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Solving parabolic partial differential equations using the cell discretization algorithm

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San Jose State University, 1989

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**Solving Parabolic Partial Differential Equations
Using the Cell Discretization Algorithm**

A Thesis

Presented to

**The Faculty of the Department of Mathematics
San Jose State University**

In Partial Fulfillment

**of the Requirements for the Degree
Master of Science**

By

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ABSTRACT

SOLVING PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS USING THE CELL DISCRETIZATION ALGORITHM

by Masahiko Nishimura

This thesis presents a method by which Greenstadt's Cell Discretization Algorithm may be applied to solving parabolic partial differential equations with homogeneous Dirichlet boundary conditions.

The mathematical foundations of the Algorithm are given, and the convergence of the method is proved by showing that the approximation can be made arbitrarily close to the actual solution in terms of a specified norm. We present a standard Galerkin conversion of an elliptic problem to one of minimizing a functional and methods of constructing the basis needed for the proof. Also a method to solve the final matrix obtained through the Algorithm is included.

Table of Contents.

Introduction.	2
1. Description of the CD Algorithm.	4
2. An Algorithm for Parabolic Equations.	12
3. Preliminaries.	15
4. Convergence of the Solutions obtained by the Algorithm for Parabolic Equations.	26
5. Appendix.	
A. Variational Methods.	49
B. Decoupling Moment Collocations Constraints.	57
C. Solving a System of Ordinary Partial Differential Equations.	62
6. References	65

Introduction.

The Cell Discretization Algorithm is a method invented by John Greenstadt to solve partial differential equations. He developed the Algorithm after encountering difficulties with then-standard methods while mathematically modeling nuclear reactor cores during the 50's.

The method he devised allows a domain to be divided into subdomains or "cells" of quite general shapes and sizes with possibly curved boundaries. It allows different basis functions to be employed on each cell, and does not demand that the boundary conditions be met exactly at any points. Instead a weaker condition is employed to make the approximations on each cell roughly match at common boundaries and with the external boundary values. The user sets the accuracy to that desired via the *moment collocation* requirements.

We offer below a proof of the convergence of the Algorithm's approximations to solve an initial value problem for parabolic operators in bounded domains in $\mathbb{R}^n$.

The proof uses results from Grisvard [6], Aubin [1] and Lions and Megenes [8]. The setting is given by Dr. Howard Swann in the previous reports of the San Jose State Center for Applied Mathematics/Computer Science IBM Cell Discretization Project [10]. The method for obtaining the new basis functions required for the proof was devised by Dr. Les Foster of San Jose State University.

I would like to thank Dr. Greenstadt for his work on the Cell Discretization Algorithm without which all of this would have not been possible.

I thank all of the people involved in the San Jose State IBM Cell Discretization Algorithm Project for their work. Among the many are: Numidee, Desmond, Jenny C., Kent, Ben, Francesca, Bing, Edric and Jenny K., as well as the technical staff at San Jose State Math/CS department: Hom, Chris and Steve.

I would also like to thank Dr. Jane Day for her helpful suggestions and Dr. Les Foster for his work on the method to obtain the new basis functions to eliminate the need for Lagrange multipliers.

I am deeply indebted to Dr. Howard Swann for his work on the Algorithm as well as suggesting the problem and patiently guiding me through the thesis.

Finally, I would like to thank my mother and father for all that they have done for me through the years.

Section 1.

Description of the Algorithm for Elliptic Equations.

The following is an informal sketch of an application of Dr. John Greenstadt's Cell Discretization Algorithm for solving elliptic partial differential equations in $\mathbb{R}^2$ and is abstracted from the previous San Jose State project reports [10].

The task is to find $u(x_1, x_2)$ that satisfies

$$E u = f$$

over a suitable domain Ω in $\mathbb{R}^2$ with boundary Γ , where operator E is a self-adjoint operator of form

$$Eu = - \sum_{i,j=1}^2 D_i(A_{ij}(x) D_j u) + A_0(x) u$$

where A_{ij} , A_0 and f are given functions of x_1 and x_2 on Ω . D_i denotes partial differentiation with respect to x_i . For the operator E to be self-adjoint we require that $A_{12} = A_{21}$.

