_{11} &= \frac{1}{2}[u(bp - aq) - v(ar - cp)] \\u_{12} &= \frac{1}{2}[v(cq - rb)] - t(bp - aq) \\u_{21} &= \frac{1}{2}[u(cq - rb)] - t(ar - cp) \\u_{22} &= -\frac{1}{2}[u(bp - aq) - v(ar - cp)].\end{aligned}$$

Let: $(\mathbf{B} \times \mathbf{C}) \times \mathbf{A} = \begin{pmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{pmatrix}$, then:

$$\begin{aligned}u'_{11} &= \frac{1}{2}[b(qt - pu) - c(pv - rt)] \\u'_{12} &= \frac{1}{2}[c(ru - vq)] - a(qt - pu) \\u'_{21} &= \frac{1}{2}[b(ru - vq)] - a(pv - rt) \\u'_{22} &= -\frac{1}{2}[b(qt - pu) - c(pv - rt)].\end{aligned}$$

Let: $(\mathbf{C} \times \mathbf{A}) \times \mathbf{B} = \begin{pmatrix} u''_{11} & u''_{12} \\ u''_{21} & u''_{22} \end{pmatrix}$, then:

$$\begin{aligned}u''_{11} &= \frac{1}{2}[q(ua - tb) - r(tc - va)] \\u''_{12} &= \frac{1}{2}[r(vb - cu)] - p(ua - tb) \\u''_{21} &= \frac{1}{2}[q(vb - cu)] - p(tc - va) \\u''_{22} &= -\frac{1}{2}[q(ua - tb) - r(tc - va)].\end{aligned}$$

Therefore:

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} + (\mathbf{B} \times \mathbf{C}) \times \mathbf{A} + (\mathbf{C} \times \mathbf{A}) \times \mathbf{B} &= \begin{pmatrix} u_{11} + u'_{11} + u''_{11} & u_{12} + u'_{12} + u''_{12} \\ u_{21} + u'_{21} + u''_{21} & u_{22} + u'_{22} + u''_{22} \end{pmatrix} \\&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}.\end{aligned}$$

This is typically referred to as the Jacobi Identity, a fundamental requirement

of the cross product, and lie algebras [71],[56], in general.

Note that using these matrices as basis, the dot product may be formed from the following expression.

$$\mathbf{x} \cdot \mathbf{y} = d \ni \frac{1}{2}(\mathbf{x}^T \mathbf{y} + \mathbf{y} \mathbf{x}^T) = d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, d \in \mathcal{R}$$

C. Example 1-3

Quaternions

The quaternions may be expressed using as product matrix multiplication, and the following basis.

$$\begin{aligned} \mathbf{q}_0 &\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \mathbf{q}_1 &\equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ \mathbf{q}_2 &\equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \mathbf{q}_3 &\equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{aligned}$$

In brief, the quaternion product may be expressed via the following Cayley table.

	$\mathbf{q}_0$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{q}_0$	$\mathbf{q}_0$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	$-\mathbf{q}_0$	$\mathbf{q}_3$	$-\mathbf{q}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$-\mathbf{q}_3$	$-\mathbf{q}_0$	$\mathbf{q}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	$-\mathbf{q}_0$

Note the similarity of the lower right 3x3 section of this table to the above cross product table.

Chapter II

Certain Features on a Class of Algebras

In this chapter, a generalization of matrix multiplication, the weighted matrix product, is defined and examined; a conjugate is defined and explored; and products arising from the basic product of the algebra are prescribed corresponding to dot and cross products in $\mathcal{R}^3$ (and therefore, to any space homeomorphic (A5-8) to it). Then, non-associative algebras are presented constructed from the weighted matrix product related to the space of special relativity.

Simple matrix multiplication may be used to construct vector product spaces, but is limited to associative vector product spaces. In order to construct non-associative vector product spaces, the weighted matrix product is introduced, which generalizes matrix multiplication. General concepts of conjugate, dot and cross products are also developed.

A check mark ($\checkmark$) denotes an identity or duplicated equation of a set.

The weighted matrix product that follows is introduced primarily as a mechanism for construction of vector product spaces of wide variety and characteristics, and examples of such constructions will be presented.

A. The Weighted Matrix Product

Definition 2-1: weighted matrix multiplication , also termed a

weighted matrix product , is a binary operation between a $n \times m$ matrix: $A_{nm} = (A_{ij})_{nm}$ and a $m \times p$ matrix: $B_{mp} = (B_{ij})_{mp}$, in combination with at least m weight matrices $\Phi_{ha\beta} = (\Phi_{hij})_{\alpha\beta} \ni h \geq m, \alpha \geq n, \beta \geq p$ (or weights) , into a matrix: $C_{np} = (C_{ij})_{np}$, such that:

$$C_{ij} = \sum_{h=1}^m \Phi_{hij} A_{ih} B_{hj} \quad , \quad \forall i, j \in \mathcal{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq p$$

The matrix product is written:

$$(C_{ij})_{np} = (A_{ij})_{nm} \overset{\Phi_{ha\beta}}{\star} (B_{ij})_{mp} \quad \text{or} \quad C_{np} = A_{nm} \overset{\Phi_{ha\beta}}{\star} B_{mp} .$$

or, where understood-in-context: $(C_{ij}) = (A_{ij}) \star (B_{ij})$ or $C = A \star B$.

Observations:

The number of Weight Matrices is greater than or equal to:

the number of columns of the left operand =

the number of rows of the right operand.

The number of rows of the Weight Matrices is:

greater than or equal to the number of rows of the left operand.

The number of columns of the Weight Matrices is:

greater than or equal to the number of columns of the right operand.

(Conformability is unchanged).

The following examples illustrate weighted matrix multiplication.

Example 2-1:

Let:

$$\Phi_1 \equiv \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Phi_2 \equiv \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ then:}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \star \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Example 2-2:

Let:

$$\Phi_1 \equiv \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad \Phi_2 \equiv \begin{pmatrix} -1 & -1 \end{pmatrix}, \quad \Phi_3 \equiv \begin{pmatrix} 1 & -1 \end{pmatrix}, \text{ then:}$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \star \begin{pmatrix} x & p \\ y & q \\ z & r \end{pmatrix} = \begin{pmatrix} (ax - by + cz) & (ap - bq - cr) \end{pmatrix}.$$

Theorem 2-1: For: $n \times n$ matrices: $A \equiv (A_{ij})$, $B \equiv (B_{ij})$, with $n \times n$ weights:

$\Phi_h \equiv (\Phi_{hij}), (\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n)$, and $\alpha \in \mathcal{R}$:

$$(\alpha A) \star B = \alpha(A \star B)$$

$$A \star (\alpha B) = \alpha(A \star B)$$

Proof:

$$\begin{aligned}
(\alpha A) \star B &= \left(\sum_{h=1}^n \Phi_{hij} (\alpha A_{ih}) B_{hj} \right) = \left(\alpha \sum_{h=1}^n \Phi_{hij} A_{ih} B_{hj} \right) \\
&= \alpha \left(\sum_{h=1}^n \Phi_{hij} A_{ih} B_{hj} \right) = \alpha (A \star B) \\
A \star (\alpha B) &= \left(\sum_{h=1}^n \Phi_{hij} A_{ih} (\alpha B_{hj}) \right) = \left(\alpha \sum_{h=1}^n \Phi_{hij} A_{ih} B_{hj} \right) \\
&= \alpha \left(\sum_{h=1}^n \Phi_{hij} A_{ih} B_{hj} \right) = \alpha (A \star B) \quad \square.
\end{aligned}$$

Theorem 2-2: For: $n \times n$ matrices: $A \equiv (A_{ij})$, $B \equiv (B_{ij})$, $C \equiv (C_{ij})$, with $n : n \times n$ weights: $\Phi_h \equiv (\Phi_{hij})$, $(\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n)$:

$$(A + B) \star C = A \star C + B \star C$$

$$A \star (B + C) = A \star B + A \star C$$

Proof:

$$\begin{aligned}
(A + B) \star C &= \left(\sum_{h=1}^n \Phi_{hij} (A_{ih} + B_{ih}) C_{hj} \right) \\
&= \left(\sum_{h=1}^n \Phi_{hij} A_{ih} C_{hj} \right) + \left(\sum_{h=1}^n \Phi_{hij} B_{ih} C_{hj} \right) \\
&= A \star C + B \star C \\
A \star (B + C) &= \left(\sum_{h=1}^n \Phi_{hij} A_{ih} (B_{hj} + C_{hj}) \right) \\
&= \left(\sum_{h=1}^n \Phi_{hij} A_{ih} B_{hj} \right) + \left(\sum_{h=1}^n \Phi_{hij} A_{ih} C_{hj} \right) \\
&= A \star B + A \star C \quad \square.
\end{aligned}$$

Theorem 2-3: For: $n \times m$ matrix: $A \equiv (A_{ij})$, $p \times q$ matrix: $B \equiv (B_{ij})$,

$r \times s$ matrix: $C \equiv (C_{ij})$, under weights: $\Phi_h \equiv (\Phi_{hij})$,

if: $A \star (B \star C)$, or $(A \star B) \star C$ exist, then:

$$m = p, q = r$$

(A is $n \times m$, B is $m \times r$, C is $r \times s$)

(Φ_h is $\eta \times \xi$, where: $h \geq \max(m, q)$),

and: $A \star (B \star C)$, $(A \star B) \star C$ are $n \times s$ matrices.

If: $A \star (B \star C)$ exists, then:

for: $\eta \times \xi$ weight matrices $\Phi_h : \eta \geq \max(n, p), \xi \geq s$.

If: $(A \star B) \star C$ exists, then:

for: $\eta \times \xi$ weight matrices $\Phi_h : \eta \geq n, \xi \geq \max(q, s)$.

Proof:

If: $A \star (B \star C)$ exists, then: $B \star C$ exists.

So, from the definition of matrix multiplication,

B must be $\pi \times \rho$, and C must be $\rho \times \sigma$,

$\therefore \pi = p, \rho = q, \rho = r, \sigma = s \Rightarrow B$ is $p \times r, C$ is $r \times s$

and: $B \star C$ is $p \times s$, ($q = r$)

and, for: $\eta \times \xi$ weight matrices Φ_h :

$$h \geq q, \eta \geq p, \xi \geq s.$$

And, again, from the definition of matrix multiplication,

A must be $\pi \times \rho$, and $B \star C$ must be $\rho \times \sigma$,

$\therefore \pi = n, \rho = m, \rho = p, \sigma = s \Rightarrow A$ is $n \times m$, ($m = p$)

$B \star C$ is $m \times s$

and, for: $\eta \times \xi$ weight matrices Φ_h :

$$h \geq m, \eta \geq n, \xi \geq s.$$

So: B is $m \times r, C$ is $r \times s$

and: $A \star (B \star C)$ is $n \times s$

and, so, for: $\eta \times \xi$ weight matrices Φ_h :

$$h \geq \max(m, q), \eta \geq \max(n, p), \xi \geq s.$$

$(A \star B) \star C$ is similar:

$A \star B$ exists: $m = p$,

and, for: $\eta \times \xi$ weight matrices Φ_h :

$$h \geq m, \eta \geq n, \xi \geq q.$$

And, again, from the definition of matrix multiplication,

$$q = r,$$

and, for: $\eta \times \xi$ weight matrices Φ_h :

$$h \geq q, \eta \geq n, \xi \geq s.$$

So, for: $\eta \times \xi$ weight matrices Φ_h :

$$h \geq \max(m, q), \eta \geq n, \xi \geq \max(q, s) \quad \square.$$

Note:

Ordinary Matrix Multiplication is the special case of Weighted Matrix Multiplication, where all the weight matrices are of arbitrarily large size, and all the matrix entries are 1s.

Ordinary Matrix Multiplication is Weighted Matrix Multiplication in a default "sea of 1s", the weight matrices formed out of the "sea" as necessary; i.e., a matrix with row count and column count large without bound.

Theorem 2-4: For: $n \times m$ matrix: $A \equiv (A_{ij})$, $m \times p$ matrix: $B \equiv (B_{ij})$,
 $p \times q$ matrix: $C \equiv (C_{ij})$, under weights: $\Phi_h \equiv (\Phi_{ha\beta})$,
 if: $A \star (B \star C)$, or $(A \star B) \star C$ exist, then:

$$(A \star B) \star C = \left(\sum_{\alpha=1}^m \sum_{\beta=1}^p A_{i\alpha} B_{\alpha\beta} C_{\beta j} [\Phi_{\beta ij} \Phi_{\alpha i\beta}] \right)$$

$$A \star (B \star C) = \left(\sum_{\alpha=1}^m \sum_{\beta=1}^p A_{i\alpha} B_{\alpha\beta} C_{\beta j} [\Phi_{\alpha ij} \Phi_{\beta \alpha j}] \right)$$

Proof:

$$\begin{aligned} (A \star B) \star C &= \left(\sum_{\alpha=1}^m \Phi_{\alpha ij} A_{i\alpha} B_{\alpha j} \right) \star C = \left(\sum_{\beta=1}^p \Phi_{\beta ij} \left[\sum_{\alpha=1}^m \Phi_{\alpha i\beta} A_{i\alpha} B_{\alpha\beta} \right] C_{\beta j} \right) \\ &= \left(\sum_{\alpha=1}^m \sum_{\beta=1}^p A_{i\alpha} B_{\alpha\beta} C_{\beta j} [\Phi_{\beta ij} \Phi_{\alpha i\beta}] \right) \\ A \star (B \star C) &= A \star \left(\sum_{\beta=1}^p \Phi_{\beta ij} B_{i\beta} C_{\beta j} \right) = \left(\sum_{\alpha=1}^m \Phi_{\alpha ij} A_{i\alpha} \left[\sum_{\beta=1}^p \Phi_{\beta \alpha j} B_{\alpha\beta} C_{\beta j} \right] \right) \\ &= \left(\sum_{\alpha=1}^m \sum_{\beta=1}^p A_{i\alpha} B_{\alpha\beta} C_{\beta j} [\Phi_{\alpha ij} \Phi_{\beta \alpha j}] \right) \quad \square. \end{aligned}$$

Corollary 2-4: The Weighted Matrix Product is not generally associative.

Proof:

If the weighted matrix product is associative, then:

$$(A \star B) \star C = A \star (B \star C)$$

implies:

$$\left(\sum_{\alpha=1}^m \sum_{\beta=1}^p A_{i\alpha} B_{\alpha\beta} C_{\beta j} [\Phi_{\beta ij} \Phi_{\alpha i\beta}] \right) = \left(\sum_{\alpha=1}^m \sum_{\beta=1}^p A_{i\alpha} B_{\alpha\beta} C_{\beta j} [\Phi_{\alpha ij} \Phi_{\beta \alpha j}] \right)$$

If this is true for arbitrary A, B, C , then:

$$\begin{aligned} \Phi_{\beta ij} \Phi_{\alpha i\beta} &= \Phi_{\alpha ij} \Phi_{\beta \alpha j}, \quad \forall i, j, \alpha, \beta \in \mathbb{Z}^+ \ni \\ &1 \leq i \leq m, 1 \leq j \leq p, \\ &1 \leq \alpha \leq m, 1 \leq \beta \leq p. \end{aligned}$$

But, this is not necessarily true.

So, the Weighted Matrix Product is not generally associative. $\square$

Example: 2-3:

Let:

$$\Phi_1 = \begin{pmatrix} \Phi_{111} & \Phi_{112} \\ \downarrow & \downarrow \\ 1 & 2 \\ \uparrow & \uparrow \\ \Phi_{121} & \Phi_{122} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \Phi_{211} & \Phi_{212} \\ \downarrow & \downarrow \\ 2 & 3 \\ \uparrow & \uparrow \\ \Phi_{212} & \Phi_{222} \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad C = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix},$$

then:

$$\begin{aligned} A \star B &= \left(\sum_{h=1}^2 \Phi_{hij} A_{ih} B_{hj} \right) \\ &= \begin{pmatrix} [\Phi_{111}A_{11}B_{11} + \Phi_{211}A_{12}B_{21}] & [\Phi_{112}A_{11}B_{12} + \Phi_{212}A_{12}B_{22}] \\ [\Phi_{121}A_{21}B_{11} + \Phi_{221}A_{22}B_{21}] & [\Phi_{122}A_{21}B_{12} + \Phi_{222}A_{22}B_{22}] \end{pmatrix}. \end{aligned}$$

Therefore:

$$A \star B = \begin{pmatrix} [(1)(5)(6) + (2)(6)(8)] & [(2)(5)(7) + (3)(6)(9)] \\ [(3)(7)(6) + (4)(8)(8)] & [(4)(7)(7) + (5)(8)(9)] \end{pmatrix} = \begin{pmatrix} 126 & 232 \\ 382 & 556 \end{pmatrix}$$

$$\begin{aligned} B \star C &= \begin{pmatrix} [(1)(6)(7) + (2)(7)(9)] & [(2)(6)(8) + (3)(7)(10)] \\ [(3)(8)(7) + (4)(9)(9)] & [(4)(8)(8) + (5)(9)(10)] \end{pmatrix} \\ &= \begin{pmatrix} 168 & 306 \\ 492 & 706 \end{pmatrix}, \end{aligned}$$

and, so:

$$\begin{aligned} (A \star B) \star C &= \begin{pmatrix} [(1)(126)(7) + (2)(232)(9)] & [(2)(126)(8) + (3)(232)(10)] \\ [(3)(382)(7) + (4)(556)(9)] & [(4)(382)(8) + (5)(556)(10)] \end{pmatrix} \\ &= \begin{pmatrix} 5058 & 8976 \\ 21162 & 40024 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A \star (B \star C) &= \begin{pmatrix} [(1)(5)(168) + (2)(6)(492)] & [(2)(5)(306) + (3)(6)(706)] \\ [(3)(7)(168) + (4)(8)(492)] & [(4)(7)(306) + (5)(8)(706)] \end{pmatrix} \\ &= \begin{pmatrix} 6744 & 15768 \\ 19272 & 36808 \end{pmatrix}. \end{aligned}$$

For: $n \times n$ matrices, and non-zero weights:

if: $n = 1$: $(A \star B) \star C = A \star (B \star C)$ is always true;

if $n = 2$: $(A \star B) \star C = A \star (B \star C)$, then:

$$\Phi_1 = \Phi_{111} \begin{pmatrix} 1 & 1 \\ 1 & \frac{\Phi_{122}}{\Phi_{111}} \end{pmatrix}, \quad \Phi_2 = \Phi_{222} \begin{pmatrix} \frac{\Phi_{122}}{\Phi_{111}} & 1 \\ 1 & 1 \end{pmatrix}$$

(See Appendix III);

if $n = 3$: $(A \star B) \star C = A \star (B \star C)$, then:

$$\Phi_1 = \Phi_{111} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{\Phi_{122}}{\Phi_{111}} & \frac{\Phi_{123}}{\Phi_{111}} \\ 1 & \frac{\Phi_{132}}{\Phi_{111}} & \frac{\Phi_{133}}{\Phi_{111}} \end{pmatrix},$$

$$\Phi_2 = \Phi_{222} \begin{pmatrix} \frac{\Phi_{122}}{\Phi_{111}} & 1 & \frac{\Phi_{122}}{\Phi_{123}} \\ 1 & 1 & 1 \\ \frac{\Phi_{122}}{\Phi_{132}} & 1 & \frac{\Phi_{122}}{\Phi_{123}} \frac{\Phi_{133}}{\Phi_{132}} \end{pmatrix},$$

$$\Phi_3 = \Phi_{333} \begin{pmatrix} \frac{\Phi_{133}}{\Phi_{111}} & \frac{\Phi_{133}}{\Phi_{132}} & 1 \\ \frac{\Phi_{133}}{\Phi_{123}} & \frac{\Phi_{133}}{\Phi_{123}} \frac{\Phi_{122}}{\Phi_{132}} & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

(as the reader may verify).

As the above examples illustrate, the associativity of matrices is completely dependent on the weight matrices.

If all entries of all weight matrices are zero, all product results are zero. The operation is still a closed mapping - all results going to zero, which is a member of every vector space, but not very interesting.

Weighted matrix multiplication may be expressed in terms of ordinary matrix multiplication, a method which eases and speeds the process, using matrices constructed from the constituent parts, from the following definitions.

Generalizations of the various forms of the Kronecker Delta are introduced here, to simplify notation that follows.

Definition 2-2: Kronecker Delta: For sets U and V :

$$\delta_j^i \equiv \begin{cases} = 1, \text{ iff: } i = j \\ = 0, \text{ iff: } i \neq j \end{cases} \quad \begin{pmatrix} i \in U \\ j \in V \end{pmatrix}$$

$$\delta_{ij} \equiv \begin{cases} = 1, \text{ iff: } i = j \\ = 0, \text{ iff: } i \neq j \end{cases} \quad \begin{pmatrix} i \in U \\ j \in V \end{pmatrix}$$

$$\delta^{ij} \equiv \begin{cases} = 1, \text{ iff: } i = j \\ = 0, \text{ iff: } i \neq j \end{cases} \quad \begin{pmatrix} i \in U \\ j \in V \end{pmatrix}$$

**Definition 2-3: For $n \times m$ matrix: $A \equiv (A_{ij})$, $A_\mu \equiv (\delta_{ij} A_{i\mu})$, $\forall \mu \in \mathcal{Z}^+ \ni 1 \leq \mu \leq n$
 A_μ are $m \times m$ matrices**

**Definition 2-4: For $n \times m$ matrix: $A \equiv (A_{ij})$, $A^\nu \equiv (\delta_{ij} A_{\nu j})$, $\forall \nu \in \mathcal{Z}^+ \ni 1 \leq \nu \leq m$
 A^ν are $n \times n$ matrices**

So, for $m \times p$ matrix: $A \equiv (A_{ij})$:

$$A_\mu \equiv \begin{pmatrix} A_{1\mu} & \cdots & \cdots & 0 \\ 0 & A_{2\mu} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_{m\mu} \end{pmatrix}, \quad \begin{matrix} (\forall \mu \in \mathcal{Z}^+ \ni 1 \leq \mu \leq m) \\ \text{are } m \times m \text{ matrices.} \end{matrix}$$

And, for $p \times n$ matrix: $B \equiv (B_{ij})$:

$$B^\nu \equiv \begin{pmatrix} B_{\nu 1} & \cdots & \cdots & 0 \\ 0 & B_{\nu 2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & B_{\nu n} \end{pmatrix}, \quad \begin{matrix} (\forall \nu \in \mathcal{Z}^+ \ni 1 \leq \nu \leq n) \\ \text{are } n \times n \text{ matrices.} \end{matrix}$$

The following theorem illustrates the process of evaluating weighted matrix multiplication, using ordinary matrix multiplication.

Theorem 2-5: For: $n \times m$ matrix: $A \equiv (A_{ij})$, and $m \times p$ matrix: $B \equiv (B_{ij})$,
and m conformable (A1-41) weights Φ_α : $A \star B = \sum_{\alpha=1}^m A_\alpha \Phi_\alpha B^\alpha$

Proof:

$$\begin{aligned}
\sum_{\alpha=1}^m A_\alpha \Phi_\alpha B^\alpha &= \sum_{\alpha=1}^m (\delta_{ij} A_{i\alpha}) (\Phi_{\alpha j}) (\delta_{ij} B_{\alpha j}) \\
&= \sum_{\alpha=1}^m \left(\sum_{h=1}^n \delta_{ih} A_{i\alpha} \Phi_{\alpha h j} \right) (\delta_{ij} B_{\alpha j}) \\
&= \sum_{\alpha=1}^m (A_{i\alpha} \Phi_{\alpha j}) (\delta_{ij} B_{\alpha j}) \\
&= \sum_{\alpha=1}^m \left(\sum_{h=1}^m A_{i\alpha} \Phi_{\alpha h} \delta_{hj} B_{\alpha j} \right) \\
&= \sum_{\alpha=1}^m (A_{i\alpha} \Phi_{\alpha j} B_{\alpha j}) \\
&= \left(\sum_{\alpha=1}^m \Phi_{\alpha j} A_{i\alpha} B_{\alpha j} \right) = A \star B. \quad \square.
\end{aligned}$$

Because this is a newly defined entity, it is altogether appropriate to present and work out examples of weighted matrix products in full detail. Such examples may be found in Appendix II.

Next, a general notion of conjugate is developed, and two products based on the product between vectors of an $\mathcal{R}$ -algebra are defined and examined.

Presuming the reader is familiar with the definition of a vector space (see [6], or [40], for example), beyond the definition, for what follows, a few further notions will be used.

For vector spaces $(\mathcal{F}, \otimes, S)$ and $(\mathcal{F}, \otimes, S_1)$, if $S_1 \subset S$, then $(\mathcal{F}, \otimes, S_1)$ is called a **subspace** of $(\mathcal{F}, \otimes, S)$; and if $(\mathcal{F}, \otimes, S_1)$ is a subspace of $(\mathcal{F}, \otimes, S)$, and $S_1 \neq S$, then $(\mathcal{F}, \otimes, S_1)$ is called a **proper subspace** of $(\mathcal{F}, \otimes, S)$.

For vector spaces $(\mathcal{F}, \otimes, S_1)$, and $(\mathcal{F}, \otimes, S_2)$, given the set:

$$S_1 + S_2 \equiv \{z \mid z = u + v, \forall u \in S_1, \forall v \in S_2\},$$

then $(\mathcal{F}, \otimes, S_1 + S_2)$ is a vector space.

The following definition generalizes upon this.

Definition 2-7: For vector spaces $(\mathcal{F}, \otimes, S_k)$, $\forall k \in \mathcal{Z} \ni 1 \leq k \leq M$,
the sum of S_k is given by:

$$\sum_{k=1}^M S_k := \begin{cases} 0 & , M < 1 \\ S_1 & , M = 1 \\ S_M + \sum_{k=1}^{M-1} S_k & , M > 1 \end{cases}$$

Clearly, $\left(\mathcal{F}, \otimes, \sum_{k=1}^M S_k \right)$ is a vector space.

Further, for vector spaces $(\mathcal{F}, \otimes, S_1), (\mathcal{F}, \otimes, S_2)$; if: $S_1 \cap S_2 = \{0\}$, then $S_1 \oplus S_2 \equiv S_1 + S_2$ is called the **direct sum** of S_1 and S_2 , and $(\mathcal{F}, \otimes, S_1)$ and $(\mathcal{F}, \otimes, S_2)$ are called **complementary subspaces** of $(\mathcal{F}, \otimes, S_1 \oplus S_2)$.

Theorem 2-6: Every subspace of a vector space has a complementary subspace.

Proof:

See [33], Ch. 10, p. 55, Theorem 21 .

Proposition 2-1: For vector spaces $(\mathcal{F}, \otimes, S)$ and $(\mathcal{F}, \otimes, S_1)$

if $(\mathcal{F}, \otimes, S_1)$ is a **proper subspace** of $(\mathcal{F}, \otimes, S)$, then $(\mathcal{F}, \otimes, S_1)$ has an infinite number of complements.

See: [35], Example: p.47 : Example 2.4.10 .

Definition 2-9: For vector spaces $(\mathcal{F}, \otimes, S_k)$, $\forall k \in \mathcal{Z} \ni 1 \leq k \leq M$,
the direct sum of S_k is given by:

$$\bigoplus_{k=1}^M S_k := \begin{cases} 0 & , M < 1 \\ S_1 & , M = 1 \\ S_M \oplus \left(\bigoplus_{k=1}^{M-1} S_k \right) & , M > 1 \end{cases}$$

so: $\bigcap_{k=1}^M S_k = \{0\}$, and: $\left(\mathcal{F}, \otimes, \bigoplus_{k=1}^M S_k \right)$ is a vector space.

Theorem 2-7: For finite dimensional vector spaces $(\mathcal{F}, \otimes, \mathbf{S})_n$ and $(\mathcal{F}, \otimes, \mathbf{S}_k)_n$,

$$\forall k \in \mathbb{Z}^+ \ni 1 \leq k \leq M,$$

the following statements are equivalent:

$$(i) \quad \mathbf{S} = \bigoplus_{k=1}^M \mathbf{S}_k$$

$$(ii) \quad \mathbf{S} = \sum_{k=1}^M \mathbf{S}_k, \text{ and, for any vectors: } \mathbf{v}_k \in \mathbf{S}_k$$

$$\sum_{k=1}^M \mathbf{v}_k = \mathbf{0} \Rightarrow \mathbf{v}_k = \mathbf{0}, \forall k \in \mathbb{Z}^+ \ni 1 \leq k \leq M$$

$$(iii) \quad \forall \mathbf{v} \in \mathbf{S}, \exists \text{ unique } \mathbf{v}_k \in \mathbf{S}_k \ni \mathbf{v} = \sum_{k=1}^M \mathbf{v}_k$$

(iv) **For ordered basis B_k of $\mathbf{S}_k$,**

$$\bigcup_{k=1}^M B_k \text{ is an ordered basis of } \mathbf{S}$$

(v) $\forall k \in \mathbb{Z}^+ \ni 1 \leq k \leq M$, **there exists an**

ordered basis of $\mathbf{S}_k \ni$

$$\bigcup_{k=1}^M B_k \text{ is an ordered basis of } \mathbf{S}$$

Proof:

See: [26], §5.2, p. 265, Theorem 5.15.

Note: Up to this point vectors have been presented exclusively in bold font. The reader is cautioned that this convention is not universal, and at this point, this practice will not be maintained, the domain (A1-7) of elements always being explicitly expressed.

Definition 2-10: An inner product space is a vector space $(\mathcal{F}, \otimes, \mathbf{S})$, within which a product is defined $\forall x, y \in \mathbf{S}$, by $x \cdot y \in \mathcal{R}$, such that:

positive definiteness: (i) $\forall \mathbf{v} \in \mathbf{S} \ni \mathbf{v} \neq \mathbf{0}, \mathbf{v} \cdot \mathbf{v} \geq 0; \mathbf{0} \cdot \mathbf{0} = 0$

symmetry: (ii) $\forall \mathbf{v}, \mathbf{w} \in \mathbf{S}, \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$

bilinearity: (iii) $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{S}, (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$
 $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{S}, \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$

homogeneity: (iv) $\forall \mathbf{v}, \mathbf{w} \in \mathbf{S}, \alpha \in \mathcal{R}, (\alpha \otimes \mathbf{v}) \cdot \mathbf{w} = \alpha \otimes (\mathbf{v} \cdot \mathbf{w})$

Definition 2-11: A metric space is a set, S , such that there exists a function

$d : S \times S \rightarrow \mathcal{R}$ (called a **metric**), where:

positive definiteness: (i) $\forall p, q \in S \ni p \neq q, d(p, q) \geq 0; d(p, p) = 0$

symmetry: (ii) $\forall p, q \in S, d(p, q) = d(q, p)$

triangle inequality (iii) $\forall p, q, r \in S, d(p, q) \leq d(p, r) + d(r, q)$

Definition 2-12: An outer product space is a vector space $(\mathcal{F}, \otimes, S)$, within which a product is defined $\forall x, y \in S$, by $x \times y \in S$, such that:

anti-symmetry: (i) $\forall v, w \in S, v \times w = -w \times v$

bilinearity: (ii) $\forall u, v, w \in S, (u + v) \times w = (u \times w) + (v \times w)$

$\forall u, v, w \in S, u \times (v + w) = (u \times v) + (u \times w)$

homogeneity: (iii) $\forall v, w \in S, \alpha \in \mathcal{P}, (\alpha \otimes v) \times w = \alpha \otimes (v \times w)$

Clearly, then, an outer product space is an F -algebra: $((\mathcal{F}, \otimes, S), \times)$.

B. Features of the Vector Space of an Algebra

Theorem 2-8: Vector space $\mathcal{R}_{n \times n}$, with scalars over field $\mathcal{R}$, and weighted matrix product under $\star : n \times n$ weights forms an $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \star)$.

Proof:

$\mathcal{R}_{n \times n}$ is a vector space.

Weighted matrix multiplication is a closed binary operation (A1-19) under $\mathcal{R}_{n \times n}$.

For: $\alpha \in \mathcal{R}$, and $n \times n$ matrices: $x, y, z \in \mathcal{R}_{n \times n}$

$(x + y) \star z = (x \star z) + (y \star z)$

$u \star (v + w) = (u \star v) + (u \star w)$

$(\alpha v) \star w = \alpha(v \star w)$

□.

Definition 2-13: For $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, S has an identity subspace whenever there exists:

a Left Handed Identity Element (I_L),

a Right Handed Identity Element (I_R), or

a Single Identity Element (I_B):

$span\{I_L\}, span\{I_R\}, span\{I_B\}$, respectively.

Whenever this identity subspace exists, it is denoted: S_I .

Note: I_X is used to represent Identity Element of any type.

Note:

Given the example referred to above in [35], the identity subspace will have an infinite number of complements.

Definition 2-14: For $\mathcal{R}$ -algebra $((\mathcal{R}, \oplus, S), \circ)$, if S has an identity subspace, and

$$A \equiv A_t + A_s \in S$$

where:

$$A_t \in \text{span}\{I_X\},$$

a **conjugate** $(*)$ of A may be defined by:

$$A^* \equiv A_t - A_s$$

where: A_t, A_s are linearly independent, and, A_s is expressed as a linear combination of linearly independent vectors, not including I_X .

So:

$$A_s \equiv \frac{1}{2} \otimes (A - A^*) \in S$$

$$A_t \equiv \frac{1}{2} \otimes (A + A^*) \in S.$$

This definition of conjugate is consistent with, arguably, the two major associative vector spaces where conjugates are defined within them, as can be seen below.

Example 2-4:

For complex field (or $\mathcal{C}$ -algebra):

$$S = \text{span}\{1, i\}; x, y \in \mathbf{P} = \mathcal{R} = \text{Reals}$$

$$\text{then: } I_X = 1 \Rightarrow A_t \in \text{span}\{1\}.$$

Let $\lambda \in \mathcal{R}$, and

$$\text{set: } x + \lambda = 1(x + \lambda) := A_t \in \text{span}\{1\}.$$

Then:

$$x + iy = z := A = A_t + A_s = x + \lambda + A_s$$

and:

$$iy = \lambda + A_s \Rightarrow A_s = iy - \lambda \text{ implying}$$

$1x, iy$ are linearly independent.

So: $A_t = 1(x + \lambda)$, $A_s = iy - 1\lambda$ are still

linearly independent, but not uniquely determined.

A_s being expressed as a linear combination of linearly independent vectors, not including I_X requires: $\lambda = 0$.

Then:

$$A_t = 1x, A_s = iy.$$

So:

$$A^* \equiv A_t - A_s = x - iy \quad \checkmark$$

Expressing A_s as a linear combination of linearly independent vectors, not including I_X specifies the basis for the complement of A_t , and, so, uniquely determines the conjugate.

Example 2-5:

For quaternions (or $\mathcal{Q}$ -algebra):

$$S = \text{span}\{\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}; A^i \in \mathbf{P} = \text{Reals},$$

$$\text{then: } I_X = I_B = \mathbf{q}_0 \Rightarrow A_t \in \text{span}\{\mathbf{q}_0\}.$$

Let $\lambda \in \mathcal{R}$, and

$$\text{set: } \mathbf{q}_0 A^0 + \mathbf{q}_0 \lambda = \mathbf{q}_0 (A^0 + \lambda) := A_t \in \text{span}\{\mathbf{q}_0\}.$$

Then:

$$\begin{aligned} \mathbf{q}_0 A^0 + \mathbf{q}_1 A^1 + \mathbf{q}_2 A^2 + \mathbf{q}_3 A^3 &= A \\ &= A_t + A_s = \mathbf{q}_0 (A^0 + \lambda) + A_s, \end{aligned}$$

and:

$$\mathbf{q}_1 A^1 + \mathbf{q}_2 A^2 + \mathbf{q}_3 A^3 = \mathbf{q}_0 \lambda + A_s.$$

Therefore,

$$\begin{aligned} A_s &= \mathbf{q}_1 A^1 + \mathbf{q}_2 A^2 + \mathbf{q}_3 A^3 - \mathbf{q}_0 \lambda \\ &\mathbf{q}_0 A^0, \text{ and } \mathbf{q}_1 A^1 + \mathbf{q}_2 A^2 + \mathbf{q}_3 A^3 \end{aligned}$$

are linearly independent,

so:

$$\begin{aligned} A_t &= \mathbf{q}_0 (A^0 + \lambda), A_s = \mathbf{q}_1 A^1 + \mathbf{q}_2 A^2 + \mathbf{q}_3 A^3 - \mathbf{q}_0 \lambda \\ &\text{are still linearly independent,} \\ &\text{but not uniquely determined.} \end{aligned}$$

A_s being expressed as a linear combination of linearly independent vectors, not including I_X requires: $\lambda = 0$.

Then:

$$A_t = q_0 A^0, A_s = q_1 A^1 + q_2 A^2 + q_3 A^3 .$$

So:

$$A^* \equiv A_t - A_s = q_0 A^0 - q_1 A^1 - q_2 A^2 - q_3 A^3 . \quad \checkmark$$

Definition 2-15: For $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, if S has an identity subspace given by S_t , the set of all vectors within S expressible as a linear combination of vectors not including I_X , is denoted: S_s .

Definition 2-16: For $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, if S has an identity subspace, then its **identity subspace type** is defined as follows:
LEFT : whenever there is a Left Handed Identity Element (I_L),
RITE : whenever there is a Right Handed Identity Element (I_R),
BOTH : whenever there is a Left Handed Identity Element (I_L),
and Right Handed Identity Element (I_R),
NONE : whenever there is NO Identity Element .

Definition 2-17: For an algebra $((\mathcal{R}, \otimes, S), \circ)$, the $\odot$ —**product** is defined such that:

$$\forall u, v \in S, u \odot v \equiv \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v + (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right],$$

where: $\delta_{I_X^{ov}}^{voI_X} = \delta_{I_X^{ou}}^{uoI_X} = 0$, whenever no identity subspace exists.

Definition 2-18: For an algebra $((\mathcal{R}, \otimes, S), \circ)$, the $\times$ —**product** is defined such that:
is defined such that:

$$\forall u, v \in S, u \times v \equiv \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v - (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right],$$

where: $\delta_{I_X^{ov}}^{voI_X} = \delta_{I_X^{ou}}^{uoI_X} = 0$, whenever no identity subspace exists.

