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Eigenvalues of graphs : algebraic connectivity and acyclic matrices

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EIGENVALUES OF GRAPHS: ALGEBRAIC CONNECTIVITY AND ACYCLIC MATRICES

A Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

Of the requirements for the Degree

Master of Science

By

Wafa Yacoub

August, 2004

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Abstract

Eigenvalues of Graphs: Algebraic Connectivity and Acyclic Matrices

By Wafa Yacoub

This thesis is an exposition of two seminal papers by Miroslav Fiedler. It includes the necessary background on graphs and matrices, and proofs of his theorems. It consists of two chapters of different topics of his work.

Chapter 1 discusses the paper “Algebraic Connectivity of Graphs.” It presents the properties of the algebraic connectivity of a graph. It explains the relation between algebraic connectivity and vertex and edge connectivity of a graph.

Chapter 2 studies the paper “Eigenvectors of Acyclic Matrices.” A matrix is acyclic if and only if its graph is a tree, and Fiedler’s main result here is that when A is acyclic, λ is an eigenvalue of A , and an eigenvector y corresponding to λ is known, the number of eigenvalues of A less than λ and the number greater than λ can be counted using the entries of y and the off diagonal entries of A .

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Chapter 1: Algebraic Connectivity of Graphs

This chapter is an exposition of the results in [F1]. Section 1.1 contains basic facts about graphs and associated matrices, and Fiedler's results are in Sections 1.2 and 1.3.

Section 1.1: Background, definitions, and notation about graphs

Contemporary graph theorists use somewhat different language and notation than Fiedler did, and we will employ contemporary definitions here. See, for example, references [HJ], [Tu], and [TS]. All our matrices will be real and all vectors will be columns unless explicitly displayed as rows.

Most of the notation we need is in the Appendix of this thesis. However, we will define a few symbols here that will be used frequently.

The $n \times n$ identity matrix will be denoted $\mathbf{I}_n$ or just $\mathbf{I}$ when its size is clear, and the

symbol $\mathbf{e}$ denotes the vector $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$, of size appropriate for the setting.

The symbol $\mathbf{J}$ denotes the $n \times n$ matrix $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} [1 \ 1 \ \dots \ 1 \ 1] = \begin{bmatrix} 1 & \dots & \dots & \dots & 1 \\ \vdots & \ddots & & & \vdots \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & \dots & 1 \end{bmatrix}$.

The cardinality of a set S of elements will be denoted $|S|$.

When $\mathbf{y}$ is an $n \times 1$ column vector, the length of $\mathbf{y}$ is $\|\mathbf{y}\| = \sqrt{\mathbf{y}^T \mathbf{y}} = \sqrt{y_1^2 + \dots + y_n^2}$.

The set of all $n \times 1$ column vectors $\mathbf{x}$ such that $\mathbf{x}^T \mathbf{e} = 0$ and $\mathbf{x}^T \mathbf{x} = 1$ will be denoted $\mathbf{W}$.

For example, when $n = 2$, $\mathbf{W} = \left\{ \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix} \right\}$. When $n = 3$, $\mathbf{W}$ is the set of all

vectors of length one which lie in the plane through the origin that is orthogonal to $\mathbf{e}$.

(1.1.1) Definition: A **graph** $G = (V, E)$ consists of a finite nonempty set V of vertices v_i and a set E of edges $\{v_i, v_j\}$ where an edge $\{v_i, v_j\}$ joins distinct vertices v_i and v_j , and where there is at most one edge joining any vertices v_i and v_j .

Thus, in this paper, a graph will not have loops or multiple edges. When necessary for clarity, we will write $V(G)$ and $E(G)$ instead of V and E .

(1.1.2) Example: Let

Then $V(G) = \{v_1, v_2, v_3, v_4\}$ and

$E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}\}$.

(1.1.3) Example: The first graph below is simple, but the other two are not.

Source: mathworld.wolfram.com/SimpleGraph.html

(1.1.4) Definition: A **path** is a sequence of distinct vertices $\{v_1, v_2, \dots, v_n\}$ for which there is an edge between each two consecutive vertices. Observe that a vertex may not appear more than once in a path. The **length** of a path is the number of edges in it.

For example, in (1.1.2), $\{v_1, v_2, v_3\}$ is a path of length 2 but $\{v_1, v_2, v_4\}$ is not a path.

(1.1.5) Definition: A **subgraph** of $G = (V, E)$ is a graph $G' = (V', E')$ where V' is a nonempty subset of V and $E' \subseteq E$. We will write $G' \subseteq G$ to mean G' is a subgraph of G .

(1.1.6) Definition: A graph is **connected** if there is a path between any pair of distinct vertices. A **component** of a graph is a largest possible connected subgraph. Thus a connected graph is a graph with a single component. An **unconnected** graph is a graph with two or more components.

In Example (1.1.2), the graph G is connected, while the following graph has two components and is unconnected.

(1.1.7) Definition: A **complete** graph G is a graph in which there is an edge joining each pair of distinct vertices. In other words, G is a complete graph when it has all possible edges between its vertices. The complete graph with n vertices is denoted K_n .

The following figure illustrates $K_2, K_3, \dots, K_7$.

Source: mathworld.wolfram.com/CompleteGraph.html

(1.1.8) Definition: Two vertices v_i and v_j are said to be **adjacent** if there is an edge connecting them.

For example, in (1.1.2) the vertices v_1 and v_2 are adjacent but v_2 and v_4 are not.

(1.1.9) Definition: The **degree** of a vertex is the number of edges incident to the vertex.

In example (1.1.2), the degree of v_2 is two and the degree of v_1 is three.

(1.1.10) Definition of adjacency matrix: If G is any graph, its **adjacency matrix**

$A(G) = [a_{ij}]$ is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \end{cases}$$

When no confusion is possible, we will write A instead of $A(G)$.

(1.1.11) Proposition: Characteristics of the adjacency matrix A of a graph G .

1. If n is the number of vertices of G , then A is $n \times n$.
2. The adjacency matrix is symmetric.
3. The adjacency matrix is a $(0, 1)$ matrix.
4. The sum of each row of A equals the degree of the corresponding vertex.
5. Each diagonal entry of A is zero (because G has no loops).

(1.1.12) Example: Consider the following graph

Its adjacency matrix is $A(G) =$

$$\begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

(1.1.13) Definition: A graph G is **directed** if each edge has a direction, denoted by an arrow.

For example, the following is a directed graph.

Sometimes we will say that a graph G is **non-directed** to emphasize that its edges have not been given directions.

All examples before (1.1.13) are non-directed.

(1.1.14) Definition of incidence matrix: If G is a directed graph, its **incidence matrix**

$N(G) = [n_{ij}]$ is defined as follows: Let $\{e_1, \dots, e_m\}$ be an ordering of the edges of G , and define

$$n_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head (start point) of the edge } e_j = \{v_i, v_k\}: & v_i \bullet \longrightarrow \bullet v_k. \\ -1 & \text{if } v_i \text{ is the tail (end point) of the edge } e_j = \{v_i, v_k\}: & v_i \blacktriangleleft \longrightarrow \bullet v_k. \\ 0 & \text{otherwise.} \end{cases}$$

(1.1.15) Proposition: Characteristics of the incidence matrix N of a directed graph G .

1. If n is the number of vertices and m is the number of edges of G , then N is $n \times m$.
2. Each column of N is zero except for one $(+1)$ and one (-1) .
3. The sum of the absolute values of the entries of row i of N equals the degree of the vertex v_i .
4. Changing the labeling of vertices in G causes a permutation of the rows of the incidence matrix.
5. Changing the labeling of edges in G causes a permutation of the columns of the incidence matrix.
6. Reversing the direction of each edge yields an incidence matrix which is opposite in sign.

(1.1.16) Example: Consider the directed graph

Its incidence matrix is $N(G) =$

$$\begin{array}{c} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}.$$

Illustration of (1.1.15.3): In Example (1.1.16), the sum of absolute values in row two of $N(G)$ is four, which is the degree of v_2 in G .

Illustration of (1.1.15.4): In Example (1.1.16), relabeling the vertices of G gives a different graph:

$$\text{and } N(G_1) = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{matrix}. \text{ Notice that the rows of } N(G_1) \text{ are}$$

those of $N(G)$ permuted.

Illustration of (1.1.15.5): In Example (1.1.16), relabeling the edges of G gives another new graph

$$\text{and } N(G_2) = \begin{matrix} & f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}. \text{ Notice that the columns of}$$

$N(G_2)$ are those of $N(G)$ permuted.

Fiedler's paper [F1] is primarily about the following concept. It had been studied before by other authors, but he discovered remarkable new properties, as we will see.

(1.1.17) Definition: The **Laplacian matrix** of a directed or nondirected graph G , is $L(G) = D(G) - A(G)$ where $A(G)$ is the adjacency matrix of the graph and $D(G)$ is the diagonal matrix in which d_{ii} is the degree of vertex i .

The Laplacian matrix got its name because matrices that look like this are used to approximate the Laplacian differential operator in many applications [15].

(1.1.18) Remark: When no confusion is likely, we will write L , D , A , and N instead of $L(G)$, $D(G)$, $A(G)$, and $N(G)$.

(1.1.19) Example: Illustration of Definition (1.1.17): Let G be the non-directed graph in Example (1.1.12) (or the graph in Example (1.1.16) with directions ignored)

Then the Laplacian matrix of G is $L(G) = D(G) - A(G) =$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}. \text{ For future reference,}$$

we record here the spectrum of $L(G)$: $0 \leq 1 \leq 3 \leq 3 \leq 5$.

For another example, let

$H =$

$$L(H) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \text{ For future reference, we record here the spectrum of}$$

$L(H)$: $0 \leq 0 \leq 2 \leq 3 \leq 3$.

(1.1.20) Proposition: Characteristics of the Laplacian matrix L of a graph G.

1. The sum of each row of L equals zero.
2. The matrix L is an $n \times n$ matrix where n is the number of vertices of G.
3. If G is a directed graph and N is its incidence matrix, then $L = NN^T$.
4. Changing the labeling of edges does not affect the Laplacian matrix of G.
5. The matrix L is a singular matrix.
6. The Laplacian matrix is a symmetric real matrix and it is positive semidefinite. In particular, its eigenvalues are real and nonnegative, and the algebraic and geometric multiplicities of any eigenvalue of L are equal.

Proof of (1.1.20.3): Fix a vertex v_i . The (i, i) diagonal entry of NN^T is the sum of all n_{ij}^2 and n_{ij} is ± 1 if and only if v_j is adjacent to v_i ; thus $(NN^T)_{ii}$ equals the degree of vertex v_i , which is also L_{ii} . When $j \neq i$, entry $(NN^T)_{ij}$ equals the sum of all terms $n_{ie}n_{je}$ where e is an edge of G. Thus $(NN^T)_{ij}$ equals -1 if (v_i, v_j) is an edge e of G, and zero otherwise, which is L_{ij} . $\square$

Proof of (1.1.20.4): Give each edge of G a direction, so $N(G)$ is defined. Permute the edges of G to get a new graph G' with incidence matrix N' ; then $N' = NP$ where P is a permutation matrix, and $L' = N'(N')^T = (NP)(NP)^T = NPP^T N^T = NN^T = L$. $\square$

Proof of (1.1.20.5): The Laplacian matrix is singular since the sum of each row is zero,

so $\mathbf{L}\mathbf{e} = \mathbf{0}$ for the nonzero vector $\mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$. $\square$

Proof of (1.1.20.6): To show that $\mathbf{L}$ is positive semidefinite, note that since $\mathbf{L} = \mathbf{N}\mathbf{N}^T$, $\mathbf{L}$ is symmetric and $\mathbf{x}^T\mathbf{L}\mathbf{x} = (\mathbf{x}^T\mathbf{N})(\mathbf{N}^T\mathbf{x}) = \|\mathbf{N}^T\mathbf{x}\|^2 \geq 0$. Therefore $\mathbf{L}$ is positive semidefinite by the Appendix, (A.4). $\square$

(1.1.21) Example: The directed graph in Example (1.1.16) has the same vertices and edges as the non-directed graph, which appears in Example (1.1.19). Its incidence matrix $\mathbf{N}$ was calculated in (1.1.16) and its Laplacian in (1.1.19). The reader can verify that $\mathbf{N}\mathbf{N}^T$ does equal $\mathbf{L}$.

(1.1.22) Definitions: The **eigenvalues of a graph** $\mathbf{G}$ are the eigenvalues of its Laplacian Matrix $\mathbf{L}$. The set of these eigenvalues is called the **spectrum of $\mathbf{G}$** as well as the spectrum of $\mathbf{L}(\mathbf{G})$. By (1.1.20.6), these eigenvalues are real and nonnegative. When $\mathbf{G}$ has n vertices, we will number them as: $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Since $\mathbf{L}$ is singular, $\lambda_1 = 0$.

(1.1.23) Definition: Two graphs are **edge-disjoint** if they have no edge in common.

(1.1.24) Definition: Two graphs are **vertex-disjoint** if they have no vertex in common.

(1.1.25) Example: Let

are edge-disjoint but not vertex-disjoint.

(1.1.26) Definition [TS]: If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the **union** of G_1 and G_2 is the graph $G = (V_1 \cup V_2, E_1 \cup E_2)$. We denote the union of G_1 and G_2 as $G_1 \cup G_2$ or $G_2 \cup G_1$.

For instance, in Example (1.1.25), G equals $G_1 \cup G_2$.

Section 1.2: Properties of the algebraic connectivity of a graph

This section explains the first part of Fiedler's results in his 1973 paper [F1] about algebraic connectivity. He begins by noting that the second smallest eigenvalue of the Laplacian matrix of a graph G is zero if and only if G is not connected (see (1.2.2) below). Because of this, he made the following definition.

(A.II.1) Definition [F1]: The second smallest eigenvalue λ_2 of the Laplacian matrix L , for a graph G is denoted $a(G)$ and called the **algebraic connectivity** of the graph G .

Later authors have called the eigenvectors of L corresponding to $a(G)$ the **Fiedler vectors** for G (for example, see [FK]).

(1.2.2) Theorem [F1]: Let G be a graph with n vertices. The algebraic connectivity $a(G)$ is zero if and only if the graph G is not connected. In fact the number of components of G equals the number of its eigenvalues, which are zero, that is, the dimension of the null space of L .

Proof: Because L is diagonalizable, the number of its zero eigenvalues equals the dimension of its null space by (A.2). So we need to describe the vectors $\mathbf{x}$ such that $L\mathbf{x} = \mathbf{0}$. If G is not directed, assign a direction to each edge so that $N = N(G)$ is defined. Since $L = NN^T$ by (1.1.20.3), $\|L\mathbf{x}\|^2 = \|N^T \mathbf{x}\|^2$.

Let $\{v_i, v_j\}$ be an edge of G . The $\{v_i, v_j\}$ entry of $N^T \mathbf{x}$ can be visualized as

$$\begin{matrix} & v_i & & v_j \\ \{v_i, v_j\} & \begin{bmatrix} 0 & \cdots & 0 & \pm 1 & 0 & \cdots & 0 & \mp 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \{v_i, v_j\} \begin{bmatrix} \pm(x_i - x_j) \end{bmatrix} \end{matrix}$$

This is because the rows of N^T are indexed by the edges $\{v_i, v_j\}$ of G , and the columns

of N^T are indexed by the vertices of G , so $N^T \mathbf{x}$ has $x_i - x_j$ in the row indexed by

$\{v_i, v_j\}$ if the $\{v_i, v_j\}$ edge is oriented as $v_i \bullet \longrightarrow \bullet v_j$, or $x_j - x_i$ if this edge is

oriented as $v_j \bullet \longrightarrow \bullet v_i$. Therefore, $\|L\mathbf{x}\|^2 = \|N^T \mathbf{x}\|^2 = \sum (x_i - x_j)^2$, so $L\mathbf{x} = \mathbf{0}$

is true if and only if $x_i = x_j$ whenever $\{v_i, v_j\}$ is an edge in G . It follows that if $L\mathbf{x} = \mathbf{0}$

and $(v_{i1}, \dots, v_{ik})$ is a path in G , then $x_{i1} - x_{i2} = 0, x_{i2} - x_{i3} = 0, \dots, x_{i,k-1} - x_{ik} = 0$.

Therefore $L\mathbf{x} = \mathbf{0}$ if and only if for each component of G and all the edges $\{v_i, v_j\}$ in

that component, $x_i = x_j$. Thus if G is a connected graph and $L\mathbf{x} = \mathbf{0}$, then $\mathbf{x}$ is a multiple

of $\mathbf{e}$, so the null space of L has dimension 1. If G has components $G_1, \dots, G_p$ then by

renumbering vertices, we can assume $L(G) = \begin{bmatrix} L(G_1) & & 0 \\ & \ddots & \\ 0 & & L(G_p) \end{bmatrix}$.

Let $z^{(i)}$ be the vector of zeros of length $|V(G_i)|$ and let $e^{(i)}$ be the vector of ones of

length $|V(G_i)|$. Then $\left\{ \begin{bmatrix} z^{(1)} \\ \vdots \\ e^{(i)} \\ z^{(i+1)} \\ \vdots \\ z^{(p)} \end{bmatrix} : i = 1, \dots, p \right\}$ is a basis for the null space of L , so it has

dimension p . Thus the spectrum of G is $0 = \lambda_1 = \dots = \lambda_p < \lambda_{p+1} \leq \dots \leq \lambda_n$. That is, the number of components of G does equal the number of its eigenvalues, which are zero. $\square$

For example, in (1.1.19), the eigenvalues of the connected graph G are: 0, 1, 3, 3, and 5, while the unconnected graph H has two components and spectrum 0, 0, 2, and 2. Thus G has one zero eigenvalue and H has two, as expected.

(1.2.3) Proposition ((3.1), [F1]): If G_1 and G_2 are edge-disjoint graphs with the same set of n vertices, then $L(G) = L(G_1) + L(G_2)$ and $a(G_1) + a(G_2) \leq a(G_1 \cup G_2)$.

Proof: Let the spectrum of $L(G_1)$ be $0 = \eta_1 \leq \eta_2 \leq \dots \eta_{n-1} \leq \eta_n$ and the spectrum of $L(G_2)$ be $0 = \lambda_1 \leq \lambda_2 \leq \dots \lambda_{n-1} \leq \lambda_n$.

Because $G_1 \cup G_2$ has the same vertices as G_1 and G_2 , $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$, and $E(G_1)$ and $E(G_2)$ are disjoint by hypothesis. It follows that $L(G_1 \cup G_2) = L(G_1) + L(G_2)$ and $a(G_1 \cup G_2) = \lambda_2(L(G_1 \cup G_2))$. Using

these facts and (A.6) of the Appendix, $\lambda_2(L(G_1 \cup G_2)) = \min_{x \in W} (x^T L(G_1 \cup G_2)x) =$

$$\min_{x \in W} (x^T L(G_1)x + x^T L(G_2)x) \geq \min_{x \in W} x^T L(G_1)x + \min_{x \in W} x^T L(G_2)x = a(G_1) + a(G_2). \square$$

(1.2.4) Example: Consider the following non-directed graphs:

Then $L(G_1) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ and its eigenvalues are $0 \leq 2 - \sqrt{2} \leq 2 \leq 2 + \sqrt{2}$, so

$$a(G_1) = 2 - \sqrt{2}.$$

In the same way, $L(G_2) = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$ and its spectrum is also

$$0 \leq 2 - \sqrt{2} \leq 2 \leq 2 + \sqrt{2}, \text{ so } a(G_2) = 2 - \sqrt{2}. \text{ Finally, } G_1 \text{ and } G_2 \text{ are edge disjoint, } G$$

$$= G_1 \cup G_2, L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \text{ and its spectrum is } 0 \leq 4 \leq 4 \leq 4, \text{ hence } a(G) = 4.$$

Therefore $a(G_1 \cup G_2) \geq a(G_1) + a(G_2)$ is true. Notice $L(G)$ does equal $L(G_1) + L(G_2)$.

(1.2.5) Corollary ((3.2), [F1]): If G_1 and G are graphs with the same set of vertices, and $G_1 \subseteq G$, then $a(G_1) \leq a(G)$.

Proof: Since G and G_1 have the same set of vertices and $G_1 \subseteq G$, we have two cases.

Case 1: If $G_1 = G$ then $a(G_1) = a(G)$.

Case 2: If $G_1 \subset G$, then the Laplacian matrix of G equals the Laplacian matrix of G_1 plus the extra 1, -1 entries that correspond to edges that are in G but not G_1 . Let G^* be the subgraph of G with the same vertices and $E(G^*) = E(G) - E(G_1)$. Then by (1.2.3), $a(G_1) + a(G^*) \leq a(G_1 \cup G^*) = a(G)$, which implies $a(G_1) + a(G^*) \leq a(G)$. Thus $a(G_1) \leq a(G)$. $\square$

For example, in (1.2.4), $G_1 \subseteq G$, and $a(G_1) = 2 - \sqrt{2} \leq 4 = a(G)$.

(1.2.6) Corollary: Let $G_1, \dots, G_r$ be edge disjoint graphs with the same vertices. Then $a(G_1) + a(G_2) + \dots + a(G_r) \leq a(G_1 \cup G_2 \cup \dots \cup G_r)$.

Proof: Let P_r denote the statement $a(G_1) + a(G_2) + \dots + a(G_r) \leq a(G_1 \cup G_2 \cup \dots \cup G_r)$.

