1989

An Application of forcing in descriptive set theory

Jeffery Johnson
San Jose State University

Follow this and additional works at: https://scholarworks.sjsu.edu/etd_theses

Recommended Citation

Johnson, Jeffery, "An Application of forcing in descriptive set theory" (1989). *Master's Theses*. 3082.
DOI: <https://doi.org/10.31979/etd.u59c-jgr5>
https://scholarworks.sjsu.edu/etd_theses/3082

This Thesis is brought to you for free and open access by the Master's Theses and Graduate Research at SJSU ScholarWorks. It has been accepted for inclusion in Master's Theses by an authorized administrator of SJSU ScholarWorks. For more information, please contact scholarworks@sjsu.edu.

INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book. These are also available as one exposure on a standard 35mm slide or as a 17" x 23" black and white photographic print for an additional charge.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

U·M·I

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0600

Order Number 1337816

An application of forcing in descriptive set theory

Johnson, Jeffery T., M.S.

San Jose State University, 1989

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

An Application of Forcing in Descriptive Set Theory

A Thesis
Presented to
the Faculty of the Department of Mathematics
San Jose State University

in Partial Fulfillment
of the Requirements for the Degree
Master of Science

By
Jeffery Johnson
May 1989

Approved for the Department of Mathematics

Dr. Maurice Stanley

Dr. Rolando Chuaqui

Dr. Jon Pearce

Approved for the University

Serena H. Stanford

Abstract

This thesis is an exposition of the following theorem: If ZFC + "there exists an inaccessible cardinal" is consistent, then ZF+ "dependent choice" + "every set of reals is Lebesgue measurable" is consistent. The exposition includes an introduction to forcing via Boolean-valued models, a presentation of a forcing construction obtained from an inaccessible cardinal (the Levy collapse), and a treatment of some basic results from descriptive set theory.

Acknowledgements

I would like to thank the members of the examining committee for there many useful suggestions and especially Dr. Mack Stanley for what has become years of patient guidance. His infectious love for this subject has encouraged me throughout the last three years and I can only hope that I might emulate his enthusiasm when I have the opportunity to guide students in the future. Finally, I would like to thank my friends and family for their encouragement and especially Glenn Dean for many useful discussions.

Contents

Introduction.....	1
Chapter 1: Boolean-valued Models and Forcing	
1. The Boolean-valued Universe.....	2
2. A Simple Example of Forcing.....	6
3. Forcing using Boolean-valued Models.....	8
4. Forcing and Independence Proofs.....	12
Chapter 2: The Levy Collapse and Its Properties	
1. The Forcing Conditions.....	14
2. Preservation Properties and the Levy Collapse.....	16
3. Homogeneity of the Levy Collapse.....	20
4. Submodels of the Levy Collapse.....	22
5. The Factor Theorem.....	29
Chapter 3: A Brief Introduction to Descriptive Set Theory	
1. Perfect Polish Spaces.....	41
2. Simply Defined Sets.....	43
3. Borel Codes.....	47
4. Absoluteness.....	49
5. Absoluteness for Borel Codes.....	52
6. Absoluteness Properties of Lebesgue Measure.....	56

Chapter 4: A Model in Which Every Set of Reals is Lebesgue Measurable	
1. Random Real Forcing.....	58
2. Solovay Sets and Random Reals.....	62
3. Lebesgue Measurability in the Levy Collapse.....	65
4. The Model.....	68
Bibliography.....	
	74

INTRODUCTION

The purpose of this thesis is to prove the following theorem of R. M. Solovay, [9]:

Theorem: If there is a model of ZFC+"there exists an inaccessible cardinal", then there is a model of ZF+"dependent choice" + "every set of reals is Lebesgue measurable".

The proof is an application of the method of forcing first introduced by Paul Cohen in the 1960's and the first chapter will be an exposition of forcing in terms of Boolean-valued models. We introduce a simple example of forcing as well. In the last section, we discuss the meta-mathematical status of forcing arguments and conclude that these arguments can be turned into finite consistency proofs.

We next go on to consider a particular example of forcing called the Levy collapse in the second chapter. We prove its basic properties in the second and third sections and go on to discuss some of its more specialized properties in the last section. These final results, while technical, will be of crucial importance in the last chapter.

Chapter 3 is a brief introduction to descriptive set theory. We introduce some of the basic classifications of simple sets of reals and fix notation first. Next, we introduce the concept of absoluteness and treat the absoluteness of various properties of sets of reals. Finally, we prove an important property of Lebesgue

measure is absolute.

The final chapter applies the material from Chapters 2 and 3 to the problem at hand. We introduce Random reals and Solovay sets of reals in order to prove our main theorem. These were the main new tools introduced by R. M. Solovay in proving this important theorem.

The reader should be familiar with the basic ideas of forcing and already know some basic examples of the method if they wish to follow this work. The third chapter of T. Jech's Set Theory would certainly suffice.

Chapter 1: Boolean Valued Models and Forcing

In this chapter, we give a brief treatment of the Forcing Method introduced by Paul Cohen. This method has shown itself to be a powerful tool in the search for relative consistency results in Set Theory.

There are different ways in which set theorists view forcing and we choose to work in the context of Boolean-valued models, a context first introduced by R. M. Solovay and D. Scott in the late sixties. These generalized models will aid in clarity and brevity of presentation and will serve to introduce the reader to a powerful meta-mathematical tool. In order to keep this treatment to a single chapter, we omit most of the proofs and content ourselves with referring the reader to the beautiful book by J. L. Bell [1].

Section 1: The Boolean-Valued Universe

The Boolean-valued universe, V^B , is defined to be a certain generalization of the usual set-theoretical universe, V , the class of all sets. This universe together with the Boolean values of membership and equality will form our Boolean-valued model.

Let V be the class of all sets

$$V = \{ x \mid x = x \}$$

Let $B \in V$ be a complete Boolean algebra. We define the Boolean-valued universe by induction as follows:

$$V_0^B = \emptyset$$

$$V_\alpha^B = \bigcup_{\beta < \alpha} V_\beta^B ; \alpha \text{ a limit ordinal}$$

$$V_\alpha^B = \{x \mid x \text{ is a function, } \text{dom}(x) \subseteq V_{\alpha-1}^B, \text{ and } \text{ran}(x) \subseteq B\}; \alpha \text{ a successor}$$

and finally,

$$V^B = \bigcup_{\alpha \in \text{Ord}} V_\alpha^B.$$

Note that we can define the rank in V^B by:

$$\rho(x) = \text{least } \alpha \text{ s.t. } x \in V_{\alpha+1}^B.$$

This allows us to state definitions by Boolean-valued induction.

Assuming the Boolean-values of equality and membership have already been defined, we define the Boolean value of ϕ ,

$$\| \phi(a_1, \dots, a_n) \| \quad (a_1, \dots, a_n \in V^B),$$

where ϕ is a first-order formula. The definition proceeds by induction on the complexity of ϕ as follows:

- (i) $\| \neg \phi(a_1, \dots, a_n) \| = - \| \phi(a_1, \dots, a_n) \|$
- (ii) $\| \phi \wedge \psi(a_1, \dots, a_n) \| = \| \phi(a_1, \dots, a_n) \| * \| \psi(a_1, \dots, a_n) \|$
- (iii) $\| \phi \vee \psi(a_1, \dots, a_n) \| = \| \phi(a_1, \dots, a_n) \| + \| \psi(a_1, \dots, a_n) \|$
- (iv) $\| \phi \rightarrow \psi(a_1, \dots, a_n) \| = \| \neg \phi \vee \psi(a_1, \dots, a_n) \|$
- (v) $\| \exists x \phi(x, a_1, \dots, a_n) \| = \sum_{a \in V^B} \| \phi(a, a_1, \dots, a_n) \|$
- (vi) $\| \forall x \phi(x, a_1, \dots, a_n) \| = \prod_{a \in V^B} \| \phi(a, a_1, \dots, a_n) \|.$

Note that this definition proceeds much as in the case of the usual satisfaction relation for standard (two-valued) models.

The definitions of the Boolean values of equality and membership are motivated by the following properties that these values must have; first,

the Axiom of Extensionality must hold for the value of equality, i. e.

$$\| u = v \| = \| \forall x \in u (x \in v) \wedge \forall y \in v (y \in u) \|$$

and second, the logical truth,

$$(u \in v \leftrightarrow \exists y \in v (u = y)),$$

which can be written

$$\| u \in v \| = \| \exists y (y \in v \wedge u = y) \|$$

must hold. The formal definitions, which actually proceed by induction on pairs $(p_B(x), p_B(y))$ under the canonical well-ordering, is as follows:

$$\| x \in y \| = \sum_{u \in \text{dom}(y)} [y(u) * \| u = y \|]$$

$$\| x = y \| = \prod_{u \in \text{dom}(x)} (-x(u) + \| u \in y \|) * \prod_{v \in \text{dom}(y)} (-y(v) + \| v \in x \|)$$

(Note that induction on pairs $(p_B(x), p_B(y))$ is necessary to ensure that the Boolean values mentioned in these definitions have already been defined.)

The notion of Boolean-valued model generalizes our notion of standard model as the following example shows:

Let B be the trivial algebra, $B = \{0, 1\}$; if we introduce the equivalence relation $x \equiv y$ iff $\| x = y \| = 1$ on V^B and define:

$$V^{B/\equiv} \models \phi(a_1, \dots, a_n) \text{ iff } \| \phi(a_1, \dots, a_n) \| = 1$$

then $V^{B/\equiv}$ becomes a standard (two-valued) model. A stronger result can be obtained, namely that $V^{B/\equiv}$ is isomorphic to V .

It can be shown that the axioms of the predicate calculus are valid in V^B and so every sentence provable in predicate calculus is valid as well. The most important result we have in this direction is the following theorem.

Theorem 1: Let B be a complete Boolean algebra. Every axiom of ZFC has Boolean value 1 in V^B . ■

This theorem will make it possible to conclude that the generic extensions introduced later in this chapter are models of ZFC.

Section 2: A Simple Example of Forcing

Now we present a simple example of forcing, first introduced by Paul Cohen.

Let $\mathfrak{M}$ be some fixed standard model of ZFC. Let $\omega_1^{\mathfrak{M}}$ be the ordinal that $\mathfrak{M}$ thinks is ω_1 , i. e.,

$$\mathfrak{M} \models "\omega_1^{\mathfrak{M}} \text{ is the first cardinal greater than } \omega".$$

We would like to extend $\mathfrak{M}$ to a model in which $\omega_1^{\mathfrak{M}}$ is a countable ordinal. The only way to accomplish this is to introduce a function, f , that witnesses to the fact that $\omega_1^{\mathfrak{M}}$ is countable. That is, f maps ω onto $\omega_1^{\mathfrak{M}}$. We call such a function a collapsing map for $\omega_1^{\mathfrak{M}}$. The crucial thing to note is that $f|n$, $n \in \omega$, is just a finite sequence of ordinals less than $\omega_1^{\mathfrak{M}}$, and so is outright definable in $\mathfrak{M}$. So the problem of finding such a function is reduced to that of defining a special set of finite sequences of ordinals less than $\omega_1^{\mathfrak{M}}$.

Let $\mathbb{P}$ be the set of all finite sequences of ordinals less than $\omega_1^{\mathfrak{M}}$. We partially order $\mathbb{P}$ by reverse inclusion as follows:

$$p \leq q \text{ iff } p \supseteq q$$

and note that $(\mathbb{P}, \leq) \in \mathfrak{M}$, since $\mathbb{P}$ is definable in $\mathfrak{M}$ without parameters. We call $\mathbb{P}$ a notion of forcing and the elements of $\mathbb{P}$ forcing conditions.

Now we want to define a special subset , G , of $\mathbb{P}$ such that $\bigcup G$ is a function from ω onto $\omega_1^{\mathfrak{M}}$. The following definitions capture the properties that G must have, as we shall show.

Definition 1: A set, $F \subseteq \mathbb{P}$, is a filter on $\mathbb{P}$ if:

- (i) F is nonempty.
- (ii) If $p \in F$ and $p \leq q$, then $q \in F$.
- (iii) If $p \in F$ and $q \in F$, then there exists an $r \in F$ such that $r \leq p$ and $r \leq q$.

Definition 2: A set, $D \subseteq \mathbb{P}$, is dense in $\mathbb{P}$ if for every $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leq p$.

Definition 3: A filter, G on $\mathbb{P}$, is a generic filter over $\mathfrak{M}$, if G intersects every dense set, D , in $\mathbb{P}$ such that $D \in \mathfrak{M}$.

As the following lemma shows, G is just the special subset of $\mathbb{P}$ we need.

Lemma 1: Let $\mathbb{P}$ be as defined above and let $G \subseteq \mathbb{P}$ be generic over $\mathfrak{M}$, then

$$\bigcup G = f \text{ is a collapsing map for } \omega_1^{\mathfrak{M}}.$$

Proof: Since G is a filter on $\mathbb{P}$, f is a well-defined function. To show that f is a collapsing map for $\omega_1^{\mathfrak{M}}$, we need to show $\text{dom}(f) = \omega$ and $\text{ran}(f) = \omega_1^{\mathfrak{M}}$. Consider the sets

$$D_n = \{ p \in \mathbb{P} \mid n \in \text{dom}(p) \} \text{ and } E_\alpha = \{ p \in \mathbb{P} \mid \alpha \in \text{ran}(p) \}.$$

Let $p \in \mathbb{P}$ and suppose $n \notin \text{dom}(p)$. We define $q \in \mathbb{P}$ as follows:

$$q(m) = p(m) \text{ for all } m \in \text{dom}(p) \text{ and } q(k) = 0 \text{ for } k \in n+1 / \text{dom}(p).$$

Then $q \leq p$ and $q \in D_n$, showing D_n is dense in $\mathbb{P}$ for each $n \in \omega$. In a similar way, we can show that E_α is dense in $\mathbb{P}$ for each $\alpha \in \omega_1^{\mathfrak{M}}$. These are used to show that $\text{dom}(f) = \omega$ and $\text{ran}(f) = \omega_1^{\mathfrak{M}}$, proving that f is a collapsing map for $\omega_1^{\mathfrak{M}}$. ■

We have no hope of finding such a $G \in \mathfrak{M}$ since $\bigcup G$ would collapse $\omega_1^{\mathfrak{M}}$. What we want to find is a generic extension of $\mathfrak{M}$, written $\mathfrak{M}[G]$, that satisfies the following:

- (i) $\mathfrak{M} \subseteq \mathfrak{M}[G]$.
- (ii) $G \in \mathfrak{M}[G]$.
- (iii) $\mathfrak{M}[G] \models \text{ZFC}$.
- (iv) $\text{Ord } \mathfrak{M} = \text{Ord } \mathfrak{M}[G]$.

Further, we want $\mathfrak{M}[G]$ to preserve almost all of the facts true in $\mathfrak{M}$; for instance, we might want $\mathfrak{M}[G]$ to preserve the fact that $\omega_2^{\mathfrak{M}}$ is a cardinal. This is the really subtle point in the construction of the extension. We will guarantee a certain amount of preservation by (i) above, along with the following properties:

- (v) $\mathfrak{M}[G]$ should be describable within the structure $\mathfrak{M}$,

and

- (vi) truth in $\mathfrak{M}[G]$ is obtainable, in a sense, from truth in $\mathfrak{M}$.

Using the Boolean-valued models of Section 1, we will construct generic extensions in such a way that (i) - (vi) above are satisfied.

Section 3: Forcing using Boolean-Valued Models.

The first step in the construction of $\mathfrak{M}[G]$ is to draw a connection between complete Boolean Algebras and the forcing conditions $(\mathbb{P}, \leq)$. From the theory of Boolean Algebras, we have the following nice result which acts as our bridge between forcing and Boolean-valued models:

Theorem 2: Let $(\mathbb{P}, \leq)$ be a partially ordered set. There exists a complete Boolean Algebra, B , unique up to isomorphism, and an embedding $E: \mathbb{P} \rightarrow B$ such that:

- (i) if $p \leq q$, then $E(p) \leq E(q)$
- (ii) p and q are compatible iff $E(p) * E(q) \neq 0$
- (iii) $E[\mathbb{P}]$ is dense in $B / \{0\}$. ■

We write $\text{comp}(\mathbb{P}) = B$, if B is the algebra of the theorem.

The generic set G on $\mathbb{P}$ corresponds in a natural way to a generic ultrafilter on B . If we let

$$G' = \{ u \in B \mid (\exists p \in G) E(p) \leq u \},$$

then G' is a generic ultrafilter on B .

With these correspondences in hand, we want to develop our model $\mathfrak{M}[G]$. Let $(\mathbb{P}, \leq)$ be some notion of forcing and let $B = \text{comp}(\mathbb{P})$ in $\mathfrak{M}$. Now, working in $\mathfrak{M}$, we define the Boolean-valued model $\mathfrak{M}^B$. That is, we relativize the construction of V^B to $\mathfrak{M}$. For instance, the universe is:

$$\mathfrak{M}^B = \bigcup_{\alpha \in \text{Ord} \cap \mathfrak{M}} \mathfrak{M}_\alpha^B, \text{ where,}$$

$$\mathfrak{M}_{\alpha+1}^B = \{ x \in \mathfrak{M} \mid x \text{ is a function, } \text{dom}(x) \subseteq \mathfrak{M}_\alpha^B, \text{ and } \text{ran}(x) \subseteq B \}.$$

In order to define $\mathfrak{M}[G]$, we construct a function, i_G , on $\mathfrak{M}^B$ called the G -interpretation of $\mathfrak{M}^B$. The definition proceeds by induction on $\rho(x)$.

Let

$$i_G(\emptyset) = \emptyset,$$

$$i_G(x) = \{ i_G(y) \mid (\exists p \in G) E(p) \leq x(y) \},$$

and finally we let,

$$\mathfrak{M}[G] = \{ i_G(x) \mid x \in \mathfrak{M}^B \}.$$

It is important to note that $\mathfrak{M}^B$ is a definable class within the ground model,

$\mathfrak{M}$, and that the relation, $E(p) \leq x(y)$, is definable in $\mathfrak{M}$ as well. The only set mentioned in this definition which is not definable in $\mathfrak{M}$ is the generic filter, G . So the generic extension is almost completely describable in the ground model, $\mathfrak{M}$.