The boundary conditions we consider here are of Dirichlet type, Neumann type or a mixture of the two. To cover all three possibilities, we consider the mixed boundary value problem.

The boundary conditions for the mixed boundary value problem are as follows: Boundary Γ is the union of two sets Γ_1 and Γ_2 ; the required condition on Γ_1 is

$$u|_{\Gamma_1} = h(s)$$

and the condition on Γ_2 is

$$D_n u|_{\Gamma_2} = t(s),$$

where $D_n u$ is the "normal derivative with respect to E ". This is defined as follows: If $n = (n_1, n_2)$ is the unit outward normal

to Γ , then

$$D_n u = \sum_{i,j=1}^2 \left(A_{ij} D_j u \right) n_i$$

When $E u = -\Delta u + A_0 u$, this is the usual normal derivative.

A standard example of such a problem is where $u(x_1, x_2)$ represents the steady-state temperature of a flat plate at point (x_1, x_2) , $h(s)$ represents the enforced temperature on the boundary segment Γ_1 , $t(s)$ represents the outward heat flux across the boundary segment Γ_2 and u satisfies Laplace's equation

$$D_1^2 u + D_2^2 u = 0.$$

Variational methods can be used to convert the general problem to an equivalent problem. (see appendix A)

Let $H_{\Gamma_1}^1(\Omega)$ be the subspace of (sufficiently smooth) functions equal to zero on Γ_1 . Define

$$a(u, v) = \iint_{\Omega} \sum_{i,j=1}^2 A_{ij}(x_1, x_2) D_i u D_j v + A_0(x_1, x_2) uv \, dx_1 dx_2.$$

Let γ be the (linear) trace map that evaluates a function v on the boundary segment Γ_2 of Ω .

The equivalent task becomes the following: Find u satisfying $u|_{\Gamma_1} = h(s)$ such that

$$a(u, v) = \iint_{\Omega} f(x_1, x_2) v(x_1, x_2) \, dx_1 dx_2 + \int_{\Gamma_2} t(s) [\gamma(v)](s) \, ds$$

for all $v \in H_{\Gamma_1}^1(\Omega)$.

This problem can be further converted to that of finding $u(x_1, x_2)$ that minimizes, over the class of sufficiently smooth functions satisfying the boundary condition $u|_{\Gamma_1} = h(s)$, the

functional:

$$F = \iint_{\Omega} (A_{11} D_1 u D_1 u + 2A_{12} D_1 u D_2 u + A_{22} D_2 u D_2 u + A_0 u^2 - 2fu) dx_1 dx_2 \\ - \int_{\Gamma_2} 2t(s) [\gamma(u)](s) ds .$$

(see appendix A)

Greenstadt's Cell Discretization Algorithm.

The following is a general description of the Algorithm as it is implemented for elliptic equations.

The cell discretization algorithm allows the user to partition the domain Ω into NC cells Ω_k , $k = 1, \dots, NC$, of the user's choice. If the boundary of Ω_k contains part of the external boundary Γ_j , denote it by $\Gamma_{j,k}$, $j = 1, 2$; the subscript j corresponds to the two types of boundary conditions.

On each cell Ω_k , the user can choose any complete set of linearly independent functions $\{ \Psi_i^k(x_1, x_2) \}$ (a basis) and form an approximation to $u(x_1, x_2)$:

$$u|_{\Omega_k} = \sum_{i=1}^m b_i^k \Psi_i^k(x_1, x_2).$$

Thus, the approximation to u is composed of these approximations on each cell.