We need to show that P_r is true for $r \geq 2$. By (1.2.3), P_2 is true, that is, $a(G_1) + a(G_2) \leq a(G_1 \cup G_2)$ is true. Now assume that for $r \geq 3$, P_{r-1} is true, so $a(G_1) + a(G_2) + \dots + a(G_{r-1}) \leq a(G_1 \cup G_2 \cup \dots \cup G_{r-1})$ is true. We need to show that P_r is also true.

The induction hypothesis, $a(G_1) + a(G_2) + \dots + a(G_{r-1}) \leq a(G_1 \cup G_2 \cup \dots \cup G_{r-1})$ implies $a(G_1) + a(G_2) + \dots + a(G_{r-1}) + a(G_r) \leq a(G_1 \cup G_2 \cup \dots \cup G_{r-1}) + a(G_r)$ and this is less than or equal to $a(G_1 \cup G_2 \cup \dots \cup G_{r-1} \cup G_r)$ by (1.2.3). $\square$

(1.2.7) Theorem ((3.3), [F1]): Let $G = (V, E)$ be a graph. If G_1 is the graph that results by removing k vertices and all their adjacent edges, then $a(G_1) \geq a(G) - k$.

Proof: Let $G = (V, E)$ be a graph on n vertices. Let $k = 1$, and let G_1 be the subgraph that results from removing the vertex v_n and its adjacent edges. If G is not connected then $a(G) = 0$, so $a(G) - 1 = -1 \leq a(G_1)$ is true. Suppose that G is connected, and let e denote the $(n-1) \times 1$ column of ones.

Define a new graph $G' = (V, E')$ where E' contains $E(G_1)$ and all edges $\{v_1, v_n\}, \dots, \{v_{n-1}, v_n\}$. Notice G' is connected because v_n is adjacent to every other vertex in G' , so $a(G') \neq 0$. Thus the degree in G' of v_n is $n-1$ and the degree in G' of any other v_i is one more than its degree in G_1 . Therefore $L(G') = \begin{pmatrix} L(G_1) + I & -e \\ -e^T & n-1 \end{pmatrix}$: the last column and row are filled with -1 in the first $n-1$ positions as indicated because v_n is

adjacent to every other vertex in G' ; and $L(G_1) + I$ is correct because G' has the edges $\{v_1, v_n\}, \dots, \{v_{n-1}, v_n\}$, in addition to the edges in G_1 .

Now let v be an eigenvector of $L(G_1)$ corresponding to the eigenvalue $a(G_1)$, so e and v are eigenvectors corresponding to different eigenvalues, thus orthogonal by (A.6.b).

$e^T v = 0$. It follows that $L(G') \begin{pmatrix} v \\ 0 \end{pmatrix} = (a(G_1) + 1) \begin{pmatrix} v \\ 0 \end{pmatrix}$. Therefore $a(G_1) + 1$ is an eigenvalue of $L(G')$ and it is greater than zero, so $a(G') \leq a(G_1) + 1$ because $a(G')$ is no larger than the smallest nonzero eigenvalue of $L(G')$. But $a(G) \leq a(G')$ by (1.2.5) since G and G' have the same set of vertices and $G \subseteq G'$, so we have $a(G) - k \leq a(G') - k \leq a(G_1)$ when $k = 1$.

Now induct. Let P_k denote the statement "If G_1 is the graph obtained from G by removing k vertices, then $a(G_1) \geq a(G) - k$ ". We have shown P_1 is true and need to show that P_k is true for every $k \geq 1$. Let $k \geq 2$ and assume P_{k-1} is true; that is, $a(G_1) \geq a(G) - (k-1)$ is true when $k-1$ vertices are removed from G , leaving G_1 .

Suppose $v_1, \dots, v_k$ are removed from G , leaving G_2 . Notice G_2 can be obtained by removing one vertex from G_1 , so by the first case above, $a(G_2) \geq a(G_1) - 1$. By the induction hypothesis we have $a(G_1) \geq a(G) - (k-1)$. Therefore $a(G_2) \geq a(G) - (k-1) - 1 = a(G) - k$. $\square$

Here are three examples. In each, we remove v_3 (and adjacent edges) to get G_1 . First consider

Then $L(G) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has spectrum $0 \leq 0 \leq 2$, $L(G_1) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has spectrum

$0 \leq 2$, and $2 = a(G_1) \geq a(G) - 1 = 0 - 1$ as expected.

Next let

Then $L(G) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ has spectrum $0 \leq 1 \leq 3$, $L(G_1) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has spectrum

$0 \leq 2$, and $2 = a(G_1) \geq a(G) - 1 = 1 - 1$ as expected.

Finally, let

Then $L(G) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ has spectrum $0 \leq 1 \leq 3$ and $L(G_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has spectrum

$0 \leq 0$. Thus $a(G_1) = 0$, $a(G) = 1$, and $0 = a(G_1) \geq a(G) - 1$. $\square$

Fiedler does not make the following observation, but it is an easy consequence of (1.2.7).

(1.2.8) Corollary: If a graph G can be unconnected by removing k vertices and all their adjacent edges, then $a(G) \leq k$.

Proof: Let G_1 be the unconnected graph that results by removing k vertices and all adjacent edges from G . Then by (1.2.7), $a(G_1) \geq a(G) - k$. But $a(G_1) = 0$ since G_1 is unconnected, so $a(G) \leq k$. $\square$

(1.2.9) **Example:** Remove v_1 and v_3 from the following graph G , getting the

unconnected graph G_1 :

Then $k = 2$, $L(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$ has spectrum $0 \leq 2 \leq 4 \leq 4$, and we see that

$a(G) = 2 \leq k$ is true.

(1.2.10) **Definition:** Let G have $V(G) = \{v_1, \dots, v_n\}$ and let H have $V(H) = \{w_1, \dots, w_m\}$.

The **product graph** $G \times H$ is the graph with vertices $V(G \times H) = V(G) \times V(H) =$

$\{(v_i, w_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$, and edges all pairs of the form $\{(v_i, w_j), (v_k, w_j)\}$

where $\{v_i, v_k\}$ is an edge of G or $\{(v_i, w_j), (v_i, w_k)\}$ where $\{w_j, w_k\}$ is an edge of H .

(1.2.11) Example:

The edges of the product graph $G \times H$ come from two copies of G and four copies of H .

Since G has 4 edges and H has one edge, $G \times H$ has $2(4) + 4(1) = 12$ edges. Since G has 4 vertices and H has two vertices, $G \times H$ will have $4(2) = 8$ vertices.

The vertices of $G \times H$, labeled in lexicographic order, are $u_1 = (v_1, w_1)$, $u_2 = (v_1, w_2)$, $u_3 = (v_2, w_1)$, $u_4 = (v_2, w_2)$, $u_5 = (v_3, w_1)$, $u_6 = (v_3, w_2)$, $u_7 = (v_4, w_1)$, and $u_8 = (v_4, w_2)$.
 $E(G \times H) = \{((v_1, w_1), (v_1, w_2)), ((v_2, w_1), (v_2, w_2)), ((v_3, w_1), (v_3, w_2)), ((v_4, w_1), (v_4, w_2)); ((v_1, w_1), (v_2, w_1)), ((v_2, w_1), (v_3, w_1)), ((v_3, w_1), (v_4, w_1)), ((v_1, w_1), (v_3, w_1)); ((v_1, w_2), (v_2, w_2)), ((v_2, w_2), (v_3, w_2)), ((v_3, w_2), (v_4, w_2)), ((v_1, w_2), (v_3, w_2)))\}.$

The product graph $G \times H$ can be drawn as:

Now we inspect the Laplacian matrices and eigenvalues of G , H and $G \times H$. Observe

$$\text{that } L(H) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ has spectrum } 0 \leq 2, \text{ and } L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \text{ has}$$

spectrum $0 \leq 1 \leq 3 \leq 4$.

$$\text{Also, } L(G \times H) = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 3 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \text{ has spectrum}$$

$0 \leq 1 \leq 2 \leq 3 \leq 3 \leq 4 \leq 5 \leq 6$.

Using the definition of Kronecker product in (A.7), observe that

$$L(G) \otimes I_2 = \begin{bmatrix} 2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 3 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\text{and } I_4 \otimes L(H) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Finally, observe that $L(G \times H)$ equals the sum of the last two matrices, which illustrates the following theorem.

(1.2.12) Proposition: Let G and H be graphs with $V(G) = \{v_1, \dots, v_n\}$ and $V(H) = \{w_1, \dots, w_m\}$. If the vertices of $G \times H$ are labeled lexicographically, then $L(G \times H) = (L(G) \otimes I_m) + (I_n \otimes L(H))$.

$$\text{Proof: Let } L(G) = \begin{bmatrix} d_1 & d_{12} & \dots & d_{1n} \\ d_{21} & d_2 & \dots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \dots & d_n \end{bmatrix} \text{ and } L(H) = \begin{bmatrix} f_1 & f_{12} & \dots & f_{1m} \\ f_{21} & f_2 & \dots & f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \dots & f_m \end{bmatrix}.$$

$$\text{Let } X = L(G) \otimes I_m = \begin{bmatrix} d_1 I_m & d_{12} I_m & \dots & d_{1n} I_m \\ d_{21} I_m & d_2 I_m & \dots & d_{2n} I_m \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} I_m & d_{n2} I_m & \dots & d_n I_m \end{bmatrix} \text{ and}$$

$$Y = I_n \otimes L(H) = \begin{bmatrix} L(H) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & L(H) \end{bmatrix}. \text{ All blocks in X and Y are mxm, so}$$

$$X + Y = \begin{bmatrix} d_1 I_m + L(H) & d_{12} I_m & \cdots & d_{1n} I_m \\ d_{21} I_m & d_2 I_m + L(H) & \cdots & d_{2n} I_m \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} I_m & d_{n2} I_m & \cdots & d_n I_m + L(H) \end{bmatrix}.$$

Case 1: A diagonal entry of the diagonal block $d_i I_m + L(H)$ in $X + Y$, is a $((v_i, w_j), (v_i, w_j))$ entry, which equals $d_i + f_j$. Clearly d_i equals the number of pairs $((v_i, w_j), (v_k, w_j))$ where (v_i, v_k) is an edge adjacent to v_i in G , and f_j equals the number of pairs $((v_i, w_j), (v_i, w_k))$ where (w_j, w_k) is an edge adjacent to w_j in H . Thus $d_i + f_j$ is the number of edges adjacent to (v_i, w_j) in $G \times H$, which is the $((v_i, w_j), (v_i, w_j))$ entry of $L(G \times H)$.

Case 2: Consider an off diagonal entry of the diagonal block $d_i I_m + L(H)$ in $X + Y$. This block has its rows and columns indexed by $(v_i, w_1), \dots, (v_i, w_m)$, so an off diagonal entry of this block is a $((v_i, w_j), (v_i, w_k))$ entry where $k \neq j$. In $d_i I_m + L(H)$ this entry is -1 if (w_j, w_k) is an edge of H , zero otherwise. Thus it equals the $((v_i, w_j), (v_i, w_k))$ entry of $L(G \times H)$.

Case 3: Consider an off diagonal block of $X + Y$, so it is of the form $d_{ik} I_m$ where $i \neq k$. Thus this block is zero if (v_i, v_k) is not an edge of G , and is $-I_m$ if (v_i, v_k) is an

edge of G . In more detail, the rows of this block are indexed by $(v_i, w_1), \dots, (v_i, w_m)$ and the columns by $(v_k, w_1), \dots, (v_k, w_m)$. So if (v_i, v_k) is not an edge of G , every $((v_i, w_j), (v_k, w_h))$ entry is zero; if (v_i, v_k) is an edge of G , the $((v_i, w_j), (v_k, w_h))$ entry is -1 when $j = h$, zero otherwise. This agrees exactly with the entries in these positions in $L(G \times H)$. $\square$

(1.2.13) Theorem ((3.4), [F1]): $a(G \times H) = \min(a(G), a(H))$.

Proof: Let $V(G) = \{v_1, \dots, v_n\}$ and $V(H) = \{w_1, \dots, w_m\}$. Put the vertices of $G \times H$ in lexicographical order: $\{(v_1, w_1), \dots, (v_1, w_m), (v_2, w_1), \dots, (v_n, w_m)\}$. Then $L(G \times H) = (L(G) \otimes I_m) + (I_n \otimes L(H))$ by (1.2.12). Thus by (A.10.b), the eigenvalues of $L(G \times H)$ are of the form $\lambda + \eta$ where λ and η are eigenvalues of G and H respectively. Hence $a(G \times H)$ is the smaller of $a(G) + 0$ and $0 + a(H)$. $\square$

For example, in (1.2.11), $a(G) = 1$, $a(H) = 2$, and $a(G \times H) = 1 = \min\{a(G), a(H)\}$.

(1.2.14) Lemma ([Tu], p.28): The sum of the degrees of all vertices of any graph is equal to twice the number of edges.

Proof: Each edge connects two vertices, so we count every edge twice when we sum the degree of all vertices. Thus the sum of degrees of all vertices in any graph is twice the number of edges. $\square$

For example, the graph G in (1.2.11) has 4 edges and the sum of vertex degrees is 8.

(1.2.15) Lemma, ([F1], p. 300): Let $M = (m_{ik})$ be a positive semidefinite $n \times n$ matrix such that $Me = 0$, let m be the smallest diagonal entry of M , and let λ_2 be the second smallest eigenvalue of M . Then

$$a(M) = \lambda_2 \leq \left(\frac{n}{n-1}\right) (m). \quad (1)$$

Proof: By (A.6.c),

$$\lambda_2 = \min_{x \in W} x^T M x. \quad (2).$$

Let J be the $n \times n$ matrix in which every entry is 1, and let $\tilde{M} = M - \lambda_2 \left(I - \frac{1}{n} J\right)$. We

claim $\tilde{M}$ is also positive semidefinite. Clearly $\tilde{M}$ is the sum of symmetric matrices, hence is symmetric. For the rest, let y be a vector in R^n . Then y can be written in the form $y = c_1 e + c_2 x$ where $x \in W$ by letting $u_1 = e$ in (A.6.c). Notice that $Je = ne$ so

$\left(I - \frac{1}{n} J\right)e = 0$, hence $\tilde{M}e = 0$ and thus $e^T \tilde{M} = 0$ also. It follows that $y^T \tilde{M} y =$

$(c_2)^2 x^T \tilde{M} x \geq (c_2)^2 (x^T M x - \lambda_2) \geq 0$, this last inequality by (2), because $x \in W$.

Therefore $\tilde{M}$ is positive semidefinite. In particular, if m is the i^{th} diagonal entry of M

and y is the i^{th} column of I then $y^T M y = m$, we have $y^T \tilde{M} y = m - \lambda_2 \left(1 - \frac{1}{n}\right) \geq 0$,

hence $\lambda_2 \leq \left(\frac{n}{n-1}\right) (m)$. $\square$

For example, the graph G in (1.2.11) has $n = 4$, $\lambda_2 = 1$ and $m = 1$; and $1 \leq \frac{4}{3}(1)$ is true. The graph $G \times H$ in the same example has $n = 8$, $\lambda_2 = 1$, $m = 2$, and $1 \leq \frac{8}{7}(2)$ is true.

(1.2.16) Theorem ((3.5), [F1]): Let $G = (V, E)$ be a graph with n vertices, $n \geq 2$, and let

d_j be the degree of vertex v_j . Then $a(G) \leq \left(\frac{n}{n-1}\right) (\min \{d_j\}) \leq \frac{2|E|}{n-1}$.

Proof: Let $d = \min \{d_j\}$. Since $nd \leq \sum d_j = 2|E|$ by (1.2.14), $\frac{n}{n-1}d \leq \frac{2|E|}{n-1}$ is true.

The matrix $L(G)$ satisfies the hypotheses of (1.2.15), so we have $a(G) \leq \frac{nd}{n-1}$ because

here, $\min d_j = d$, the smallest diagonal entry of $L(G)$. Thus $a(G) \leq \frac{nd}{n-1} \leq \frac{2|E|}{n-1}$. $\square$

For example, the graph $G \times H$ in (1.2.11) has $n = 8$, $a(G \times H) = 1$, $\min \{d_j\} = 2$, and $|E| = 12$, and we see that $1 \leq \frac{8}{7}(2) \leq \frac{2(12)}{7}$ is true. The graph G in (1.1.19) has $n = 5$,

$a(G) = 1$, $\min \{d_j\} = 2$ and $|E| = 6$; again $1 \leq \frac{5(2)}{4} \leq \frac{2(6)}{4}$ is true.

(1.2.17) Proposition ([F1], p. 300-301): A positive semidefinite real matrix $M = (m_{ik})$

such that $Me = 0$ has the property that $2 \max_i \sqrt{m_{ii}} \leq \sum_i \sqrt{m_{ii}}$. Also, if λ_2 is the second

smallest eigenvalue of M , then $2 \max_i \sqrt{m_{ii} - \frac{n-1}{n} \lambda_2} \leq \sum_{i=1}^n \sqrt{m_{ii} - \frac{n-1}{n} \lambda_2}$.

Proof: Because M is real positive semidefinite, there exists a real matrix $U = [u_1 \dots u_n]$

such that $M = U^T U$ by (A.6.c). Therefore $e^T M e = \|Ue\|^2$ and $Me = 0$ so $Ue = 0$. That is

$u_1 + \dots + u_n = 0$. Then we have $u_i = -\sum_{k \neq i} u_k$ so $\|u_i\| \leq \sum_{k \neq i} \|u_k\|$ for each $i = 1, \dots, n$. In

particular if $\|u_i\| = \max_k \|u_k\|$, then $2\|u_i\| \leq \sum_{k=1}^n \|u_k\|$. Since $M = U^T U$, $m_{kk} = u_k^T u_k$,

so $\|u_k\| = \sqrt{m_{kk}}$ for each k , and we have proved the first inequality,

$$2 \max_i \sqrt{m_{ii}} \leq \sum_i \sqrt{m_{ii}}.$$

For the second assertion, recall that in the proof of Lemma (1.2.15), it was shown that

$\tilde{M} = M - \lambda_2 (\mathbf{I} - 1/n)J$ is positive semidefinite and $\tilde{M} e = 0$. Applying the above

inequality to $\tilde{M}$ yields exactly what we want:

$$2 \max_i \sqrt{m_{ii} - [(n-1)/n] \lambda_2} \leq \sum_{i=1}^n \sqrt{m_{ii} - [(n-1)/n] \lambda_2}. \quad \square$$

To illustrate the inequalities in (1.2.17), the graph G in (1.2.9) has $n = 4$, $\lambda_2 = 2$, and

the diagonal entries of L are 3, 2, 3, 2; the first inequality says $2(3)^{\frac{1}{2}} \leq 2(2)^{\frac{1}{2}} + 2(3)^{\frac{1}{2}}$

which is true, and the second inequality says $2(3 - (\frac{3}{4})(2))^{\frac{1}{2}} \leq 2(3 - (\frac{3}{4})(2))^{\frac{1}{2}} +$

$2(2 - (\frac{3}{4})(2))^{\frac{1}{2}}$, which is also true.

For another example, the graph G in Example (1.1.19) has $n = 5$, $\lambda_2 = 1$, and diagonal entries of L are 2, 4, 2, 2, 2; the first inequality says $2(4)^{\frac{1}{2}} \leq 4(2)^{\frac{1}{2}} + (4)^{\frac{1}{2}}$ which is true, and the second inequality says $2(4 - (\frac{4}{5})(1))^{\frac{1}{2}} \leq 4(2 - (\frac{4}{5})(1))^{\frac{1}{2}} + (4 - (\frac{4}{5})(1))^{\frac{1}{2}}$, which is also true.

(1.2.18) Corollary ([F1], p. 301): If G is a graph and d_i is the degree of vertex v_i , then

$$2 \max_i [nd_i - (n-1)a(G)]^{\frac{1}{2}} \leq \sum_{i=1}^n (nd_i - (n-1)a(G))^{\frac{1}{2}}.$$

Proof: Let $M = L(G)$ and apply the second inequality of (1.2.17) to get

$$2 \max_i \sqrt{d_i - [(n-1)/n]a(G)} \leq \sum_{i=1}^n \sqrt{d_i - [(n-1)/n]a(G)}; \text{ then multiply by } \sqrt{n}. \quad \square$$

For example, the graph in (1.1.19) has $n = 5$, the diagonal entries of L are 2, 4, 2, 2, 2, and $a(G) = 1$; then $2(5(4) - (5-1)(1))^{\frac{1}{2}} \leq 4(5(2) - (5-1)(1))^{\frac{1}{2}} + (5(4) - (5-1)(1))^{\frac{1}{2}}$ reduces to $8 \leq 4(6)^{\frac{1}{2}} + 4$ which is true.

(1.2.19) Theorem ((3.6), [F1]): The complete graph K_n has spectrum $0 < n \leq \dots \leq n$,

hence $a(K_n) = n$.

