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On the Integer-antimagic Spectra of Non-Hamiltonian Graphs

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Cover Page Footnote

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Abstract

Let A be a nontrivial abelian group. A connected simple graph $G = (V, E)$ is A-**antimagic** if there exists an edge labeling $f : E(G) \to A \setminus \{0\}$ such that the induced vertex labeling $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma \{f(u, v) : (u, v) \in E(G)\}$, is a one-to-one map. In this paper, we analyze the group-antimagic property for Cartesian products, hexagonal nets and theta graphs.

1 Introduction

In this paper, we only consider simple graphs G with p vertices and q edges. Any undefined notation used in this paper can be found in [1]. For any nontrivial abelian group A (written additively), let $A^* = A \setminus \{0\}$, where 0 is the additive identity of A. Let a map $f : E(G) \to A^*$ be an edge labeling of G. Any such edge labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \sum_{uv \in E(G)} f(uv)$. If there exists an edge labeling f whose induced map f^+ on $V(G)$ is one-to-one, we say that f is an *A-antimagic labeling* and that G is an *A-antimagic* graph. The *integer-antimagic spectrum* of a graph G is the set $\mathrm{IAM}(G) = \{k : G$ is $\mathbb{Z}_k$-antimagic and $k \geq 2\}$.

For integers $a \leq b$, let $[a, b]$ denote the set of integers from a to b, inclusive. The notations (a, b) and $[a, \infty)$ are defined in similar fashion.

A $\mathbb{Z}$-antimagic labeling f of G is also called a *weak antimagic labeling*. Such a labeling f of G is called a *graceful weak antimagic labeling* of G if the range of f is a subset of $(-p, p) \setminus \{0\}$ and the range of f^+ is $[a + 1, a + p]$ when $p \not\equiv 2 \pmod 4$ or is $[a + 1, a + p + 1] \setminus \{b\}$ when $p \equiv 2 \pmod 4$, for some $a, b \in \mathbb{Z}$ with $a + 1 < b < a + p + 1$.

The concept of the A-antimagicness property for a graph G (introduced independently in [2, 7]) naturally arises as a variation of the A-magic labeling problem (where the induced vertex labeling is a constant map). $\mathbb{Z}$-magic (or $\mathbb{Z}_1$-magic) graphs were considered by Stanley [39, 40], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. Doob [3–5] and others [11, 13, 19, 20, 31, 35] have studied A-magic graphs and $\mathbb{Z}_k$-magic graphs were investigated in [8, 10, 12, 14–18, 22, 26–28, 32, 34].

As for A-antimagic graphs (which is the focus of our paper), many different classes of graphs have been analyzed and can be found within the mathematical literature.

2 Useful Known Results and their Applications

The following lemma is found in [2].

Lemma 2.1 ([2, Lemma 1]). *For $m \geq 0$, a graph of order $4m + 2$ is not $\mathbb{Z}_{4m+2}$-antimagic.*

A graph G of order p is called *full integer-antimagic* if

$$\mathrm{IAM}(G) = \begin{cases} [p, \infty) & \text{if } p \not\equiv 2 \pmod 4, \\ [p + 1, \infty) & \text{if } p \equiv 2. \end{cases}$$

We will use the following labelings g and f for paths and cycles defined in [2], respectively. Those particular descriptions have a minor defect. So, we rewrite the labelings in the following way.

Remark 2.1. *Let $P_n = v_1v_2\cdots v_n$, and $e_1 = v_1v_2, e_2 = v_2v_3, \ldots, e_{n-1} = v_{n-1}v_n$ be its edges.*

Case 1. $n = 4m, m \geq 1$.

$$g(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m-2. \end{cases}$$

The range of g is $[1, 2m]$.

Case 2. $n = 4m + 1$ *with* $m \geq 1$.

$$g(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-3; \ (m \neq 1) \\ \frac{i+5}{2} & \text{if } i \text{ is odd and } 2m-1 \leq i \leq 4m-1. \end{cases}$$

The range of g is $[1, 2m+2]$.

Case 3. $n = 4m + 2, m \geq 1$.

$$g(e_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd;} \\ \frac{i+2}{2} & \text{if } i \text{ is even and } 2 \leq i \leq 2m-2; \ (m \neq 1) \\ \frac{i+4}{2} & \text{if } i \text{ is even and } 2m \leq i \leq 4m. \end{cases}$$

The range of g is $[1, 2m+2]$.

Case 4. $n = 4m + 3, m \geq 1$.

$$g(e_i) = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even;} \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq 2m-1; \\ \frac{i+3}{2} & \text{if } i \text{ is odd and } 2m+1 \leq i \leq 4m+1. \end{cases}$$

The range of g is $[1, 2m+2]$.

Case 5. $n = 2$. *Even though P_2 is not antimagic, we will define $g(v_1v_2) = 1$ in this paper.*

Case 6. $n = 3$. *We define $g(v_1v_2) = 1$ and $g(v_2v_3) = 2$.*

Remark 2.2. *Let $C_p = u_1u_2\cdots u_pu_1$ and $e_1 = u_1u_2$, $e_2 = u_2u_3$, $\ldots$, $e_p = u_pu_1$ be its edges.*

Case 1. $p = 4n, n \geq 1$.

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n; \\ 3 + 2(2n - \lceil \frac{i}{2} \rceil) & \text{if } 2n+1 \leq i \leq 4n. \end{cases}$$

The range of f is $[1, 2n+1]$.

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Case 2. $p = 4n+1, n \geq 1$.

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n; \\ 3 + 2(2n - \lceil \frac{i}{2} \rceil) & \text{if } 2n+1 \leq i \leq 4n+1. \end{cases}$$

The range of f *is* $[1, 2n+1]$.

Case 3. $p = 4n+2, n \geq 1$.

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n+3; \\ 3 + 2(2n - \lceil \frac{i-2}{2} \rceil) & \text{if } 2n+4 \leq i \leq 4n+2. \end{cases}$$

The range of f *is* $[1, 2n+3]$.

Case 4. $p = 4n-1, n \geq 2$.

$$f(e_i) = \begin{cases} i & \text{if } 1 \leq i \leq 2n+1; \\ 3 + 2(2n - \lceil \frac{i+1}{2} \rceil) & \text{if } 2n+2 \leq i \leq 4n-1. \end{cases}$$

The range of f *is* $[1, 2n+1]$.

Case 5. $p = 3$. *We label the edges of* C_3 *by* $1, 2$ *and* 3. *Hence, the image of* f^+ *is* $[3, 5]$ *and so* C_3 *is* $\mathbb{Z}_k$*-antimagic for* $k \geq 4$.

Let ϕ be an edge labeling of G and ϕ^+ be its induced vertex labeling. Let the image of ϕ^+ be

$$I_\phi(G) = \{\phi^+(v) \mid v \in V(G)\},$$

where G is the graph being considered.

Corollary 2.2 ([38, Corollary 1.3]). *For* $m \geq 1$ *and labeling* g *for paths provided in Remark 2.1, we have* $I_g(P_{4m}) = [1, 4m]$, $I_g(P_{4m+1}) = [2, 4m+2]$, $I_g(P_{4m+2}) = [1, 4m+3] \setminus \{2\}$, $I_g(P_{4m-1}) = [1, 4m-1]$.

Corollary 2.3 ([38, Corollary 1.4]). *For* $n \geq 1$ *and labeling* f *for cycles provided in Remark 2.2, we have* $I_f(C_{4n-1}) = [3, 4n+1]$, $I_f(C_{4n}) = [3, 4n+2]$, $I_f(C_{4n+1}) = [2, 4n+2]$ *and* $I_f(C_{4n+2}) = [3, 4n+5] \setminus \{4n+2\}$.