Theorem 2-9: For an algebra $((\mathcal{R}, \otimes, S), \circ)$, with $\odot$ -product and $\times$ -product:

$$\forall u, v \in S, u \odot v = v \odot u$$

$$\forall u, v \in S, u \times v = -v \times u$$

Proof:

$$\begin{aligned}
\forall u, v \in \mathbf{S}, u \odot v &\equiv \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. + (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
\forall u, v \in \mathbf{S}, v \odot u &\equiv \frac{1}{2} \left[(v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right. \\
&\quad \left. + (u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right] \\
&= \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. + (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
&= u \odot v \\
\forall u, v \in \mathbf{S}, u \times v &\equiv \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. - (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
\forall u, v \in \mathbf{S}, v \times u &\equiv \frac{1}{2} \left[(v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right. \\
&\quad \left. - (u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right] \\
&= \frac{1}{2} \left[-(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. + (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
&= -\frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. - (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
&= -u \times v \quad \square.
\end{aligned}$$

Theorem 2-10: For an algebra $((\mathcal{R}, \otimes, \mathbf{S}), \circ)$, with $\odot$ -product and $\times$ -product:

$$\forall u, v \in \mathbf{S}, u \circ v = u \odot v + u \times v - ([u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v$$

Proof:

$$\forall u, v \in \mathbf{S}, u \odot v \equiv \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\ \left. + (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right]$$

$$\forall u, v \in \mathbf{S}, u \times v \equiv \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\ \left. - (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right]$$

so:

$$\begin{aligned}
\forall u, v \in \mathbf{S}, u \odot v + u \times v &= \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. + (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
&\quad + \frac{1}{2} \left[(u + [u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. - (v + [v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
&= \frac{1}{2} \left[(u \circ v) + ([u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. + (v \circ u) + ([v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right. \\
&\quad \left. + (u \circ v) + ([u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \right. \\
&\quad \left. - (v \circ u) - ([v^* - v] \delta_{I_X^{ov}}^{voI_X}) \circ u \right] \\
&= (u \circ v) + ([u^* - u] \delta_{I_X^{ou}}^{uoI_X}) \circ v \quad \square.
\end{aligned}$$

Theorem 2-11: For an $\mathcal{R}$ -algebra $((\mathcal{R}, \oplus, \mathbf{S}), \circ)$, with $\odot$ -product and $\times$ -product:

if the identity subspace type is BOTH, then:

$$\forall u, v \in \mathbf{S}, u \circ v = u^* \odot v + u^* \times v$$

$$\forall u, v \in \mathbf{S}, u \odot v = \frac{1}{2}(u^* \circ v + v^* \circ u)$$

$$\forall u, v \in \mathbf{S}, u \times v = \frac{1}{2}(u^* \circ v - v^* \circ u),$$

if the identity subspace type is NOT BOTH, then:

$$\forall u, v \in \mathbf{S}, u \circ v = u \odot v + u \times v$$

$$\forall u, v \in \mathbf{S}, u \odot v = \frac{1}{2}(u \circ v + v \circ u)$$

$$\forall u, v \in \mathbf{S}, u \times v = \frac{1}{2}(u \circ v - v \circ u)$$

Proof:

$$\forall u, v \in \mathbf{S},$$

if the identity subspace type is BOTH, then:

by the last theorem:

$$\begin{aligned} u^* \circ v &= u^* \odot v + u^* \times v - [(u^*)^* - u^*] \circ v \\ &= u^* \odot v + u^* \times v - u \circ v + u^* \circ v. \end{aligned}$$

So:

$$u^* \circ v = u \odot v + u \times v.$$

By definition:

$$\begin{aligned} u \odot v &= \frac{1}{2}[(u + [u^* - u]) \circ v \\ &\quad + (v + [v^* - v]) \circ u] \\ &= \frac{1}{2}[u^* \circ v + v^* \circ u]. \end{aligned}$$

By definition:

$$\begin{aligned} u \times v &= \frac{1}{2}[(u + [u^* - u]) \circ v \\ &\quad - (v + [v^* - v]) \circ u] \\ &= \frac{1}{2}[u^* \circ v - v^* \circ u]. \end{aligned}$$

If the identity subspace type is NOT BOTH, then:

by the last theorem:

$$\begin{aligned} u \circ v &= u \odot v + u \times v - 0 \\ &= u \odot v + u \times v. \end{aligned}$$

By definition:

$$\begin{aligned} u \odot v &= \frac{1}{2}[(u + 0) \circ v + (v + 0) \circ u] \\ &= \frac{1}{2}[u \circ v + v \circ u] \\ u \times v &= \frac{1}{2}[(u + 0) \circ v - (v + 0) \circ u] \\ &= \frac{1}{2}[u \circ v - v \circ u] \quad \square. \end{aligned}$$

Theorem 2-12: For an $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, with $\times$ -product,

if its identity subspace type is BOTH, then:

$$\forall \mathbf{u}, \mathbf{v} \in S, \mathbf{u} \times \mathbf{v} = u_t \otimes v_s - v_t \otimes u_s + u_s \times v_s.$$

Proof:

$$\begin{aligned} \forall \mathbf{u}, \mathbf{v} \in S, \mathbf{u} \times \mathbf{v} &= \frac{1}{2}(\mathbf{u}^* \circ \mathbf{v} - \mathbf{v}^* \circ \mathbf{u}) \\ &= \frac{1}{2}[(\mathbf{u}_t + \mathbf{u}_s)^* \circ (\mathbf{v}_t + \mathbf{v}_s) \\ &\quad - (\mathbf{v}_t + \mathbf{v}_s)^* \circ (\mathbf{u}_t + \mathbf{u}_s)] \\ &= \frac{1}{2}[(\mathbf{u}_t - \mathbf{u}_s) \circ (\mathbf{v}_t + \mathbf{v}_s) \\ &\quad - (\mathbf{v}_t - \mathbf{v}_s) \circ (\mathbf{u}_t + \mathbf{u}_s)] \\ &= \frac{1}{2}[\mathbf{u}_t \circ \mathbf{v}_t + \mathbf{u}_t \circ \mathbf{v}_s - \mathbf{u}_s \circ \mathbf{v}_t - \mathbf{u}_s \circ \mathbf{v}_s \\ &\quad - \mathbf{v}_t \circ \mathbf{u}_t - \mathbf{v}_t \circ \mathbf{u}_s + \mathbf{v}_s \circ \mathbf{u}_t + \mathbf{v}_s \circ \mathbf{u}_s] \\ &= \frac{1}{2}[(u_t \otimes I_B) \circ (v_t \otimes I_B) + (u_t \otimes I_B) \circ v_s \\ &\quad - u_s \circ (v_t \otimes I_B) - u_s \circ v_s - (v_t \otimes I_B) \circ (u_t \otimes I_B) \\ &\quad - (v_t \otimes I_B) \circ u_s + v_s \circ (u_t \otimes I_B) + v_s \circ u_s] \\ &= \frac{1}{2}[(u_t v_t) \otimes I_B + u_t \otimes v_s - u_s \otimes v_t - u_s \circ v_s \\ &\quad - (v_t u_t) \otimes I_B - v_t \otimes u_s + v_s \otimes u_t + v_s \circ u_s] \\ &= \frac{1}{2}(2u_t \otimes v_s - 2v_t \otimes u_s - u_s \circ v_s + v_s \circ u_s) \\ &= \frac{1}{2}(2u_t \otimes v_s - 2v_t \otimes u_s + u_s^* \circ v_s - v_s^* \circ u_s) \\ &= u_t \otimes v_s - v_t \otimes u_s + u_s \times v_s \end{aligned} \quad \square.$$

Theorem 2-13: For an $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, with $\odot$ -product,

if its identity subspace type is BOTH, then:

$$\forall \mathbf{u}, \mathbf{v} \in S, \mathbf{u} \odot \mathbf{v} = (u_t v_t) \otimes I_B + u_s \odot v_s.$$

Proof:

$$\begin{aligned} \forall \mathbf{u}, \mathbf{v} \in S, \mathbf{u} \odot \mathbf{v} &= \frac{1}{2}(\mathbf{u}^* \circ \mathbf{v} + \mathbf{v}^* \circ \mathbf{u}) \\ &= \frac{1}{2}[(\mathbf{u}_t + \mathbf{u}_s)^* \circ (\mathbf{v}_t + \mathbf{v}_s) \\ &\quad + (\mathbf{v}_t + \mathbf{v}_s)^* \circ (\mathbf{u}_t + \mathbf{u}_s)] \\ &= \frac{1}{2}[(\mathbf{u}_t - \mathbf{u}_s) \circ (\mathbf{v}_t + \mathbf{v}_s) \\ &\quad + (\mathbf{v}_t - \mathbf{v}_s) \circ (\mathbf{u}_t + \mathbf{u}_s)] \\ &= \frac{1}{2}[\mathbf{u}_t \circ \mathbf{v}_t + \mathbf{u}_t \circ \mathbf{v}_s - \mathbf{u}_s \circ \mathbf{v}_t - \mathbf{u}_s \circ \mathbf{v}_s \\ &\quad + \mathbf{v}_t \circ \mathbf{u}_t + \mathbf{v}_t \circ \mathbf{u}_s - \mathbf{v}_s \circ \mathbf{u}_t - \mathbf{v}_s \circ \mathbf{u}_s] \\ &= \frac{1}{2}[(u_t \otimes I_B) \circ (v_t \otimes I_B) + (u_t \otimes I_B) \circ v_s \\ &\quad - u_s \circ (v_t \otimes I_B) - u_s \circ v_s + (v_t \otimes I_B) \circ (u_t \otimes I_B) \\ &\quad + (v_t \otimes I_B) \circ u_s - v_s \circ (u_t \otimes I_B) - v_s \circ u_s] \\ &= \frac{1}{2}[(u_t v_t) \otimes I_B + u_t \otimes v_s - u_s \otimes v_t - u_s \circ v_s \\ &\quad + (v_t u_t) \otimes I_B + v_t \otimes u_s - v_s \otimes u_t - v_s \circ u_s] \\ &= \frac{1}{2}(2(u_t v_t) \otimes I_B - u_s \circ v_s - v_s \circ u_s) \\ &= \frac{1}{2}(2(u_t v_t) \otimes I_B + u_s^* \circ v_s + v_s^* \circ u_s) \\ &= (u_t v_t) \otimes I_B + u_s \odot v_s \end{aligned} \quad \square.$$

Definition 2-19: For an $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, with $\odot$ -product and $\times$ -product:

if $\forall u, v \in S_s$:

$u \times v \in S_s$, and

$u \odot v \in S_t$,

then the $\mathcal{R}$ -algebra has a space-time structure.

Theorem 2-14: For an $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S), \circ)$, with $\odot$ -product and $\times$ -product:

if its identity subspace type is BOTH, and

the $\mathcal{R}$ -algebra has a space-time structure, then:

$\forall u, v \in S, u \odot v \in S_t$, and

$\forall u, v \in S, u \times v \in S_s$.

Proof:

$\forall u, v \in S,$

$u_s \times v_s \in S_s$,

$u_s \odot v_s \in S_t$,

$u \odot v = (u_t v_t) \otimes I_B + u_s \odot v_s \in S_t$

$u \times v = u_t \otimes v_s - v_t \otimes u_s + u_s \times v_s \in S_s$ $\square$.

Example 2-6: The Complex Field ($\mathcal{C}$ -algebra):

$S := \mathcal{R}^2$,

$P := \mathcal{R}, \oplus \equiv \text{addition} : +$, $\otimes \equiv \text{multiplication} : \cdot$,

and, $\forall (s_1^0, s_1^1), (s_2^0, s_2^1) \in S$:

$(s_1^0, s_1^1) \circ (s_2^0, s_2^1) := (s_1^0 s_2^0 - s_1^1 s_2^1, s_1^0 s_2^1 + s_1^1 s_2^0)$

Identity Subspace Type: BOTH

conjugate:

$(s_1^0, s_1^1)^* = (s_1^0, -s_1^1)$

$\times$ -product:

$$\begin{aligned} (s_1^0, s_1^1) \times (s_2^0, s_2^1) &= \frac{1}{2} [(s_1^0, s_1^1)^* \circ (s_2^0, s_2^1) - (s_2^0, s_2^1)^* \circ (s_1^0, s_1^1)] \\ &= \frac{1}{2} [(s_1^0, -s_1^1) \circ (s_2^0, s_2^1) - (s_2^0, -s_2^1) \circ (s_1^0, s_1^1)] \\ &= \frac{1}{2} [(s_1^0 s_2^0 + s_1^1 s_2^1, s_1^0 s_2^1 - s_1^1 s_2^0) \\ &\quad - (s_1^0 s_2^0 + s_1^1 s_2^1, -s_1^0 s_2^1 + s_1^1 s_2^0)] \\ &= (0, s_1^0 s_2^1 - s_1^1 s_2^0) \end{aligned}$$

The interested reader might try

verifying this with Theorem 2-12 .

Traditionally, the complex cross product is

defined as: $\text{Im}(z_1^* z_2) = s_1^0 s_2^1 - s_1^1 s_2^0$.

The magnitude matches the above, but that it must be imaginary: see Appendix IV.

Traditionally, the complex product ($\times_t$) is defined as follows:

for $z_1, z_2 \in C : i(z_1 \times_t z_2) \equiv z_1 \times z_2$, or:

$z_1 \times_t z_2$ is the component of $z_1 \times z_2$

in the identity complement subspace.

$\odot$ -product:

$$\begin{aligned} (s_1^0, s_1^1) \odot (s_2^0, s_2^1) &= \frac{1}{2} [(s_1^0, s_1^1) \circ (s_2^0, s_2^1)^* + (s_2^0, s_2^1) \circ (s_1^0, s_1^1)^*] \\ &= (s_1^0 s_2^0 + s_1^1 s_2^1, 0) \end{aligned}$$

These may be compared most easily with [61].

Example: 2-7: The Cross Product $\mathcal{R}$ -algebra:

$\mathbf{S} := \mathcal{R}^3$,

$\mathbf{P} := \mathcal{R}, \oplus \equiv \text{addition} : +, \otimes \equiv \text{multiplication} : \cdot$,

and, $\forall (s_1^1, s_1^2, s_1^3), (s_2^1, s_2^2, s_2^3) \in \mathbf{S}$:

$$(s_1^1, s_1^2, s_1^3) \circ (s_2^1, s_2^2, s_2^3) := (s_1^2 s_2^3 - s_1^3 s_2^2, s_1^3 s_2^1 - s_1^1 s_2^3, s_1^1 s_2^2 - s_1^2 s_1^1)$$

there is no right identity, for if there were, then:

for arbitrary (s_1^1, s_1^2, s_1^3) :

$$\begin{aligned} (s_1^1, s_1^2, s_1^3) \circ (I_R^1, I_R^2, I_R^3) &:= (s_1^2 I_R^3 - s_1^3 I_R^2, s_1^3 I_R^1 - s_1^1 I_R^3, s_1^1 I_R^2 - s_1^2 I_R^1) \\ &= (s_1^1, s_1^2, s_1^3) . \end{aligned}$$

$$\begin{aligned} -s_1^1 + I_R^3 s_1^2 - I_R^2 s_1^3 &= 0 \\ -I_R^3 s_1^1 - s_1^2 + I_R^1 s_1^3 &= 0 \\ I_R^2 s_1^1 - I_R^1 s_1^2 - s_1^3 &= 0 . \end{aligned}$$

$$s_1^1 = \frac{\begin{vmatrix} 0 & I_R^3 & -I_R^2 \\ 0 & -1 & I_R^1 \\ 0 & -I_R^1 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & I_R^3 & -I_R^2 \\ -I_R^3 & -1 & I_R^1 \\ I_R^2 & -I_R^1 & -1 \end{vmatrix}} = \frac{0}{-1 - (I_R^1)^2 - (I_R^2)^2 - (I_R^3)^2}$$

$$I_R^1, I_R^2, I_R^3 \in \mathcal{R} \Rightarrow -1 - (I_R^1)^2 - (I_R^2)^2 - (I_R^3)^2 < 0.$$

So: $s_1^1 = 0$.

Similarly, $s_1^2 = 0, s_1^3 = 0$.

But (s_1^1, s_1^2, s_1^3) is arbitrary,

so this is a contradiction

Similarly, there is no left identity.

So, there is no identity subspace.

Identity Subspace Type: NONE

conjugate:

There is no identity subspace, so: no conjugate.

$\times$ -product:

$$\begin{aligned} (s_1^1, s_1^2, s_1^3) \times (s_2^1, s_2^2, s_2^3) &= \frac{1}{2} [(s_1^1, s_1^2, s_1^3) \circ (s_2^1, s_2^2, s_2^3) + \\ &\quad - (s_2^1, s_2^2, s_2^3) \circ (s_1^1, s_1^2, s_1^3)] \\ &= (s_1^2 s_2^3 - s_1^3 s_2^2, s_1^3 s_2^1 - s_1^1 s_2^3, s_1^1 s_2^2 - s_1^2 s_2^1) \end{aligned}$$

$\odot$ -product:

$$\begin{aligned} (s_1^1, s_1^2, s_1^3) \odot (s_2^1, s_2^2, s_2^3) &= \frac{1}{2} [(s_1^1, s_1^2, s_1^3) \circ (s_2^1, s_2^2, s_2^3) + \\ &\quad - (s_2^1, s_2^2, s_2^3) \circ (s_1^1, s_1^2, s_1^3)] \\ &= (0, 0, 0) \end{aligned}$$

Example: 2-8: The Quaternions ($\mathcal{Q}$ -algebra):

$$\mathbf{S} := \mathcal{R}^4, \mathbf{P} := \mathcal{R},$$

$$\oplus := \text{addition} : +, \otimes := \text{multiplication} : \cdot,$$

$$\text{and, } \forall (s_1^0, s_1^1, s_1^2, s_1^3), (s_2^0, s_2^1, s_2^2, s_2^3) \in \mathbf{S} :$$

$$\begin{aligned} (s_1^0, s_1^1, s_1^2, s_1^3) \circ (s_2^0, s_2^1, s_2^2, s_2^3) &:= (s_1^0 s_2^0 - s_1^1 s_2^1 - s_1^2 s_2^2 - s_1^3 s_2^3, \\ &\quad s_1^0 s_2^1 + s_1^1 s_2^0 + s_1^2 s_2^3 - s_1^3 s_2^2, \\ &\quad s_1^0 s_2^2 - s_1^1 s_2^3 + s_1^2 s_2^0 + s_1^3 s_2^1, \\ &\quad s_1^0 s_2^3 + s_1^1 s_2^2 - s_1^2 s_2^1 + s_1^3 s_2^0) \end{aligned}$$

Identity Subspace Type: BOTH

conjugate:

$$(s_1^0, s_1^1, s_1^2, s_1^3)^* = (s_1^0, -s_1^1, -s_1^2, -s_1^3)$$

$\times$ -product:

$$\begin{aligned} (s_1^0, s_1^1, s_1^2, s_1^3) \times (s_2^0, s_2^1, s_2^2, s_2^3) &= \\ &= \frac{1}{2} [(s_1^0, s_1^1, s_1^2, s_1^3)^* \circ (s_2^0, s_2^1, s_2^2, s_2^3) \\ &\quad - (s_2^0, s_2^1, s_2^2, s_2^3)^* \circ (s_1^0, s_1^1, s_1^2, s_1^3)] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} [(s_1^0, -s_1^1, -s_1^2, -s_1^3) \circ (s_2^0, s_2^1, s_2^2, s_2^3) \\
&\quad - (s_2^0, -s_2^1, -s_2^2, -s_2^3) \circ (s_1^0, s_1^1, s_1^2, s_1^3)] \\
&= \frac{1}{2} [(s_1^0 s_2^0 + s_1^1 s_2^1 + s_1^2 s_2^2 + s_1^3 s_2^3, \\
&\quad s_1^0 s_2^1 - s_1^1 s_2^0 - s_1^2 s_2^3 + s_1^3 s_2^2, \\
&\quad s_1^0 s_2^2 + s_1^1 s_2^3 - s_1^2 s_2^0 - s_1^3 s_2^1, \\
&\quad s_1^0 s_2^3 - s_1^1 s_2^2 + s_1^2 s_2^1 - s_1^3 s_2^0) \\
&\quad - (s_1^0 s_2^0 + s_1^1 s_2^1 + s_1^2 s_2^2 + s_1^3 s_2^3, \\
&\quad -s_1^0 s_2^1 + s_1^1 s_2^0 - s_1^2 s_2^3 + s_1^3 s_2^2, \\
&\quad -s_1^0 s_2^2 + s_1^1 s_2^3 + s_1^2 s_2^0 - s_1^3 s_2^1, \\
&\quad -s_1^0 s_2^3 - s_1^1 s_2^2 + s_1^2 s_2^1 + s_1^3 s_2^0)] \\
&= (0, s_1^0 s_2^1 - s_1^1 s_2^0, s_1^0 s_2^2 - s_1^2 s_2^0, s_1^0 s_2^3 - s_1^3 s_2^0)
\end{aligned}$$

$\odot$ -product:

$$\begin{aligned}
&(s_1^0, s_1^1, s_1^2, s_1^3) \odot (s_2^0, s_2^1, s_2^2, s_2^3) = \\
&= \frac{1}{2} [(s_1^0, s_1^1, s_1^2, s_1^3)^* \circ (s_2^0, s_2^1, s_2^2, s_2^3) \\
&\quad + (s_2^0, s_2^1, s_2^2, s_2^3)^* \circ (s_1^0, s_1^1, s_1^2, s_1^3)] \\
&= (s_1^0 s_2^0 + s_1^1 s_2^1 + s_1^2 s_2^2 + s_1^3 s_2^3, 0, 0, 0)
\end{aligned}$$

Definition 2-20: A lie algebra or lie space is a $\mathcal{F}$ -algebra $((\mathcal{F}, \otimes, \mathbf{S}), [\cdot, \cdot])$, such that:

Axiom L-1: $\forall \alpha, \beta \in \mathcal{F}, \forall X, Y, Z \in \mathbf{V}$, (Bilinearity)

$$[X, (\alpha \otimes Y + \beta \otimes Z)] = \alpha \otimes [X, Y] + \beta \otimes [X, Z]$$

$$[(\alpha \otimes X + \beta \otimes Y), Z] = \alpha \otimes [X, Z] + \beta \otimes [Y, Z]$$

Axiom L-2: $\forall X, Y \in \mathbf{S}, [Y, X] = -[X, Y]$ (Skew Symmetry)

Axiom L-3: $\forall X, Y, Z \in \mathbf{S}$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \quad (\text{Jacobi Identity})$$

Definition 2-21: In an $\mathcal{F}$ -algebra $((\mathcal{F}, \otimes, \mathbf{S}), \circ)$ with identity subspace, and $\forall u, v \in \mathbf{S}$,

$u \odot v \in \mathbf{S}$, the dot product is given by:

if: $\exists D \in \mathcal{F} \ni \forall u, v \in \mathbf{S}, D \otimes I_X = u \odot v$

then: $u \cdot v \equiv D$

C. The Vectric Space

Definition 2-22: A vectric space is an $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, \mathbf{S}), \circ)$ with dot product, and metric/pseudometric, d , defined as follows:

if:

$$\exists x \in \mathcal{R} \forall u, v \in \mathbf{S}, x^2 - [(u - v) \cdot (u - v)] = 0$$

then: $d(u, v) \equiv x$; and denoted by: $(((\mathcal{R}, \otimes, \mathbf{S}), \circ), d)$.

Definition 2-23: Shorthand notation:

$$\delta s^i \equiv (s_1^i - s_2^i),$$

$$\delta \|s\|^2 \equiv d((s_1^i)_n, (s_2^i)_n) = d((s_1^0, s_1^1, \dots, s_1^n), (s_2^0, s_2^1, \dots, s_2^n))$$

Example 2-9: The Complex Vectric Space

For the $\mathcal{C}$ -algebra,
the Complex Vectric Space follows, with:

dot product:

$$(s_1^0, s_1^1) \cdot (s_2^0, s_2^1) = s_1^0 s_2^0 + s_1^1 s_2^1$$

metric:

$$d((s_1^0, s_1^1), (s_2^0, s_2^1)) \equiv (s_1^0 - s_2^0)^2 + (s_1^1 - s_2^1)^2$$

In shorthand notation:

$$\delta \|s\|^2 \equiv \delta s^0 \delta s^0 + \delta s^1 \delta s^1$$

Example 2-10: For the Cross Product $\mathcal{R}$ -algebra:

There is no identity subspace, so, the dot
product is undefined; and, so, no
vectric space is defined.

Example 2-11: The $\mathcal{Q}$ -algebra

For the $\mathcal{Q}$ -algebra,
the Quaternion Vectric Space follows, with:

dot product:

$$(s_1^0, s_1^1, s_1^2, s_1^3) \cdot (s_2^0, s_2^1, s_2^2, s_2^3) = s_1^0 s_2^0 + s_1^1 s_2^1 + s_1^2 s_2^2 + s_1^3 s_2^3$$

metric:

$$d((s_1^0, s_1^1, s_1^2, s_1^3), (s_2^0, s_2^1, s_2^2, s_2^3)) \equiv (s_1^0 - s_2^0)^2 + (s_1^1 - s_2^1)^2 + (s_1^2 - s_2^2)^2 + (s_1^3 - s_2^3)^2$$

In shorthand notation:

$$\delta \|s\|^2 \equiv \delta s^0 \delta s^0 + \delta s^1 \delta s^1 + \delta s^2 \delta s^2 + \delta s^3 \delta s^3$$

Because the complex product is associative, it is a simple matter to verify the dot and cross products are as generally defined in a C -algebra:

$$u \cdot 1 = 1 \cdot u \Rightarrow \delta_{1,u}^{u,1} = 1$$

so:

$$\begin{aligned} u \odot v &= \frac{1}{2} [u \circ (v + [v^* - v]) + v \circ (u + [u^* - u])] \\ &= \frac{1}{2} [u \circ v + u \circ v^* - u \circ v + v \circ u + v \circ u^* - v \circ u] \\ &= \frac{1}{2} [u \circ v^* + v \circ u^*] \end{aligned}$$

and:

$$\begin{aligned} u \times v &= \frac{1}{2} [u \circ (v + [v^* - v]) - v \circ (u + [u^* - u])] \\ &= \frac{1}{2} [u \circ v + u \circ v^* - u \circ v - v \circ u - v \circ u^* + v \circ u] \\ &= \frac{1}{2} [u \circ v^* - v \circ u^*]. \end{aligned}$$

These same formulas hold in a Q -algebra.

Note:

Under the Einstein Summation Convention, in any given term, where any index is repeated, that term is summed over the range (A1-7) of that repeated index.

Generally, for an associative $\mathcal{R}$ -algebra $(\mathcal{R}, \otimes, (S, \circ))$

where, using the Einstein Summation Convention:

$$\forall A, B \in S, \exists u_k \in S \ni A = u_k A^k, B = u_k B^k$$

(written in terms of vector components for a basis)

and:

$$\exists b_{ij}^m \in P \ni u_i \circ u_j = b_{ij}^m u_m;$$

then:

$$A \circ B = u_m b_{ij}^m A^i B^j.$$

Now:

$$\begin{aligned} A \circ (B \circ C) &= A \circ (u_m b_{ij}^m B^i C^j) \\ &= u_n b_{hm}^n b_{ij}^m A^h B^i C^j \\ &= (A \circ B) \circ C \\ &= (u_m b_{ij}^m A^i B^j) \circ C \\ &= u_n b_{ij}^m b_{mh}^n A^i B^j C^h \end{aligned}$$

$$\begin{aligned}
&= \mathbf{u}_n b_{ij}^m b_{mi}^n A^h B^j C^i \\
&= \mathbf{u}_n b_{hi}^m b_{mj}^n A^h B^i C^j .
\end{aligned}$$

$\therefore$

$$\mathbf{u}_n b_{hm}^n b_{ij}^m A^h B^i C^j = \mathbf{u}_n b_{hi}^m b_{mj}^n A^h B^i C^j ,$$

so:

$$b_{hm}^n b_{ij}^m = b_{hi}^m b_{mj}^n .$$

For the non-associative $\mathcal{R}$ -algebras previously introduced,

they have a Right Hand Identity, but not a left handed one, so:

$$u \circ I_R \neq I_R \circ u \Rightarrow \delta_{I_R \circ u}^{u \circ I_R} = 0 .$$

$\therefore$

$$u \odot v = \frac{1}{2}[u \circ v + v \circ u] ,$$

and:

$$u \times v = \frac{1}{2}[u \circ v - v \circ u] .$$

Now, a **permutation matrix** is a square matrix with all zero entries, except for a 1 in a single column of each row, and a single row of each column. (See: [33], §1.6 , p.75).

Examples of permutation matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ,$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Permutation matrices are called such, because each permutation matrix may be obtained through a set of permutations of two rows and/or two columns of an $n \times n$ identity matrix.

So, $\forall n \in \mathbb{Z}^+$, there are $n! : n \times n$ permutation matrices.

$\forall n \in \mathbb{Z}^+$, any n distinct permutation matrices span $\mathcal{R}^n$.

Any $n + 1$ permutation matrices are linearly dependent.

So, $\forall n \in \mathbb{Z}^+$, any n distinct permutation matrices form a basis for $\mathcal{R}^n$.

Example 2-12:

The $\mathcal{C}$ -algebra may be constructed using permutation matrices and suitable weight matrices.

Let:

$$\mathbf{u}^{2;0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}^{2;1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and weights: $(\Phi_h \equiv \varphi^{2;h})$

$$\varphi^{2;0} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \varphi^{2;1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

then:

$$\mathbf{u}^{2;0}|_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u}^{2;0}|^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{u}^{2;0}|_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}^{2;0}|^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(Note: here, indices begin with 0 , whereas all previous have begun with 1.

For consistency with standard notation this notation is used in these examples.)

Consequently:

$$\begin{aligned} \mathbf{u}^{2;0} \circ \mathbf{u}^{2;0} &= \sum_{h=0}^1 \mathbf{u}^{2;0}|_h \varphi^{2;h} \mathbf{u}^{2;0}|^h \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{u}^{2;0} \end{aligned}$$

$$\begin{aligned} \mathbf{u}^{2;0} \circ \mathbf{u}^{2;1} &= \sum_{h=0}^1 \mathbf{u}^{2;0}|_h \varphi^{2;h} \mathbf{u}^{2;1}|^h \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \mathbf{u}^{2;1} \end{aligned}$$

$$\mathbf{u}^{2;1} \circ \mathbf{u}^{2;0} = \sum_{h=0}^1 \mathbf{u}^{2;1}|_h \varphi^{2;h} \mathbf{u}^{2;0}|^h$$

$$\begin{aligned}
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \mathbf{u}^{2;1} \\
\mathbf{u}^{2;1} \circ \mathbf{u}^{2;1} &= \sum_{h=0}^1 \mathbf{u}^{2;1} |_h \varphi^{2;h} \mathbf{u}^{2;1} |^h \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&= -\mathbf{u}^{2;0}
\end{aligned}$$

Thus, is shown another way of constructing a $\mathcal{C}$ -algebra.

(However, with this construction, product associativity must be shown, separately.)

Example 2-13:

The $\mathcal{Q}$ -algebra may be constructed using permutation matrices and suitable weight matrices.

Let:

$$\begin{aligned}
\mathbf{u}^{4;0} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{u}^{4;1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\mathbf{u}^{4;2} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{u}^{4;3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The weight matrices are fully determined by:

$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;0}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;0} = \mathbf{u}^{4;1}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$
$\mathbf{u}^{4;0} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$

$u^{4;2} \circ u^{4;0} = u^{4;2}$	$u^{4;3} \circ u^{4;0} = u^{4;3}$
$u^{4;2} \circ u^{4;1} = -u^{4;3}$	$u^{4;3} \circ u^{4;1} = u^{4;2}$
$u^{4;2} \circ u^{4;2} = -u^{4;0}$	$u^{4;3} \circ u^{4;2} = -u^{4;1}$
$u^{4;2} \circ u^{4;3} = u^{4;1}$	$u^{4;3} \circ u^{4;3} = -u^{4;0}$

equivalent to the Cayley table:

	$u^{4;0}$	$u^{4;1}$	$u^{4;2}$	$u^{4;3}$
$u^{4;0}$	$u^{4;0}$	$u^{4;1}$	$u^{4;2}$	$u^{4;3}$
$u^{4;1}$	$u^{4;1}$	$-u^{4;0}$	$u^{4;3}$	$-u^{4;2}$
$u^{4;2}$	$u^{4;2}$	$-u^{4;3}$	$-u^{4;0}$	$u^{4;1}$
$u^{4;3}$	$u^{4;3}$	$u^{4;2}$	$-u^{4;1}$	$-u^{4;0}$

This determines the weight matrix entries, as follows.

$u^{4;0} \circ u^{4;0} = u^{4;0} :$ $1 = \Phi_{000} = \Phi_{111} = \Phi_{222} = \Phi_{333}$ $= \Phi_{00}^{4;0} = \Phi_{11}^{4;1} = \Phi_{22}^{4;2} = \Phi_{33}^{4;3}$	$u^{4;1} \circ u^{4;0} = u^{4;1} :$ $1 = \Phi_{101} = \Phi_{010} = \Phi_{323} = \Phi_{232}$ $= \Phi_{01}^{4;1} = \Phi_{10}^{4;0} = \Phi_{23}^{4;3} = \Phi_{32}^{4;2}$
$u^{4;0} \circ u^{4;1} = u^{4;1} :$ $1 = \Phi_{001} = \Phi_{110} = \Phi_{223} = \Phi_{332}$ $= \Phi_{01}^{4;0} = \Phi_{10}^{4;1} = \Phi_{23}^{4;2} = \Phi_{32}^{4;3}$	$u^{4;1} \circ u^{4;1} = -u^{4;0} :$ $-1 = \Phi_{100} = \Phi_{011} = \Phi_{322} = \Phi_{233}$ $= \Phi_{00}^{4;1} = \Phi_{11}^{4;0} = \Phi_{22}^{4;3} = \Phi_{33}^{4;2}$
$u^{4;0} \circ u^{4;2} = u^{4;2} :$ $1 = \Phi_{002} = \Phi_{113} = \Phi_{220} = \Phi_{331}$ $= \Phi_{02}^{4;0} = \Phi_{13}^{4;1} = \Phi_{20}^{4;2} = \Phi_{31}^{4;3}$	$u^{4;1} \circ u^{4;2} = u^{4;3} :$ $1 = \Phi_{103} = \Phi_{012} = \Phi_{321} = \Phi_{230}$ $= \Phi_{03}^{4;1} = \Phi_{12}^{4;0} = \Phi_{21}^{4;3} = \Phi_{30}^{4;2}$
$u^{4;0} \circ u^{4;3} = u^{4;3} :$ $1 = \Phi_{003} = \Phi_{112} = \Phi_{221} = \Phi_{330}$ $= \Phi_{03}^{4;0} = \Phi_{12}^{4;1} = \Phi_{21}^{4;2} = \Phi_{30}^{4;3}$	$u^{4;1} \circ u^{4;3} = -u^{4;2} :$ $-1 = \Phi_{102} = \Phi_{013} = \Phi_{320} = \Phi_{231}$ $= \Phi_{02}^{4;1} = \Phi_{13}^{4;0} = \Phi_{20}^{4;3} = \Phi_{31}^{4;2}$
$u^{4;2} \circ u^{4;0} = u^{4;2} :$ $1 = \Phi_{202} = \Phi_{313} = \Phi_{020} = \Phi_{131}$ $= \Phi_{02}^{4;2} = \Phi_{13}^{4;3} = \Phi_{20}^{4;0} = \Phi_{31}^{4;1}$	$u^{4;3} \circ u^{4;0} = u^{4;3} :$ $1 = \Phi_{303} = \Phi_{212} = \Phi_{121} = \Phi_{030}$ $= \Phi_{03}^{4;3} = \Phi_{12}^{4;2} = \Phi_{21}^{4;1} = \Phi_{30}^{4;0}$

$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$ $-1 = \Phi_{203} = \Phi_{312} = \Phi_{021} = \Phi_{130}$ $= \varphi_{03}^{4;2} = \varphi_{12}^{4;3} = \varphi_{21}^{4;0} = \varphi_{30}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;1} = \mathbf{u}^{4;2} :$ $1 = \Phi_{302} = \Phi_{213} = \Phi_{120} = \Phi_{031}$ $= \varphi_{02}^{4;3} = \varphi_{13}^{4;2} = \varphi_{20}^{4;1} = \varphi_{31}^{4;0}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;0} :$ $-1 = \Phi_{200} = \Phi_{311} = \Phi_{022} = \Phi_{133}$ $= \varphi_{00}^{4;2} = \varphi_{11}^{4;3} = \varphi_{22}^{4;0} = \varphi_{33}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;2} = -\mathbf{u}^{4;1} :$ $-1 = \Phi_{301} = \Phi_{210} = \Phi_{123} = \Phi_{032}$ $= \varphi_{01}^{4;3} = \varphi_{10}^{4;2} = \varphi_{23}^{4;1} = \varphi_{32}^{4;0}$
$\mathbf{u}^{4;2} \circ \mathbf{u}^{4;3} = \mathbf{u}^{4;0} :$ $1 = \Phi_{201} = \Phi_{310} = \Phi_{023} = \Phi_{132}$ $= \varphi_{01}^{4;2} = \varphi_{10}^{4;3} = \varphi_{23}^{4;0} = \varphi_{32}^{4;1}$	$\mathbf{u}^{4;3} \circ \mathbf{u}^{4;3} = -\mathbf{u}^{4;0} :$ $-1 = \Phi_{300} = \Phi_{211} = \Phi_{122} = \Phi_{033}$ $= \varphi_{00}^{4;3} = \varphi_{11}^{4;2} = \varphi_{22}^{4;1} = \varphi_{33}^{4;0}$

Therefore:

$$\varphi^{4;0} = \Phi_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \quad \varphi^{4;1} = \Phi_1 = \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix},$$

$$\varphi^{4;2} = \Phi_2 = \begin{pmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \quad \varphi^{4;3} = \Phi_3 = \begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Thus, is shown another way of constructing a $\mathcal{Q}$ -algebra.

(Again, with this construction, the associativity of the product needs to be shown, separately.)

A 2^M -dimensional basis may be constructed from permutation matrices.

(See Appendix VI for an illustration.)

These $\mathcal{R}$ -algebras with spacetime Minkowski pseudo-metric, and cross product reducing to the ordinary cross product under zeroed out time (x^0) components. This is referred to as a Lorentz-Minkowski Vectric Space.