Proof: Let $L = L(K_n)$ and let J be the $n \times n$ matrix with every entry 1, so $L = nI - J$. For $i \geq 2$, let $\mathbf{x}^{(i)}$ be the vector with 1 as first component, -1 as i^{th} component and zero

elsewhere. To see that $L\mathbf{x}^{(i)} = n\mathbf{x}^{(i)}$, let L_j denote row j of L ; then $L_1\mathbf{x}^{(i)} = (n-1) + 1 = n$, $L_i\mathbf{x}^{(i)} = -1-(n-1) = -n$, and if $j \neq i$, $L_j\mathbf{x}^{(i)} = -1 + 1 = 0$. Clearly $\{\mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}\}$ is an independent set, so n is an eigenvalue of L with multiplicity at least $n-1$.

We know that zero is an eigenvalue of L also, hence the multiplicity of the eigenvalue n must be $n-1$. $\square$

For example, $G = K_4$ has eigenvalues $0 \leq 4 \leq 4 \leq 4$, so $a(G) = 4$.

(1.2.20) Theorem: When $n \geq 2$, the complete graph K_n is the only connected graph on n vertices, which has exactly two distinct eigenvalues, and they are 0 and n .

Proof: Suppose G is connected, has n vertices and two distinct eigenvalues, $0 < \lambda$. Then, because G is connected, the spectrum of L is $0 < \lambda = \lambda = \dots = \lambda$, so $\text{trace}(L) = (n-1)\lambda$.

Let d_i be the degree of vertex i . Then $\text{trace}(L) = \sum d_i$, hence

$$(a) \sum d_i = (n-1)\lambda.$$

The spectrum of L^2 is $0 < \lambda^2 = \dots = \lambda^2$, so $\text{trace}(L^2) = (n-1)\lambda^2$. One can calculate directly that $\text{trace}(L^2) = \sum d_i^2 + \sum d_i$. Therefore $(n-1)\lambda^2 = \sum d_i^2 + (n-1)\lambda$, hence

$$(b) \sum d_i^2 = (n-1)\lambda(\lambda-1).$$

Applying the Cauchy Schwarz Inequality (A.12) to $(d_1, \dots, d_n)$ and $(1, \dots, 1)$ we get:

$$(\sum d_i)^2 \leq (\sum d_i^2)(n).$$

Using (a) and (b) this yields: $((n-1)\lambda)^2 \leq ((n-1)\lambda(\lambda-1))n$. The latter simplifies to $(n-1)\lambda \leq (\lambda-1)n$, hence $\lambda \geq n$. Then (a) implies $\sum d_i \geq (n-1)(n)$; but for each i , $d_i \leq n-1$ is true; hence $\sum d_i = (n-1)n$. Therefore $(n-1)n = (n-1)\lambda$ so $\lambda = n$ and each $d_i = n-1$; hence G is complete. $\square$

(1.2.21) Definition: The **complement** of a graph G is the graph $\overline{G}$ that has the same set of vertices as G and whose edges are all the edges missing in G . That is, for each $i \neq j$, v_i and v_j are adjacent in $\overline{G}$ if and only if v_i and v_j are not adjacent in G .

For example,

As another example, the complement graph of $G \times H$ of (1.2.11) is

(1.2.22) Definition: A **void** or **discrete** graph is a graph with no edges. Clearly the discrete graph on n vertices and K_n are complements of each other.

For example, the complement graph of K_4 is

(1.2.23) Definition [F1]: For a graph G with n vertices, define $\mathbf{b}(G) = \mathbf{n} - \mathbf{a}(\overline{G})$ where $\overline{G}$ is the complement graph of G .

(1.2.24) Theorem ((3.7), [F1]): Let G be a graph with $n \geq 2$ vertices. Define $W =$

$\{\mathbf{x} \in R^n : \mathbf{x}^T \mathbf{e} = 0 \text{ and } \mathbf{x}^T \mathbf{x} = 1\}$. Then $b(G)$ has the following properties:

1. $b(G)$ is $\lambda_n(G)$, the maximum eigenvalue of $L(G)$. Thus $b(G) = \max_{\mathbf{x} \in W} \mathbf{x}^T L(G) \mathbf{x}$

and $a(G) \leq b(G)$. If G is a complete graph or a discrete graph, then $a(G) = b(G)$.

2. $b(G) = \lambda_n(G) = \max_{i=1, \dots, r} \{b(G_i); G_1, \dots, G_r \text{ are the components of } G\}$.

3. If $G_1 \subseteq G_2$ then $b(G_1) \leq b(G_2)$.

4. If G_1 and G_2 have the same set of vertices, then

$$b(G_1 \cup G_2) \leq b(G_1) + b(G_2) - a(G_1 \cap G_2).$$

5. If d_i is the degree of vertex v_i in G , then $(\frac{n}{n-1}) \max\{d_i\} \leq b(G) \leq 2 \max\{d_i\}$.

Proof of (1): Let $\overline{G}$ be the complement of G . We know $\lambda_2(\overline{G}) = \min_{\mathbf{x} \in W} \mathbf{x}^T L(\overline{G}) \mathbf{x}$ by

(A.6.b).

Let $\mathbf{x} \in W$. Since $\mathbf{e}^T \mathbf{x} = 0$, $J\mathbf{x} = 0$, so $\mathbf{x}^T (n\mathbf{I} - J)\mathbf{x} = n$. Since $\overline{G}$ has exactly the edges missing in G , $L(\overline{G}) + L(G) = n\mathbf{I} - J$. Therefore by (A.6.b), we have $\lambda_n(G) =$

$$\max_{\mathbf{x} \in W} \mathbf{x}^T L(G) \mathbf{x} = \max_{\mathbf{x} \in W} (\mathbf{x}^T (n\mathbf{I} - J)\mathbf{x} - \mathbf{x}^T L(\overline{G}) \mathbf{x}) = \max_{\mathbf{x} \in W} (n - \mathbf{x}^T L(\overline{G}) \mathbf{x}) =$$

$n - \min_{\mathbf{x} \in W} \mathbf{x}^T L(\overline{G}) \mathbf{x} = n - a(\overline{G}) = b(G)$. Thus $b(G) = \lambda_n(G)$ as claimed, and it follows at

once that $a(G) \leq b(G)$.

In the complete graph $G = K_n$, $a(G) = b(G)$ because the spectrum is $0 < n = \dots = n$. In a discrete graph G , $a(G) = b(G)$ because the spectrum of $\overline{K_n}$ is $0 = \dots = 0$. $\square$

Proof of (2): If G is unconnected, then we can order the vertices so that $L(G) =$

$$\begin{bmatrix} L(G_1) & & 0 \\ & \ddots & \\ 0 & & L(G_q) \end{bmatrix} \text{ where } G_1, \dots, G_q \text{ are the components of } G. \text{ Thus } \lambda_n(G) \text{ is the}$$

largest eigenvalue occurring among those of $G_1, \dots, G_q$.

Proof of (3): Since $G_1 \subseteq G_2$ then $\overline{G_2} \subseteq \overline{G_1}$, hence $a(\overline{G_2}) \leq a(\overline{G_1})$ by (1.2.5); that is, $n - b(G_2) \leq n - b(G_1)$, hence $b(G_1) \leq b(G_2)$. $\square$

Proof of (4): It is true that $L(G_1 \cup G_2) = L(G_1) + L(G_2) - L(G_1 \cap G_2)$ because if G_1 and G_2 have an edge (i, j) in common, the corresponding contributions in each entry will appear once in $L(G_1)$ and again in $L(G_2)$; subtracting $L(G_1 \cap G_2)$ from $L(G_1) + L(G_2)$ will get rid of the repeated contributions in each entry, making each entry of $L(G_1) + L(G_2) - L(G_1 \cap G_2)$ the same as the corresponding entry in $L(G_1 \cup G_2)$.

Therefore $\max_{x \in W} \mathbf{x}^T L(G_1 \cup G_2) \mathbf{x} \leq \max_{x \in W} \mathbf{x}^T L(G_1) \mathbf{x} + \max_{x \in W} \mathbf{x}^T L(G_2) \mathbf{x} - \min_{x \in W} \mathbf{x}^T (G_1 \cap G_2) \mathbf{x}$. That is, by (A.6.c), $\lambda_n(G_1 \cup G_2) \leq \lambda_n(G_1) + \lambda_n(G_2) - \lambda_2(G_1 \cap G_2)$, which implies $b(G_1 \cup G_2) \leq b(G_1) + b(G_2) - a(G_1 \cap G_2)$ by (1) above. $\square$

Proof of (5): Let d_i be the degree of vertex v_i in G . By (2), $\lambda_n = \max_{x \in W} x^T Lx = b(G)$.

By ([HJ], 5.6.9, p.297), $\lambda_n \leq \|L\|_\infty$, which is $\max\{2d_i\}$. That is $b(G) \leq 2 \max\{d_i\}$,

which proves the right side inequality of (5).

By (1.2.15) applied to $\overline{G}$, we have $a(\overline{G}) \leq \frac{n}{n-1} \min\{d_i(\overline{G})\}$ and $d_i(\overline{G}) =$

$(n-1) - d_i(G)$, so this last inequality can be restated as $n - b(G) \leq \frac{n}{n-1} ((n-1) - \max\{d_i\}) =$

$n - (\frac{n}{n-1}) \max\{d_i\}$. This implies $\frac{n}{n-1} \max\{d_i\} \leq b(G)$, which proves the left side

inequality of (5).

(1.2.25) Example illustrating (1.2.24.1): Obviously $\lambda_2 = a(G) \leq \lambda_n = b(G)$ for any graph

with $n \geq 2$ vertices. Also we know that the complete graph K_n has $a(K_n) = n = b(K_n)$,

and the discrete graph $\overline{K_n}$ has $a(\overline{K_n}) = 0 = b(\overline{K_n})$.

(1.2.26) Example illustrating (1.2.24.2): Let $G = G_1 \cup G_2$ where

Then $L(G) = \begin{bmatrix} L(G_1) & 0 \\ 0 & L(G_2) \end{bmatrix}$, the eigenvalues of G_2 are $0 \leq 2$, and of G_1 are $0 \leq 1 \leq 3$, so

the eigenvalues of $L(G)$ are $0 \leq 0 \leq 1 \leq 2 \leq 3$. Therefore we have $b(G_2) = 2$, $b(G_1) = 3$ and

$b(G) = 3 = \max\{b(G_1), b(G_2)\}$.

(1.2.27) Example illustrating (1.2.24.3): Consider the following graphs, where $G_1 \subseteq G_2$:

The eigenvalues of G_2 are $0 \leq 3 \leq 3$, and of G_1 are $0 \leq 1 \leq 3$. Therefore we have $b(G_2) = 3$, $b(G_1) = 3$ and $b(G_1) = 3 \leq b(G_2)$.

(1.2.28) Example illustrating (1.2.24.4): Let $G = G_1 \cup G_2$ where

$$G_1 \cap G_2 =$$

We have $b(G) = 3$, $b(G_1) = 3$, $b(G_2) = 2$ and $a(G_1 \cap G_2) = 0$. Thus $3 \leq 3 + 2 - 0$ is true.

For another example let $H = H_1 \cup H_2$ where

$$H_1 \cap H_2 =$$

From Example (1.2.9), $L(H) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$ and the spectrum of H is $0 \leq 2 \leq 4 \leq 4$.

$$\text{Also, } L(H_1) = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, L(H_2) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}, L(H_1 \cap H_2) =$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}; \text{ the spectrum of } H_1 \text{ is } 0 \leq 1 \leq 3 \leq 4, \text{ the spectrum of } H_2 \text{ is}$$

$0 \leq 2 \leq 2 \leq 4$ and the spectrum of $H_1 \cap H_2$ is $0 \leq 2 - \sqrt{2} \leq 2 \leq 2 + \sqrt{2}$.

Thus $b(H) = 4$, $b(H_1) = 4$, and $b(H_2) = 4$, and $4 \leq 4 + 4 - (2 - \sqrt{2})$ is true.

(1.2.29) Example illustrating (1.2.24.5): Consider the graph G_1 in (1.2.28). We have

$b(G_1) = 3$ and $\frac{n}{n-1} d_1 = \frac{3}{2}(2) \leq 3 \leq 2d_1 = 4$, which is true. Using H_2 above for

another example, $b(H_2) = 4$ and $\frac{n}{n-1} d_1 = \frac{4}{3}(2) \leq 4 \leq 2d_1 = 4$ is true.

(1.2.30) Theorem ((3.8) [F1]): $\lambda_2 = a(G) \geq 2 \min \{d_i\} - n + 2$.

Proof: We know from (1.2.24.5) that $b(\overline{G}) \leq 2 \max d_i(\overline{G})$, which can be rewritten as

$n - a(G) \leq 2((n-1) - \min \{d_i(G)\})$. Therefore $a(G) \geq 2 \min \{d_i(G)\} - n + 2$. $\square$

For example, the graph $G \times H$ of (1.2.11) has $a(G \times H) = 1$, $\min \{d_i\} = 2$ and $n = 8$, and $1 \geq 2(2) - 8 + 2$ is true. For the graph H_1 in (1.2.28) we have: $1 \geq 2(1) - 4 + 2$ which is true. And for the graph

$a(G) = 2 - \sqrt{2} \geq 2(1) - 4 + 2$ is true.

(1.2.31) Definition: A subset of vertices $S \subseteq V$ is called an **independent set** of a graph G if no two of the vertices in S are adjacent. That is, the only subgraph of G with vertices S is discrete.

For example, in the graph G of (1.2.21), the set $\{v_2, v_3, v_4\}$ is an independent set.

(1.2.32) Theorem ((3.9) [F1]): If G is a graph on n vertices that contains an independent set of m vertices, then $\lambda_2 = a(G) \leq n - m$.

Proof: Suppose G contains an independent set of m vertices. Since the complement of a discrete graph is a complete graph, the complement of G contains a complete graph K_m .

By (1.2.19), we have $b(K_m) = m$. Then by (1.2.24.3), $b(\overline{G}) \geq b(K_m) = m$ since

$K_m \subset \overline{G}$. Therefore $a(G) = \lambda_2 = n - b(\overline{G}) \leq n - m$. $\square$

For example, the graph $G \times H$ of (1.2.11) has an independent set $\{u_3, u_6\}$; $n = 8$, $a(G) = 1$, $m = 2$, and $1 \leq 8 - 2$ is true.

(1.2.33) Theorem ((3.10) [F1]): If G is a non-complete graph on n vertices, then

$$\lambda_2 = a(G) \leq n - 2.$$

Proof: Because G is not complete, there exists an independent set with $m \geq 2$ vertices.

Since $a(G) = \lambda_2 \leq n - m$, we have $a(G) \leq n - 2$. $\square$

For example, the non-complete graph H in (1.2.28) has $n = 4$ and $a(G) = 2$, and $2 \leq 4 - 2$ is true.

Now we will study algebraic connectivity for a very special type of graph.

(1.2.34) Definition: A graph G is **bipartite** if the vertex set V of G can be partitioned into two subsets V_1 and V_2 such that the only edges in G are of the form (a, b) where $a \in V_1$ and $b \in V_2$. We will call G a **complete bipartite graph** if each vertex in V_1 is

adjacent to each vertex in V_2 . The notation $K_{m,n}$ denotes the complete bipartite graph in which V_1 has m vertices and V_2 has n vertices.

(1.2.35) Example: The following graph is bipartite if one takes $V_1 = \{v_1, v_2\}$ and

$V_2 = \{v_3, v_4, v_5\}$:

The complete bipartite graph $K_{2,3}$ can be visualized as follows:

Observe that $L(K_{2,3}) = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$ and its spectrum is $0 \leq 2 \leq 2 \leq 3 \leq 5$.

(1.2.36) Theorem ((3.11) [F1]): The complete bipartite graph $K_{p,q}$ has $a(K_{p,q}) = \min(p, q)$.

Proof: Let $n = p + q$, the number of vertices in $K_{p,q}$. Let J_{pq} be the $p \times q$ matrix of all ones. It is possible to order the vertices so that $v_1, \dots, v_p$ are in V_1 and $v_{p+1}, \dots, v_n$ are in V_2 , where V_1 and V_2 are defined in (1.2.34). Then the complement graph $\overline{K_{p,q}}$ is the union of disjoint complete graphs K_p and K_q , so $L(\overline{K_{p,q}}) = \begin{bmatrix} L(K_p) & 0 \\ 0 & L(K_q) \end{bmatrix}$. We know $b(K_r) = r$ for any r by (1.2.19) and (1.2.20), so by (1.2.24.2), we have $b(\overline{K_{p,q}}) = \max\{p, q\}$. That is by definition of $b(G)$, $n - a(K_{p,q}) = \max\{p, q\}$, hence $a(K_{p,q}) = n - \max\{p, q\} = n - \max\{p, n - p\} = \min\{p, q\}$. $\square$

For example, the spectrum of $L(K_{2,3})$ is $0 \leq 2 \leq 2 \leq 3 \leq 5$ and $a(K_{2,3}) = 2 = \min\{2, 3\}$ is true.

(1.2.37) Theorem ((3.12) [F1]): Let G be a graph with vertices $V = V_1 \cup V_2$ where V_1 and V_2 are disjoint and $|V_1|, |V_2| \geq 2$. Let G_1 be the subgraph of G obtained by removing

the vertices in V_2 and adjacent edges. Define G_2 in the same way, discarding the vertices in V_1 . Then $a(G) \leq \min(a(G_1) + |V_2|, a(G_2) + |V_1|)$.

Proof: By (1.2.7), we have $a(G_1) \geq a(G) - |V_2|$ and $a(G_2) \geq a(G) - |V_1|$. Therefore $a(G) \leq a(G_1) + |V_2|$ and $a(G) \leq a(G_2) + |V_1|$. $\square$

For example, consider the following graph G , and let $V_1 = \{v_1, v_2\}$ and $V_2 = \{v_3, v_4, v_5\}$. Then G_1 and G_2 are as shown:

The eigenvalues of G are $0 \leq 1 \leq 1 \leq 3 \leq 5$, of G_1 are $0 \leq 2$ and of G_2 are $0 \leq 1 \leq 3$.

Therefore we have $a(G) = 1$ and $a(G_1) = 2$, so $a(G) \leq a(G_1) + |V_2| = 5$; $a(G_2) = 1$ so $a(G) \leq 1 + 2 = 3$ also.

Section 1.3: Relations between algebraic connectivity, edge connectivity, and vertex connectivity

In this section we present the rest of Fiedler's results in [F1] about the connection between algebraic connectivity and other types of connectivity. Theorem (1.3.8) and (1.3.9) prove that $a(G) \leq v(G) \leq e(G)$ when G is not complete, where $v(G)$ is the vertex connectivity of G and $e(G)$ is its edge connectivity. Lemma (1.3.11) is needed for (1.3.14) and (1.3.15) where Fiedler gives more precise bounds for $a(G)$.

(1.3.1) Definition: The **edge connectivity** of a graph G , denoted $e(G)$ is the least number of edges whose removal would yield a graph with the same vertices but unconnected. If G has one vertex, we define $e(G) = 0$.

(1.3.2) Example:

Let $G =$

Removal of e_3 and e_5 of G yields the unconnected graph:

Removal of any one edge would not disconnect the graph G , so $e(G) = 2$. For future reference we record here the spectrum of G , which is $0 \leq 2 \leq 4 \leq 4$.

(1.3.3) Theorem: If G is a complete graph on n vertices and $n > 1$, then $e(G) = n - 1$.

Proof [E]: Let K_n be a complete graph on n vertices where $n \geq 2$. We need to show that $e(K_n) = n - 1$ for every $n \geq 2$. Since removal of all $(n-1)$ edges from any one vertex will disconnect that vertex from the rest of the graph, $e(K_n)$ can not exceed $(n-1)$.

Let P_n denote the statement “ $e(K_n) = n-1$.” We need to show that P_n is true for $n \geq 2$. By inspecting K_2 and K_3 we can see P_2 and P_3 are true.

Now assume that $n \geq 4$ and P_{n-1} is true. So $e(K_{n-1}) = n-2$ is true. We need to show that P_n is also true. We can show that by showing that removal of any $(n-2)$ edges from K_n will result in a connected graph.

Suppose edges $e_1, \dots, e_{n-2}$ have been removed from K_n , getting a new graph G on n vertices.

Case 1: No edge has been removed from some vertex v_i . Then G is connected because every vertex in G has an edge joining it to vertex v_i .

Case 2: At least one edge has been removed from every vertex of K_n . Assume that $G = K_n - \{e_1, \dots, e_{n-2}\} = G_1 \cup G_2$, where G_1 and G_2 are disjoint graphs. Each of G_1 and G_2 has at least two vertices because if G_1 or G_2 has only one isolated vertex then $n-1$ edges have been removed from that vertex which contradicts the assumption that $n-2$ edges have been removed. Create another subgraph K_{n-1} of K_n by removing one vertex v_j of K_n and all its adjacent edges.

Observe that $K_{n-1} \cap G_1$ and $K_{n-1} \cap G_2$ are nonempty since each of G_1 and G_2 has at least two vertices so each must share at least one vertex with K_{n-1} , and they are disjoint since G_1 and G_2 are. At least one of the edges $e_1, \dots, e_{n-2}$ is adjacent to v_j , so at most $n-3$ of them lie in K_{n-1} . Removal of those edges that lie in K_{n-1} would disconnect K_{n-1} because those edges are the ones connecting the two disjoint subgraphs of K_{n-1} , $K_{n-1} \cap G_1$ and $K_{n-1} \cap G_2$.