The Boolean-valued model, $\mathfrak{M}^B$, provides a language, called the *forcing language*, for the description of the model, $\mathfrak{M}[G]$. We call any $x \in \mathfrak{M}^B$, such that $i_G(x) = x$ a *name* for $x \in \mathfrak{M}[G]$.

To see that $\mathfrak{M} \subseteq \mathfrak{M}[G]$, we define canonical names for elements of $\mathfrak{M}$ and show that the G -interpretations of these names are just these sets in $\mathfrak{M}$. The definition proceeds by induction. Let

$$\emptyset^* = \emptyset$$

and for $x \in \mathfrak{M}$, let $x^* \in \mathfrak{M}^B$ be the function with domain $\{y^* \mid y \in x\}$ and $x^*(y^*) = 1$ for all y^* . Note that x^* is indeed a name for $x \in \mathfrak{M}$:

$$\begin{aligned} i_G(x^*) &= \{i_G(y^*) \mid (\exists p \in G) E(p) \leq x^*(y^*)\} \\ &= \{y \mid (\exists p \in G) E(p) \leq x^*(y^*)\} \\ &= \{y \mid y \in x\}. \end{aligned}$$

In a similar way, we can show that $G \in \mathfrak{M}[G]$. Let $G \in \mathfrak{M}^B$ be the function with

$$\text{dom}(G) = \{x^* \mid x \in B\} \text{ and } G(x^*) = x, \forall x \in B.$$

Then

$$\begin{aligned} i_G(G) &= \{i_G(x^*) \mid (\exists p \in G) E(p) \leq G(x^*)\} \\ &= \{i_G(x^*) \mid (\exists p \in G) E(p) \leq x\} \\ &= G', \end{aligned}$$

where G' is the generic ultrafilter on B which corresponds to G . But G is definable in $\mathfrak{M}[G]$, from G' , by

$$G = \{p \in \mathbb{P} \mid E(p) \in G'\},$$

and so $G \in \mathfrak{M}[G]$.

We now address the issue of truth in the model $\mathfrak{M}[G]$. The following theorem is our main result.

Theorem 3: (Forcing Theorem) Let G be a generic filter on the forcing, $\mathbb{P}$, and let G' be the generic ultrafilter on $B = \text{comp}(\mathbb{P})$ which corresponds to G . Let ϕ be a first order formula in the language of ZFC, then

$$\mathfrak{M}[G] \models \phi(x_1, \dots, x_n) \quad \text{iff} \quad \|\phi(x_1, \dots, x_n)\| \in G',$$

where $x_1, \dots, x_n$ are names for $x_1, \dots, x_n$. ■

This theorem is not surprising if we look at the definition of the interpretation function and recall the properties of a generic ultrafilter. For all the details, see Section 18 of Jech [].

Definition 4: A condition, $p \in \mathbb{P}$, *forces ϕ at $x_1, \dots, x_n$* if:

$$E(p) \leq \|\phi(x_1, \dots, x_n)\|$$

and we write

$$p \Vdash \phi(x_1, \dots, x_n)$$

where $x_1, \dots, x_n$ are names for $x_1, \dots, x_n$.

Theorem 1 of this chapter, in connection with the Forcing Theorem, allows us to conclude that

$$\mathfrak{M}[G] \models \text{ZFC}$$

since every axiom of ZFC has Boolean value 1 in $\mathfrak{M}^B$ and $1 \in G'$.

Finally, to see that $\text{Ord } \mathfrak{M}[G] = \text{Ord } \mathfrak{M}$, we have only to note that every set in $\mathfrak{M}[G]$ has rank equal to some ordinal in $\mathfrak{M}$ and so $\text{Ord } \mathfrak{M}[G] \subseteq \text{Ord } \mathfrak{M}$. The opposite inclusion follows from the absoluteness of Ord, and the fact that $\mathfrak{M} \subseteq \mathfrak{M}[G]$.

We have managed to define a generic extension of $\mathfrak{M}$ in such a way that

the points mentioned in Section 2, above, are satisfied. The machinery so far developed will make it possible to force an inaccessible to become a successor cardinal in the next chapter. Before discussing this example of forcing, however, we must discuss the existence of generic sets and the use of generic extensions in independence proofs.

Section 4: Forcing and Independence Proofs

In this section, we explain how to use forcing to arrive at consistency results.

We will assume the existence of countable transitive models (CTM's) of ZFC throughout this thesis. The reason for this is the following theorem.

Theorem 4: Let $\mathbb{P}$ be a notion of forcing and let $p \in \mathbb{P}$. If the ground model $\mathfrak{M}$ contains only a countable number of dense subsets of $\mathbb{P}$, then we can find a set, G , generic over $\mathfrak{M}$, such that $p \in G$.

Proof: Let $D_1, \dots, D_n$ be an enumeration of the dense subsets of $\mathbb{P}$ in $\mathfrak{M}$. Let $p_0 = p$ and for each $n \in \omega$, $n > 0$, pick $p_n \in D_n$ such that $p_n \leq p_{n-1}$, which is possible since each D_n is dense in $\mathbb{P}$. Finally, let

$$G = \{ p' \in \mathbb{P} \mid p' \geq p_n \text{ for some } n \},$$

then G is a generic filter on $\mathbb{P}$ over $\mathfrak{M}$. ■

Thus, assuming there exists a CTM of ZFC, we can always find an $\mathfrak{M}$ -generic G for any forcing $\mathbb{P} \in \mathfrak{M}$.

Suppose we want to prove the consistency of some first-order statement, A , with the axioms of ZFC. Using the theorems of Section 3, Theorem 4, and,

most importantly, the properties of our forcing conditions we can show that:

$$\text{CTM } \mathfrak{M} \models \text{ZFC} \text{ implies } \mathfrak{M}[G] \models \text{ZFC}+A$$

for any generic G on $\mathbb{P}$. Now, existence of CTM's for ZFC is not provable in ZFC, so we convert the proof of this statement to a proof of:

$$\text{Con (ZFC)} \text{ implies } \text{Con (ZFC}+A\text{)},$$

as follows:

If we analyze the proof of

$$\text{CTM } \mathfrak{M} \models \text{ZFC} \text{ implies } \mathfrak{M}[G] \models \text{ZFC}+A$$

we can see that it requires only a finite number of axioms of ZFC, $\phi_1, \dots, \phi_n$ to show $\mathfrak{M}[G]$ is a model of A and that $\mathbb{P}$ exists.

Now, to get a contradiction, suppose that $\text{ZFC} \vdash \text{not-}A$. Then, by the Compactness Theorem of first-order logic, there is a finite set of axioms, F , such that F proves $\text{not-}A$. Further, using the results of Section 3, we can see that we need a finite set of axioms, $\pi_1, \dots, \pi_m$, in $\mathfrak{M}$, to show that $\mathfrak{M}[G] \models \phi$, for every $\phi \in F$.

Given the finite set of sentences:

$$\{ \phi_1, \dots, \phi_n, \pi_1, \dots, \pi_m \}$$

and our assumption that ZFC is consistent, it is a theorem of ZFC that there is a countable set, M , such that

$$M \models \phi_1 \wedge \dots \wedge \phi_n \wedge \pi_1 \wedge \dots \wedge \pi_m.$$

But, then we can use Theorem 4 above to find a generic G on $\mathbb{P}$ over M , and build the model $M[G]$, which would model A and $\text{not-}A$, a contradiction.

We conclude that the CTM assumption poses no problems, and we will make this assumption throughout.

Chapter 2: The Levy Collapse and Its Properties

In this chapter, we describe a forcing due to Azriel Levy which collapses all ordinals below an inaccessible cardinal. Further, we describe some of the properties this forcing has which we will take advantage of in the development of the model in which every set of reals is Lebesgue measurable.

Section 1: The Forcing Conditions

In Chapter 1, we introduced a simple method of collapsing $\aleph_1^{\mathfrak{M}}$ to $\aleph_0$. The forcing we introduce here can be viewed as a product of forcings, each of which collapses an ordinal, $\alpha < \lambda$, where λ is an inaccessible in the ground model, $\mathfrak{M}$.

Let $\mathfrak{M}$ be a CTM of the theory ZFC+I, where I is the statement an inaccessible cardinal exists. Let λ be an inaccessible cardinal in $\mathfrak{M}$, i.e.

$$\mathfrak{M} \models \text{"}\lambda \text{ is regular"} \quad \text{and} \quad \mathfrak{M} \models \forall \alpha (|\alpha| < \lambda \rightarrow 2^{|\alpha|} < \lambda).$$

It is necessary to introduce the axiom I into ZFC because existence of inaccessibles is independent of ZFC.

Definition 1: The Levy conditions, $\mathbb{P}_L$, are all functions, p , ordered by inverse inclusion, such that

- (i) $\text{dom}(p)$ is a finite subset of $\lambda \times \omega$,
- (ii) $\text{ran}(p) \subseteq \lambda$
- (iii) $p(\alpha, n) < \alpha$ whenever $(\alpha, n) \in \text{dom}(p)$

The condition, (iii), will ensure that the indexed family of functions introduced in the next theorem will collapse only those ordinals which should be collapsed.

Theorem 1: Let G be an $\mathfrak{m}$ -generic filter on $\mathbb{P}_L$. Let $0 < \alpha < \lambda$ and define

$$p_\alpha = \{ (n, \gamma) \mid (\alpha, n, \gamma) \in UG \},$$

then p_α is a function collapsing α to ω .

Proof: First, to see that p_α is a function, let (n, γ) and (n, γ') be in p_α , then, since G is generic, there is a $q \in G$ such that

$$\{(\alpha, n, \gamma)\} \cup \{(\alpha, n, \gamma')\} \subseteq q,$$

but q is a function, so $\gamma = \gamma'$.

Now, let $n \in \omega$ and

$$Q = \{ q \in \mathbb{P}_L \mid (\alpha, n) \in \text{dom}(q) \},$$

then Q is dense, as in Lemma 1.1. So

$$Q \cap G \neq \emptyset,$$

which implies $\exists p \in G$ and $\beta < \lambda$ such that $\{(\alpha, n, \beta)\} \subseteq p$. But, in that case, $p \leq \{(\alpha, n, \beta)\}$, and by Definition 1.1(ii), $\{(\alpha, n, \beta)\} \in G$, which implies $n \in \text{dom}(p_\alpha)$.

In a similar way, we can show, for $\beta < \alpha$,

$$R = \{ r \in \mathbb{P}_L \mid \beta \in \text{ran}(r) \}$$

is dense and conclude that $\beta \in \text{ran}(p_\alpha)$; and so p_α collapses α to ω . ■

Corollary: In $\mathfrak{m}[G]$, $\lambda \leq \aleph_1^{\mathfrak{m}[G]}$.

Proof: Any $\alpha < \lambda$ is countable in $\mathfrak{m}[G]$ by the theorem and so λ is at most the successor of $\aleph_0$ in the extension. ■

We actually want to show that $\lambda = \aleph_1^{\mathfrak{m}[G]}$, i.e. that λ is not somehow collapsed in the generic extension. This leads us to a discussion of preservation properties.

Section 2: Preservation Properties and the Levy Collapse

In this section we discuss the strong connection which exists between properties of the algebra, $B = \text{comp}(\mathbb{P})$, and truth in the generic extension. We first discuss an important property of forcing conditions and then show how this property relates to preservation properties in generic extensions.

We say a set of conditions, W , is pairwise incompatible, if, for every pair, $p, q \in \mathbb{P}$, there does not exist an r such that

$$r \leq p \text{ and } r \leq q.$$

Definition 2: A forcing, $\mathbb{P}$, is said to satisfy the κ -chain condition if every set, $W \subseteq \mathbb{P}$, of pairwise incompatible elements is of size less than κ .

Since $E[\mathbb{P}]$ is dense in $B \setminus \{0\} = \text{comp}(\mathbb{P}) \setminus \{0\}$ and $E: \mathbb{P} \rightarrow B$ preserves compatibility, we see that $\mathbb{P}$ satisfies the κ -chain condition if and only if $B \setminus \{0\}$ does.

The importance of this property of forcing conditions lies in the following theorem:

Theorem 2: If $\mathbb{P}$ satisfies the λ -chain condition, where λ is a regular, uncountable cardinal in $\mathfrak{M}$, then λ remains a regular cardinal in $\mathfrak{M}[G]$.

Proof: We work in the ground model, $\mathfrak{M}$. Suppose λ is not regular in $\mathfrak{M}[G]$, then there is a $\gamma < \lambda$, a name $\dot{f} \in \mathfrak{M}^B$, and a condition $p \in B$ such that p forces the following two properties:

- (i) $\dot{f}$ is a function,
- (ii) $\dot{f}: \gamma^* \rightarrow \lambda^*$ and the range of $\dot{f}$ is cofinal in λ^* ,

where γ^* and λ^* are the canonical names for γ and λ , respectively.

For each $\delta < \gamma$ and $\kappa < \lambda$, define:

$$p(\delta, \kappa) = p * \parallel f(\delta^*) = \kappa^* \parallel.$$

Note that, since $p * \parallel f(\delta^*) = \kappa^* \parallel \leq p$, and p forces (i), we must have

$$p(\delta, \kappa) * p(\delta, \kappa') = 0 \text{ for } \kappa \neq \kappa'.$$

Since, otherwise, we could find a condition $q \leq p$ such that q forces f not to be a function. We'll use this fact to get an incompatible subset of B of size λ .

Now, since p forces (ii), we must have, for every $\alpha < \lambda$, a $\beta > \alpha$ and a $\delta < \gamma$ such that

$$p(\delta, \beta) \neq 0.$$

So we have:

$$| \{ \beta < \lambda \mid \exists \delta < \gamma \ p(\delta, \beta) \neq 0 \} | = \lambda,$$

but

$$\{ \beta < \lambda \mid \exists \delta < \gamma \ p(\delta, \beta) \neq 0 \} = \bigcup_{\delta < \gamma} \{ \beta < \lambda \mid p(\delta, \beta) \neq 0 \}.$$

Now recall that λ is a regular cardinal in $\mathfrak{M}$ and $\gamma < \text{cf}(\lambda) = \lambda$, so there must be some set in this union of size λ . In other words, $\exists \delta_0 < \gamma$ such that

$$| \{ \beta < \lambda \mid p(\delta_0, \beta) \neq 0 \} | = \lambda.$$

But, as was shown above,

$$W = \{ p(\delta_0, \beta) \mid \beta < \lambda \text{ and } p(\delta_0, \beta) \neq 0 \}$$

is pairwise incompatible, contradicting the assumption that $B \setminus \{0\}$ satisfies the λ -chain condition. ■

Corollary: Every cardinal, $\kappa > \lambda$, in $\mathfrak{M}$, is a cardinal in $\mathfrak{M}[G]$, if $\mathbb{P}$ satisfies the κ -chain condition.

Proof: If κ is regular in $\mathfrak{M}$, then $\mathbb{P}$ satisfies the κ -chain condition, so κ remains a regular cardinal in $\mathfrak{M}[G]$. If $\kappa > \lambda$ is singular, it is a limit of regular cardinals in $\mathfrak{M}$, and hence in $\mathfrak{M}[G]$, and so remains a singular cardinal in $\mathfrak{M}[G]$. ■

So, in order to show that $\aleph_1^{\mathfrak{M}[G]} \leq \aleph_1$, where $\mathfrak{M}[G]$ is Levy's model, we will

show that the Levy conditions satisfy the λ -chain condition, then, by the Corollary, λ remains a regular cardinal in $\mathfrak{M}[G]$. This fact, together with the Corollary to Theorem 1, implies

$$\lambda = \aleph_1^{\mathfrak{M}[G]}$$

as desired.

Theorem 3: Let $\mathbb{P}_L$ be the Levy conditions, collapsing all ordinals below λ , an inaccessible cardinal in $\mathfrak{M}$. Then $\mathbb{P}_L$ satisfies the λ -chain condition.

Proof: Let W be a set of pairwise incompatible elements from $\mathbb{P}_L$. We will show $|W| < \lambda$. Define two sequences of sets:

$$A_0 \subseteq \dots \subseteq A_n \subseteq \dots \quad \text{and} \quad W_0 \subseteq \dots \subseteq W_n \subseteq \dots \subseteq W,$$

where $A_n = \bigcup \{ \text{dom}(p) \mid p \in W_n \}$.

Pick $q \in W$, and let $W_0 = \{q\}$ and $A_0 = \text{dom}(q)$. Given W_n and A_n , define A_{n+1} and W_{n+1} as follows:

For each $p \in \mathbb{P}_L$ with $\text{dom}(p) \subseteq A_n$, pick some $q \in W$ such that $q|_{A_n} = p$ (if such a q exists). Let

$$W_{n+1} = W_n \cup \{ \text{chosen } q\text{'s} \}$$

and

$$A_{n+1} = A_n \cup \{ \text{dom}(q) \mid q \in W_{n+1} \}.$$

We want to show that $W = \bigcup_{n < \omega} W_n$, so let $p \in W$. Now, since $\text{dom}(p)$ is finite, there exists an $n \in \omega$ such that:

$$\text{dom}(p) \cap A = \text{dom}(p) \cap A_n.$$

By construction, there is an $r \in W_{n+1}$ such that:

$$r|_{A_n} = p|_{A_n} = p|_A,$$

with $\text{dom}(r) \subseteq A_{n+1} \subseteq A$.

It follows that r and p are compatible members of W , hence equal, since W

consists of pairwise incompatible elements. But, in this case, $p \in \bigcup_{n < \omega} W_n$ as desired.

Now, to verify that $|W| < \lambda$, we will show that $|W_n| < \lambda$, for each n , and so:

$$|W| = \left| \bigcup_{n < \omega} W_n \right| < \aleph_0 * \lambda = \lambda.$$

Clearly, $|W_0| < \lambda$ and $|A_0| < \lambda$, so suppose $|W_n| < \lambda$ and $|A_n| < \lambda$. We obtain W_{n+1} from W_n by adding at most one $q \in W$ for each $p \in \mathbb{P}_L$, with $\text{dom}(p) \subseteq A_n$. Now, $p(\alpha, n) < \alpha$ for every $p \in \mathbb{P}_L$ and $\sup A_n = \delta < \lambda$, since λ is regular and $|A_n| < \lambda$, so

$$|W_{n+1}| \leq |W_n| * \delta^\delta.$$

But $|W_n| < \lambda$ by assumption, and $\delta^\delta < \lambda$, since λ is inaccessible. ■

We will need the following result on preservation of inaccessibles in sub-models of the Levy Collapse.