We then substitute the above approximation into the functional F .

$$\iint_{\Omega} (A_{11} D_1 u D_1 u + 2A_{12} D_1 u D_2 u + A_{22} D_2 u D_2 u + A_0 u^2 - 2fu) dx_1 dx_2 \\ - \int_{\Gamma_2} 2 t(s) [\gamma(u)](s) ds$$

$$\begin{aligned}
&= \sum_{k=1}^{NC} \left[\iint_{\Omega_k} (A_{11} D_1 u D_1 u + 2A_{12} D_1 u D_2 u + A_{22} D_2 u D_2 u + A_0 u^2 - 2fu) dx_1 dx_2 \right. \\
&\quad \left. - \int_{\Gamma_{2,k}} 2 t(s) [\gamma(u)](s) ds \right] \\
&= \sum_{k=1}^{NC} \left[\iint_{\Omega_k} \left[A_{11} (D_1 \sum_i b_i^k \Psi_i^k)^2 + 2A_{12} (D_1 \sum_i b_i^k \Psi_i^k) (D_2 \sum_i b_i^k \Psi_i^k) \right. \right. \\
&\quad \left. \left. + A_{22} (D_2 \sum_i b_i^k \Psi_i^k)^2 + A_0 (\sum_i b_i^k \Psi_i^k)^2 - 2f(\sum_i b_i^k \Psi_i^k) \right] dx_1 dx_2 \right. \\
&\quad \left. - \int_{\Gamma_{2,k}} 2 t(s) [\sum_i b_i^k \gamma(\Psi_i^k)](s) ds \right] \\
&= \sum_{k=1}^{NC} \left[\iint_{\Omega_k} \left[A_{11} (\sum_i b_i^k D_1 \Psi_i^k) (\sum_j b_j^k D_1 \Psi_j^k) + 2A_{12} (\sum_i b_i^k D_1 \Psi_i^k) (\sum_j b_j^k D_2 \Psi_j^k) \right. \right. \\
&\quad \left. \left. + A_{22} (\sum_i b_i^k D_2 \Psi_i^k) (\sum_j b_j^k D_2 \Psi_j^k) + A_0 (\sum_i b_i^k \Psi_i^k) (\sum_j b_j^k \Psi_j^k) \right. \right. \\
&\quad \left. \left. - 2f(\sum_i b_i^k \Psi_i^k) \right] dx_1 dx_2 \right. \\
&\quad \left. - \sum_i b_i^k \left[\int_{\Gamma_{2,k}} t(s) \gamma(\Psi_i^k)(s) ds \right] \right] \\
&= \sum_{k=1}^{NC} \left[\sum_i^m \sum_j^m b_i^k b_j^k \iint_{\Omega_k} \left[A_{11} D_1 \Psi_i^k D_1 \Psi_j^k + 2A_{12} D_1 \Psi_i^k D_2 \Psi_j^k + A_{22} D_2 \Psi_i^k D_2 \Psi_j^k \right. \right. \\
&\quad \left. \left. + A_0 \Psi_i^k \Psi_j^k \right] dx_1 dx_2 \right. \\
&\quad \left. - \sum_i^m b_i^k \left[\iint_{\Omega_k} 2f \Psi_i^k dx_1 dx_2 + \int_{\Gamma_{2,k}} t(s) \gamma(\Psi_i^k)(s) ds \right] \right].
\end{aligned}$$

Note that this is a quadratic polynomial (or "form") in terms of the undetermined coefficients $\{b_i^k\}$, $k = 1, \dots, NC$ and $i = 1, \dots, m^k$. The integrated terms will be constants determined by the known functions A_{11} , A_{12} , A_{22} , A_0 , f and $\{\Psi_i^k\}$. (The ellipticity of the equation will ensure that as a function of $\{b_i^k\}$, it will be bounded below.)

The approximations to the solution, $u|_{\Omega_k} = \sum_{i=1}^m b_i^k \Psi_i^k$ on each cell, need to be "approximately in" the class of functions over which the form F is to be minimized. To do this, they need to approximate the boundary data and "match up" across the boundaries of adjacent cells. Denote the exterior of the domain Ω by Ω_0 . Let Γ_{kj} stand for the intersection between cells Ω_k and Ω_j with Γ_{k0} standing for the intersection of cell Ω_k and the exterior Ω_0 . As in figure 1a, there may be more than one segment involved in the Γ_{k0} ; for simplicity of notation, we will ignore the distinction, acknowledging that there may be multiplicity involved. If two cells intersect at only one point, we do not treat them as adjacent.