Definition 2-24: A Lorentz-Minkowski Vectric Space is a vectric space,

$(((\mathcal{R}, \otimes, S), \odot), d)$, with space-time structure, such that:

$$\forall \mathbf{u}, \mathbf{v} \in S, \quad \mathbf{u} \times \mathbf{v} \in S_s,$$

$$\mathbf{u} \odot \mathbf{v} \in S_t, \text{ and}$$

$$\mathbf{u}_t \odot \mathbf{v}_s + \mathbf{u}_s \odot \mathbf{v}_t = \mathbf{0}$$

Note:

The reader may readily verify that, whenever the identity subspace type is BOTH, then: $\mathbf{u}_t \odot \mathbf{v}_s + \mathbf{u}_s \odot \mathbf{v}_t = \mathbf{0}$, identically.

Under this definition, the inner and outer product are separated into disjoint parts, and the inner product is the sum of a bilinear form of identity (time) components alone and a bilinear form of non-identity (space) components alone (and the why of using subscripts s and t for the identity subspace and the complement, makes itself clear).

Example 2-14:

The Lorentz-Minkowski Vectric Space of 2 dimensions constructed using the above mechanism follows:

For $M = 1$, (2-D): (see Appendix VI)

$$\Phi^{2;0} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \Phi^{2;1} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Yielding:

$\mathbf{u}^{2;0} \star \mathbf{u}^{2;0} = -\mathbf{u}^{2;0}$	$\mathbf{u}^{2;1} \star \mathbf{u}^{2;0} = -\mathbf{u}^{2;1}$
$\mathbf{u}^{2;0} \star \mathbf{u}^{2;1} = \mathbf{u}^{2;1}$	$\mathbf{u}^{2;1} \star \mathbf{u}^{2;1} = -\mathbf{u}^{2;0}$

So, with:

$$\mathbf{A} \equiv \mathbf{u}^{2;0} A^0 + \mathbf{u}^{2;1} A^1, \quad \mathbf{B} \equiv \mathbf{u}^{2;0} B^0 + \mathbf{u}^{2;1} B^1, \text{ then:}$$

$$\mathbf{A} \star \mathbf{B} = \mathbf{u}^{2;0} (-A^0 B^0 - A^1 B^1) + \mathbf{u}^{2;1} (A^0 B^1 - A^1 B^0)$$

and, so:

$$\mathbf{A} \odot \mathbf{B} \equiv \frac{1}{2} (\mathbf{A} \star \mathbf{B} + \mathbf{B} \star \mathbf{A}) = \mathbf{u}^{2;0} (-A^0 B^0 - A^1 B^1), \odot\text{-product}$$

$$\mathbf{A} \times \mathbf{B} \equiv \frac{1}{2} (\mathbf{A} \star \mathbf{B} - \mathbf{B} \star \mathbf{A}) = \mathbf{u}^{2;1} (A^0 B^1 - A^1 B^0), \times\text{-product}$$

$$\mathbf{A} \star (-\mathbf{u}^{2;0}) = \mathbf{A}, \quad \forall \mathbf{A}.$$

So:

$$\exists I_R (= -\mathbf{u}^{2;0}) \ni \mathbf{A} \circ I_R = \mathbf{A}, \quad \forall \mathbf{A} : \text{Right Hand Identity}.$$

And, therefore:

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &\equiv D \in \mathcal{R} \ni \mathbf{A} \odot \mathbf{B} = I_R \cdot D && : \text{dot product} \\ &= A^0 B^0 + A^1 B^1 \end{aligned}$$

Example 2-15:

The Lorentz-Minkowski Vectric Space of 4 dimensions constructed using the above mechanism follows:

For $M = 2$, (4-D): (see Appendix VI)

$$\begin{aligned} \Phi^{4;0} &= \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}, & \Phi^{4;1} &= \begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}, \\ \Phi^{4;2} &= \begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, & \Phi^{4;3} &= \begin{pmatrix} -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \end{aligned}$$

Yielding:

$\mathbf{u}^{4;0} \star \mathbf{u}^{4;0} = -\mathbf{u}^{4;0}$	$\mathbf{u}^{4;1} \star \mathbf{u}^{4;0} = -\mathbf{u}^{4;1}$
$\mathbf{u}^{4;0} \star \mathbf{u}^{4;1} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;1} \star \mathbf{u}^{4;1} = -\mathbf{u}^{4;0}$
$\mathbf{u}^{4;0} \star \mathbf{u}^{4;2} = \mathbf{u}^{4;2}$	$\mathbf{u}^{4;1} \star \mathbf{u}^{4;2} = \mathbf{u}^{4;3}$
$\mathbf{u}^{4;0} \star \mathbf{u}^{4;3} = \mathbf{u}^{4;3}$	$\mathbf{u}^{4;1} \star \mathbf{u}^{4;3} = -\mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \star \mathbf{u}^{4;0} = -\mathbf{u}^{4;2}$	$\mathbf{u}^{4;3} \star \mathbf{u}^{4;0} = -\mathbf{u}^{4;3}$
$\mathbf{u}^{4;2} \star \mathbf{u}^{4;1} = -\mathbf{u}^{4;3}$	$\mathbf{u}^{4;3} \star \mathbf{u}^{4;1} = \mathbf{u}^{4;2}$
$\mathbf{u}^{4;2} \star \mathbf{u}^{4;2} = -\mathbf{u}^{4;0}$	$\mathbf{u}^{4;3} \star \mathbf{u}^{4;2} = -\mathbf{u}^{4;1}$
$\mathbf{u}^{4;2} \star \mathbf{u}^{4;3} = \mathbf{u}^{4;1}$	$\mathbf{u}^{4;3} \star \mathbf{u}^{4;3} = -\mathbf{u}^{4;0}$

So, with:

$$\mathbf{A} \equiv \mathbf{u}^{4;0} A^0 + \mathbf{u}^{4;1} A^1 + \mathbf{u}^{4;2} A^2 + \mathbf{u}^{4;3} A^3$$

$$\mathbf{B} \equiv \mathbf{u}^{4;0} B^0 + \mathbf{u}^{4;1} B^1 + \mathbf{u}^{4;2} B^2 + \mathbf{u}^{4;3} B^3$$

then:

$$\begin{aligned} \mathbf{A} \star \mathbf{B} &= \mathbf{u}^{4;1} (A^0 B^1 - A^1 B^0 + A^2 B^3 - A^3 B^2) \\ &\quad + \mathbf{u}^{4;2} (A^0 B^2 - A^1 B^3 - A^2 B^0 + A^3 B^1) \\ &\quad + \mathbf{u}^{4;3} (A^0 B^3 + A^1 B^2 - A^2 B^1 - A^3 B^0) \\ &\quad + \mathbf{u}^{4;0} (-A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3), \end{aligned}$$

and, so:

$$\mathbf{A} \odot \mathbf{B} \equiv \frac{1}{2}(\mathbf{A} \star \mathbf{B} + \mathbf{B} \star \mathbf{A}) = \mathbf{u}^{4;0}(-A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3) , \odot -product$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} \equiv \frac{1}{2}(\mathbf{A} \star \mathbf{B} - \mathbf{B} \star \mathbf{A}) &= \mathbf{u}^{4;1}(A^0 B^1 - A^1 B^0 + A^2 B^3 - A^3 B^2) \\ &+ \mathbf{u}^{4;2}(A^0 B^2 - A^1 B^3 - A^2 B^0 + A^3 B^1) \\ &+ \mathbf{u}^{4;3}(A^0 B^3 + A^1 B^2 - A^2 B^1 - A^3 B^0) , \times -product \end{aligned}$$

$$\mathbf{A} \star (-\mathbf{u}^{4;0}) = \mathbf{A} , \quad \forall \mathbf{A} ,$$

so:

$$\exists I_R (= -\mathbf{u}^{4;0}) \ni \mathbf{A} \star I_R = \mathbf{A} , \quad \forall \mathbf{A} : \text{Right Hand Identity.}$$

Therefore,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &\equiv D \in \mathcal{R} \ni \mathbf{A} \odot \mathbf{B} = I_R \cdot D : \text{dot product} \\ &= A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 . \end{aligned}$$

Using the terminology of [10], this four dimensional manifold (A5-15) [49],[71] with Lorentzian metric (spacetime) [10],[18] with metric form as above is called a Minkowski spacetime.

Terminology varies widely among authors, so, here, the signs of the coefficients are neglected, and this pseudo-metric [2],[49] is herein given the name: Lorentz-Minkowski Vectric Space.

The Lorentz-Minkowski Vectric Space of 8 dimensions constructed using the above mechanism follows:

For $M = 3$, (8-D): (see Appendix VI)

$$\Phi^{8;0} \equiv \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix} ,$$

$$\phi^{8;1} \equiv \begin{pmatrix} -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

$$\phi^{8;2} \equiv \begin{pmatrix} -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix},$$

$$\phi^{8;3} \equiv \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix},$$

$$\phi^{8;4} \equiv \begin{pmatrix} -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

$$\varphi^{8,5} \equiv \begin{pmatrix} -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 \end{pmatrix},$$

$$\varphi^{8,6} \equiv \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \end{pmatrix},$$

$$\varphi^{8,7} \equiv \begin{pmatrix} -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

$\mathbf{u}^{8,0} \star \mathbf{u}^{8,0} = -\mathbf{u}^{8,0}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,0} = -\mathbf{u}^{8,1}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,0} = -\mathbf{u}^{8,2}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,0} = -\mathbf{u}^{8,3}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,1} = \mathbf{u}^{8,1}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,1} = -\mathbf{u}^{8,0}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,1} = -\mathbf{u}^{8,3}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,1} = \mathbf{u}^{8,2}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,2} = \mathbf{u}^{8,2}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,2} = \mathbf{u}^{8,3}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,2} = -\mathbf{u}^{8,0}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,2} = -\mathbf{u}^{8,1}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,3} = \mathbf{u}^{8,3}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,3} = -\mathbf{u}^{8,2}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,3} = \mathbf{u}^{8,1}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,3} = -\mathbf{u}^{8,0}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,4} = \mathbf{u}^{8,4}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,4} = -\mathbf{u}^{8,5}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,4} = \mathbf{u}^{8,6}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,4} = \mathbf{u}^{8,7}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,5} = \mathbf{u}^{8,5}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,5} = -\mathbf{u}^{8,4}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,5} = \mathbf{u}^{8,7}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,5} = \mathbf{u}^{8,6}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,6} = \mathbf{u}^{8,6}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,6} = -\mathbf{u}^{8,7}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,6} = \mathbf{u}^{8,4}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,6} = \mathbf{u}^{8,5}$
$\mathbf{u}^{8,0} \star \mathbf{u}^{8,7} = \mathbf{u}^{8,7}$	$\mathbf{u}^{8,1} \star \mathbf{u}^{8,7} = -\mathbf{u}^{8,6}$	$\mathbf{u}^{8,2} \star \mathbf{u}^{8,7} = \mathbf{u}^{8,5}$	$\mathbf{u}^{8,3} \star \mathbf{u}^{8,7} = \mathbf{u}^{8,4}$

$u^{8,4} \star u^{8,0} = -u^{8,4}$	$u^{8,5} \star u^{8,0} = -u^{8,5}$	$u^{8,6} \star u^{8,0} = -u^{8,6}$	$u^{8,7} \star u^{8,0} = -u^{8,7}$
$u^{8,4} \star u^{8,1} = u^{8,5}$	$u^{8,5} \star u^{8,1} = u^{8,4}$	$u^{8,6} \star u^{8,1} = u^{8,7}$	$u^{8,7} \star u^{8,1} = u^{8,6}$
$u^{8,4} \star u^{8,2} = -u^{8,6}$	$u^{8,5} \star u^{8,2} = -u^{8,7}$	$u^{8,6} \star u^{8,2} = -u^{8,4}$	$u^{8,7} \star u^{8,2} = -u^{8,5}$
$u^{8,4} \star u^{8,3} = -u^{8,7}$	$u^{8,5} \star u^{8,3} = -u^{8,6}$	$u^{8,6} \star u^{8,3} = -u^{8,5}$	$u^{8,7} \star u^{8,3} = -u^{8,4}$
$u^{8,4} \star u^{8,4} = -u^{8,0}$	$u^{8,5} \star u^{8,4} = -u^{8,1}$	$u^{8,6} \star u^{8,4} = -u^{8,2}$	$u^{8,7} \star u^{8,4} = -u^{8,3}$
$u^{8,4} \star u^{8,5} = u^{8,1}$	$u^{8,5} \star u^{8,5} = -u^{8,0}$	$u^{8,6} \star u^{8,5} = -u^{8,3}$	$u^{8,7} \star u^{8,5} = -u^{8,2}$
$u^{8,4} \star u^{8,6} = u^{8,2}$	$u^{8,5} \star u^{8,6} = u^{8,3}$	$u^{8,6} \star u^{8,6} = -u^{8,0}$	$u^{8,7} \star u^{8,6} = -u^{8,1}$
$u^{8,4} \star u^{8,7} = u^{8,3}$	$u^{8,5} \star u^{8,7} = u^{8,2}$	$u^{8,6} \star u^{8,7} = u^{8,1}$	$u^{8,7} \star u^{8,7} = -u^{8,0}$

So,with:

$$A \equiv u^{8,0}A^0 + u^{8,1}A^1 + u^{8,2}A^2 + u^{8,3}A^3 + u^{8,4}A^4 + u^{8,5}A^5 + u^{8,6}A^6 + u^{8,7}A^7$$

$$B \equiv u^{8,0}B^0 + u^{8,1}B^1 + u^{8,2}B^2 + u^{8,3}B^3 + u^{8,4}B^4 + u^{8,5}B^5 + u^{8,6}B^6 + u^{8,7}B^7$$

then:

$$\begin{aligned} A \star B &= u^{8,0}(-A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3 - A^4B^4 - A^5B^5 - A^6B^6 - A^7B^7) \\ &+ u^{8,1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2 + A^4B^5 - A^5B^4 + A^6B^7 - A^7B^6) \\ &+ u^{8,2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1 + A^4B^6 + A^5B^7 - A^6B^4 - A^7B^5) \\ &+ u^{8,3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0 + A^4B^7 + A^5B^6 - A^6B^5 - A^7B^4) \\ &+ u^{8,4}(A^0B^4 - A^1B^5 + A^2B^6 + A^3B^7 - A^4B^0 + A^5B^1 - A^6B^2 - A^7B^3) \\ &+ u^{8,5}(A^0B^5 - A^1B^4 + A^2B^7 + A^3B^6 + A^4B^1 - A^5B^0 - A^6B^3 - A^7B^2) \\ &+ u^{8,6}(A^0B^6 - A^1B^7 + A^2B^4 + A^3B^5 - A^4B^2 - A^5B^3 - A^6B^0 + A^7B^1) \\ &+ u^{8,7}(A^0B^7 - A^1B^6 + A^2B^5 + A^3B^4 - A^4B^3 - A^5B^2 + A^6B^1 - A^7B^0) \end{aligned}$$

$$\begin{aligned} A \odot B &\equiv \frac{1}{2}(A \star B + B \star A) \quad , \odot\text{-product} \\ &= u^{8,0}(-A^0B^0 - A^1B^1 - A^2B^2 - A^3B^3 - A^4B^4 - A^5B^5 - A^6B^6 - A^7B^7) \end{aligned}$$

$$\begin{aligned} A \times B &\equiv \frac{1}{2}(A \star B - B \star A) \quad , \times\text{-product} \\ &= u^{8,1}(A^0B^1 - A^1B^0 + A^2B^3 - A^3B^2 + A^4B^5 - A^5B^4 + A^6B^7 - A^7B^6) \\ &+ u^{8,2}(A^0B^2 - A^1B^3 - A^2B^0 + A^3B^1 + A^4B^6 + A^5B^7 - A^6B^4 - A^7B^5) \\ &+ u^{8,3}(A^0B^3 + A^1B^2 - A^2B^1 - A^3B^0 + A^4B^7 + A^5B^6 - A^6B^5 - A^7B^4) \\ &+ u^{8,4}(A^0B^4 - A^1B^5 + A^2B^6 + A^3B^7 - A^4B^0 + A^5B^1 - A^6B^2 - A^7B^3) \\ &+ u^{8,5}(A^0B^5 - A^1B^4 + A^2B^7 + A^3B^6 + A^4B^1 - A^5B^0 - A^6B^3 - A^7B^2) \\ &+ u^{8,6}(A^0B^6 - A^1B^7 + A^2B^4 + A^3B^5 - A^4B^2 - A^5B^3 - A^6B^0 + A^7B^1) \\ &+ u^{8,7}(A^0B^7 - A^1B^6 + A^2B^5 + A^3B^4 - A^4B^3 - A^5B^2 + A^6B^1 - A^7B^0) \end{aligned}$$

$$A \star (-u^{8,0}) = A \quad , \quad \forall A \quad ,$$

so:

$$\exists I_R (= -u^{8,0}) \ni A \star I_R = A \quad , \quad \forall A \quad : \text{Right Hand Identity}$$

$$\begin{aligned} \therefore \mathbf{A} \cdot \mathbf{B} &\equiv D \in \mathcal{R} \ni \mathbf{A} \odot \mathbf{B} = I_R \cdot D && : \text{dot product} \\ &= A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 + A^4 B^4 + A^5 B^5 + A^6 B^6 + A^7 B^7 \end{aligned}$$

The alert reader may observe the similarity of the skew-symmetric $\times$ -product to the cross product in these Lorentz-Minkowski Vectric Spaces. In [58] it is shown that a generalization of the vector product must be of dimension 0, 1, 3, or 7. Skew symmetry (anti-commutativity) is not one of the defining relations (as, arguably, it should be); and the reader may verify that for $n > 3$ the above constructions do not generally satisfy:

$$(\mathbf{A} \times \mathbf{B}) \odot \mathbf{A} = (\mathbf{A} \times \mathbf{B}) \odot \mathbf{B} = \mathbf{0} \ , \ (A^0 = 0 = B^0).$$

So, there is no discrepancy, here; although for a modified choice of weight entries the above 8-D construction may be transformed into the vector product generated by the octonions referred to in [58].

Chapter III

Analysis On Finite Dimensional R-Algebras

Preliminary and common definitions, and supporting material appear in Appendix V, including: limit, variation, differential, partial derivative, mean value theorem in space, Taylor's formula in space, homeomorphism, chart, atlas, n -form.

In this chapter, the main thrust is to examine $\mathcal{F}$ -algebras using criteria and techniques similar to those used in complex variable analysis, to obtain similar results for a large variety of spaces. Among these results will be a generalization of the Cauchy-Riemann equations, and a generalization of the Cauchy Integral Theorem.

A convenient place to begin this analysis is with the following notation definition (in standard usage [62]).

Definition 3-1: For n -dimensional (A1-40) vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$,

a **Contravariant Vector Basis** is expressed using the notation:

$$(\hat{\mathbf{u}}_i)_n = (\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3, \dots, \hat{\mathbf{u}}_n)$$

The following theorems and definitions provide groundwork from which some interesting and useful results may be developed.

Theorem 3-1 : For contravariant bases of n -dimensional vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$:

$$(\mathbf{u}_i)_n, (\hat{\mathbf{u}}_i)_n : \exists L_i^j, \Gamma_i^j \in \mathcal{F} \ni$$

$$\mathbf{u}_i = \sum_{j=1}^n L_i^j \otimes \hat{\mathbf{u}}_j, \quad \hat{\mathbf{u}}_i = \sum_{j=1}^n \Gamma_i^j \otimes \mathbf{u}_j$$

$$\text{and: } \sum_{h=1}^n L_i^h \otimes \Gamma_h^j = \delta_i^j, \quad (\forall i, j \in \mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq n).$$

Proof:

Because $(\mathbf{u}_i)_n$, and $(\hat{\mathbf{u}}_i)_n$ are bases:

$$\forall v \in \mathbf{S}, \exists c_1^j, c_2^j \in \mathcal{F}, 1 \leq j \leq n : \ni v = \sum_{j=1}^n c_1^j \otimes \mathbf{u}_j, \quad v = \sum_{j=1}^n c_2^j \otimes \hat{\mathbf{u}}_j.$$

So, for $v =$ each $\hat{\mathbf{u}}_i$, $\forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n$:

$$\hat{\mathbf{u}}_i = v|_{\hat{\mathbf{u}}_i} = \sum_{j=1}^n c_1^j|_{\hat{\mathbf{u}}_i} \otimes \mathbf{u}_j,$$

and, for $v = \text{each } \mathbf{u}_i$, $\forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n$:

(each vector is a linear combination of its base vectors)

$$\mathbf{u}_i = v|_{\mathbf{u}_i} = \sum_{j=1}^n c_2^j|_{\mathbf{u}_i} \otimes \hat{\mathbf{u}}_j .$$

So, let:

$$L_i^j \equiv c_2^j|_{\mathbf{u}_i}, \quad \Gamma_i^j \equiv c_1^j|_{\hat{\mathbf{u}}_i} \Rightarrow \mathbf{u}_i = \sum_{j=1}^n L_i^j \otimes \hat{\mathbf{u}}_j, \quad \hat{\mathbf{u}}_i = \sum_{j=1}^n \Gamma_i^j \otimes \mathbf{u}_j .$$

Therefore,

$$\mathbf{u}_i = \sum_{j=1}^n \left[L_i^j \otimes \left(\sum_{h=1}^n \Gamma_j^h \otimes \mathbf{u}_h \right) \right] = \sum_{h=1}^n \left(\sum_{j=1}^n L_i^j \otimes \Gamma_j^h \right) \otimes \mathbf{u}_h ,$$

implying:

$$\sum_{j=1}^n (L_i^j \otimes \Gamma_j^h) = \delta_i^h \quad \square.$$

Note 3-1

a: $(\mathbf{u}_i)_n$ may be an arbitrary basis, as used here, but will generally be used herein as a constant basis.

b: L_i^j , and Γ_j^h relate contravariant bases .

Theorem 3-2 : For contravariant basis $(\hat{\mathbf{u}}_i)_n$ of n-dimensional vector space:

$(\mathcal{F}, \otimes, \mathbf{S})_n$, a **Covariant Vector Basis**, $(\hat{\mathbf{u}}^i)_n$, exists $\ni$

$$\hat{\mathbf{u}}^j \equiv \sum_{i=1}^n \left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) \otimes \hat{\mathbf{u}}_i, \quad (\forall j \in \mathbb{Z}^+ \ni 1 \leq i \leq n)$$

whenever the inverse matrix: $\left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right)_n^{-1}$ exists.

Proof:

$$\text{Consider vectors: } \hat{\mathbf{u}}^j \equiv \sum_{i=1}^n \left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) \otimes \hat{\mathbf{u}}_i .$$

Then:

$$\sum_{j=1}^n c_j \otimes \hat{\mathbf{u}}^j = \mathbf{0} \Rightarrow \mathbf{0} = \sum_{j=1}^n c_j \otimes \sum_{i=1}^n \left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) \otimes \hat{\mathbf{u}}_i .$$

Therefore:

$$\sum_{i=1}^n \sum_{j=1}^n c_j \otimes \left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) \otimes \hat{\mathbf{u}}_i = \mathbf{0} \Rightarrow \sum_{j=1}^n \left(c_j \otimes \sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) = \mathbf{0}$$

$$\text{So: } c_j = 0, \quad (\forall j \in \mathbb{Z}^+ \ni 1 \leq i \leq n)$$

whenever: $\left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right)_n^{-1}$ exists,

and $\hat{\mathbf{u}}^j$ as defined, are linearly independent.

Now:

$$\hat{\mathbf{u}}^j \equiv \sum_{i=1}^n \left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) \otimes \hat{\mathbf{u}}_i$$

implies

$$\begin{aligned} \sum_{j=1}^n L_j^m \otimes \hat{\mathbf{u}}^j &= \left(\sum_{j=1}^n L_j^m \otimes \sum_{i=1}^n \left(\sum_{h=1}^n \Gamma_h^j \otimes \Gamma_h^i \right) \right) \otimes \hat{\mathbf{u}}_i \\ &= \left(\sum_{h=1}^n \sum_{j=1}^n L_j^m \otimes \Gamma_h^j \otimes \sum_{i=1}^n (\Gamma_h^i) \right) \otimes \hat{\mathbf{u}}_i \\ &= \left(\sum_{h=1}^n \delta_h^m \otimes \sum_{i=1}^n (\Gamma_h^i) \right) \otimes \hat{\mathbf{u}}_i \\ &= \sum_{i=1}^n \Gamma_m^i \otimes \hat{\mathbf{u}}_i, \text{ by Theorem 3-1.} \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{j=1}^n \left(\sum_{m=1}^n L_h^m \otimes L_j^m \right) \otimes \hat{\mathbf{u}}^j &= \sum_{m=1}^n L_h^m \otimes \sum_{i=1}^n \Gamma_m^i \otimes \hat{\mathbf{u}}_i \\ &= \sum_{i=1}^n \left(\sum_{m=1}^n L_h^m \otimes \Gamma_m^i \right) \otimes \hat{\mathbf{u}}_i \\ &= \sum_{i=1}^n \delta_h^i \otimes \hat{\mathbf{u}}_i = \hat{\mathbf{u}}_h, \text{ by Theorem 3-1.} \end{aligned}$$

For any: $\mathbf{v} \in \mathbf{S}$, $\exists \mathcal{C}^j \in \mathcal{F} \ni$

$$\begin{aligned} \mathbf{v} &= \sum_{j=1}^n \mathcal{C}^j \otimes \hat{\mathbf{u}}_i \\ &= \sum_{j=1}^n \mathcal{C}^j \otimes \sum_{i=1}^n \left(\sum_{m=1}^n L_j^m \otimes L_i^m \right) \otimes \hat{\mathbf{u}}^i \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n \sum_{m=1}^n \mathcal{C}^j \otimes (L_j^m \otimes L_i^m) \right) \otimes \hat{\mathbf{u}}^i. \end{aligned}$$

For any : $\mathbf{v} \in \mathbf{S}$, $\exists k_i \in \mathcal{F} \ni \mathbf{v} = \sum_{j=1}^n k_i \otimes \hat{\mathbf{u}}^i$

$$\text{namely: } k_i \equiv \left(\sum_{j=1}^n \sum_{m=1}^n \mathcal{C}^j \otimes (L_j^m \otimes L_i^m) \right)$$

(i.e. the covariant basis spans $\mathbf{S}$).

Therefore,

$(\hat{\mathbf{u}}^j)_n$ is a basis for $(\mathcal{F}, \otimes, \mathbf{S})_n$ □.

Corollary 3-2a : For covariant basis $(\hat{u}^i)_n$ of n -dimensional vector space:

$(\mathcal{F}, \otimes, S)_n$, a contravariant basis, $(\hat{u}_i)_n$, exists $\ni$

$$\hat{u}_i \equiv \sum_{j=1}^n \left(\sum_{h=1}^n L_j^h \otimes L_i^h \right) \otimes \hat{u}^j, \quad (\forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n)$$

where the inverse matrix: $(L_j^h \otimes L_i^h)_n^{-1}$ exists.

Proof:

For covariant basis: $(\hat{u}^j)$, from within the main theorem:

$$\begin{aligned} \sum_{j=1}^n \left(\sum_{m=1}^n L_h^m \otimes L_j^m \right) \otimes \hat{u}^j &= \sum_{m=1}^n L_h^m \otimes \sum_{i=1}^n \Gamma_m^i \otimes \hat{u}_i \\ &= \sum_{i=1}^n \left(\sum_{m=1}^n L_h^m \otimes \Gamma_m^i \right) \otimes \hat{u}_i \\ &= \sum_{i=1}^n \delta_h^i \otimes \hat{u}_i = \hat{u}_h \quad \square. \end{aligned}$$

The following definitions are developed from the above work.

Definition 3-2: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$ with

contravariant and covariant bases: $(\hat{u}_i)_n$, $(\hat{u}^i)_n$, respectively, then:

the co-contr-coeffs are given by: Γ_h^j , and

the contra-co-coeffs are given by: $L_h^j \ni$

$$\sum_{m=1}^n L_m^h \otimes \hat{u}^m \equiv \sum_{m=1}^n \Gamma_h^m \otimes \hat{u}_m, \quad \begin{cases} (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n) \\ (\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n) \end{cases}$$

Definition 3-3: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$ with

contravariant and covariant bases: $(\hat{u}_i)_n$, $(\hat{u}^i)_n$, respectively, then:

the metric tensors are given by:

$$\begin{aligned} g_{ij} &\equiv \sum_{h=1}^n (L_j^h \otimes L_i^h), \\ g^{ij} &\equiv \sum_{h=1}^n (\Gamma_h^j \otimes \Gamma_h^i), \\ &(\forall i, j \in \mathcal{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq n) \end{aligned}$$

The following identities result from the above definitions.

$$\hat{u}_i = \sum_{j=1}^n g_{ij} \hat{u}^j, \quad \hat{u}^i = \sum_{j=1}^n g^{ij} \hat{u}_j, \quad (\forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n)$$

Theorem 3-3: For n -dimensional vector space: $(\mathcal{F}, \otimes, \mathbf{S})_n$ with contravariant and covariant bases: $(\hat{\mathbf{u}}_i)_n$, $(\hat{\mathbf{u}}^i)_n$, respectively, then:

$$\forall \mathbf{A} \in \mathbf{S}, \exists A^j, A_j \in \mathbf{P} \ni \mathbf{A} \equiv (A^i)_n \equiv \sum_{j=1}^n A^j \otimes \hat{\mathbf{u}}_j = (A_i)_n \equiv \sum_{j=1}^n A_j \otimes \hat{\mathbf{u}}^j ,$$

and:

$$A_j = \sum_{i=1}^n \sum_{h=1}^n (A^i \otimes L_j^h \otimes L_i^h) , \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq i \leq n)$$

$$A^j = \sum_{i=1}^n \sum_{h=1}^n (A_i \otimes \Gamma_h^i \otimes \Gamma_h^j) ,$$

Proof:

As: $(\hat{\mathbf{u}}_i)_n$, $(\hat{\mathbf{u}}^i)_n$, are bases for: $(\mathcal{F}, \otimes, \mathbf{S})_n$, clearly:

$$\forall \mathbf{A} \in \mathbf{S}, \exists A^j, A_j \in \mathcal{F} \ni \mathbf{A} = \sum_{j=1}^n A^j \otimes \hat{\mathbf{u}}_j = \sum_{j=1}^n A_j \otimes \hat{\mathbf{u}}^j .$$

Therefore:

$$\begin{aligned} \sum_{j=1}^n A_j \otimes \hat{\mathbf{u}}^j &= \mathbf{A} = \sum_{i=1}^n A^i \otimes \sum_{j=1}^n \sum_{h=1}^n (L_j^h \otimes L_i^h) \otimes \hat{\mathbf{u}}^j , \text{ by Corollary 3-2a,} \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n \sum_{h=1}^n A^i \otimes L_j^h \otimes L_i^h \right) \otimes \hat{\mathbf{u}}^j \end{aligned}$$

So:

$$A_j = \sum_{i=1}^n \sum_{h=1}^n (A^i \otimes L_j^h \otimes L_i^h) .$$

Similarly:

$$\begin{aligned} \sum_{j=1}^n A^j \otimes \hat{\mathbf{u}}_j &= \mathbf{A} = \sum_{i=1}^n A_i \otimes \sum_{j=1}^n \sum_{h=1}^n (\Gamma_h^i \otimes \Gamma_h^j) \otimes \hat{\mathbf{u}}_j , \text{ by Theorem 3-2.} \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n \sum_{h=1}^n A_i \otimes \Gamma_h^i \otimes \Gamma_h^j \right) \otimes \hat{\mathbf{u}}_j \end{aligned}$$

so:

$$A^j = \sum_{i=1}^n \sum_{h=1}^n (A_i \otimes \Gamma_h^i \otimes \Gamma_h^j) \quad \square.$$

The following relationships follow, immediately.

$$A_i = \sum_{j=1}^n g_{ji} A^j , \quad A^i = \sum_{j=1}^n g^{ji} A_j , \quad (\forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n) .$$

As the previous theorems have shown, co-contra-coeffs (Γ_h^j) relate covariant components to contravariant components (or contravariant basis vectors to covariant

basis vectors), and contra-co-coeffs (L_j^h) relate contravariant components to covariant components (or covariant basis vectors to contravariant basis vectors).

The following definitions are developed from the above.

Definition 3-4: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$,

a Contravariant Representation

of a vector is expressed using the notation:

$$A = \sum_{i=1}^n A^i \otimes \hat{u}_i$$

with **Contravariant Components:** A^i

Definition 3-5: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$

a Covariant Representation

of a vector is expressed using the notation:

$$A = \sum_{j=1}^n A_j \otimes \hat{u}^j$$

with **Covariant Components:** A_j

Theorem 3-4: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$ with contravariant and covariant bases: $(\hat{u}_i)_n$, $(\hat{u}^i)_n$, respectively, then:

$$\sum_{m=1}^n \left(L_i^m \frac{\partial}{\partial x^k} \Gamma_m^h \right) = - \sum_{m=1}^n \left(\Gamma_m^h \frac{\partial}{\partial x^k} L_i^m \right) , \quad \begin{cases} (\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n) \\ (\forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n) \\ (\forall k \in \mathcal{Z}^+ \ni 1 \leq k \leq n) \end{cases}$$

Proof:

Because:

$$\sum_{m=1}^n (L_i^m \Gamma_m^h) = \delta_i^h ,$$

then:

$$0 = \sum_{m=1}^n \frac{\partial}{\partial x^k} (L_i^m \Gamma_m^h) = \sum_{m=1}^n \left(\Gamma_m^h \frac{\partial}{\partial x^k} L_i^m + L_i^m \frac{\partial}{\partial x^k} \Gamma_m^h \right) .$$

So:

$$\sum_{m=1}^n \left(L_i^m \frac{\partial}{\partial x^k} \Gamma_m^h \right) = - \sum_{m=1}^n \left(\Gamma_m^h \frac{\partial}{\partial x^k} L_i^m \right) \quad \square.$$

Theorem 3-5: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$ and continuous vector in

Contravariant Representation: $\mathbf{A} = \sum_{i=1}^n A^i \otimes \hat{\mathbf{u}}_i$:

$$\frac{\partial}{\partial x^h} \mathbf{A} = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \left(\sum_{j=1}^n A^j \Gamma_j^i \right) \right] \right)$$

Proof:

Note from Appendix I:

$$((x^i)_{h-1}, x^h + \delta x^h, (x^{h+i})_n) \equiv (x^1, x^2, \dots, x^{h-1}, x^h + \delta x^h, x^{h+1}, \dots, x^n)$$

So:

$$\frac{\partial}{\partial x^h} \mathbf{A} = \lim_{\delta x^h \rightarrow 0} \left\{ (\delta x^h)^{-1} \otimes \sum_{j=1}^n \left[(A^j \hat{\mathbf{u}}_j)(x^1, x^2, \dots, x^h + \delta x^h, \dots, x^n) - (A^j \hat{\mathbf{u}}_j)(x^1, x^2, \dots, x^h, \dots, x^n) \right] \right\}.$$

For: $\mathbf{u}_i$ constants, then:

$$\frac{\partial}{\partial x^h} \mathbf{A} = \lim_{\delta x^h \rightarrow 0} \left\{ (\delta x^h)^{-1} \otimes \sum_{j=1}^n \left[\left(\sum_{i=1}^n A^j \Gamma_j^i \otimes \mathbf{u}_i \right) (x^1, x^2, \dots, x^h + \delta x^h, \dots, x^n) - \left(\sum_{i=1}^n A^j \Gamma_j^i \otimes \mathbf{u}_i \right) (x^1, x^2, \dots, x^h, \dots, x^n) \right] \right\},$$

by Theorem 3-1.

$$\begin{aligned} &= \lim_{\delta x^h \rightarrow 0} \left\{ (\delta x^h)^{-1} \otimes \sum_{j=1}^n \left[\left(\sum_{i=1}^n \left\{ A^j \Gamma_j^i (x^1, x^2, \dots, x^h + \delta x^h, \dots, x^n) - A^j \Gamma_j^i (x^1, x^2, \dots, x^h, \dots, x^n) \right\} \otimes \mathbf{u}_i \right) \right] \right\} \\ &= \sum_{i=1}^n \mathbf{u}_i \otimes \lim_{\delta x^h \rightarrow 0} \left\{ (\delta x^h)^{-1} \left[\sum_{j=1}^n \left\{ A^j \Gamma_j^i (x^1, x^2, \dots, x^h + \delta x^h, \dots, x^n) - A^j \Gamma_j^i (x^1, x^2, \dots, x^h, \dots, x^n) \right\} \right] \right\} \\ &= \sum_{i=1}^n \left[\mathbf{u}_i \otimes \frac{\partial}{\partial x^h} \left(\sum_{j=1}^n A^j \Gamma_j^i \right) \right] \\ &= \sum_{i=1}^n \left[\sum_{m=1}^n (L_i^m \otimes \hat{\mathbf{u}}_m) \otimes \frac{\partial}{\partial x^h} \left(\sum_{j=1}^n A^j \Gamma_j^i \right) \right] \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \left(\sum_{j=1}^n A^j \Gamma_j^i \right) \right] \right) \quad \square. \end{aligned}$$

Corollary 3-5a: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$ with continuous

contravariant basis: $(\hat{\mathbf{u}}_i)_n \ni \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n$,

$(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$:

$$\frac{\partial}{\partial x^h} \hat{\mathbf{u}}_j = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i \right] \right)$$

Proof:

$$\mathbf{A} = \hat{\mathbf{u}}_j \Rightarrow A^k = \delta^{jk},$$

$$\begin{aligned}
\frac{\partial}{\partial x^h} \hat{u}_j &= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \left(\sum_{k=1}^n \delta^{jk} \Gamma_k^i \right) \right] \right) \\
&= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i \right] \right) \quad \square.
\end{aligned}$$

Corollary 3-5b: For n -dimensional vector space $(\mathcal{F}, \otimes, S)_n$ with continuous

covariant basis: $(\hat{u}^i)_n \ni \hat{u}^j = \hat{u}^j(x^i)_n$,

$(\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n)$:

$$\frac{\partial}{\partial x^h} \hat{u}^j = \sum_{m=1}^n \left(\hat{u}^m \otimes \left[\sum_{i=1}^n \sum_{p=1}^n \sum_{q=1}^n L_m^q L_p^q L_i^p \frac{\partial}{\partial x^h} \left(\sum_{k=1}^n \sum_{t=1}^n (\Gamma_t^j \Gamma_t^k) \Gamma_k^i \right) \right] \right)$$

or:

$$\frac{\partial}{\partial x^h} \hat{u}^j = \sum_{m=1}^n \left(\hat{u}^m \otimes \left[\sum_{i=1}^n \sum_{p=1}^n g_{pm} L_i^p \frac{\partial}{\partial x^h} \left(\sum_{k=1}^n g^{kj} \Gamma_k^i \right) \right] \right)$$

Proof:

By Theorem 3-3:

$$A = \hat{u}^j \Rightarrow A^k = \sum_{m=1}^n (\Gamma_m^j \otimes \Gamma_m^k).$$

So, by Theorem 3-5:

$$\frac{\partial}{\partial x^h} \hat{u}^j = \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \left(\sum_{k=1}^n \sum_{t=1}^n (\Gamma_t^j \Gamma_t^k) \Gamma_k^i \right) \right] \right).$$

Then, by Corollary 3-2a:

$$\begin{aligned}
\frac{\partial}{\partial x^h} \hat{u}^j &= \sum_{m=1}^n \left(\left(\sum_{p=1}^n \sum_{q=1}^n L_p^q L_m^q \hat{u}^p \right) \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \left(\sum_{k=1}^n \sum_{t=1}^n (\Gamma_t^j \Gamma_t^k) \Gamma_k^i \right) \right] \right) \\
&= \sum_{m=1}^n \left(\hat{u}^m \otimes \left[\sum_{i=1}^n \sum_{p=1}^n \sum_{q=1}^n L_m^q L_p^q L_i^p \frac{\partial}{\partial x^h} \left(\sum_{k=1}^n \sum_{t=1}^n (\Gamma_t^j \Gamma_t^k) \Gamma_k^i \right) \right] \right) \quad \square.
\end{aligned}$$

Theorem 3-6: For n -dimensional vector space: $(\mathcal{F}, \otimes, S)_n$ with continuous

contravariant and covariant bases: $(\hat{u}_i)_n$, $(\hat{u}^i)_n$, respectively, $\ni$

$\hat{u}_j = \hat{u}_j(x^i)_n$, $(\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n)$:

$$\delta \hat{u}_i = \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\sum_{h=1}^n \sum_{j=1}^n L_j^m \left(\frac{\partial}{\partial x^h} \Gamma_i^j \right) \Big|_{\mathbf{x}=\xi} \delta x^h \right] \right),$$

$\exists \xi \in \mathcal{R}^n \ni (x^i < \xi^i < x^i + \delta x^i, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n)$

Proof 1:

By Corollary A5-2a (to the Mean Value Theorem in space (A5-2)),

if the partial derivatives of $u_j(\mathbf{x})$ exist at $\mathbf{x}$ and $\mathbf{x} + \delta \mathbf{x}$, $\forall \delta \mathbf{x} > 0$;

and all $u_j(\mathbf{x})$ are continuous (A5-3) at $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$, then there is an ξ ($x^i < \xi^i < x^i + \delta x^i, \forall i \in Z^+ \ni 1 \leq i \leq n$), such that:

$$\begin{aligned}\delta \hat{\mathbf{u}}_i &= \sum_{h=1}^n \left[\delta x^h \otimes \left(\frac{\partial}{\partial x^h} \hat{\mathbf{u}}_i \Big|_{\mathbf{x}=\xi} \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n) \\ &= \sum_{h=1}^n \left[\delta x^h \otimes \left(\frac{\partial}{\partial x^h} \left(\sum_{j=1}^n \Gamma_i^j \otimes \mathbf{u}_j \right) \Big|_{\mathbf{x}=\xi} \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n), \\ &\quad \text{by Theorem 3-1.}\end{aligned}$$

For u_j independent of all x^h :

$$\delta \hat{\mathbf{u}}_i = \sum_{h=1}^n \left[\sum_{j=1}^n \mathbf{u}_j \otimes \left(\frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \delta x^h \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n).$$

So, by Theorem 3-1:

$$\begin{aligned}\delta \hat{\mathbf{u}}_i &= \sum_{h=1}^n \left[\sum_{j=1}^n \left(\sum_{m=1}^n L_j^m \otimes \hat{\mathbf{u}}_m \right) \otimes \left(\frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \delta x^h \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{h=1}^n \sum_{j=1}^n L_j^m \left(\frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \delta x^h \right) \right] \right), (\forall i \in Z^+ \ni 1 \leq i \leq n)\end{aligned}$$

□.