Lemma 1: Let $\mathfrak{M} \models "$ λ is an inaccessible" and let $\mathbb{P}$ be a forcing in $\mathfrak{M}$ with $|\mathbb{P}| < \lambda$. Then, if G is a $\mathbb{P}$ -generic filter over $\mathfrak{M}$, we have

$$\mathfrak{M}[G] \models "\lambda \text{ is an inaccessible}."$$

Proof: Since λ is inaccessible, there is a regular $\kappa < \lambda$ such that $|\mathbb{P}| < \kappa$ so $\mathbb{P}$ satisfies the κ -cc. By Theorem 2 then, λ is preserved as a regular limit cardinal in $\mathfrak{M}[G]$. We want to see that

$$2^\gamma \mathfrak{M}[G] < \lambda \mathfrak{M}[G] \text{ for every } \gamma < \lambda$$

to conclude that λ is inaccessible in $\mathfrak{M}[G]$. In order to prove this, we will count names for subsets of γ in the Boolean-valued model $\mathfrak{M}^{\text{comp}(\mathbb{P})}$. Now, while there are class many names for any particular set, the total number of names for distinct subsets of γ is limited. It is sufficient to consider functions from γ into the Boolean algebra $\text{comp}(\mathbb{P})$ as names for subsets of γ . An easy estimate in terms of maximal incompatible subset of $\mathbb{P}$, allows us to conclude that $|\text{comp}(\mathbb{P})| < \kappa^\kappa$. But λ is inaccessible in $\mathfrak{M}$, so $|\text{comp}(\mathbb{P})| < \lambda$. We conclude that the maximum number

of new subsets of γ is less than $\gamma^{\text{comp}(\mathbb{P})}$ which is less than λ , so 2^γ is less than λ in $\mathfrak{M}[G]$, as desired. ■

Section 3: Homogeneity of the Levy Collapse

We now explore further properties of the Levy model, $\mathfrak{M}[G]$.

The first of these, introduced in this section, is the homogeneity of the Levy algebra, $\mathbb{B} = \text{comp}(\mathbb{P}_L)$. In the next section, we will discuss submodels of the model, which will be important in our proofs concerning Lebesgue measurability.

We wish to show that, for any formula, ϕ , and for all $x_1, \dots, x_n \in \mathfrak{M}$, the Boolean value of ϕ at $x_1, \dots, x_n$ in the Levy algebra is 0 or 1. In order to do this, we must introduce the notion of permutations of forcing conditions.

A permutation of the conditions, $\mathbb{P}_L$, is an order-isomorphism of $\mathbb{P}_L$ with itself. Let π be any permutation of ω lying in $\mathfrak{M}$. We extend π to a permutation, π^* , of $\mathbb{P}_L$ as follows:

$$\pi^* : \mathbb{P}_L \rightarrow \mathbb{P}_L$$

where

$$\pi^*(h)(\alpha, n) = h(\alpha, \pi(n))$$

when h is defined at $\pi(n)$.

Now, in order to prove our main theorem, we'll need the following easy lemma on isomorphisms between forcings.

Lemma 2: Let $\mathfrak{M}$ be a transitive model of ZFC and $\mathbb{P}$ and $\mathbb{Q}$ be forcings in $\mathfrak{M}$. If there is an order-isomorphism, $\pi: \mathbb{P} \rightarrow \mathbb{Q}$, $\pi \in \mathfrak{M}$, then π induces a 1-1 correspondence between $\mathbb{P}$ -generic and $\mathbb{Q}$ -generic filters over $\mathfrak{M}$.

Proof: Let G be $\mathbb{P}$ -generic over $\mathfrak{M}$ and suppose π is an isomorphism from $\mathbb{P}$ to $\mathbb{Q}$. First note that, since π preserves the ordering, $\pi[G]$ is a filter on $\mathbb{Q}$. Now let $D \in \mathfrak{M}$ be a dense subset of $\mathbb{Q}$. We want to see that $D \cap \pi[G] \neq \emptyset$. Since π is an order-isomorphism, $\pi^{-1}[D] \in \mathfrak{M}$ will be dense in $\mathbb{P}$ and so $\pi^{-1}[D]$ intersects G . We conclude that $D \cap \pi[G] \neq \emptyset$; so $\pi[G]$ is $\mathbb{Q}$ -generic over $\mathfrak{M}$. The same argument can be applied to an H which is $\mathbb{Q}$ -generic, by considering $\pi^{-1} \in \mathfrak{M}$. ■

Let π^* be the automorphism of $\mathbb{P}_L$ introduced earlier, we will apply Lemma 2 to π^* to prove the main result on Homogeneity.

Lemma 3: Let ϕ be a formula and $x_1, \dots, x_n \in \mathfrak{M}$. We claim

$$p \Vdash \phi(x_1^*, \dots, x_n^*) \quad \text{if and only if} \quad \pi^*(p) \Vdash \phi(x_1^*, \dots, x_n^*).$$

Proof: Let $p \Vdash \phi(x_1^*, \dots, x_n^*)$. This is equivalent, by the Forcing Theorem, to

$$\mathfrak{M}[G] \models \phi(x_1, \dots, x_n)$$

for every generic G on $\mathbb{P}_L$, containing p . Since π^* is definable in $\mathfrak{M}$, we have

$$\mathfrak{M}[G] = \mathfrak{M}[\pi^*(G)],$$

and, since ϕ mentions only elements of the ground model, we have

$$\mathfrak{M}[\pi^*(G)] \models \phi$$

for every $\mathbb{P}_L$ -generic filter, G . Now, by Lemma 2, every generic H containing $\pi^*(p)$ is of the form $\pi^*(G) = H$ for some $\mathbb{P}_L$ -generic filter, G , so $\mathfrak{M}[H] \models \phi$, for every generic set containing $\pi^*(p)$. Then, by the Forcing Theorem again,

$$\pi^*(p) \Vdash \phi.$$

For the other implication, the argument is reversible by simply considering the well-defined inverse of π^* . ■

Now for our main theorem:

Theorem 4: If ϕ is any formula and $x_1, \dots, x_n$ are elements of $\mathfrak{M}$, then $\phi(x_1, \dots, x_n)$ has value 0 or 1 in the Boolean-valued model generated by the Levy algebra.

Proof: Assume not, then, by the Forcing Theorem, there exist elements p and p' in $\mathbb{P}_L$ such that

$$p \Vdash \phi(x_1^*, \dots, x_n^*) \text{ and } p' \Vdash \neg \phi(x_1^*, \dots, x_n^*).$$

Now, the domains of p and p' are finite subsets of $\lambda \times \omega$, so we can find a permutation, $\pi \in \mathfrak{M}$, such that

$$\text{dom}(\pi(p)) \cap \text{dom}(p') = \emptyset.$$

By the previous lemma,

$$\pi(p) \Vdash \phi(x_1^*, \dots, x_n^*),$$

but, in this case, $\pi(p)$ and p' would have to be incompatible, which is impossible, since $\pi(p) \cup p' \in \mathbb{P}_L$. ■

This theorem, while perhaps not very interesting in and of itself, will be of importance in Chapter 4 when it comes time to prove some results involved with Lebesgue measurability. Our next topic, while more interesting in its own right, will also be of use in our later work.

Section 4: Submodels of the Levy Collapse

Let s be a countable sequence of ordinals which is definable in the Levy model, $\mathfrak{M}[G]$. We write $\mathfrak{M}[s]$ for the minimal model of ZFC which extends $\mathfrak{M}$, contains s , and has the same ordinals as $\mathfrak{M}$. In this section, we will see that $\mathfrak{M}[s]$ can be obtained as a generic extension of the ground model if s comes from a

generic extension of $\mathfrak{M}$.

Let $\mu < \lambda$ be any ordinal. We define two special subsets of $\mathbb{P}_L$ as follows:

$$\mathbb{P}_\mu = \{ p \in \mathbb{P}_L \mid \text{dom}(p) \subseteq \omega \times \mu \}$$

and

$$\mathbb{P}^\mu = \{ p \in \mathbb{P}_L \mid \text{dom}(p) \cap \omega \times \mu = \emptyset \}.$$

Note that there is a natural order-isomorphism, m , between $\mathbb{P}_\mu \times \mathbb{P}^\mu$ and $\mathbb{P}_L$ given by:

$$m(p, q) = p \cup q.$$

Now, because $s \in \mathfrak{M}[G]$, we assume for simplicity, that

$$\emptyset \Vdash s: \omega^* \rightarrow \text{OR}^*.$$

For each $n \in \omega$, let $A_n \subseteq \mathbb{P}_L$ be a maximal incompatible set of conditions such that

$$p \in A_n \rightarrow \exists \alpha \in \text{OR} \quad p \Vdash s(n^*) = \alpha^*.$$

Intuitively, note that the sequence of A_n 's is rich enough to decide every question that can be asked about the values of s . Further, since $\mathbb{P}_L$ satisfies the λ -chain condition, we have that $|A_n| < \lambda$, for every $n \in \omega$. Using the fact that λ is inaccessible in $\mathfrak{M}$, we can find a $\mu < \lambda$ such that

$$\bigcup_{n \in \omega} A_n \subseteq \mathbb{P}_\mu.$$

Given all this, the following lemma is not surprising.

Lemma 4: Let J be a $\mathbb{P}_\mu$ -generic filter over $\mathfrak{M}$, then

$$(i) \quad s \in \mathfrak{M}[J]$$

$$(ii) \quad \mathfrak{M}[J] \models "\lambda \text{ is an inaccessible}."$$

Proof: To see that s is definable in $\mathfrak{M}[J]$, we note that, for each $n \in \omega$,

$$A_n' = \{ p \in \mathbb{P}_\mu \mid \exists q \in A_n \quad p \leq q \}$$

is dense in $\mathbb{P}_\mu$, so intersects J , therefore s is decided for every $n \in \omega$.

For (ii), note that, since $|\mathbb{P}_\mu| < \lambda$, λ being inaccessible in $\mathfrak{M}$, we have, by Lemma 1, that λ is an inaccessible in $\mathfrak{M}[J]$. ■

In the next section, we will be studying extensions of $\mathfrak{M}[s]$ which collapse ω_1 to a countable ordinal, so let κ be such that

$$\mathfrak{M}[J] \models "\kappa = \omega_1"$$

and note that by Lemma 4(ii), $\kappa < \lambda$.

Now let

$$B = \text{comp}(\mathbb{P}_{\kappa+1})$$

and noting that $\mathfrak{M}[J] \models "\mu \leq \omega_1"$, $\mathbb{P}_\mu \subseteq \mathbb{P}_{\kappa+1}$, let B_μ be the complete subalgebra of B generated by $\mathbb{P}_\mu$. Furthermore, for $n \in \omega$ and $\alpha \in \text{OR}$, set

$$b_{n,\alpha} = \sum^B \{ p \in \mathbb{P}_\mu \mid p \Vdash s(n^*) = a^* \text{ (forcing in } \mathbb{P}_\mu) \}.$$

Let B_S be the complete subalgebra of B generated by the $b_{n,\alpha}$'s. Note that, from these definitions, we have

$$B_S \subseteq B_\mu \subseteq B.$$

We will prove that, given G , a $B/\{0\}$ -generic filter over $\mathfrak{M}$, $G_S = G \cap B_S$ is a $B_S/\{0\}$ -generic filter over $\mathfrak{M}$ and $\mathfrak{M}[s] = \mathfrak{M}[G_S]$. First we need to discuss dense embeddings and generic filters.

Definition 3: A forcing, $\mathbb{R}$, is densely embedded in a forcing, $\mathbb{S}$, if there is an isomorphism, j , mapping $\mathbb{R}$ onto a dense subset of $\mathbb{S}$.

Unless stated otherwise, we will assume that the isomorphism is definable in the ground model under discussion. Our main result on dense embeddings is the following lemma

Lemma 5: If a forcing, $\mathbb{R}$, is densely embedded in a forcing, $\mathbb{S}$, and K is an $\mathbb{R}$ -generic set over $\mathfrak{M}$, then there is an $H \in \mathfrak{M}[K]$ such that, H is $\mathbb{S}$ -generic over $\mathfrak{M}$, and $\mathfrak{M}[K] = \mathfrak{M}[H]$.

Proof: We assume, of course, that $\mathbb{R}$ and $\mathbb{S}$ are in the ground model. Define H as follows:

$$H = \{ s \in \mathbb{S} \mid \exists k \in K \ j(k) \leq s \}$$

where j is the dense embedding of $\mathbb{R}$ into $\mathbb{S}$. Since K is a filter, and j is order-preserving, H is a filter on $\mathbb{S}$. Let D be a dense subset of $\mathbb{S}$ in $\mathfrak{M}$; we want to show that $D \cap H$ is nonempty, proving that H is generic over $\mathfrak{M}$.

We define a subset, X , of $\mathbb{R}$ as follows:

$$X = \{ r \in \mathbb{R} \mid j(r) \leq d \text{ for some } d \in D \}.$$

First, note that, since $\mathbb{R}$, D , and j are in $\mathfrak{M}$, so is X .

We claim X is dense in $\mathbb{R}$. Let $r \in \mathbb{R}$. Then, since D is dense in $\mathbb{S}$, there is a $d \in D$ such that $d \leq j(r)$. Now, since j is a dense embedding, there is an $r' \in \mathbb{R}$ such that $j(r') \leq d \leq j(r)$ but, in this case, $r' \in X$, by definition, and $r' \leq r$, so X is dense.

Since K is generic on $\mathbb{R}$, $K \cap X \neq \emptyset$. Let $r \in K \cap X$, then $j(r) \leq d$ for some $d \in D$. So, since $r \in K$, we have $d \in H$ by definition, and so $K \cap H$ is nonempty.

To see that $\mathfrak{M}[K] = \mathfrak{M}[H]$, we note that $H \in \mathfrak{M}[K]$ by definition and that K can be obtained from H by

$$K = \{ r \in \mathbb{R} \mid j(r) \in H \}. \blacksquare$$

The effect of this lemma is that, if we have two forcings and we can densely embed one of them in the other, we can consider them equivalent in terms of the extensions they generate. Although we first use Lemma 5 in connection with our work in this section, it will also be used extensively in the next section.

We will apply Lemma 5 to the results in the next lemma.

Lemma 6: Let $B = \text{comp}(\mathbb{P}_{\kappa+1})$, B_μ , and $\mathbb{P}_\mu$ be defined as above. Then $\mathbb{P}_\mu$ can be densely embedded in $B_\mu/\{0\}$.

Proof: Let $B_\mu' = \text{comp}(\mathbb{P}_\mu)$. Define

$$\mathbb{P}_{\kappa+1}^\mu = \{ p \in \mathbb{P}_{\kappa+1} \mid \text{dom}(p) \cap \omega \times \mu = \emptyset \}$$

and note that there is a natural isomorphism, m , between $\mathbb{P}_\mu \times \mathbb{P}_{\kappa+1}^\mu$ and $\mathbb{P}_{\kappa+1}$, as above. Letting $B^{\mu'} = \text{comp}(\mathbb{P}_{\kappa+1}^\mu)$, m lifts to a natural order-isomorphism, $m^*: B_\mu' \times B^{\mu'} \rightarrow B$ given by:

$$m^*(c, d) = \sum \{ r \in \mathbb{P}_{\kappa+1} \mid m(p, q) = r \text{ and } p \leq c \text{ and } q \leq d \}.$$

Let B_μ^* be the image of B_μ' in B under m^* . Since m^* is an order-isomorphism and B_μ' is complete, B_μ^* is a complete subalgebra of B . Furthermore, since $\mathbb{P}_\mu$ is a dense subset of $B_\mu'/\{0\}$, $m^*(\mathbb{P}_\mu)$ is a dense subset of $B_\mu^*/\{0\}$. But, by definition, B_μ is the smallest complete subalgebra of B containing $\mathbb{P}_\mu$, so

$$\mathbb{P}_\mu \subseteq B_\mu \subseteq B_\mu^*.$$

Therefore, $\mathbb{P}_\mu$ being dense in $B_\mu^*/\{0\}$, it is dense in $B_\mu/\{0\}$, as desired. ■

Our final result before our main theorem draws the connection between generic sets on the various algebras under discussion.

Lemma 7: Let $\mathbb{P}_\mu$, B_μ , and B_S be as defined earlier. Given a G which is $B/\{0\}$ -generic over $\mathfrak{M}$, we have:

(i) $G_\mu = B_\mu \cap G$ is $B_\mu/\{0\}$ -generic over $\mathfrak{M}$

(ii) $G_S = B_S \cap G$ is $B_S/\{0\}$ -generic over $\mathfrak{M}$

and, furthermore, there is a G_μ' which is $\mathbb{P}_\mu$ -generic over $\mathfrak{M}$ with

$$\mathfrak{M}[G_\mu] = \mathfrak{M}[G_\mu'].$$

Proof: (i) and (ii) are proved in exactly the same way so we will prove (i). First off, it is clear that G_μ is a filter on B_μ , since B_μ is a subalgebra. To see that

G_μ is generic, let $D \subseteq B_\mu/\{0\}$ be dense with $D \in \mathfrak{M}$. Set

$$D' = \{ b \in B/\{0\} \mid \exists d \in D \ b \leq d \}$$

then D' is dense in $B/\{0\}$ and so

$$D' \cap G_\mu \neq \emptyset,$$

which implies that $D \cap G_\mu \neq \emptyset$, from the definition of D' and G_μ .

For the rest, recall that, by Lemma 6, $\mathbb{P}_\mu$ is densely embedded in $B_\mu/\{0\}$, therefore, by Lemma 5, there is such a G_μ' . ■

We are now ready to prove our main theorem.

Theorem 5: Let s be a countable sequence of ordinals in the Levy collapse and let B, B_S, B_μ be as above. If $G_S = G \cap B_S$, then

$$\mathfrak{M}[s] = \mathfrak{M}[G_S].$$

Proof: We will show that s is definable in $\mathfrak{M}[G_S]$ and that G_S is definable in $\mathfrak{M}[s]$ to conclude the theorem.