That means $e(K_{n-1}) \leq n-3$, which contradicts our induction hypothesis that $e(K_{n-1}) = n-2$. Thus G is connected. $\square$

(1.3.4) Definition: The **vertex connectivity** of a non-complete graph G is the least number of vertices whose removal (together with their adjacent edges) would yield an

unconnected graph. If G is complete and has n vertices, we define $\kappa(G) = n - 1$. Vertex connectivity of G is denoted by $\kappa(G)$.

In Example (1.3.2), removal of v_1 and v_2 of the graph G (together with adjacent edges) will yield the unconnected graph:

By inspection one can see that removing any one vertex will not disconnect G , so $\kappa(G) = 2$.

(1.3.5) Example:

Let $H =$

Removal of v_5 (together with adjacent edges) will yield the unconnected graph:

Thus $v(H) = 1$. For future reference we record here the spectrum of H , which is $0 \leq 1 \leq 3 \leq 3 \leq 5$.

(1.3.6) Example:

Removal of v_3 (together with adjacent edges) will yield the unconnected graph:

Thus $v(G) = 1$. For future reference we record here the spectrum of G , which is

$$0 \leq 2 - \sqrt{2} \leq 2 \leq 2 + \sqrt{2}.$$

(1.3.7) Definition: A **vertex cut** for a graph G is a set of vertices in G whose removal causes the number of components of G to increase.

For example, $\{v_3\}$ is a vertex cut for the graph H in Example (1.3.5), and $\{v_1, v_2\}$ is a vertex cut for the graph G in Example (1.3.2). Observe that $v(G)$ is the minimum size of a vertex cut for G .

(1.3.8) Theorem ((4.1) [F1]): If G is a graph on n vertices and not complete, then

$$a(G) \leq v(G) \leq n - 2.$$

Proof: Let $G = (V, E)$ be a non-complete graph. Because G is not complete, $n \geq 2$. If G is not connected, $a(G) = v(G) = 0 \leq n - 2$ is true. Suppose G is connected. Let V_1 be a

vertex cut for G (V_1 exists because there exist vertices v_i and v_j that are not adjacent since G is not complete).

Thus G^* , the subgraph remaining with vertices $V_2 = V - V_1$, is unconnected. We know from (1.2.4) that $a(G^*) \geq a(G) - |V_1|$, and $a(G^*) = 0$ since G^* is unconnected, so $a(G) \leq |V_1|$. This is true for any vertex cut, hence $a(G) \leq v(G)$.

Now we need to show that $v(G) \leq n-2$. Since G is not complete, there exist two vertices which are not adjacent, say v_1 and v_2 , so there are at most $n-2$ vertices adjacent to v_1 . If those vertices and their adjacent edges are removed, getting a subgraph G' , then $\{v_1\}$ is a component of G' , and there must be another component containing v_2 , so G' is not connected. $\square$

For example, the graph H in Example (1.3.5) has $a(G) = 1$, $v(G) = 1$, and $n - 2 = 3$, the graph G in Example (1.3.2) has $a(G) = 2$, $v(G) = 2$, and $n - 2 = 2$, and the graph G in Example (1.3.6) has $a(G) = 2 - \sqrt{2}$, $v(G) = 1$, and $n - 2 = 2$.

(1.3.9) Theorem ((4.2) [F1]): If G is a graph, then $v(G) \leq e(G)$.

Proof ([BR], 2.5.6, p. 41): If G is complete, this is true by definition. Suppose now that G is not complete, and thus $n \geq 2$. Let $e(G) = k$ be the edge connectivity of G . If G is not connected, $v(G) = e(G) = 0$. Suppose G is connected, so $k \geq 1$. Then by definition, there are k edges $e_1, \dots, e_k$ whose removal will result in an unconnected graph with two components, say G_1 and G_2 . Because $e(G)$ is the least number of edges that if removed

will disconnect G , each of these k edges connects a vertex in G_1 with a vertex in G_2 . Let $v_1, \dots, v_k$ be the vertices in G_1 which are in the edges $e_1, \dots, e_k$. Observe that $v_1, \dots, v_k$ need not be distinct.

Case 1: If removal of $\{v_1, \dots, v_k\}$ disconnects G , then $v(G) \leq k = e(G)$.

Case 2: If removal of $\{v_1, \dots, v_k\}$ does not disconnect G , then G_1 contains only $\{v_1, \dots, v_k\}$ because otherwise G_1 contains more than k vertices so removing $v_1, \dots, v_k$ from G would leave G_2 and a subgraph of G_1 , hence would disconnect G .

Let $\{v_1, \dots, v_r\}$ denote the distinct vertices in G_1 . Then $v_i \in G_1$ could have degree $(r-1)$ at most in G_1 . Since each vertex in $\{v_1, \dots, v_r\}$ has at least one edge connecting it to G_2 , and there are exactly k edges connecting G_1 and G_2 , the maximum number of edges connecting v_i with G_2 is $k-(r-1)$. Thus v_i can have degree $(r-1) + k-(r-1) = k$ at most.

The degree of v_i cannot be less than k , else $e(G) < k$ which contradicts $e(G) = k$. Thus the degree in G of each vertex in G_1 is exactly k . The same argument applies to G_2 , so every vertex of G has degree k . Therefore $k < n-1$ since G is not complete, so deleting the k vertices adjacent to any one vertex v of G leaves v and at least one more vertex, and v is isolated in the graph remaining, hence $v(G) \leq k = e(G)$. $\square$

For example, the graph H in Example (1.3.5) has $v(G) = 1$ and $e(G) = 2$; the graph G in Example (1.3.2) has $v(G) = e(G) = 2$; and the graph G in Example (1.3.6) has $v(G) = e(G) = 1$.

(1.3.10) Definition: A **nonnegative** matrix is a matrix in which all entries are real and nonnegative.

(1.3.11) Lemma: Let G be a graph with n vertices, let q be the maximum degree of a vertex, and let $L = L(G)$. Define $S = I - \frac{1}{q}L$. Then

(a) S is symmetric, nonnegative, and each row sum and column sum of S is one. In

fact, if $i \neq j$, $s_{ij} = \frac{1}{q}$ when (v_i, v_j) is an edge of G , and $s_{ij} = 0$ otherwise.

(b) If $0 = \lambda_1 \leq \dots \leq \lambda_n$ are the eigenvalues of L , then the eigenvalues of S

are $\sigma_1 \geq \dots \geq \sigma_n$, where $\sigma_i = 1 - \frac{\lambda_i}{q}$.

(c) Let $N = \{1, \dots, n\}$ and define $\eta = \min \left\{ \sum_{i \in M, j \in N-M} s_{ij} : M \text{ is a nonempty, proper subset} \right.$

$\left. \text{of } N \right\}$. Then $\eta = \frac{e(G)}{q}$.

Proof of (1.3.11) (a): Because L is symmetric, S is the sum of symmetric matrices, hence

is symmetric. The diagonal entries of S are of the form $(1 - \frac{d_i}{q})$ where d_i is the degree

of v_i , and these are nonnegative since $d_i \leq q$. The off diagonal entries of S are 0 or $\frac{1}{q}$,

also nonnegative. Thus S is a nonnegative matrix. The sum of each row or column of I is one and the sum of each column or row of L is zero, so the sum of each column or row of

S is one. The last assertion is true because $S = I - \frac{1}{q}L$ and s_{ij} is zero or $\frac{1}{q}$ since L_{ij} is

always zero and L_{ij} is either zero or -1. $\square$

Proof of (1.3.11) (b): Because L is real and symmetric, it has n independent eigenvectors

by (A.2) and (A.3). If $\mathbf{x}$ is an eigenvector of L and $L\mathbf{x} = \lambda \mathbf{x}$, then $S\mathbf{x} = (I - \frac{1}{q}L)\mathbf{x} =$

$(1 - \frac{1}{q}\lambda)\mathbf{x}$. Similarly one can see that any eigenvector of S is an eigenvector of L, so L

and S have the same eigenvectors, hence $\{1 - \frac{1}{q}\lambda_i : 1 \leq i \leq n\}$ is the set of all eigenvalues

of S. $\square$

Proof of (1.3.11) (c): Each off diagonal entry of S is $\frac{1}{q}$ or zero. If a proper, nonempty

subset M of N has the property that $\sum_{i \in M, j \in N-M} s_{ij}$ is minimal, that means that a different

choice of M could not decrease the number of nonzero entries in rows “M” and columns “N – M”. The nonzero entries in the “M, N-M” positions correspond to a set of edges in

G, and removal of that set of edges will disconnect G because removal of that set of

edges will isolate the vertices with “M” subscripts from those with “N-M” subscripts.

Since the number of those edges cannot be made smaller by choosing a different M, the

number η of nonzero entries in the “M, N-M” positions is $e(G)$. Since each of those

entries equals $\frac{1}{q}$, η equals $(\frac{1}{q})$ times the number of nonzero entries in the M, N-M

positions of S, which is $e(G)$. Thus $\eta = e(G)(\frac{1}{q})$. $\square$

(1.3.12) Example:

$$\text{Then } q = 4, L(G) = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

The spectrum of L is $0 \leq 1.5858 \leq 3 \leq 4.4142 \leq 5$, the spectrum of S is $1 \geq 0.6036 \geq 0.25$

$\geq -0.1036 \geq -0.25$, and $q = 4$. Thus $\sigma_i = 1 - \frac{\lambda_i}{4}$ is true for each i . Clearly $e(G) = 2$, and

$$\eta(S) = \min\left\{\sum_{i \in M, j \in N-M} s_{ij} : i \in M, j \in N-M\right\} = \frac{1}{2} \text{ because one can see that choosing } M = \{2\}$$

and $N - M = \{1, 3, 4, 5\}$ gives the minimum such sum: $s_{21} + s_{23} + s_{24} + s_{25} = \frac{1}{2}$. Thus

$$\eta = \frac{e(G)}{q} \text{ is true.}$$

For another example, let

$$\text{Here } q = 3, L(G) = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}, \text{ and } S = \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}.$$

Then $\eta(S) = \frac{e(G)}{q} = 0$ and one can see that choosing $M = \{2\}$ and $N - M = \{1, 3, 4, 5\}$

$$\text{gives } \min\left\{\sum_{i \in M, j \in N-M} s_{ij} : i \in M, j \in N-M\right\} = s_{21} + s_{23} + s_{24} + s_{25} = 0.$$

(1.3.13) Definition: A matrix is called **stochastic** if it is nonnegative and each column sum is one.

If S is an $n \times n$ stochastic matrix and $N = \{1, 2, \dots, n\}$, define the **measure of irreducibility** of S as $\eta(S) = \min\left\{\sum_{i \in M, j \in N-M} s_{ij} : M \text{ is a nonempty, proper subset of } N\right\}$.

We need the following theorem from an earlier paper by Fiedler.

(1.3.14) Theorem ([F2], 3.2): Let S be an $n \times n$ symmetric stochastic matrix with eigenvalues $\sigma_1 \geq \dots \geq \sigma_n$, and let $\eta = \eta(S)$. Then $0 \leq \eta \leq 1$ and

$$1 - \sigma_2 \geq 2(1 - \cos \frac{\pi}{n})\eta \text{ if } 0 \leq \eta \leq \frac{1}{2},$$

$$1 - \sigma_2 \geq 1 - 2(1 - \eta)\cos \frac{\pi}{n} - (2\eta - 1)\cos \frac{2\pi}{n} \text{ if } \frac{1}{2} \leq \eta \leq 1.$$

(1.3.15) Theorem (4.3)[F1]: Let G be a graph with n vertices and let q be the maximum degree of a vertex. Define $C_1 = 2(\cos \frac{\pi}{n} - \cos \frac{2\pi}{n})$ and $C_2 = 2(\cos \frac{\pi}{n})(1 - \cos \frac{\pi}{n})$. Then

$$(A) \quad a(G) \geq 2e(G)(1 - \cos \frac{\pi}{n})$$

$$(B) \quad a(G) \geq C_1 e(G) - C_2 q$$

Further, the second bound is better than the first if and only if $2e(G) > q$.

Proof: Let $L = L(G)$ and $S = I - \frac{1}{q}L$. By (1.3.11)(a), S is symmetric and stochastic.

Let $\sigma_1 \geq \dots \geq \sigma_n$ be the eigenvalues of S and let η be the measure of irreducibility of S .

Then by Theorem (1.3.14) we have:

$$(1) (1 - \sigma_2) \geq 2(1 - \cos(\frac{\pi}{n}))\eta \text{ when } 0 \leq \eta \leq \frac{1}{2},$$

$$(2) (1 - \sigma_2) \geq 1 - 2(1 - \eta)\cos(\frac{\pi}{n}) - (2\eta - 1)\cos\frac{2\pi}{n} \text{ when } \frac{1}{2} \leq \eta \leq 1.$$

From (b) and (c) of Lemma (1.3.11) we have $\sigma_2 = 1 - \frac{\lambda_2}{q} = 1 - \frac{a(G)}{q}$ and $\eta = \frac{e(G)}{q}$. So

$$\frac{a(G)}{q} = 1 - \sigma_2 \text{ and using (1) we have } \frac{a(G)}{q} \geq 2(1 - \cos(\frac{\pi}{n}))\frac{e(G)}{q}, \text{ hence } a(G) \geq 2e(G)(1$$

$$- \cos(\frac{\pi}{n})) \text{ when } 0 \leq \eta \leq \frac{1}{2}, \text{ so (A) is proved.}$$

To derive (B), use (2) and $\sigma_2 = 1 - \frac{a(G)}{q}$ to get

$$\frac{a(G)}{q} = 1 - \sigma_2 \geq 1 - 2(1 - \eta)\cos(\frac{\pi}{n}) - (2\eta - 1)\cos(\frac{2\pi}{n}) \text{ when } \frac{1}{2} \leq \eta \leq 1. \text{ Now expand}$$

the right side of the inequality and use the identity $1 + \cos(2\Theta) = 2\cos^2(\Theta)$ to get

$$1 - 2(1 - \eta)\cos(\frac{\pi}{n}) - (2\eta - 1)\cos(\frac{2\pi}{n}) = 1 - 2\cos(\frac{\pi}{n}) + [2\cos\frac{\pi}{n}]\eta - 2\eta\cos(\frac{2\pi}{n}) +$$

$$\cos(\frac{2\pi}{n}) = 1 + \cos(\frac{2\pi}{n}) - 2\cos(\frac{\pi}{n}) + 2\eta[\cos(\frac{\pi}{n}) - \cos(\frac{2\pi}{n})] = 2\cos^2(\frac{\pi}{n}) - 2\cos(\frac{\pi}{n})$$

$$+ 2\eta[\cos(\frac{\pi}{n}) - \cos(\frac{2\pi}{n})] = -2\cos(\frac{\pi}{n})[1 - \cos(\frac{\pi}{n})] + 2\eta[\cos(\frac{\pi}{n}) - \cos(\frac{2\pi}{n})] = -C_2 +$$

ηC_1 . Thus we have $\frac{a(G)}{q} \geq -C_2 + \eta C_1 = -C_2 + \frac{e(G)}{q} C_1$ when $\frac{1}{2} \leq \eta \leq 1$, and

multiplying by q gives $a(G) \geq e(G) C_1 - q C_2$, so (B) is proved.

To say that bound (B) is better than bound (A) is to say $C_1 e(G) - C_2 q >$

$$2e(G)(1 - \cos(\frac{\pi}{n})).$$

Rearrange this last inequality as $e(G)[C_1 - 2 + 2\cos(\frac{\pi}{n})] > q C_2$. To show this is true

if and only if $2e(G) > q$, we will show that $C_1 - 2 + 2\cos(\frac{\pi}{n}) = 2 C_2$:

$$C_1 - 2 + 2\cos(\frac{\pi}{n}) = 2(\cos(\frac{\pi}{n}) - \cos(\frac{2\pi}{n})) - 2 + 2\cos(\frac{\pi}{n}) = 2[\cos(\frac{\pi}{n}) - \cos(\frac{2\pi}{n}) - 1 +$$

$$\cos(\frac{\pi}{n})] = 2[2 \cos(\frac{\pi}{n}) - (\cos(\frac{2\pi}{n}) + 1)] = 2[2 \cos(\frac{\pi}{n}) - 2 \cos^2(\frac{\pi}{n})] = [2(2 \cos(\frac{\pi}{n})(1 -$$

$$\cos(\frac{\pi}{n}))] = 2 C_2.$$

Therefore $e(G)[C_1 - 2 + 2\cos(\frac{\pi}{n})] > q C_2$ is true if and only if $e(G)(2 C_2) > q C_2$ is

true, i.e, if and only if $2e(G) > q$. $\square$

Chapter 2: An Application of Graph Theory to Acyclic Matrices

An acyclic matrix is a special kind of symmetric matrix, one whose graph is a tree. In this chapter we will use graph theory and eigenvectors to study the placement of an eigenvalue in the natural order, for such matrices.

Trees are also used in many applications such as designing and finding shortest paths in computer networks, freeways, and telephone networks.

Section 2.1: Background, definitions, and notation

This section presents the basic facts about acyclic matrices. We will deal here only with real numbers, vectors and matrices.

(2.1.1) Definition: In a graph G , a **path** is a sequence of distinct vertices $\{v_1, v_2, \dots, v_n\}$ for which there is an edge between each two consecutive vertices. Observe that a vertex may not appear more than once in a path. The **length** of a path with n vertices is $n-1$, the number of edges in it.

(2.1.2) Definition: In a graph G , a **circuit** is a sequence of vertices $\{v_1, v_2, \dots, v_k\}$ where $k \geq 3$ and the vertices are distinct except $v_1 = v_k$. Notice a circuit is not a path because it has a repeated vertex, and a circuit does not self-intersect.

(2.1.3) Example: Let

The sequence $\{v_1, v_2, v_3, v_4\}$ is a path of length 3 but the sequence $\{v_1, v_2, v_4\}$ is not a path. The sequence $\{v_1, v_2, v_3, v_4, v_1\}$ is a circuit.

(2.1.4) Definition: A **tree** is a connected graph that does not contain any circuit.

(2.1.5) Example: The following graph is a tree.

(2.1.6) Definition: In a graph G , a **leaf** is a vertex that is adjacent to a single edge. In [F3], Fiedler calls this an end-vertex but we will say leaf. For example, vertices v_3, v_4 and v_5 of the graph T of Example (2.1.5) are leaves, but v_1 and v_2 are not.

(2.1.7) Proposition: Characteristics of a tree: Let G be a tree with $n \geq 2$ vertices.

1. There is at least one leaf in G .
2. There are exactly $n-1$ edges in G .
3. There is a unique path between any two vertices of G .
4. If we remove one edge from G (but not the adjacent vertices), the resulting graph will have exactly two components and each is a tree, possibly with a single vertex.
5. If we remove a vertex of degree s and all its adjacent edges from G (but not the adjacent vertices) the resulting subgraph has exactly s components.
6. If we remove a leaf and its adjacent edge from G , the resulting subgraph will be a tree with $n-1$ vertices. In general, any connected subgraph of a tree is a tree.

Proof of (2.1.7.1): Choose a path of maximum length, which exists since the graph is finite. Such a path will end at some vertex of degree one, which is therefore a leaf. $\square$

Proof of (2.1.7.2): Let P_n denote the statement “A tree on n vertices has $n-1$ edges.” The statement P_2 is true since a tree on two vertices has one edge. Assume that $n \geq 2$, and P_n is true, that is, every tree on n vertices has $n-1$ edges. We need to show that P_{n+1} is also true; that is, a tree on $n+1$ vertices has n edges. Let G be a tree with $n+1$ vertices. Then G has at least one leaf by (2.1.7.1). Since a leaf is a vertex adjacent to a single edge, by (2.1.6), then removing one leaf v and its adjacent edge from G will result in a graph G' on n vertices. Because G is connected, each component of G' must contain a vertex v' such that (v, v') is an edge of G . Since $\text{degree}(v) = 1$, there is only one component in G' , that

is, G' is connected. Since G has no circuits, G' cannot. Thus G' is a tree on n vertices so has $n-1$ edges by the induction hypothesis, hence G has n edges, so P_{n+1} is true. $\square$

Proof of (2.1.7.3): If there were two different paths between two vertices, their union would contain one or more circuits and that contradicts the definition of a tree. Thus there is only one path between any two vertices of a tree. $\square$

Proof of (2.1.7.4): Removal of an edge (v_i, v_j) creates a subgraph G' with no path between v_i and v_j . All vertices still connected to v_i form a connected subgraph C_1 and all those still connected to v_j form a connected subgraph C_2 . These are disjoint because the path from v_i to v_j in G is unique. We claim that every vertex $v \notin \{v_i, v_j\}$ is in one of these components for there is a path in G from v to v_i , if v_j is not on this path then $v \in C_1$. If v_j is on this path, as $\{v, \dots, v_j, v_i\}$, then $\{v, \dots, v_j\}$ does not contain v_i so $v \in C_2$. $\square$

Proof of (2.1.7.5): Suppose w is a vertex of degree s in G . If $s = 1$, then removal of the single edge adjacent to w will leave a two component graph by (2.1.7.4), where one of these components is the single vertex w because $\text{degree}(w) = 1$. Then removal of w also will leave a graph of one component. Assume $s \geq 2$ and w is adjacent to $v_1, \dots, v_s$. Let G' be the subgraph obtained after removing w and the edges $(w, v_1), \dots, (w, v_s)$. Let

C_i be the subgraph of G' containing v_i and all the vertices lying on a path to v_i that does not contain w . Thus each C_i is connected. We claim $G' = C_1 \cup \dots \cup C_s$ and the graphs C_i are disjoint. To see this, first note $C_1 \cup \dots \cup C_s \subseteq G'$ by definition. Let v be a vertex of G' other than $v_1, \dots, v_s$. There is a unique path in G from v to each v_j . Choose i so that $\{v, \dots, v_i\}$ is the shortest of these. Then $v \in C_i$ so $G' \subseteq C_1 \cup \dots \cup C_s$. Also, if $j \neq i$, then $\{v, \dots, v_i, w, v_j\}$ has to be the unique path in G from v to v_j . Therefore v is not in C_j , so C_i and C_j are disjoint. Therefore G' is a graph with s components. $\square$

Proof of (2.1.7.6): By (2.1.7.5), removal of a leaf and its one adjacent edge from a tree on n vertices will result in a one-component graph with $n-1$ vertices, and it has no circuits since it is a subgraph of a tree, hence it is a tree. By the same argument, any connected subgraph of a tree is itself a tree. $\square$

(2.1.8) Definition: Let A be an $n \times n$ symmetric matrix. Define the **graph** of A , $G(A)$, to be the graph on n vertices $v_1, \dots, v_n$ with (v_i, v_k) as an edge if and only if $i \neq k$ and $a_{ik} \neq 0$. Define A to be **acyclic** if $G(A)$ contains no circuits.