Proof 2:

$$\begin{aligned}\delta \hat{\mathbf{u}}_i &= \sum_{h=1}^n \left[\delta x^h \otimes \left(\frac{\partial}{\partial x^h} \hat{\mathbf{u}}_i \Big|_{\mathbf{x}=\xi} \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n) \\ &= \sum_{h=1}^n \left[\delta x^h \otimes \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \right] \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n). \\ &= \sum_{h=1}^n \left[\delta x^h \otimes \sum_{j=1}^n \sum_{m=1}^n (\hat{\mathbf{u}}_m \otimes L_j^m \Big|_{\mathbf{x}=\xi} \left(\frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n), \\ &\quad \text{by Corollary 3-5a.}\end{aligned}$$

But:

$$\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes L_j^m = \mathbf{u}_j \text{ is the same for all } \mathbf{x},$$

so:

$$\left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes L_j^m \Big|_{\mathbf{x}=\xi} \right) = \sum_{m=1}^n \hat{\mathbf{u}}_m \otimes L_j^m.$$

Therefore,

$$\begin{aligned}\delta \hat{\mathbf{u}}_i &= \sum_{h=1}^n \left[\delta x^h \otimes \sum_{j=1}^n \sum_{m=1}^n (\hat{\mathbf{u}}_m \otimes L_j^m) \left(\frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \right) \right], (\forall i \in Z^+ \ni 1 \leq i \leq n) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{h=1}^n \sum_{j=1}^n L_j^m \left(\frac{\partial}{\partial x^h} \Gamma_i^j \Big|_{\mathbf{x}=\xi} \delta x^h \right) \right] \right), (\forall i \in Z^+ \ni 1 \leq i \leq n) \quad \square.\end{aligned}$$

Theorem 3-7: For n -dimensional vector space: $(\mathcal{F}, \otimes, \mathcal{S})_n$ with continuous contravariant and covariant bases: $(\hat{\mathbf{u}}_i)_n$, $(\hat{\mathbf{u}}^i)_n$, respectively, $\ni$
 $\hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n$, $(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$, and
continuous vector in Contravariant Representation: $\mathbf{A} = \sum_{i=1}^n A^i \otimes \hat{\mathbf{u}}_i$:

$$\delta \mathbf{A} = \sum_{h=1}^n \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left(\frac{\partial A^m}{\partial x^h} + \left[\sum_{i=1}^n A^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h$$

Proof:

$$\begin{aligned} \delta \mathbf{A} &= \sum_{i=1}^n \delta(A^i \otimes \hat{\mathbf{u}}_i) \\ &= \sum_{i=1}^n \sum_{h=1}^n \delta x^h \otimes \left(\frac{\partial}{\partial x^h} (A^i \otimes \hat{\mathbf{u}}_i) \right) \Big|_{\mathbf{x}=\xi}, \\ &\quad \xi: (x^i < \xi^i < x^i + \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n) \\ &= \sum_{i=1}^n \sum_{h=1}^n \delta x^h \otimes \left(\frac{\partial}{\partial x^h} (A^i) \otimes \hat{\mathbf{u}}_i + A^i \otimes \frac{\partial}{\partial x^h} \hat{\mathbf{u}}_i \right) \Big|_{\mathbf{x}=\xi} \\ &= \sum_{i=1}^n \sum_{h=1}^n \delta x^h \otimes \left(\frac{\partial A^i}{\partial x^h} \otimes \hat{\mathbf{u}}_i + A^i \otimes \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \\ &= \sum_{h=1}^n \delta x^h \otimes \left(\sum_{i=1}^n \frac{\partial A^i}{\partial x^h} \otimes \hat{\mathbf{u}}_i + \sum_{i=1}^n A^i \otimes \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \\ &= \sum_{h=1}^n \delta x^h \otimes \left(\sum_{m=1}^n \frac{\partial A^m}{\partial x^h} \otimes \hat{\mathbf{u}}_m + \sum_{i=1}^n A^i \otimes \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \\ &= \sum_{h=1}^n \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left(\frac{\partial A^m}{\partial x^h} + \left[\sum_{i=1}^n A^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h \end{aligned}$$

□

Definition 3-6: For n -dimensional vector space $(\mathcal{F}, \otimes, \mathcal{S})_n$, and for some basis:

$(\hat{\mathbf{u}}_i)_n = \{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3, \dots, \hat{\mathbf{u}}_n\}$, the radius or position vector is specified by:

$$\mathbf{r} \equiv \sum_{j=1}^n C_r^j \otimes \hat{\mathbf{u}}_j \ni$$

$$\delta \mathbf{r} \equiv \sum_{j=1}^n \delta x^j \otimes (\hat{\mathbf{u}}_j)_{\mathbf{x}=\xi}, \quad \forall \mathbf{x}, \xi \in \mathcal{S} \ni |\xi^i - x^i| < \delta x^i, \quad \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n \quad (\text{locally})$$

Theorem 3-8 : For n -dimensional vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$ with continuous contravariant and covariant bases: $(\hat{\mathbf{u}}_i)_n$, $(\hat{\mathbf{u}}^i)_n$, respectively, $\ni$

$$\begin{aligned} \mathbf{r} &\equiv \sum_{j=1}^n C_r^j \otimes \hat{\mathbf{u}}_j, \quad \delta \mathbf{r} \equiv \sum_{j=1}^n \delta x^j \otimes (\hat{\mathbf{u}}_j|_{\mathbf{x}=\xi}, \\ &(\forall \mathbf{x}, \xi \in \mathbf{S} \ni |\xi^i - x^i| < \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n) : \\ &\frac{\partial C_r^j}{\partial x^h} + \sum_{i=1}^n C_r^i \sum_{m=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^m = \delta_h^j, \\ &(\forall j, h \in \mathcal{Z}^+ \ni 1 \leq j \leq n, 1 \leq h \leq n) \end{aligned}$$

Proof:

$$\delta \mathbf{r} = \sum_{h=1}^n \delta x^h \otimes (\hat{\mathbf{u}}_h|_{\mathbf{x}=\xi}, \quad \forall \mathbf{x}, \xi \in \mathbf{S} \ni |\xi^i - x^i| < \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n,$$

and, by Theorem 3-7:

$$\delta \mathbf{r} = \sum_{h=1}^n \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left(\frac{\partial C_r^m}{\partial x^h} + \left[\sum_{i=1}^n C_r^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h,$$

$$\exists \xi \ni x^i < \xi^i < x^i + \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n;$$

or:

$$\exists \xi \ni -\delta x^i < 0 < |\xi^i - x^i| < \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n.$$

So, for this: ξ :

$$\begin{aligned} \delta \mathbf{r} &= \sum_{m=1}^n \delta x^m \otimes (\hat{\mathbf{u}}_m|_{\mathbf{x}=\xi} \\ &= \sum_{m=1}^n \sum_{h=1}^n (\delta_h^m \delta x^h) \otimes (\hat{\mathbf{u}}_m|_{\mathbf{x}=\xi} = \sum_{h=1}^n \sum_{m=1}^n (\hat{\mathbf{u}}_m|_{\mathbf{x}=\xi_1} \otimes (\delta_h^m \delta x^h) \\ &= \sum_{h=1}^n \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \delta_h^m \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h, \text{ for arbitrary } \delta x^h. \end{aligned}$$

It follows that:

$$\left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \delta_h^m \right) \Big|_{\mathbf{x}=\xi} = \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left(\frac{\partial C_r^m}{\partial x^h} + \left[\sum_{i=1}^n C_r^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi}$$

and, so, as: $\delta \mathbf{x} \rightarrow \mathbf{0} \Rightarrow \xi \rightarrow \mathbf{x}$:

$$\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \delta_h^m = \sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left(\frac{\partial C_r^m}{\partial x^h} + \left[\sum_{i=1}^n C_r^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right)$$

Therefore,

$$\begin{aligned} \delta_h^m &= \frac{\partial C_r^m}{\partial x^h} + \left[\sum_{i=1}^n C_r^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \\ &(\forall m, h \in \mathcal{Z}^+ \ni 1 \leq m \leq n, 1 \leq h \leq n) \quad \square. \end{aligned}$$

Theorem 3-9: For n -dimensional vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$ with continuous contravariant and covariant bases: $(\hat{\mathbf{u}}_i)_n$, $(\hat{\mathbf{u}}^i)_n$, respectively, $\ni$

$$\mathbf{r} \equiv \sum_{j=1}^n C_r^j \otimes \hat{\mathbf{u}}_j, \quad \delta \mathbf{r} \equiv \sum_{j=1}^n \delta x^j \otimes (\hat{\mathbf{u}}_j|_{\mathbf{x}=\xi},$$

$$(\forall \mathbf{x}, \xi \in \mathbf{S} \ni |\xi^i - x^i| < \delta x^i, \quad \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n) :$$

$$\frac{\partial \mathbf{r}}{\partial x^h} = \hat{\mathbf{u}}_h, \quad (\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n)$$

Proof:

For vector space: $(\mathcal{F}, \otimes, \mathbf{S})_n$

with radius / position vector, satisfying:

$$\mathbf{r} \equiv \sum_{j=1}^n C_r^j \otimes \hat{\mathbf{u}}_j, \quad \text{so:}$$

$$\delta \mathbf{r} \equiv \sum_{j=1}^n \delta x^j \otimes (\hat{\mathbf{u}}_j|_{\mathbf{x}=\xi}, \quad \forall \mathbf{x}, \xi \in \mathbf{S} \ni |\xi^i - x^i| < \delta x^i,$$

$$\forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n,$$

and, therefore, by Theorem 3-8:

$$\frac{\partial C_r^j}{\partial x^h} + \sum_{i=1}^n C_r^i \sum_{m=1}^n L_m^i \frac{\partial}{\partial x^h} \Gamma_i^m = \delta_h^j, \quad (\forall j, h \in \mathcal{Z}^+ \ni 1 \leq j \leq n, 1 \leq h \leq n).$$

But, by Corollary 3-5a:

$$\begin{aligned} \frac{\partial}{\partial x^h} \mathbf{r} &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \left(\sum_{j=1}^n C_r^j \Gamma_j^i \right) \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{i=1}^n L_i^m \left\{ \sum_{j=1}^n C_r^j \frac{\partial}{\partial x^h} \Gamma_j^i + \sum_{j=1}^n \Gamma_j^i \frac{\partial}{\partial x^h} C_r^j \right\} \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{i=1}^n \left\{ \sum_{j=1}^n C_r^j L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i + \sum_{j=1}^n L_i^m \Gamma_j^i \frac{\partial}{\partial x^h} C_r^j \right\} \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n \left\{ \sum_{i=1}^n C_r^j L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i + \sum_{i=1}^n L_i^m \Gamma_j^i \frac{\partial}{\partial x^h} C_r^j \right\} \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n \left\{ \sum_{i=1}^n C_r^j L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i + \delta_j^m \frac{\partial}{\partial x^h} C_r^j \right\} \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n \sum_{i=1}^n C_r^j L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i + \sum_{j=1}^n \delta_j^m \frac{\partial}{\partial x^h} C_r^j \right] \right), \\ &\quad \text{(by Theorem 3-1)}, \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{j=1}^n \sum_{i=1}^n C_r^j L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i + \frac{\partial}{\partial x^h} C_r^m \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\frac{\partial C_r^m}{\partial x^h} + \sum_{j=1}^n C_r^j \sum_{i=1}^n L_i^m \frac{\partial}{\partial x^h} \Gamma_j^i \right] \right) \\ &= \sum_{m=1}^n (\hat{\mathbf{u}}_m \otimes \delta_h^m), \quad \text{(by Theorem 3-8)} \end{aligned}$$

$$= \hat{\mathbf{u}}_h$$

□.

Corollary 3-9a: For n -dimensional vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$ with continuous contravariant and covariant bases: $(\hat{\mathbf{u}}_i)_n$, $(\hat{\mathbf{u}}^i)_n$, respectively, $\ni$

$$\hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n, \quad (\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n) :$$

$$\left(\frac{\partial \mathbf{r}}{\partial x^j} \right)_n \text{ is a basis for: } (\mathcal{F}, \otimes, \mathbf{S})_n$$

which is, here, termed the standard basis.

Proof:

$(\hat{\mathbf{u}}_i)_n$ is a contravariant basis of $(\mathcal{F}, \otimes, \mathbf{S})_n$,

and from the main theorem:

$$\frac{\partial \mathbf{r}}{\partial x^j} = \hat{\mathbf{u}}_j, \quad \forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n.$$

Consequently, so is: $\left(\frac{\partial \mathbf{r}}{\partial x^j} \right)_n$ □.

Definition 3-7: The Christoffel symbols of the 2nd kind are:

$$\left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \equiv \sum_{m=1}^n L_m^j \frac{\partial}{\partial x^h} \Gamma_k^m$$

This definition of Christoffel symbols helps limit the size of expressions.

Any vector space isomorphic to $\mathcal{R}^n$ will have a basis:

$$\left\{ \left(\frac{\partial \mathbf{r}}{\partial x^h} \right)_n \mid \forall h \in \mathbb{Z}^+ \ni 1 \leq h \leq n \right\},$$

and coordinate system n -tuple (A1-2).

However, even some of the most used and important coordinate systems (A5-10) are beyond this simplistic model.

Example 3-1:

Consider, the mapping:

$$\left. \begin{matrix} x = r \cos \theta \\ y = r \sin \theta \end{matrix} \right\} r = 0 \Rightarrow (x, y) = (0, 0), \forall \theta.$$

Consequently, this is not an isomorphism.

The inverse mapping:

$$\left. \begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \right\} \begin{cases} y = 0 \Rightarrow \theta = 0, \forall x \neq 0 \\ x = 0 \Rightarrow \theta = 0, \forall y \neq 0 \end{cases}$$

is clearly not continuous at $(x, y) = (0, 0)$.

The mapping:

$$\begin{aligned} r \in U_0 &\equiv \{z \mid z = \sqrt{x^2 + y^2}, \forall (x, y) \in \mathcal{R} \times \mathcal{R}\}, \\ \theta \in U_1 &\equiv \{z \mid z = \tan^{-1}\left(\frac{y}{x}\right), \forall (x, y) \in \mathcal{R} \times \mathcal{R} \setminus (0, 0) \vee (x, y) = (0, 0) \Rightarrow z = 0\} \end{aligned}$$

is an isomorphism, but it is not continuous, and there is no

$$(x, y) = (x_0, y_0) \Rightarrow z = z_0$$

that would make it so.

However, the two charts (See Appendix V, Definition A5-9.):

$$\begin{aligned} C_0 : (U_0, h_0, 2, M_0) : &\begin{cases} U_0 \equiv \{(x, y) \mid \forall x, y \in \mathcal{R} \ni (x, y) \neq (0, 0)\} \\ M_0 \equiv \{(r, \theta) \mid \forall r, \theta \in \mathcal{R} \ni (r > 0) \wedge (0 < \theta < 2\pi)\} \\ h_0 : U_0 \rightarrow M_0 : (r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right) \\ h_0^{-1} : U_0 \leftarrow M_0 : (x, y) = (r \cos \theta, r \sin \theta) \end{cases} \\ C_1 : (U_1, h_1, 2, M_1) : &\begin{cases} U_1 \equiv \{(0, 0)\} \\ M_1 \equiv \{(0, 0)\} \\ h_1 : U_1 \rightarrow M_1 : (r, \theta) = \left(\sqrt{x^2 + y^2}, \tan^{-1}(y)\right) \\ h_1^{-1} : U_1 \leftarrow M_1 : (x, y) = (r \cos \theta, r \sin \theta) \end{cases} \end{aligned}$$

$$U_0 \cup U_1 = \mathcal{R} \times \mathcal{R} = \mathcal{R}^2, \quad M_0 \cup M_1 = (\mathcal{R}, [0, 2\pi)) \subset \mathcal{R}^2$$

are isomorphisms; and homeomorphisms which form an atlas
(See Appendix V, Definition A5-11.).

In C_0 :

$$\begin{aligned} \mathbf{r} &= \hat{\mathbf{i}}x + \hat{\mathbf{j}}y = \hat{\mathbf{i}}(r \cos \theta) + \hat{\mathbf{j}}(r \sin \theta) \\ &= \alpha_r r + \alpha_\theta \theta = \alpha_r(r) + \alpha_\theta(\theta) \\ \alpha_r &= \frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{i}}(\cos \theta) + \hat{\mathbf{j}}(\sin \theta) = \hat{\mathbf{i}}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \hat{\mathbf{j}}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) \\ \alpha_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} = \hat{\mathbf{i}}(-r \sin \theta) + \hat{\mathbf{j}}(r \cos \theta) = \hat{\mathbf{i}}(-y) + \hat{\mathbf{j}}(x) \\ \hat{\mathbf{i}} &= \frac{\partial \mathbf{r}}{\partial x} = \alpha_r(\cos \theta) + \alpha_\theta\left(-\frac{1}{r} \sin \theta\right) \\ \hat{\mathbf{j}} &= \frac{\partial \mathbf{r}}{\partial y} = \alpha_r(\sin \theta) + \alpha_\theta\left(\frac{1}{r} \cos \theta\right), \end{aligned}$$

relates the bases.

In C_1 :

C_1 is the Zero Vector Space, so has no basis,
and no dimension, but for compatibility:

$$\begin{aligned}\alpha_r &= \frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{i}}(\cos 0) + \hat{\mathbf{j}}(\sin 0) = \hat{\mathbf{i}} \\ \alpha_\theta &= \frac{\partial \mathbf{r}}{\partial \theta} = \hat{\mathbf{i}}(-0 \sin 0) + \hat{\mathbf{j}}(0 \cos 0) = \mathbf{0} \\ \hat{\mathbf{i}} &= \frac{\partial \mathbf{r}}{\partial x} = \alpha_r \left(\lim_{y \rightarrow 0} \frac{x}{\sqrt{x^2 + y^2}} \right) + \alpha_\theta \left(- \lim_{y \rightarrow 0} \frac{y}{x^2 + y^2} \right) = \alpha_r \\ \hat{\mathbf{j}} &= \frac{\partial \mathbf{r}}{\partial y} = \alpha_r \left(\lim_{y \rightarrow 0} \frac{y}{\sqrt{x^2 + y^2}} \right) + \alpha_\theta \left(\lim_{y \rightarrow 0} \frac{x}{x^2 + y^2} \right) = \alpha_r \cdot \mathbf{0} + \mathbf{0} \cdot \infty, \\ &\text{is not determined in } C_1.\end{aligned}$$

So, in C_0 :

$$\begin{aligned}\frac{\partial}{\partial r} \alpha_r &= \frac{\partial^2 \mathbf{r}}{\partial r^2} = \mathbf{0} \\ \frac{\partial}{\partial r} \alpha_\theta &= \frac{\partial^2 \mathbf{r}}{\partial r \partial \theta} = \frac{\partial}{\partial r} [\hat{\mathbf{i}}(-r \sin \theta) + \hat{\mathbf{j}}(r \cos \theta)] \\ &= \hat{\mathbf{i}}(-\sin \theta) + \hat{\mathbf{j}}(\cos \theta) \\ &= [\alpha_r(\cos \theta) + \alpha_\theta \left(-\frac{1}{r} \sin \theta \right) (-\sin \theta) \\ &\quad + [\alpha_r(\sin \theta) + \alpha_\theta \left(\frac{1}{r} \cos \theta \right)]](\cos \theta) \\ &= \alpha_\theta \left(\frac{1}{r} \right) \\ \frac{\partial}{\partial \theta} \alpha_r &= \frac{\partial^2 \mathbf{r}}{\partial \theta \partial r} = \frac{\partial^2 \mathbf{r}}{\partial r \partial \theta} = \frac{\partial}{\partial r} \alpha_\theta = \alpha_\theta \left(\frac{1}{r} \right) \\ \frac{\partial}{\partial \theta} \alpha_\theta &= \frac{\partial^2 \mathbf{r}}{\partial \theta^2} = \hat{\mathbf{i}}(-r \cos \theta) + \hat{\mathbf{j}}(-r \sin \theta) \\ &= [\alpha_r(\cos \theta) + \alpha_\theta \left(-\frac{1}{r} \sin \theta \right) (-r \cos \theta) \\ &\quad + [\alpha_r(\sin \theta) + \alpha_\theta \left(\frac{1}{r} \cos \theta \right)](-r \sin \theta)] \\ &= \alpha_r(-r)\end{aligned}$$

$\therefore$

$$\begin{aligned}d\alpha_r &= dr \otimes \frac{\partial}{\partial r} \alpha_r + d\theta \otimes \frac{\partial}{\partial \theta} \alpha_r = \alpha_\theta \left(\frac{1}{r} \right) d\theta \\ &= \lim_{\delta r \rightarrow 0} \delta \alpha_r = \sum_{m=1}^2 \sum_{h=1}^2 \alpha_m \left\{ \begin{matrix} m \\ h1 \end{matrix} \right\} dx^h \text{ (Corollary 3-5a)} \\ &= \alpha_r \left(\left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} dr + \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} d\theta \right) + \alpha_\theta \left(\left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} dr + \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} d\theta \right) \\ d\alpha_\theta &= dr \otimes \frac{\partial}{\partial r} \alpha_\theta + d\theta \otimes \frac{\partial}{\partial \theta} \alpha_\theta = \alpha_\theta \left(\frac{1}{r} \right) dr + \alpha_r(-r) d\theta \\ &= \lim_{\delta r \rightarrow 0} \delta \alpha_\theta = \sum_{m=1}^2 \sum_{h=1}^2 \alpha_m \left\{ \begin{matrix} m \\ h2 \end{matrix} \right\} dx^h \\ &= \alpha_r \left(\left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} dr + \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} d\theta \right) + \alpha_\theta \left(\left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} dr + \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} d\theta \right).\end{aligned}$$

So:

$$\begin{cases} 1 \\ 11 \end{cases} = 0, \begin{cases} 1 \\ 12 \end{cases} = 0, \begin{cases} 1 \\ 21 \end{cases} = 0, \begin{cases} 1 \\ 22 \end{cases} = -r \\ \begin{cases} 2 \\ 11 \end{cases} = 0, \begin{cases} 2 \\ 12 \end{cases} = \frac{1}{r}, \begin{cases} 2 \\ 21 \end{cases} = \frac{1}{r}, \begin{cases} 2 \\ 22 \end{cases} = 0$$

i.e.: α_r and α_θ are not constants.

Setting:

$$\hat{\mathbf{i}} \equiv (1, 0), \hat{\mathbf{j}} \equiv (0, 1)$$

then:

$$\alpha_r = \frac{\partial \mathbf{r}}{\partial r} = \hat{\mathbf{i}}(\cos \theta) + \hat{\mathbf{j}}(\sin \theta) = (\cos \theta, \sin \theta)$$

$$\alpha_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \hat{\mathbf{i}}(-r \sin \theta) + \hat{\mathbf{j}}(r \cos \theta) = (-r \sin \theta, r \cos \theta).$$

Therefore

$$\begin{aligned} \mathbf{r} &= \hat{\mathbf{i}}x + \hat{\mathbf{j}}y = (x, y) = (1, 0) \otimes x + (0, 1) \otimes y. \\ &= \alpha_r C_r^r + \alpha_\theta C_r^\theta = (\cos \theta, \sin \theta) \otimes C_r^r + (-r \sin \theta, r \cos \theta) \otimes C_r^\theta. \end{aligned}$$

and

$$\begin{aligned} \mathbf{r} &= \hat{\mathbf{i}}x + \hat{\mathbf{j}}y = (x, y) = (r \cos \theta, r \sin \theta) \\ &= (C_r^r \cos \theta - C_r^\theta r \sin \theta, C_r^r \sin \theta + C_r^\theta r \cos \theta). \end{aligned}$$

It follows that:

$$\begin{aligned} C_r^r \cos \theta - C_r^\theta r \sin \theta &= r \cos \theta \\ C_r^r \sin \theta + C_r^\theta r \cos \theta &= r \sin \theta, \end{aligned}$$

Hence:

$$\begin{aligned} C_r^r &= \frac{\begin{vmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{vmatrix}}{\begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}} = \frac{r^2 \cos^2 \theta + r^2 \sin^2 \theta}{r \cos^2 \theta + r \sin^2 \theta} = r \\ C_r^\theta &= \frac{\begin{vmatrix} \cos \theta & r \cos \theta \\ \sin \theta & r \sin \theta \end{vmatrix}}{\begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}} = \frac{r \sin \theta \cos \theta - r \cos \theta \sin \theta}{r \cos^2 \theta + r \sin^2 \theta} = 0. \end{aligned}$$

$$\text{Here, the equations: } \frac{\partial C_r^m}{\partial x^h} + \sum_{i=1}^n C_r^i \left\{ \begin{matrix} m \\ hi \end{matrix} \right\} = \delta_h^m,$$

as obtained in Theorem 3-8, are easily verified:

$$\begin{aligned} \frac{\partial C_r^r}{\partial r} + C_r^r \left\{ \begin{matrix} 1 \\ 11 \end{matrix} \right\} + C_r^\theta \left\{ \begin{matrix} 1 \\ 12 \end{matrix} \right\} &= \delta_1^1 = 1 \\ &= 1 + (r)(0) + (0)(0) = 1 \quad \checkmark \end{aligned}$$

$$\begin{aligned}
\frac{\partial C_r^r}{\partial \theta} + C_r^r \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} + C_r^\theta \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} &= \delta_2^1 = 0 \\
&= 0 + (r)(0) + (0)(-r) = 0 \quad \checkmark \\
\frac{\partial C_r^\theta}{\partial r} + C_r^r \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} + C_r^\theta \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} &= \delta_1^2 = 0 \\
&= 0 + (r)(0) + (0)\left(\frac{1}{r}\right) = 0 \quad \checkmark \\
\frac{\partial C_r^\theta}{\partial \theta} + C_r^r \left\{ \begin{matrix} 2 \\ 21 \end{matrix} \right\} + C_r^\theta \left\{ \begin{matrix} 2 \\ 22 \end{matrix} \right\} &= \delta_2^2 = 1 \\
&= 0 + (r)\left(\frac{1}{r}\right) + (0)(0) = 1 \quad \checkmark.
\end{aligned}$$

The above discussion implies that for at least some class of atlases, the above theorems will hold true. Certainly, so, if the atlas is also a vector space, but this goes outside the scope of this work.

Theorem 3-10: For n -dimensional vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$ with continuous

contravariant basis: $(\hat{\mathbf{u}}_i)_n \ni \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n$,

$(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$, and

vector function $\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\not{f}(x^i)_n)_n = \mathbf{f}(\mathbf{x}) :$

$$d\mathbf{f} = \sum_{h=1}^n \left(\sum_{j=1}^n \hat{\mathbf{u}}_j \otimes \left[\frac{\partial \not{f}}{\partial x^h} + \sum_{m=1}^n \not{f}^m \left\{ \begin{matrix} j \\ hm \end{matrix} \right\} \right] \right) \otimes dx^h$$

Proof:

For any vector function f (contravariant representation):

$$\begin{aligned}
\delta \mathbf{f} &= \sum_{h=1}^n \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left(\frac{\partial \not{f}^m}{\partial x^h} + \left[\sum_{i=1}^n \not{f}^i \sum_{j=1}^n L_j^m \frac{\partial}{\partial x^h} \Gamma_i^j \right] \right) \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h \\
&\quad (\forall \mathbf{x}, \xi \in \mathcal{R}^n \ni |\xi^i - x^i| < \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n) \\
&= \sum_{h=1}^n \left(\sum_{m=1}^n \hat{\mathbf{u}}_m \otimes \left[\frac{\partial \not{f}^m}{\partial x^h} + \sum_{i=1}^n \not{f}^i \left\{ \begin{matrix} m \\ hi \end{matrix} \right\} \right] \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h \\
&= \sum_{h=1}^n \left(\sum_{j=1}^n \hat{\mathbf{u}}_j \otimes \left[\frac{\partial \not{f}}{\partial x^h} + \sum_{k=1}^n \not{f}^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right] \right) \Big|_{\mathbf{x}=\xi} \otimes \delta x^h
\end{aligned}$$

$$d\mathbf{f} = \lim_{\delta \mathbf{x} \rightarrow 0} [(\mathbf{f} + \delta \mathbf{f}) - \mathbf{f}] = \lim_{\delta \mathbf{x} \rightarrow 0} (\delta \mathbf{f}) ,$$

$$d\mathbf{x} = \lim_{\delta \mathbf{x} \rightarrow 0} [(\mathbf{x} + \delta \mathbf{x}) - \mathbf{x}] = \lim_{\delta \mathbf{x} \rightarrow 0} (\delta \mathbf{x}) .$$

So:

$$d\mathbf{f} = \sum_{h=1}^n \lim_{\delta \mathbf{x} \rightarrow 0} \left(\sum_{j=1}^n \hat{\mathbf{u}}_j \otimes \left[\frac{\partial \not{f}}{\partial x^h} + \sum_{k=1}^n \not{f}^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right] \right) \Big|_{\mathbf{x}=\xi} \otimes \left(\lim_{\delta \mathbf{x} \rightarrow 0} \delta x^h \right) ,$$

and, as $\delta \mathbf{x} \rightarrow 0$, $\xi \rightarrow \mathbf{x}$, so:

$$= \sum_{h=1}^n \left(\sum_{j=1}^n \hat{u}_j \otimes \left[\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n f^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right] \right) \otimes dx^h \quad \square.$$

A. The Covariant Derivative

Definition 3-8: The Covariant Derivative of a vector function

$$\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(f^i(x^i)_n)_n = \mathbf{f}(\mathbf{x}),$$

in Contravariant Representation, is:

$$f^j_{,h} \equiv \frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(f^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right)$$

And, so (contravariant representation):

Theorem 3-11: For n -dimensional vector space $(\mathcal{F}, \otimes, \mathbf{S})_n$ with continuous

contravariant basis: $(\hat{u}_i)_n \ni \hat{u}_j = \hat{u}_j(x^i)_n$,

$(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$, and

vector function $\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(f^i(x^i)_n)_n = \mathbf{f}(\mathbf{x})$:

$$d\mathbf{f} = \sum_{j=1}^n \left(\hat{u}_j \otimes \sum_{h=1}^n f^j_{,h} dx^h \right)$$

Proof:

$$\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n :$$

$$d\mathbf{f} = \sum_{j=1}^n \left(\sum_{h=1}^n \left[\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(f^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right) \right] dx^h \otimes \hat{u}_j \right)$$

$\therefore$

$$d\mathbf{f} = \sum_{j=1}^n \left(\hat{u}_j \otimes \sum_{h=1}^n f^j_{,h} dx^h \right) \quad \square.$$

The form of the results that follow are significantly simpler in contravariant representation, so what follows will be analyzed using contravariant representation, only.

B. β_{ij}^m Product Coeffs for a Finite Dimensional $\mathcal{R}$ -algebra

Theorem 3-12: For n -dimensional $\mathcal{F}$ -algebra $((\mathcal{F}, \otimes, \mathbf{S})_n, \circ)$

with contravariant basis:

$$(\hat{\mathbf{u}}_i)_n \ni \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x_i)_n, \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n) :$$

$\exists \beta_{ij}^m \in \mathcal{F}$, called **product coeffs**, $\ni$

$$\hat{\mathbf{u}}_i \circ \hat{\mathbf{u}}_j = \sum_{m=1}^n \beta_{ij}^m \otimes \hat{\mathbf{u}}_m, \quad (\forall i, j \in \mathcal{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq n)$$

Proof:

Because this is an $\mathcal{F}$ -algebra $\circ$ is a closed binary operation, and, since $(\hat{\mathbf{u}}_i)_n$ is a basis:

$$\hat{\mathbf{u}}_i \circ \hat{\mathbf{u}}_j \in \mathbf{S} \Rightarrow \exists \beta_{ij}^m \in \mathcal{F} \ni \hat{\mathbf{u}}_i \circ \hat{\mathbf{u}}_j = \sum_{m=1}^n \beta_{ij}^m \otimes \hat{\mathbf{u}}_m \quad \square.$$

Now, the machinery is in place to produce some fascinating results.