Figure 1a.

We require a weak match-up across mutual boundaries and a weak agreement with the boundary values using Greenstadt's method called *moment collocation*. Suppose that Γ_{kj} is parametrically represented as $L_{kj}(s)$, with s being arc length. Along each Γ_{kj} we choose a complete set of linearly independent functions $\{w_{kj}^p(s)\}$, called weight functions. Then, to force $u|_{\Omega_k}$ to weakly match $u|_{\Omega_j}$

on Γ_{kj} , we require, for $p = 1, \dots, n_{kj}$, that

$$\int \sum_{i=1}^m b_i^k \Psi_i^k(L_{kj}(s)) w_{kj}^p(s) ds = \int \sum_{i=1}^m b_i^j \Psi_i^j(L_{kj}(s)) w_{kj}^p(s) ds$$

Using linearity of the integral, the above requirement becomes,

$$\sum_{i=1}^m b_i^k \int \Psi_i^k(L_{kj}(s)) w_{kj}^p(s) ds = \sum_{i=1}^m b_i^j \int \Psi_i^j(L_{kj}(s)) w_{kj}^p(s) ds$$

We call $\sum_{i=1}^m b_i^k \int \Psi_i^k(L_{kj}(s)) w_{kj}^p(s) ds$, the p^{th} moment of $u|_{\Omega_k}$ on Γ_{kj} .

Similarly, to force $u|_{\Omega_k}$ to weakly match the boundary data $h(s)$ on the part of Γ_{k0} comprising Γ_1 , we require, for $p = 1, \dots, n_{k0}$, that

$$\sum_{i=1}^m b_i^k \int \Psi_i^k(L_{k0}(s)) w_{k0}^p(s) ds = \int h(s) w_{k0}^p(s) ds$$

i.e. the first n_{k0} moments of u on Γ_{k0} agree with the first n_{k0} moments of the boundary values $h(s)$ on Γ_{k0} .

Note that both of these types of moment collocation constraints are linear expressions in the $\{b_i^k\}$, since the integrals involved are all integrals of known functions.

We form the set of functions $\{u\}$, expressed on each cell as $u|_{\Omega_k} = \sum_{i=1}^m b_i^k \Psi_i^k$, that satisfy the moment collocation constraints forcing a weak match-up across inter-cell boundaries Γ_{kj} and a weak match with the boundary values on that part of Γ_{k0} comprising Γ_1 . This is the class of functions over which functional F is to be minimized. The functions are expressed in terms of the unknown coefficients $\{b_i^k\}$, $k=1, \dots, NC$; $i=1, \dots, m^k$.

The function from this class that minimizes F can now be computed: We minimize F , quadratic in the $\{b_i^k\}$, subject to the linear moment collocation constraints, by using Lagrange multipliers. Thus, with $\{t_{kj}^p\}$ denoting the Lagrange multipliers used to enforce moment collocation while minimizing F , we must minimize the following unconstrained functional:

$$\begin{aligned} & \sum_k \left[\sum_i^m \sum_j^m b_i^k b_j^k \iint_{\Omega_k} \left[A_{11} D_1 \Psi_i^k D_1 \Psi_j^k + 2A_{12} D_1 \Psi_i^k D_2 \Psi_j^k \right. \right. \\ & \quad \left. \left. + A_{22} D_2 \Psi_i^k D_2 \Psi_j^k + A_0 \Psi_i^k \Psi_j^k \right] dx_1 dx_2 \right. \\ & \quad \left. - \sum_i^m b_i^k \left[\iint_{\Omega_k} 2f \Psi_i^k dx_1 dx_2 + \int_{\Gamma_{2,k}} t(s) \gamma(\Psi_i^k)(s) ds \right] \right] \\ & + \sum_{t_{kj}^p} t_{kj}^p \left[\sum_{i=1}^m b_i^k \int_{\Gamma_{kj}} \Psi_i^k(L_{kj}(s)) w_{kj}^p(s) ds - \sum_{i=1}^m b_i^j \int_{\Gamma_{kj}} \Psi_i^j(L_{kj}(s)) w_{kj}^p(s) ds \right] \\ & + \sum_{t_{k0}^p} t_{k0}^p \left[\sum_{i=1}^m b_i^k \int_{\Gamma_{k0}} \Psi_i^k(L_{k0}(s)) w_{k0}^p(s) ds - \int_{\Gamma_{k0}} h(s) w_{k0}^p(s) ds \right] \end{aligned}$$