(2.1.9) Example: Let $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Then B is acyclic but

A is not since

contains no circuits.

(2.1.10) Theorem: An $n \times n$ matrix A is acyclic if and only if for each set of distinct indices, $i_1, i_2, \dots, i_k$, $k \geq 3$, the product of the $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k), (i_k, i_1)$ entries of A is zero.

Proof: Let $i_1, i_2, \dots, i_k$ be distinct indices. If A is an acyclic matrix then $G(A)$ is a tree, so $(i_1, i_2, \dots, i_k, i_1)$ is not a circuit in $G(A)$. Thus there must be a missing edge between two vertices in that sequence so at least one of the $(i_1, i_2), \dots, (i_k, i_1)$ entries of A equals zero, hence the product of the $(i_1, i_2), \dots, (i_k, i_1)$ entries of A equals zero.

Conversely, A has the property that for each set of distinct indices $i_1, i_2, \dots, i_k, i_1$, the product of the $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k), (i_k, i_1)$ entries of A equals zero. Then at least one of those entries equals zero which means there is at least one missing edge between two vertices of the sequence $i_1, i_2, \dots, i_k, i_1$. That means no circuit exists in $G(A)$, so by Definition (2.1.8), A is acyclic. $\square$

For example, in Example (2.1.9), A is not acyclic and we have $a_{12} a_{23} a_{34} a_{41} = 1$.

(2.1.11) Definition: A **permutation matrix** is a matrix P obtained by rearranging the

rows of I_n : that is, if $I_n = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$, then $P = \begin{bmatrix} e_{\sigma_1} \\ e_{\sigma_2} \\ \vdots \\ e_{\sigma_n} \end{bmatrix}$ where σ is a permutation of $\{1, \dots, n\}$.

A matrix A is called **reducible** if there is a permutation matrix P such that the matrix

PAP^T is of the form $\begin{bmatrix} X & B \\ 0 & Z \end{bmatrix}$ where X and Z are square matrices. Otherwise, A is

called **irreducible**. When A is reducible and symmetric, $B = 0$ because PAP^T is symmetric since A is.

(2.1.12) Example: It is easy to see that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is irreducible, but

$B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ is reducible because if we choose $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $PAP^T =$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 3 \end{bmatrix}.$$

(2.1.13) Theorem: Let $G = (V, E)$ be a graph on n vertices. Then G is connected if and only if $A(G)$ is irreducible.

Proof: Let $A = A(G)$. If A is reducible, then $PAP^T = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ for some permutation

matrix P . Suppose A_{11} is $k \times k$. Let $S_1 = \{ v_{i_1}, \dots, v_{i_k} \}$ be the vertices corresponding to rows and columns $1, \dots, k$ of PAP^T , and $S_2 = \{ v_{i_{k+1}}, \dots, v_{i_n} \}$ be those corresponding to the remaining rows and columns of PAP^T . Then G is the union of two disjoint subgraphs, one with vertices S_1 and one with vertices S_2 , so G is not connected.

Conversely, suppose G is not connected and G is the union of two disjoint subgraphs, G_1 and G_2 . Let $V(G_1) = \{ v_{i_1}, \dots, v_{i_k} \}$ and $V(G_2) = \{ v_{i_{k+1}}, \dots, v_{i_n} \}$. Let σ be the permutation $\sigma(r) = i_r$ and let P be the permutation matrix in which row r equals row

$\sigma(r)$ of I_n ; then PAP^T has the form $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$, so A is reducible. $\square$

Observe that performing a permutation similarity on $G(A)$ is equivalent to reordering the vertices of $G(A)$. For example, consider the reducible matrix $B = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{bmatrix}$ in

Example (2.1.12). Then

Letting $S_1 = \{2\}$ and $S_2 = \{1, 3\}$, we get $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $PAP^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 4 & 3 \end{bmatrix}$,

which corresponds to reordering the vertices of $G(B)$ as

Notice the block form of matrix PAP^T makes it obvious that the graph is not connected.

(2.1.15) Theorem: A symmetric matrix A is irreducible and acyclic if and only if $G(A)$ is a tree.

Proof: Let G be a tree. Then G is connected, hence A is irreducible by (2.1.13). Also G has no circuits since it is a tree so A is acyclic.

Conversely, let A be irreducible and acyclic. Then $G(A)$ is connected by (2.1.13) and has no circuits, so $G(A)$ is a tree. $\square$

The concept of inertia of a matrix A , which is denoted $\text{In}(A)$, will be used in the remaining theory. This is defined in (A.13) and its basic properties are recorded there.

(2.1.16) Lemma (I.12, [F3]): Let A be an $n \times n$ partitioned symmetric matrix of the form

$A = \begin{bmatrix} B & c \\ c^T & d \end{bmatrix}$ where B is $(n-1) \times (n-1)$, and suppose there exists $\mathbf{u} \in R^{n-1}$ such that $B\mathbf{u} = \mathbf{0}$

and $c^T \mathbf{u} \neq 0$. Then $\text{In}(A) = \text{In}(B) + (1, 1, -1)$.

Proof: Since $\mathbf{u} \neq \mathbf{0}$, we can assume its first component u_1 is not zero and partition the

vector as $\mathbf{u} = \begin{bmatrix} u_1 \\ \tilde{\mathbf{u}} \end{bmatrix}$. Partition B and c conformally as $B = \begin{bmatrix} b_{11} & \tilde{b}^T \\ \tilde{b} & \tilde{B} \end{bmatrix}$, $c = \begin{bmatrix} c_1 \\ \tilde{c} \end{bmatrix}$ where

b_{11} and c_1 are scalars, and let $\beta = c^T \mathbf{u} = \begin{bmatrix} c_1 & \tilde{c}^T \end{bmatrix} \begin{bmatrix} u_1 \\ \tilde{\mathbf{u}} \end{bmatrix} = [c_1 u_1 + \tilde{c}^T \tilde{\mathbf{u}}] \neq 0$.

We will show (1) A is congruent to $\begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix}$, hence $\text{In}(A) = \text{In}(\tilde{B}) + (1, 1, 0)$ by

(A.14) and (A.16.1). Then we will show (2) B is congruent to $\begin{bmatrix} 0 & 0 \\ 0 & \tilde{B} \end{bmatrix}$, hence $\text{In}(B) =$

$\text{In}(\tilde{B}) + (0, 0, 1)$ by (A.16.1). It follows at once from (1) and (2) that $\text{In}(A) =$

$\text{In}(B) + (1, 1, -1)$.

To prove (1), observe first that, by hypothesis,

$$B\mathbf{u} = \begin{bmatrix} b_{11} & \tilde{b}^T \\ \tilde{b} & \tilde{B} \end{bmatrix} \begin{bmatrix} u_1 \\ \tilde{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} b_{11}u_1 + \tilde{b}^T \tilde{\mathbf{u}} \\ \tilde{b}u_1 + \tilde{B}\tilde{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (*)$$

Let $P = \begin{bmatrix} u_1 & \tilde{\mathbf{u}}^T & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so P is $n \times n$, invertible and

$$\begin{aligned}
PAP^T &= \begin{bmatrix} u_1 & \tilde{u}^T & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & \tilde{b}^T & c_1 \\ \tilde{b} & \tilde{B} & \tilde{c} \\ c_1 & \tilde{c}^T & d \end{bmatrix} \begin{bmatrix} u_1 & 0 & 0 \\ \tilde{u} & I & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} u_1 & \tilde{u}^T & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11}u_1 + \tilde{b}^T\tilde{u} & \tilde{b}^T I & c_1 \\ \tilde{b}u_1 + \tilde{B}\tilde{u} & \tilde{B}I & \tilde{c} \\ c_1u_1 + \tilde{c}^T\tilde{u} & \tilde{c}^T I & d \end{bmatrix}.
\end{aligned}$$

But $b_{11}u_1 + \tilde{b}^T\tilde{u} = 0$ and $\tilde{b}u_1 + \tilde{B}\tilde{u} = 0$ by (*). Then we have

$$\begin{bmatrix} u_1 & \tilde{u}^T & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \tilde{b}^T & c_1 \\ 0 & \tilde{B} & \tilde{c} \\ c_1u_1 + \tilde{c}^T\tilde{u} & \tilde{c}^T & d \end{bmatrix} = \begin{bmatrix} 0 & u_1\tilde{b}^T + \tilde{u}^T\tilde{B} & u_1c_1 + \tilde{u}^T\tilde{c} \\ 0 & \tilde{B} & \tilde{c} \\ c_1u_1 + \tilde{c}^T\tilde{u} & \tilde{c}^T & d \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & \tilde{c} \\ \beta & \tilde{c}^T & d \end{bmatrix}. \text{ Notice } u_1\tilde{b}^T + \tilde{u}^T\tilde{B} = 0 \text{ because this equals } (\tilde{b}u_1 + \tilde{B}\tilde{u})^T, \text{ which is zero}$$

by (*).

$$\text{Let } Q = \begin{bmatrix} 1 & 0 & 0 \\ -\beta^{-1}\tilde{c} & I & 0 \\ -(2\beta)^{-1}d & 0 & 1 \end{bmatrix}. \text{ Then } (QP)A(QP)^T = Q(PAP^T)Q^T =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\beta^{-1}\tilde{c} & I & 0 \\ -(2\beta)^{-1}d & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & \tilde{c} \\ \beta & \tilde{c}^T & d \end{bmatrix} \begin{bmatrix} 1 & -\beta^{-1}\tilde{c}^T & -(2\beta)^{-1}d \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\beta^{-1}\tilde{c} & I & 0 \\ -(2\beta)^{-1}d & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B}I & \tilde{c} \\ \beta & -\beta(\beta^{-1}\tilde{c}^T) + \tilde{c}^T I & \beta(-2\beta)^{-1}d + d \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\beta^{-1}\tilde{c} & I & 0 \\ -(2\beta)^{-1}d & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & \tilde{c} \\ \beta & 0 & \frac{d}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & -\beta^{-1}\tilde{c}\beta + \tilde{c} \\ \beta & 0 & -(2\beta)^{-1}d\beta + \frac{d}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \beta \\ 0 & \tilde{B} & 0 \\ \beta & 0 & 0 \end{bmatrix}.$$

Now let $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $R = SQP$. Then R is invertible and $RAR^T =$

$$S[(QP)A(QP)^T]S^T = \begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix}. \text{ Using (A.15), we have } \text{In}(A) = \text{In}(RAR^T), \text{ and this}$$

equals $\text{In}(\tilde{B}) + \text{In} \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$ by (A.16.1). The inertia of $\begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}$ is $(1,1,0)$ because its

eigenvalues are $\pm \beta$ and $\beta \neq 0$. Thus $\text{In}(A) = \text{In}(\tilde{B}) + (1, 1, 0)$.

To prove (2), let $\tilde{P} = \begin{bmatrix} u_1 & \tilde{u}^T \\ 0 & I \end{bmatrix}$. Because $Bu = 0$, $\tilde{P}B\tilde{P}^T =$

$$\begin{bmatrix} u_1 & \tilde{u}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} b_{11} & \tilde{b}^T \\ \tilde{b} & \tilde{B} \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ \tilde{u} & I \end{bmatrix} = \begin{bmatrix} u_1 & \tilde{u}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & \tilde{b}^T \\ 0 & \tilde{B} \end{bmatrix} = \begin{bmatrix} 0 & u_1\tilde{b}^T + \tilde{u}^T\tilde{B} \\ 0 & \tilde{B} \end{bmatrix} \text{ which equals}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \tilde{B} \end{bmatrix} \text{ by (*).}$$

Because $\tilde{P}$ is invertible, by (A.14) we have $\text{In } B = \text{In}(\tilde{P}B\tilde{P}^T)$, and this equals

$\text{In}(\tilde{B}) + (0, 0, 1)$ by (A.16.1). Therefore $\text{In}(\tilde{B}) = \text{In}(B) - (0, 0, 1)$.

Finally, by (A.16.1) we have $\text{In}(A) = \text{In}(\tilde{B}) + (1, 1, 0)$ and with (1) and (2) this gives

$$\text{In}(A) = \text{In}(B) - (0, 0, 1) + (1, 1, 0) = \text{In}(B) + (1, 1, -1). \quad \square$$

Section 2.2: Counting the eigenvalues of an acyclic matrix using an eigenvector

The main result of Fiedler's paper [F3] will be presented in this section: if A is acyclic, λ is an eigenvalue of A , and y is an associated eigenvector, then the entries of y and the off diagonal entries of A can be used to count the number of eigenvalues of A which are less than λ and the number which are greater than λ .

The first goal is Proposition (2.2.5), which establishes how to do the counting when $\lambda = 0$ and y is the column of ones. This seems very special but is the key to proving Theorem (2.2.6), which shows how to do the counting when $\lambda = 0$, y has no zero coordinates and A is any acyclic irreducible matrix.

Following (2.2.6) are definitions and propositions, which are needed for Theorems (2.2.11) and (2.2.12). Those last two theorems establish the counting method for any acyclic matrix and any eigenvalue. Examples are given in (2.2.13) and (2.2.14).

(2.2.1) Definition ([BR], p. 238): If a is a scalar, define

$$\text{sign}(a) = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0 \end{cases}$$

(2.2.2) Definition: A matrix A is called **essentially nonnegative** if $a_{ik} \geq 0$ for every $i \neq k$.

(2.2.3) Theorem ((2.1), [F3]): Let A be an $n \times n$ acyclic matrix. Then there exists a diagonal matrix D such that $D^2 = I$, and DAD is essentially nonnegative.

Proof: We will consider two cases.

Case 1. Suppose that A is irreducible and acyclic, and let $G = G(A)$, so G is a tree by (2.1.17). Define $d_1 = 1$. If $k > 1$, let $(v_1, v_j, v_m, \dots, v_p, v_k)$ be the unique path in G from v_1 to v_k . Define $d_k = \text{sign}(a_{1j} a_{jm} \dots a_{pk})$, so $d_k = \pm 1$. Then let

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & d_n \end{bmatrix}.$$

If $n = 2$ then $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ and $a_{12} \neq 0$ because A is irreducible. Also $d_1 = 1$ and

$$d_2 = \text{sign}(a_{12}) \text{ so } d_1 d_2 a_{12} = |a_{12}|, \text{ hence } DAD = \begin{bmatrix} a_{11} & |a_{12}| \\ |a_{12}| & a_{22} \end{bmatrix}. \text{ Thus } DAD \text{ is}$$

essentially nonnegative.

Now assume $n \geq 3$. Let $i \neq k$. We need to show that $a_{ik} d_i d_k \geq 0$. This is true if $a_{ik} = 0$, which happens when there is no edge between vertices v_i and v_k . Suppose $a_{ik} \neq 0$, so (i, k) is an edge in G . By (2.1.7.4), removal of the edge (i, k) from G will result in two components G_1 and G_2 , such that v_i is in G_1 and v_k is in G_2 . We can suppose that v_1 belongs to the component G_1 and also $i, k \neq 1$ because $n \geq 3$. Then there is a unique path $(v_1, v_j, \dots, v_p, v_i)$ in G_1 from v_1 to v_i , and $d_i = \text{sign}(a_{1j} \dots a_{pi})$. Also $(v_1, v_j, \dots, v_p, v_i, v_k)$ is the path in G from v_1 to v_k , so $d_k = \text{sign}(a_{1j} \dots a_{pi} a_{ik}) =$

$(\text{sign}(a_{1j} \dots a_{pi}))(\text{sign } a_{ik}) = d_i \text{ sign } a_{ik}$ by definition of d_i . Then $a_{ik} d_i d_k =$

$d_i^2 a_{ik} (\text{sign } a_{ik}) = a_{ik} (\text{sign } a_{ik}) \geq 0$. Thus DAD is essentially nonnegative so we have

proved the theorem when A is irreducible.

Case 2. If A is acyclic and reducible, then after a permutation similarity we may

suppose $A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_k \end{bmatrix}$ where each A_i is irreducible and also acyclic because A

is acyclic. Case 1 applies to each A_i , so there is a diagonal matrix D_i with diagonal

entries either +1 or -1 so that $D_i^2 = I$ and $D_i A_i D_i$ is essentially nonnegative. Define

$D = \begin{bmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & D_n \end{bmatrix}$; then $D^2 = I$, and DAD is essentially nonnegative. $\square$

(2.2.4) Lemma: Let C be nxn and symmetric. Suppose $C\mathbf{e} = \mathbf{0}$ is true, where $\mathbf{e}$ is the nx1

column of ones, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where $x_1, \dots, x_n$ are independent variables. Then

$$\mathbf{x}^T C \mathbf{x} = - \sum_{j < k} c_{jk} (x_j - x_k)^2.$$

Proof: Observe that $C\mathbf{e} = \mathbf{0}$ means that each row sum of C is zero, hence

$$(a) \text{ For each row } j, c_{jj} = -\sum_{k \neq j} c_{jk}.$$

Now calculate:

$$\mathbf{x}^T C \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \sum_{k=1, n} c_{1k} x_k \\ \sum_{k=1, n} c_{2k} x_k \\ \vdots \\ \sum_{k=1, n} c_{nk} x_k \end{bmatrix} = x_1 \sum_{k=1, n} c_{1k} x_k + x_2 \sum_{k=1, n} c_{2k} x_k + \cdots + x_n \sum_{k=1, n} c_{nk} x_k$$

$$= c_{11} x_1^2 + \cdots + c_{nn} x_n^2 + \sum_{k \neq 1} c_{1k} x_1 x_k + \cdots + \sum_{k \neq n} c_{nk} x_n x_k. \text{ By (a) the latter equals}$$

$$(b) - \left[\sum_{k \neq 1} c_{1k} x_1^2 + \cdots + \sum_{k \neq n} c_{nk} x_n^2 \right] + \left[\sum_{k \neq 1} c_{1k} x_1 x_k + \cdots + \sum_{k \neq n} c_{nk} x_n x_k \right].$$

In the preceding expression, for each j, k such that $j < k$, the terms in the first brace with coefficient c_{jk} or c_{kj} are $-(c_{jk} x_j^2 + c_{kj} x_k^2)$ and the terms in the second brace with coefficient c_{jk} or c_{kj} are $(c_{jk} x_j x_k + c_{kj} x_k x_j)$; because C is symmetric, the sum of these four terms equals $-c_{jk}(x_j^2 + x_k^2 - 2x_j x_k)$, which equals $-c_{jk}(x_j - x_k)^2$. Therefore the expression in (b) equals $-\sum_{j < k} c_{jk}(x_j - x_k)^2$ and the proof of the Lemma is finished. $\square$

(2.2.5) Proposition ((2.2), [F3]): Let C be an $n \times n$ acyclic matrix such that

$C\mathbf{e} = \mathbf{0}$ is true, where $\mathbf{e}$ is the $n \times 1$ column of ones. Then $\text{In}(C) = (p, q, n-p-q)$ where p is the number of negative c_{jk} , $j < k$, and q is the total number of positive c_{jk} , $j < k$.

Proof: Case 1. Suppose that C is irreducible. Let G be the graph of C , so G is a tree by (2.1.17). Let $\{1, 2, \dots, n\}$ denote the set of vertices of G . By (2.2.4), $\mathbf{x}^T C \mathbf{x} =$

$$- \sum_{j < k} c_{jk} (x_j - x_k)^2 \text{ for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \text{ Since } G \text{ is a tree, it has } n-1 \text{ edges, hence there are}$$

exactly $n-1$ nonzero terms $c_{jk} (x_j - x_k)^2$ in the previous sum. Let $\mathbf{Y} = \{x_j - x_k : j < k \text{ and } c_{jk} \neq 0\}$, so $\mathbf{Y}$ has $n-1$ elements, and order its elements as $\mathbf{Y} = \{y_1, \dots, y_{n-1}\}$. Define a

$$\text{function } P: R^n \rightarrow R^n \text{ such that for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, P(\mathbf{x}) = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \\ x_n \end{bmatrix}. \text{ Then each coordinate of}$$

$P(\mathbf{x})$ is linear in the coordinates of $\mathbf{x}$, so P is a linear function, and thus P can be represented by a matrix.

$$\text{For example, if } C = \begin{bmatrix} -1 & 5 & -4 & 0 & 0 \\ 5 & -2 & 0 & 1 & -4 \\ -4 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 & 4 \end{bmatrix}, \text{ then } \mathbf{Y} = \{x_1 - x_2, x_1 - x_3, x_2 - x_4,$$

$$x_2 - x_5\} \text{ and } P = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We want to show that P is invertible, and will do that by showing that its null space is trivial. So consider the equation $P\mathbf{x} = \mathbf{0}$, and suppose $\mathbf{x}$ is a solution. Then $x_n = 0$ and

each of the linear forms $x_j - x_k$ in Y has value zero for this $\mathbf{x}$. We will show that $x_j = 0$ for each $j = 1, \dots, n-1$.