C. Analyticity on Finite Dimensional Associative $\mathcal{R}$ -algebra

Definition 3-9: An associative $\mathcal{F}$ -algebra $((\mathcal{F}, \otimes, \mathbf{S})_n, \circ)^A$ is an $\mathcal{F}$ -algebra $((\mathcal{F}, \otimes, \mathbf{S})_n, \circ)$ such that:

$$\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{S}, \quad (\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z} = \mathbf{x} \circ (\mathbf{y} \circ \mathbf{z}).$$

1. Generalized Right-Handed Analyticity Equation

Right-Handed: $((\delta \mathbf{r})^{-1}$ on the Right)

Theorem 3-13a: In an n -dimensional associative $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, \mathbf{S})_n, \circ)^A$, with

product coeffs β_{ij}^m , contravariant basis: $(\hat{\mathbf{u}}_i)_n \ni$

$$\hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n, \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n), \text{ and vector function}$$

$$\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\beta(x^i)_n)_n = \mathbf{f}(\mathbf{x});$$

if: $\exists \mathbf{f}_1 \in \mathcal{R}^n \ni \mathbf{f}_1 = \delta \mathbf{f} \circ (\delta \mathbf{r})^{-1}$, is continuous for arbitrary $\delta \mathbf{r}$,

$$\text{then: } \sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$$

Proof:

$$\mathbf{f}_1 \circ \delta \mathbf{r} = (\delta \mathbf{f} \circ (\delta \mathbf{r})^{-1}) \circ \delta \mathbf{r} = \delta \mathbf{f} \circ ((\delta \mathbf{r})^{-1} \circ \delta \mathbf{r}) = \delta \mathbf{f}.$$

So:

$$\mathbf{f}_1 \circ \sum_{h=1}^n (\hat{\mathbf{u}}_h \delta x^h) = \sum_{j=1}^n \left(\sum_{h=1}^n \left(\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(\sum_{m=1}^n \left(f^k L_m^j \left(\frac{\partial}{\partial x^h} \Gamma_k^m \Big|_{\mathbf{x}=\xi} \right) \right) \right) \right) \delta x^h \otimes \hat{\mathbf{u}}_j \right),$$

for arbitrary δx^h , so, $\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n$:

$$\mathbf{f}_1 \circ \hat{\mathbf{u}}_h = \sum_{j=1}^n \left(\sum_{h=1}^n \left(\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(\sum_{m=1}^n \left(f^k L_m^j \left(\frac{\partial}{\partial x^h} \Gamma_k^m \Big|_{\mathbf{x}=\xi} \right) \right) \right) \right) \otimes \hat{\mathbf{u}}_j \right),$$

and, so, in the limit (A5-1 & 2), as $\mathbf{x} \rightarrow \xi$:

$$\mathbf{f}_1 \circ \hat{\mathbf{u}}_h = \sum_{j=1}^n \left(\left(\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(f^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right) \right) \otimes \hat{\mathbf{u}}_j \right),$$

(using Definition 3-7a),

or:

$$\mathbf{f}_1 \circ \hat{\mathbf{u}}_h = \sum_{j=1}^n (f^j_{;h} \otimes \hat{\mathbf{u}}_j).$$

Now, $\forall h, k \in \mathcal{Z}^+ \ni 1 \leq h, k \leq n$:

$$(\mathbf{f}_1 \circ \hat{\mathbf{u}}_h) \circ \hat{\mathbf{u}}_k = \mathbf{f}_1 \circ (\hat{\mathbf{u}}_h \circ \hat{\mathbf{u}}_k).$$

$\therefore$

$$\begin{aligned} \left(\sum_{j=1}^n f^j_{;h} \otimes \hat{\mathbf{u}}_j \right) \circ \hat{\mathbf{u}}_k &= \mathbf{f}_1 \circ \left(\sum_{m=1}^n (\beta_{hk}^m \otimes \hat{\mathbf{u}}_m) \right) \\ &= \sum_{m=1}^n (\beta_{hk}^m \otimes (\mathbf{f}_1 \circ \hat{\mathbf{u}}_m)) \\ &= \sum_{m=1}^n \left(\beta_{hk}^m \otimes \left(\sum_{j=1}^n f^j_{;m} \otimes \hat{\mathbf{u}}_j \right) \right) \\ &= \sum_{j=1}^n \left(\sum_{m=1}^n (\beta_{hk}^m f^j_{;m} \otimes \hat{\mathbf{u}}_j) \right) \quad (*1) \end{aligned}$$

$$\begin{aligned} \left(\sum_{j=1}^n f^j_{;h} \otimes \hat{\mathbf{u}}_j \right) \circ \hat{\mathbf{u}}_k &= \sum_{j=1}^n (f^j_{;h} \otimes (\hat{\mathbf{u}}_j \circ \hat{\mathbf{u}}_k)) \\ &= \sum_{j=1}^n \left(f^j_{;h} \otimes \left(\sum_{m=1}^n \beta_{jk}^m \otimes \hat{\mathbf{u}}_m \right) \right) \\ &= \sum_{j=1}^n \left(\sum_{m=1}^n (f^j_{;h} \beta_{jk}^m \otimes \hat{\mathbf{u}}_m) \right) \\ &= \sum_{m=1}^n \left(\sum_{j=1}^n (f^m_{;h} \beta_{mk}^j \otimes \hat{\mathbf{u}}_j) \right) \\ &= \sum_{j=1}^n \left(\sum_{m=1}^n (f^m_{;h} \beta_{mk}^j \otimes \hat{\mathbf{u}}_j) \right) \quad (*2) \end{aligned}$$

$\therefore$

$$\sum_{j=1}^n \left(\sum_{m=1}^n ((\beta_{hk}^m f^j_{;m} - \beta_{mk}^j f^m_{;h}) \otimes \hat{\mathbf{u}}_j) \right) = 0$$

and, so:

$$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$$

□.

2. Generalized Left-Handed Analyticity Equation

Left-Handed: $((\delta \mathbf{r})^{-1}$ on the Left)

Theorem 3-13b: In an n -dimensional associative $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, \mathbf{S})_n, \circ)^A$, with product coeffs β_{ij}^m , contravariant basis: $(\hat{\mathbf{u}}_i)_n \ni \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n$, $(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$, and vector function $\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(f^j(x^i)_n)_n = \mathbf{f}(\mathbf{x})$;
if: $\exists \mathbf{f}_2 \in \mathcal{R}^n \ni \mathbf{f}_2 = \lim_{dr \rightarrow 0} (dr)^{-1} \circ d\mathbf{f}$, is continuous,
then: $\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{hm}^j f_{,k}^m) = 0$

Proof:

If $\exists \mathbf{f}_2 \in \mathbf{S} \ni \mathbf{f}_2 = \lim_{r \rightarrow 0} (dr)^{-1} \circ d\mathbf{f}$

then

$$\lim_{r \rightarrow 0} (dr \circ \mathbf{f}_2) = \lim_{r \rightarrow 0} \left(dr \circ \lim_{r \rightarrow 0} ((dr)^{-1} \circ d\mathbf{f}) \right) = \lim_{r \rightarrow 0} ((dr \circ (dr)^{-1}) \circ d\mathbf{f}) = \lim_{r \rightarrow 0} d\mathbf{f}$$

$\therefore$

$$\begin{aligned} \lim_{r \rightarrow 0} \sum_{h=1}^n (\hat{\mathbf{u}}_h dx^h) \circ \mathbf{f}_2 &= \\ &= \lim_{r \rightarrow 0} \sum_{j=1}^n \left(\sum_{h=1}^n \left(\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(f^k L_m^j \left(\frac{\partial}{\partial x^h} \Gamma_k^m \Big|_{\mathbf{x}=\xi} \right) \right) \right) dx^h \otimes \hat{\mathbf{u}}_j \right) \\ &= \lim_{r \rightarrow 0} \sum_{j=1}^n \left(\sum_{h=1}^n \left(\frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(f^k \left\{ \begin{smallmatrix} j \\ hk \end{smallmatrix} \right\} \right) \right) dx^h \otimes \hat{\mathbf{u}}_j \right) \end{aligned}$$

for arbitrary dx^h , which are linearly independent, so:

$$\hat{\mathbf{u}}_h \circ \mathbf{f}_2 = \sum_{j=1}^n (f_{,h}^j \otimes \hat{\mathbf{u}}_j) .$$

And, so, in the limit, as $\mathbf{x} \rightarrow \xi$:

$$\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq n :$$

$$\hat{\mathbf{u}}_h \circ \mathbf{f}_2 = \sum_{j=1}^n (f_{,h}^j \otimes \hat{\mathbf{u}}_j)$$

Now, $\forall h, k \in \mathcal{Z}^+ \ni 1 \leq h, k \leq n$:

$$\hat{\mathbf{u}}_k \circ (\hat{\mathbf{u}}_h \circ \mathbf{f}_2) = (\hat{\mathbf{u}}_k \circ \hat{\mathbf{u}}_h) \circ \mathbf{f}_2 .$$

$\therefore$

$$\hat{\mathbf{u}}_k \circ \left(\sum_{j=1}^n f_{,h}^j \otimes \hat{\mathbf{u}}_j \right) = \left(\sum_{m=1}^n (\beta_{kh}^m \otimes \hat{\mathbf{u}}_m) \right) \circ \mathbf{f}_2$$

$$\begin{aligned}
&= \sum_{m=1}^n (\beta_{kh}^m \otimes (\hat{\mathbf{u}}_m \circ \mathbf{f}_2)) \\
&= \sum_{m=1}^n \left(\beta_{kh}^m \left(\sum_{j=1}^n f_{:,m}^j \otimes \hat{\mathbf{u}}_j \right) \right) \\
&= \sum_{j=1}^n \left(\sum_{m=1}^n (\beta_{kh}^m f_{:,m}^j \otimes \hat{\mathbf{u}}_j) \right) \quad (*) \\
\hat{\mathbf{u}}_k \circ \left(\sum_{j=1}^n f_{:,h}^j \otimes \hat{\mathbf{u}}_j \right) &= \sum_{j=1}^n (f_{:,h}^j \otimes (\hat{\mathbf{u}}_k \circ \hat{\mathbf{u}}_j)) \\
&= \sum_{j=1}^n \left(f_{:,h}^j \otimes \left(\sum_{m=1}^n \beta_{kj}^m \otimes \hat{\mathbf{u}}_m \right) \right) \\
&= \sum_{j=1}^n \left(\sum_{m=1}^n (f_{:,h}^j \beta_{kj}^m \otimes \hat{\mathbf{u}}_m) \right) \\
&= \sum_{m=1}^n \left(\sum_{j=1}^n (f_{:,h}^m \beta_{km}^j \otimes \hat{\mathbf{u}}_j) \right) \\
&= \sum_{j=1}^n \left(\sum_{m=1}^n (f_{:,h}^m \beta_{km}^j \otimes \hat{\mathbf{u}}_j) \right) \quad (*)
\end{aligned}$$

Therefore,

$$\sum_{j=1}^n \left(\sum_{m=1}^n ((\beta_{kh}^m f_{:,m}^j - \beta_{km}^j f_{:,h}^m) \otimes \hat{\mathbf{u}}_j) \right) = \mathbf{0} ,$$

and, so:

$$\sum_{m=1}^n (\beta_{hk}^m f_{:,m}^j - \beta_{hm}^j f_{:,k}^m) = 0 \quad \square.$$

D. Contour Integrals

1. Generalization of the Cauchy Integral Theorem for 'smooth' functions within Commutative Associative $\mathcal{R}$ -algebras

To evaluate contour integrals in n -dimensional vector spaces, definitions of contours are required. Extending from complex function theory is the most obvious path to take.

Definition 3-10: For n -dimensional vector space: $(\mathcal{R}, \oplus, \mathcal{S})_n$, an **arc** is a parametrized vector function:

$$\gamma : [a, b] \subset \mathcal{R} \mapsto \mathcal{R}^n, \gamma = \gamma(\gamma^j(x^i(t)))_n = \gamma(t) ;$$

a **smooth arc** is an arc, such that:

- (i) $\gamma(t)$ is continuous ,
- (ii) $\frac{d}{dt}\gamma(t) \equiv \gamma'(t)$ is continuous $\forall t \in [a, b]$,
- (iii) $\gamma'(t)$ never vanishes $\forall t \in [a, b]$, and
- (iv) $\gamma(t)$ is injective (A1-9);

a **smooth closed curve** is an smooth arc, such that:

$$\gamma(b) = \gamma(a) , \gamma'(b) = \gamma'(a) ;$$

a **smooth curve** is a smooth arc, or smooth closed curve.

(see: [55])

In order to quantify direction, the following definition is constructed:

Definition 3-11: For n -dimensional vector space: $(\mathcal{R}, \oplus, \mathcal{S})_n$, a

directed smooth curve is a smooth curve:

$$\gamma : [a, b] \subset \mathcal{R} \mapsto \mathcal{R}^n, \gamma = \gamma(\gamma^j(x^i(t)))_n = \gamma(t) ,$$

$$\gamma^j : [a, b] \subset \mathcal{R} \mapsto [A^j, B^j] \subset \mathcal{R} , \quad (\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n)$$

from: $A \equiv (A^j)_n$, to: $B \equiv (B^j)_n$, such that:

$$\mathcal{D}(\gamma) \equiv (-1)^{(1-\delta_A^{f(a)})} \cdot (\delta_A^{f(a)} + \delta_A^{f(b)}) \cdot (\delta_B^{f(a)} + \delta_B^{f(b)}) \neq 0 .$$

Note:

$$\mathcal{D}(\gamma) = \begin{cases} 1 & , f(a) = A \text{ and } f(b) = B \\ -1 & , f(a) = B \text{ and } f(b) = A \\ 0 & , \text{otherwise} \end{cases}$$

Definition 3-12: For n -dimensional vector space: $(\mathcal{R}, \otimes, S)_n$, a **contour** is a single point, or a finite sequence of directed smooth curves $\Gamma \equiv (\gamma_i)_m$:

$$\begin{aligned} \gamma_h : [a_h, b_h] \subset \mathcal{R} &\mapsto \mathcal{R}_{n \times n}, \gamma_h = \gamma_h(\gamma^j(x^i(t)))_n = \gamma_h(t), \\ &(\forall h \in \mathbb{Z}^+ \ni 1 \leq h \leq m) \\ \gamma_h^j : [a_h, b_h] \subset \mathcal{R} &\mapsto [A_h^j, B_h^j] \subset \mathcal{R}, \\ &(\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n), (\forall h \in \mathbb{Z}^+ \ni 1 \leq h \leq m) \\ \text{from: } A_h &\equiv (A_h^j)_n, \text{ to: } B_h \equiv (B_h^j)_n, \\ \text{such that:} \\ A_h &= B_{h-1}, (\forall h \in \mathbb{Z}^+ \ni 2 \leq h \leq m). \end{aligned}$$

And so, the contour integral may be defined within an n -dimensional $\mathcal{R}$ -algebra, as follows:

Definition 3-13: For n -dimensional $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S)_n, \circ)$,

contravariant basis: $(\hat{u}_i)_n \ni \hat{u}_j = \hat{u}_j(x^i)_n, (\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n)$,

and vector function $f : \mathcal{R}^n \mapsto \mathcal{R}^n, f = f(\beta^j(x^i)_n) = f(x)$;

the contour integral of f , along contour $\Gamma \equiv (\gamma_i)_m$:

$$\begin{aligned} \gamma_h : [a_h, b_h] \subset \mathcal{R} &\mapsto \mathcal{R}^n, \gamma_h = \gamma_h(\beta^j(x^i(t)))_n = \gamma_h(t), \\ &(\forall h \in \mathbb{Z}^+ \ni 1 \leq h \leq m) \\ \gamma_h^j : [a_h, b_h] \subset \mathcal{R} &\mapsto [A_h^j, B_h^j] \subset \mathcal{R}, \\ &(\forall j \in \mathbb{Z}^+ \ni 1 \leq j \leq n), (\forall h \in \mathbb{Z}^+ \ni 1 \leq h \leq m) \end{aligned}$$

is given by:

$$\int_{\Gamma} f \circ dr.$$

The contour integral just defined may be evaluated by evaluating ordinary integrals of real-valued functions (A1-11) using the following theorem.

Theorem 3-14: For n -dimensional $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, \mathbf{S})_n, \circ)$, with product coeffs β_{ij}^m ,

contravariant basis: $(\hat{\mathbf{u}}_i)_n \ni \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n$, $(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$,

and vector function $\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n$, $\mathbf{f} = \mathbf{f}(\hat{\mathbf{f}}(x^i)_n)_n = \mathbf{f}(\mathbf{x})$;

the contour integral of $\mathbf{f}$, along contour $\Gamma \equiv (\gamma_i)_m$:

$$\gamma_h : [a_h, b_h] \subset \mathcal{R} \mapsto \mathcal{R}^n, \gamma_h = \gamma_h(\hat{\mathbf{f}}(x^i(t))_n)_n = \gamma_h(t),$$

$$(\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq m)$$

$$\gamma_h^j : [a_h, b_h] \subset \mathcal{R} \mapsto [A_h^j, B_h^j] \subset \mathcal{R},$$

$$(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n), (\forall h \in \mathcal{Z}^+ \ni 1 \leq h \leq m)$$

may be found from:

$$\int_{\Gamma} \mathbf{f} \circ d\mathbf{r} = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \left(\int_{\Gamma} \beta_{ij}^h \hat{\mathbf{f}}^i \Gamma_h^k dx^j \right) \right) \right] \right).$$

Proof:

$$\begin{aligned} \int_{\Gamma} \mathbf{f} \circ d\mathbf{r} &= \int_{\Gamma} \left(\sum_{i=1}^n \hat{\mathbf{f}}^i \otimes \hat{\mathbf{u}}_i \right) \circ \left(\sum_{j=1}^n \hat{\mathbf{u}}_j \otimes dx^j \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(\int_{\Gamma} (\hat{\mathbf{f}}^i \otimes (\hat{\mathbf{u}}_i \circ \hat{\mathbf{u}}_j)) \otimes dx^j \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(\int_{\Gamma} \left(\hat{\mathbf{f}}^i \otimes \left[\sum_{m=1}^n (\beta_{ij}^m \otimes \hat{\mathbf{u}}_m) \right] \right) \otimes dx^j \right) \\ &= \sum_{m=1}^n \int_{\Gamma} \left(\hat{\mathbf{u}}_m \otimes \sum_{i=1}^n \sum_{j=1}^n (\beta_{ij}^m \hat{\mathbf{f}}^i dx^j) \right) \\ &= \sum_{m=1}^n \int_{\Gamma} \left(\left(\sum_{k=1}^n (\Gamma_m^k \otimes \mathbf{u}_k) \right) \otimes \sum_{i=1}^n \sum_{j=1}^n (\beta_{ij}^m \hat{\mathbf{f}}^i dx^j) \right) \\ &= \sum_{k=1}^n \mathbf{u}_k \otimes \left(\sum_{m=1}^n \int_{\Gamma} \left(\Gamma_m^k \sum_{i=1}^n \sum_{j=1}^n (\beta_{ij}^m \hat{\mathbf{f}}^i dx^j) \right) \right), \text{ for constant } \mathbf{u}_k \\ &= \sum_{k=1}^n \left(\sum_{h=1}^n (L_k^h \otimes \hat{\mathbf{u}}_h) \right) \otimes \left(\sum_{m=1}^n \int_{\Gamma} \left(\Gamma_m^k \sum_{i=1}^n \sum_{j=1}^n (\beta_{ij}^m \hat{\mathbf{f}}^i dx^j) \right) \right) \\ &= \sum_{k=1}^n \left(\sum_{m=1}^n (L_k^m \otimes \hat{\mathbf{u}}_m) \right) \otimes \left(\sum_{h=1}^n \int_{\Gamma} \left(\Gamma_h^k \sum_{i=1}^n \sum_{j=1}^n (\beta_{ij}^h \hat{\mathbf{f}}^i dx^j) \right) \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \left(\int_{\Gamma} \beta_{ij}^h \hat{\mathbf{f}}^i \Gamma_h^k dx^j \right) \right) \right] \right) \quad \square. \end{aligned}$$

Green's Theorem in the plane:

if R is a closed region of xy -plane bounded by a single closed curve ∂R , and if $f(x, y)$ and $g(x, y)$ are continuous functions of x and y having continuous derivatives in $\mathcal{R}$, then:

$$\oint_{\partial R} f(x,y)dx + g(x,y)dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy .$$

The analogue of this, in an $\mathcal{R}$ -algebra, is the following theorem.

Theorem 3-15: For n -dimensional $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, \mathbf{S})_n, \circ)$ with product coeffs β_{ij}^m ,
 contravariant basis: $(\hat{\mathbf{u}}_i)_n \ni \hat{\mathbf{u}}_j = \hat{\mathbf{u}}_j(x^i)_n$, $(\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n)$,
 and vector function $\mathbf{f} : \mathcal{R}^n \mapsto \mathcal{R}^n$, $\mathbf{f} = \mathbf{f}(f(x^i)_n)_n = \mathbf{f}(\mathbf{x})$,
 continuously differentiable on closed region $\mathbf{R}$ in $\mathcal{R}^n$ bounded by
 closed contour $\partial \mathbf{R}$:

$$\oint_{\partial \mathbf{R}} \mathbf{f} \circ d\mathbf{r} = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \sum_{r=1}^{s-1} \sum_{s=1}^n \left\{ \iint_{\mathbf{R}_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ik}^h f^i \Gamma_h^k) - \frac{\partial}{\partial x^r} (\beta_{is}^h f^i \Gamma_h^k) \right] dx^r dx^s \right\} \right) \right] \right) \right]$$

Proof:

By Theorem 3-14:

$$\oint_{\partial \mathbf{R}} \mathbf{f} \circ d\mathbf{r} = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \left(\oint_{\partial \mathbf{R}} \beta_{ij}^h f^i \Gamma_h^k dx^j \right) \right) \right] \right]$$

The number of coordinate planes in the n -dimensional vector space is:

$$C_{,2}^n = \frac{n!}{(n-2)!2!} .$$

Note:

$$\sum_{s=1}^n \sum_{r \geq 1}^{s-1} \varphi_{rs} = \sum_{s=1}^{n-1} \sum_{r \geq 1}^{s-1} \varphi_{rs} + \sum_{r \geq 1}^n \varphi_{rn}$$

$$\text{For } n = 2 : \sum_{s=1}^2 \sum_{r \geq 1}^{s-1} \varphi_{rs} = \varphi_{12}$$

$$\text{1-term} = C_{,2}^2$$

$$\begin{aligned} \text{For } n = 3 : \sum_{s=1}^3 \sum_{r \geq 1}^{s-1} \varphi_{rs} &= \varphi_{12} + [\varphi_{13} + \varphi_{23}] \\ &= \sum_{s=1}^2 \sum_{r \geq 1}^{s-1} \varphi_{rs} + \sum_{r \geq 1}^2 \varphi_{r3} \end{aligned}$$

$$\text{3-terms} = C_{,2}^3$$

$$\text{For } n = 4 : \sum_{s=1}^4 \sum_{r \geq 1}^{s-1} \varphi_{rs} = \varphi_{12} + [\varphi_{13} + \varphi_{23}] + [\varphi_{14} + \varphi_{24} + \varphi_{34}]$$

$$= \sum_{s=1}^3 \sum_{r \geq 1}^{s-1} \varphi_{rs} + \sum_{r \geq 1}^3 \varphi_{r4}$$

$$\text{6-terms} = C_{;2}^3$$

So, if $\sum_{s=1}^N \sum_{r \geq 1}^{s-1} \varphi_{rs}$ has $C_{;2}^N$ terms, then

$$\sum_{s=1}^{N+1} \sum_{r \geq 1}^{s-1} \varphi_{rs} \text{ has:}$$

$$\begin{aligned} C_{;2}^N + N &= \frac{N(N-1)}{2} + N = \frac{N^2 - N + 2N}{2} \\ &= \frac{N(N+1)}{2} = C_{;2}^{N+1} \text{ terms.} \end{aligned}$$

Hence, this covers all terms for all coordinate planes.

And, so, expanding all the terms of $\oint_{\partial R} \beta_{ij}^h f^j \Gamma_h^k dx^j$

using Green's Theorem:

$$\oint_{\partial R} f \circ dr = \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \sum_{s=1}^{s-1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ik}^h f^j \Gamma_h^k) - \frac{\partial}{\partial x^r} (\beta_{is}^h f^j \Gamma_h^k) \right] dx^r dx^s \right\} \right) \right] \right)$$

Definition 3-14: For n -dimensional $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S)_{n, \circ})$ with contravariant basis:

$$(\hat{u}_i)_n \ni \hat{u}_j = \hat{u}_j(x_i)_n, \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n):$$

its product coeffs $\beta_{ij}^m \in \mathcal{R}$ are **symmetric**, whenever:

$$\beta_{ij}^m = \beta_{ji}^m, \quad (\forall i, j, m \in \mathcal{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq m \leq n)$$

Theorem 3-16: For n -dimensional $\mathcal{R}$ -algebra $((\mathcal{R}, \otimes, S)_{n, \circ})$

with constant symmetric product coeffs β_{ij}^m , and constant

contravariant basis: $(\hat{u}_i)_n \ni \hat{u}_j = \hat{u}_j(x^i)_n, \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq n),$

and vector function $f : \mathcal{R}^n \mapsto \mathcal{R}^n, f = f(f^j(x^i)_n)_n = f(\mathbf{x}),$

continuously differentiable on closed region R in $\mathcal{R}^n$ bounded by closed contour ∂R , such that:

$\lim_{dr \rightarrow 0} df \circ (dr)^{-1}$ and $\lim_{dr \rightarrow 0} (dr)^{-1} \circ df$ exist and are continuous on R :

$$\oint_{\partial R} f \circ dr = 0$$

Proof:

$$\oint_{\partial R} \mathbf{f} \circ d\mathbf{r} = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ir}^h f^k) - \frac{\partial}{\partial x^r} (\beta_{is}^h f^k) \right] dx^r dx^s \right\} \right) \right] \right) .$$

Because the $\hat{\mathbf{u}}_m$ are constant vectors, and, by Theorem 3-1:

$$\mathbf{u}_i = \sum_{j=1}^n L_i^j \otimes \hat{\mathbf{u}}_j, \quad \hat{\mathbf{u}}_i = \sum_{j=1}^n \Gamma_i^j \otimes \mathbf{u}_j ;$$

then: L_i^j and Γ_i^j are constants, so they pass through $\frac{\partial}{\partial x^s}$ and $\iint_{R_{rs}}$.

$\therefore$

$$\begin{aligned} \oint_{\partial R} \mathbf{f} \circ d\mathbf{r} &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{k=1}^n \sum_{h=1}^n L_k^m \Gamma_h^k \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ir}^h f^i) - \frac{\partial}{\partial x^r} (\beta_{is}^h f^i) \right] dx^r dx^s \right\} \right) \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\sum_{h=1}^n \delta_h^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ir}^h f^i) - \frac{\partial}{\partial x^r} (\beta_{is}^h f^i) \right] dx^r dx^s \right\} \right) \right] \right) \\ &= \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\left(\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ir}^m f^i) - \frac{\partial}{\partial x^r} (\beta_{is}^m f^i) \right] dx^r dx^s \right\} \right) \right] \right) . \end{aligned}$$

And, since β_{ir}^m are constants, then:

$$\oint_{\partial R} \mathbf{f} \circ d\mathbf{r} = \sum_{m=1}^n \left(\hat{\mathbf{u}}_m \otimes \left[\left(\sum_{i=1}^n \sum_{j=1}^n \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\beta_{ir}^m \frac{\partial f^i}{\partial x^s} - \beta_{is}^m \frac{\partial f^i}{\partial x^r} \right] dx^r dx^s \right\} \right) \right] \right) .$$

$$f_{,h}^j \equiv \frac{\partial f^j}{\partial x^h} + \sum_{k=1}^n \left(f^k \left\{ \begin{matrix} j \\ hk \end{matrix} \right\} \right) ,$$

But since $\hat{\mathbf{u}}_m$ are constant vectors: $\left\{ \begin{matrix} j \\ hk \end{matrix} \right\} = 0$,

$$\text{so: } f_{,h}^j \equiv \frac{\partial f^j}{\partial x^h} .$$

From continuity of: $\lim_{d\mathbf{r} \rightarrow 0} d\mathbf{f} \circ (d\mathbf{r})^{-1} \Rightarrow \sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$

$$\text{i.e.: } \sum_{m=1}^n \beta_{hk}^m \frac{\partial f^j}{\partial x^m} = \sum_{m=1}^n \beta_{mk}^j \frac{\partial f^m}{\partial x^h} \Rightarrow \sum_{i=1}^n \beta_{rs}^i \frac{\partial f^m}{\partial x^i} = \sum_{i=1}^n \beta_{is}^m \frac{\partial f^i}{\partial x^r} .$$

From continuity of: $\lim_{d\mathbf{r} \rightarrow 0} (d\mathbf{r})^{-1} \circ d\mathbf{f}$, by Theorem 3-13b:

$$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{hm}^j f_{,k}^m) = 0$$

$$\text{i.e.: } \sum_{m=1}^n \beta_{hk}^m \frac{\partial f^j}{\partial x^m} = \sum_{m=1}^n \beta_{hm}^j \frac{\partial f^m}{\partial x^k} \Rightarrow \sum_{i=1}^n \beta_{rs}^i \frac{\partial f^m}{\partial x^i} = \sum_{i=1}^n \beta_{ri}^m \frac{\partial f^i}{\partial x^s} .$$

So:

$$\sum_{i=1}^n \beta_{is}^m \frac{\partial f^i}{\partial x^r} = \sum_{i=1}^n \beta_{rs}^i \frac{\partial f^m}{\partial x^i} = \sum_{i=1}^n \beta_{ri}^m \frac{\partial f^i}{\partial x^s}$$

And, thus:

$$\begin{aligned} \oint_{\partial R} f \circ dr &= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\left(\sum_{j=1}^n \sum_{s=1}^n \sum_{r \geq 1}^{s-1} \left\{ \iint_{R,rs} \left[\sum_{i=1}^n \beta_{ir}^m \frac{\partial f^i}{\partial x^s} - \sum_{i=1}^n \beta_{ri}^m \frac{\partial f^i}{\partial x^s} \right] dx^r dx^s \right\} \right) \right] \right) \\ &= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\left(\sum_{j=1}^n \sum_{s=1}^n \sum_{r \geq 1}^{s-1} \left\{ \iint_{R,rs} \sum_{i=1}^n (\beta_{ir}^m - \beta_{ri}^m) \frac{\partial f^i}{\partial x^s} dx^r dx^s \right\} \right) \right] \right) . \end{aligned}$$

Therefore, if the product coeffs are symmetric:

$$\begin{aligned} \oint_{\partial R} f \circ dr &= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\left(\sum_{j=1}^n \sum_{s=1}^n \sum_{r \geq 1}^{s-1} \left\{ \iint_{R,rs} \sum_{i=1}^n (0) \frac{\partial f^i}{\partial x^s} dx^r dx^s \right\} \right) \right] \right) \\ &= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\left(\sum_{j=1}^n \sum_{s=1}^n \sum_{r \geq 1}^{s-1} \left\{ \iint_{R,rs} 0 dx^r dx^s \right\} \right) \right] \right) \\ &= \sum_{m=1}^n \left(\hat{u}_m \otimes \left[\left(\sum_{j=1}^n \sum_{s=1}^n \sum_{r \geq 1}^{s-1} \{0\} \right) \right] \right) = \sum_{m=1}^n \left(\hat{u}_m \otimes [0] \right) = 0 \quad \square. \end{aligned}$$

Now, consider examples of the preceeding in several $\mathcal{F}$ -algebras.

Example 3-2:

**Analysis on the Complex Field
($\mathcal{C}$ -algebra)**

$$u_1 \equiv 1 \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_2 \equiv i \text{ or } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The Right-Handed Analyticity Equations become:

(h,j,k)	$\sum_{m=1}^n (\beta_{hkf,m}^m f_{,m}^j - \beta_{mkf,h}^j f_{,h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hkf,m}^m f_{,m}^j - \beta_{mkf,h}^j f_{,h}^m) = 0$
$(1,1,1)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^1}{\partial x^1} = 0 \quad \checkmark$	$(2,1,1)$	$-\frac{\partial f^1}{\partial x^2} + \frac{\partial f^1}{\partial x^2} = 0 \quad \checkmark$
$(1,1,2)$	$\frac{\partial f^1}{\partial x^2} + \frac{\partial f^2}{\partial x^1} = 0$	$(2,1,2)$	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} = 0$
$(1,2,1)$	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^2}{\partial x^1} = 0 \quad \checkmark$	$(2,2,1)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^2}{\partial x^2} = 0 \quad \checkmark$
$(1,2,2)$	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} = 0$	$(2,2,2)$	$-\frac{\partial f^2}{\partial x^1} - \frac{\partial f^1}{\partial x^2} = 0$

These are known as the Cauchy-Riemann equations.

The Left-Handed Analyticity Equations become:

(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{;m}^j - \beta_{hm}^j f_{;k}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{;m}^j - \beta_{hm}^j f_{;k}^m) = 0$
$(1,1,1)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^1}{\partial x^1} = 0 \checkmark$	$(2,1,1)$	$\frac{\partial f^1}{\partial x^2} + \frac{\partial f^2}{\partial x^1} = 0$
$(1,1,2)$	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^1}{\partial x^2} = 0 \checkmark$	$(2,1,2)$	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} = 0$
$(1,2,1)$	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^2}{\partial x^1} = 0 \checkmark$	$(2,2,1)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^1}{\partial x^1} = 0$
$(1,2,2)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^2}{\partial x^2} = 0 \checkmark$	$(2,2,2)$	$-\frac{\partial f^2}{\partial x^1} - \frac{\partial f^1}{\partial x^2} = 0$

Again, these are known as the Cauchy-Riemann equations.

Surely, it would not be inappropriate to refer to Theorems 3-13 a & b as generalizations of the Cauchy-Riemann equations.

The reader may consult [38] for many references to generalizations of the Cauchy-Riemann equations, as well as [72] for one recent article.

Green's Theorem in the plane becomes:

$$\begin{aligned}
\oint_{\partial R} f \circ dr &= \sum_{m=1}^n (\hat{u}_m \otimes \left[\sum_{k=1}^n L_k^m \left(\sum_{i=1}^n \sum_{j=1}^n \sum_{h=1}^n \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ik}^h f^j \Gamma_h^k) \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{\partial}{\partial x^r} (\beta_{is}^h f^j \Gamma_h^k) \right] dx^r dx^s \right\} \right) \right] \right) \\
&= \sum_{m=1}^2 (\hat{u}_m \otimes \left[\left(\sum_{i=1}^2 \sum_{j=1}^2 \sum_{s=1}^{s-1} \sum_{r \geq 1} \left\{ \iint_{R_{rs}} \left[\frac{\partial}{\partial x^s} (\beta_{ik}^h f^j) - \frac{\partial}{\partial x^r} (\beta_{is}^h f^j) \right] dx^r dx^s \right\} \right) \right] \right) \\
&= \sum_{m=1}^2 (\hat{u}_m \otimes \left[\left(\sum_{i=1}^2 \sum_{j=1}^2 \left\{ \iint_{R_{12}} \left[\frac{\partial}{\partial x^2} (\beta_{i1}^m f^j) - \frac{\partial}{\partial x^1} (\beta_{i2}^m f^j) \right] dx^1 dx^2 \right\} \right) \right] \right) \\
&= \hat{u}_1 \otimes \left[\iint_{R_{12}} \sum_{i=1}^2 \sum_{j=1}^2 \left\{ \frac{\partial}{\partial x^2} (\beta_{i1}^1 f^j) - \frac{\partial}{\partial x^1} (\beta_{i2}^1 f^j) \right\} dx^1 dx^2 \right] \\
&\quad + \hat{u}_2 \otimes \left[\iint_{R_{12}} \sum_{i=1}^2 \sum_{j=1}^2 \left\{ \frac{\partial}{\partial x^2} (\beta_{i1}^2 f^j) - \frac{\partial}{\partial x^1} (\beta_{i2}^2 f^j) \right\} dx^1 dx^2 \right] \\
&= \hat{u}_1 \otimes \left[\iint_{R_{12}} \left\{ \frac{\partial f^1}{\partial x^2} + \frac{\partial f^2}{\partial x^1} \right\} dx^1 dx^2 \right] \\
&\quad + \hat{u}_2 \otimes \left[\iint_{R_{12}} \left\{ \frac{\partial f^2}{\partial x^2} - \frac{\partial f^1}{\partial x^1} \right\} dx^1 dx^2 \right] \\
&= \hat{u}_1 \otimes \left[\iint_{R_{12}} \{0\} dx^1 dx^2 \right] + \hat{u}_2 \otimes \left[\iint_{R_{12}} \{0\} dx^1 dx^2 \right] = 0 .
\end{aligned}$$

This is known as the Cauchy Integral Theorem

This is as expected, since the complex field is associative, commutative (so has symmetric β_{ij}^m), and has identity subspace type BOTH, so if $\lim_{dr \rightarrow 0} df \circ (dr)^{-1}$ exists and is continuous, then so is $\lim_{dr \rightarrow 0} (dr)^{-1} \circ df$; so Theorem 3-16 applies.

So, Theorem 3-16 is a generalization of the Cauchy Integral Theorem.

The reader is referred to [51] for a recent article containing another proof yielding the same results.

Example 3-3:

A Four Dimensional Associative Example:

$$\mathbf{u}_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u}_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{u}_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{u}_4 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{If } c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 = \mathbf{0}$$

then:

$$\begin{pmatrix} (c_1 + c_3) & (c_2 + c_4) \\ (c_2 - c_4) & (c_1 - c_3) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent

And, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ spans the set of all matrices:

$$\begin{pmatrix} (x_1 + x_3) & (x_2 + x_4) \\ (x_2 - x_4) & (x_1 - x_3) \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

i.e., $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ spans the set: (4-D vector space):

$$V = \left\{ x \mid x = \begin{pmatrix} W+X & Y+Z \\ Y-Z & W-X \end{pmatrix}, \forall W, X, Y, Z \in \mathcal{R} \right\}$$

$$x_1 + x_3 = u_{11} \quad , \quad x_2 + x_4 = u_{12} \quad , \quad x_2 - x_4 = u_{21} \quad , \quad x_1 - x_3 = u_{22} \quad .$$

$\therefore$

$$x_1 = \frac{u_{11} + u_{22}}{2} \quad , \quad x_2 = \frac{u_{12} + u_{21}}{2} \quad , \quad x_3 = \frac{u_{11} - u_{22}}{2} \quad , \quad x_4 = \frac{u_{12} - u_{21}}{2}$$

(isomorphic to all 4-D real/complex)

The reader may verify the following tables, under ordinary matrix multiplication.

	u_1	u_2	u_3	u_4	$\beta_{11}^1 = 1$	$\beta_{11}^2 = 0$	$\beta_{11}^3 = 0$	$\beta_{11}^4 = 0$
u_1	u_1	u_2	u_3	u_4	$\beta_{12}^1 = 0$	$\beta_{12}^2 = 1$	$\beta_{12}^3 = 0$	$\beta_{12}^4 = 0$
u_2	u_2	u_1	$-u_4$	$-u_3$	$\beta_{13}^1 = 0$	$\beta_{13}^2 = 0$	$\beta_{13}^3 = 1$	$\beta_{13}^4 = 0$
u_3	u_3	u_4	u_1	u_2	$\beta_{14}^1 = 0$	$\beta_{14}^2 = 0$	$\beta_{14}^3 = 0$	$\beta_{14}^4 = 1$
u_4	u_4	u_3	$-u_2$	$-u_1$	$\beta_{21}^1 = 0$	$\beta_{21}^2 = 1$	$\beta_{21}^3 = 0$	$\beta_{21}^4 = 0$
					$\beta_{22}^1 = 1$	$\beta_{22}^2 = 0$	$\beta_{22}^3 = 0$	$\beta_{22}^4 = 0$
					$\beta_{23}^1 = 0$	$\beta_{23}^2 = 0$	$\beta_{23}^3 = 0$	$\beta_{23}^4 = -1$
					$\beta_{24}^1 = 0$	$\beta_{24}^2 = 0$	$\beta_{24}^3 = -1$	$\beta_{24}^4 = 0$
					$\beta_{31}^1 = 0$	$\beta_{31}^2 = 0$	$\beta_{31}^3 = 1$	$\beta_{31}^4 = 0$
					$\beta_{32}^1 = 0$	$\beta_{32}^2 = 0$	$\beta_{32}^3 = 0$	$\beta_{32}^4 = 1$
					$\beta_{33}^1 = 1$	$\beta_{33}^2 = 0$	$\beta_{33}^3 = 0$	$\beta_{33}^4 = 0$
					$\beta_{34}^1 = 0$	$\beta_{34}^2 = 1$	$\beta_{34}^3 = 0$	$\beta_{34}^4 = 0$
					$\beta_{41}^1 = 0$	$\beta_{41}^2 = 0$	$\beta_{41}^3 = 0$	$\beta_{41}^4 = 1$
					$\beta_{42}^1 = 0$	$\beta_{42}^2 = 0$	$\beta_{42}^3 = 1$	$\beta_{42}^4 = 0$
					$\beta_{43}^1 = 0$	$\beta_{43}^2 = -1$	$\beta_{43}^3 = 0$	$\beta_{43}^4 = 0$
					$\beta_{44}^1 = -1$	$\beta_{44}^2 = 0$	$\beta_{44}^3 = 0$	$\beta_{44}^4 = 0$

Let $Z_1 = u_1 Z_1^1 + u_2 Z_1^2 + u_3 Z_1^3 + u_4 Z_1^4$

$Z_2 = u_1 Z_2^1 + u_2 Z_2^2 + u_3 Z_2^3 + u_4 Z_2^4$,

then $Z_1 \circ Z_2 = u_1(Z_1^1 Z_2^1 + Z_1^2 Z_2^2 + Z_1^3 Z_2^3 - Z_1^4 Z_2^4)$
 $+ u_2(Z_1^1 Z_2^2 + Z_1^2 Z_2^1 + Z_1^3 Z_2^4 - Z_1^4 Z_2^3)$
 $+ u_3(Z_1^1 Z_2^3 - Z_1^2 Z_2^4 + Z_1^3 Z_2^1 + Z_1^4 Z_2^2)$
 $+ u_4(Z_1^1 Z_2^4 - Z_1^2 Z_2^3 + Z_1^3 Z_2^2 + Z_1^4 Z_2^1)$.