The following lemma is obvious.

Lemma 2.4. *Let* G *be a graph of order* p, *where* $p \geq 3$. *Suppose there is a graceful weak antimagic labeling of* G. *Then,* G *is full integer-antimagic.*

By Lemma 2.4, we restate some theorems.

Theorem 2.5 ([2, Theorem 4]). P_m, $m \geq 3$, *and* C_n, $n \geq 4$, *are full integer-antimagic. Moreover,* $\mathrm{IAM}(C_3) = [4, \infty)$.

For $S \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, we let the set $a + S = \{a + s \mid s \in S\}$.

Lemma 2.6 ([37, Lemma 3.1])**.** *Suppose $n \geq 3$ and let $g : E(C_n) \to \mathbb{Z}$ be a labeling and $c \in \mathbb{Z}$. Then, there is a labeling h such that $I_h(C_n) = 2c + I_g(C_n)$. Note that the range of h is a subset of $[c+1, c+\lfloor n/2 \rfloor + 2]$.*

Lemma 2.7 ([37, Lemma 3.2])**.** *Suppose $n \geq 2$ and let $g : E(C_{2n}) \to \mathbb{Z}$, $c \in \mathbb{Z}$. Then, there is a labeling h such that $I_h(C_{2n}) = c + I_g(C_{2n})$. Note that the range of h is a subset of $[1, n+2] \cup [c+2, c+n+1]$.*

Theorem 2.8 ([30, Theorem 6.2])**.** *Suppose G is a disjoint union of cycles, where $|V(G)| \geq 4$. Then, G is full integer-antimagic.*

Note that the labelings involving in the proof of Theorem 2.8 are graceful weak antimagic.

Suppose G and H are two disjoint graphs. Let $G+H$ denote the disjoint union of G and H.

By the labeling method provided in the proof of [30, Lemma 3.1], we have

Corollary 2.9. *Suppose $G = \sum_{i=1}^{2n} C_{2k_i+1}$ of order N. Then, there is a labeling ϕ such that the range of ϕ is a subset of $[1, N/2+1]$ and $I_\phi(G) = [2, N+1]$.*

Corollary 2.10. *Any 2-connected cubic graph of order p is full integer-antimagic.*

Proof. Any 2-connected cubic graph is a union of disjoint cycles with a perfect matching. By Theorem 2.8, we obtain a graceful weak antimagic of union of disjoint cycles and label all edges of the perfect matching with 1. By Lemma 2.4, we conclude that every 2-connected cubic graph is full integer-antimagic. □

Theorem 2.11 ([23, Theorem 6.1])**.** *Any Hamiltonian graph of order $p \geq 4$ is full integer-antimagic.*

Let $P_n = v_1 v_2 \cdots v_n$ be the path of order n. The k-th power of P_n, denoted by P_n^k, is the path with the same vertex set of P_n and the edge set is $\{v_i v_j \mid 1 \leq j - i \leq k\}$, where $n > k \geq 2$. Note that, $P_n^{n-1} = K_n$, the complete graph of order n.

Since P_n^2 is Hamiltonian, P_n^k is also Hamiltonian. By Theorem 2.11, P_n^k is full integer-antimagic.

Example 2.1. *Consider P_{10}^3, where $P_{10} = v_1 v_2 \cdots v_{10}$. Label all edges $v_i v_j$ by 1 if $j - i = 3$ and by -1 if $j - i = 2$. Then the edges of the path P_{10} are labeled by $1, 1, 3, 1, 4, 3, 5, 5, 6$ in the natural order. It is easy to check that this labeling is a graceful weak antimagic labeling of P_{10}^3. Hence, P_{10}^3 is full integer-antimagic.*

Since $P_m \times P_n$ for even mn, $P_m \times C_n$ for $m \geq 2, n \geq 3$ and $C_m \times C_n$ for $m, n \geq 3$ are Hamiltonian, they are full integer-antimagic. In the following sections, we focus on non-Hamiltonian graphs. For example, $P_m \times P_n$, where m, n are odd, is such a class of graphs.

3 Cartesian Product Graphs

Let G and H be graphs of orders m and n respectively, where $m, n \geq 3$. Consider the Cartesian product graph $G \times H$. Each copy of G and H is called a *horizontal* graph and a *vertical* graph, respectively.

Lemma 3.1. *Let G and H be graphs of orders m and n respectively, where $m, n \geq 3$. Suppose there are two labelings g and h of G and H, respectively, satisfying the following conditions:*

1. *$g : E(G) \to (-m, m) \setminus \{0\}$ such that $I_g(G)$ is a set of m consecutive integers.*

2. *$h : E(H) \to (-n, n) \setminus \{0\}$ such that $I_h(H)$ is a set of n consecutive integers.*

Then, there is a graceful weak antimagic labeling for $G \times H$. Hence, $G \times H$ is full integer-antimagic.

Proof. Let $I_g(G) = [a+1, a+m]$ and $I_h(H) = [b+1, b+n]$, for some $a, b \in \mathbb{Z}$. Let $\phi = mh$. Define a labeling $F : E(G \times H) \to \mathbb{Z}$ by $F(e) = g(e)$ if e is an edge of a horizontal graph, and $F(e) = \phi(e) = mh(e)$ if e is an edge of a vertical graph.

Then,

$$F^+(u, v) = g^+(u) + \phi^+(v) = g^+(u) + mh^+(v),$$

where$(u, v) \in V(G \times H)$.

Clearly, the range of F is a subset of $(-mn, mn) \setminus \{0\}$ and

$$I_F(G \times H) \subseteq [(a+1) + m(b+1), (a+m) + m(b+n)] = [mb + a + m + 1, mb + a + m + mn].$$

Suppose $F^+(u, v) = F^+(x, y)$ for some vertices $(u, v), (x, y) \in V(G \times H)$. Then, $g^+(u) - g^+(x) = m(h^+(y) - h^+(v))$. Since $m > |g^+(u) - g^+(x)| = m|(h^+(y) - h^+(v))|$, we have that $|(h^+(y) - h^+(v))| < 1$. Thus, $h^+(y) - h^+(v) = 0$. Hence, $g^+(u) - g^+(x) = 0$. Since g^+ and h^+ are injections, F^+ is an injection. Thus, $I_F(G \times H) = [mb + a + m + 1, mb + a + m + mn]$. By Lemma 2.4, the result is established. □

There are several known graphs having labelings which satisfy the conditions of Lemma 3.1. Examples of this include complete graphs, complete bipartite graph except stars, tadpole and lollipop graphs, fans, wheels, and dumbbell graphs. (see [2, 29, 30, 36–38]). So, there are many Cartesian product graphs with order $n \geq 3$ and $n \not\equiv 2 \pmod 4$ which are full integer-antimagic.