Fix $j < n$. Since G is connected, there is a path in G from vertex j to vertex n , say $j, j_1, j_2, \dots, j_k, j_n$. That means the linear forms $x_j - x_{j_1}, x_{j_1} - x_{j_2}, \dots, x_{j_k} - x_n$ (or their negatives) are all elements of Y , hence are some of the coordinates of $P\mathbf{x}$. Thus each has value zero when $P\mathbf{x} = \mathbf{0}$ is true: $x_j - x_{j_1} = 0, x_{j_1} - x_{j_2} = 0, \dots, x_{j_k} - x_n = 0$. Thus $x_j = x_{j_1}, x_{j_1} = x_{j_2}, \dots, x_{j_k} = x_n$. But $x_n = 0$ so all are zero, in particular $x_j = 0$. Since j was any vertex other than n , we have shown that $x_j = 0$ is true for every j . Thus we have shown that if $P\mathbf{x} = \mathbf{0}$ is true then $\mathbf{x} = \mathbf{0}$, hence P is invertible.

Now we have $\mathbf{x}^T C \mathbf{x} = \sum_{j < k} (-c_{jk}) y_{p_j}^2$, for some ordering $y_{p_1}, \dots, y_{p_{n-1}}$ of

$y_1, \dots, y_{n-1}$. Since $\mathbf{y} = P\mathbf{x}$, if we let $Q = P^{-1}$, then $\mathbf{x} = Q\mathbf{y}$, hence $\mathbf{x}^T C \mathbf{x} = (Q\mathbf{y})^T C Q\mathbf{y} =$

$\mathbf{y}^T (Q^T C Q) \mathbf{y}$. So $\mathbf{y}^T (Q^T C Q) \mathbf{y} = \sum_{j < k} (-c_{jk}) y_{p_j}^2$ which is of the form $\sum_{i=1}^{n-1} a_i y_i^2$ since

there are $n-1$ nonzero $c_{jk}, j < k$. By Sylvester's Inertia Theorem for Quadratic Forms

(A.16), the number of positive eigenvalues of $Q^T C Q$ equals the number of positive a_i , and

the number of negative eigenvalues of $Q^T C Q$ equals the number of negative a_i .

Finally, because Q is invertible, C and $Q^T C Q$ have the same inertia by (A.14).

So, when C is irreducible and $C\mathbf{e} = \mathbf{0}$ is true, we have shown that the number of positive eigenvalues of C equals the number of negative off-diagonal c_{jk} which are in the upper triangle of C , and the number of negative eigenvalues of C is the number of positive off-diagonal c_{jk} which are in the upper triangle of C .

Case 2. Suppose C is reducible. After a permutation similarity, C can be written as

$$C = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & C_k \end{bmatrix} \text{ where each } C_i \text{ is irreducible. Each } C_i \text{ is acyclic because } C \text{ is}$$

acyclic. Since each C_i is irreducible, by Case 1 the number of positive eigenvalues of C_i equals the number of negative off-diagonal c_{jk} , which are in the upper triangle of C_i , and the number of negative eigenvalues of C_i is the number of positive off-diagonal c_{jk} ,

which are in the upper triangle of C_i . By (A.15.1), $\text{In } C = \sum_{i=1}^k \text{In } C_i$. Therefore the

theorem is also true when C is reducible. $\square$

(2.2.6) Theorem (Proposition 1, [F3]): Suppose A is acyclic, irreducible, and $Ay = 0$ is true for a vector y which has each component $y_i \neq 0$. Then

(a) $\text{In}(A) = (a^+, a^-, n - a^+ - a^-)$, where a^+ equals the number of $a_{ik}y_iy_k$ such that $i < k$ and $a_{ik}y_iy_k < 0$ and a^- equals the number of $a_{ik}y_iy_k$ such that $i < k$ and $a_{ik}y_iy_k > 0$.

(b) $\text{In}(A) = (a^+, a^-, 1)$.

(c) Every $(n-1) \times (n-1)$ submatrix of A is invertible.

Proof: (a) Let $\text{In}(A) = (\omega^+, \omega^-, \omega^{(0)})$. Thus ω^+ equals the number of positive eigenvalues of A and ω^- equals the number of negative eigenvalues of A , by definition of inertia (A.13).

Let $Y = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & y_n \end{bmatrix}$ and let $B = YAY$. Then B has the same inertia as A by

Sylvester's Law of Inertia (A.14), because A is Hermitian and Y is invertible since all $y_i \neq 0$.

Observe that $b_{ij} = y_i a_{ij} y_j$ so $b_{ij} = 0$ if and only if $a_{ij} = 0$, for all i, j . It follows that B is irreducible and acyclic because A is, and also that $G(B) = G(A)$.

Also observe that $Be = \mathbf{0}$ when $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, because $Ye = \mathbf{y}$ so $Be = YAYe = YAy$ and

$Ay = \mathbf{0}$. Therefore by Theorem (2.2.5), ω^+ equals the number of b_{ik} such that $i < k$ and $b_{ik} < 0$ and ω^- equals the number of b_{ik} such that $i < k$ and $b_{ik} > 0$. Since $b_{ik} = y_i a_{ik} y_k$, $\omega^+ = a^+$ and $\omega^- = a^-$.

(b) Because A is irreducible, $G(A)$ is connected; so by (1.2.2), the null space of A has dimension one, i.e., A has only one zero eigenvalue. Thus $\text{In}(A) = (a^+, a^-, 1)$.

(c) Partition A by columns, $A = [c_1 \dots c_n]$. Now $Ay = \mathbf{0}$ and $Ay = y_1 c_1 + \dots + y_n c_n = \mathbf{0}$, so each column of A equals a linear combination of the other columns because each $y_i \neq 0$, so we can solve the identity $y_1 c_1 + \dots + y_n c_n = \mathbf{0}$ for any c_i . Because A is symmetric, it follows that each row of A equals a linear combination of the other rows. Now choose any row r and column s of A . Let A' be the submatrix of A gotten by omitting row r and let A'' be the submatrix gotten by omitting both row r and column s

from A. We wish to show that A'' is invertible, so suppose $A''\mathbf{x} = \mathbf{0}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$.

We will show that $\mathbf{x} = \mathbf{0}$.

First, define $\mathbf{z} = \begin{bmatrix} x_1 \\ \vdots \\ x_{s-1} \\ 0 \\ x_{s+1} \\ \vdots \\ x_{n-1} \end{bmatrix}$ and notice $A'\mathbf{z} = \mathbf{0}$ since $A''\mathbf{x} = \mathbf{0}$. To see this, let $[a_{p1} \dots a_{pn}]$

be any row of A' . Then $[a_{p1} \dots a_{p,s-1} \ a_{p,s+1} \dots a_{pn}]$ is a row of A'' . Because $A''\mathbf{x} = \mathbf{0}$, $a_{p1}x_1 + \dots + a_{p,s-1}x_{s-1} + a_{p,s+1}x_{s+1} + \dots + a_{pn}x_{n-1} = 0$ is true. Since $z_s = 0$, this identity can be rewritten as $a_{p1}x_1 + \dots + a_{p,s-1}x_{s-1} + a_{ps}0 + a_{p,s+1}x_{s+1} + \dots + a_{pn}x_{n-1} = 0$. That is, $[a_{p1} \dots a_{pn}]\mathbf{z} = 0$, and thus $A'\mathbf{z} = \mathbf{0}$.

Finally, row r of A equals some linear combination of the rows of A' as explained

above, hence $A\mathbf{z} = \mathbf{0}$. In detail, let $A = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$ so $A' = \begin{bmatrix} d_1 \\ \vdots \\ d_{r-1} \\ d_{r+1} \\ \vdots \\ d_n \end{bmatrix}$. We know $\mathbf{d}_r =$

$a_1\mathbf{d}_1 + \dots + a_{r-1}\mathbf{d}_{r-1} + a_{r+1}\mathbf{d}_{r+1} + \dots + a_n\mathbf{d}_n$ for some scalars a_i , so $\mathbf{d}_r\mathbf{z} = a_1\mathbf{d}_1\mathbf{z} + \dots + a_{r-1}\mathbf{d}_{r-1}\mathbf{z} + a_{r+1}\mathbf{d}_{r+1}\mathbf{z} + \dots + a_n\mathbf{d}_n\mathbf{z}$ and each term in this sum is zero because $\mathbf{d}_i\mathbf{z} = \mathbf{0}$ for each $i \neq r$.

Therefore we do have $A\mathbf{z} = \mathbf{0}$. Now we can say that $\mathbf{z}$ must be a multiple of $\mathbf{y}$, because the null space of A has dimension one so it equals $\text{span}\{\mathbf{y}\}$. But $\mathbf{z}$ has a zero coordinate

and $\mathbf{y}$ does not, hence $\mathbf{z}$ must be zero. Therefore $\mathbf{x} = \mathbf{0}$, so $\mathbf{C}''$ is invertible. This completes the proof that every $(n-1) \times (n-1)$ submatrix of $\mathbf{A}$ is invertible. $\square$

The following definitions and notation will be used in all remaining proofs.

(2.2.7) Definitions: Suppose $\mathbf{A}$ is $n \times n$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. A coordinate y_k will be called

isolated in $\mathbf{y}$ if $y_k = 0$ and $a_{kj}y_j = 0$ for all j .

Define $M = \{ k : y_k = 0 \text{ and } a_{kj}y_j = 0 \text{ for all } j \}$, so the elements of M are the indices of the isolated coordinates of $\mathbf{y}$.

Define $M_2 = \{ k : y_k = 0 \text{ and } a_{kj}y_j \neq 0 \text{ for some } j \neq k \}$. Thus k in M_2 means y_k is a nonisolated zero coordinate of $\mathbf{y}$.

Define $M_1 = \{ k : y_k \neq 0 \}$. Thus M_1 is not empty if and only if $\mathbf{y} \neq \mathbf{0}$.

(2.2.8) Proposition (Proposition 2, [F3]): Suppose $\mathbf{A}$ is $n \times n$, irreducible, acyclic, $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{A}\mathbf{y} = \mathbf{0}$. If $k \in M_2$, there exist at least two u, v in M_1 such that (u, k) and (v, k) are edges of G .

Proof: Because $k \in M_2$, there exists $u \neq k$ such that $a_{ku}y_u \neq 0$. Suppose $a_{kv}y_v = 0$ for all other $v \neq k$.

Because $A\mathbf{y} = \mathbf{0}$, we have $a_{kk}y_k + \sum_{j \neq k} a_{kj}y_j = 0$, hence $a_{kk}y_k = - \sum_{j \neq k} a_{kj}y_j$. Therefore

$a_{kk}y_k = -a_{ku}y_u$ since $a_{kv}y_v = 0$ for all $v \neq k, u$. We also have $a_{kk}y_k = 0$ because $y_k = 0$ since $k \in M_2$, hence $a_{ku}y_u = 0$. But this is a contradiction. Therefore there must exist some $v \neq u, k$ such that $a_{kv}y_v \neq 0$. Then both $a_{ku}y_u \neq 0$ and $a_{kv}y_v \neq 0$ so both y_v and y_u are nonzero, hence both u and v are in M_1 . $\square$

(2.2.9) Proposition (Proposition 3, [F3]): Suppose that A is $n \times n$, irreducible, acyclic, $\mathbf{y} \neq \mathbf{0}$, $A\mathbf{y} = \mathbf{0}$, and there exists at least one zero coordinate in $\mathbf{y}$. Then there exists an element in M_2 such that if we let G' be the graph obtained by removing that vertex and all edges adjacent to it, then the indices of the vertices of at least one component of G' are contained in M_1 .

Proof: Let $u \in M_1$ and $v \notin M_1$. These exist because by hypothesis, $\mathbf{y}$ has both nonzero and zero coordinates.

First we will show that M_2 is not empty. Since A is irreducible, G is a tree so it has a unique path between any two vertices. Because $u \in M_1$ and $v \notin M_1$, on the path from u to v there is a first vertex w such that $w \notin M_1$. Let $(u, \dots, k, w)$ be the path from u to w . Note that $a_{wk} \neq 0$ and $y_k \neq 0$ because k is before w in the path $(u, \dots, k, w)$ and w is the first vertex on that path which is not in M_1 . So $a_{wk}y_k \neq 0$, hence $w \notin M$. Thus $w \notin M_1 \cup M$, so w must be in M_2 , hence M_2 is not empty.

To finish the proof, consider two cases.

Case 1. Suppose M_2 contains only one element. Let G' be the subgraph of G obtained by removing that vertex and all adjacent edges. Then all vertices of G' are contained in $M_1 \cup M$. Let C denote the set of vertices of the component of G' which contains u . To see that C is contained in M_1 , suppose it is not, so there exists $j \in C \cap M$. Since j is in M and u is not, there is a first vertex r on the path in C from j to u , such that r is not in M ; therefore r must be in M_1 . Let $(j, \dots, k, r)$ be the path from j to r . Then k is in M so $y_k = 0$, and also $a_{kr} \neq 0$ and $y_r \neq 0$, where $a_{kr} \neq 0$ because there exists an edge (k, r) , and $y_r \neq 0$ since $r \in M_1$.

Therefore $a_{kr}y_r \neq 0$, but since $y_k = 0$, this means k is in M_2 , which is a contradiction.

Therefore $C \subseteq M_1$.

Case 2. The set M_2 has more than one element. Let p and q be elements of M_2 such that the path in G from p to q has maximal length. Because p is in M_2 , by (2.2.8) there are two different vertices r, s in M_1 such that (r, p) and (s, p) are edges of G . We claim that at least one of r or s is not in the path from p to q . For if r and s are both on the path from p to q , then we may suppose the path is $(p, \dots, r, \dots, s, \dots, q)$; but (s, p) is an edge of G and that with the path $(p, \dots, r, \dots, s)$ creates a circuit in G , which cannot exist since G is a tree.

Therefore one of r or s is not on the path from p to q , say r is not. Let G' be the subgraph obtained by removing p and its adjacent edges, and let C be the set of vertices of the component of G' which contains r .

To show that C is contained in M_1 , let $j \in C$. First we will show that j is not in M_2 . Because C is connected there is a path $(j, \dots, r)$ in C , and $(r, p, \dots, q)$ is a path since (r, p) is an edge of G and r is not in the path $(p, \dots, q)$. Thus if j is in M_2 , then both end vertices of

the path $(j, \dots, r, p, \dots, q)$ are in M_2 and this path is longer than $(p, \dots, q)$, a contradiction.

Therefore j cannot be in M_2 . Since j was an arbitrary element of C , we now know that C is contained in $M \cup M_1$.

We can also show that j is not in M . For suppose j is in M . Because $G'(C)$ is connected and r is not in M , the path from j to r in C contains a first z not in M , so z is in M_1 . Let $(j, \dots, k, z)$ be the path from j to z . Then k precedes z in the path $(j, \dots, z)$ so $k \notin M_1$, hence k is in M because $C \subseteq M \cup M_1$. In particular, $y_k = 0$. But we also have $y_z \neq 0$ and $a_{kz} \neq 0$ so $a_{kz}y_z \neq 0$, so k must be in M_2 . This is a contradiction. Therefore j is not in M , hence must be in M_1 , and we have shown $C \subseteq M_1$, which completes Case 2. $\square$

(2.2.10) Proposition (Proposition 4, [F3]): Suppose A is an $n \times n$ acyclic irreducible matrix and $Ay = 0$ where $y \neq 0$ but y has at least one zero coordinate. Then a permutation

similarity can be done so that $A = \begin{bmatrix} B & c \\ c^T & d \end{bmatrix}$ where d is a scalar, c an appropriately sized

vector, B is acyclic, and $\text{In}(A) = \text{In } B + (1, 1, -1)$. Also, $B = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ where A_{11} is

irreducible and A_{22} appears only if B is reducible. Partition y conformally, as $y =$

$\begin{bmatrix} y^{(1)} \\ y^{(2)} \\ 0 \end{bmatrix}$, and then all coordinates of $y^{(1)}$ are nonzero and $y^{(2)} \neq 0$ if it appears.

Proof: Let M , M_1 and M_2 be defined as in (2.2.7). By Proposition (2.2.9), there is a vertex of $G(A)$ in M_2 so that the graph G' obtained by removing that vertex and its

adjacent edges has a component whose set of vertices C is contained in M_1 . Order the vertices in $G(A)$ so that the vertex removed is n and $C = \{1, \dots, m\}$. This is equivalent to

doing a permutation similarity to A so that A now looks like
$$\begin{bmatrix} A_{11} & 0 & \mathbf{a}^{(1)} \\ 0 & A_{22} & \mathbf{a}^{(2)} \\ (\mathbf{a}^{(1)})^T & (\mathbf{a}^{(2)})^T & a_{nn} \end{bmatrix}$$

where C is the set of vertices of $G(A_{11})$ and $G(A_{22})$ is the union of the components of G' other than $G(A_{11})$, if there are any. Thus $y_n = 0$ and A_{22} and $\mathbf{a}^{(2)}$ will not appear if $G(A_{11})$

is the only component of G' . Partition $\mathbf{y}$ conformally, as $\mathbf{y} = \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \\ 0 \end{bmatrix}$.

Let $B = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \end{bmatrix}$; then A is of the form $\begin{bmatrix} B & \mathbf{c} \\ \mathbf{c}^T & a_{nn} \end{bmatrix}$. We wish to

apply Lemma (2.1.18). For that purpose, define $\mathbf{u} = \begin{bmatrix} \mathbf{y}^{(1)} \\ 0 \end{bmatrix}$ so that $\mathbf{u}$ is in $\mathbb{R}^{n-1}$.

We claim that $\mathbf{c}^T \mathbf{u} \neq 0$. To see this, first notice that all coordinates of $\mathbf{y}^{(1)}$ are nonzero because by the conformal partition, $\mathbf{y}^{(1)}$ corresponds to A_{11} and $G(A_{11}) = C \subseteq M_1$.

Also note that some $a_{np}y_p \neq 0$ for some p in C : for $G(A)$ is connected since A is irreducible, and C is contained in G' , so there exists a vertex p in C such that (p, n) is an edge of $G(A)$. Thus $y_p \neq 0$. Further, there is only one such p in C . For suppose there are different vertices p and q in C such that (p, n) and (q, n) are both edges in G ; because C is connected, there is a path from p to q in C , but then there is a circuit $\{q, \dots, p, n, q\}$ in G , which cannot happen because G is a tree. Thus only one coordinate a_{pn} of $\mathbf{a}^{(1)}$ is not zero, so $\mathbf{c}^T \mathbf{u} = (\mathbf{a}^{(1)})^T \mathbf{y}^{(1)} = a_{np}y_p \neq 0$.

Since $Ay = 0$ implies $A_{11}y^{(1)} + (0)y^{(2)} + a^{(1)}(0) = 0$, we see that $A_{11}y^{(1)} = 0$. Thus

Lemma (2.1.16) can be applied to get $\text{In}(A) = \text{In } B + (1, 1, -1)$.

Finally, if $y^{(2)}$ appears then $y^{(2)} \neq 0$. To see this, first notice that $(a^{(1)})^T y^{(1)} + (a^{(2)})^T y^{(2)}$

$$= 0 \text{ because } Ay = 0, \text{ hence } \begin{bmatrix} A_{11} & 0 & a^{(1)} \\ 0 & A_{22} & a^{(2)} \\ (a^{(1)})^T & (a^{(2)})^T & a_{nn} \end{bmatrix} \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{ Thus } (a^{(1)})^T y^{(1)} +$$

$(a^{(2)})^T y^{(2)} = 0$, and we saw above that $(a^{(1)})^T y^{(1)} \neq 0$, so $y^{(2)}$ cannot be zero. $\square$

Now we can prove Fiedler's main theorem. For simplicity of notation, we will first prove it when the eigenvalue is zero.

(2.2.11) (Theorem (2.3), [F3], with $\lambda = 0$): Suppose A is $n \times n$ and acyclic, $y \neq 0$ and $Ay = 0$. Let $\text{In}(A) = (\omega^+, \omega^-, \omega^{(0)})$.

Let a^+ denote the number of a_{ik} such that $i < k$ and $a_{ik}y_i y_k < 0$, and let a^- denote the number of a_{ik} such that $i < k$ and $a_{ik}y_i y_k > 0$.

Let m denote the total number of zero coordinates in y .