The Identity element is: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = u_1$.

Inverses:

If $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ then $x \circ x^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $x^{-1} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$, then $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\therefore$

$y_{11} = \frac{x_{22}}{x_{11}x_{22} - x_{12}x_{21}}$, $y_{12} = \frac{-x_{12}}{x_{11}x_{22} - x_{12}x_{21}}$

$y_{21} = \frac{-x_{21}}{x_{11}x_{22} - x_{12}x_{21}}$, $y_{22} = \frac{x_{11}}{x_{11}x_{22} - x_{12}x_{21}}$

The Right-Handed Analyticity Equations become:

(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$
(1,1,1)	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^1}{\partial x^1} = 0 \quad \checkmark$	(2,1,1)	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^1}{\partial x^2} = 0 \quad \checkmark$
(1,1,2)	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^2}{\partial x^1} = 0$	(2,1,2)	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^2}{\partial x^2} = 0$
(1,1,3)	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^3}{\partial x^1} = 0$	(2,1,3)	$-\frac{\partial f^1}{\partial x^4} - \frac{\partial f^3}{\partial x^2} = 0$
(1,1,4)	$\frac{\partial f^1}{\partial x^4} + \frac{\partial f^4}{\partial x^1} = 0$	(2,1,4)	$-\frac{\partial f^1}{\partial x^3} + \frac{\partial f^4}{\partial x^2} = 0$
(1,2,1)	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^2}{\partial x^1} = 0 \quad \checkmark$	(2,2,1)	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^2}{\partial x^2} = 0 \quad \checkmark$
(1,2,2)	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^1}{\partial x^1} = 0$	(2,2,2)	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^1}{\partial x^2} = 0$
(1,2,3)	$\frac{\partial f^2}{\partial x^3} + \frac{\partial f^4}{\partial x^1} = 0$	(2,2,3)	$-\frac{\partial f^2}{\partial x^4} + \frac{\partial f^4}{\partial x^2} = 0$
(1,2,4)	$\frac{\partial f^2}{\partial x^4} - \frac{\partial f^3}{\partial x^1} = 0$	(2,2,4)	$-\frac{\partial f^2}{\partial x^3} - \frac{\partial f^3}{\partial x^2} = 0$
(1,3,1)	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^3}{\partial x^1} = 0 \quad \checkmark$	(2,3,1)	$\frac{\partial f^3}{\partial x^2} - \frac{\partial f^3}{\partial x^2} = 0 \quad \checkmark$
(1,3,2)	$\frac{\partial f^3}{\partial x^2} - \frac{\partial f^4}{\partial x^1} = 0$	(2,3,2)	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^4}{\partial x^2} = 0$
(1,3,3)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^1}{\partial x^1} = 0$	(2,3,3)	$-\frac{\partial f^3}{\partial x^4} - \frac{\partial f^1}{\partial x^2} = 0$
(1,3,4)	$\frac{\partial f^3}{\partial x^4} + \frac{\partial f^2}{\partial x^1} = 0$	(2,3,4)	$-\frac{\partial f^3}{\partial x^3} + \frac{\partial f^2}{\partial x^2} = 0$
(1,4,1)	$\frac{\partial f^4}{\partial x^1} - \frac{\partial f^4}{\partial x^1} = 0 \quad \checkmark$	(2,4,1)	$\frac{\partial f^4}{\partial x^2} - \frac{\partial f^4}{\partial x^2} = 0 \quad \checkmark$
(1,4,2)	$\frac{\partial f^4}{\partial x^2} - \frac{\partial f^3}{\partial x^1} = 0$	(2,4,2)	$\frac{\partial f^4}{\partial x^1} - \frac{\partial f^3}{\partial x^2} = 0$
(1,4,3)	$\frac{\partial f^4}{\partial x^3} + \frac{\partial f^2}{\partial x^1} = 0$	(2,4,3)	$-\frac{\partial f^4}{\partial x^4} + \frac{\partial f^2}{\partial x^2} = 0$
(1,4,4)	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^1}{\partial x^1} = 0$	(2,4,4)	$-\frac{\partial f^4}{\partial x^3} - \frac{\partial f^1}{\partial x^2} = 0$

(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$
(3,1,1)	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^1}{\partial x^3} = 0 \quad \checkmark$	(4,1,1)	$\frac{\partial f^1}{\partial x^4} - \frac{\partial f^1}{\partial x^4} = 0 \quad \checkmark$
(3,1,2)	$\frac{\partial f^1}{\partial x^4} - \frac{\partial f^2}{\partial x^3} = 0$	(4,1,2)	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^2}{\partial x^4} = 0$
(3,1,3)	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^3}{\partial x^3} = 0$	(4,1,3)	$-\frac{\partial f^1}{\partial x^2} - \frac{\partial f^3}{\partial x^4} = 0$
(3,1,4)	$\frac{\partial f^1}{\partial x^2} + \frac{\partial f^4}{\partial x^3} = 0$	(4,1,4)	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^4}{\partial x^4} = 0$
(3,2,1)	$\frac{\partial f^2}{\partial x^3} - \frac{\partial f^2}{\partial x^3} = 0 \quad \checkmark$	(4,2,1)	$\frac{\partial f^2}{\partial x^4} - \frac{\partial f^2}{\partial x^4} = 0 \quad \checkmark$
(3,2,2)	$\frac{\partial f^2}{\partial x^4} - \frac{\partial f^1}{\partial x^3} = 0$	(4,2,2)	$\frac{\partial f^2}{\partial x^3} - \frac{\partial f^1}{\partial x^4} = 0$
(3,2,3)	$\frac{\partial f^2}{\partial x^1} + \frac{\partial f^4}{\partial x^3} = 0$	(4,2,3)	$-\frac{\partial f^2}{\partial x^2} + \frac{\partial f^4}{\partial x^4} = 0$
(3,2,4)	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^3}{\partial x^3} = 0$	(4,2,4)	$-\frac{\partial f^2}{\partial x^1} - \frac{\partial f^3}{\partial x^4} = 0$
(3,3,1)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^3}{\partial x^3} = 0 \quad \checkmark$	(4,3,1)	$\frac{\partial f^3}{\partial x^4} - \frac{\partial f^3}{\partial x^4} = 0 \quad \checkmark$
(3,3,2)	$\frac{\partial f^3}{\partial x^4} - \frac{\partial f^4}{\partial x^3} = 0$	(4,3,2)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^4}{\partial x^4} = 0$
(3,3,3)	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^1}{\partial x^3} = 0$	(4,3,3)	$-\frac{\partial f^3}{\partial x^2} - \frac{\partial f^1}{\partial x^4} = 0$
(3,3,4)	$\frac{\partial f^3}{\partial x^2} + \frac{\partial f^2}{\partial x^3} = 0$	(4,3,4)	$-\frac{\partial f^3}{\partial x^1} + \frac{\partial f^2}{\partial x^4} = 0$
(3,4,1)	$\frac{\partial f^4}{\partial x^3} - \frac{\partial f^4}{\partial x^3} = 0 \quad \checkmark$	(4,4,1)	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^4}{\partial x^4} = 0 \quad \checkmark$
(3,4,2)	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^3}{\partial x^3} = 0$	(4,4,2)	$\frac{\partial f^4}{\partial x^3} - \frac{\partial f^3}{\partial x^4} = 0$
(3,4,3)	$\frac{\partial f^4}{\partial x^1} + \frac{\partial f^2}{\partial x^3} = 0$	(4,4,3)	$-\frac{\partial f^4}{\partial x^2} + \frac{\partial f^2}{\partial x^4} = 0$
(3,4,4)	$\frac{\partial f^4}{\partial x^2} - \frac{\partial f^1}{\partial x^3} = 0$	(4,4,4)	$-\frac{\partial f^4}{\partial x^1} - \frac{\partial f^1}{\partial x^4} = 0$

So:

$$\begin{aligned} \frac{\partial f^1}{\partial x^1} &= \frac{\partial f^2}{\partial x^2} = \frac{\partial f^3}{\partial x^3} = \frac{\partial f^4}{\partial x^4} \quad , \quad \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1} = -\frac{\partial f^3}{\partial x^4} = -\frac{\partial f^4}{\partial x^3} \\ \frac{\partial f^1}{\partial x^3} &= \frac{\partial f^2}{\partial x^4} = \frac{\partial f^3}{\partial x^1} = \frac{\partial f^4}{\partial x^2} \quad , \quad \frac{\partial f^1}{\partial x^4} = \frac{\partial f^2}{\partial x^3} = -\frac{\partial f^3}{\partial x^2} = -\frac{\partial f^4}{\partial x^1} \end{aligned}$$

$\therefore$

$f_{,4}^4 = f_{,3}^3 = f_{,2}^2 = f_{,1}^1$ $-f_{,4}^3 = -f_{,3}^4 = f_{,2}^1 = f_{,1}^2$	$f_{,4}^2 = f_{,3}^1 = f_{,2}^4 = f_{,1}^3$ $-f_{,4}^1 = -f_{,3}^2 = f_{,2}^3 = f_{,1}^4$
---	---

Green's Theorem in the plane becomes:

$$\begin{aligned}
\oint_{\partial R} \mathbf{f} \circ d\mathbf{r} &= \oint_{\partial R} \mathbf{u}_1(f^1 dx^1 + f^2 dx^2 + f^3 dx^3 - f^4 dx^4) \\
&\quad + \mathbf{u}_2(f^1 dx^2 + f^2 dx^1 + f^3 dx^4 - f^4 dx^3) \\
&\quad + \mathbf{u}_3(f^1 dx^3 - f^2 dx^4 + f^3 dx^1 + f^4 dx^2) \\
&\quad + \mathbf{u}_4(f^1 dx^4 - f^2 dx^3 + f^3 dx^2 + f^4 dx^1) \\
&= \iint_R \mathbf{u}_1[(\partial_2 f^1 - \partial_1 f^2) dx^1 dx^2 + (\partial_4 f^3 + \partial_3 f^4) dx^3 dx^4] \\
&\quad + \mathbf{u}_2[(\partial_1 f^1 - \partial_2 f^2) dx^1 dx^2 + (\partial_3 f^3 + \partial_4 f^4) dx^3 dx^4] \\
&\quad + \mathbf{u}_3[(\partial_4 f^1 + \partial_3 f^2) dx^3 dx^4 + (\partial_2 f^3 - \partial_1 f^4) dx^1 dx^2] \\
&\quad + \mathbf{u}_4[(\partial_3 f^1 + \partial_4 f^2) dx^3 dx^4 + (\partial_1 f^3 - \partial_2 f^4) dx^1 dx^2] \quad (\text{see A5 - 7}) \\
&= 2 \iint_R \mathbf{u}_1 \partial_3 f^4 dx^3 dx^4 + \mathbf{u}_2 \partial_3 f^3 dx^3 dx^4 \\
&\quad + \mathbf{u}_3 \partial_3 f^2 dx^3 dx^4 + \mathbf{u}_4 \partial_3 f^1 dx^3 dx^4 \\
&= 2 \iint_R \partial_3 (\mathbf{u}_1 f^4 + \mathbf{u}_2 f^3 + \mathbf{u}_3 f^2 + \mathbf{u}_4 f^1) dx^3 dx^4 \\
&= 2 \int (\mathbf{u}_1 f^4 + \mathbf{u}_2 f^3 + \mathbf{u}_3 f^2 + \mathbf{u}_4 f^1) dx^4
\end{aligned}$$

Note that this integral does not generally vanish.

Example 3-4:

Another Four Dimensional Associative Example

$$\mathbf{u}_1 \equiv (1, 0), \mathbf{u}_2 \equiv (0, 1), \mathbf{u}_3 \equiv (i, 0), \mathbf{u}_4 \equiv (0, i)$$

$$\text{If } c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 = \mathbf{0}$$

then:

$$(c_1 + ic_3, c_2 + ic_4) = (0, 0), \text{ so the } \mathbf{u}_i \text{ are linearly independent,}$$

$$\text{and spans the set of all matrices: } (x_1 + ix_3, x_2 + ix_4).$$

Defining the circle product as follows:

$\circ$:

$$(a, b) \circ (c, d) = (ac + bd, bc + ad)$$

So:

$\mathbf{u}_1 \circ \mathbf{u}_1 = (1, 0)(1, 0) = (1, 0) = \mathbf{u}_1$	$\mathbf{u}_3 \circ \mathbf{u}_1 = (i, 0)(1, 0) = (i, 0) = \mathbf{u}_3$
$\mathbf{u}_1 \circ \mathbf{u}_2 = (1, 0)(0, 1) = (0, 1) = \mathbf{u}_2$	$\mathbf{u}_3 \circ \mathbf{u}_2 = (i, 0)(0, 1) = (0, i) = \mathbf{u}_4$
$\mathbf{u}_1 \circ \mathbf{u}_3 = (1, 0)(i, 0) = (i, 0) = \mathbf{u}_3$	$\mathbf{u}_3 \circ \mathbf{u}_3 = (i, 0)(i, 0) = (-1, 0) = -\mathbf{u}_1$
$\mathbf{u}_1 \circ \mathbf{u}_4 = (1, 0)(0, i) = (0, i) = \mathbf{u}_4$	$\mathbf{u}_3 \circ \mathbf{u}_4 = (i, 0)(0, i) = (0, -1) = -\mathbf{u}_2$
$\mathbf{u}_2 \circ \mathbf{u}_1 = (0, 1)(1, 0) = (0, 1) = \mathbf{u}_2$	$\mathbf{u}_4 \circ \mathbf{u}_1 = (0, i)(1, 0) = (0, i) = \mathbf{u}_4$

$\mathbf{u}_2 \circ \mathbf{u}_2 = (0, 1)(0, 1) = (1, 0) = \mathbf{u}_1$	$\mathbf{u}_4 \circ \mathbf{u}_2 = (0, i)(0, 1) = (i, 0) = \mathbf{u}_3$
$\mathbf{u}_2 \circ \mathbf{u}_3 = (0, 1)(i, 0) = (0, i) = \mathbf{u}_4$	$\mathbf{u}_4 \circ \mathbf{u}_3 = (0, i)(i, 0) = (0, -1) = -\mathbf{u}_2$
$\mathbf{u}_2 \circ \mathbf{u}_4 = (0, 1)(0, i) = (i, 0) = \mathbf{u}_3$	$\mathbf{u}_4 \circ \mathbf{u}_4 = (0, i)(0, i) = (-1, 0) = -\mathbf{u}_1$

More compactly:

$\mathbf{u}_1 \circ \mathbf{u}_1 = \mathbf{u}_1$	$\mathbf{u}_2 \circ \mathbf{u}_1 = \mathbf{u}_2$	$\mathbf{u}_3 \circ \mathbf{u}_1 = \mathbf{u}_3$	$\mathbf{u}_4 \circ \mathbf{u}_1 = \mathbf{u}_4$
$\mathbf{u}_1 \circ \mathbf{u}_2 = \mathbf{u}_2$	$\mathbf{u}_2 \circ \mathbf{u}_2 = \mathbf{u}_1$	$\mathbf{u}_3 \circ \mathbf{u}_2 = \mathbf{u}_4$	$\mathbf{u}_4 \circ \mathbf{u}_2 = \mathbf{u}_3$
$\mathbf{u}_1 \circ \mathbf{u}_3 = \mathbf{u}_3$	$\mathbf{u}_2 \circ \mathbf{u}_3 = \mathbf{u}_4$	$\mathbf{u}_3 \circ \mathbf{u}_3 = -\mathbf{u}_1$	$\mathbf{u}_4 \circ \mathbf{u}_3 = -\mathbf{u}_2$
$\mathbf{u}_1 \circ \mathbf{u}_4 = \mathbf{u}_4$	$\mathbf{u}_2 \circ \mathbf{u}_4 = \mathbf{u}_3$	$\mathbf{u}_3 \circ \mathbf{u}_4 = -\mathbf{u}_2$	$\mathbf{u}_4 \circ \mathbf{u}_4 = -\mathbf{u}_1$

This leads to the following table:

$\beta_{11}^1 = 1$	$\beta_{11}^2 = 0$	$\beta_{11}^3 = 0$	$\beta_{11}^4 = 0$
$\beta_{12}^1 = 0$	$\beta_{12}^2 = 1$	$\beta_{12}^3 = 0$	$\beta_{12}^4 = 0$
$\beta_{13}^1 = 0$	$\beta_{13}^2 = 0$	$\beta_{13}^3 = 1$	$\beta_{13}^4 = 0$
$\beta_{14}^1 = 0$	$\beta_{14}^2 = 0$	$\beta_{14}^3 = 0$	$\beta_{14}^4 = 1$
$\beta_{21}^1 = 0$	$\beta_{21}^2 = 1$	$\beta_{21}^3 = 0$	$\beta_{21}^4 = 0$
$\beta_{22}^1 = 1$	$\beta_{22}^2 = 0$	$\beta_{22}^3 = 0$	$\beta_{22}^4 = 0$
$\beta_{23}^1 = 0$	$\beta_{23}^2 = 0$	$\beta_{23}^3 = 0$	$\beta_{23}^4 = 1$
$\beta_{24}^1 = 0$	$\beta_{24}^2 = 0$	$\beta_{24}^3 = 1$	$\beta_{24}^4 = 0$
$\beta_{31}^1 = 0$	$\beta_{31}^2 = 0$	$\beta_{31}^3 = 1$	$\beta_{31}^4 = 0$
$\beta_{32}^1 = 0$	$\beta_{32}^2 = 0$	$\beta_{32}^3 = 0$	$\beta_{32}^4 = 1$
$\beta_{33}^1 = -1$	$\beta_{33}^2 = 0$	$\beta_{33}^3 = 0$	$\beta_{33}^4 = 0$
$\beta_{34}^1 = 0$	$\beta_{34}^2 = -1$	$\beta_{34}^3 = 0$	$\beta_{34}^4 = 0$
$\beta_{41}^1 = 0$	$\beta_{41}^2 = 0$	$\beta_{41}^3 = 0$	$\beta_{41}^4 = 1$
$\beta_{42}^1 = 0$	$\beta_{42}^2 = 0$	$\beta_{42}^3 = 1$	$\beta_{42}^4 = 0$
$\beta_{43}^1 = 0$	$\beta_{43}^2 = -1$	$\beta_{43}^3 = 0$	$\beta_{43}^4 = 0$
$\beta_{44}^1 = -1$	$\beta_{44}^2 = 0$	$\beta_{44}^3 = 0$	$\beta_{44}^4 = 0$

(Note that these product coeffs are symmetric.)

Let: $\mathbf{Z}_1 = \mathbf{u}_1 Z_1^1 + \mathbf{u}_2 Z_1^2 + \mathbf{u}_3 Z_1^3 + \mathbf{u}_4 Z_1^4$
 $\mathbf{Z}_2 = \mathbf{u}_1 Z_2^1 + \mathbf{u}_2 Z_2^2 + \mathbf{u}_3 Z_2^3 + \mathbf{u}_4 Z_2^4$,

then: $\mathbf{Z}_1 \circ \mathbf{Z}_2 = \mathbf{u}_1(Z_1^1 Z_2^1 + Z_1^2 Z_2^2 - Z_1^3 Z_2^3 - Z_1^4 Z_2^4)$
 $+ \mathbf{u}_2(Z_1^1 Z_2^2 + Z_1^2 Z_2^1 + Z_1^3 Z_2^4 - Z_1^4 Z_2^3)$
 $+ \mathbf{u}_3(Z_1^1 Z_2^3 + Z_1^2 Z_2^4 + Z_1^3 Z_2^1 + Z_1^4 Z_2^2)$
 $+ \mathbf{u}_4(Z_1^1 Z_2^4 + Z_1^2 Z_2^3 + Z_1^3 Z_2^2 + Z_1^4 Z_2^1)$.

Associativity is easily shown:

$$(a, b) \circ (c, d) = (ac + bd, bc + ad)$$

$\therefore$

$$\begin{aligned} ((a, b) \circ (c, d)) \circ (p, q) &= (ac + bd, bc + ad) \circ (p, q) \\ &= ((ac + bd)p + (bc + ad)q, (bc + ad)p + (ac + bd)q) \\ &= (acp + bdp + bcq + adq, bcp + adp + acq + bdq) \end{aligned}$$

and, so:

$$\begin{aligned} (a, b) \circ ((c, d) \circ (p, q)) &= (a, b) \circ (cp + dq, dp + cq) \\ &= (a(cp + dq) + b(dp + cq), b(cp + dq) + a(dp + cq)) \\ &= (acp + adq + bdp + bcq, bcp + bdq + adp + acq) \\ &= ((a, b) \circ (c, d)) \circ (p, q) \end{aligned}$$

From the above tables, clearly the identity element is: $(1, 0)$

Inverses may be shown to exist as follows:

$$\text{If } \mathbf{x} = (x_1 + ix_3, x_2 + ix_4) \text{ then } \mathbf{x} \circ \mathbf{x}^{-1} = (1, 0)$$

$$\text{Let } \mathbf{x}^{-1} = (y_1 + iy_3, y_2 + iy_4),$$

$$\text{then } (x_1 + ix_3, x_2 + ix_4) \circ (y_1 + iy_3, y_2 + iy_4) = (1, 0).$$

The Right-Handed Analyticity Equations become:

(h, j, k)	$\sum_{m=1}^n (\beta_{hkl}^m f_{;m}^j - \beta_{mkl}^j f_{;h}^m) = 0$	(h, j, k)	$\sum_{m=1}^n (\beta_{hkl}^m f_{;m}^j - \beta_{mkl}^j f_{;h}^m) = 0$
$(1, 1, 1)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^1}{\partial x^1} = 0 \checkmark$	$(2, 1, 1)$	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^1}{\partial x^2} = 0 \checkmark$
$(1, 1, 2)$	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^2}{\partial x^1} = 0$	$(2, 1, 2)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^2}{\partial x^2} = 0$
$(1, 1, 3)$	$\frac{\partial f^1}{\partial x^3} + \frac{\partial f^3}{\partial x^1} = 0$	$(2, 1, 3)$	$\frac{\partial f^1}{\partial x^4} + \frac{\partial f^3}{\partial x^2} = 0$
$(1, 1, 4)$	$\frac{\partial f^1}{\partial x^4} + \frac{\partial f^4}{\partial x^1} = 0$	$(2, 1, 4)$	$\frac{\partial f^1}{\partial x^3} + \frac{\partial f^4}{\partial x^2} = 0$
$(1, 2, 1)$	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^2}{\partial x^1} = 0 \checkmark$	$(2, 2, 1)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^2}{\partial x^2} = 0 \checkmark$
$(1, 2, 2)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^1}{\partial x^1} = 0$	$(2, 2, 2)$	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^1}{\partial x^2} = 0$
$(1, 2, 3)$	$\frac{\partial f^2}{\partial x^3} + \frac{\partial f^4}{\partial x^1} = 0$	$(2, 2, 3)$	$\frac{\partial f^2}{\partial x^4} + \frac{\partial f^4}{\partial x^2} = 0$
$(1, 2, 4)$	$\frac{\partial f^2}{\partial x^4} + \frac{\partial f^3}{\partial x^1} = 0$	$(2, 2, 4)$	$\frac{\partial f^2}{\partial x^3} + \frac{\partial f^3}{\partial x^2} = 0$
$(1, 3, 1)$	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^3}{\partial x^1} = 0 \checkmark$	$(2, 3, 1)$	$\frac{\partial f^3}{\partial x^2} - \frac{\partial f^3}{\partial x^2} = 0 \checkmark$

(h,j,k)	$\sum_{m=1}^n (\beta_{hkf,m}^m - \beta_{mkf,h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hkf,m}^m - \beta_{mkf,h}^m) = 0$
(1,3,2)	$\frac{\partial f^3}{\partial x^2} - \frac{\partial f^4}{\partial x^1} = 0$	(2,3,2)	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^4}{\partial x^2} = 0$
(1,3,3)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^1}{\partial x^1} = 0$	(2,3,3)	$\frac{\partial f^3}{\partial x^4} - \frac{\partial f^1}{\partial x^2} = 0$
(1,3,4)	$\frac{\partial f^3}{\partial x^4} - \frac{\partial f^2}{\partial x^1} = 0$	(2,3,4)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^2}{\partial x^2} = 0$
(1,4,1)	$\frac{\partial f^4}{\partial x^1} - \frac{\partial f^4}{\partial x^1} = 0 \checkmark$	(2,4,1)	$\frac{\partial f^4}{\partial x^2} - \frac{\partial f^4}{\partial x^2} = 0 \checkmark$
(1,4,2)	$\frac{\partial f^4}{\partial x^2} - \frac{\partial f^3}{\partial x^1} = 0$	(2,4,2)	$\frac{\partial f^4}{\partial x^1} - \frac{\partial f^3}{\partial x^2} = 0$
(1,4,3)	$\frac{\partial f^4}{\partial x^3} - \frac{\partial f^2}{\partial x^1} = 0$	(2,4,3)	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^2}{\partial x^2} = 0$
(1,4,4)	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^1}{\partial x^1} = 0$	(2,4,4)	$\frac{\partial f^4}{\partial x^3} - \frac{\partial f^1}{\partial x^2} = 0$
(h,j,k)	$\sum_{m=1}^n (\beta_{hkf,m}^m - \beta_{mkf,h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hkf,m}^m - \beta_{mkf,h}^m) = 0$
(3,1,1)	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^1}{\partial x^3} = 0 \checkmark$	(4,1,1)	$\frac{\partial f^1}{\partial x^4} - \frac{\partial f^1}{\partial x^4} = 0 \checkmark$
(3,1,2)	$\frac{\partial f^1}{\partial x^4} - \frac{\partial f^2}{\partial x^3} = 0$	(4,1,2)	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^2}{\partial x^4} = 0$
(3,1,3)	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^3}{\partial x^3} = 0$	(4,1,3)	$-\frac{\partial f^1}{\partial x^2} + \frac{\partial f^3}{\partial x^4} = 0$
(3,1,4)	$-\frac{\partial f^1}{\partial x^2} + \frac{\partial f^4}{\partial x^3} = 0$	(4,1,4)	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^4}{\partial x^4} = 0$
(3,2,1)	$\frac{\partial f^2}{\partial x^3} - \frac{\partial f^2}{\partial x^3} = 0 \checkmark$	(4,2,1)	$\frac{\partial f^2}{\partial x^4} - \frac{\partial f^2}{\partial x^4} = 0 \checkmark$
(3,2,2)	$\frac{\partial f^2}{\partial x^4} - \frac{\partial f^1}{\partial x^3} = 0$	(4,2,2)	$\frac{\partial f^2}{\partial x^3} - \frac{\partial f^1}{\partial x^4} = 0$
(3,2,3)	$-\frac{\partial f^2}{\partial x^1} + \frac{\partial f^4}{\partial x^3} = 0$	(4,2,3)	$-\frac{\partial f^2}{\partial x^2} + \frac{\partial f^4}{\partial x^4} = 0$
(3,2,4)	$-\frac{\partial f^2}{\partial x^2} + \frac{\partial f^3}{\partial x^3} = 0$	(4,2,4)	$-\frac{\partial f^2}{\partial x^1} + \frac{\partial f^3}{\partial x^4} = 0$
(3,3,1)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^3}{\partial x^3} = 0 \checkmark$	(4,3,1)	$\frac{\partial f^3}{\partial x^4} - \frac{\partial f^3}{\partial x^4} = 0 \checkmark$
(3,3,2)	$\frac{\partial f^3}{\partial x^4} - \frac{\partial f^4}{\partial x^3} = 0$	(4,3,2)	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^4}{\partial x^4} = 0$
(3,3,3)	$-\frac{\partial f^3}{\partial x^1} - \frac{\partial f^1}{\partial x^3} = 0$	(4,3,3)	$-\frac{\partial f^3}{\partial x^2} - \frac{\partial f^1}{\partial x^4} = 0$
(3,3,4)	$-\frac{\partial f^3}{\partial x^2} - \frac{\partial f^2}{\partial x^3} = 0$	(4,3,4)	$-\frac{\partial f^3}{\partial x^1} - \frac{\partial f^2}{\partial x^4} = 0$

(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{,m}^j - \beta_{mk}^j f_{,h}^m) = 0$
$(3,4,1)$	$\frac{\partial f^4}{\partial x^3} - \frac{\partial f^4}{\partial x^3} = 0 \checkmark$	$(4,4,1)$	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^4}{\partial x^4} = 0 \checkmark$
$(3,4,2)$	$\frac{\partial f^4}{\partial x^4} - \frac{\partial f^3}{\partial x^3} = 0$	$(4,4,2)$	$\frac{\partial f^4}{\partial x^3} - \frac{\partial f^3}{\partial x^4} = 0$
$(3,4,3)$	$-\frac{\partial f^4}{\partial x^1} - \frac{\partial f^2}{\partial x^3} = 0$	$(4,4,3)$	$-\frac{\partial f^4}{\partial x^2} - \frac{\partial f^2}{\partial x^4} = 0$
$(3,4,4)$	$-\frac{\partial f^4}{\partial x^2} - \frac{\partial f^1}{\partial x^3} = 0$	$(4,4,4)$	$-\frac{\partial f^4}{\partial x^1} - \frac{\partial f^1}{\partial x^4} = 0$

In brief:

$$\begin{aligned} \frac{\partial f^1}{\partial x^1} &= \frac{\partial f^2}{\partial x^2} = \frac{\partial f^3}{\partial x^3} = \frac{\partial f^4}{\partial x^4} \quad , \quad \frac{\partial f^1}{\partial x^2} = \frac{\partial f^2}{\partial x^1} = \frac{\partial f^3}{\partial x^4} = \frac{\partial f^4}{\partial x^3} \\ \frac{\partial f^1}{\partial x^3} &= \frac{\partial f^2}{\partial x^4} = -\frac{\partial f^3}{\partial x^1} = -\frac{\partial f^4}{\partial x^2} \quad , \quad \frac{\partial f^1}{\partial x^4} = \frac{\partial f^2}{\partial x^3} = -\frac{\partial f^3}{\partial x^2} = -\frac{\partial f^4}{\partial x^1} \end{aligned}$$

Green's Theorem in the plane becomes:

$$\begin{aligned} \oint_{\partial R} \mathbf{f} \circ d\mathbf{r} &= \oint_{\partial R} \mathbf{u}_1 (f^1 dx^1 + f^2 dx^2 - f^3 dx^3 - f^4 dx^4) \\ &\quad + \mathbf{u}_2 (f^1 dx^2 + f^2 dx^1 - f^3 dx^4 - f^4 dx^3) \\ &\quad + \mathbf{u}_3 (f^1 dx^3 + f^2 dx^4 + f^3 dx^1 + f^4 dx^2) \\ &\quad + \mathbf{u}_4 (f^1 dx^4 + f^2 dx^3 + f^3 dx^2 + f^4 dx^1) \\ &= \iint_R \mathbf{u}_1 [(\partial_2 f^1 - \partial_1 f^2) dx^1 dx^2 + (-\partial_4 f^3 + \partial_3 f^4) dx^3 dx^4] \\ &\quad + \mathbf{u}_2 [(\partial_1 f^1 - \partial_2 f^2) dx^1 dx^2 + (-\partial_3 f^3 + \partial_4 f^4) dx^3 dx^4] \\ &\quad + \mathbf{u}_3 [(\partial_4 f^1 - \partial_3 f^2) dx^3 dx^4 + (\partial_2 f^3 - \partial_1 f^4) dx^1 dx^2] \\ &\quad + \mathbf{u}_4 [(\partial_3 f^1 - \partial_4 f^2) dx^3 dx^4 + (\partial_1 f^3 - \partial_2 f^4) dx^1 dx^2] = 0 \end{aligned}$$

as Theorem 3-16 states.

Example 3-5:

**A Third Four Dimensional Associative Example
(Quaternions)**

$$\mathbf{q}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad \mathbf{q}_1 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{q}_2 \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{q}_3 \equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Quaternion Cayley table:

	$\mathbf{q}_0$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{q}_0$	$\mathbf{q}_0$	$\mathbf{q}_1$	$\mathbf{q}_2$	$\mathbf{q}_3$
$\mathbf{q}_1$	$\mathbf{q}_1$	$-\mathbf{q}_0$	$\mathbf{q}_3$	$-\mathbf{q}_2$
$\mathbf{q}_2$	$\mathbf{q}_2$	$-\mathbf{q}_3$	$-\mathbf{q}_0$	$\mathbf{q}_1$
$\mathbf{q}_3$	$\mathbf{q}_3$	$\mathbf{q}_2$	$-\mathbf{q}_1$	$-\mathbf{q}_0$

So:

$\beta_{00}^0 = 1$	$\beta_{00}^1 = 0$	$\beta_{00}^2 = 0$	$\beta_{00}^3 = 0$
$\beta_{01}^0 = 0$	$\beta_{01}^1 = 1$	$\beta_{01}^2 = 0$	$\beta_{01}^3 = 0$
$\beta_{02}^0 = 0$	$\beta_{02}^1 = 0$	$\beta_{02}^2 = 1$	$\beta_{02}^3 = 0$
$\beta_{03}^0 = 0$	$\beta_{03}^1 = 0$	$\beta_{03}^2 = 0$	$\beta_{03}^3 = 1$
$\beta_{10}^0 = 0$	$\beta_{10}^1 = 1$	$\beta_{10}^2 = 0$	$\beta_{10}^3 = 0$
$\beta_{11}^0 = -1$	$\beta_{11}^1 = 0$	$\beta_{11}^2 = 0$	$\beta_{11}^3 = 0$
$\beta_{12}^0 = 0$	$\beta_{12}^1 = 0$	$\beta_{12}^2 = 0$	$\beta_{12}^3 = 1$
$\beta_{13}^0 = 0$	$\beta_{13}^1 = 0$	$\beta_{13}^2 = -1$	$\beta_{13}^3 = 0$
$\beta_{20}^0 = 0$	$\beta_{20}^1 = 0$	$\beta_{20}^2 = 1$	$\beta_{20}^3 = 0$
$\beta_{21}^0 = 0$	$\beta_{21}^1 = 0$	$\beta_{21}^2 = 0$	$\beta_{21}^3 = -1$
$\beta_{22}^0 = -1$	$\beta_{22}^1 = 0$	$\beta_{22}^2 = 0$	$\beta_{22}^3 = 0$
$\beta_{23}^0 = 0$	$\beta_{23}^1 = 1$	$\beta_{23}^2 = 0$	$\beta_{23}^3 = 0$
$\beta_{30}^0 = 0$	$\beta_{30}^1 = 0$	$\beta_{30}^2 = 0$	$\beta_{30}^3 = 1$
$\beta_{31}^0 = 0$	$\beta_{31}^1 = 0$	$\beta_{31}^2 = 1$	$\beta_{31}^3 = 0$
$\beta_{32}^0 = 0$	$\beta_{32}^1 = -1$	$\beta_{32}^2 = 0$	$\beta_{32}^3 = 0$
$\beta_{33}^0 = -1$	$\beta_{33}^1 = 0$	$\beta_{33}^2 = 0$	$\beta_{33}^3 = 0$

$\therefore$

$$\begin{aligned} \mathbf{Z}_1 \circ \mathbf{Z}_2 &= \mathbf{q}_0(\mathbf{Z}_1^0 \mathbf{Z}_2^0 - \mathbf{Z}_1^1 \mathbf{Z}_2^1 - \mathbf{Z}_1^2 \mathbf{Z}_2^2 - \mathbf{Z}_1^3 \mathbf{Z}_2^3) \\ &\quad + \mathbf{q}_1(\mathbf{Z}_1^0 \mathbf{Z}_2^1 + \mathbf{Z}_1^1 \mathbf{Z}_2^0 + \mathbf{Z}_1^2 \mathbf{Z}_2^3 - \mathbf{Z}_1^3 \mathbf{Z}_2^2) \\ &\quad + \mathbf{q}_2(\mathbf{Z}_1^0 \mathbf{Z}_2^2 - \mathbf{Z}_1^1 \mathbf{Z}_2^3 + \mathbf{Z}_1^2 \mathbf{Z}_2^0 + \mathbf{Z}_1^3 \mathbf{Z}_2^1) \\ &\quad + \mathbf{q}_3(\mathbf{Z}_1^0 \mathbf{Z}_2^3 + \mathbf{Z}_1^1 \mathbf{Z}_2^2 - \mathbf{Z}_1^2 \mathbf{Z}_2^1 + \mathbf{Z}_1^3 \mathbf{Z}_2^0) \end{aligned}$$

The identity element is: $\mathbf{q}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Inverses exist, shown as follows.