As a particular example, we consider paths and cycles. Combining Corollaries 2.2, 2.3 and Lemma 3.1, we have

Theorem 3.2. *Suppose $m, n \geq 3$ and $m, n \not\equiv 2 \pmod 4$. Then,*

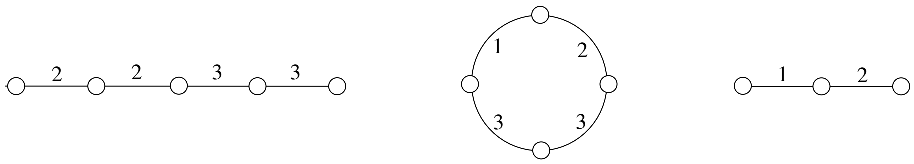
1. *$P_m \times P_n$ admits a graceful weak antimagic labeling and hence, it is full integer-antimagic.*

2. *$P_m \times C_n$ admits a graceful weak antimagic labeling and hence, it is full integer-antimagic, for $n \geq 4$.*

3. *$C_m \times C_n$ admits a graceful weak antimagic labeling and hence, it is full integer-antimagic, for $m, n \geq 4$.*

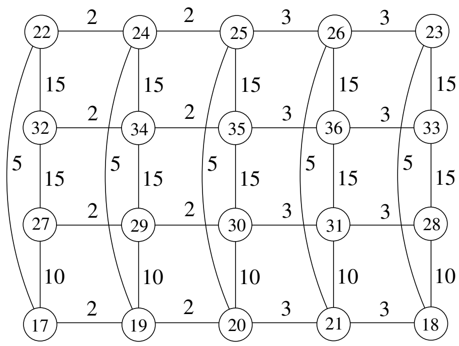
Remark 3.1. *Even if $P_m \times P_n$ (m even), $P_m \times C_n$ and $C_m \times C_n$ are Hamiltonian, Theorem 3.2 provides an explicit labeling for each of grid, cylinder and torus, for some suitable m and n.*

Example 3.1. *The labelings for P_5 (not the same labeling defined in Remark 2.1), C_4 and for P_3 are*

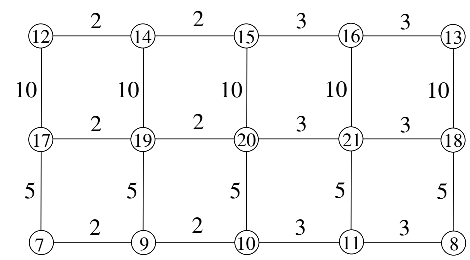


Labelings for P_5, C_4 and P_3, respectively.

Then, we have



A labeling F for $P_5 \times C_4$, with $I_F = [17, 36]$.



A labeling F for $P_5 \times P_3$, with $I_F = [7, 21]$.

4 Hexagonal Nets

Let $n \geq 2$ and H_n be an *n-hexagonal net*, which is a hexagonal system consisting of one central hexagon and is surrounded by $n-1$ layers of hexagonal cells. We call the central hexagon the 1-st layer, the 6 hexagons surrounding the central hexagon form the 2-nd layer and is called the 2-nd hexagonal ring, and so on. Precisely, the m-th layer consists of $6(m-1)$ hexagons and is called the m-th *hexagonal ring*, where $2 \leq m \leq n$.

Note that H_n is a molecular graph, corresponding to benzene ($n = 1$), coronene ($n = 2$), circumcoronene ($n = 3$), circum-circumcoronene ($n = 4$), etc. (see Figure 1).

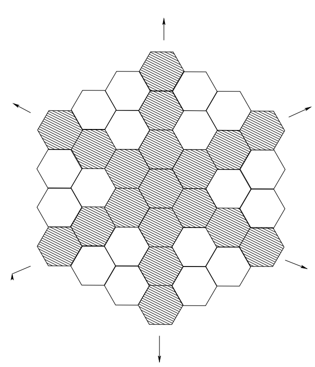


Figure 1: H_4 (circum-circumcoronene).

Observe the following structures of H_n. The outer cycle of the m-th hexagonal ring is a $6(2m-1)$-cycle and is denoted by O_m, $1 \le m \le n$. Hence, the order of H_n is $6n^2$. Moreover in O_n, there are $6n$ vertices of degree 2 and $6(n-1)$ vertices of degree 3. Other vertices of H_n are of degree 3. Each vertex in O_m is incident with exactly one edge whose other end lies in $V(O_{m-1}) \cup V(O_{m+1})$, $1 \le m \le n-1$. These edges are called *spokes*.

Let

$$O_n = z_{1,0}y_{1,1}z_{1,1}\cdots y_{1,n-1}z_{1,n-1}z_{2,0}y_{2,1}z_{2,1}\cdots y_{2,n-1}z_{2,n-1}\cdots z_{6,0}y_{6,1}z_{6,1}\cdots y_{6,n-1}z_{6,n-1}z_{1,0}$$

such that $\deg(z_{i,l}) = 2$ and $\deg(y_{i,j}) = 3$, where $1 \le i \le 6$, $0 \le l \le n-1$ and $1 \le j \le n-1$. For convenience, we let $z_{7,0} = z_{1,0}$.

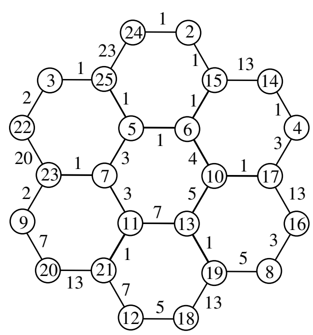
When $n \ge 2$, if we delete all $y_{i,j}$'s ($1 \le i \le 6$, $1 \le j \le n-1$) from H_n, then the resulting graph contains $6n-5$ components. So, H_n is non-Hamiltonian (see [1, Theorem 18.1]).

Let us try to find a graceful weak antimagic labeling for H_n.

Suppose G_1 and G_2 are edge-disjoint graphs with labelings g_1 and g_2. We use $g_1 \cup g_2 = g_2 \cup g_1$ to denote the combined labeling for $G_1 \cup G_2$. That is, $(g_1 \cup g_2)(e) = g_i(e)$ if $e \in E(G_i)$, $i = 1, 2$.

Theorem 4.1. *Let $n \ge 1$. Then, there is a graceful weak antimagic labeling for H_n. Hence, it is full integer-antimagic.*

Proof. This is a known result for $H_1 = C_6$. The following is a graceful weak antimagic labeling for H_2:



So, we assume $n \ge 3$. For each case, we label all spokes with 1 and denote this labeling by α.

(1) Suppose $n = 2k+1$, $k \ge 1$.

Step 1: We label the subgraph $G_1 = H_n - E(O_n)$. By Corollary 2.9, there is a labeling η of $\mathcal{O} = \sum_{i=1}^{2k} O_i$ such that $I_\eta(\mathcal{O}) = [2, |V(\mathcal{O})|+1]$. So, $I_{\alpha\cup\eta}(\mathcal{O}) = [3, |V(\mathcal{O})|+2]$.

Step 2: We define a labeling ϕ for O_n. We make use of the labeling f for $C_6 = z_1z_2z_3z_4z_5z_6z_1$, defined in Remark 2.2. Namely, the edges of C_6 are labeled by 1,2,3,4,5,3 in natural order. Hence, $I_f(C_6) = [3, 9] \setminus \{6\}$. For convenience, we let $z_7 = z_1$.

Let $\phi(z_{i,l}y_{i,l+1}) = \phi(z_{i,n-1}z_{i+1,0}) = f(z_i)$, $1 \le i \le 6$ and $0 \le l \le n-2$. Then the induced labels for $z_{1,0}$, $z_{2,0}$, $z_{3,0}$, $z_{4,0}$, $z_{5,0}$ and $z_{6,0}$ are 4, 3, 5, 7, 9, 8, respectively. Only (disjoint) edges $y_{1,1}z_{1,1}$, $y_{1,2}z_{1,2}$, $\ldots$, $y_{1,n-1}z_{1,n-1}$, $y_{3,1}z_{3,1}$, $y_{3,2}z_{3,2}$, $\ldots$, $y_{3,n-1}z_{3,n-1}$, $y_{5,1}z_{5,1}$, $y_{5,2}z_{5,2}$, $\ldots$, $y_{5,n-1}z_{5,n-1}$ have not been labeled. When we label the edge $y_{i,l}z_{i,l}$, the induced labels of $y_{i,l}$ and $z_{i,l}$ will be the same under ϕ.