To see that $s \in \mathfrak{M}[G_S]$, recall the definition of B_S as the complete subalgebra of B generated by the $b_{n,\alpha}$'s, where

$$b_{n,\alpha} = \sum \{ p \in \mathbb{P}_\mu \mid p \Vdash s(n^*) = \alpha^* \}.$$

Using the fact that μ was chosen so that $s \in \mathfrak{M}[G_\mu]$, we will show that

$$b_{n,\alpha} \in G_S \leftrightarrow s(n) = \alpha \text{ in the Levy collapse.}$$

Suppose $b_{n,\alpha} \in G_S$. This happens if and only if $b_{n,\alpha} \in G_\mu$ since $G_S \subseteq G_\mu$. Now, regarding $\mathbb{P}_\mu$ as embedded in $B_\mu/\{0\}$, and applying the proof of Lemma 5, we have $b_{n,\alpha} \in G_\mu$ if and only if there is a $p \in G_\mu \cap \mathbb{P}_\mu$ such that $p \leq b_{n,\alpha}$. Then, using the definition of the $b_{n,\alpha}$'s, we have

$$b_{n,\alpha} \in G_S \leftrightarrow \exists p \in G_\mu \cap \mathbb{P}_\mu \ p \Vdash s(n^*) = \alpha^*,$$

but, using the Forcing Theorem, this is equivalent to:

$$\mathfrak{M}[G_\mu] \models s(n) = \alpha \quad \text{as desired.}$$

For the other direction, working in $\mathfrak{M}[s]$, we will show how to construct G_S using transfinite recursion. So, in $\mathfrak{M}[s]$, define:

$$G_0 = \{ b_{n,\alpha} \mid s(n)=a \} \cup \{ \neg b_{n,\alpha} \mid s(n) \neq \alpha \}$$

$$G_{\delta+1} = \{ \prod_i b_i \mid \forall i \ b_i \in G_\delta \} \cup \{ \sum_i b_i \mid \exists i \ b_i \in G_\delta \} \cup G_\delta$$

and, for limit δ ,

$$G_\delta = \bigcup_{\gamma < \delta} G_\gamma.$$

Since $\mathfrak{M}[s]$ models Replacement, there is a δ' such that $G_{\delta'} = G_{\delta'+1}$. Set $G' = G_{\delta'}$. We will show that $G' = G_S$.

We obviously have $G_0 \subseteq G_S$. Also, since G_S is generic, assuming $G_\delta \subseteq G_S$, we have $G_{\delta+1} \subseteq G_S$. Limits are obvious, so we have $G' \subseteq G_S$.

To see that $G' = G_S$, we will show that G' is an ultrafilter. We want to see that $b \in G'$ or $\neg b \in G'$ for every $b \in B_S$. We proceed by induction on Boolean combinations of the $b_{n,\alpha}$'s.

Note that, first of all, we have $b_{n,\alpha} \in G'$ or $\neg b_{n,\alpha} \in G'$ for every $b_{n,\alpha}$. Now assume that $D \subseteq B_S$ and $\forall d \in D$ we have $d \in G'$ or $\neg d \in G'$. We want to see that the same holds true for $\sum D$ and $\prod D$. Suppose $\sum D \notin G'$, then for every $d \in D$, since $d \leq \sum D$, we have $d \notin G'$. So by the induction hypothesis, we have $\neg d \in G'$ for every $d \in D$. So, by construction, we have $\neg \sum D = \prod \neg d \in G'$, as desired. The same idea works for $\prod D$. We conclude then that for every $b \in B_S$, either $b \in G'$ or $\neg b \in G'$, as desired. This finishes the theorem. ■

Section 5: The Factor Theorem

Our goal in this section is to prove that, given a G which is $\mathbb{P}_L$ -generic over $\mathfrak{M}$ and an $s \in \mathfrak{M}[G]$, which is a countable sequence of ordinals, there is an H which is $\mathbb{P}_L$ -generic over $\mathfrak{M}[s]$ such that

$$\mathfrak{M}[s][H] = \mathfrak{M}[G].$$

That is, $\mathfrak{M}[G]$ is "factorable" as $\mathfrak{M}[s][H]$.

To simplify some of the proofs in this section, we prove two lemmas concerning the genericity of filters.

We say a subset, U , of a forcing $\mathbb{R}$, is open if $p \in U$ and $q \leq p$ imply $q \in U$. We define a set $D \subseteq \mathbb{R}$ to be dense below p if, for every $p' \leq p$, there is a $q \in D$ such that $q \leq p'$.

Lemma 8: A filter, G , on $\mathbb{R}$ is generic over $\mathfrak{M}$ if and only if G intersects every open and dense subset of $\mathbb{R}$ such that $D \in \mathfrak{M}$.

Proof: If G is generic, then, obviously, G intersects every dense and open set, D , $D \in \mathfrak{M}$.

Let G satisfy the second part of the lemma; we need to show G intersects every dense subset of $\mathbb{R}$ in $\mathfrak{M}$. Let D be a dense subset of $\mathbb{R}$ in $\mathfrak{M}$; define a dense and open extension of D as follows:

$$C = \{ q \in \mathbb{R} \mid q \leq p \text{ for some } p \in D \}.$$

Now, clearly C is dense (since D was) and open, and $C \in \mathfrak{M}$, therefore $C \cap G \neq \emptyset$. Let $q \in C \cap G$, then $q \leq p$ for some $p \in D$. Recall that G is a filter, so $p \in G$; therefore $G \cap D$ is nonempty, as desired. ■

Lemma 9: G on $\mathbb{R}$ is generic over $\mathfrak{M}$ if and only if for every $p \in G$, if $D \in \mathfrak{M}$ is dense below p , $G \cap D$ is nonempty.

Proof: Let D be dense below some $p \in G$ and definable over $\mathfrak{M}$. We extend D as follows:

$$C = D \cup \{ r \in \mathbb{R} \mid r \text{ is incompatible with } p \}.$$

We claim $C \in \mathfrak{M}$ is dense in $\mathbb{R}$. Let $r \in \mathbb{R}$. If r is incompatible with p , then $r \in C$; otherwise r is compatible with p and there is a $d \in D$ such that

$$d \leq r \text{ and } d \leq p.$$

But $d \in C$, so C is dense in $\mathbb{R}$.

We conclude that $C \cap G \neq \emptyset$. Now, since G is a filter and $p \in G$, every $q \in C \cap G$ is compatible with p , therefore is in D , as desired.

Now, for the other direction simply note that every D which is dense in $\mathbb{R}$ is dense below every $p \in G$. ■

The following well-known result will be used to initially factor the Levy collapse by applying it to the isomorphism between $\mathbb{P}_\mu \times \mathbb{P}^\mu$ and $\mathbb{P}_L$.

Lemma 10: Given two forcings, P_1 and P_2 in $\mathfrak{M}$, G is $P_1 \times P_2$ -generic over $\mathfrak{M}$ if and only if $G = G_1 \times G_2$ where G_1 is P_1 -generic over $\mathfrak{M}$ and G_2 is P_2 -generic over $\mathfrak{M}[G_1]$. Moreover, we have

$$\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2].$$

Proof: First, let G be $P_1 \times P_2$ -generic over $\mathfrak{M}$. Let

$$G_1 = \{ p_1 \in P_1 \mid \exists p_2 \in P_2 (p_1, p_2) \in G \}$$

$$G_2 = \{ p_2 \in P_2 \mid \exists p_1 \in P_1 (p_1, p_2) \in G \}.$$

Clearly, G_1 and G_2 are both filters and $G \subseteq G_1 \times G_2$. So, suppose $(p_1, p_2) \in G_1 \times G_2$, then there is a $p_2' \in G_2$ such that $(p_1, p_2') \in G$ and a $p_1' \in G_1$ such that $(p_1', p_2) \in G$. Since G is a filter, there is a q_1 and a q_2 such that

$$q_1 \leq p_1, p_1' \quad \text{and} \quad q_2 \leq p_2, p_2'$$

with $(q_1, q_2) \in G$. But $(q_1, q_2) \leq (p_1, p_2)$, so $(p_1, p_2) \in G$. Therefore $G = G_1 \times G_2$.

To see that G_1 is P_1 -generic over $\mathfrak{M}$, let $D \in \mathfrak{M}$ be dense in P_1 , then $D \times P_2$ is in $\mathfrak{M}$ and obviously dense in $P_1 \times P_2$, so G intersects $D_1 \times P_2$. This implies $G_1 \cap D_1$, as desired.

Further, to see that G_2 is P_2 -generic over $\mathfrak{M}[G_1]$, let $D_2 \subseteq P_2$ be dense with $D_2 \in \mathfrak{M}[G_1]$. Let $p_1 \in G_1$ be such that

$$p_1 \Vdash "D_2 \text{ is dense in } P_2".$$

We want to apply Lemma 9, so let $p_2 \in G_2$ be arbitrary. Since p_1 forces D_2 to be dense in P_2 , for every $q_1 \leq p_1$ and $q_2 \leq p_2$ there exist $r_1 \leq q_1$ and $r_2 \leq q_2$ such that

$$r_1 \Vdash r_2^* \in D_2$$

and so

$$D = \{ (r_1, r_2) \mid r_1 \leq p_1 \text{ and } r_1 \Vdash r_2^* \in D_2 \}$$

is in $\mathfrak{M}$ and dense below $(p_1, p_2) \in G$; thus $D \cap G \neq \emptyset$, by Lemma 9. So, there are r_1, r_2 with $r_1 \in G_1$ and $r_1 \Vdash r_2^* \in D_2$. Hence $r_2 \in D_2 \cap G_2$ by the Forcing Theorem applied in $\mathfrak{M}[G_1]$.

For the other direction, suppose $G_1 \subseteq P_1$ is generic over $\mathfrak{M}$ and $G_2 \subseteq P_2$ is $\mathfrak{M}[G_1]$ generic. Let $G = G_1 \times G_2$ and note that G is a filter on the product of P_1 and P_2 . To see that G is generic over $\mathfrak{M}$, let $D \in \mathfrak{M}$ be dense in $P_1 \times P_2$. Then set

$$D_2 = \{ p_2 \in P_2 \mid (p_1, p_2) \in D \text{ for some } p_1 \in G_1 \}.$$

The set D_2 is in $\mathfrak{M}[G_1]$ and we will show that it is dense, implying that $D_2 \cap G_2 \neq \emptyset$ and thus $D \cap (G_1 \times G_2) \neq \emptyset$, as desired.

Let $q_2 \in P_2$ be arbitrary. Set

$$D_1 = \{ p_1 \in P_1 \mid \exists p_2 \leq q_2 (p_1, p_2) \in D \}.$$

Then, since D is dense in $P_1 \times P_2$, we have D_1 is dense in P_1 , so there is a

$p_1 \in D_1 \cap G_1$ and so a $p_2 \in D_2$ with $p_2 \leq q_2$, implying D_2 is dense.

Finally, $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$ is obvious from our constructions. ■

We will be applying this lemma to the full Levy collapse, but first we need to discuss the relationship between certain submodels of the full collapse.

To recall the context, we let s be a countable sequence of ordinals in the full collapse; for each $n \in \omega$, A_n be a maximal incompatible set of conditions each of which decide s at n ; $\mathbb{P}_\mu$ be such that $\bigcup A_n \subseteq \mathbb{P}_\mu$, and κ be such that $\kappa = \omega_1$ in $\mathfrak{M}[J]$, where J is $\mathbb{P}_\mu$ -generic over $\mathfrak{M}$. Next, we defined $B = \text{comp}(\mathbb{P}_{\kappa+1})$, B_μ and B_S to be complete subalgebras of B generated by $\mathbb{P}_\mu$ and the $b_{n,\alpha}$'s, respectively. Then for G a B -generic filter, $G_S = B_S \cap G$ is a generic filter on B_S such that $\mathfrak{M}[s] = \mathfrak{M}[G_S]$.

Now, before factoring the full collapse, we will show how to go from a $\mathbb{P}_{\kappa+1}$ -generic filter, G , over $\mathfrak{M}$ to a $\mathbb{P}_{\kappa+1}$ -generic filter over $\mathfrak{M}[s]$, H , in such a way that

$$\mathfrak{M}[G] = \mathfrak{M}[s][H].$$

In order to do this, we next introduce the concept of a neat embedding and show that $B/\{0\}$ and $B_S/\{0\}$ are related by it.

Definition 4: A forcing, $\mathbb{R}$, is neatly embedded in a forcing, $\mathbb{S}$, if there exist two maps, f , mapping $\mathbb{R}$ into $\mathbb{S}$, and h , mapping $\mathbb{S}$ into $\mathbb{R}$, such that:

- (i) f and h preserve the orderings of $\mathbb{R}$ and $\mathbb{S}$, respectively.
- (ii) h composed with f is the identity on $\mathbb{R}$.
- (iii) If $r \in \mathbb{R}$, $s \in \mathbb{S}$, and r and $h(s)$ are compatible in $\mathbb{R}$, then $f(r)$ and s are compatible in $\mathbb{S}$.

Lemma 11: $B_S/\{0\}$ is neatly embedded in $B/\{0\}$.

Proof: We use the same notation as in Definition 4. Since, by construction, $B_S \subseteq B$, it is natural to let $f: B_S/\{0\} \rightarrow B/\{0\}$ simply be inclusion. For the map $h: B/\{0\} \rightarrow B_S/\{0\}$, we take the natural projection,

$$h(b) = \prod \{ d \in B_S/\{0\} \mid b \leq d \}$$

and note that h is well-defined because B_S is a complete subalgebra of B . Also, note that Definition 4(i) is satisfied by f and h . For 4(ii), simply note that, for $b \in B_S/\{0\}$,

$$h(f(b)) = h(b) = b.$$

Finally, for the third clause, note that, if $f(b)=b$ and d are incompatible in $B/\{0\}$, then

$$b * d = 0,$$

which implies $d \leq \neg b$. But, $\neg b \in B_S/\{0\}$, so $h(d) * b = 0$, as desired. ■

Now, working relative to $\mathfrak{M}[s] = \mathfrak{M}[G_S]$, we are ready to prove our main lemma concerning neat embeddings.

Lemma 12: (i) If G is $B/\{0\}$ -generic over $\mathfrak{M}$ and $G_S = G \cap B_S$, then, letting,

$$B^* = \{ b \in B \mid h(b) \in G_S \},$$

G is $B^*/\{0\}$ -generic over $\mathfrak{M}[G_S] = \mathfrak{M}[s]$.

(ii) If H is $B^*/\{0\}$ -generic over $\mathfrak{M}[G_S]$, then H is $B/\{0\}$ -generic over $\mathfrak{M}$.

Proof: (i) First, note that by definition, $G \subseteq B^*$. Let, let $D \in \mathfrak{M}[G_S]$, with D dense and open in $B^*/\{0\}$. We want to show that $D \cap G$ is nonempty. We claim it suffices to see that

$$A = \{ b \in B/\{0\} \mid h(b) \Vdash "b \in D" \text{ (forcing in } B_S) \}$$

is dense in $B/\{0\}$ below a condition, $b' \in G$ such that

$$h(b') \Vdash "D \text{ is dense and open in } B^*/\{0\}."$$

For, in this case, by Lemma 9, we would have $A \cap G$ being nonempty, and so a condition $d \leq b'$, such that

$$h(d) \Vdash "d \in D" \text{ (forcing in } B^*)$$

and

$$h(d) \in G_S.$$

But, then, by the Forcing Theorem,

$$\mathfrak{M}[G_S] \models d \in D$$

and so $d \in D \cap G$, implying that G is $B^*/\{0\}$ -generic over $\mathfrak{M}[G_S]$.

So, we have to show that A is dense in $B/\{0\}$, below a condition $b' \in G$ such that

$$h(b') \Vdash "D \text{ is dense and open in } B^*/\{0\}."$$

Such a condition certainly exists, by the definition of G_S , and the fact that $D \in \mathfrak{M}[G_S]$ is assumed dense and open.

Since $D \in \mathfrak{M}[G_S]$ is dense and open in $B^*/\{0\}$, we can find a condition, $g \in G_S$, such that, for some $r \leq b'$,

$$g \Vdash "r \in D \cap B^*."$$

Now, $g \Vdash "r \in B^*"$ implies that $g \Vdash "h(r) \in G_S"$, by the definition of B^* . But, in this case, since $h(g) \in G_S$, we must have $h(r) \in G_S$, by the Forcing Theorem. So, g and $h(r)$ are compatible, since G_S is a filter on B_S . But, B_S is neatly embedded in B , so $f(g)$ and r must be compatible in B . Let $s \in B$, be such that

$$s \leq r \quad \text{and} \quad s \leq f(g).$$

In this case, $h(s) \leq h(f(g)) = g$, so

$$h(s) \Vdash "r \in D."$$

But, by the first relation, $s \leq r$, $h(s) \leq h(r)$, which implies, since $h(s) \leq h(b')$, and

$h(b')$ forces D to be open, that

$$h(s) \Vdash "s \in D".$$

Therefore, s witnesses that A is dense below $b' \in G$, as desired.

(ii) Now, let H be $B^*/\{0\}$ -generic over $\mathfrak{M}[G_S]$. We want to show that H is $B/\{0\}$ -generic over $\mathfrak{M}$. H obviously consist of pairwise compatible elements of B and it is easy to see, from the definition of B^* , that H is a filter on $B/\{0\}$. Now, suppose $D \subseteq B$ is dense and open. We will show that $D \cap B^*$ is dense in B^* , thereby concluding that $D \cap H$, as desired.

For any $b_s \in B_S$, it is clear that, if $b_s \Vdash "b' \in B^*",$ then $b_s \Vdash "h(b') \in G_S",$ so b_s and $h(b')$ are compatible in B_S . Now let

$$D^\# = \{ h(b) \in B_S \mid h(b) \leq b_s \wedge b \leq b' \wedge b \in D \}.$$

We will show that $D^\#$ is dense below $b_s \in B_S$. Pick $d \leq b_s$, then b_s is compatible with $h(b')$ implies d is compatible with $h(b')$. So, since $B_S/\{0\}$ is neatly embedded in $B/\{0\}$, we have $f(d)$ is compatible with b' in B . Since D is dense and open in $B/\{0\}$, pick $b^* \leq f(d), b'$ with $b^* \in D$. Then $h(b^*) \leq d$ and $h(b^*) \in D^\#$, implying that $D^\#$ is dense below b_s .