Each of the partial derivatives with respect to the b_{ij} and the t_{kj}^p will be a linear expression in the $\{b_i^k\}$ and Lagrange multipliers $\{t_{kj}^p\}$. When all these partial derivatives are simultaneously equated to zero, as the use of Lagrange multipliers requires, we will obtain a system of simultaneous linear equations. Expressed as a matrix equation, it can take the following form by a suitable rearrangement of the ordering of the Lagrange multipliers:

$$\begin{pmatrix}
 \boxed{C_{ij}^1} & & 0 & \boxed{\phantom{C_{ij}^1}} & 0 \\
 & \boxed{C_{ij}^2} & & \boxed{\phantom{C_{ij}^2}} & \\
 0 & & \boxed{\phantom{C_{ij}^2}} & \text{sparse} & \boxed{\phantom{C_{ij}^2}} \\
 \hline
 \boxed{D_{1k}^{ip}} & \text{sparse} & & & \\
 0 & \boxed{D_{2k}^{ip}} & & 0 & \\
 & & \boxed{\phantom{D_{2k}^{ip}}} & &
 \end{pmatrix}
 \begin{pmatrix}
 b_1^1 \\
 b_2^1 \\
 \vdots \\
 b_1^2 \\
 b_2^2 \\
 \vdots \\
 t_{1k}^p \\
 \vdots \\
 \vdots
 \end{pmatrix}
 =
 \begin{pmatrix}
 F_1^1 \\
 \vdots \\
 F_1^2 \\
 \vdots \\
 \vdots \\
 E_1^p \\
 \vdots \\
 \vdots
 \end{pmatrix}$$

Figure 1b.

Note that the matrix has the following properties:

1. It can be shown to be symmetric;
2. It will be quite sparse;
3. It will be banded.

Therefore, this matrix equation can be solved by standard methods to obtain the $\{b_i^k\}$. These $\{b_i^k\}$ in the solution, when combined with the basis functions $\{\psi_i^k\}$, give our approximation on each cell Ω_k as $u|_{\Omega_k} = \sum_{i=1}^m b_i^k \psi_i^k$.

Section 2.

The Algorithm for Parabolic equations.

We approximate the solution to the following parabolic problem:

Find $u(x,t)$ such that

$$\frac{\partial u}{\partial t} - \sum_{i,j}^K \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial u}{\partial x_j} \right) + A_0(x) u = f(x,t),$$

$$u(x,0) = u^0(x), \quad u(x,t) \Big|_{x \in \partial\Omega} = 0, \quad t \in [0,T]$$

and Ω is some LPC^1 domain in $\mathbb{R}^k$

(See Section 3 for LPC^1 domains).

Note that A_{ij} and A_0 are functions of x on Ω whereas f is a function of both x and t . Again for self-adjointness, we require that $A_{ij} = A_{ji}$.

This is not as general as the equations considered by Coffey [3] and Greenstadt [5], since the coefficients, A_{ij} and A_0 are functions of x only, not of t , and we are only considering here the case where $u(x,t) = 0$ when $x \in \partial\Omega$. This last requirement is not necessary; the argument below can be adapted to handle more general boundary conditions on $\partial\Omega$.

$$\text{Let } a(u,v) = \int_{\Omega} \left[\sum_{i,j} A_{ij}(x) D_i u D_j v + A_0(x) uv \right] dx$$

Note that this is the same as the previous definition for $a(u,v)$ in section 1.