But the induced label of $y_{i,l}$ is greater than that of $z_{i,l}$ by 1 under $\alpha\cup\phi$. Thus, we may choose the labels for all $y_{i,l}z_{i,l}$'s such that $I_{\alpha\cup\phi}(O_n) = [3, 6(2n-1)+3]\setminus\{6\}$ (namely, the set of induced vertex labels of $\{y_{i,l} \mid 1 \le i \le 6, 1 \le l \le n-1\} \cup \{z_{i,l} \mid 1 \le i \le 6, 1 \le l \le n-1\}$ is $[10, 6(2n-1)+3]$ under $\alpha\cup\phi$). Currently, the maximum edge label of O_n is less than $6(2n-1)+3 < 3n^2+6n-3 = |V(H_n)| - |V(\mathcal{O})|/2$. Now, we add the label of each edge of O_n by $|V(\mathcal{O})|/2$. Let this new labeling still be denoted by ϕ.

Clearly, the labels assigned to all edges of H_n are less than $|V(H_n)|$. Hence, $\alpha\cup\eta\cup\phi$ is a graceful weak antimagic of H_n.

(2) Suppose $n = 2k$, $k \ge 2$.

Step 1: We label the subgraph $G_1 = (H_n - O_n) - E(O_{n-1})$. Similar to Case 1, there is a labeling η of $\mathcal{O} = \sum_{i=1}^{2k-2} O_i$ such that $I_\eta(\mathcal{O}) = [2, |V(\mathcal{O})|+1]$. So, $I_{\eta\cup\alpha}[3, |V(\mathcal{O})|+2]$.

Step 2: By Corollary 2.9, there is a labeling $\eta_1\cup\eta_2$ of $C_6 + C_{6(2n-3)}$ such that $I_{\eta_1\cup\eta_2}(C_6 + C_{6(2n-3)}) = [2, 12n-11]$. Namely, the edges of $C_6 = z_1z_2z_3z_4z_5z_6z_1$ are labeled with 1,3,5,7,2,1 in the natural order. Denote this labeling by η_1.

We shall define a labeling ϕ for $O_n = C_{6(2n-1)}$. We keep the labeling η_2 for $O_{n-1} = C_{6(2n-3)}$ and make use of the labeling η_1 of C_6 to define the labeling ϕ. When we combine the labeling α, the induced vertex labels of O_{n-1} will increase by 1, but some of labels of vertices in O_n will not, namely $I_{\alpha\cup\eta_2} = [6, 12n-10]\setminus\{9, 10, 13\}$. So, we have to increase the labels of vertices of C_6 by 1. That is, increase the 1-st, 3-rd and 5-th edges of C_6 by 1. So the new labeling ψ of C_6 is $2, 3, 6, 7, 3, 1$. The induced labels for z_1, z_2, z_3, z_4, z_5 and z_6 under ψ are 3, 5, 9, 13, 10, 4, respectively.

Similar to Case 1, let $\phi(z_{i,l}y_{i,l+1}) = \phi(z_{i,n-1}z_{i+1,0}) = \psi(z_i)$, $1 \le i \le 6$ and $0 \le l \le n-2$. Then the induced labels for $z_{1,0}$, $z_{2,0}$, $z_{3,0}$, $z_{4,0}$, $z_{5,0}$ and $z_{6,0}$ under ϕ are 3, 5, 9, 13, 10, 4, respectively.

Similar to Case 1, we choose the labels of all $y_{i,l}z_{i,l}$'s such that the set of induced vertex labels of $V(O_{n-1}+O_n)$ is $[3, 24n-22]$ under $\alpha\cup\phi\cup\eta_2$. That is, the set of induced vertex labels of $\{y_{i,l} \mid 1 \le i \le 6, 1 \le l \le n-1\} \cup \{z_{i,l} \mid 1 \le i \le 6, 1 \le l \le n-1\}$ is $[12n-9, 24n-22]$ under $\alpha\cup\phi\cup\eta_2$.

Add the label of each edge of $O_{n-1}+O_n$ by $|V(\mathcal{O})|/2$ and denote this new labeling by λ. Then the set of induced vertex labels of $V(O_{n-1}+O_n)$ is $[3+|V(\mathcal{O})|, |V(\mathcal{O})|+|V(O_{n-1}+O_n)|+2] = [3+|V(\mathcal{O})|, |V(H_n)|+2]$ under $\alpha\cup\lambda$.

Clearly, the labels assigned to all edges of H_n are less than $|V(H_n)|$. Hence, $\alpha\cup\eta\cup\lambda$ is a graceful weak antimagic of H_n.

□

Example 4.1. *Consider H_3. Now $O_1 = C_6$, $O_2 = C_{18}$ and $O_3 = C_{30}$.*

By the labeling method provided in the proof of [30, Lemma 3.1], we have a labeling η for C_6+C_{18} such that $I_\eta(C_6+C_{18}) = [2, 25]$. Namely, the edges of C_6 are labeled by $1, 3, 5, 7, 2, 1$ in *the natural order; the edges of* C_{18} *are labeled with* $2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 11, 11, 9, 9, 7, 7, 4, 3$ *in the natural order. Note that* $I_{\eta\cup\alpha}(C_6 + C_{18}) = [3, 26]$*, where* α *is the labeling defined in the proof of Theorem 4.1.*

Now, we define the labeling for $O_3 = C_{30}$*, as in the proof of Theorem 4.1. Label the edges of* $C_{30} = z_{1,0}y_{1,1}z_{1,1} \cdots y_{6,2}z_{6,2}z_{1,0}$ *with*

$$1, 9, 1, 11, 1, 2, 12, 2, 14, 2, 3, 15, 3, 17, 3, 4, 18, 4, 20, 4, 5, 21, 5, 23, 5, 3, 27, 3, 29, 3.$$

Then, add the label of each edge of C_{30} *by* 12*. Namely, the edges of* C_{30} *are labeled with*

$$13, 21, 13, 23, 13, 14, 24, 14, 26, 14, 15, 27, 15, 29, 15, 16, 30, 16, 32, 16, 17, 33, 17, 35, 17, 15, 39, 15, 41, 15.$$

This labeling is denoted by ϕ*. So,* $I_{\alpha\cup\phi}(C_{30})$ *is* $[27, 57] \setminus \{30\}$*.*

We have $I_{\alpha\cup\eta\cup\phi}(H_3) = [3, 57] \setminus \{30\}$*. So,* $\alpha \cup \eta \cup \phi$ *is a graceful weak antimagic labeling for* H_3 *and hence,* H_3 *is full integer-antimagic.*

Example 4.2. *Consider* H_4*. Now* $O_1 = C_6$*,* $O_2 = C_{18}$*,* $O_3 = C_{30}$ *and* $O_4 = C_{42}$*. The labeling* η *of* $C_6 + C_{18}$ *is the same as Example 4.1. So,* $I_{\eta\cup\alpha}(C_6 + C_{18}) = [3, 26]$*, where* α *is the labeling defined in the proof of Theorem 4.1.*

The labeling ψ *of* C_6 *is* $2, 3, 6, 7, 3, 1$*. The induced labels for* z_1*,* z_2*,* z_3*,* z_4*,* z_5 *and* z_6 *under* ψ *are* $3, 5, 9, 13, 10, 4$*, respectively.*

The labeling η_2 *for the edges of* C_{30} *is*

$$2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 17, 17, 15, 15, 13, 13, 11, 11, 9, 9, 7, 7, 4, 3.$$

It is straightforward to check that $I_{\eta_2\cup\alpha}(C_{30}) = [6, 38] \setminus \{9, 10, 13\}$*.*