As G_S is $B_S/\{0\}$ -generic over $\mathfrak{M}$, we have $D^\# \cap G_S \neq \emptyset$, so there is an $h(b) \in G_S$ with $b \leq b'$, $b \in D$, and $h(b) \Vdash "b \in B^*",$ so by the Forcing theorem, $b \in B^*$ and $D \cap B^*$ is dense, as desired. ■

So, given a G which is $B/\{0\}$ -generic over $\mathfrak{M}$, we conclude that H is $B^*/\{0\}$ -generic over $\mathfrak{M}[s]$, and conversely. We want to prove the existence of a $B/\{0\}$ -generic filter over $\mathfrak{M}[s]$. In order to do this, we will show that $B^*/\{0\}$ and B have isomorphic dense subsets in $\mathfrak{M}[s]$ and apply Lemma 5.

First off, we have the following easy result.

Lemma 13: The set

$$\mathbb{P}^* = \{ p \in \mathbb{P}_{\kappa+1} \mid p \in B^* \}$$

is dense in $B^*/\{0\}$.

Proof: Pick any $b \in B^*/\{0\}$. We can choose H which is $B^*/\{0\}$ -generic over $\mathfrak{M}[s]$ with $b \in H$. By the previous lemma, H is $B/\{0\}$ -generic over $\mathfrak{M}$.

Now, $\mathbb{P}_{\kappa+1}$ is dense in $B/\{0\}$, so there is a $p \leq b$ with $p \in \mathbb{P}_{\kappa+1} \cap H$. But $H \subseteq B^*$, so $p \in B^*$; we conclude that $\mathbb{P}^*$ is dense in $B^*/\{0\}$. ■

We want to show how to go from a $\mathbb{P}^*$ -generic filter over $\mathfrak{M}[s]$ to a $\mathbb{P}_{\kappa+1}$ -generic filter over $\mathfrak{M}[s]$. First we need a definition:

Definition 5: We define the collapsing forcing, $\mathbb{C}_\alpha$, as in Chapter 1, to be, for $\alpha \in \text{OR}$, the set of all finite sequences of ordinals less than α with the ordering

$$f \leq g \leftrightarrow f \supseteq g.$$

Our next lemma is a strong result concerning relatively "small" forcings which collapse a cardinal to ω . We call a forcing, $\mathbb{Q}$, separative if, for every $p, q \in \mathbb{Q}$, if p is not stronger than q , then there is an r , extending q , such that r and p are incompatible.

Lemma 14: Let $\kappa > \aleph_0$ be a regular cardinal. If $\mathbb{Q}$ is any forcing of size κ which is separative and collapses κ to ω , then $\mathbb{C}_\kappa$ can be densely embedded in $\mathbb{Q}$.

Proof: Without loss of generality, we assume there is a name, $\dot{f} \in \mathfrak{M}^B$, where $B = \text{comp}(\mathbb{Q})$, such that, if G is the canonical name of the generic filter on $\mathbb{Q}$,

$$\| \dot{f} \text{ maps } \omega^* \text{ onto } G \|_B = 1.$$

For each $c \in \mathbb{C}_\kappa$, we shall construct, by induction, an element $t(c) \in \mathbb{Q}$, such that the set $D = \{ t(c) \mid c \in \mathbb{C}_\kappa \}$ is dense in $\mathbb{Q}$, and $c \rightarrow t(c)$ is an embedding.

If $c \in \mathbb{C}_\kappa$ is a 1-place function, i.e., $c = \langle c(0) \rangle$, we construct $t(c)$ as follows:

Since $\mathbb{Q}$ collapses κ , by Theorem 2, there is a maximal incompatible subset, $W \subseteq \mathbb{Q}$ of size κ . Moreover, by taking extensions if necessary, we can pick W in such a way that each $w \in W$ decides $f(0)$; that is, there is a $q_w \in \mathbb{Q}$ such that

$$w \Vdash f(0^*) = q_w^*.$$

Put W into a sequence, $W_\emptyset = \{ w_\xi \mid \xi < \kappa \}$, and let

$$t(c) = w_\xi \text{ where } \xi = c(0).$$

Now, assume $t(c)$, where $c = \langle c(0), \dots, c(n-1) \rangle$, has already been constructed. We construct $t(c \cup (n, \xi))$, for $\xi < \kappa$ as follows:

Let $W_c = \{ w_\xi \mid \xi < \kappa \}$ be a maximal incompatible subset of conditions such that each properly extends $t(c)$ and each decides f at n . Then we let

$$t(c \cup (n, \xi)) = w_\xi, \text{ for all } w_\xi \in W_c.$$

Now, by the way we've chosen the image points, $t(c)$, t is an embedding of all of $\mathbb{C}_\kappa$ in $\mathbb{Q}$. We have only to see that $D = \{ t(c) \mid c \in \mathbb{C}_\kappa \}$ is dense in $\mathbb{Q}$.

Let $q \in \mathbb{Q}$, then, by the definition of $\mathbf{G}$,

$$q \Vdash q^* \in \mathbf{G},$$

so $q \Vdash "q \in \text{ran}(f)"$, by the way f was chosen. So $q \Vdash "\exists n (q = f(n))"$, which implies there is a $q' \leq q$ such that $q' \Vdash q^* = f(n^*)$, for some n . Now, by the construction of t , there is a $c \in \mathbb{C}_\kappa$ of length $n+1$ such that $t(c)$ is compatible with q ; since $t(c)$ decides $f(n)$, by construction, we must have

$$t(c) \Vdash q^* = f(n^*).$$

Therefore, by the choice of f ,

$$t(c) \Vdash q^* \in \mathbf{G}.$$

From this fact and since $\mathbb{Q}$ is separative, we have

$$t(c) \leq q$$

as desired. Thus D is dense in $\mathbb{Q}$. ■

We wish to apply this lemma to the forcing $\mathbb{P}^*$ introduced earlier, but it is not necessarily separative. The following lemma indicates how to remedy this situation.

Lemma 15: Given any partially ordered set, $\mathbb{P}$, there is a separative partial ordering, $\mathbb{Q}$, and a map h of $\mathbb{P}$ onto $\mathbb{Q}$ such that

(i) $p \leq q$ in $\mathbb{P}$ implies $h(p) \leq h(q)$ in $\mathbb{Q}$.

(ii) p and q are compatible $\leftrightarrow h(p)$ and $h(q)$ are compatible

Furthermore, h sets up a one-to-one correspondence between generic filter on $\mathbb{P}$ and $\mathbb{Q}$.

Proof: Let $\mathbb{P}$ be a forcing and set up the equivalence relation on $\mathbb{P}$ given by:

$$p \equiv q \leftrightarrow \forall r (r \text{ is compatible with } p \leftrightarrow r \text{ is compatible with } q).$$

Now, let $\mathbb{Q} = \{ [p]_{\equiv} \mid p \in \mathbb{P} \}$ ordered by,

$$[p] \leq [q] \leftrightarrow \forall r \leq p (r \text{ is compatible with } q).$$

Finally, define $h: \mathbb{P} \rightarrow \mathbb{Q}$ by $h(x) = [x]$. h is obviously onto and is easily seen to satisfy (i) and (ii). Furthermore $\mathbb{Q}$ is separative since, if $[p] \leq [q]$ then there is an element, r , such that $r \leq q$ and r is incompatible with p .

Now to conclude that h sets up a correspondence between generics, we will show that it does so between dense sets. Let $D \subseteq \mathbb{P}$ be dense. Given any $q \in \mathbb{Q}$ there is a $q' \in \mathbb{P}$ such that $h(q') = q$. Since D is dense in $\mathbb{P}$, there is an $r \leq q'$ with $r \in D$. But, in this case, $h(r) \in h[D]$, with $h(r) \leq q$, and $h[D]$ is dense in $\mathbb{Q}$.

For the other direction, let $D \subseteq \mathbb{Q}$ be a dense and open set and let $y \in \mathbb{P}$ be arbitrary. Since D is dense in $\mathbb{Q}$, there is a $d \in D$ such that $d \leq h(y)$. But, by the definition of the ordering on $\mathbb{Q}$, this implies there is a $r' \in h^{-1}(d)$ with r' compatible with y . Pick $d' \leq r', y$, then $h(d') \leq r'$, but D is open, so $h(d') \in D$, and we conclude

that $h^{-1}[D]$ is dense in $\mathbb{P}$. That h carries generics to generics follows from an argument like that in Lemma 2. ■

We are now ready to factor $\mathfrak{M}[G]$ where G is $\mathbb{P}_{\kappa+1}$ -generic over $\mathfrak{M}$.

Theorem 6: Let G be $\mathbb{P}_{\kappa+1}$ -generic over $\mathfrak{M}$ and let $s \in \mathfrak{M}[G]$ be a countable sequence of ordinals. Then, there exists a $\mathbb{P}_{\kappa+1}$ -generic filter over $\mathfrak{M}[s]$, H , such that

$$\mathfrak{M}[G] = \mathfrak{M}[s][H].$$

Proof: Let G' be the generic filter on $B/\{0\}$ obtained from G . By Lemma 5, $\mathfrak{M}[G] = \mathfrak{M}[G']$. Furthermore, applying Lemma 12 to G' , we see that G' is B^* -generic over $\mathfrak{M}[s]$. So we have:

$$\mathfrak{M}[G] = \mathfrak{M}[G'] = \mathfrak{M}[s][G'].$$

Finally, Lemma 13 in conjunction with Lemma 5 provides an H' which is $\mathbb{P}^*$ -generic over $\mathfrak{M}[s]$ with

$$\mathfrak{M}[G] = \mathfrak{M}[s][H'].$$

Let us work relative to $\mathfrak{M}[s]$. By the choice of κ and the fact that $\mathfrak{M}[s] \subseteq \mathfrak{M}[G_\mu]$, we have that κ is an uncountable cardinal in $\mathfrak{M}[s]$. Furthermore, by the last equality, $\mathbb{P}^*$ collapses κ to ω and, since $\mathbb{P}^* \subseteq \mathbb{P}_{\kappa+1}$, we have

$$|\mathbb{P}^*| \leq |\mathbb{P}_{\kappa+1}| \leq \kappa.$$

So, we have $\mathbb{P}^*$ and $\mathbb{P}_{\kappa+1}$ each of size at most κ , both collapsing κ to ω , and, by Lemma 15, without loss of generality, we can consider both separative.

Therefore, by Lemma 14, $\mathbb{C}_\kappa$ can be densely embedded in both $\mathbb{P}^*$ and $\mathbb{P}_{\kappa+1}$. So, applying Lemma 5 twice, we have an H which is $\mathbb{P}_{\kappa+1}$ -generic over $\mathfrak{M}[s]$ with

$$\mathfrak{M}[G] = \mathfrak{M}[s][H]$$

as desired. ■

Our main theorem is an easy consequence of Theorem 6 and Lemma 10.

Theorem 7: (The Factor Theorem) Let G be a $\mathbb{P}_L$ -generic filter over $\mathfrak{M}$ and let $s \in \mathfrak{M}[G]$ be a countable sequence of ordinals. Then, there exists a $\mathbb{P}_L$ -generic filter, H , over $\mathfrak{M}[s]$ such that

$$\mathfrak{M}[G] = \mathfrak{M}[s][H].$$

Proof: Let $\kappa < \lambda$ have the properties specified for it earlier. We have a natural isomorphism between $\mathbb{P}_{\kappa+1} \times \mathbb{P}_{\kappa+1}$ and $\mathbb{P}_L$, given by

$$m(p, q) = p \cup q.$$

Applying Lemma 10, we have a $\mathbb{P}_{\kappa+1}$ -generic filter, G_1 , over $\mathfrak{M}$ and a $\mathbb{P}_{\kappa+1}$ -generic filter over $\mathfrak{M}[G_1]$, G_2 , so that

$$\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2].$$

Now, by Theorem 6, there is a G' which is $\mathbb{P}_{\kappa+1}$ -generic over $\mathfrak{M}[s]$ such that

$$\mathfrak{M}[G_1] = \mathfrak{M}[s][G'].$$

Finally, applying Lemma 10 to G' and G_2 and the isomorphism, m , we have an H which is $\mathbb{P}_L$ -generic over $\mathfrak{M}[s]$ with

$$\mathfrak{M}[s][G'][G_2] = \mathfrak{M}[s][H],$$

but, $\mathfrak{M}[s][G'][G_2] = \mathfrak{M}[G]$, so

$$\mathfrak{M}[G] = \mathfrak{M}[s][H],$$

as desired. ■

We have finished deriving the two basic properties of the Levy model which we will make use of in our proofs involving Lebesgue measurability. We need to go on to develop the tools from descriptive set theory we will need in order to talk about special sets of reals. Our next chapter will be devoted to this topic.

Chapter 3: A Brief Introduction to Descriptive Set Theory

In this chapter, we will introduce some basic ideas of Descriptive Set Theory. This material is central to our study of Lebesgue measurability.

Our goal is to investigate properties of sets of reals which have a simple structure or simple definition. As many authors have noted, questions which are hard or impossible to answer for arbitrary sets of reals, can often be answered in a nice way if we restrict attention to simple subsets of the real line. This is the spirit of Descriptive Set Theory.

Section 1: Perfect Polish Spaces

While we could develop this material with reference to a model of ZFC, it is not necessary for our work to assume a full choice axiom. We will assume the existence of an ambient model of ZF plus the following weaker choice axiom:

Definition 1: The Principle of Dependent Choices (D.C.)

If ρ is a binary relation over a nonempty set, A , and if for every $a \in A$ there is a $b \in A$ such that $b\rho a$, then there is a countable sequence, $a_0, a_1, \dots$ in A such that

$$a_{n+1}\rho a_n$$

for all $n \in \omega$.

This axiom has the following important consequences:

- (i) Choice for a countable sequence of sets.
- (ii) The union of countably many countable sets is countable.
- (iii) A binary relation without infinite descending chains is well-founded.

We will need all of these results in the following, so the reader is encouraged to keep them in mind.

Now, although ultimately we are interested in sets of reals and Lebesgue measure, our results can be applied to a more general class of spaces. Indeed, Descriptive set theory is usually developed with reference to a perfect Polish Space, that is a topological space which is homeomorphic to a complete, separable metric space.

Besides the reals, the most important example of such a space is the following:

Definition 2: The Baire space is the set of all infinite sequences of natural numbers, $\mathcal{N} = \omega^\omega$, with the topology determined by the basic open sets,

$$U_s = \{ f \in \mathcal{N} \mid s \subseteq f \}$$

where $s = \langle a_k \mid k < n \rangle$ is a sequence of natural numbers.

We can put a metric on the Baire space as follows: Let $x, y \in \mathcal{N}$ and define

$$d(x, y) = 2^{-n}, \text{ where } n \text{ is least such that } x(n) \neq y(n).$$

There are certain advantages to working with the Baire space as opposed to working with the reals. First, note that we have a nice homeomorphism, h , of $\mathcal{N} \times \mathcal{N}$ with $\mathcal{N}$. Namely, $h(x, y) = z$ where $z(2n) = x(n)$ and $z(2n+1) = y(n)$ so that dimensionality, which is unimportant for the theory, can be disregarded in a natural way in the Baire space. The second advantage of the Baire space is the following nice characterization of closed sets.

Let $\text{Seq} = \bigcup \{ {}^n\omega \mid n=0,1,\dots \}$, be the set of all finite sequences from ω . A tree is a set $T \subseteq \text{Seq}$ such that, if $t \in T$ and $s = t \upharpoonright n$, for some n , then $s \in T$. If T is a tree, define $[T]$ to be the set of all infinite paths through T :

$$[T] = \{ f \in \mathcal{N} \mid f \upharpoonright n \in T \text{ for all } n \in \omega \}.$$

We have the following simple lemma.

Lemma 1: The set $[T]$ is a closed set in the Baire space and, if F is any closed set in the Baire space, then

$$T_F = \{ s \in \text{Seq} \mid s \subseteq f \text{ for some } f \in F \}$$

is a tree, and we have

$$[T_F] = F.$$

Proof: Let $f \in \mathcal{N}$ be such that $f \notin [T]$. Then there is an $n \in \omega$ such that $f|n = s$ is not in T . Then

$$U_s = \{ g \in \mathcal{N} \mid s \subseteq g \}$$

is an open set containing f such that $[T] \cap U_s = \emptyset$. Hence $[T]$ is closed in $\mathcal{N}$. Now let F be closed. It is clear that T_F is a tree and that $F \subseteq [T_F]$. To see that $[T_F] \subseteq F$, let $f \in [T_F]$. In this case $f|n \in T_F$, for every $n \in \omega$, so, for each n , there is a $g \in F$ such that $g|n = f|n$. But, then, since F is closed, we have $f \in F$, as desired. ■

So, we will work in the Baire space with the understanding that most of our results hold for the real line, or any perfect Polish space.

Section 2: Simply Defined Sets

In this section, we introduce the main hierarchies of simply defined sets, the Borel, Projective, and Effective hierarchies of subsets of $\mathcal{N}$.

The Borel sets are those sets in the smallest σ -algebra containing the closed sets. More explicitly, we define collections of subsets of $\mathcal{N}$, Σ^0_α and Π^0_α , for $\alpha < \omega_1$, as follows:

Σ^0_1 = all open sets

$\Pi^0_\alpha = \{ A \subseteq \mathcal{N} \mid A^c \in \Sigma^0_\alpha \}$, where $A^c = \mathcal{N} - A$,

$\Sigma^0_\alpha = \{ A \subseteq \mathcal{N} \mid A = \bigcup_n A_n \text{ where each } A_n \in \Pi^0_\beta \text{ for some } \beta < \alpha \}$.

By induction, we can see that every set in a Σ^0_α or Π^0_α is a Borel set. In fact, the set of all Borel sets is $\bigcup_{\alpha < \omega_1} \Sigma^0_\alpha = \bigcup_{\alpha < \omega_1} \Pi^0_\alpha$. Note that this hierarchy is closed under complementation, countable unions, and inverse images of continuous functions.

While the Borel sets give us a large collection of nice sets to study, it fails to have some important closure properties. For instance, the Borel sets are not closed under continuous images, that is, the continuous image of a Borel set need not be a Borel set itself. Our next hierarchy, the Projective sets, will have this closure property.

A subset, A , of $\mathcal{N}$ is called analytic if there is a continuous function, $f: \mathcal{N} \rightarrow \mathcal{N}$, such that $f[\mathcal{N}] = A$. It can be shown that this is equivalent to A being a projection of a closed set in $\mathcal{N} \times \mathcal{N}$. We build the Projective hierarchy starting with analytic sets, much as we begin with the closed sets in the Borel hierarchy.