Following, for example, Teman [11], the solution u will also satisfy

$$\frac{d}{dt} (u, v)_{L_2} + a(u, v) = (f, v)_{L_2} \text{ for all } v \in H_0^1(\Omega). \quad (2.1)$$

Using the notation from section 3, we can weakly approximate $H_0^1(\Omega)$ by $G_0[n]$, the set of functions satisfying the moment collocation constraints. We show later that we can construct an L_2 orthonormal basis $\{B_i^n\}$ for $G_0[n]$. Thus any linear combination of the B_i^n will satisfy the moment collocation constraints. (Again, such a basis need not be orthonormal; this assumption is made to simplify the following exposition.)

Following the Galerkin method, we require that an approximation $u_m \equiv \sum_{i=1}^m b_i(t) B_i^n$ satisfy the equation

$$\frac{d}{dt} \left(\sum_{i=1}^m b_i(t) B_i^n, B_j^n \right)_{L_2} + a \left(\sum_{i=1}^m b_i(t) B_i^n, B_j^n \right) = (f, B_j^n)_{L_2} \quad (2.2)$$

for all $j = 1, \dots, m$.

Using the linearity of the inner product, equation (2.2) can be written in vector notation as

$$(B_i^n, B_j^n)_{L_2}^T \underline{b} + a(B_i^n, B_j^n)_{L_2}^T \underline{b} = \underline{(f, B_j^n)}_{L_2} \quad (2.3)$$

with the initial data $\underline{b}^0$ obtained by finding the L_2 projection of u^0 onto the span of the $\{B_i^n\}_{i=1}^m$. Therefore $u_m(0) = \sum_{i=1}^m b_i^0 B_i^n$ where $b_i^0 = (u^0, B_i^n)_{L_2}$.

Since the $\{B_i^n\}$ are orthonormal, the system becomes

$$\begin{aligned} \underline{\dot{b}} + a(B_i^n, B_j^n)_{L_2}^T \underline{b} &= \underline{(f, B_j^n)}_{L_2} \\ \underline{b}(0) &= \underline{b}^0. \end{aligned} \quad (2.4)$$

We will show that $a(B_i^n, B_j^n)$ is a symmetric and positive definite matrix of size of about 140 by 140 when there are ten subdomains. (see appendix B) This linear system of first order ordinary differential equations can then be solved by conventional methods to obtain $b_i(t)$. (see appendix C)

We show here that the $b_i(t)$, when combined with the $B_i^n(x)$, will give a function that will converge to the solution of the parabolic equation as the number of moment collocations enforced becomes large when the number of basis functions utilized becomes suitably larger.

Section 3.

Preliminaries.

We start the discussion of the proof with the definitions and basic lemmas from the proof of the elliptic case necessary for the proof of the parabolic case.

Definition 3.1.

A domain $\Omega \subset \mathbb{R}^K$ has a boundary Γ that is Lipschitz and piecewise C^1 , denoted LPC^1 , if it satisfies the following:

(i) Ω is open, bounded and connected;

(ii) Ω is the interior of its closure;

(iii) Γ is Lipschitz; i.e., for any $x \in \Gamma$, there exists a neighborhood V of x in $\mathbb{R}^K$ and new orthogonal coordinates $(y_1, y_2, \dots, y_K)$ such that

$$(a) \quad V = \left\{ (y_1, \dots, y_K) \mid -a_i < y_i < a_i, \quad 1 \leq i \leq K \right\}$$

(b) There exists a uniformly Lipschitz-continuous function h defined in

$$V^1 = \left\{ (y_1, \dots, y_{K-1}) \mid -a_i < y_i < a_i, \quad 1 \leq i \leq K-1 \right\}$$

such that $|h(y^1)| \leq (1/2)a_K$ for any $y^1 \in V^1$,

$$\Omega \cap V = \left\{ y = (y^1, y_K) \in V \mid y_K < h(y^1) \right\}, \text{ and}$$

$$\Gamma \cap V = \left\{ y = (y^1, y_K) \in V \mid y_K = h(y^1) \right\}.$$

Thus, in a neighborhood of $x \in \Gamma$, Ω is below the graph of h and Γ is the graph of h .