The labeling ϕ *for the edges of* $O_4 = C_{42} = z_{1,0}y_{1,1}z_{1,1} \cdots y_{6,3}z_{6,3}z_{1,0}$ *is*

$$2, 37, 2, 39, 2, 41, 2, 3, 42, 3, 44, 3, 46, 3, 6, 45, 6, 47, 6, 49, 6, 7, 50, 7, 52, 7, 54, 7, 3, 60, 3, 62, 3, 64, 3, 1, 68, 1, 70, 1, 72, 1.$$

Again, it is straightforward to check that $I_{\phi\cup\alpha}(C_{42}) = [39, 74] \cup \{3, 4, 5, 9, 10, 13\}$*.*

After adding each edge of $O_3 + O_4$ *by* 12 *to obtain the new labeling* λ *for* $O_3 \cup O_4$*, we have* $I_{\eta\cup\lambda\cup\alpha}(H_4) = [3, 98]$*. So,* $\alpha \cup \eta \cup \lambda$ *is a graceful weak antimagic labeling for* H_4 *and hence,* H_4 *is full integer-antimagic.*

5 Theta Graphs

For $r, s, t \geq 2$, let $G_1 = u_0 \cdots u_{r-1}$, $G_2 = v_0 \cdots v_{s-1}$ and $G_3 = w_0 \cdots w_{t-1}$ be three disjoint paths of order r, s and t, respectively.

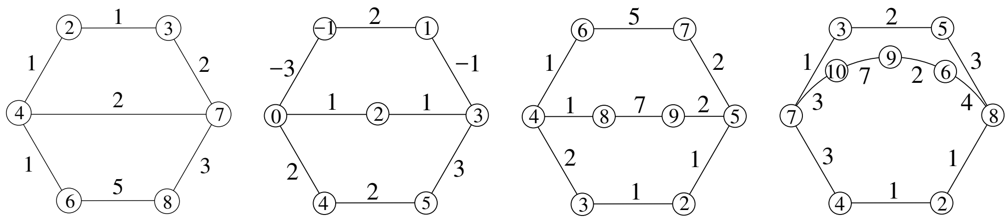
The *theta* graph $\Theta(r, s, t)$ is obtained from $G_1 + G_2 + G_3$ by identifying u_0, v_0, w_0 into a vertex and $u_{r-1}, v_{s-1}, w_{t-1}$ into another vertex. Note that the order of $\Theta(r, s, t)$ is $r+s+t-4$.

If at least two of r, s, t are 2, then $\Theta(r, s, t)$ is not simple, which is not considered in this paper. When one of r, s, t is 2, say $s = 2$, then $\Theta(r, 2, t)$ is an $(r + t)$-cycle with a chord. In [21, Lemma 2.2 and Theorem 2.3], graceful weak antimagic labelings of $\Theta(r, 2, t)$ are provided. However, some of these labelings cannot be extended to graceful weak antimagic labelings of $\Theta(r, s, t)$. We will make some modifications, as illustrated in Theorem 5.6.

In this section, we establish Theorem 5.1.

Theorem 5.1. *The theta graph* $\Theta(r, s, t)$ *admits a graceful weak antimagic labeling for* $r, t \geq 3$ *and* $s \geq 2$*. Hence,* $\Theta(r, s, t)$ *is full integer-antimagic.*

Example 5.1. *Some graceful weak antimagic labelings for theta graphs of small orders.*


$\Theta(4,2,4)$ $\Theta(4,3,4)$ $\Theta(4,4,4)$ $\Theta(4,5,4)$

Before considering the integer-antimagicness of theta graphs, we introduce a useful technique (see Lemma 5.2 and Corollary 5.3).

Lemma 5.2. *Let G be a graph of order p. Suppose there is a graceful weak antimagic labeling σ of G. Let*

1. *uw be an edge of G such that $\sigma(uw) = a$;*

2. *M be the maximum value of σ^+;*

3. *H be a graph obtained from G by replacing the edge uw by a $(4m+2)$-path $P = uv_1v_2\cdots v_{4m}w$, $m \geq 1$.*

Then, there is a labeling ϕ of a graph H such that $I_\phi(H) = I_\sigma(G) \cup [M+1, M+4m]$. Hence, ϕ ia a graceful weak antimagic labeling of H if the range of ϕ is a subset of $(-|V(H)|, |V(H)|) \setminus \{0\}$.

Proof. From the assumption, $I_\sigma(G) = [M-p+1, M]$ if $p \not\equiv 2 \pmod 4$ and $I_\sigma(G) = [M-p, M] \setminus \{b\}$ for some $b < M$ if $p \equiv 2 \pmod 4$, $|V(H)| = p + 4m$.

Let g be the graceful weak antimagic labeling of $Q = v_1v_2\cdots v_{4m}$. From Corollary 2.2, we know $I_g(Q) = [1, 4m]$. Now we extend the labeling g to P by setting $g(uv_1) = 0 = g(v_{4m}w)$. Then, add a and $M-a$ (in alternating fashion) to the labels of each edge of P, starting from uv_1. Let this new labeling of P be $\hat{g}$. Then, $\{\hat{g}^+(v_i) \mid 1 \leq i \leq 4m\} = [M+1, M+4m]$.

Combining σ and $\hat{g}$, we have a labeling ϕ ($\phi = \sigma \cup \hat{g}$) of H such that $I_\phi(H) = I_\sigma(G) \cup [M+1, M+4m]$. □

By a similar argument above, we have

Corollary 5.3. *Let G be a graph of order $p \equiv 0 \pmod 4$. Suppose there is a graceful weak antimagic labeling σ of G. Let*

1. *uw be an edge of G such that $\sigma(uw) = a$;*

2. *M be the maximum value of σ^+;*

3. *H be a graph obtained from G by replacing the edge uw by a $(4m+4)$-path*

$$P = uv_1v_2\cdots v_{4m+2}w, \quad m \geq 1.$$

Then, there is a labeling ϕ of a graph H such that $I_\phi(H) = I_\sigma(G) \cup [M+1, M+4m+3] \setminus \{b\}$, for some b. Hence, ϕ ia a graceful weak antimagic labeling of H if the range of ϕ is a subset of $(-|V(H)|, |V(H)|) \setminus \{0\}$.

Remark 5.1. *In the proof of Lemma 5.2, the maximum value of g is $2m$, which is the label of the last two edges $v_{4m-2}v_{4m-1}$ and $v_{4m-1}v_{4m}$. So the maximum value of $\hat{g}$ is $\max\{M-a+2m, a+2m\}$.*

In the proof of Corollary 5.3, the maximum value of g is $2m+2$ only labels the second to last edge $v_{4m}v_{4m+1}$. Moreover, the last edge $v_{4m+1}v_{4m+2}$ is $2m+1$. So the maximum value of $\hat{g}$ is $\max\{M-a+2m+1, a+2m+2\}$.

We now examine the integer-antimagicness of theta graphs.