If $\mathbf{x} = x_0\mathbf{q}_0 + x_1\mathbf{q}_1 + x_2\mathbf{q}_2 + x_3\mathbf{q}_3 = \begin{pmatrix} x_0 & x_1 & -x_3 & -x_2 \\ -x_1 & x_0 & -x_2 & x_3 \\ x_3 & x_2 & x_0 & x_1 \\ x_2 & -x_3 & -x_1 & x_0 \end{pmatrix}$,

then $\mathbf{x} \circ \mathbf{x}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Let $\mathbf{x}^{-1} = y_0\mathbf{q}_0 + y_1\mathbf{q}_1 + y_2\mathbf{q}_2 + y_3\mathbf{q}_3 = \begin{pmatrix} y_0 & y_1 & -y_3 & -y_2 \\ -y_1 & y_0 & -y_2 & y_3 \\ y_3 & y_2 & y_0 & y_1 \\ y_2 & -y_3 & -y_1 & y_0 \end{pmatrix}$,

then:

$$\begin{pmatrix} x_0 & x_1 & -x_3 & -x_2 \\ -x_1 & x_0 & -x_2 & x_3 \\ x_3 & x_2 & x_0 & x_1 \\ x_2 & -x_3 & -x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 & y_1 & -y_3 & -y_2 \\ -y_1 & y_0 & -y_2 & y_3 \\ y_3 & y_2 & y_0 & y_1 \\ y_2 & -y_3 & -y_1 & y_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So:

$$y_0 = \frac{x_0}{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad y_1 = \frac{-x_1}{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

$$y_2 = \frac{-x_2}{x_0^2 + x_1^2 + x_2^2 + x_3^2}, \quad y_3 = \frac{-x_3}{x_0^2 + x_1^2 + x_2^2 + x_3^2},$$

and, as show previously:

$$\mathbf{A}^* \equiv \mathbf{A}_t - \mathbf{A}_s = \mathbf{q}_0 A^0 - \mathbf{q}_1 A^1 - \mathbf{q}_2 A^2 - \mathbf{q}_3 A^3.$$

So:

$$\mathbf{x}^{-1} = \frac{x_0\mathbf{q}_0 - x_1\mathbf{q}_1 - x_2\mathbf{q}_2 - x_3\mathbf{q}_3}{x_0^2 + x_1^2 + x_2^2 + x_3^2} = -\frac{\mathbf{x}^*}{\mathbf{x} \cdot \mathbf{x}}$$

The Right-Handed Analyticity Equations become:

(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{;m}^j - \beta_{mk}^j f_{;h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{;m}^j - \beta_{mk}^j f_{;h}^m) = 0$
$(0,0,0)$	$\frac{\partial f^0}{\partial x^0} - \frac{\partial f^0}{\partial x^0} = 0 \checkmark$	$(1,0,0)$	$\frac{\partial f^1}{\partial x^0} - \frac{\partial f^1}{\partial x^0} = 0 \checkmark$
$(0,0,1)$	$\frac{\partial f^0}{\partial x^1} + \frac{\partial f^1}{\partial x^0} = 0$	$(1,0,1)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^0}{\partial x^0} = 0$
$(0,0,2)$	$\frac{\partial f^0}{\partial x^2} + \frac{\partial f^2}{\partial x^0} = 0$	$(1,0,2)$	$\frac{\partial f^1}{\partial x^2} + \frac{\partial f^3}{\partial x^0} = 0$
$(0,0,3)$	$\frac{\partial f^0}{\partial x^3} + \frac{\partial f^3}{\partial x^0} = 0$	$(1,0,3)$	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^2}{\partial x^0} = 0$
$(0,1,0)$	$\frac{\partial f^0}{\partial x^1} - \frac{\partial f^0}{\partial x^1} = 0 \checkmark$	$(1,1,0)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^1}{\partial x^1} = 0 \checkmark$
$(0,1,1)$	$-\frac{\partial f^0}{\partial x^0} + \frac{\partial f^1}{\partial x^1} = 0$	$(1,1,1)$	$-\frac{\partial f^1}{\partial x^0} - \frac{\partial f^0}{\partial x^1} = 0$
$(0,1,2)$	$\frac{\partial f^0}{\partial x^3} + \frac{\partial f^2}{\partial x^1} = 0$	$(1,1,2)$	$\frac{\partial f^1}{\partial x^3} + \frac{\partial f^3}{\partial x^1} = 0$
$(0,1,3)$	$-\frac{\partial f^0}{\partial x^2} + \frac{\partial f^3}{\partial x^1} = 0$	$(1,1,3)$	$-\frac{\partial f^1}{\partial x^2} - \frac{\partial f^2}{\partial x^1} = 0$
$(0,2,0)$	$\frac{\partial f^0}{\partial x^2} - \frac{\partial f^0}{\partial x^2} = 0 \checkmark$	$(1,2,0)$	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^1}{\partial x^2} = 0 \checkmark$
$(0,2,1)$	$-\frac{\partial f^0}{\partial x^3} + \frac{\partial f^1}{\partial x^2} = 0$	$(1,2,1)$	$-\frac{\partial f^1}{\partial x^3} - \frac{\partial f^0}{\partial x^2} = 0$
$(0,2,2)$	$-\frac{\partial f^0}{\partial x^0} + \frac{\partial f^2}{\partial x^2} = 0$	$(1,2,2)$	$-\frac{\partial f^1}{\partial x^0} + \frac{\partial f^3}{\partial x^2} = 0$
$(0,2,3)$	$\frac{\partial f^0}{\partial x^1} + \frac{\partial f^3}{\partial x^2} = 0$	$(1,2,3)$	$\frac{\partial f^1}{\partial x^1} - \frac{\partial f^2}{\partial x^2} = 0$
$(0,3,0)$	$\frac{\partial f^0}{\partial x^3} - \frac{\partial f^0}{\partial x^3} = 0 \checkmark$	$(1,3,0)$	$\frac{\partial f^1}{\partial x^3} - \frac{\partial f^1}{\partial x^3} = 0 \checkmark$
$(0,3,1)$	$\frac{\partial f^0}{\partial x^2} + \frac{\partial f^1}{\partial x^3} = 0$	$(1,3,1)$	$\frac{\partial f^1}{\partial x^2} - \frac{\partial f^0}{\partial x^3} = 0$
$(0,3,2)$	$-\frac{\partial f^0}{\partial x^1} + \frac{\partial f^2}{\partial x^3} = 0$	$(1,3,2)$	$-\frac{\partial f^1}{\partial x^1} + \frac{\partial f^3}{\partial x^3} = 0$
$(0,3,3)$	$-\frac{\partial f^0}{\partial x^0} + \frac{\partial f^3}{\partial x^3} = 0$	$(1,3,3)$	$-\frac{\partial f^1}{\partial x^0} - \frac{\partial f^2}{\partial x^3} = 0$
$(2,0,0)$	$\frac{\partial f^2}{\partial x^0} - \frac{\partial f^2}{\partial x^0} = 0 \checkmark$	$(3,0,0)$	$\frac{\partial f^3}{\partial x^0} - \frac{\partial f^3}{\partial x^0} = 0 \checkmark$
$(2,0,1)$	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^3}{\partial x^0} = 0$	$(3,0,1)$	$\frac{\partial f^3}{\partial x^1} + \frac{\partial f^2}{\partial x^0} = 0$
$(2,0,2)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^0}{\partial x^0} = 0$	$(3,0,2)$	$\frac{\partial f^3}{\partial x^2} - \frac{\partial f^1}{\partial x^0} = 0$
$(2,0,3)$	$\frac{\partial f^2}{\partial x^3} + \frac{\partial f^1}{\partial x^0} = 0$	$(3,0,3)$	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^0}{\partial x^0} = 0$

(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{;m}^j - \beta_{mk}^j f_{;h}^m) = 0$	(h,j,k)	$\sum_{m=1}^n (\beta_{hk}^m f_{;m}^j - \beta_{mk}^j f_{;h}^m) = 0$
$(2,1,0)$	$\frac{\partial f^2}{\partial x^1} - \frac{\partial f^2}{\partial x^1} = 0 \quad \checkmark$	$(3,1,0)$	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^3}{\partial x^1} = 0 \quad \checkmark$
$(2,1,1)$	$-\frac{\partial f^2}{\partial x^0} - \frac{\partial f^3}{\partial x^1} = 0$	$(3,1,1)$	$-\frac{\partial f^3}{\partial x^0} + \frac{\partial f^2}{\partial x^1} = 0$
$(2,1,2)$	$\frac{\partial f^2}{\partial x^3} - \frac{\partial f^0}{\partial x^1} = 0$	$(3,1,2)$	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^1}{\partial x^1} = 0$
$(2,1,3)$	$-\frac{\partial f^2}{\partial x^2} + \frac{\partial f^1}{\partial x^1} = 0$	$(3,1,3)$	$-\frac{\partial f^3}{\partial x^2} - \frac{\partial f^0}{\partial x^1} = 0$
$(2,2,0)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^2}{\partial x^2} = 0 \quad \checkmark$	$(3,2,0)$	$\frac{\partial f^3}{\partial x^2} - \frac{\partial f^3}{\partial x^2} = 0 \quad \checkmark$
$(2,2,1)$	$-\frac{\partial f^2}{\partial x^3} - \frac{\partial f^3}{\partial x^2} = 0$	$(3,2,1)$	$-\frac{\partial f^3}{\partial x^3} + \frac{\partial f^2}{\partial x^2} = 0$
$(2,2,2)$	$-\frac{\partial f^2}{\partial x^0} - \frac{\partial f^0}{\partial x^2} = 0$	$(3,2,2)$	$-\frac{\partial f^3}{\partial x^0} - \frac{\partial f^1}{\partial x^2} = 0$
$(2,2,3)$	$\frac{\partial f^2}{\partial x^1} + \frac{\partial f^1}{\partial x^2} = 0$	$(3,2,3)$	$\frac{\partial f^3}{\partial x^1} - \frac{\partial f^0}{\partial x^2} = 0$
$(2,3,0)$	$\frac{\partial f^2}{\partial x^3} - \frac{\partial f^2}{\partial x^3} = 0 \quad \checkmark$	$(3,3,0)$	$\frac{\partial f^3}{\partial x^3} - \frac{\partial f^3}{\partial x^3} = 0 \quad \checkmark$
$(2,3,1)$	$\frac{\partial f^2}{\partial x^2} - \frac{\partial f^3}{\partial x^3} = 0$	$(3,3,1)$	$\frac{\partial f^3}{\partial x^2} + \frac{\partial f^2}{\partial x^3} = 0$
$(2,3,2)$	$-\frac{\partial f^2}{\partial x^1} - \frac{\partial f^0}{\partial x^3} = 0$	$(3,3,2)$	$-\frac{\partial f^3}{\partial x^1} - \frac{\partial f^1}{\partial x^3} = 0$
$(2,3,3)$	$-\frac{\partial f^2}{\partial x^0} + \frac{\partial f^1}{\partial x^3} = 0$	$(3,3,3)$	$-\frac{\partial f^3}{\partial x^0} - \frac{\partial f^0}{\partial x^3} = 0$

$\therefore$

$\begin{aligned} \partial_0 f^0 &= \partial_1 f^1 = \partial_2 f^2 = \partial_3 f^3 \\ \partial_1 f^0 &= -\partial_0 f^1 = -\partial_2 f^3 = \partial_3 f^2 \end{aligned}$	$\begin{aligned} \partial_2 f^0 &= -\partial_3 f^1 = -\partial_0 f^2 = \partial_1 f^3 \\ \partial_3 f^0 &= \partial_2 f^1 = -\partial_1 f^2 = -\partial_0 f^3 \end{aligned}$
--	--

Green's Theorem in the plane becomes: (n=3)

$$\begin{aligned}
\oint_{\partial R} \mathbf{f} \circ d\mathbf{r} &= \oint_{\partial R} \mathbf{q}_0 (f^0 dx^0 - f^1 dx^1 - f^2 dx^2 - f^3 dx^3) \\
&\quad + \mathbf{q}_1 (f^0 dx^1 + f^1 dx^0 + f^2 dx^3 - f^3 dx^2) \\
&\quad + \mathbf{q}_2 (f^0 dx^2 - f^1 dx^3 + f^2 dx^0 + f^3 dx^1) \\
&\quad + \mathbf{q}_3 (f^0 dx^3 + f^1 dx^2 - f^2 dx^1 + f^3 dx^0) \\
&= \iint_R \mathbf{q}_0 [(-\partial_2 f^1 + \partial_1 f^2) dx^1 dx^2 + (-\partial_0 f^3 - \partial_3 f^0) dx^3 dx^0] \\
&\quad + \mathbf{q}_1 [(\partial_2 f^0 + \partial_1 f^3) dx^1 dx^2 + (\partial_0 f^2 - \partial_3 f^1) dx^3 dx^0] \\
&\quad + \mathbf{q}_2 [(-\partial_0 f^1 - \partial_3 f^2) dx^3 dx^0 + (\partial_2 f^3 - \partial_1 f^0) dx^1 dx^2] \\
&\quad + \mathbf{q}_3 [(\partial_0 f^0 - \partial_3 f^3) dx^3 dx^0 + (-\partial_2 f^2 - \partial_1 f^1) dx^1 dx^2]
\end{aligned}$$

$$\begin{aligned}
&= \iint_{\mathbf{R}} \mathbf{q}_0 [(-\partial_2 f^1 + \partial_1 f^2) dx^1 dx^2 + 0] + \mathbf{q}_1 [(\partial_2 f^0 + \partial_1 f^3) dx^1 dx^2 + 0] \\
&\quad + \mathbf{q}_2 [0 + (\partial_2 f^3 - \partial_1 f^0) dx^1 dx^2] + \mathbf{q}_3 [0 + (-\partial_2 f^2 - \partial_1 f^1) dx^1 dx^2] \\
&= 2 \iint_{\mathbf{R}} \partial_2 [\mathbf{q}_0(-f^1) + \mathbf{q}_1(f^0) + \mathbf{q}_2(f^3) + \mathbf{q}_3(-f^2)] dx^2 dx^1 \\
&= 2 \int (-\mathbf{q}_0 f^1 + \mathbf{q}_1 f^0 + \mathbf{q}_2 f^3 - \mathbf{q}_3 f^2) dx^1 \\
&= 2 \int \mathbf{f} \circ d\mathbf{x}^1 \\
&= 2 \iint_{\mathbf{R}} \partial_1 [\mathbf{q}_0(f^2) + \mathbf{q}_1(f^3) + \mathbf{q}_2(-f^0) + \mathbf{q}_3(-f^1)] dx^1 dx^2 \\
&= 2 \int (\mathbf{q}_0 f^2 + \mathbf{q}_1 f^3 - \mathbf{q}_2 f^0 - \mathbf{q}_3 f^1) dx^2 \\
&= -2 \int \mathbf{f} \circ d\mathbf{x}^2
\end{aligned}$$

Note that this integral does not generally vanish.

Appendix A

Common Definitions and Supporting Material, Algebra

Definition A1-1: The **Cartesian Product** of two sets, S and T is the ordered pair set $S \times T$, whose elements are the elements of S and T :

$$S \times T = \{(x, y) \mid \forall x, y \ni x \in S, y \in T\}$$

Definition A1-2: For $X_i \in S_i, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n$:

$$(X \mid S_i)_n \equiv \begin{cases} \{X_1\} & , n = 1 \\ \{(X \mid S_i)_{n-1}, X_n\} & , n > 1 \end{cases}, \text{ and}$$

$(X \mid S_i)_n$ is called an **n-tuple** over sets, S_i .

X_i is called a variable of the n-tuple.

Therefore, an n-tuple is an ordered set of variables.

Definition A1-3: For $X_i \in S, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n$:

$$(X \mid S)_n \equiv \begin{cases} \{X_1\} & , n = 1 \\ \{(X \mid S)_{n-1}, X_n\} & , n > 1 \end{cases}, \text{ and}$$

$(X \mid S)_n$ is called a **homogeneous n-tuple** of set, S .

X_i is called a variable of the n-tuple.

Therefore, an n-tuple is an ordered set of variables.

$X_n \equiv (X_i)_n \equiv (X \mid C)_n$ are alternative descriptors of n-tuples of C .

Note:

X_n and $(X_i)_n$ are shorthand for: $(X_1, X_2, \dots, X_n)$.

Let: $S_1 = (x^i)_{h-1}, S_2 = \varphi^h, S_3 = (x^{h+i})_n$.

Then for: $X_i \in S_i, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq 3$

$$\begin{aligned} (X \mid S_i)_3 &\equiv \{\{X_1\}, X_2, X_3\} \\ &= \{\{\{(x^i)_{h-1}\}, \varphi^h\}, (x^{h+i})_n\} \end{aligned}$$

In particular, for: $\varphi^h \equiv x^h + \delta x^h$:

$$\begin{aligned} (X \mid S_i)_3 &= \{\{\{(x^i)_{h-1}\}, x^h + \delta x^h\}, (x^{h+i})_n\} \\ &\equiv ((x^i)_{h-1}, x^h + \delta x^h, (x^{h+i})_n) \\ &\equiv (x^1, x^2, x^3, \dots, x^h, x^h + \delta x^h, x^{h+1}, \dots, x^n) \end{aligned}$$

Definition A1-4: For: $n \in \mathbb{Z}^+$, and sets: $S_i, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n$:

$$[X_i \mid S_i]_n \equiv \{(X_i)_n \mid \forall X_i \in S_i, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n\}$$

Definition A1-5: For: $n \in \mathbb{Z}^+$, and set: S :

$$[X \mid S]_n \equiv \{(X_i)_n \mid \forall X_i \in S, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n\}$$

And, let: $[X_i]_n \equiv [X \mid C]_n$.

Definition A1-6: For $X_{ij} \in S, \forall i, j \in \mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq m$:

$$((X \mid S)_n \mid S)_m \equiv \begin{cases} \{X_{i1}\} & , m = 1 \\ \{(X_i \mid S)_{m-1}, X_{im}\} & , m > 1 \end{cases} ,$$

$$\forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n$$

and, $((X \mid S)_n \mid S)_m$ is called an **$n \times m$ -frame** of set, S .

X_{ij} is called an entry of the $n \times m$ -frame.

$$\mathbf{X}_{nm} \equiv (X_{ij})_{nm} \equiv ((X \mid C)_n \mid C)_m$$

are alternative descriptors of $n \times m$ -frames of C .

Definition A1-7: A **mapping** is the resultant set of all elements satisfying the map conditions on the elements from the domain set to the codomain set, which contains the resultant set.

Note:

A **mapping** is a correspondence between a **Domain Set**, and a **Range Set**.
(a Codomain is a subset of a Range set)

(see: [68],[70]).

Definition A1-8: A **surjective mapping / onto mapping** is a mapping

$m : X \mapsto Y$, such that:

$$m : X \twoheadrightarrow Y \equiv m : X \mapsto Y \ni (\forall y \in Y, \exists x \in X \ni m(x) = y)$$

See: [49],[70]

Definition A1-9: An **injective mapping / one-to-one mapping** is a mapping

$m : X \mapsto Y$, such that:

$$m : X \not\Leftarrow Y \equiv m : X \mapsto Y \ni (\forall a, b \in X, (m(a) = m(b)) \Rightarrow (a = b))$$

See: [68],[70]

Definition A1-10: A bijective mapping / oneto mapping is a mapping

$$m : X \mapsto Y, \text{ such that:}$$

$$m : X \not\Leftarrow Y \equiv m : X \rightarrow Y \ni m : X \Leftarrow Y$$

See: [68],[70]

i.e.: a oneto mapping is one-to-one and onto.

Definition A1-11: A function is a mapping $m : X \mapsto Y$, such that

$$m : X \rightarrow_f Y \equiv m : X \rightarrow Y \ni$$

$$(\forall x \in X, \exists y \subset Y \ni (y = \emptyset) \vee (\exists a, b \in y \ni m(a) = m(b) \Rightarrow (a = b)))$$

See: [70]

Definition A1-12: For bijective mapping: $f : X \not\Leftarrow Y$, the **inverse function** of f

$$\text{is the function: } f^{-1} \equiv \{(y, x) \in Y \times X \mid (x, y) \in f\}$$

See: [9],[41]

Definition A1-13: A homomorphism is a mapping $m : X \mapsto Y$, such that:

$$\exists n \in \mathbb{Z}^+ \ni (\exists \varphi_i : X \times X \rightarrow X) \wedge (\exists \psi_i : Y \times Y \rightarrow Y) \ni$$

$$\forall a, b \in X, m(\varphi_i(a, b)) = \psi_i(m(a), m(b)) \in Y, (\forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n)$$

See: [25],[46]

Definition A1-14: An isomorphism is an injective homomorphism, i.e.: a mapping

$$m : X \not\Leftarrow Y, \text{ such that:}$$

$$\exists n \in \mathbb{Z}^+ \ni (\exists \varphi_i : X \times X \rightarrow X) \wedge (\exists \psi_i : Y \times Y \rightarrow Y) \ni$$

$$\forall a, b \in X, m(\varphi_i(a, b)) = \psi_i(m(a), m(b)) \in Y, (\forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n)$$

See: [25],[34]

Definition A1-15: (notation)

$$(S, T) \equiv \{S, \{T\}\}$$

Definition A1-16: (notation)

$$m(\varphi, S, T) \equiv \{(x, y) \mid (\forall x \in S) \wedge (y \in T) \ni \varphi\}$$

Definition A1-17: (notation)

$$T \supset m(S), (m : S \mapsto T) \equiv (m(S) \subset T)$$

Definition A1-18: (notation)

$$(\circ(S \times S) \subset S) \equiv (S = \{u \mid \forall s \in S, \forall t \in S, u = \circ(s, t)\} \subset S)$$

Definition A1-19: A closed binary operation is a closed binary mapping.

Homomorphism may be restated using the language of binary operation, as a mapping that conserves the binary operation across the mapping.

Definition A1-20: A groupoid is a closed binary mapping of a non-empty set.

Definition A1-21: A semigroup is an associative closed binary mapping.

$$\Sigma(S, \oplus) \equiv \{(\oplus(S \times S) \subset S) \ni \forall r, s, t \in S, r \oplus (s \oplus t) = (r \oplus s) \oplus t\}$$

Definition A1-22: A ring is a Cartesian product of two closed binary mappings with the same set. $\rho(S, +, \cdot) \equiv (+(S \times S) \subset S) \times (\cdot(S \times S) \subset S)$, such that:

$$\Sigma(S, +) \equiv \{(+ (S \times S) \subset S) \ni \forall r, s, t \in S, r + (s + t) = (r + s) + t\},$$

$$\Sigma(S, \cdot) \equiv \{(\cdot (S \times S) \subset S) \ni \forall r, s, t \in S, r \cdot (s \cdot t) = (r \cdot s) \cdot t\};$$

$$\text{Axiom R.A-1: } \forall X, Y \in S, X + Y = Y + X \quad (\text{Commutativity})$$

$$\text{Axiom R.A-2: } \forall X, Y \in S, \exists 0 \ni X + 0 = X \quad (\text{Additive Identity})$$

$$\text{Axiom R.A-3: } \forall X \in S, \exists Y \in S \ni X + Y = 0 \quad (\text{Additive Inverses})$$

and

$$\text{Axiom R.D-1: } \forall X, Y, Z \in S, (X + Y) \cdot Z = X \cdot Z + Y \cdot Z \quad (\text{Distributive})$$

Definition A1-23: A field is a ring $\Psi(S, +, \cdot) \equiv \rho(S, +, \cdot)$ such that:

$$\text{Commutativity: } (i) \quad \forall X, Y \in S, X \cdot Y = Y \cdot X$$

$$\text{Multiplicative Identity: } (ii) \quad \exists e \in S \ni \forall X \in S, e \cdot X = X \cdot e = X$$

$$\text{Multiplicative Inverses: } (iii) \quad \forall X \in S \ni X \neq 0, \exists X^{-1} \in S \ni X \cdot X^{-1} = e$$

Typically, a matrix is defined as a rectangular array of numbers [6],[40], but

here the definition will be more general, inspired by [25] and [59].

Definition A1-24: For non-empty set, X , and collection of subsets of X , J , $\ni$

- (i) $X \in J$,
- (ii) $\emptyset \in J$,
- (iii) $\forall X, Y \in J, X \cap Y \in J$,
- (iv) $\forall \alpha \in I, X_\alpha \in J \Rightarrow \bigcup_{\alpha \in I} X_\alpha \in J$, (for some index set I) :

J is a **topology** on X , and:

(X, J) is a **topological space**.

See: [27],[47]

Definition A1-25: For $X_{ij} \in \Psi(P, \oplus, \odot), \forall i, j \in \mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq m$

$$((X | P)_n | P)_m \equiv \begin{cases} \{X_{i1}\} & , m = 1 \\ \{(X_i | P)_{m-1}, X_{im}\} & , m > 1 \end{cases},$$

$$\forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq n :$$

is called an **$n \times m$ -matrix** of field $\Psi(P, \oplus, \odot)$;

where the 1st operand, n , is called the row count,

and the 2nd operand, m , is called the column count.

X_{ij} is called the entry of the $n \times m$ -matrix

at the i -th row and j -th column, or the ij -th entry.

An $n \times m$ -matrix is an $n \times m$ -frame, where its set is a field.

$X_{nm} \equiv (X_{ij})_{nm} \equiv ((X | C)_n | C)_m$ are alternative descriptors
of $n \times m$ -matrices of C .

Note:

The plural of $n \times m$ -matrix is $n \times m$ -matrices.

$n \times m$ -matrices are displayed visually as a rectangular array of elements
of P .

Definition A1-26: The **matrix**, X_{nm} , is an $n \times m$ -matrix, with field made up of the set
of complex numbers:

$$X_{nm} \equiv (X_{ij})_{nm} \equiv ((X | C)_n | C)_m$$

the domain of: i and j is: $\mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq m$

An entry of $\mathbf{X}_{nm}$ is an entry of the $n \times m$ -matrix: X_{ij} ,

$$(\forall i, j \in \mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq m).$$

For an $n \times m$ - matrix where the row count and column count are, in

context, understood: $\mathbf{X} \equiv (X_{ij}) \equiv ((X \mid C)_n \mid C)_m$

the domain of: i and j is: $\mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq m$

(i.e., for: $X \equiv (X_{ij})$, ij represents the row/column, respectively)

Examples of matrices:

$$1 \times 3 : \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad 2 \times 2 : \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Note 1:

A matrix denoted with a boldface character is also denoted with the same character within parens in normal face having two subscripts which run the domain of the row/column counts, respectively.

The entry of the matrix is the same character in normal face having two subscripts which run the domain of the row/column counts, respectively.

So, $\mathbf{X}_{nm}$ and $(X_{ij})_{nm}$ are shorthand for:

$$\begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & & X_{2m} \\ \vdots & & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nm} \end{pmatrix}.$$

Definition A1-27: A **vector space** is a set S , along with a field $\mathcal{F}$, and an operator $\oplus$, between them (denoted: $(S, \oplus, \mathcal{F})$), such that:

- (i) **Axiom V.A-1:** $\forall X, Y \in S, X + Y \in S$ (Closure)
- (ii) **Axiom V.A-2:** $\forall X, Y, Z \in S, (X + Y) + Z = X + (Y + Z)$ (Associativity)
- (iii) **Axiom V.A-3:** $\forall X, Y \in S, X + Y = Y + X$ (Commutativity)
- (iv) **Axiom V.A-4:** $\forall X, Y \in S, \exists 0 \ni X + 0 = X$ (Additive Identity)
- (v) **Axiom V.A-5:** $\forall X \in S, \exists Y \in S \ni X + Y = 0$ (Additive Inverses)
- (vi) **Axiom V.M-1:** $\forall \alpha \in \mathcal{F}, \forall X \in S, \alpha \oplus X \in S$ (Closure)

(vii) **Axiom V.M-2:** $(\exists 1 \in \mathbf{P} \ni \forall \alpha \in \mathbf{P}, 1 \otimes \alpha = \alpha)$

$$\forall X \in \mathbf{S}, 1 \otimes X = X \quad (\text{Scalar Multiplicative Identity})$$

(viii) **Axiom V.M-3:** $\forall \alpha, \beta \in \mathbf{P}, \forall X \in \mathbf{S}, \alpha \otimes (\beta \otimes X) = (\alpha \otimes \beta) \otimes X$ (Associativity)

and

(ix) **Axiom V.D-1:** $\forall \alpha, \beta \in \mathbf{P}, \forall X \in \mathbf{S},$

$$(\alpha \oplus \beta) \otimes X = \alpha \otimes X + \beta \otimes X \quad (\mathbf{P} \text{ Distributive})$$

(x) **Axiom V.D-2:** $\forall \alpha \in \mathbf{P}, \forall X, Y \in \mathbf{S},$

$$\alpha \otimes (X + Y) = \alpha \otimes X + \alpha \otimes Y \quad (\mathbf{S} \text{ Distributive})$$

As an example of a vector space, consider the set:

$$V = \left\{ x \mid x = \begin{pmatrix} X & Y \\ Z & -X \end{pmatrix}, \forall X, Y, Z \in \mathcal{R} \right\}$$

lf:

$$u, v, w \in V$$

then, $\exists u_{11}, u_{12}, u_{21}, v_{11}, v_{12}, v_{21}, w_{11}, w_{12}, w_{21} \in \mathcal{R} \ni$

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix}, v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix}, w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & -w_{11} \end{pmatrix}$$

and, for: $r, s \in \mathcal{R} :$

$$i) \quad u + v = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & -v_{11} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & -(u_{11} + v_{11}) \end{pmatrix} \in V$$

$$ii) \quad (u + v) + w = \begin{pmatrix} u_{11} + v_{11} + w_{11} & u_{12} + v_{12} + w_{12} \\ u_{21} + v_{21} + w_{21} & -(u_{11} + v_{11} + w_{11}) \end{pmatrix} = u + (v + w)$$

$$iii) \quad u + v = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & -(u_{11} + v_{11}) \end{pmatrix} = \begin{pmatrix} v_{11} + u_{11} & v_{12} + u_{12} \\ v_{21} + u_{21} & -(v_{11} + u_{11}) \end{pmatrix} = v + u$$

$$iv) \quad \exists 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -0 \end{pmatrix} \in V \ni u + 0 = u$$

$$v) \quad \forall u \in V, \exists u_{11}, u_{12}, u_{21} \in \mathcal{R} \ni u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix}, \text{so :}$$

$$-u = \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix} \in V \ni u + (-u) = 0$$

$$vi) \quad ru = \begin{pmatrix} ru_{11} & ru_{12} \\ ru_{21} & -ru_{11} \end{pmatrix} \in V$$

$$vii) \quad 1u = 1 \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix} = \begin{pmatrix} 1u_{11} & 1u_{12} \\ 1u_{21} & -1u_{11} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix} = u$$

$$viii) \quad r(su) = r \begin{pmatrix} su_{11} & su_{12} \\ su_{21} & -su_{11} \end{pmatrix} = \begin{pmatrix} rsu_{11} & rsu_{12} \\ rsu_{21} & -rsu_{11} \end{pmatrix} = (rs) \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & -u_{11} \end{pmatrix} = (rs$$

$$ix) \quad (r+s)u = \begin{pmatrix} (r+s)u_{11} & (r+s)u_{12} \\ (r+s)u_{21} & -(r+s)u_{11} \end{pmatrix} \\ = \begin{pmatrix} ru_{11} + su_{11} & ru_{12} + su_{12} \\ ru_{21} + su_{21} & -(ru_{11} + su_{11}) \end{pmatrix} \\ = \begin{pmatrix} ru_{11} & ru_{12} \\ ru_{21} & -ru_{11} \end{pmatrix} + \begin{pmatrix} su_{11} & su_{12} \\ su_{21} & -su_{11} \end{pmatrix} = ru + su$$

$$x) \quad r(u+v) = r \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & -(u_{11} + v_{11}) \end{pmatrix} \\ = \begin{pmatrix} ru_{11} + rv_{11} & ru_{12} + rv_{12} \\ ru_{21} + rv_{21} & -(ru_{11} + rv_{11}) \end{pmatrix} \\ = \begin{pmatrix} ru_{11} & ru_{12} \\ ru_{21} & -ru_{11} \end{pmatrix} + \begin{pmatrix} rv_{11} & rv_{12} \\ rv_{21} & -rv_{11} \end{pmatrix} = ru + rv$$

So, V , is a Vector Space.

Definition A1-28: For: $m, n \in \mathbb{Z}^+$, the set of all $m \times n$ matrices with entries from a field, $\mathcal{F}$, is a vector space, denoted by: $\mathcal{F}_{m \times n}$.

The notation on this vector space varies widely [26],[35],[70].

The notation used here, follows [35].

Verification that the set: $\mathcal{F}_{m \times n}$ is a vector space is left to the reader.

Definition A1-29: For: $m, n \in \mathbb{Z}^+$, the set of all $m \times n$ matrices with entries from the field of real numbers, $\mathcal{R}$, is a vector space, denoted by: $\mathcal{R}_{m \times n}$.

The notation on this vector space varies widely.[6],[33],[35],[53].

The notation used here, follows [35].

Definition A1-30: Negative vectors: For vector space $(S, \otimes, \mathcal{F})$:

$$\forall X \in S, -X \equiv \{ Y \ni X + Y = 0 \}$$

Definition A1-31: (notation)

For vector space: $(S, \otimes, \mathcal{F})$, $\forall \alpha \in \mathbf{P}, \forall X \in S, X \otimes \alpha \equiv \alpha \otimes X$

Definition A1-32: An isomorphism_{VS} is a bijective mapping between two vector spaces, for:

$$\left(V_1, \overset{1}{\otimes}, \mathcal{F} \right), \overset{1}{+}; \text{ and } \left(V_2, \overset{2}{\otimes}, \mathcal{F} \right), \overset{2}{+};$$

$m : V_1 \rightleftharpoons V_2$, such that:

$$\forall x, y \in V_1, m\left(x \overset{1}{+} y\right) = m(x) \overset{2}{+} m(y) \in V_2 ,$$

$$\forall x \in V_1, \forall \alpha \in \mathcal{F}, m\left(\alpha \overset{1}{\otimes} x\right) = \alpha \overset{2}{\otimes} m(x) \in V_2 .$$

See: [53],[60]

Definition A1-33: Generalized multiple operation notation:

$$\sum_{i=m}^{\oplus:n} X_i \equiv \begin{cases} = X_m & , n = m \\ = \sum_{i=m}^{\oplus:n-1} X_i \oplus X_n & , n > m \end{cases} ,$$

that is:

$$\sum_{i=m}^{\oplus:n} X_i \equiv X_m \oplus X_{m+1} \oplus X_{m+2} \oplus \cdots \oplus X_n$$

So:

Definition A1-34: $\sum_{i=1}^n \equiv \sum_{i=1}^{+:n}$ (the usual sum syntax/notation)

Definition A1-35: $\prod_{i=1}^n \equiv \sum_{i=1}^{\cdot:n}$ (the usual product syntax/notation)

Definition A1-36: vector-sum notation is the same used for real functions/variables

i.e.: $\sum_{i=1}^n V_i \equiv V_1 + V_2 + V_3 + \cdots + V_n$, where:

$$V_1, V_2, V_3, \dots, V_n \in (S, \otimes, \mathcal{F})$$

Definition A1-37: A **vector space spanning set** of a vector space $(S, \otimes, \mathcal{F})$, is a set of the vector-sum of its of its members for all coefficients from the field set.

$$\mathcal{L}(V_1, V_2, V_3, \dots, V_n; (S, \otimes, \mathcal{F})) \equiv \left\{ V \mid V = \sum_{i=1}^n (a_i \otimes V_i), \forall a_1, a_2, a_3, \dots, a_n \in \mathcal{F} \right\}, \text{ where:}$$

$$V_1, V_2, V_3, \dots, V_n \in S$$

A set $\{V_1, V_2, V_3, \dots, V_n\}$ **spans** a vector space $(S, \otimes, \mathcal{F})$, whenever:

$$\mathcal{L}(V_1, V_2, V_3, \dots, V_n; (S, \otimes, \mathcal{F})) = (S, \otimes, \mathcal{F}).$$

Definition A1-38: n vectors, V_i , are **linearly independent**, whenever:

$$\left\{ \text{if } \sum_{i=1}^n (a_i \otimes V_i) = 0 \text{ then } a_i = 0 \right\}$$

Definition A1-39: A **vector space basis** is a set of vectors: $V_1, V_2, V_3, \dots \in S$, such that:

$$\mathcal{L}(V_1, V_2, V_3, \dots; (S, \otimes, \mathcal{F})) = (S, \otimes, \mathcal{F})$$

and

$$\left\{ \text{if: } a_1 \otimes V_1 + a_2 \otimes V_2 + a_3 \otimes V_3 + \dots = 0, \text{ then } a_i = 0, \forall i \right\}$$

Definition A1-40: A vector space: $(S, \otimes, \mathcal{F})$, is called **n-dimensional**, whenever

$$\exists n \in \mathbb{Z}^+ \ni \text{for: } V_1, V_2, V_3, \dots, V_n \in S,$$

$$\{V_1, V_2, V_3, \dots, V_n\} \text{ is a basis of: } (S, \otimes, \mathcal{F})$$

$$\text{and is denoted by: } (S, \otimes, \mathcal{F})_n$$

Theorem A1-1: Any n -dimensional vector space $(S, \otimes, \mathcal{F})_n$, is isomorphic_{VS} to:

$$(\mathcal{F}_n, \otimes, \mathcal{F}), \text{ where:}$$

$$\mathcal{F}_n \equiv \{(X \mid \mathcal{F})_n \mid \forall X \in \mathcal{F}\}, \left(\begin{array}{l} \text{the set of all (homogeneous)} \\ \text{n-tuples over the field } \mathcal{F} \end{array} \right)$$

Proof:

See [25], §2.6, p. 43, Theorem 2.15.

As noted in [25], since any n -dimensional vector space over a field is

isomorphic_{VS} to an (homogeneous) n-tuple over that field, all n-dimensional vector spaces are isomorphic_{VS}.

Definition A1-41: matrix multiplication, also termed a **matrix product**, is

a binary mapping of a matrix: $\mathbf{A}_{nm} = (A_{ij})_{nm}$ and a matrix:

$\mathbf{B}_{mp} = (B_{ij})_{mp}$ into a matrix: $\mathbf{C}_{np} = (C_{ij})_{np}$, such that:

$$C_{ij} \equiv \sum_{h=1}^m A_{ih} B_{hj}, \quad \forall i, j \in \mathbb{Z}^+ \ni 1 \leq i \leq n, 1 \leq j \leq p.$$

The matrix product is written:

$$(C_{ij})_{np} = (A_{ij})_{nm} (B_{ij})_{mp} \quad \text{or} \quad \mathbf{C}_{np} = \mathbf{A}_{nm} \mathbf{B}_{mp},$$

or, where understood-in-context:

$$(C_{ij}) = (A_{ij})(B_{ij}) \quad \text{or} \quad \mathbf{C} = \mathbf{AB}.$$

Example A1-1:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example A1-2:

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} x & p \\ y & q \\ z & r \end{pmatrix} = \begin{pmatrix} (ax + by + cz) & (ap + bq + cr) \end{pmatrix}$$

The matrix product $\mathbf{AB}$ exists if and only if, the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$, in which case, $\mathbf{A}$ and $\mathbf{B}$ are called **conformable matrices** [52].