We define collections, for $n > 0$, Σ^1_n , Π^1_n , and Δ^1_n , as follows:

Σ^1_1 = the collection of all analytic sets;

Π^1_n = complements of Σ^1_n sets;

Σ^1_n = projections of all Π^1_n sets in $\mathcal{N} \times \mathcal{N}$;

$\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.

Since every Borel set is its own image under the identity map, and the Borel sets are closed under complementation, it is clear that,

Borel sets $\subseteq \Delta^1_1$.

The first important theorem in the subject is due to Suslin, who proved

$$\text{Borel sets} = \Delta^1_1.$$

In order to obtain a nice characterization of projective sets, we need to describe closed sets in the Baire product space, $\mathfrak{N}^r$, $r > 0$. It turns out closed sets in these spaces can be described in terms of trees, as in the one dimensional case.

Let

$$\text{Seq}_r = \{ (s_1, \dots, s_r) \in \mathfrak{N}^r \mid \text{length}(s_1) = \dots = \text{length}(s_r) \}.$$

A set, $T \subseteq \text{Seq}_r$ is an r -dimensional tree, if, for $(s_1, \dots, s_r) \in T$ and each $n \leq \text{length}(s_1)$

$$(s_1|n, \dots, s_r|n) \in T.$$

Let, as in the one dimensional case,

$$[T] = \{ (a_1, \dots, a_r) \in \mathfrak{N}^r \mid \forall n (a_1|n, \dots, a_r|n) \in T \}.$$

Then $[T]$ is a closed set in $\mathfrak{N}^r$ and every closed set can be described in this way; the proof is exactly as in the one dimensional case.

This tree description for closed sets allows us to describe analytic sets in a nice way. Since $A \subseteq \mathfrak{N}^r$ is analytic just in case it is the projection of a closed set in $\mathfrak{N}^{r+1}$, we have,

$$(x_1, \dots, x_r) \in A \leftrightarrow (\exists z \in \mathfrak{N}) (\forall n \in \omega) (x_1|n, \dots, x_r|n, z|n) \in T$$

for some $r+1$ dimensional tree, T . This is called the normal form for the description of analytic sets. Note that, since each Π^1_1 set is the complement of an analytic set, we have, for any Π^1_1 set, $B \subseteq \mathfrak{N}^r$,

$$(x_1, \dots, x_r) \in B \leftrightarrow (\forall z \in \mathfrak{N}) (\exists n \in \omega) (x_1|n, \dots, x_r|n, z|n) \notin T$$

for some $r+1$ dimensional tree, T .

Similar normal forms, with alternating quantifiers can be developed for every level in the projective hierarchy, although we will not be concerned with

describing sets more complicated than those of type Π^1_1 .

These normal forms lead to our third hierarchy, the Effective, Descriptive Set Hierarchy. We will not construct the full hierarchy here, but content ourselves with the introduction of the simplest level of the hierarchy, the Σ^1_1 and Π^1_1 sets.

Definition 3: A predicate on ω^k , for $k < \omega$, R , is arithmetical if it is definable in the model $V_\omega = HF$, the set of all hereditarily finite sets.

Equivalently, these are the predicates that are definable from ZF without the infinity axiom, since V_ω is the minimal model of ZF-'infinity'. Alternatively, arithmetical relations are those definable in the structure, $\langle \omega, +, \cdot \rangle$. We now define a subset, A , of $\mathcal{N}$ to be Σ^1_1 if there is an arithmetical R such that,

$$x \in A \leftrightarrow (\exists z \in \mathcal{N})(\forall n) (x|n, z|n) \in R.$$

We define a set, B , to be Π^1_1 if it is the complement of a Σ^1_1 set, or, equivalently, if there is an arithmetical R such that,

$$x \in B \leftrightarrow (\forall z \in \mathcal{N})(\exists n) (x|n, z|n) \notin R.$$

It is clear from the tree characterization of Σ^1_1 and Π^1_1 sets introduced earlier, that every Σ^1_1 set is Σ^1_1 and similarly for Π^1_1 sets.

We have introduced a very restricted definition of Σ^1_1 and Π^1_1 sets in order to simplify certain proofs of absoluteness results later in this chapter. It turns out that we can allow for more general quantification in the defining relations for these sets. We will discuss these more general types of allowable quantification later on.

We will now go on to discuss a special coding of Borel sets by members of the Baire space, $\mathcal{N}$.

Section 3: Borel Codes

In this section, we will introduce a way of coding Borel subsets of the reals by members of the Baire space, $\mathcal{N}$. This coding is possible because each Borel set is obtained from the basic open sets by a countable number of operations.

For each level of the Borel hierarchy, Σ_n^0 , Π_n^0 , we will define corresponding subsets of the Baire space, written Σ_n and Π_n , respectively, such that the members of any Σ_n (respectively, Π_n) act as codes for sets in Σ_n^0 (respectively Π_n^0). The codes actually describe how the sets are built up from the basic open sets. We will denote the set coded by $c \in \mathcal{N}$ by A_c .

Let $I_1, \dots, I_k, \dots$ be an enumeration of the open intervals with rational endpoints. We define the set of Borel codes, BC , by induction on $\alpha < \omega_1$. First, define a function, $u: \mathcal{N} \rightarrow \mathcal{N}$, by,

$$u(c)=d \leftrightarrow d(n)=c(n+1),$$

and a family of functions, v_i , one for each i , as follows:

$$v_i(c)=d \leftrightarrow d(n)=c(\Gamma(i,n)+1)$$

where Γ is the one-to-one canonical correspondence between $\omega \times \omega$ and ω .

Notice that the v_i 's 'decode' c into a countable sequence of Borel codes, $v_1(c), v_2(c)$, etc, and $u(c)$ is the sequence obtained by dropping the first entry of the sequence, c .

Now, we put $c \in \Sigma_1$, if $c(0) > 1$ and let c code:

$$A_c = \bigcup \{ I_n \mid c(n)=1 \},$$

that is, c codes the at most countable union of open rational intervals at which c has value 1.

Next, we put $c \in \Pi_\alpha$, if, either, $c \in \bigcup_{\beta < \alpha} \Sigma_\beta \cup \Pi_\beta$, or $c(0)=0$, and $u(c) \in \Sigma_\alpha$.

In this case, if $c \in \Pi_\alpha$ and $c(0)=0$, then c codes:

$$A_c = \mathbb{R} - A_{u(c)}.$$

(Note that $A_{u(c)}$ is well-defined, by induction.)

Finally, we put $c \in \Sigma_\alpha$, if, either, $c \in \bigcup_{\beta < \alpha} \Sigma_\beta \cup \Pi_\beta$,

or, $c(0)=1$, and $v_i(c) \in \bigcup_{\beta < \alpha} \Sigma_\beta \cup \Pi_\beta$, for all i .

In this case, c codes:

$$A_c = \bigcup_i A_{v_i(c)}$$

and again note that $A_{v_i(c)}$ is well-defined by induction.

We define the set BC to be the union of all the Σ_α 's, or, equivalently, Π_α 's.

We want to argue that the Borel sets are just that family of sets coded by elements of BC, i.e. we want to show that

$$\text{Borel sets} = \{ A_c \mid c \in \text{BC} \}.$$

If $\alpha > 0$ and $c \in \Sigma_\alpha$, then it is clear that c codes a Borel set, and that $A_c \in \Sigma_\alpha^0$; similarly for Π_α -codes. To see that every Borel set is coded by a Borel code, we need to argue by induction on α .

If $\alpha=1$ and $B \in \Sigma_\alpha^0$, then B is an open sets and thus a union of rational open intervals, $I_{a_1}, \dots, I_{a_n}, \dots$; we simply let $c \in \text{BC}$, with $c(0)>1$, be some code of the union of these rational intervals, and we'll have

$$A_c = B.$$

The other cases work by construction of codes as well. For instance, if c_i , $i \in \omega$, is a sequence of elements from $\bigcup_{\beta < \alpha} \Pi_\beta$, then we need to see that there is a $c \in \Sigma_\alpha$ such that

$$v_i(c) = c_i, \text{ for all } i.$$

We let c be defined as follows:

$$c(0)=1,$$

and

$$c(\Gamma(i,n)+1)=c_i(n), \text{ for every } i \text{ and } n.$$

Then c is of the required form, and c codes the union of the sets coded by the c_i 's.

We conclude, then, that every Borel set, B , is coded by some member of BC , as desired.

We want to show that certain properties of Borel codes are preserved in all transitive models of $ZF+DC$. In order to do this, a discussion of Absoluteness, and specifically Π^1_1 -absoluteness, will occupy the next section.

Section 4: Absoluteness

In this section, we will discuss one of the most basic ideas of mathematical logic, the concept of absoluteness. The definition is as follows:

Definition 4: A formula, ϕ , is absolute for a model, $\mathfrak{M}$, if for every $x_1, \dots, x_n$ from $\mathfrak{M}$,

$$\mathfrak{M} \models \phi(x_1, \dots, x_n) \quad \text{iff} \quad \phi(x_1, \dots, x_n) \text{ holds in the universe.}$$

We can go on to define absoluteness for properties, relations, functions, and et cetera, using this basic definition. For instance, we will show that the property, " c is a Borel code", is absolute for all transitive models of $ZF+DC$. This means that, given any transitive model of $ZF+DC$, we have, for $c \in \mathfrak{M}$,

$$c \in BC \text{ iff } \mathfrak{M} \models c \in BC,$$

or, if we write $BC^{\mathfrak{M}}$ for the set of codes in $\mathfrak{M}$, we have

$$BC^{\mathfrak{M}} = BC \cap \mathfrak{M}.$$

We will now show that the property of well-foundedness is absolute for a

certain class of models. We will use this result to prove that any Π^1_1 relation is absolute for transitive models of ZF+DC.

Lemma 2: The property,

"E is a well-founded relation"

is absolute for all transitive models of ZF.

Proof: We will need to use the following result, which is provable in ZF:

E is a well-founded relation on P if and only if there is a function, f, from P into Ord, such that, xEy implies $f(x) < f(y)$.

Let $\mathfrak{M}$ be a transitive model of ZF. First, note that "E is well-founded on P" can be expressed as:

$$\forall X [(X \neq \emptyset \wedge X \subseteq P) \rightarrow (\exists a \in X) a \text{ is E-minimal in } X].$$

If this formula holds in the universe, then it holds for every subset of P, and so holds for every subset, $X \in \mathfrak{M}$. This implies the formula holds in $\mathfrak{M}$.

For the other direction, we use the characterization of well-foundedness stated above. That is, we express well-foundedness as:

$$\begin{aligned} &E \text{ is a relation on } P \wedge \exists f [f \text{ is a function} \wedge (\forall u \in \text{ran}(f)) u \text{ is an ordinal} \\ &\wedge (\forall x, y \in P) (xEy \rightarrow f(x) < f(y))]. \end{aligned}$$

Now, if this formula holds in $\mathfrak{M}$, then f exists in the universe as well. Further, f carries these same properties in the universe because ordinals in $\mathfrak{M}$ are ordinals in the universe and being a function is absolute; therefore E will be well-founded in the universe. We have shown that well-foundedness is absolute for transitive models of ZF. ■

This proof has a marked model-theoretic flavor to it and we will find several opportunities to use the idea of universal and existential description in concluding other absoluteness theorems. We will now use the lemma to prove absoluteness

for Π^1_1 relations.

Theorem 1: (Π^1_1 -absoluteness) If A is a Π^1_1 relation and $\mathfrak{M}$ is a transitive model of ZF+DC, then for $x \in \mathfrak{M}$

$$x \in A \quad \text{iff} \quad \mathfrak{M} \models x \in A.$$

Proof: We prove the result for Π^1_1 relations over the Baire space. We will show that being Π^1_1 is equivalent to the well-foundedness of a certain relation.

Let A be a Π^1_1 -set in $\mathfrak{N}$. Then there is an arithmetical relation, R , such that:

$$\alpha \in A \leftrightarrow \forall \beta \exists n R(\alpha|n, \beta|n).$$

For this R , define, for $\alpha \in \mathfrak{N}$:

$$X(\alpha, R) = \{ \beta | m \in \text{Seq} \mid \forall n \leq m \neg R(\alpha|n, \beta|n) \}.$$

We partial-order Seq by,

$$\gamma E \delta \leftrightarrow \gamma \supseteq \delta,$$

and claim,

$$\alpha \in A \leftrightarrow E \text{ is well-founded on } X(\alpha, R).$$

To see this, note that, if $\alpha \in A$, then there is, for every $\beta \in \mathfrak{N}$, an $n \in \omega$, such that $R(\alpha|n, \beta|n)$. We conclude that E has no infinite descending chains, so, by DC, E is well-founded.

Now, assume $\alpha \notin A$, then there is a $\beta \in \mathfrak{N}$ such that, for every $n \in \omega$, $\neg R(\alpha|n, \beta|n)$. But, in this case,

$$\beta|1, \beta|2, \dots$$

will be an infinite descending chain in $X(\alpha, R)$, and we've proved E is not well-founded on $X(\alpha, R)$.

We conclude that membership in A is absolute for transitive models of ZF+DC, by Lemma 2. ■

With this result, we are ready to discuss absoluteness properties for the

Borel codes. For the most part, we will show that properties are Π^1_1 and conclude absoluteness for models of ZF+DC by Theorem 1.

Section 5: Absoluteness for Borel codes

We will prove some important properties of Borel codes are absolute for transitive models of ZF+DC. These results will be used in the next section to conclude that "c codes a measure zero Borel set" is absolute. First, we show that the property of being a Borel code is absolute.

Lemma 3: The set, BC, of all Borel codes is absolute for transitive models of ZF+DC; that is,

$$c \text{ is a Borel code} \leftrightarrow \mathfrak{M} \models c \text{ is a Borel code}$$

for any transitive model of ZF+DC containing c.

Proof: We define a relation, E, over $\mathcal{N}$, in $\mathfrak{M}$ as follows:

$$\begin{aligned} xEy &\leftrightarrow \text{either, } y(0)=0 \text{ and } x=u(y), \\ &\text{or, } y(0)=1 \text{ and } x=v_i(y), \text{ for some } i. \end{aligned}$$

We want to show that $y \in BC$ if and only if E is well-founded below y. We can then conclude that $y \in BC$ is absolute.

Now, if $y \in \Sigma_1$, then y is E-minimal, since $y \in \Sigma_1$ implies $y(0) > 1$. Also, note that, if y is E-minimal, then, by the definition of E, $y(0) > 1$, and $y \in \Sigma_1$. Next, note that, if $y \in \Pi_\alpha$ and xEy , we have $x=u(y)$, and therefore $x \in \Sigma_\alpha$. On the other hand, if $y \in \Sigma_\alpha$ ($\alpha > 1$) and xEy , we have $x=v_i(y)$ for some i, and $x \in \bigcup_{\beta < \alpha} \Sigma_\beta \cup \Pi_\beta$. We can conclude, using DC, that, if $y \in BC$, there can be no infinite descending chain below y. That is,

$$y \in BC \rightarrow E \text{ is well-founded below } y.$$

Let E be well-founded below an element, $y \in \mathcal{N}$. We can then define the rank function, $\rho_E(y) = \{ \rho_E(x) \mid x E y \}$, of E below y and proceed by induction on ρ_E to show that every $x \in \text{ext}_E(y)$ is a Borel code, and, finally, that y itself is a Borel code. We have, then,

$$E \text{ is well-founded below } y \rightarrow y \in BC$$

as desired. ■

We want to show that the relations, $A_c \subseteq A_d$, $A_c = A_d$, and $A_c = \emptyset$ are absolute, that is, for instance, "c and d code the same Borel set" is absolute for transitive models of ZF+DC. To do this we will find Π^1_1 and Σ^1_1 predicates that are equivalent to $a \in A_c$, where we regard the real, a , as a countable sequence of ordinals. We want to argue that the definition:

$$A_c \subseteq A_d \leftrightarrow c, d \in BC \wedge \forall a ((a, c) \in Q \rightarrow (a, d) \in P)$$

where Q is Σ^1_1 and P is Π^1_1 , can be written in a Π^1_1 way. For instance, note that,

$$((a, c) \in Q \rightarrow (a, d) \in P) \leftrightarrow ((a, c) \notin Q \vee (a, d) \in P),$$

which is Π^1_1 , since P and $\neg Q$ are.

We have the further definitions,

$$A_c = A_d \leftrightarrow c, d \in BC \wedge A_c \subseteq A_d \wedge A_d \subseteq A_c,$$

$$A_c = \emptyset \leftrightarrow c \in BC \wedge \forall a (a, c) \notin Q,$$

which can be written in Π^1_1 ways as well. The next lemma defines the predicates, P and Q , mentioned above.

Lemma 4: There are predicates, P and Q , which are Π^1_1 and Σ^1_1 , respectively, such that,

$$a \in A_c \leftrightarrow (a, c) \in P \leftrightarrow (a, c) \in Q.$$

Proof: Let $x \in \mathcal{N}$ be fixed. Define T to be the smallest set, $T \subseteq \mathcal{N}$, such that,

$$(I) \quad x \in T, \text{ and if } y \in T \text{ and } z E y, \text{ then } z \in T,$$

where E is the relation defined in Lemma 3. Using DC, the set, T , is at most countable, and note that there is a unique smallest countable set, T , with these properties. Also, since T is a countable, we can code it into a single member of the Baire space, b' , using the arithmetically definable functions, v_i , introduced earlier.

Next we define a 'decision function' which will assign the value 1 to a code, $x \in T$ if and only if $a \in A_x$, for every real, a . Specifically, let $h: T \rightarrow \{0,1\}$ be a function with the following property, for every $y \in T$,

- (II) if $y(0) > 1$, then $h(y) = 1$ iff for some n , $y(n) = 1$ and $a \in I_n$;
- if $y(0) = 0$, then $h(y) = 1$ iff $h(u(y)) = 0$;
- if $y(0) = 1$, then $h(y) = 1$ iff for some i , $h(v_i(y)) = 1$.

Note, again, that, if $x \in BC$, then there is a unique countable function, $h: T \rightarrow \{0,1\}$ with these properties, and so h can be coded by a single element of the Baire space, just as T has been. In this case, we will have, for every $y \in T$, $h(y) = 1$ just in case $a \in A_y$.