Lemma 5.4. *For $s \geq 2$, there is a graceful weak antimagic labeling for $\Theta(4, s, 4)$.*

Proof. From Example 5.1, there is a graceful weak antimagic labeling ϕ for each $\Theta(4, i, 4)$, $2 \leq i \leq 5$. The maximum value M of ϕ^+ is 8, 5, 9 and 10, for $i = 2, 3, 4, 5$, respectively. We may choose the label of an edge e in each G_2 so that $a = 2, 1, 2, 4$. Replacing e by a $(4m+2)$-path, we obtained a labeling for $\Theta(4, 4m+i, 4)$, $2 \leq i \leq 5$, $m \geq 1$. All labels for $\Theta(4, 4m+i, 4)$ are positive and the maximum label is $6+2m$, $4+2m$, $7+2m$ and $6+2m$, for $2 \leq i \leq 5$, $m \geq 1$, respectively. □

Theorem 5.5. *For $s \geq 2$, $r \equiv 0 \pmod 4$ and $t \equiv 0 \pmod 4$, there is a graceful weak antimagic labeling for $\Theta(r, s, t)$.*

Proof. Let $r = 4k$ and $t = 4n$, where $k, n \geq 1$. We may replace the edge of G_1 by a $(4k-2)$-path with and G_3 $(4n-2)$-path in $\Theta(4, 4m+i, 4)$ labeled as in the proof of Lemma 5.4. All labels for $\Theta(4k, 4m+i, 4n)$ are positive and the maximum label is $6+2m+(-1+2k-2)+(-5+2n-2)$, $4+2m+(-2+2k-2)+(-2+2n-2)$, $7+2m+(-5+2k-2)+(-1+2n-2)$ and $6+2m+(-2+2k-2)+(-1+2n-2)$, for $i = 2, 3, 4, 5$ and $m \geq 1$, respectively. For $m = 0$ and $i = 5$, the maximum label of $\Theta(4k, 5, 4n)$ is $7+(-2+2k-2)+(-1+2n-2)$. Clearly the maximum label is less than $4(k+m+k)+i-4 = |V(\Theta(4k, 4m+i, 4n))|$ for each $1 \leq i \leq 4$ and $m \geq 1$. By Lemma 5.2, the claim is established. □

Suppose that $C_k = (u_1, u_2, \dots, u_k)$, where $k \geq 4$. Let $2 \leq l \leq \lfloor k/2 \rfloor$. Define $C_k(l)$ to be the graph obtained from C_k by adding an edge $c = u_iu_j$, where $l = \min\{|i-j|, k-|i-j|\}$. We call $C_k(l)$ an *n-cycle with a chord c of perimeter l*. In other words, the shortest cycle in $C_k(l)$ is an $(l+1)$-cycle.

Theorem 5.6. *For $k \geq 4$ and $2 \leq l \leq \lfloor k/2 \rfloor$, $C_k(l)$ admits a graceful weak antimagic labeling. Moreover, the chord is labeled with 1.*

Proof. Let f be the labeling defined in Remark 2.2. We modify the labeling f of C_{4n+2} for further use in the proof of Theorem 5.8. The new labeling is to add the label of each edge by 0 and -1 on the original labeling (in alternating fashion), starting from u_1u_2. We denote this labeling by f'. Here, $I_{f'}(C_{4n+2}) = [2, 4n+4] \setminus \{4n+1\}$. Now, $f'(u_1u_2) = f'(u_2u_3) = 1$ and the other labels of edges are greater than 1.

(1) Suppose there is an even subcycle H of $C_k(l)$ containing the chord c, say. Let H be an r-cycle, where r is even. Note that H may not be the shortest cycle. By renaming the vertices of the cycle C_k, we may assume that $H = (u_{r+2}, u_3, u_4, \dots, u_{r+1})$ and $c = u_{r+2}u_3$.

    Label the edges of H by 1 and -1 in alternating fashion, starting from c. The remaining edges of $C_k(l)$ are labeled with 0. Let this labeling be η. Hence, η^+ is a zero mapping. Then, the labeling $\sigma = f + \eta$ (or $f' + \eta$) is a graceful weak antimagic labeling of $C_k(l)$.

(2) Suppose there is no even subcycle of $C_k(l)$. In this case, k must be even. Let H be an r-cycle containing the chord c of the shortest length, where r is odd. In fact, $r = l + 1$. There are two subcases to consider.

(a) Suppose $k = 4n$. Note that $f(u_{2n}u_{2n+1}) = 2n$, $f(u_{2n+1}u_{2n+2}) = 2n + 1$, and $f(u_{2n+2}u_{2n+3}) = 2n + 1$. Thus, $f^+(u_{2n+1}) = 4n + 1$ and $f^+(u_{2n+2}) = 4n + 2$. Moreover, only the edge u_1u_2 in C_{4n} is labeled with 1. After renaming the vertices, we may assume that $H = (u_{2n+1}, u_{2n+2}, \ldots, u_{2n+l+1})$, where $c = u_{2n+1}u_{2n+l+1}$, $2 \le l \le 2n$. By convention, $u_{4n+1} = u_1$. Label the edges of the path

$$H - u_{2n+1}u_{2n+2} = u_{2n+1}u_{2n+l+1}u_{2n+l} \cdots u_{2n+3}u_{2n+2}$$

with 1 and -1 in alternating fashion, starting from c. The remaining edges of $C_k(l)$ are labeled with 0. Denote this labeling by η. Then, $\eta^+(u_{2n+1}) = 1$, $\eta^+(u_{2n+2}) = -1$ and $\eta^+(u_j) = 0$ otherwise. Then, the labeling $\sigma = f + \eta$ is a graceful weak antimagic labeling of $C_k(l)$.

(b) Suppose $k = 4n + 2$. Note that $f(u_{2n+2}u_{2n+3}) = 2n + 2$, $f(u_{2n+3}u_{2n+4}) = 2n + 3$, $f(u_{2n+4}u_{2n+5}) = 2n+1$. Thus, $f^+(u_{2n+3}) = 4n+5$ and $f^+(u_{2n+4}) = 4n+4$. Moreover, only the edge u_1u_2 in C_{4n+2} is labeled by 1. After renaming the vertices, we may assume that $H = (u_{2n+4-l}, u_{2n+5-l}, \ldots, u_{2n+3}, u_{2n+4})$, where $c = u_{2n+4}u_{2n+4-l}$, $2 \le l \le 2n$. Label the edges of the path $H - u_{2n+3}u_{2n+4} = u_{2n+4}u_{2n+4-l}u_{2n+5-l} \cdots u_{2n+2}u_{2n+3}$ with 1 and -1 in alternating fashion, starting from c. The remaining edges of $C_k(l)$ are labeled with 0. Denote this labeling by η. Then, $\eta^+(u_{2n+4}) = 1$, $\eta^+(u_{2n+3}) = -1$ and $\eta^+(u_j) = 0$ otherwise. Then, the labeling $\sigma = f + \eta$ (or $f' + \eta$) is a graceful weak antimagic labeling of $C_k(l)$.

□

Remark 5.2. *From the proof above, we see that $I_\sigma(C_k(l)) = I_f(C_k)$ (or $I_{f'}(C_k)$). So, the maximum value of σ^+ is at most $k + 2$. From Remark 2.2, we have the following.*

(1) *If $k = 4m$, then the range of σ is a subset of $[1, 2m + 2]$.*

(2) *If $k = 4m + 2$, then the range of σ is a subset of $[1, 2m + 3]$.*

(3) *If $k = 4m + 1$, then the range of σ is a subset of $[1, 2m + 2]$.*

(4) *If $k = 4m - 1$, then the range of σ is a subset of $[1, 2m + 2]$.*

Corollary 5.7. *There is a graceful weak antimagic labeling for $\Theta(r, 3, t)$, if $r+t \not\equiv 0 \pmod 4$.*

Proof. Since $\Theta(r, 2, t)$ is an $(r + t - 2)$-cycle with a chord, by Theorem 5.6 there is a graceful weak antimagic labeling σ such that chord is labeled with 1. From Remark 5.2 and Corollary 2.3, we know that $I_\sigma(\Theta(r, 2, t)) = [3, r + t]$ if $r + t - 2 \equiv 0, 3 \pmod 4$ and $I_\sigma(\Theta(r, 2, t)) = [2, r + t - 1]$ if $r + t - 2 \equiv 1 \pmod 4$.