(1) If there are no isolated coordinates in y , then

$$\omega^+ = a^+ + m \text{ and } \omega^- = a^- + m.$$

(2) Suppose there is at least one isolated coordinate in y . Let A' be the submatrix of A obtained by removing the rows and columns k for each isolated coordinate y_k , and let $\text{In}(A') = (\alpha^+, \alpha^-, \alpha^{(0)})$. Then there exist nonnegative integers c_0, c_1, c_2 such that

$c_0 + c_1 + c_2$ equals the number of isolated coordinates in $\mathbf{y}$ and $\text{In}(\mathbf{A}) = \text{In}(\mathbf{A}') + (c_0, c_1, c_2)$. That is,

$$\omega^+ = \alpha^+ + c_1, \omega^- = \alpha^- + c_2, \text{ and } \omega^{(0)} = \alpha^{(0)} + c_0.$$

Proof: By a permutation similarity we may assume that $\mathbf{A} = \begin{bmatrix} A_1 & & O \\ & \ddots & \\ O & & A_s \end{bmatrix}$ where each

A_j is irreducible or zero. Partition $\mathbf{y}$ conformally, as $\mathbf{y} = \begin{bmatrix} \mathbf{y}^{(1)} \\ \vdots \\ \mathbf{y}^{(s)} \end{bmatrix}$. We may also assume

$\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(r)}$ are all nonzero vectors and if any more appear, they are zero vectors. In particular, $\mathbf{y}^{(1)} \neq \mathbf{0}$ since $\mathbf{y} \neq \mathbf{0}$.

We will induct on m .

First let $m = 0$, that is, suppose that every coordinate of $\mathbf{y}$ is nonzero. Note this is a special case of (1). Define a_j^+ and a_j^- for each A_j as a^+ and a^- were defined for $\mathbf{A}$, and note that by Theorem (2.2.6) applied to A_j and $\mathbf{y}^{(j)}$, $\text{In}(A_j) = (a_j^+, a_j^-, 1)$. Since $\text{In}(\mathbf{A})$ equals the sum of all $\text{In}(A_j)$ by Theorem (A.15.1), we have $\omega^+ = a^+$ and $\omega^- = a^-$. Thus the theorem is true when $m = 0$.

Now assume $m \geq 1$, that is, let $\mathbf{y}$ have at least one zero coordinate, and suppose the theorem is true when such a vector has fewer than m zero coordinates. Re-partition the matrix and vector as $\mathbf{A} = \begin{bmatrix} A_1 & O \\ O & A_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} \mathbf{y}^{(1)} \\ \mathbf{y}^{(2)} \end{bmatrix}$ where A_1 is irreducible and A_1 and A_2 are acyclic.

Case 1. Suppose $\mathbf{y}^{(1)}$ has fewer than m zero coordinates and $\mathbf{y}^{(2)}$ is also nonzero. Then each has fewer than m zero coordinates, $A_1\mathbf{y}^{(1)} = \mathbf{0}$ and $A_2\mathbf{y}^{(2)} = \mathbf{0}$, so the inductive hypothesis applies to each. That is, if we let A_1' and A_2' be the submatrices obtained by deleting the rows and columns of A_1 and A_2 which correspond to the coordinates that are isolated in $\mathbf{y}^{(1)}$ and isolated in $\mathbf{y}^{(2)}$ respectively, and let $\text{In}(A_1') = (\alpha_1^+, \alpha_1^-, \alpha_1^{(0)})$ and $\text{In}(A_2') = (\alpha_2^+, \alpha_2^-, \alpha_2^{(0)})$, then there exist nonnegative integers c_{10}, c_{11}, c_{12} , and c_{20}, c_{21}, c_{22} such that $c_{10} + c_{11} + c_{12}$ equals the number of coordinates isolated in $\mathbf{y}^{(1)}$, $c_{20} + c_{21} + c_{22}$ equals the number of coordinates isolated in $\mathbf{y}^{(2)}$, and $\text{In}(A_1) = (\alpha_1^+ + c_{11}, \alpha_1^- + c_{12}, \alpha_1^{(0)} + c_{10})$ and $\text{In}(A_2) = (\alpha_2^+ + c_{21}, \alpha_2^- + c_{22}, \alpha_2^{(0)} + c_{20})$.

By (A.15.1), we know that $\text{In}(A) = \text{In}(A_1) + \text{In}(A_2)$ so $\text{In}(A) = ((\alpha_1^+ + \alpha_2^+) + (c_{11} + c_{21}), (\alpha_1^- + \alpha_2^-) + (c_{12} + c_{22}), (\alpha_1^{(0)} + \alpha_2^{(0)}) + (c_{10} + c_{20}))$. Finally, $(c_{11} + c_{21}) + (c_{12} + c_{22}) + (c_{10} + c_{20})$ is the number of coordinates isolated in $\mathbf{y}$ because the block structure of A implies that a coordinate y_k is isolated in $\mathbf{y}$ if and only if y_k is a coordinate of $\mathbf{y}^{(1)}$ and isolated in $\mathbf{y}^{(1)}$ or y_k is a coordinate of $\mathbf{y}^{(2)}$ and isolated in $\mathbf{y}^{(2)}$. Therefore the theorem is valid in Case 1.

Case 2. Suppose $\mathbf{y}^{(1)}$ has fewer than m zero coordinates and $\mathbf{y}^{(2)} = \mathbf{0}$. Each coordinate of $\mathbf{y}^{(2)}$ is isolated in $\mathbf{y}$ because of the block structure of A , so $A' = A_1'$. Let $\text{In}(A_1') = (\alpha_1^+, \alpha_1^-, \alpha_1^{(0)})$ and $\text{In}(A_2') = (\alpha_2^+, \alpha_2^-, \alpha_2^{(0)})$. By the inductive hypothesis applied to A_1 and $\mathbf{y}^{(1)}$, there exist nonnegative integers c_0, c_1, c_2 such that $c_1 + c_2 + c_0$ equals the number of coordinates isolated in $\mathbf{y}^{(1)}$ and $\text{In}(A_1) = (\alpha_1^+ + c_1, \alpha_1^- + c_2, \alpha_1^{(0)} + c_0)$. Then $(c_1 + c_2 + c_0) + (\alpha_2^+, \alpha_2^-, \alpha_2^{(0)})$ is the total number of isolated coordinates in $\mathbf{y}$, and by (A.15.1), $\text{In}(A) = \text{In}(A_1) + \text{In}(A_2)$, so we have

$$\text{In}(A) = (\alpha_1^+ + (c_1 + \alpha_2^+), \alpha_1^- + (c_2 + \alpha_2^-), \alpha_1^{(0)} + (c_0 + \alpha_2^{(0)})),$$

which proves the theorem in Case 2.

Case 3. Suppose $\mathbf{y}^{(1)}$ has m zero coordinates. Then all zero coordinates of $\mathbf{y}$ are in $\mathbf{y}^{(1)}$, so if A_2 and $\mathbf{y}^{(2)}$ appear, all coordinates of $\mathbf{y}^{(2)}$ are nonzero.

Since A_1 is acyclic and irreducible, we may apply Proposition (2.2.10) to A_1 and $\mathbf{y}^{(1)}$.

As shown there, we may assume that $A_1 = \begin{bmatrix} B & c \\ c^T & d \end{bmatrix}$, that $B = \begin{bmatrix} B_{11} & O \\ O & B_{22} \end{bmatrix}$ where B_{11} is

acyclic and irreducible, that the last coordinate of $\mathbf{y}^{(1)}$ is zero but not isolated in $\mathbf{y}^{(1)}$, and

that, when $\mathbf{y}^{(1)}$ is partitioned conformally with B as $\mathbf{y}^{(1)} = \begin{bmatrix} \mathbf{z}^{(1)} \\ \mathbf{z}^{(2)} \\ 0 \end{bmatrix}$, then $\mathbf{z}^{(1)}$ has no zero

coordinate and $\mathbf{z}^{(2)} \neq \mathbf{0}$ if it appears. Given this notation, (2.2.10) says

$$\text{In}(A_1) = \text{In}(B) + (1, 1, -1) \text{ hence } \text{In}(A_1) = \text{In}(B_{11}) + \text{In}(B_{22}) + (1, 1, -1).$$

Let B_{22}' and $(\mathbf{z}^{(2)})'$ be obtained from B_{22} and $\mathbf{z}^{(2)}$ by deleting the rows and columns corresponding to isolated coordinates of $\mathbf{z}^{(2)}$. Let $\text{In}(B_{22}') = (\beta_2^+, \beta_2^-, \beta_2^{(0)})$.

Before proceeding, observe that both $\mathbf{z}^{(1)}$ and in $\mathbf{z}^{(2)}$ have fewer than m zero coordinates. Also, all isolated coordinates of $\mathbf{y}^{(1)}$ are in $\mathbf{z}^{(2)}$ so A_1' has the form

$$\begin{bmatrix} \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22}' \end{bmatrix} & c' \\ c'^T & d \end{bmatrix}. \text{ Using this and (2.2.10), we have}$$

$$\text{In}(A_1') = \text{In}(B_{11}) + \text{In}(B_{22}') + (1, 1, -1).$$

Because the last coordinate of $\mathbf{y}$ is zero, $\mathbf{z}^{(2)}$ has fewer than m zero coordinates if it appears, and $B_{22}\mathbf{z}^{(2)} = \mathbf{0}$, then by the inductive hypothesis, $\text{In}(B_{22}) = \text{In}(B_{22}') + (c_1, c_2, c_0)$ where $c_0 + c_1 + c_2$ equals the number of isolated coordinates in $\mathbf{z}^{(2)}$. Therefore

$$\begin{aligned}\text{In}(A_1) &= \text{In}(B_{11}) + \text{In}(B_{22}') + (c_1, c_2, c_0) + (1, 1, -1) \\ &= \text{In}(A_1') + (c_1, c_2, c_0).\end{aligned}$$

Note $A' = \begin{bmatrix} A_1' & O \\ O & A_2 \end{bmatrix}$ so $\text{In}(A') = \text{In}(A_1') + \text{In}(A_2)$ by (A.15.1), and $c_1 + c_2 + c_0$ is

actually the total number of isolated coordinates in $\mathbf{y}$, because y_n is not isolated and there are no zero coordinates in $\mathbf{y}^{(2)}$. We now have

$$\text{In}(A) = \text{In}(A_1) + \text{In}(A_2) = \text{In}(A_1') + (c_1, c_2, c_0) + \text{In}(A_2) = \text{In}(A') + (c_1, c_2, c_0).$$

Thus the theorem is proved in Case 3. $\square$

(2.2.12) (Theorem (2.3), [F3]): Suppose A is $n \times n$, acyclic and λ is any eigenvalue of A .

Let $A\mathbf{y} = \lambda\mathbf{y}$, $\mathbf{y} \neq \mathbf{0}$, and let ω^+ denote the number of eigenvalues greater than λ and ω^- denote the number of eigenvalues less than λ . Let a^+ denote the number of a_{ik} such that $i < k$ and $a_{ik}y_i y_k < 0$, and let a^- denote the number of a_{ik} such that $i < k$ and $a_{ik}y_i y_k > 0$.

Let m denote the total number of zero coordinates in $\mathbf{y}$.

(a) If there are no isolated coordinates in $\mathbf{y}$, then

$$\omega^+ = a^+ + m \text{ and } \omega^- = a^- + m.$$

(b) Suppose there is at least one isolated coordinate in $\mathbf{y}$. Let A' be the submatrix of A obtained by removing the rows and columns k for each isolated coordinate y_k , and let

$\text{In}(A') = (\alpha^+, \alpha^-, \alpha^{(0)})$. Then there exist nonnegative integers c_0, c_1, c_2 such that $c_0 + c_1 + c_2$ equals the number of isolated coordinates in $\mathbf{y}$ and $\text{In}(A) = \text{In}(A') + (c_0, c_1, c_2)$.

That is,

$$\omega^+ = \alpha^+ + c_1, \omega^- = \alpha^- + c_2, \text{ and } \omega^{(0)} = \alpha^{(0)} + c_0.$$

Proof: Let $B = A - \lambda I$, which is an $n \times n$ acyclic matrix such that $B\mathbf{y} = \mathbf{0}$. Observe that $B' = A' - \lambda I$, the off diagonal elements of B and A are the same, and the off diagonal elements of B' and A' are the same. Thus we can immediately apply the previous theorem to B and $\mathbf{y}$. $\square$

(2.2.13) Example: Let

$$\text{Then } A(G) = \begin{bmatrix} -2 & 1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -3 & -3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & -5 & 1 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \text{ is an}$$

acyclic matrix.

For the purposes of our example, it will be useful to know all the eigenvalues and a complete set of corresponding eigenvectors for A. Using Matlab we find:

-9.1916	λ_1
-3.7042	λ_2
-2.0229	λ_3
-1.3700	λ_4
-0.7701	λ_5
-0.4095	λ_6
0.0000	λ_7
1.2257	λ_8
1.3976	λ_9
2.6079	λ_{10}
4.6677	λ_{11}
5.5694	λ_{12}

and $V =$

$$\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 & \mathbf{y}_5 & \mathbf{y}_6 & \mathbf{y}_7 & \mathbf{y}_8 & \mathbf{y}_9 & \mathbf{y}_{10} & \mathbf{y}_{11} & \mathbf{y}_{12} \end{bmatrix} =$$

-0.1721	-0.6673	0.3645	-0.1724	-0.0861	0.3485	-0.2887	-0.0175	0.3567	0.0213	-0.1408	-0.0510
0.0210	0.2467	-0.3563	0.4659	-0.3743	0.5902	-0.2887	-0.0079	0.1488	0.0059	-0.0248	-0.0078
-0.0169	-0.1418	0.1206	-0.0727	-0.0486	0.2472	-0.2887	0.0776	-0.8973	-0.0132	0.0384	0.0112
0.5999	0.3743	0.2343	-0.3236	0.1099	0.1056	-0.2887	0.0145	0.0830	0.0395	-0.4377	-0.1834
0.1476	0.1675	0.1399	-0.2222	0.0875	0.0930	-0.2887	0.0245	0.1553	0.3020	0.7873	0.2141
-0.7318	0.3933	-0.0201	-0.2123	0.1699	-0.0361	-0.2887	0.0424	0.0293	0.2712	-0.1781	-0.2068
0.0893	-0.1455	0.0196	0.5737	0.7390	-0.0611	-0.2887	0.0191	0.0122	0.0752	-0.0314	-0.0315
-0.1308	0.1379	-0.0100	-0.1260	0.1227	-0.0300	-0.2887	0.1096	0.0973	-0.8921	0.1335	0.1159
0.1585	-0.2830	-0.5182	-0.1509	-0.1073	-0.2935	-0.2887	0.2032	0.0190	0.0229	0.1636	-0.5946
-0.0194	0.1047	0.5066	0.4078	-0.4667	-0.4971	-0.2887	0.0913	0.0079	0.0064	0.0289	-0.0905
0.0390	-0.1267	-0.3095	-0.1036	-0.0854	-0.2583	-0.2887	0.3436	0.0356	0.1754	-0.2942	0.6942
0.0156	-0.0602	-0.1714	-0.0637	-0.0606	-0.2083	-0.2887	-0.9005	-0.0478	-0.0143	-0.0446	0.1301

where column $\mathbf{y}_j$ of V is an eigenvector corresponding to λ_j , for each $j = 1, \dots, 12$.

It is easy to check that the 12×1 column $\mathbf{e}$ of ones is an eigenvector corresponding to $\lambda_7 = 0$. (Of course $\mathbf{y}_7$ is another, but clearly $\mathbf{y}_7$ is a multiple of $\mathbf{e}$.)

Theorem (2.2.6) says the number of negative eigenvalues of A equals a^+ , the number of negative entries in the upper triangle of A , and the number of positive eigenvalues of A equals a^- , the number of positive entries in the upper triangle of A ; by inspection these numbers are 5 and 6. This is verified by examining the set of all eigenvalues of A in the list above.

To illustrate Theorem (2.2.12), choose a different eigenvalue, say λ_9 . Using the complete set of eigenvalues of A above, we see that the number of eigenvalues greater

than λ_9 is three and the number less than λ_9 is 8. To use Theorem (2.2.12) and $\mathbf{y}_9$ to calculate these numbers, first let $\mathbf{B} = \mathbf{A} - \lambda_9 \mathbf{I}$ and calculate $\mathbf{YBY}$, where $\mathbf{Y}$ is the diagonal matrix with the coordinates of $\mathbf{y}_9$ on its diagonal:

$$\mathbf{Y} = \begin{pmatrix} 0.3567 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1488 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8973 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0830 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1553 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0293 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0122 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0973 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0190 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0079 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0356 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0478 \end{pmatrix}$$

Then $\mathbf{YBY} =$

$$\begin{pmatrix} -0.2545 & 0.0531 & 0.3201 & 0.0592 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0531 & -0.0221 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3201 & 0 & 0.8051 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0592 & 0 & 0 & -0.0207 & -0.0387 & 0.0097 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.0387 & 0.0724 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0097 & 0 & -0.0043 & 0.0004 & -0.0057 & 0.0011 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0004 & -0.0001 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.0057 & 0 & 0.0189 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0011 & 0 & 0 & 0.0004 & 0.0002 & -0.0020 & 0.0009 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0002 & -0.0001 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0020 & 0 & 0.0038 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0009 & 0 & 0 & 0.0023 \end{pmatrix}$$

Counting a^+ and a^- in this matrix YBY gives 8 and 3, which agrees with what we counted directly above.

For one more application of Theorem (2.2.12), choose another eigenvalue of A , say λ_2 , and let $B = A - \lambda_2 I$. Using y_2 , define Y as above. Then $YBY =$

-2.5397	-0.1646	-0.0946	-0.4995	0	0	0	0	0	0	0	0
-0.1646	-0.2864	0	0	0	0	0	0	0	0	0	0
-0.0946	0	-0.0544	0	0	0	0	0	0	0	0	0
-0.4995	0	0	-0.9391	-0.1880	0.5889	0	0	0	0	0	0
0	0	0	-0.1880	-0.0198	0	0	0	0	0	0	0
0	0	0	0.5889	0	-1.3467	-0.0572	-0.1085	-0.2227	0	0	0
0	0	0	0	0	-0.0572	-0.0995	0	0	0	0	0
0	0	0	0	0	-0.1085	0	-0.0324	0	0	0	0
0	0	0	0	0	-0.2227	0	0	-0.2166	-0.0296	-0.1076	-0.0170
0	0	0	0	0	0	0	0	-0.0296	-0.0515	0	0
0	0	0	0	0	0	0	0	-0.1076	0	-0.0113	0
0	0	0	0	0	0	0	0	-0.0170	0	0	-0.0098

Counting a^+ and a^- , the number of positive and negative entries in the upper triangle of YBY , we get 1 and 10. Since λ_2 is the second eigenvalue of A , this is correct.

(2.2.14) Example: Consider again the example which followed Definition (2.2.7), of a 4x4 acyclic irreducible matrix A and nonzero y such that $Ay = 0$, where y has both a zero

but non-isolated coordinate y_2 and one isolated coordinate, y_4 : $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ and

$$\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \text{ Omitting row and column four in } A \text{ and } \mathbf{y} \text{ gives } A' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \text{ and}$$

$$\mathbf{y}' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Using Matlab we find that the eigenvalues of A are:

$$\begin{aligned} (1 - \sqrt{5})/2 \approx -0.6180 &= \lambda_1 \\ 0 &= \lambda_2 \\ (1 + \sqrt{5})/2 \approx 1.6180 &= \lambda_3 \\ 2 &= \lambda_4 \end{aligned}$$

and column $\mathbf{y}_j$ of the following matrix is an eigenvector corresponding to λ_j , for each j :

$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$
0.4472	0.7071	-0.4472	0.3162
-0.7236	0	-0.2764	0.6325
-0.4472	0.7071	0.4472	-0.3162
0.2764	0	0.7236	0.6325

We will use $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ as an eigenvector associated with $\lambda_2 = 0$. (Note $\mathbf{y}_2$ is a multiple

of this).

Clearly $\text{In}(A) = (2, 1, 1)$. However, if we want to find $\text{In}(A)$ using Fiedler's algorithm, we cannot use $\mathbf{y}$ to count a^+ and a^- for A : if Y is the diagonal matrix with $\mathbf{y}$ on its diagonal then YAY is zero, so $a^+ = a^- = 0$ for this matrix; $\mathbf{y}$ has $m = 2$ zero coordinates, and $\text{In}(A)$ is not equal to $(a^+ + m, a^- + m, 1) = (2, 2, 1)$. The problem is that there is an isolated coordinate in $\mathbf{y}$ so we must apply Fiedler's algorithm to A' and $\mathbf{y}'$.

The diagonal matrix Y with $\mathbf{y}'$ on its diagonal is

$$Y' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; Y'A'Y' \text{ is zero so } a^+ = a^- = 0 \text{ for this matrix; } \mathbf{y}' = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ has } m = 1 \text{ zero}$$

coordinates, so Theorem (2.2.12) says $\text{In}(A')$ should equal $(a^+ + m, a^- + m, 1) = (1, 1, 1)$.

This is correct so far, since the eigenvalues of A' are:

$$-\sqrt{2} \approx -1.4142 = \alpha_1$$

$$0 = \alpha_2$$

$$\sqrt{2} \approx 1.4142 = \alpha_3$$

Finally, letting $c_1 = 1$ and $c_2 = c_0 = 0$, we get $\text{In}(A) = \text{In}(A') + (c_1, c_2, c_0) = (2, 1, 1)$.