Now, define P and Q as follows:

$$(a, x) \in P \leftrightarrow (\forall \text{countable } T \subseteq \mathcal{N}) (\forall h: T \rightarrow \{0,1\}) [\text{if } (I) \wedge (II), \text{ then } h(x) = 1]$$

and

$$(a, x) \in Q \leftrightarrow (\exists \text{countable } T \subseteq \mathcal{N}) (\exists h: T \rightarrow \{0,1\}) [(I) \wedge (II) \wedge h(x) = 1].$$

Using the codings for T and h mentioned above and codings of pairs of the Baire space, it can be verified that P can be written in a Π^1_1 , and Q in a Σ^1_1 way. ■

We'll use the following absoluteness results to show the absoluteness of null measure sets.

Lemma 5: The following operations are absolute for all transitive models of ZF+DC:

$$A_e = A_c \cup A_d \quad \text{and} \quad A_e = \bigcup_n A_{c_n}.$$

Proof: Let $A_e^{\mathfrak{M}}$ denote the Borel set coded in $\mathfrak{M}$ by e , where $\mathfrak{M}$ is a transitive model of ZF+DC. First, let $c_0, c_1, \dots, c_n, \dots$ be a countable sequence of codes in $\mathfrak{M}$ and let $c \in \mathfrak{M}$ be such that

$$c(0)=1 \quad \text{and} \quad v_i(c)=c_i, \text{ for all } i.$$

Then, by definition, $c \in BC^{\mathfrak{M}}$. For any Borel code, $e \in \mathfrak{M}$, we have

$$A_e^{\mathfrak{M}} = \bigcup_n A_{c_n}^{\mathfrak{M}} \quad \text{iff} \quad A_e^{\mathfrak{M}} = A_c^{\mathfrak{M}},$$

but, by the absoluteness of $A_e = A_c$, we have,

$$A_e^{\mathfrak{M}} = A_c^{\mathfrak{M}} \quad \text{iff} \quad A_e = A_c.$$

This last holds if and only if $A_e = \bigcup_n A_{c_n}$, by the definition of c . We conclude that countable unions are absolute.

For the complement, let d be a Borel code in $\mathfrak{M}$ and let $c \in \mathfrak{M}$ be such that $c(0)=0$ and $u(c)=d$. Then $c \in \mathfrak{M}$, and we have, for every Borel code, $e \in \mathfrak{M}$,

$$A_e^{\mathfrak{M}} = (\mathbb{R} - A_d)^{\mathfrak{M}} \quad \text{iff} \quad A_e^{\mathfrak{M}} = A_c^{\mathfrak{M}} \quad \text{iff} \quad A_e = A_c \quad \text{iff} \quad A_e = \mathbb{R} - A_d,$$

as desired. ■

We conclude that the properties:

$$A_e = A_c \cap A_d \quad \text{and} \quad A_e = A_c \Delta A_d$$

are absolute, since they are definable from the absolute operations above.

In the next section, we will investigate the absoluteness properties of the Lebesgue measure.

Section 6: Absoluteness Properties of Lebesgue Measure

Using the absoluteness properties of the last section, we'll go on to show that the property:

"c codes a measure zero Borel set"

is absolute for transitive models of ZF+DC. We first prove that the Lebesgue measure is absolute for all Σ_1 and Π_1 codes, and thus for all Σ^0_1 and Π^0_1 sets.

Lemma 6: For every Borel code, $c \in \Sigma_1 \cup \Pi_1$,

$$\mu^{\mathfrak{M}}(A_c^{\mathfrak{M}}) = \mu(A_c)$$

for every transitive model of ZF+DC.

Proof: First, let $c \in \Sigma_1$ be a Σ^0_1 -code. Let $k_0, k_1, \dots, k_n, \dots$ be all the $k \in \omega$ such that $c(k)=1$, that is A_c is the union of the I_{k_n} 's which are rational open intervals. For each n, let

$$X_n = I_{k_n} - (I_{k_0} \cup \dots \cup I_{k_{n-1}}),$$

then we have,

$$A_c = \bigcup X_n$$

and

$$\mu(A_c) = \sum_n \mu(X_n).$$

Each X_n is a finite union of disjoint rational intervals, each of whose measure, being its length, is absolute. We conclude that

$$\mu^{\mathfrak{M}}(A_c^{\mathfrak{M}}) = \mu(A_c).$$

If $c \in \Pi_1$, then $A_c = \mathbb{R} - A_{u(c)}$ and $u(c) \in \Sigma_1$. But, in this case, we have,

$$\mu^{\mathfrak{M}}(A_c) = 1 - \mu^{\mathfrak{M}}(A_{u(c)}) = 1 - \mu(A_{u(c)}) = \mu(A_c),$$

as desired. ■

Now, in order to show, " A_c is measure zero" is absolute, we will use the following characterizations of a set being null:

- (1) X is null if and only if, for every $n \in \omega$, there is an open set, $G \supseteq X$ of measure less than $1/n$,

and

- (2) $\mu(X) > 0$ if and only if there is a closed set, $F \subseteq X$, of positive measure.

Theorem 3: The property:

" e codes a null Borel set"

is absolute for all transitive models of ZF+DC.

Proof: Let $\mathfrak{M}$ be a transitive model of ZF+DC. First, assume $\mathfrak{M} \models "A_c \text{ is null}"$, then $\mathfrak{M}$ satisfies,

$$\forall n \exists e (e \in \Sigma_1 \wedge A_e \supseteq A_c \wedge \mu(A_e) < 1/n)$$

which is characterization (1), above. Note that, by the previous lemma, and the absoluteness results of the previous section, the $(\dots)$ part of this formula is absolute. We conclude that it holds universally, since n and e are members of ω and ω^ω , respectively, in the universe.

Next, suppose $\mathfrak{M} \models "A_c \text{ is not null}"$, then, using characterization (2), we have,

$$\mathfrak{M} \models \exists e (e \in \Pi_1 \wedge A_e \subseteq A_c \wedge \mu(A_e) > 0).$$

Again, $(\dots)$ is absolute from previous results, and the formula will hold universally, being existential.

We conclude that,

$$\mathfrak{M} \models "A_e \text{ is null}" \quad \text{iff} \quad A_e \text{ is null.} \blacksquare$$

These absoluteness results will be of great importance in the next Chapter, in which we introduce the model in which every set of reals is Lebesgue

measurable.

Chapter 4: A Model in Which Every Set of Reals is Lebesgue Measurable

It is our purpose in this chapter to prove the existence of a model in which every set of reals is Lebesgue measurable. The first two sections introduce the concepts of random reals and Solovay sets of reals; the main tools introduced by Solovay in proving this result. In the third section, we show that, in Levy's model, every set of reals definable from a sequence of ordinals is Lebesgue measurable using the results of Chapter 2. In the last section, we use the absoluteness results of Chapter 3 to prove that a certain inner model of the Levy collapse contains only Lebesgue measurable sets.

Section 1: Random Real Forcing

We first introduce the concept of a random real and provide two characterizations of this property.

Let $\mathfrak{B}$ be the algebra of Borel sets. We define the equivalence relation:

$$B \sim_m C \leftrightarrow \mu(B \Delta C) = 0,$$

and let,

$$\mathfrak{B}_m = \mathfrak{B} / \sim_m = \{ [B]_m \mid B \in \mathfrak{B} \},$$

be the algebra of Borel sets modulo the sets of measure zero. $\mathfrak{B}_m$ is a complete Boolean algebra under the partial ordering:

$$[B]_m \leq [C]_m \leftrightarrow B-C \text{ is null.}$$

Since this is the only equivalence relation we need to introduce, we will write $[B]$ instead of $[B]_m$.

Let us work relative to some transitive model of ZFC, $\mathfrak{M}$. Let B be a Borel set in $\mathfrak{M}$; B has a Borel code, $c \in \mathfrak{M}$, so that $A_c = B$. Let B^* be the Borel set in the universe coded by c . By the absoluteness results of the previous chapter, the definition of B^* is independent of which code, c , we choose for B and we have,

$$B = B^* \cap \mathfrak{M},$$

for every $B \in \mathfrak{B}$.

We refer to forcing using the algebra of Borel sets modulo measure zero sets as random real forcing because of the following correspondence between certain reals, called random reals, and generic ultrafilters on $\mathfrak{B}_m$.

Theorem 1: If G is a generic ultrafilter over $\mathfrak{M}$ on $\mathfrak{B}_m$, there is a unique real, x_G , such that, for all $B \in \mathfrak{B}$,

$$x_G \in B^* \leftrightarrow [B] \in G.$$

This formula uniquely characterizes x_G and the generic filter, G , so that,

$$\mathfrak{M}[G] = \mathfrak{M}[x_G].$$

We say x_G is random over $\mathfrak{M}$.

Proof: First, we show uniqueness. Suppose

$$x \in B^* \leftrightarrow [B] \in G \text{ (for all } B \in \mathfrak{B}).$$

Suppose $x < y$, the proof works the same for $y < x$. Let r be a rational such that $x < r < y$ and let A be the interval (r, ∞) . Since G is an ultrafilter, either $[A]$ or $[\mathbb{R}-A]$, but not both, belongs to G . But $x \notin A^*$ and $y \notin (\mathbb{R}-A)^*$, so both cannot be random for G .

To prove existence of such a real, let

$$x = \sup\{r \in \mathbb{Q} \mid [(r, \infty)] \in G\}.$$

This supremum exists, since, otherwise,

$$[\emptyset] = \bigcap_{r \in \mathbb{Q}} [(r, \infty)] \in G,$$

contradicting the fact that G is a filter.

We'll show by induction on codes, that, for every $c \in BC^{\mathfrak{M}}$,

$$x \in A_c^* \leftrightarrow [A_c] \in G.$$

Let us begin with Σ_1 codes that code a single rational interval, that is $c(n)=1$ for exactly one n . Then, c codes $I_n=(p,q)$. We have,

$$x \in A_c^* \leftrightarrow p < x < q,$$

which is equivalent, by the definition of x , to:

$$[(p, \infty)] \in G \quad \text{and} \quad [(q, \infty)] \notin G.$$

But, from the fact that G is an ultrafilter, this is equivalent to,

$$[(p, \infty)] \in G \quad \text{and} \quad [(-\infty, q)] \in G.$$

Lastly, since G is a filter, this holds if and only if

$$[(p, q)] \in G,$$

or,

$$[A_c] \in G,$$

as desired.

Now, if $c \in \Sigma_1$ is a general code, then $A_c = \bigcup_n I_{k_n}$, where $\{k_n \mid n \in \omega\}$ is the set of indices at which c is 1. We have,

$$x \in A_c^* \leftrightarrow x \in \bigcup_n I_{k_n}^* \leftrightarrow \exists n (x \in I_{k_n}^*).$$

This last is equivalent to:

$$\exists n [I_{k_n}] \in G,$$

by the induction hypothesis.

Clearly, since G is a filter, we have,

$$\exists n [I_{k_n}] \in G \text{ implies } [A_c] = [\bigcup_n I_{k_n}] = \Sigma_n [I_{k_n}] \in G.$$

On the other hand, since G is an ultrafilter, we have, by considering complements

$$\forall k_n [I_{k_n}] \notin G \rightarrow \forall k_n [I_{k_n}^c] \in G,$$

which implies, since G is generic,

$$\prod [I_{k_n}^c] \in G,$$

which, implies, by DeMorgan's Law, and that $0 \notin G$,

$$\Sigma [I_{k_n}] \notin G.$$

We conclude that, for $c \in \Sigma_1$,

$$x \in A_c^* \leftrightarrow [A_c] \in G.$$

Let $c \in \Pi_\alpha \cap \mathfrak{M}$, $\alpha < \omega_1^{\mathfrak{M}}$, and assume the equivalence holds for all $d \in \Sigma_\alpha \cap \mathfrak{M}$. Without loss of generality, assume $c(0)=0$, and $u(c) \in \Sigma_\alpha \cap \mathfrak{M}$. We then have,

$$A_{u(c)} = \mathbb{R} - A_c,$$

and so,

$$x \in A_c^* \leftrightarrow x \notin A_{u(c)}^* \leftrightarrow [A_{u(c)}] \notin G \leftrightarrow [A_c] \in G,$$

since G is an ultrafilter and assuming the equivalence holds for $u(c)$.

The proof for Σ_α , for $1 < \alpha < \omega_1^{\mathfrak{M}}$, is exactly the same as the Σ_1 case. ■

The description of the reals added by generic filters on $\mathfrak{B}_m$ as random is explained by the characterization is the next theorem.

Theorem 2: A real, x , is random over $\mathfrak{M}$ if and only if it does not belong to any null Borel set with code in $\mathfrak{M}$. So that, denoting the set of reals random over $\mathfrak{M}$ by $Ra(\mathfrak{M})$, we have,

$$Ra(\mathfrak{M}) = \mathbb{R}^* - \bigcup \{ A_c^* \mid c \in BC^{\mathfrak{M}} \text{ and } A_c^* \text{ is null} \}.$$

Proof: Recall from the last chapter that A_c is null in $\mathfrak{M}$ if and only if A_c^* is null. If x is a random real for the generic filter, G , on $\mathfrak{B}_m$, and A_c is null, we have

$[A_d] \notin G$, and so $x \notin A_c^*$, by Theorem 1.

Now, suppose $x \notin A_c^*$, whenever $c \in \mathfrak{M}$ and A_c is null. Define,

$$G = \{ [A_d] \mid x \in A_c^* \}$$

and note that G is well-defined, since, if $[A_c] = [A_d]$, then $A_c^* \Delta A_d^*$ is null, and so $x \in A_c^*$ if and only if $x \in A_d^*$.

G is clearly a filter on $\mathfrak{B}_m$. To see that G is $\mathfrak{M}$ -generic, let $X \subseteq G$; $X \in \mathfrak{M}$. Then, by definition, $x \in A_c^*$ for every A_c^* such that $[A_c] \in X$. We have,

$$\prod X = \bigcap [A_c] = [\bigcap A_c],$$

and since $x \in \bigcap A_c^*$, we have $\prod X \in G$. ■

We will go on to introduce the concept of Solovay sets of reals and draw a strong connection between Random reals, Solovay sets, and Borel sets.

Section 2: Solovay sets and Random reals

In this section, we will define what it means for a set of reals to be Solovay over some model, $\mathfrak{M}$, of ZFC and draw a relation between these sets and the random reals introduced earlier.

Let $\mathfrak{M}$ be a transitive model of ZFC and let x be a set with $x \in \mathfrak{M}[G]$ for some generic extension of $\mathfrak{M}$. Recall that we defined the model, $\mathfrak{M}[x]$, to be the least transitive model of ZFC, $\mathfrak{N}$, having the same ordinals as $\mathfrak{M}$, such that $\mathfrak{M} \subseteq \mathfrak{N}$ with $x \in \mathfrak{N}$.

Definition 1: Let $\mathfrak{M}$ be as above. A set of reals, S , is called Solovay over $\mathfrak{M}$ if there is a formula, $\phi(x)$, with parameters from $\mathfrak{M}$, such that, for all reals, x ,

$$x \in S \leftrightarrow \mathfrak{M}[x] \models \phi(x),$$

with $\mathfrak{M}[x]$ defined as above.

There is a strong connection between Solovay sets and random reals. This connection is the key property in investigating Lebesgue measurability in the Levy collapse. We will show in the next theorem that any Solovay set differs from some Borel set only in the nonrandom reals it contains.

Theorem 3: Let S be Solovay over $\mathfrak{M}$. There is a Borel set, A , such that,

$$S \cap \text{Ra}(\mathfrak{M}) = A \cap \text{Ra}(\mathfrak{M}).$$

Proof: To prove the theorem, we do random real forcing in $\mathfrak{M}$. Consider the forcing language in $\mathfrak{M}$ associated with the complete Boolean algebra, $\mathfrak{B}_m$. There is a canonical name, $\mathbf{H}$, for a generic ultrafilter on $\mathfrak{B}_m$ over $\mathfrak{M}$. For this $\mathbf{H}$, there is a random real, $x_{\mathbf{H}}$, by Theorem 1; let it be named, in the forcing language, by $\mathbf{a}$, that is,

$$\|\mathbf{a} = x_{\mathbf{H}}\| = 1.$$

Now, let S be a Solovay set of reals over $\mathfrak{M}$ and let ϕ be a formula, with parameters from $\mathfrak{M}$, such that,

$$x \in S \text{ iff } \mathfrak{M}[x] \models \phi(x).$$

Let $A_c \in \mathfrak{B}$ be such that,

$$[A_c] = \|\phi(\mathbf{a})\|,$$

that is $[A_c]$ is the Boolean value of ϕ at the random real named by $\mathbf{a}$. We claim that, for every $x \in \text{Ra}(\mathfrak{M})$, we have,

$$x \in S \text{ iff } x \in A_c^*.$$

If x is random over $\mathfrak{M}$, then $\mathbf{a}$ is a name for x , and we have,

$$x \in S \leftrightarrow \mathfrak{M}[x] \models \phi(x),$$

but, since $x = x_{\mathbf{H}}$, for some $\mathfrak{B}_m$ -generic filter over $\mathfrak{M}$, we have,

$$\mathfrak{M}[x] \models \phi(x) \leftrightarrow \mathfrak{M}[\mathbf{H}] \models \phi(x).$$

Now, by the forcing theorem, we have

$$\mathfrak{M}[H] \models \phi(x) \leftrightarrow \|\phi(\mathfrak{a})\| \in H,$$

but, by definition,

$$\|\phi(\mathfrak{a})\| \in H \leftrightarrow [A_c] \in H.$$

So, using the fact that x , with name, $\mathfrak{a}$, is random, we have, by Theorem 1,

$$[A_c] \in H \leftrightarrow x \in A_c^*,$$

and we conclude that,

$$x \in S \leftrightarrow x \in A_c^*,$$

for every real, x , which is random over $\mathfrak{M}$. ■

This theorem will enable us to prove that every set of reals definable in a certain way in the Levy model is Lebesgue measurable. The key Corollary to this theorem is the following result.

Corollary: If the set of all reals that are not random over $\mathfrak{M}$ is a null set, then every Solovay set of reals is Lebesgue measurable.

Proof: By the theorem, given a Solovay set, S , there is a Borel set, A , such that,

$$S \cap \text{Ra}(\mathfrak{M}) = A \cap \text{Ra}(\mathfrak{M}).$$

But, in this case, we have,

$$S \Delta A \subseteq \mathbb{R} - \text{Ra}(\mathfrak{M}).$$

Now, if the set of reals that are not random over $\mathfrak{M}$ is null, we have that S differs from a Borel set by a set of measure zero, and hence S itself is measurable. ■

With these results, we will go on to investigate Solovay sets, Random reals, and Lebesgue measurability in the Levy collapse.