Suppose $r + t \equiv 2, 1 \pmod 4$. We add an extra vertex v into the chord so that the chord becomes a 3-path and the graph becomes $\Theta(r, 3, t)$. We label the edges of this 3-path with

1. As a result, the induced label of v is 2. Since $r + t - 2 \equiv 0, 3 \pmod 4$, the new labeling of $\Theta(r, 3, t)$ is a graceful weak antimagic labeling.

Suppose $r + t \equiv 3 \pmod 4$. For each non-chord edge label, we add 1 to the label in $\Theta(r, 2, t)$. Let this new labeling be $\hat{\sigma}$. Clearly $\hat{\sigma}$ is a graceful weak antimagic labeling for $\Theta(r, 2, t)$ and $I_{\hat{\sigma}}(\Theta(r, 2, t) = [4, r + t + 1]$. As in the previous case, we add an extra vertex v into the chord and label the new edges with 1. Let this new labeling be ϕ. Then, $I_\phi(\Theta(r, 3, t)) = [2, r+t+1] \setminus \{3\}$. Since $|V(\Theta(r, 3, t)| = r+t-1 \equiv 2 \pmod 4$, ϕ is a graceful weak antimagic labeling of $\Theta(r, 3, t)$. □

Theorem 5.8. *Let $s = 4m + 2 \geq 6$. Then, $\Theta(r, s, t)$ admits a graceful weak antimagic labeling.*

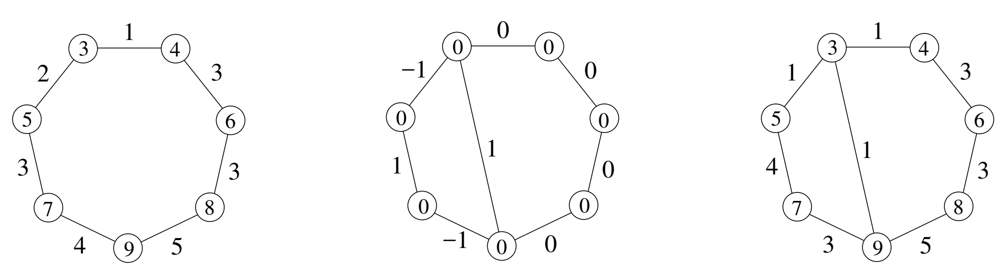
Proof. Let ϕ be the graceful weak antimagic labeling of $\Theta(r, 2, t)$, as in the proof of Theorem 5.6. When $r + t \equiv 0 \pmod 4$, we use the labeling f' which was defined at the proof of Theorem 5.6. The maximum value M of ϕ^+ is at most $r + t$. Now $a = 1$ and the maximum value of g is $2m$. Thus, the maximum value of $M - a + 2m \leq r + t - 1 + 2m = r + t + s - 2m - 3 < r + s + t - 4 = |V(\Theta(r, s, t))|$. Clearly, $a + 2m < |V(\Theta(r, s, t))|$. So the range of all edge labels is a subset of $(0, |V(\Theta(r, s, t))|)$. Applying Lemma 5.2 by replacing the chord by a $(4m + 2)$-path, the claim is established. □

Theorem 5.9. *Let $s = 4m+3 \geq 3$ and $r+t \not\equiv 0 \pmod 4$. Then, $\Theta(r, s, t)$ admits a graceful weak antimagic labeling.*

Proof. The proof is similar to that of Theorem 5.8, making use of Corollary 5.7 and applying Lemma 5.2. □

Example 5.2. *Let us demonstrate the modified labelings of the graphs $C_7(3)$, $C_8(4)$ and $C_6(2)$, which are provided in [21], and extend to labelings of the graph $\Theta(4, 6, 5)$, $\Theta(5, 6, 5)$ and $\Theta(5, 6, 3)$, respectively.*

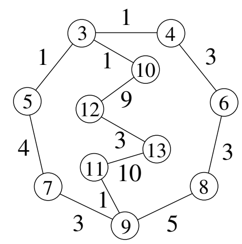
(a)



Labeling f for C_7, labeling η for $C_7(3)$, labeling $\sigma = f + \eta$ for $C_7(3) = \Theta(4, 2, 5)$.

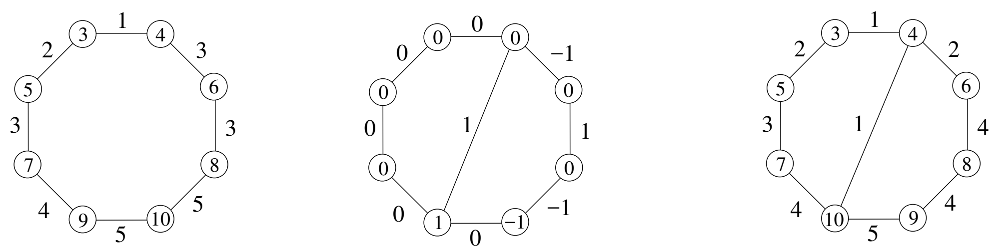
Extended labeling g and labeling $\hat{g}$ for $G_2 \cong P_6$.

We obtain



$I_\phi(\Theta(4,6,5)) = [3,13]$.

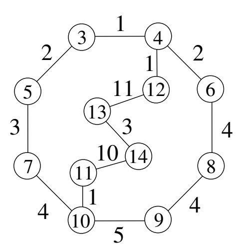
(b)



Labeling f *for* C_8, *labeling* η *for* $C_8(4)$, *labeling* $\sigma = f + \eta$ *for* $C_8(4) = \Theta(5,2,5)$.

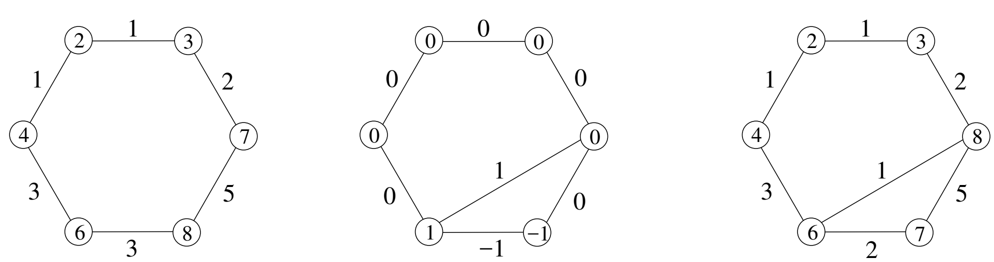
Extended labeling g *and labeling* $\hat{g}$ *for* $G_2 \cong P_6$.

We obtain



$I_\phi(\Theta(5,6,5)) = [3,14]$.

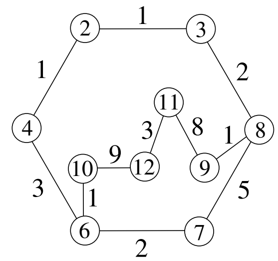
(c)



Labeling f' *for* C_6, *labeling* η *for* $C_6(2)$, *labeling* $\phi = f' + \eta$ *for* $C_6(2) = \Theta(5,2,3)$.

Extended labeling g *and labeling* $\hat{g}$ *for* $G_2 \cong P_6$.

We obtain



$$I_\phi(\Theta(5,6,3)) = [2,12] \setminus \{5\}.$$

Theorem 5.10. *Let $s = 4m+1 \geq 5$. Then, $\Theta(r,s,t)$ admits a graceful weak antimagic labeling if $r+t \not\equiv 0 \pmod 4$.*

Proof. Let σ be the graceful weak antimagic labeling of $\Theta = \Theta(r,2,t)$, (as found in the proof of Theorem 5.6) such that $\sigma(c) = 1$, where c is the chord. Consider $s = 5$ first.