For one more example, consider λ_1 , for which the associated eigenvector $\mathbf{y}_1$ of A has no zero coordinates. Since we know λ_1 is the smallest eigenvalue of A , $\text{In}(A - \lambda_1 I) = (3, 0, 1)$. To verify this using Theorem (2.2.12), first note there are no zero coordinates in $\mathbf{y}_1$, that is, $m = 0$ for $\mathbf{y}_1$. Letting Y be the diagonal matrix with $\mathbf{y}_1$ on its diagonal, we find

$$Y(A - \lambda_1 I)Y = \begin{bmatrix} 0.3236 & -0.3236 & 0 & 0 \\ -0.3236 & 0.8472 & -0.3236 & -0.2000 \\ 0 & -0.3236 & 0.3236 & 0 \\ 0 & -0.2000 & 0 & 0.2000 \end{bmatrix}$$

Thus $a^+ = 3$, $a^- = 0$, and $m = 0$, so by Theorem (2.2.12), $\text{In}(A - \lambda_1 I)$ should be $(a^+ + 0, a^- + 0, 1) = (3, 0, 1)$, which is correct.

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Appendix

Here we collect basic definitions and theorems about matrices, which are used in the preceding chapters.

(A.1) Gershgorin Theorem ([RR], 3.6.2 p.89): If A is an $n \times n$ matrix, then the eigenvalues of A lie in the region of the complex plane determined by the union of the n closed discs $Z_i = \{ z: |z - a_{ii}| \leq R_i \}$, $i = 1, 2, \dots, n$.

(A.2) Definition: A square matrix A is **diagonalizable** if there exists a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

(A.3) Theorem [SIF], p.277]: A matrix is diagonalizable if and only if the multiplicity of each distinct eigenvalue equals the dimension of the corresponding eigenspace.

(A.4) Definition: Let R^n denote the set of real n -tuples. An $n \times n$ real matrix A is said to be **positive semidefinite** if $A = A^T$ and $\mathbf{x}^T A \mathbf{x}$ is nonnegative for all $\mathbf{x}$ in R^n .

(A.5) Definition: For any complex matrix A , A^* will denote its conjugate transpose. Thus if A is actually real, $A^* = A^T$. If $A = A^*$, A is called **Hermitian**.

(A.6) Theorem: Suppose A is $n \times n$, real and positive semidefinite. Then

(a) The eigenvalues of A are real and nonnegative. We will order them as

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

(b) There are n orthonormal real eigenvectors for A . Furthermore, any unit eigenvector $\mathbf{u}_1$ of A can be extended to an orthonormal set of n eigenvectors of A which is a basis for $\mathbb{R}^n$. If $\mathbf{u}_1, \dots, \mathbf{u}_n$ are real orthonormal eigenvectors corresponding to $\lambda_1, \lambda_2, \dots, \lambda_n$ respectively, and we define

$$W = \{\mathbf{x} \text{ in } \mathbb{R}^n : \mathbf{x}^T \mathbf{x} = 1 \text{ and } \mathbf{u}_1^T \mathbf{x} = 0\}, \text{ then } \lambda_2 = \min \{\mathbf{x}^T A \mathbf{x} : \mathbf{x} \text{ in } W\} \text{ and}$$

$$\lambda_n = \max \{\mathbf{x}^T A \mathbf{x} : \mathbf{x} \text{ in } W\}.$$

Also, for any $\mathbf{y}$ in $\mathbb{R}^n$, $\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{x}$ for some scalars c_1, c_2 and some $\mathbf{x}$ in W .

(c) There exists a real $n \times n$ matrix U such that $A = U^T U$.

Proof: (a) Let λ be an eigenvalue of A and let $\mathbf{u}$ be an associated eigenvector of length one. Then $A\mathbf{u} = \lambda\mathbf{u}$ implies $\mathbf{u}^* A \mathbf{u} = \lambda \mathbf{u}^* \mathbf{u}$ and this equals λ since $\mathbf{u}$ is a unit vector and $\mathbf{u}^* \mathbf{u} = \mathbf{I}$; because $A = A^*$ here, taking conjugate transposes of both sides gives $(\mathbf{u}^* A \mathbf{u})^* = \mathbf{u}^* A \mathbf{u} = \bar{\lambda}$. Therefore $\lambda = \bar{\lambda}$ so λ is real.

Because A and λ are both real, we may suppose $\mathbf{u}$ is real and $\mathbf{u}^T \mathbf{u} = 1$. It follows that λ is nonnegative because A is positive semidefinite and $A\mathbf{u} = \lambda \mathbf{u}$, so $0 \leq \mathbf{u}^T A \mathbf{u} = \mathbf{u}^T \lambda \mathbf{u} = \lambda \mathbf{u}^T \mathbf{u} = \lambda$. Therefore the eigenvalues of A are real and nonnegative and we can order them as indicated. Hence (a) is proved.

(b) By Schur's Theorem as presented in ([HJ] (2.3.1)), since both A and its eigenvalues are real, there exists a real orthogonal matrix U such that $U^T A U$ is upper triangular, and U can be chosen to have an arbitrary unit eigenvector as its the first column, i.e. the first

column is chosen first. Because $A = A^T$ here, $U^T A U = (U^T A U)^T$ is also true. Therefore $U^T A U$ is actually diagonal, hence the set of columns of U is an orthonormal set of eigenvectors of A which a basis for $\mathbb{R}^n$. Thus the first column of U can be extended to a basis for $\mathbb{R}^n$ as claimed.

Now let $\mathbf{u}_1, \dots, \mathbf{u}_n$ and W be as described. To see that $\lambda_2 = \min \{\mathbf{x}^T A \mathbf{x} : \mathbf{x} \text{ in } W\}$, let $\mathbf{x} =$

$$\sum_{i=2}^n c_i \mathbf{u}_i \text{ be an element of } W. \text{ Then } A \mathbf{x} = \sum_{i=2}^n c_i \lambda_i \mathbf{u}_i, \text{ hence } \mathbf{x}^T A \mathbf{x} = \sum_{i=2}^n (c_i \mathbf{u}_i)^T \sum_{i=2}^n c_i \lambda_i \mathbf{u}_i =$$

$$\sum_{i=2}^n c_i^2 \lambda_i \text{ because the } \mathbf{u}_i \text{ are orthonormal vectors, and this is greater than or equal to}$$

$$\lambda_2 \sum_{i=2}^n (c_i)^2. \text{ Also because the } \mathbf{u}_i \text{ are orthonormal, } \sum_{i=2}^n (c_i)^2 = \mathbf{x}^T \mathbf{x}, \text{ and } \mathbf{x}^T \mathbf{x} = 1 \text{ since}$$

$\mathbf{x} \in W$. Therefore $\mathbf{x}^T A \mathbf{x} \geq \lambda_2$. Because $\mathbf{u}_2$ is in W and $\mathbf{u}_2^T A \mathbf{u}_2 = \lambda_2$, we have shown

that $\lambda_2 = \min_{\mathbf{x} \in W} \{\mathbf{x}^T A \mathbf{x} : \mathbf{x} \text{ in } W\}$ as claimed.

$$\text{Using the same idea, we have } \mathbf{x}^T A \mathbf{x} = \sum_{i=2}^n (c_i \mathbf{u}_i)^T \sum_{i=2}^n c_i \lambda_i \mathbf{u}_i = \sum_{i=2}^n c_i^2 \lambda_i \leq \lambda_n \sum_{i=2}^n (c_i)^2 = \lambda_n.$$

Therefore $\mathbf{x}^T A \mathbf{x} \leq \lambda_n$. Because $\mathbf{u}_n$ is in W and $\mathbf{u}_n^T A \mathbf{u}_n = \lambda_n$, we have shown that $\lambda_n =$

$\max_{\mathbf{x} \in W} \{\mathbf{x}^T A \mathbf{x} : \mathbf{x} \text{ in } W\}$ as claimed.

We know $\mathbf{y} = d_1 \mathbf{u}_1 + \dots + d_n \mathbf{u}_n$ for some scalars $d_1, \dots, d_n$. Let $c_1 = d_1$ and

$c_2 = \|d_2 \mathbf{u}_2 + \dots + d_n \mathbf{u}_n\|$. If $c_2 \neq 0$, let $\mathbf{x} = (d_2 \mathbf{u}_2 + \dots + d_n \mathbf{u}_n)/c_2$. If $c_2 = 0$, let $\mathbf{x} = \mathbf{u}_2$. In

either case, $\mathbf{x}$ is in W and $\mathbf{y} = c_1 \mathbf{u}_1 + c_2 \mathbf{x}$ is true. Hence (b) is proved. $\square$

(c) Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be orthonormal eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$ and define

$$P = [\mathbf{u}_1 \dots \mathbf{u}_n]. \text{ Then } P^{-1} = P^T \text{ so } A = P \begin{bmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{bmatrix} P^T. \text{ Because each } \lambda_j \geq 0, D =$$

$$\begin{bmatrix} \sqrt{\lambda_1} & & O \\ & \ddots & \\ O & & \sqrt{\lambda_n} \end{bmatrix} \text{ is real. Clearly } A = (PD)(DP^T) \text{ so } U = DP^T \text{ satisfies } U^T U = A. \text{ Hence}$$

(c) is proved.

(A.7) Definition: If A is $n \times q$ and B is $p \times m$, the **Kronecker product** $A \otimes B$ is the $np \times qm$

$$\text{matrix } A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & & & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix}.$$

$$\textbf{Example:} \text{ Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}. \text{ Then } A \otimes B = \begin{bmatrix} 1B & 1B & 3B \\ 0B & 2B & 1B \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 3 & 1 & 3 & 3 & 9 \\ 2 & 5 & 2 & 5 & 6 & 15 \\ 0 & 0 & 2 & 6 & 1 & 3 \\ 0 & 0 & 4 & 10 & 2 & 5 \end{bmatrix}.$$

The identities in the following result are standard; see ([16], 4.2.10).

(A.8) Theorem [(HJ2], 4.2.10]: Let A, B, C, and D be matrices such that the sums and products in the following statements can be calculated. Let c be a scalar. Then

$$(a) (A \otimes B)(C \otimes D) = AC \otimes BD.$$

$$(b) (A+B) \otimes C = (A \otimes C) + (B \otimes C).$$

$$(c) c(A \otimes B) = (cA) \otimes B.$$

(A.9) Theorem: Suppose $\{y^{(1)}, \dots, y^{(k)}\}$ is an independent subset of C^m .

$$(a) \text{ If } z^{(1)}, \dots, z^{(k)} \text{ are vectors in } C^n \text{ such that } \sum_{j=1}^k z^{(j)} \otimes y^{(j)} = 0 \text{ is true, then each } z^{(j)}$$

must be 0.

$$(b) \text{ If } \{x^{(1)}, \dots, x^{(k)}\} \text{ is an independent subset of } C^n, \text{ then } \{x^{(j)} \otimes y^{(j)} : j = 1, \dots, k\} \text{ is independent.}$$

Proof:

$$(a) \text{ Let } z^{(j)} = \begin{bmatrix} z_{1j} \\ \vdots \\ z_{pj} \\ \vdots \\ z_{nj} \end{bmatrix}, \text{ so by (A.7), we have } \sum_{j=1}^k z^{(j)} \otimes y^{(j)} = \sum_{j=1}^k \begin{bmatrix} z_{1j} y^{(j)} \\ \vdots \\ z_{pj} y^{(j)} \\ \vdots \\ z_{nj} y^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ by}$$

assumption, where the size of each of those zero matrices is $n \times 1$. Therefore for each p , we have $z_{pj} y^{(j)} = 0$. But $y^{(j)}$ is not zero, so z_{pj} must be zero for each p . Thus $z^{(j)} =$

0. $\square$

(b) Suppose $\sum_{i=1}^k \sum_{j=1}^k c_{ij} (\mathbf{x}^{(i)} \otimes \mathbf{y}^{(j)}) = \mathbf{0}$. By (A.8.c), $c_{ij} (\mathbf{x}^{(i)} \otimes \mathbf{y}^{(j)}) = (c_{ij} \mathbf{x}^{(i)}) \otimes \mathbf{y}^{(j)}$. Thus we

have $\sum_{i=1}^k \left(\sum_{j=1}^k c_{ij} \mathbf{x}^{(i)} \right) \otimes \mathbf{y}^{(j)} = \mathbf{0}$. But $\sum_{j=1}^k c_{ij} \mathbf{x}^{(i)} = \mathbf{0}$ by part (a), and $\mathbf{x}^{(i)}$ are nonzero

vectors because they are independent sets of C^n . Thus all c_{ij} must be zero. Then

$\mathbf{x}^{(i)} \otimes \mathbf{y}^{(j)} \neq \mathbf{0}$ for each j , thus it is an independent set. $\square$

(A.10) Theorem: Suppose A and B are diagonalizable, where A is $n \times n$ with spectrum $\{\lambda_1, \dots, \lambda_n\}$ and B is $m \times m$ with spectrum $\{\mu_1, \dots, \mu_m\}$.

(a) Suppose $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{y} = \mu\mathbf{y}$. Then $((A \otimes I_m) + (I_n \otimes B))(\mathbf{x} \otimes \mathbf{y}) = (\lambda + \mu)(\mathbf{x} \otimes \mathbf{y})$.

(b) If λ is an eigenvalue of A with multiplicity r and μ is an eigenvalue of B with multiplicity s , then $\lambda + \mu$ is an eigenvalue of $(A \otimes I_m) + (I_n \otimes B)$ with multiplicity rs . Thus the eigenvalues of $A \otimes I_m + I_n \otimes B$ are the nm numbers $\lambda_i + \mu_j$.

Proof:

(a): By (A.8.b), we have $((A \otimes I_m) + (I_n \otimes B))(\mathbf{x} \otimes \mathbf{y}) = (A \otimes I_m)(\mathbf{x} \otimes \mathbf{y}) +$

$(I_n \otimes B)(\mathbf{x} \otimes \mathbf{y})$ and this equals $(A\mathbf{x}) \otimes (I_m \mathbf{y}) + (I_n \mathbf{x}) \otimes (B\mathbf{y})$ by (A.8.a). That last

expression equals $(\lambda \mathbf{x}) \otimes (I_m \mathbf{y}) + (I_n \mathbf{x}) \otimes (\mu \mathbf{y})$; then by (A.8.c) we have

$\mathbf{x} \otimes (\lambda \mathbf{y}) + (\mu \mathbf{x}) \otimes \mathbf{y} = \lambda (\mathbf{x} \otimes \mathbf{y}) + \mu (\mathbf{x} \otimes \mathbf{y}) = (\lambda + \mu)(\mathbf{x} \otimes \mathbf{y})$. $\square$

(b): By Theorem (A.6), there exist r independent eigenvectors $x^{(1)}, \dots, x^{(r)}$ of A corresponding to the eigenvalue λ and there exist s independent eigenvectors $y^{(1)}, \dots, y^{(s)}$ of B corresponding to the eigenvalue μ . The rs vectors $\{x^{(i)} \otimes y^{(j)} : i = 1, \dots, r; j = 1, \dots, s\}$ are independent by (A.9.b), and by (a) each is an eigenvector corresponding to the eigenvalue $\lambda + \mu$ of $(A \otimes I_m) + (I_n \otimes B)$. It follows that there are nm independent eigenvectors of $(A \otimes I_m) + (I_n \otimes B)$, one for each $\lambda_i + \mu_j$, $i = 1, \dots, n; j = 1, \dots, m$, so these numbers are the only eigenvalues of $(A \otimes I_m) + (I_n \otimes B)$. $\square$

(A.11) Definition ([SIF], p.313): Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in R^n . Then their **dot product** is defined as: $\mathbf{x} \bullet \mathbf{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.

(A.12) Cauchy Schwarz Inequality ([SIF], 6.3): Let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors in R^n , then $|\mathbf{x} \bullet \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$ where $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$ is the length of the vector $\mathbf{u}$.

(A.13) Definition: Let A be symmetric. The **inertia** of A , $\text{In}(A)$, is the triple (p, q, z) where p is the number of positive eigenvalues of A , q is the number of negative eigenvalues of A , and z is the number of zero eigenvalues of A (including multiplicities).

For example, if $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, A has spectrum $0 \leq 3 \leq 3$, hence $\text{In}(A) = (2, 0, 1)$.

(A.14) Sylvester's law of inertia ([HJ], 4.5.8): Let A and B be two Hermitian matrices, then exists an invertible matrix S such that $A = SBS^*$ if and only if A and B have the same inertia.

(A.15) Proposition: Characteristics of inertia for a symmetric matrix A.

1. If A is block-diagonal, as $A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_s \end{bmatrix}$, then $\text{In}(A) = \sum_{k=1}^s \text{In}(A_k)$.
2. Suppose B is a principal submatrix of A. Let $\text{In}(A) = (p, q, z)$ and $\text{In}(B) = (p_1, q_1, z_1)$. Then $p_1 \leq p$ and $q_1 \leq q$.

Proof of (A.15.1): Since the set of eigenvalues of A equals the union of the eigenvalues of $A_1, A_2, \dots, A_s$, then $\text{In}(A) = \sum_{k=1}^s \text{In}(A_k)$. $\square$

Proof of (A.15.2): Since $\text{In}(A) = (p, q, z)$ we may label the eigenvalues of A as

$\alpha_1 \leq \dots \leq \alpha_q < 0 = \alpha_{q+1} = \dots = \alpha_{q+z} < \alpha_{q+z+1} \leq \dots \leq \alpha_n$. Suppose first that B is

$(n-1) \times (n-1)$ with eigenvalues $\beta_1 \leq \dots \leq \beta_{n-1}$. Because B is a principal submatrix of A,

its eigenvalues interlace those of A: $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \dots \leq \alpha_{n-1} \leq \beta_{n-1} \leq \alpha_n$ ([HJ],

Theorem 4.3.8). We also know $\alpha_q < 0 \leq \alpha_{q+1}$ and

$$\alpha_1 \leq \beta_1 \leq \dots \leq \alpha_q \leq \beta_q \leq \alpha_{q+1} \leq \beta_{q+1}.$$

Thus β_q could be negative, zero, hence B has either q-1 or q negative eigenvalues, hence

$$q_1 \leq q.$$

In the same way, we have $\alpha_{q+z} \leq 0 < \alpha_{q+z+1}$ and $\alpha_{q+z} \leq \beta_{q+z} \leq \alpha_{q+z+1} \leq \dots \leq \beta_{n-1} \leq \alpha_n$,

so β_{q+z} could be zero or positive, hence B could have p-1 or p positive eigenvalues. In

any case, $p_1 \leq p$ is true.

So far we have proved that (A.15.2) is true when A is nxn and B is a principal submatrix one size smaller. Now let B_0 be a principal submatrix of A of size kxk, where $k \leq n-2$.

There exist principal submatrices of A,

$$B_0, B_1, \dots, B_{n-k-1}, B_{n-k} = A$$

where each B_i is a principal submatrix of B_{i+1} and B_{i+1} is one size larger than B_i . Let

$p(B_i)$ and $q(B_i)$ denote the number of positive and the number of negative eigenvalues

of B_i , respectively. By the argument above, $p(B_i) \leq p(B_{i+1})$ and $q(B_i) \leq q(B_{i+1})$.

Therefore we have

$$p_1 = p(B_0) \leq p(B_1) \leq \dots \leq p(B_{n-k}) = p$$

$$q_1 = q(B_0) \leq q(B_1) \leq \dots \leq q(B_{n-k}) = q$$

This shows that (A.15.2) is true for any principal submatrix of A. $\square$

To illustrate (A.15.1), consider the graph G in Example (1.1.14). The spectrum of $A(G)$ is $0 \leq 0 \leq 2 \leq 2$, the spectrum of A_{11} is $0 \leq 2$, and the spectrum of A_{22} is $0 \leq 2$. Thus

$$\text{In } A(G) = (2, 0, 2) = (1, 0, 1) + (1, 0, 1) = \text{In}(A_{11}) + \text{In}(A_{22}).$$

To illustrate (A.15.2), let $A = \begin{bmatrix} -1 & -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}$, $A_2 =$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, and $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$. Notice that each of A , A_1 , A_2 and A_3 has the next one

as a principal submatrix one size smaller. The spectrum of A is $-2.6554 \leq -1.2108 \leq 0$

$\leq 0 \leq 1.8662$, the spectrum of A_1 is $-2.4812 \leq -0.6889 \leq 0 \leq 1.1701$, the spectrum of

A_2 is $-1 \leq 0 \leq 0$, and the spectrum of A_3 is $-1 \leq 0$. Therefore $\text{In}(A_3) = (0, 1, 1)$,

$\text{In}(A_2) = (0, 1, 2)$, $\text{In}(A_1) = (1, 2, 1)$, and $\text{In}(A) = (1, 2, 2)$. Thus $p_3 \leq p_2 \leq p_1 \leq p$, and

$q_3 \leq q_2 \leq q_1 \leq q$.

(A.16) Sylvester's Inertia Theorem for Quadratic Forms ([LT], p 204): Let C be an

$n \times n$ symmetric real matrix such that $\mathbf{y}^T C \mathbf{y} = a_1 y_1^2 + \dots + a_n y_n^2$, where $y_1, \dots, y_n$ are

independent variables and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Let p be the number of positive eigenvalues of C

and q be the number of negative eigenvalues of C . Then the number of positive a_j equals

p and the number of negative a_j equals q .