Appendix B

Weighted Matrix Product Examples

Example A2-1:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 2 \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} 0 & -1 & -2 \\ -3 & -4 & -5 \end{pmatrix}$$

$$\Phi_0 \equiv \begin{pmatrix} 6 & 7 & 8 \end{pmatrix}, \quad \Phi_1 \equiv \begin{pmatrix} 9 & 10 & 11 \end{pmatrix}$$

then:

$$\begin{aligned} (\mathbf{A} \star \mathbf{B})_{00} &= \Phi_{000}A_{00}B_{00} + \Phi_{100}A_{01}B_{10} \\ &= (6 \cdot 1 \cdot 0) + (9 \cdot 2 \cdot (-3)) = 0 - 54 = -54 \end{aligned}$$

$$\begin{aligned} (\mathbf{A} \star \mathbf{B})_{01} &= \Phi_{001}A_{00}B_{01} + \Phi_{101}A_{01}B_{11} \\ &= (7 \cdot 1 \cdot (-1)) + (10 \cdot 2 \cdot (-4)) = -7 - 80 = -87 \end{aligned}$$

$$\begin{aligned} (\mathbf{A} \star \mathbf{B})_{02} &= \Phi_{002}A_{00}B_{02} + \Phi_{102}A_{01}B_{12} \\ &= (8 \cdot 1 \cdot (-2)) + (11 \cdot 2 \cdot (-5)) = -16 - 110 = -126 \end{aligned}$$

$\therefore$

$$\mathbf{A} \star \mathbf{B} = \begin{pmatrix} -54 & -87 & -126 \end{pmatrix}$$

or:

$$\begin{aligned} \mathbf{A} \star \mathbf{B} &= (1) \begin{pmatrix} 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} + (2) \begin{pmatrix} 9 & 10 & 11 \end{pmatrix} \begin{pmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -5 \end{pmatrix} \\ &= (1) \begin{pmatrix} 0 & -7 & -16 \end{pmatrix} + (2) \begin{pmatrix} -27 & -40 & -55 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -7 & -16 \end{pmatrix} + \begin{pmatrix} -54 & -80 & -110 \end{pmatrix} = \begin{pmatrix} -54 & -87 & -126 \end{pmatrix} \quad \checkmark \end{aligned}$$

Example A2-2:

$$\mathbf{A} \equiv \begin{pmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{pmatrix}, \quad \mathbf{B} \equiv \begin{pmatrix} 0 & 2 & 4 & 8 \\ 1 & 3 & 5 & 7 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

$$\Phi_0 \equiv \begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix}, \quad \Phi_1 \equiv \begin{pmatrix} 8 & 10 & 12 & 14 \\ 9 & 11 & 13 & 15 \end{pmatrix}, \quad \Phi_2 \equiv \begin{pmatrix} 16 & 18 & 20 & 22 \\ 17 & 19 & 21 & 23 \end{pmatrix}$$

then:

$$\begin{aligned} (\mathbf{A} \star \mathbf{B})_{00} &= \Phi_{000}A_{00}B_{00} + \Phi_{100}A_{01}B_{10} + \Phi_{200}A_{02}B_{20} \\ &= (0 \cdot 1 \cdot 0) + (8 \cdot 2 \cdot 1) + (16 \cdot 3 \cdot 1) = 0 + 16 + 48 = 64 \end{aligned}$$

$$\begin{aligned} (\mathbf{A} \star \mathbf{B})_{01} &= \Phi_{001}A_{00}B_{01} + \Phi_{101}A_{01}B_{11} + \Phi_{201}A_{02}B_{21} \\ &= (2 \cdot 1 \cdot 2) + (10 \cdot 2 \cdot 3) + (18 \cdot 3 \cdot 0) = 4 + 60 + 0 = 64 \end{aligned}$$

$$\begin{aligned}
(\mathbf{A} \star \mathbf{B})_{02} &= \Phi_{002}A_{00}B_{02} + \Phi_{102}A_{01}B_{12} + \Phi_{202}A_{02}B_{22} \\
&= (4 \cdot 1 \cdot 4) + (12 \cdot 2 \cdot 5) + (20 \cdot 3 \cdot -1) = 16 + 120 - 60 = 76 \\
(\mathbf{A} \star \mathbf{B})_{03} &= \Phi_{003}A_{00}B_{03} + \Phi_{103}A_{01}B_{13} + \Phi_{203}A_{02}B_{23} \\
&= (6 \cdot 1 \cdot 8) + (14 \cdot 2 \cdot 7) + (22 \cdot 3 \cdot 0) = 48 + 196 + 0 = 244 \\
(\mathbf{A} \star \mathbf{B})_{10} &= \Phi_{010}A_{10}B_{00} + \Phi_{110}A_{11}B_{10} + \Phi_{210}A_{12}B_{20} \\
&= (1 \cdot 5 \cdot 0) + (9 \cdot 6 \cdot 1) + (17 \cdot 7 \cdot 1) = 0 + 54 + 119 = 173 \\
(\mathbf{A} \star \mathbf{B})_{11} &= \Phi_{011}A_{10}B_{01} + \Phi_{111}A_{11}B_{11} + \Phi_{211}A_{12}B_{21} \\
&= (3 \cdot 5 \cdot 2) + (11 \cdot 6 \cdot 3) + (19 \cdot 7 \cdot 0) = 30 + 198 + 0 = 228 \\
(\mathbf{A} \star \mathbf{B})_{12} &= \Phi_{012}A_{10}B_{02} + \Phi_{112}A_{11}B_{12} + \Phi_{212}A_{12}B_{22} \\
&= (5 \cdot 5 \cdot 4) + (13 \cdot 6 \cdot 5) + (21 \cdot 7 \cdot -1) = 100 + 390 - 147 = 343 \\
(\mathbf{A} \star \mathbf{B})_{13} &= \Phi_{013}A_{10}B_{03} + \Phi_{113}A_{11}B_{13} + \Phi_{213}A_{12}B_{23} \\
&= (7 \cdot 5 \cdot 8) + (15 \cdot 6 \cdot 7) + (23 \cdot 7 \cdot 0) = 280 + 630 + 0 = 910
\end{aligned}$$

$\therefore$

$$\mathbf{A} \star \mathbf{B} = \begin{pmatrix} 64 & 64 & 76 & 244 \\ 173 & 228 & 343 & 910 \end{pmatrix}$$

or:

$$\begin{aligned}
\mathbf{A} \star \mathbf{B} &= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 2 & 4 & 6 \\ 1 & 3 & 5 & 7 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \\
&+ \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 8 & 10 & 12 & 14 \\ 9 & 11 & 13 & 15 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix} \\
&+ \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 16 & 18 & 20 & 22 \\ 17 & 19 & 21 & 23 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 2 & 4 & 6 \\ 5 & 15 & 25 & 35 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix} \\
&+ \begin{pmatrix} 16 & 20 & 24 & 28 \\ 54 & 66 & 78 & 90 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& + \begin{pmatrix} 48 & 54 & 60 & 66 \\ 119 & 133 & 147 & 161 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
& = \begin{pmatrix} 0 & 4 & 16 & 48 \\ 0 & 30 & 100 & 280 \end{pmatrix} + \begin{pmatrix} 16 & 60 & 120 & 196 \\ 54 & 198 & 390 & 630 \end{pmatrix} \\
& + \begin{pmatrix} 48 & 0 & -60 & 0 \\ 119 & 0 & -147 & 0 \end{pmatrix} \\
& = \begin{pmatrix} 64 & 64 & 76 & 244 \\ 173 & 228 & 343 & 910 \end{pmatrix} \quad \checkmark \quad \square.
\end{aligned}$$

Appendix C

Associative Weighted Matrix Product

$n = 2$

α	β	i	j	$\Phi_{\beta ij} \Phi_{ai\beta} = \Phi_{aij} \Phi_{\beta aj}$
1	1	1	1	$\Phi_{111} \Phi_{111} = \Phi_{111} \Phi_{111} \quad \checkmark$
1	1	1	2	$\Phi_{112} \Phi_{111} = \Phi_{112} \Phi_{112} \Rightarrow \Phi_{111} = \Phi_{112}$
1	1	2	1	$\Phi_{121} \Phi_{121} = \Phi_{121} \Phi_{111} \Rightarrow \Phi_{121} = \Phi_{111}$
1	1	2	2	$\Phi_{122} \Phi_{121} = \Phi_{122} \Phi_{112} \Rightarrow \Phi_{121} = \Phi_{112}$
1	2	1	1	$\Phi_{211} \Phi_{112} = \Phi_{111} \Phi_{211} \Rightarrow \Phi_{112} = \Phi_{111}$
1	2	1	2	$\Phi_{212} \Phi_{112} = \Phi_{112} \Phi_{212} \quad \checkmark$
1	2	2	1	$\Phi_{221} \Phi_{122} = \Phi_{121} \Phi_{211}$
1	2	2	2	$\Phi_{222} \Phi_{122} = \Phi_{122} \Phi_{212} \Rightarrow \Phi_{222} = \Phi_{212}$
2	1	1	1	$\Phi_{111} \Phi_{211} = \Phi_{211} \Phi_{121} \Rightarrow \Phi_{111} = \Phi_{121}$
2	1	1	2	$\Phi_{112} \Phi_{211} = \Phi_{212} \Phi_{122}$
2	1	2	1	$\Phi_{121} \Phi_{221} = \Phi_{221} \Phi_{121} \quad \checkmark$
2	1	2	2	$\Phi_{122} \Phi_{221} = \Phi_{222} \Phi_{122} \Rightarrow \Phi_{221} = \Phi_{222}$
2	2	1	1	$\Phi_{211} \Phi_{212} = \Phi_{211} \Phi_{221} \Rightarrow \Phi_{212} = \Phi_{221}$
2	2	1	2	$\Phi_{212} \Phi_{212} = \Phi_{212} \Phi_{222} \Rightarrow \Phi_{212} = \Phi_{222}$
2	2	2	1	$\Phi_{221} \Phi_{222} = \Phi_{221} \Phi_{221} \Rightarrow \Phi_{222} = \Phi_{221}$
2	2	2	2	$\Phi_{222} \Phi_{222} = \Phi_{222} \Phi_{222} \quad \checkmark$

Appendix D

Complex Cross Product

As shown, the complex field (under complex multiplication) is isomorphic to the matrix pair (under ordinary matrix multiplication):

$$\mathbf{u}_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

or, more generally:

$$\mathbf{u}_0 \equiv \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}, \quad \mathbf{u}_1 \equiv \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

So, for $n = 2$:

$$\mathbf{u}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{u}_1 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Comparing to the quaternion matrices given:

$$\mathbf{q}_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{q}_1 \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{q}_2 \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{q}_3 \equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

only $\mathbf{u}_0$ and $\mathbf{q}_0$ match.

However, a look at the quaternion Cayley Table shows that the pairs:

$(\mathbf{q}_0, \mathbf{q}_1), (\mathbf{q}_0, \mathbf{q}_2), (\mathbf{q}_0, \mathbf{q}_3)$ under ordinary matrix multiplication, are all, also, isomorphic to the complex field.

Thus, the complex field is a subspace of the quaternion field.

So, the cross product of the complex field may be given by that of the quaternion pairs:

$$(s_1^0, s_1^1, s_1^2, s_1^3) \times (s_2^0, s_2^1, s_2^2, s_2^3) = (0, s_1^0 s_2^1 - s_1^1 s_2^0, s_1^0 s_2^2 - s_1^2 s_2^0 - s_1^3 s_2^3 - s_1^1 s_2^2)$$

∴

$$(q_0, q_1, 0, 0) :$$

$$\begin{aligned} (s_1^0, s_1^1, 0, 0) \times (s_2^0, s_2^1, 0, 0) &= \frac{1}{2} [(s_1^0, s_1^1, 0, 0) \circ (s_2^0, s_2^1, 0, 0)^* \\ &\quad - (s_2^0, s_2^1, 0, 0) \circ (s_1^0, s_1^1, 0, 0)^*] \\ &= (0, -s_1^0 s_2^1 - s_1^1 s_2^0, 0, 0) \end{aligned}$$

$$(q_0, 0, q_1, 0) :$$

$$\begin{aligned} (s_1^0, 0, s_1^2, 0) \times (s_2^0, 0, s_2^2, 0) &= \frac{1}{2} [(s_1^0, 0, s_1^2, 0) \circ (s_2^0, 0, s_2^2, 0)^* \\ &\quad - (s_2^0, 0, s_2^2, 0) \circ (s_1^0, 0, s_1^2, 0)^*] \\ &= (0, 0, s_1^0 s_2^2 - s_1^2 s_2^0, 0) \end{aligned}$$

$$(q_0, 0, 0, q_3) :$$

$$\begin{aligned} (s_1^0, 0, 0, s_1^3) \times (s_2^0, 0, 0, s_2^3) &= \frac{1}{2} [(s_1^0, 0, 0, s_1^3) \circ (s_2^0, 0, 0, s_2^3)^* \\ &\quad - (s_2^0, 0, 0, s_2^3) \circ (s_1^0, 0, 0, s_1^3)^*] \\ &= (0, 0, 0, s_1^0 s_2^3 - s_1^3 s_2^0) \end{aligned}$$

Note that the non-zero part of the cross product is in the same dimension as the non- q_0 dimension.

This makes it clear that the complex cross product should be imaginary.

Note:

The three dimensional cross product is simply defined in terms of the quaternion product:

$$\forall u, v \in \mathbb{S}, u \times_3 v \equiv \frac{1}{2}(u \circ v - v \circ u)$$

So:

$$(s_1^0, s_1^1, s_1^2, s_1^3) \times_3 (s_2^0, s_2^1, s_2^2, s_2^3) := (0, s_1^2 s_2^3 - s_1^3 s_2^2, -s_1^1 s_2^3 + s_1^3 s_2^1, s_1^1 s_2^2 - s_1^2 s_2^1)$$

The three dimensional dot product is also simply defined in terms of the quaternion product:

$$\forall u, v \in \mathbb{S}_s, u \cdot_3 v \equiv -\frac{1}{2}(u \circ v + v \circ u)$$

So:

$$(0, s_1^1, s_1^2, s_1^3) \cdot_3 (0, s_2^1, s_2^2, s_2^3) := (s_1^1 s_2^1 + s_1^2 s_2^2 + s_1^3 s_2^3, 0, 0, 0)$$

Appendix E

Common Definitions and Supporting Material, Analysis

Definition A5-1: The **Limit** of real-valued function $f(x)$ as the real variable x approaches real number a is real number L ,

$$\text{written: } \lim_{x \rightarrow a} f(x) = L,$$

$$\text{if: } \forall \epsilon \in \mathcal{R} > 0, \exists \delta \in \mathcal{R} > 0 \ni$$

$$|f(x) - L| < \epsilon \quad \text{whenever: } 0 < |x - a| < \delta$$

See: [9],[41].

Note A5-1: For vector function $\mathbf{f} : \mathcal{R}_{m \times m} \mapsto \mathcal{R}_{n \times n}$:

$$\mathbf{f} = \mathbf{f}(\boldsymbol{\beta}(x^i)_m)_n \equiv \mathbf{f} = \mathbf{f}(f^1, f^2, \dots, f^m) \ni$$

$$\boldsymbol{\beta} = \boldsymbol{\beta}(x^1, x^2, \dots, x^m), \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq m);$$

$$\mathbf{f} = \mathbf{f}(f_j(x^i)_m)_n \equiv \mathbf{f} = \mathbf{f}(f_1, f_2, \dots, f_n) \ni$$

$$f_j = f_j(x^1, x^2, \dots, x^m), \quad (\forall j \in \mathcal{Z}^+ \ni 1 \leq j \leq m);$$

$$\mathbf{f} = \mathbf{f}(\mathbf{x}) \equiv \mathbf{f} = \mathbf{f}(\boldsymbol{\beta}(x^i)_m)_n \vee \mathbf{f} = \mathbf{f}(f_j(x^i)_m)_n$$

Definition A5-2: The **Limit** of vector function $\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n$

as the vector variable $\mathbf{x} = (x^i)_m \in \mathcal{R}^m$ approaches vector

$$\mathbf{a} = (a^i)_m \in \mathcal{R}^m \text{ is the vector } \mathbf{L} = (L^i)_n \in \mathcal{R}^n,$$

$$\text{written: } \lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L},$$

$$\text{if: } \forall \epsilon^j \in \mathcal{R} > 0, \exists \delta^i \in \mathcal{R} > 0 \ni$$

$$|\boldsymbol{\beta}(x^i)_m - L^j| < \epsilon^j \quad \text{whenever: } 0 < |x^i - a^i| < \delta^i$$

$$(\forall i, j \in \mathcal{Z}^+ \ni 1 \leq i \leq m, 1 \leq j \leq n)$$

See: [45].

Definition A5-3: A vector function $\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n$ is

continuous as the vector variable $\mathbf{x} = (x^i)_m \in \mathcal{R}^m$:

approaches vector $\mathbf{a} = (a^i)_m \in \mathcal{R}^m$ if and only if: $\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$

See: [45].

Definition A5-4: The **variation** of a vector function $\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x})$ is given

by:

$$\delta \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x} + \delta \mathbf{x}) - \mathbf{f}(\mathbf{x})$$

Theorem A5-1: For vector function $\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \equiv \mathbf{u}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x}) \ni$

$$\mathbf{x} \in \mathcal{R}^m ; \mathbf{u}(\mathbf{x}) \in \mathcal{R} ; \mathbf{v}(\mathbf{x}) \in \mathcal{R}^n :$$

$$\begin{aligned} \delta \mathbf{f}(\mathbf{x}) &= \delta \mathbf{u}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x} + \delta \mathbf{x}) + \mathbf{u}(\mathbf{x}) \otimes \delta \mathbf{v}(\mathbf{x}) = \delta \mathbf{u} \otimes \mathbf{v}|_{\mathbf{x}+\delta \mathbf{x}} + \mathbf{u} \otimes \delta \mathbf{v} \\ &= \mathbf{u}(\mathbf{x} + \delta \mathbf{x}) \otimes \delta \mathbf{v}(\mathbf{x}) + \delta \mathbf{u}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x}) = \mathbf{u}|_{\mathbf{x}+\delta \mathbf{x}} \otimes \delta \mathbf{v} + \delta \mathbf{u} \otimes \mathbf{v} \end{aligned}$$

Definition A5-5: The **differential** of a vector function

$$\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n$$

at the point $\mathbf{x} = \mathbf{a} \in \mathcal{R}^m$, is given by:

$$d\mathbf{f}(\mathbf{a}) = \lim_{\mathbf{x} \rightarrow \mathbf{a}} [\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})]$$

or:

$$d\mathbf{f}(\mathbf{a}) = \lim_{\delta \mathbf{x} \rightarrow 0} \delta \mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{a}}$$

Definition A5-6:(notation)

$$\begin{aligned} \mathbf{f}(\mathbf{a}, \mathbf{b}; j, n) &\equiv \begin{cases} \mathbf{f}(\mathbf{b}) = \mathbf{f}(b^i)_n & , (0 = j) \\ \mathbf{f}((a^i)_j, (b^{j+i})_n) & , (0 < j \leq n) \end{cases} \\ &= \mathbf{f}(a^1, a^2, a^3, \dots, a^j, b^{j+1}, b^{j+2}, b^{j+3}, \dots, b^n) . \end{aligned}$$

Definition A5-7: partial derivative syntax:

$$\partial_i^m \mathbf{f}(\mathbf{x}) \equiv \begin{cases} \mathbf{f}(\mathbf{x}) & , m = 0 \\ \frac{\partial}{\partial x^i} \partial_i^{m-1} \mathbf{f}(\mathbf{x}) & , m > 0 \end{cases}$$

So:

$$\partial_i^0 \mathbf{f}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}) , \partial_i^1 \mathbf{f}(\mathbf{x}) \equiv \frac{\partial}{\partial x^i} \mathbf{f}(\mathbf{x}) ,$$

$$\forall m \in \mathbb{N} > 1 : \partial_i^m \mathbf{f}(\mathbf{x}) \equiv \frac{\partial}{\partial x^i} \partial_i^{m-1} \mathbf{f}(\mathbf{x})$$

Theorem A5-2: The Mean Value Theorem in space:

If the partial derivatives of vector function:

$$\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n \text{ exist}$$

$$\forall \mathbf{x} \ni a^i < x^i < b^i, \forall i \in \mathbb{Z}^+ \ni 1 \leq i \leq m , \text{ and}$$

$\mathbf{f}(\mathbf{x})$ is continuous at $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$, then
there is an $\xi \in \mathcal{R}^m$, $(a^i < \xi^i < b^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq n)$,
such that:

$$\mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a}) = \sum_{i=1}^n (b^i - a^i) \otimes \frac{\partial}{\partial x^i} \mathbf{f}(\xi)$$

Proof:

See [8],[75].

Corollary A5-2a: If the partial derivatives of vector function:

$\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n$ exist at $\mathbf{x}$ and
 $\mathbf{x} + \delta \mathbf{x}$, $\forall \delta \mathbf{x} > \mathbf{0}$; and $\mathbf{f}(\mathbf{x})$ is continuous at $\mathbf{x}$
and $\mathbf{x} + \delta \mathbf{x}$, then there is an
 $\xi \in \mathcal{R}^m$ ($x^i < \xi^i < x^i + \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq m$),
such that:

$$\delta \mathbf{f}(\mathbf{x}) = \sum_{i=1}^n \left[\delta x^i \otimes \frac{\partial \mathbf{f}(\xi)}{\partial x^i} \right].$$

Theorem A5-3: Taylor's Formula in space:

If the partial derivatives of vector function:

$\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n$ of order h exist
everywhere $\forall \mathbf{x} \ni a^i < x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq m$,
then there is an $\xi \in \mathcal{R}^m$ ($a^i < \xi^i < x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq m$),
such that:

$$\begin{aligned} \mathbf{f}(\mathbf{x}) = & \sum_{k=0}^h \left[\frac{1}{k!} \sum_{i=1}^m \{ (x^i - a^i)^k \otimes \partial_i^k \mathbf{f}(\mathbf{x}) \} \right] \\ & + \frac{1}{(h+1)!} \sum_{i=1}^m \{ (x^i - a^i)^{h+1} \otimes \partial_i^{h+1} \mathbf{f}(\xi) \} \end{aligned}$$

Proof:

See [45],[75].

Corollary A5-3a: If the partial derivatives of vector function:

$\mathbf{f} : \mathcal{R}^m \mapsto \mathcal{R}^n, \mathbf{f} = \mathbf{f}(\mathbf{x}) \in \mathcal{R}^n$ of order h exist
at $\mathbf{x}$ and $\mathbf{x} + \delta \mathbf{x}$, $\forall \delta \mathbf{x} > \mathbf{0}$, then there is an
 $\xi \in \mathcal{R}^m$ ($x^i < \xi^i < x^i + \delta x^i, \forall i \in \mathcal{Z}^+ \ni 1 \leq i \leq m$),
such that:

$$\mathbf{f}(\mathbf{x}) = \sum_{k=0}^h \left[\frac{1}{k!} \sum_{i=1}^m \{(\delta x^i)^k \otimes \partial_i^k \mathbf{f}(\mathbf{x})\} \right] + \frac{1}{(h+1)!} \sum_{i=1}^m \{(\delta x^i)^{h+1} \otimes \partial_i^{h+1} \mathbf{f}(\xi)\}$$

Definition A5-8: A **homeomorphism** is a bijective mapping: $f : X \rightleftharpoons Y \ni f$ and f^{-1} are continuous.

See: [44],[49] .

Definition A5-9: A **chart** of dimension n of a set M , including set U is the 4-tuple (U, h, n, M) , whenever, $\exists n \in \mathbb{Z}^+$, there exists a homeomorphism: $h : U \rightleftharpoons M \subset \mathcal{R}_{n \times n}$

See: [2],[44] .

Definition A5-10: A chart (U, h, n, M) is a **local coordinate system** of neighborhood U of space M .

Definition A5-11: An **atlas of dimension n** on set M is a collection of charts, such that, for some index set I :

$$\{(U_m, h_m, n, M) \mid \forall m \in I\} \ni \bigcup_{m \in I} U_m = M$$

See: [29],[44],[56] .

Definition A5-12: An atlas on set M is a **global coordinate system** of space M .

Definition A5-13: A **Hausdorff space**, M , is a topological space, such that:

$$\forall p, q \in M, \exists A, B \subset M \ni p \neq q \Rightarrow A \cap B = \emptyset .$$

See: [44],[47] .

A Hausdorff space is a topological space such that, distinct points have disjoint neighborhoods.

Definition A5-14: A set, M , can be covered by an indexed set $I : \{U_m \mid \forall m \in I\}$,

$$\text{whenever: } \bigcup_{m \in I} U_m = M .$$

Definition A5-15: A **manifold** is a Hausdorff space that can be covered by an atlas.

See: [18],[29],[44],[56],[57] .

Appendix F

A 2^M -dimensional $\mathcal{R}$ -algebra basis may be constructed from permutation matrices

$$\text{Let: } \mathbf{u}_1^{2^1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{u}_2^{2^1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

and generally, recursively:

$$\begin{aligned} \mathbf{u}_i^{2^k} &= \begin{pmatrix} \mathbf{u}_i^{2^{k-1}} & 0 \\ 0 & \mathbf{u}_i^{2^{k-1}} \end{pmatrix}, & (1 \leq i \leq 2^{k-1}) \\ \mathbf{u}_i^{2^k} &= \begin{pmatrix} 0 & \mathbf{u}_{i-2^{k-1}}^{2^{k-1}} \\ \mathbf{u}_{i-2^{k-1}}^{2^{k-1}} & 0 \end{pmatrix}, & (2^{k-1} + 1 \leq i \leq 2^k) \end{aligned}$$

The elegant and compact definition above expresses a set of permutation matrices suitable for constructing $\mathcal{R}$ -algebra bases of dimension: 2^M .

The following set of functions provide a framework for numbering these bases (as well as a less elegant way of expressing them) in an invariant manner, each embedded within the next higher power-of-two dimension, without change of indices.

Definition A6-1: (notation)

$$\sum_{i \geq X}^Y A(i) \equiv \begin{cases} = \sum_{i=X}^Y A(i), & \forall Y \geq X \\ = 0, & \forall Y < X \end{cases}$$

Definition A6-2:

$$T_0(i) \equiv \frac{1}{2} \left(i - 1 + \delta_{(-1)^i}^1 \right), \quad (i \in \mathbb{Z}^+)$$

Definition A6-3:

$$S_n(i) \equiv i + 2^n \cdot (-1)^{T_{n-1}(i)} \cdot (1 - \delta_0^i), \quad (i, n \in \mathbb{Z}^+)$$

Definition A6-4:

$$I(h) \equiv \sum_{r \geq 0}^{h-1} \delta_{T_r(h)}^1 \cdot (r+1), \quad (h \in \mathbb{Z}^+)$$

Definition A6-5:

$$J_h \equiv \begin{cases} = \{\}, & (h \in \{0, 1\}) \\ = \{I(h)\}, & (h \in \{2, 3, 4, 5\}) \\ = \{I(h), y \mid y \in I(h - 2^{I(h)})\}, & (h \in \mathbb{Z}^+ \geq 6) \end{cases}$$

Definition A6-6:

$$L_h \equiv \begin{cases} = \{\}, & (h = 0) \\ = J_{h-2^{I(h)}}, & (h \in \mathbb{N} \geq 1) \end{cases}$$

Definition A6-7:

$$\begin{aligned} m_0(k, h, i) &\equiv S_k(i) + \sum_{r \in L_{2^k+h}} 2^r \cdot (-1)^{T_{r-1}(i)} + (1 - \delta_{(-1)^h}^1) \cdot (-1)^i, \\ (\forall i, j, h, k \in \mathbb{Z}^+ \ni k \geq 1, 0 \leq j \leq k-1, I(h+2^j \cdot [1 - \delta_j^0]) = j, 0 \leq i \leq 2^k - 1) \\ m(k, j, h, i) &\equiv m_0(j, h, i) \\ (\forall i, j, h, k \in \mathbb{Z}^+ \ni k \geq 1, 0 \leq j \leq k-1, I(h+2^j \cdot [1 - \delta_j^0]) = j, 0 \leq i \leq 2^k - 1) \end{aligned}$$

Definition A6-8:

$$w(k, i) \equiv \begin{cases} 0, & (\text{iff: } i = 0) \\ w(k, i-1) + (1 - \delta_2^k) \cdot (1 - \delta_{(-1)^i}^1) \cdot 2^{k-1} \cdot (-1)^{\left(\frac{i-2^0}{2^1}\right)} \\ \quad + \sum_{j \geq 0}^{k-4} \delta_{(-1)^{\frac{j}{2^j}}}^1 \cdot \left(1 - \delta_{(-1)^{\frac{j}{2^{j+1}}}}^1\right) \cdot 2^{k-1-(1+j)} \cdot (-1)^{\left(\frac{j-2^{0+(1+j)}}{2^{1+(1+j)}}\right)} \\ \quad + \sum_{j \geq 1}^3 \delta_{j-2^{k-2}}^i \cdot (1 - \delta_1^k), & (\text{iff: } i \neq 0) \\ (\forall i, k \in \mathbb{N} \ni k \geq 1, 0 \leq i \leq 2^k - 1; \text{Dimension} = 2^k) \end{cases}$$

Definition A6-9:

$$\mathbf{u}^{2^k; w(k,h)} \equiv \left(u_{ij}^{2^k; w(k,h)} \right) \ni u_{ij}^{2^k; w(k,h)} = \delta_{m(k, I(h), h-2^{I(h)} \cdot (1-\delta_0^{I(h)}), i)}^j$$

$$(\forall h, k \in N \ni 0 \leq j \leq k, 0 \leq h \leq 2^k - 1)$$

These base vectors form a basis for $\mathcal{R}^{2^k}$.

Here, these basis permutation matrices are determined entry by entry, and the indexing of the matrices is chosen for "embedding invariance."

For $M = 1$, (2-D): (see Appendix VI)

i	0	1
$w(1, i)$	0	1
$\left(\mathbf{u}_{ij}^{2;0} \right) = \delta_{m_0(0,0,i)}^j = \delta_{ij}$	$\left(\mathbf{u}_{ij}^{2;1} \right) = \delta_{m_0(0,1,i)}^j$	

as given above.

For $M = 2$, (4-D): (See Appendix VI)

i	0	1	2	3
$w(2, i)$	0	1	2	3
$\left(\mathbf{u}_{ij}^{4;0} \right) = \delta_{m_0(0,0,i)}^j = \delta_{ij}$	$\left(\mathbf{u}_{ij}^{4;1} \right) = \delta_{m_0(0,1,i)}^j$			
$\left(\mathbf{u}_{ij}^{4;2} \right) = \delta_{m_0(1,0,i)}^j$	$\left(\mathbf{u}_{ij}^{4;3} \right) = \delta_{m_0(1,1,i)}^j$			

as given above.

For $M = 3$, (8-D): (see Appendix VI)

i	0	1	2	3	4	5	6	7
$w(3,i)$	0	4	5	1	2	6	7	3

$\left(\mathbf{u}_{ij}^{8;0}\right)=\delta_{m_0(0,0,i)}^j=\delta_{ij}$	$\left(\mathbf{u}_{ij}^{8;4}\right)=\delta_{m_0(0,1,i)}^j$	$\left(\mathbf{u}_{ij}^{8;5}\right)=\delta_{m_0(1,0,i)}^j$	$\left(\mathbf{u}_{ij}^{8;1}\right)=\delta_{m_0(1,1,i)}^j$
$\left(\mathbf{u}_{ij}^{8;2}\right)=\delta_{m_0(2,0,i)}^j$	$\left(\mathbf{u}_{ij}^{8;6}\right)=\delta_{m_0(2,1,i)}^j$	$\left(\mathbf{u}_{ij}^{8;7}\right)=\delta_{m_0(2,2,i)}^j$	$\left(\mathbf{u}_{ij}^{8;3}\right)=\delta_{m_0(2,3,i)}^j$

so:

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and so on, for 2^M -dimensional space.

Other interesting $\mathcal{R}$ -algebras may be constructed using these permutation matrix bases of $\mathcal{R}^n$, as follows:

For dimension: 2^k (in this last case: $k = 3 \Rightarrow 8 - D$), $\mathcal{R}$ -algebras where dot and cross product of the usual form arise naturally using weighted matrix multiplication, with weights given by:

Definition A6-10:

$$\lambda_{rs}^{2^k} \equiv (-1)^{[\delta_s^r + \delta_2^r \delta_1^s + \delta_1^r (1 - \delta_2^s)(1 - \delta_s^r)] + [1 - \delta_1^r (1 - \delta_2^s)](1 - \delta_1^s)} \left(\sum_{h=s+1}^{2^k} \delta_r^h \right) =$$

$$= \sum_{h=0}^{2^k-1} \Phi_{hrs} \delta_{m_0(I(r), r-2^{I(r)} \cdot (1 - \delta_0^{I(r)}), r)}^h \delta_{m_0(I(s), s-2^{I(s)} \cdot (1 - \delta_0^{I(s)}), h)}^j$$

So:

$$\lambda_{rs}^{2^k} \equiv (-1)^{[\delta_s^r + \delta_2^r \delta_1^s + \delta_1^r (1 - \delta_2^s)(1 - \delta_s^r)] + [1 - \delta_1^r (1 - \delta_2^s)](1 - \delta_1^s)} \left(\sum_{h=s+1}^{2^k} \delta_r^h \right) =$$

$$= \Phi_{m_0(I(r), r-2^{I(r)} \cdot (1 - \delta_0^{I(r)}), r)} m_0(I(s), s-2^{I(s)} \cdot (1 - \delta_0^{I(s)}), m_0(I(r), r-2^{I(r)} \cdot (1 - \delta_0^{I(r)}), r))$$

$$= \Phi_{m_0(I(r), J(r), r)} m_0(I(m_0(I(s), J(s), s)), m_0(I(s), J(s), s) - 2^{I(m_0(I(s), J(s), s))} \cdot (1 - \delta_0^{I(m_0(I(s), J(s), s))}), r))$$

$$= \Phi_{2^k; m_0(I(r), r-2^{I(r)} \cdot (1 - \delta_0^{I(r)}), r)} m_0(I(s), s-2^{I(s)} \cdot (1 - \delta_0^{I(s)}), m_0(I(r), r-2^{I(r)} \cdot (1 - \delta_0^{I(r)}), r))$$

$$= \Phi_{2^k; m_0(I(r), J(r), r)} m_0(I(m_0(I(s), J(s), s)), m_0(I(s), J(s), s) - 2^{I(m_0(I(s), J(s), s))} \cdot (1 - \delta_0^{I(m_0(I(s), J(s), s))}), r))$$

(where: $J(t) \equiv t - 2^{I(t)} \cdot (1 - \delta_0^{I(t)})$)

Note:

with:

$$H(s, t) \equiv s + 2^t \cdot (1 - \delta_0^t)$$

then:

$$m_0(p, q, m_0(r, s, h)) =$$

$$= m_0(I(m_0(p, q, H(s, r))), m_0(p, q, H(s, r)) - 2^{I(m_0(p, q, H(s, r)))} \cdot (1 - \delta_0^{I(m_0(p, q, H(s, r)))}), h)$$

$$\forall p, q, r, s, h \in N \geq 0$$

Here is the definition of the basis product:

Definition A6-11:

$$\mathbf{u}^{2^k;r} \circ \mathbf{u}^{2^k;s} = \lambda_{rs}^{2^k} \mathbf{u}^{2^k;m_0(I(r),r-2^{I(r)} \cdot (1-\delta_0^{I(r)}),r)}$$

$w(k,h)$ and $\lambda_{rs}^{2^k}$ serve to embed each $\mathcal{R}^{2^{k-1}}$ within each $\mathcal{R}^{2^k}$, without change for some set of weights, under a weighted matrix product, resulting in an $\mathcal{R}$ -algebra with spacetime Minkowski pseudo-metric, and cross product reducing to the ordinary cross product under zeroed out time (x^0) components. This is referred to as a Lorentz-Minkowski Vectric Space.

For M , (2^M-D) , with:

$$\mathbf{A} \equiv \sum_{k=0}^{2^M-1} \mathbf{u}^{2^k;k} A^k, \quad \mathbf{B} \equiv \sum_{k=0}^{2^M-1} \mathbf{u}^{2^k;k} B^k$$

then:

$$\begin{aligned} \mathbf{A} \circ \mathbf{B} &= \sum_{r=0}^{2^M-1} \sum_{s=0}^{2^M-1} \mathbf{u}^{2^M;m_0(I(r),J(r),r)} \lambda_{rs} A^r B^s \\ &= \mathbf{u}^{2^M;0} \left(- \sum_{r=0}^{2^M-1} A^r B^r \right) + \sum_{r=1}^{2^M-1} \mathbf{u}^{2;r} \left[\sum_{s=0}^{2^M-1} \lambda_{sm_0(I(s),J(s),r)}^{2^M} A^s B^{m_0(I(s),J(s),r)} \right] \\ &= \mathbf{u}^{2^M;0} \left(- \sum_{r=0}^{2^M-1} A^r B^r \right) \\ &\quad + \frac{1}{2} \sum_{r=1}^{2^M-1} \mathbf{u}^{2;r} \left[\sum_{s=0}^{2^M-1} \lambda_{sm_0(I(s),J(s),r)}^{2^M} (A^s B^{m_0(I(s),J(s),r)} - A^{m_0(I(s),J(s),r)} B^s) \right] \end{aligned}$$

and, so:

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &\equiv \frac{1}{2} (\mathbf{A} \circ \mathbf{B} + \mathbf{B} \circ \mathbf{A}) = \mathbf{u}^{2;0} \left(- \sum_{k=0}^{2^M-1} A^k B^k \right), \odot\text{-product} \\ \mathbf{A} \times \mathbf{B} &\equiv \frac{1}{2} (\mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}), \times\text{-product} \\ &= \frac{1}{2} \sum_{r=1}^{2^M-1} \mathbf{u}^{2;r} \left[\sum_{s=0}^{2^M-1} \lambda_{sm_0(I(s),J(s),r)}^{2^M} (A^s B^{m_0(I(s),J(s),r)} - A^{m_0(I(s),J(s),r)} B^s) \right] \end{aligned}$$

This analysis describes the process of constructing a 2^M -dimensional Lorentz-Minkowski vectric space.

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