Section 3: Lebesgue Measurability in the Levy Collapse

We want to prove the following theorem:

Theorem 4: In the Levy model, $\mathfrak{M}[G]$, every set of reals definable from a countable sequence of ordinals is Lebesgue measurable.

In order to prove the theorem, we need to investigate submodels of the Levy collapse constructed from countable sequences of ordinals. We will make use of the following important results from Chapter 2 about these models.

Let s be a countable sequence of ordinals in $\mathfrak{M}[G]$. Then:

(i) the inaccessible, λ , in $\mathfrak{M}$, remains inaccessible in $\mathfrak{M}[s]$.

and,

(ii) $\mathfrak{M}[G] = \mathfrak{M}[s][H]$, where H is a $\mathbb{P}_L$ -generic filter over $\mathfrak{M}[s]$.

Lemma 1: Let $s \in \mathfrak{M}[G]$ be a countable sequence of ordinals. The set of all reals, in $\mathfrak{M}[G]$, that are not random over $\mathfrak{M}[s]$ is null and so every set of reals in $\mathfrak{M}[G]$ which is Solovay over $\mathfrak{M}[s]$ is Lebesgue measurable.

Proof: Working relative to $\mathfrak{M}[G]$, we use the characterization of random reals from Theorem 2, that is,

$$\text{Ra}(\mathfrak{M}[s]) = \mathbb{R}^{\mathfrak{M}[G]} - \bigcup \{A_c^{\mathfrak{M}[G]} \mid c \in \mathfrak{M}[s] \text{ and } A_c \text{ is null}\}.$$

Since λ is inaccessible in $\mathfrak{M}[s]$, we have,

$$|2^\omega|^{\mathfrak{M}[s]} < \lambda,$$

and since $\lambda = \aleph_1^{\mathfrak{M}[G]}$, $\mathfrak{M}[s]$ has only a countable number of subsets of ω from $\mathfrak{M}[G]$. So, there are only a countable number of Borel codes in $\mathfrak{M}[s]$. Therefore, the complement of $\text{Ra}(\mathfrak{M}[s])$ is a countable union of null sets and hence null, as

desired. The second statement follows from the Corollary to Theorem 3. ■

Our next step, which will be no surprise, is to show that every set in $\mathfrak{M}[G]$ which is definable from a countable sequence of ordinals, s , is Solovay over $\mathfrak{M}[s]$. To do this, we use the Factor Theorem together with the Homogeneity result from Chapter 2.

Lemma 2: If $X \in \mathfrak{M}[G]$, is a set of reals definable from a countable sequence of ordinals, $s \in \mathfrak{M}[G]$, then X is, in $\mathfrak{M}[G]$, Solovay over $\mathfrak{M}[s]$.

Proof: For $X \in \mathfrak{M}[G]$ we need to find a formula, ϕ , with parameters from $\mathfrak{M}[s]$, such that, for every real, $x \in \mathfrak{M}[G]$, we have,

$$x \in X \leftrightarrow \mathfrak{M}[s][x] \models \phi(x).$$

First, we will show that, given a formula, ψ , there is a formula, ϕ , such that for every sequence of ordinals, $x \in \mathfrak{M}[G]$,

$$\mathfrak{M}[G] \models \psi(x) \leftrightarrow \mathfrak{M}[x] \models \phi(x).$$

Since $\mathfrak{M}[x]$ is a model of ZFC, and the definition of $\mathbb{P}_L$ is absolute, we can build a Boolean-valued model in $\mathfrak{M}[x]$ using the Levy forcing. Let $\phi(x)$ be the formula which says,

$$\|\psi(x^*)\| = 1,$$

where the value is computed in $\mathfrak{M}[x]$, using the Levy algebra, and x^* is the canonical name, in $\mathfrak{M}[x]$, for x . We will show that,

$$\mathfrak{M}[G] \models \psi(x) \leftrightarrow \mathfrak{M}[x] \models \phi(x).$$

Recall, from the Factor Theorem of Chapter 2, that there is a $\mathbb{P}_L$ -generic filter, H , over $\mathfrak{M}[x]$, such that,

$$\mathfrak{M}[G] = \mathfrak{M}[x][H],$$

so that,

$$\mathfrak{M}[G] \models \psi(x) \leftrightarrow \mathfrak{M}[x][H] \models \psi(x).$$

Now, using the homogeneity of the Levy collapse in $\mathfrak{M}[x]$, we have $\psi(x)$ being true in $\mathfrak{M}[x][H]$ if and only if $\|\psi(x^*)\|=1$. We conclude, then, that

$$\mathfrak{M}[G] \models \psi(x) \leftrightarrow \mathfrak{M}[x] \models \phi(x),$$

by the definition of ϕ .

Now, let $X \in \mathfrak{M}[G]$ be definable from a sequence of ordinals, s , then there is a formula, ψ , such that,

$$x \in X \leftrightarrow \mathfrak{M}[G] \models \psi(x, s).$$

We can regard the members of X , which are reals, as countable sequences of ordinals. Then, using a suitably definable coding of pairs, (x, s) , we can find another formula, $\psi'(u)$, such that,

$$\mathfrak{M}[G] \models \psi(x, s) \leftrightarrow \mathfrak{M}[G] \models \psi'(u).$$

Then, by the argument above, we have a formula, ϕ , such that,

$$\mathfrak{M}[G] \models \psi'(u) \leftrightarrow \mathfrak{M}[u] \models \phi(u).$$

But, since u is definable from x and s , and vice versa, we have

$$\mathfrak{M}[u] = \mathfrak{M}[s][x],$$

and a formula, $\phi'(x, s)$, such that,

$$\mathfrak{M}[G] \models \psi(x, s) \leftrightarrow \mathfrak{M}[s][x] \models \phi'(x, s).$$

We conclude, then, that:

$$x \in X \leftrightarrow \mathfrak{M}[s][x] \models \phi'(x, s),$$

and X is Solovay over $\mathfrak{M}[s]$, as desired. ■

Now, to prove Theorem 4, we let $X \in \mathfrak{M}[G]$ be a set of reals definable from a sequence of ordinals, $s \in \mathfrak{M}[G]$. By Lemma 2, X is Solovay over $\mathfrak{M}[s]$ and so, in $\mathfrak{M}[G]$, differs from a Borel set, $A \in \mathfrak{M}[G]$, by a null set, namely,

$$\mathbb{R}^{\mathfrak{M}[G]} - \text{Ra}(\mathfrak{M}[s])^{\mathfrak{M}[G]},$$

so that,

$\mathfrak{M}[G] \models \text{"X is Lebesgue measurable"},$

as desired. ■

In the next section we develop an inner model of $\mathfrak{M}[G]$, which is a model of ZF+DC, and consists solely of sets definable from countable sequences of ordinals; thus, every set of reals in this model will be Lebesgue measurable.

Section 4: The Model

We will now develop an inner model of the Levy collapse in which every set of reals is Lebesgue measurable. We know that every set of reals definable from a countable sequence of ordinals in $\mathfrak{M}[G]$ is measurable. The idea, then, is to find an inner model of ZF consisting of sets definable in precisely this way. In addition, this model will have to satisfy DC, allowing us to apply the absoluteness results of the last chapter.

To build this inner model, we use many of the tools used to build Godel's constructible universe. We will be omitting many details and content ourselves with referring the reader to Section 15 of Jech[4]. First, some definitions:

Definition 2: The Godel operations are the following ten operations on sets:

$$G_1(x,y) = \{x,y\}$$

$$G_2(x,y) = x \times y$$

$$G_3(x,y) = \in (x,y) = \{ (u,v) \mid u \in x \wedge v \in y \wedge u \in v \}$$

$$G_4(x,y) = x - y$$

$$G_5(x,y) = x \cap y$$

$$G_6(x) = \bigcup x$$

$$G_7(x) = \text{dom}(x)$$

$$G_8(x) = \{ (u,v) \mid (v,u) \in x \}$$

$$G_9(x) = \{ (u,v,w) \mid (u,w,v) \in x \}$$

$$G_{10}(x) = \{ (u,v,w) \mid (v,w,u) \in x \}.$$

We call any set closed under all Godel operations Godel-closed. For any set, x , there is a smallest Godel-closed set, $\text{cl}(x)$, containing x , constructed as follows:

Let $W_0 = x$, and

$$W_{n+1} = W_n \cup \{ G_i(x,y) \mid x, y \in W_n, i=1, \dots, 10 \},$$

where we define $G_i(x,y) = G_i(x)$, for $i=6, \dots, 10$. Finally, we let,

$$\text{cl}(x) = \bigcup_n W_n.$$

We next define what it means for a set, x , to be ordinal-definable over a countable sequence of ordinals, s , written $x \in \text{OD}(s)$. We would like x to be ordinal-definable over s , if there is a formula, ϕ , such that,

$$x = \{ u \mid \phi(u,s) \}.$$

However, we do not take this as our definition of the concept because it is not directly expressible in the language of set theory. Instead, we use the Godel operations to define $\text{OD}(s)$ and then show the constructive definition gives us the same sets as those satisfying the condition above.

We define:

$$\text{OD}(s) = \text{cl}(\{ V_\alpha \mid \alpha \in \text{Ord} \} \cup \{s\} \cup s).$$

That is, $\text{OD}(s)$ consists of all sets obtained from V_α 's, s , and members of s , by application of Godel operations.

Using the following lemma, we can show that every set in $\text{OD}(s)$ is of the form,

$$\{ u \mid \phi(u,s) \},$$

for some formula, ϕ .

Lemma 3: There is a well-ordering of the class $OD(s)$ definable from s and, thus, a definable mapping, F , from s , of Ord onto $OD(s)$.

Proof: For every β , we will show how to go from the obvious well-ordering of $s \cup \{s\} \cup \{V_\alpha \mid \alpha < \beta\}$ to a well-ordering of $cl(s \cup \{s\} \cup \{V_\alpha \mid \alpha < \beta\})$, call it $<_\beta$, in such a way that, for every $\gamma < \beta$, $<_\gamma \subseteq <_\beta$. Finally, we will let

$$<_{OD(s)} = \bigcup_{\beta \in Ord} <_\beta,$$

and call the definable mapping obtained from this well-ordering F .

We'll show how to go from $<_\beta$ to $<_{\beta+1}$. Recall that we defined $cl(x) = \bigcup W_n$, where W_{n+1} was obtained by Godel operations from W_n . Given, $<_\beta$, define $<_{\beta+1}^n$, for $n \in \omega$, as follows:

(i) $<_{\beta+1}^0$ is the well-ordering of $cl(s \cup \{s\} \cup \{V_\alpha \mid \alpha < \beta\}) \cup V_\beta$ which extends $<_\beta$, with V_β being the last element.

(ii) $<_{\beta+1}^{n+1}$ is the following well-ordering of W_{n+1} :

$$x <_{\beta+1}^{n+1} y \text{ iff either: } x <_{\beta+1}^n y$$

$$\text{or: } x \in W_n \text{ and } y \notin W_n$$

$$\text{or: } x \notin W_n \text{ and } y \notin W_n \text{ and}$$

$$(a) \text{ least } i \text{ s.t. } \exists u, v \in W_n (x = G_i(u, v)) < \text{least } j \text{ s.t. } \exists s, t \in W_n (y = G_j(s, t))$$

$$\text{or } (b) \text{ least } i = \text{least } j \text{ and}$$

$$[<_{\beta+1}^{n-\text{least } u \in W_n \text{ s.t. } \exists v \in W_n (x = G_i(u, v)) }] <_{\beta+1}^n [<_{\beta+1}^{n-\text{least } s \in W_n \text{ s.t.}}$$

$$\exists t \in W_n (y = G_j(s, t))]$$

$$\text{or } (c) \text{ least } i = \text{least } j \text{ and least } u = \text{least } v \text{ and}$$

$$[<_{\beta+1}^{n-\text{least } v \in W_n \text{ s.t. } x = G_i(u, v)}] <_{\beta+1}^n [<_{\beta+1}^{n-\text{least } t \in W_n \text{ s.t. } y = G_j(u, t)}].$$

Finally, let,

$$<_{\beta+1} = \bigcup_n <_{\beta+1}^n.$$

Now, the well-ordering, $<_{OD(s)}$ is a definable class (from s) and we can obtain F from it. ■

Now, let $x \in OD(s)$, then

$$x = \{ u \mid u \in F(\alpha) \},$$

where $F(\alpha)=x$. So, we conclude that x is of the form,

$$\{ u \mid \phi(u,s) \},$$

since F is definable from s .

Proving the converse, that every set of this form is in $OD(s)$ would take us too far afield. The result requires the Reflection Principle and the normal form theorem for restricted formulas. For further details, see Jech, Sections 11, 12, and 13.

We are now ready to define our inner model. First of all, we define $OD(S)$ to be the class of all sets ordinals definable over some $s \in {}^\omega \text{Ord}$. Next, let $TC(X)$ be the smallest transitive set containing X . We define the class of all sets hereditarily ordinal-definable over S , $HOD(S)$, to be,

$$HOD(S) = \{ x \mid TC(\{x\}) \subseteq OD(S) \}.$$

From the argument above, we can see that $OD(S)$ is just the class of sets in $\mathfrak{M}[G]$ which are definable from countable sequences of ordinals. It can be shown, although we will not do so, that $HOD(S)$ is a model of ZF. Instead, we will concentrate on showing that it is a model of DC. This is our next result, which relies on Lemma 3, proved above.

Theorem 5: (a) If $f \in \mathfrak{M}[G]$ is a function from ω into $\text{HOD}(\mathbf{S})$, then $f \in \text{HOD}(\mathbf{S})$.

(b) $\text{HOD}(\mathbf{S})$ models dependent choice.

Proof: (a) For every $s \in \mathbf{S}$, we have, by Lemma 3, a definable function, F_s from Ord onto $\text{OD}(s)$, uniformly for every s . Using this function, we can define a function, F , from $\text{Ord} \times \mathbf{S}$ onto $\text{OD}(\mathbf{S})$, such that $F(x, s) = F_s(x)$, for all $x \in \text{Ord}$.

Now, given $f \in \mathfrak{M}[G]$, mapping ω into $\text{OD}(\mathbf{S})$, we can choose, for each n , α_n and s_n , such that,

$$f(n) = F(\alpha_n, s_n).$$

Since F is a definable class, it is clear that f is definable from the sequences $\langle \alpha_n \mid n \in \omega \rangle$ and $\langle s_n \mid n \in \omega \rangle$. Using a definable coding, such as that developed for the Borel codes earlier, we can find a single sequence, $u \in {}^\omega \text{Ord}$, such that $\langle \alpha_n \rangle$ and $\langle s_n \rangle$ are definable from u . Hence f is definable from u and so is in $\text{OD}(\mathbf{S})$.

(b) In $\text{HOD}(\mathbf{S})$, let p be a relation on the nonempty set A such that, for every $x \in A$, there is a y such that ypx . Since $\mathfrak{M}[G]$ satisfies AC, there is, in $\mathfrak{M}[G]$, a sequence, $a_0, \dots, a_n, \dots$ such that $a_{n+1}pa_n$, for all n . But, by part (a), the sequence $\langle a_n \rangle \in \text{HOD}(\mathbf{S})$, so $\text{HOD}(\mathbf{S})$ satisfies DC. ■

We are now ready to show that $\text{HOD}(\mathbf{S})$ satisfies:

"every set of reals is Lebesgue measurable".

Theorem 6: In $\text{HOD}(\mathbf{S})$, defined in $\mathfrak{M}[G]$, every set of reals is Lebesgue measurable.

Proof: Since reals can be considered as countable sequences of

ordinals, $\text{HOD}(\mathbf{S})$ has the same reals (and Borel codes) as $\mathfrak{M}[G]$, so they have the same Borel sets.

Let $X \in \text{HOD}(\mathbf{S})$, be a set of reals. Then, X is definable, in $\mathfrak{M}[G]$, from a sequence of ordinals and, by Theorem 4,

$$\mathfrak{M}[G] \models (X \text{ is Lebesgue measurable}).$$

Hence, there are, in $\mathfrak{M}[G]$, Borel sets, A and N such that

$$X \Delta A = N$$

and N is null in $\mathfrak{M}[G]$. By the argument above, A and N are in $\text{HOD}(\mathbf{S})$.

Furthermore, since $\text{HOD}(\mathbf{S})$ is a model of $\text{ZF} + \text{DC}$, we have, by Theorem 3 of Chapter 3, that $\text{HOD}(\mathbf{S}) \models "N \text{ is null}"$. Therefore,

$$\text{HOD}(\mathbf{S}) \models (X \Delta A = N \wedge N \text{ is null}),$$

so,

$$\text{HOD}(\mathbf{S}) \models (X \text{ is Lebesgue measurable}). \blacksquare$$

We have accomplished the goal of this chapter and this thesis.

A natural question to ask at this point is whether the inaccessible is *necessary* in order to prove this consistency result. It is a recent result of Shelah [8] that the inaccessible is indeed necessary. A simpler treatment of Shelah's result can be found in Raisonier [7].

Bibliography

- [1] Bell, J.L. Boolean-Valued Models and Independence Proofs in Set Theory. Oxford: Clarendon Press, 1985.
- [2] Burgess, J.P. "Forcing," in The Handbook of Mathematical Logic, J. Barwise, Editor. Amsterdam: North-Holland, 1977.
- [3] Halmos, P.R. Lectures on Boolean Algebras. Princeton: Van Nostrand, 1963.
- [4] Jech, T. Set Theory. New York: Academic Press, 1978.
- [5] Martin, D.A. "Descriptive Set Theory: projective sets," in The Handbook of Mathematical Logic. J. Barwise, Editor. Amsterdam: North-Holland, 1977.
- [6] Moschovakis, Y.N. Descriptive Set Theory. Amsterdam: North-Holland, 1980.
- [7] Raisonnier, J. "A Mathematical Proof of S. Shelah's Theorem on the Measure Problem and Related Results," Israel Journal of Mathematics, vol. 48, pp. 48-56, 1984.
- [8] Shelah, S. "Can You Take Solovay's Inaccessible Away?" Israel Journal of Mathematics, vol. 48, pp. 1-47, 1984.
- [9] Solovay, R.M. "A Model of Set Theory in which Every Set of Reals is Lebesgue Measurable," Annals of Mathematics, vol. 92, pp. 1-56, 1970.