(iv) $\Gamma = A_1 \cup \dots \cup A_p \cup B$, where,

(a) A_i is a relatively open subset of Γ and A_i is a simply-connected compact subset of a $C^1(K-1)$ manifold.

(b) B is a compact set contained in finite union of $(K-2)$ manifolds and $\bar{A}_i \cap \bar{A}_j \subset B$ if $i \neq j$.

This definition is that of Grisvard [6] for Lipschitz boundaries and follows Fleming [4] for the definition of piecewise C^1 .

Let Ω be a LPC^1 domain in $\mathbb{R}^K$ with domain Γ .

Let $(.,.)$ denote the $L_2(\Omega)$ inner product, with norm denoted $\| . \|_0$.

The Hilbert spaces we will use are the following:

$$H^1(\Omega) \equiv \left\{ u : \Omega \rightarrow \mathbb{R} \mid u \in L_2(\Omega); D_i u \in L_2(\Omega) \text{ for } i=1, \dots, k \right\}$$

where partial derivatives $D_i u$ are distribution derivatives with respect to x_i . This is a Hilbert space when endowed with the inner product $(u,v)_1 = \sum_{i=1}^K (D_i u, D_i v)_0 + (u,v)_0$. The $H^1(\Omega)$ norm is denoted $\| . \|_1$.

Following Greenstadt's Cell Discretization method, we allow domain Ω to be partitioned into NC LPC^1 subdomains $\Omega_1, \dots, \Omega_{NC}$, with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and $\bar{\Omega} = \bigcup_{i=1}^{NC} \bar{\Omega}_i$. The Ω_i are called *cells*.

Let $\Omega_0 = \mathbb{R}^K \setminus \bar{\Omega}$.

Let $(.,.)_{1,i}$ denote the $H^1(\Omega_i)$ inner product on cell Ω_i . The norm is denoted $\| . \|_{1,i}$. The $L_2(\Omega_i)$ inner product will be denoted $(.,.)_i$.

The parent space for much of our discussion will be

$$H_{CD} = \left\{ u \in L_2(\Omega) \mid u|_{\Omega_i} \in H^1(\Omega_i); i = 1, \dots, NC \right\}.$$

H_{CD} is a Hilbert space with inner product

$$(u, v)_{H_{CD}} = \sum_{i=1}^{NC} (u, v)_{1,i}. \quad (3.1)$$

The H_{CD} norm is denoted $\| \cdot \|_{H_{CD}}$.

Let $\Gamma_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$. Assume that Γ_{ij} is the finite union of $\bar{A}_q$ where $\{ A_q \}$ have the properties of definition 3.1. Where no A_q exist, Γ_{ij} is ignored. To simplify notation, we will refer to such A_q as Γ_{ij} , acknowledging that there may be multiplicity involved. Γ_{i0} is a boundary segment between Ω_i and the exterior Ω_0 .

We denote by γ_{ij} the trace operator restricting $u|_{\Omega_i}$ to its values on Γ_{ij} . From Grisvard [6], theorem 1.5.1.3, we can take it to be a bounded linear operator from $H^1(\Omega_i)$ to $L_2(\Gamma_{ij})$. The inner product on $L_2(\Gamma_{ij})$ will be denoted by $\langle \cdot, \cdot \rangle_{ij}$, with norm represented as $\| \cdot \|_{ij}$.

Weak continuity across interfaces Γ_{ij} is enforced by Greenstadt's method called *moment collocation*.

Definition 3.2.