(1) Suppose $r+t-2 \equiv 0 \pmod 4$. By Lemma 2.7, there is a graceful weak antimagic labeling h for C_{r+t-2} with $I_h(C_{r+t-2}) = [6, r+t+3]$. As in the proof of Theorem 5.6, we have a graceful weak antimagic labeling σ for $\Theta(r,2,t)$ with $I_\sigma(\Theta(r,2,t)) = [6, r+t+3]$.

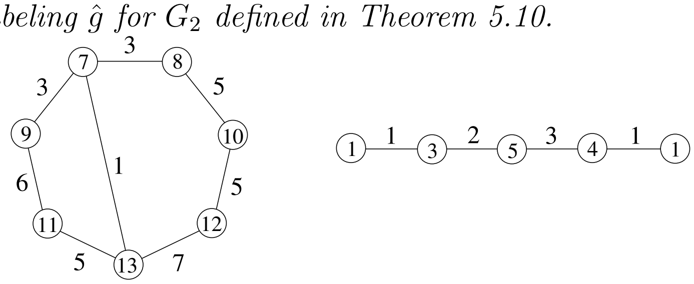
Define $\hat{g}$ for $G_2 = v_0v_1v_2v_3v_4$ in the following way: $\hat{g}(v_0v_1) = 1$, $\hat{g}(v_1v_2) = 2$, $\hat{g}(v_2v_3) = 3$ and $\hat{g}(v_3v_4) = 1$. Replacing the chord c by G_2, we have a graceful weak antimagic labeling $\hat{\sigma}$ for $\Theta(r,5,t)$. Here, $I_{\hat{\sigma}}(\Theta(r,5,t)) = [3, r+t+3]$.

(2) Suppose $r+t-2 \equiv 1 \pmod 4$. By Lemma 2.6, there is a graceful weak antimagic labeling h for C_{r+t-2} with $I_h(C_{r+t-2}) = [6, r+t+3]$. As in the proof of case (1), we have a graceful weak antimagic labeling $\hat{\sigma}$ for $\Theta(r,5,t)$. Here, $I_{\hat{\sigma}}(\Theta(r,5,t)) = [3, r+t+3]$.

(3) Suppose $r+t-2 \equiv -1 \pmod 4$. By Lemma 2.6, there is a graceful weak antimagic labeling h for C_{r+t-2} with $I_h(C_{r+t-2}) = [7, r+t+4]$. Note that the order of $\Theta(r,5,t)$ is congruent to 2, modulo 4. By the same procedure of case (1), we have a graceful weak antimagic labeling $\hat{\sigma}$ for $\Theta(r,5,t)$ with $I_{\hat{\sigma}}(\Theta(r,5,t)) = [3, r+t+4] \setminus \{6\}$.

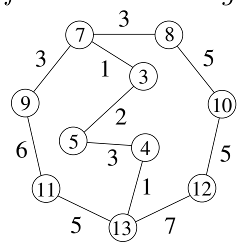
To apply Lemmas 5.2, we may choose the edge in v_2v_3 whose label is 3, and replace this edge with a $(4m-2)$-path, $m \geq 2$. Now, the maximum induced vertex label M of $\Theta(r,5,t)$ of each case is at most $r+t+4$, the label of the chord is $a = 3$ and the maximum value of g is $2m$. So, $M - a + 2(m-1) \leq r+t+1+2m-2 = r+s+t-4-2m+3 < r+s+t-4$. Clearly, $a + 2m < r+s+t-4$. Hence, the claim is established. □

Example 5.3. *Consider $C_7(3) = \Theta(4,2,5)$. By Lemma 2.6, we have the labeling h for C_7. Applying the approach of the proof of Theorem 5.6, we have the labeling σ for $\Theta(4,2,5)$. Also, we have the labeling $\hat{g}$ for G_2 defined in Theorem 5.10.*



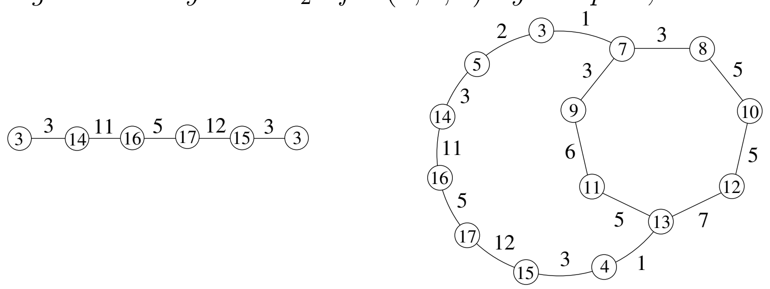
Labelings σ for $\Theta(4,2,5)$ and $\hat{g}$ for $G_2 \cong P_5$.

Combining them, we have a graceful weak antimagic labeling $\hat{\sigma}$ for $\Theta(4,5,5)$.



$I_{\hat{\sigma}}(\Theta(4,5,5)) = [3,13] \setminus \{6\}$.

Replacing the edge labeled by 3 in G_2 of $\Theta(4,5,5)$ by a 6-path, we have



A graceful weak antimagic labeling of $(\Theta(4,9,5))$.

Since G_2 of $\Theta(4,9,5)$ too "long", we move it outside the cycle formed by G_1 and G_3.

From Theorems 5.8, 5.9 and 5.10, we see that the remaining cases are $s \equiv 0 \pmod 4$, or $r+t \equiv 0 \pmod 4$ and $s \not\equiv 2 \pmod 4$.

Theorem 5.11. *Let $r+t \equiv 0 \pmod 4$ and suppose $s \geq 2$. Then, $\Theta(r,s,t)$ admits a graceful weak antimagic labeling.*

Proof. By Theorem 5.8, we may assume $r,s,t \not\equiv 2 \pmod 4$. Without loss of generality, there are two cases: $(r,t) \equiv (0,0), \pmod 4$ and $(r,t) \equiv (1,3), \pmod 4$.

The case $(r,t) \equiv (0,0), \pmod 4$ was addressed by Theorem 5.5.

When $(r,t) \equiv (1,3) \pmod 4$, then either $t \equiv 3 \pmod 4$ with $r+s \equiv 1$ or $2 \pmod 4$, or $r \equiv 1 \pmod 4$ with $t+s \equiv 2 \pmod 4$. These cases were addressed by Theorems 5.9 and 5.10. □

Theorem 5.12. *Let $s = 4m \geq 4$. Then, $\Theta(r,s,t)$ admits a graceful weak antimagic labeling.*

Proof. By Theorem 5.8, we may assume that $r,t \not\equiv 2 \pmod 4$.

(1) Suppose $r+t \equiv 0 \pmod 4$. Theorem 5.11 takes care of this case.

(2) Suppose $r+t \equiv 1$ or $3 \pmod 4$. Without loss of generality, $r \equiv 0 \pmod 4$ and $t \equiv 1$ or $3 \pmod 4$. Then, $r+s \equiv 0 \pmod 4$. Theorem 5.11 takes care of this case.

(3) Suppose $r+t \equiv 2 \pmod 4$. Then, $(r,t) \equiv (1,1), \pmod 4$ or $(r,t) \equiv (3,3), \pmod 4$. So, $s+t \equiv 1 \pmod 4$ with $r \equiv 1 \pmod 4$ or $s+t \equiv 3 \pmod 4$ with $r \equiv 3 \pmod 4$. These cases were addressed by Theorems 5.10 and 5.9.

□

Combining Theorems 5.8, 5.9, 5.10, 5.11 and 5.12, we have proven Theorem 5.1.

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