For each Γ_{ij} , choose $\left\{ w_q^{ij} \right\}_{q=1}^{\infty}$ to be a Schauder basis for $L_2(\Gamma_{ij})$. For $u \in H_{CD}$, we define the q -th moment of $u|_{\Omega_i}$ on Γ_{ij} to be $M_q^{ij}(u) = \langle \gamma_{ij}(u), w_q^{ij} \rangle_{ij}$.

We set moments of approximations equal across interfaces Γ_{ij} in the following fashion:

Let NI be the number of interfaces Γ_{ij} , and $[n]$ denote a multi-index, an NI -vector of non-negative integers $(..., n_{ij}, ...)$. A partial order is $[n'] \geq [n]$ if and only if for any i and j , $n'_{ij} \geq n_{ij}$. We say that $[n^k] \rightarrow [\infty]$ if $[n^k] \leq [n^{k+1}]$ and $\inf\{n_{ij}^k\} \rightarrow \infty$ as $k \rightarrow \infty$.

Set $G[n] = \left\{ u \in H_{CD} \mid \text{for any } ij, i, j = 1, \dots, NI \text{ and for any } q \leq n_{ij}, \text{ we have } M_q^{ij}(u) = M_q^{ji}(u) \right\}$. In this case, $[n]$ is the multi-index described above, with all $n_{i0} = 0$, where the n_{i0} refer to the Γ_{i0} . Thus $G[n]$ is the set of functions in H_{CD} that have n_{ij} moments equal on internal interface Γ_{ij} , giving a notion of weak continuity across interfaces that we call *moment collocation*.

We also set

$$G_0[n] = \left\{ u \in G[n] \mid \text{for any } i, \text{ for any } k \leq n_{i0}, M_k^{i0}(u) = 0 \right\}.$$

Thus $G_0[n]$ is the set of functions in $G[n]$ that are weakly 0 on the external interfaces Γ_{i0} making up Γ .

Both $G[n]$ and $G_0[n]$ are closed subspaces of H_{CD} due to the continuity of the trace operator. In either definition we have the inclusions $[n'] \geq [n] \Rightarrow G[n'] \subset G[n]$.

Note that $G[n] \supset C^\infty(\Omega)$, $G_0[n] \supset C_0^\infty(\Omega)$ and $C^\infty(\Omega_1)$ is dense in $H^1(\Omega_1)$ (Grisvard [6], 1.4.2.1, 1.4.2.2 and 1.4.4.5.)

Lemma 3.3.

$$\bigcap_{[n]} G_0[n] = H_0^1(\Omega).$$

The proof of this theorem is in the San Jose State IBM CDA project Fall 1988 report [10].

We now review our method for forming the basis used to solve elliptic problems. The method will have to be altered somewhat for use in the parabolic case.

NC is the number of cells in the domain decomposition. For each j -th cell, choose any Schauder basis $\{\Psi_i^j\}$ for $H^1(\Omega_j)$. We assume that each $\Psi_i^j \in C^1(\Omega_j)$. Let $[m]$ be a NC-dimensional multi-index indicating the number of basis functions used in the approximation on each cell.

Let $G_0[n][m] = \left\{ u \in G_0[n] \mid u|_{\Omega_j} = \sum_{k=1}^m b_k^j \Psi_k^j \right\}$. This will be a finite dimensional space, non-trivial if $[m]$ is sufficiently large. We also have the inclusion $[m'] \geq [m] \Rightarrow G_0[n][m'] \supset G_0[n][m]$.

A crucial lemma is the following:

Lemma 3.4.

If P_n^m is the orthogonal projection operator of $G_0[n]$ onto $G_0[n][m]$, then

$$\lim_{[m] \rightarrow [\infty]} \|u - P_n^m u\|_{H_{CD}} = 0 \text{ for any } u \in G_0[n].$$

The proof is in [10].

The numerical methods employed to generate functions in $G_0[n][m]$ to approximate solutions of elliptic problems use Lagrange multipliers to enforce the moment collocation constraints. Since this will be difficult to implement for parabolic equations, we need a method to enforce the moment

collocation constraints without Lagrange multipliers. The following is a method for constructing a basis that will automatically satisfy the moment collocation constraints.

For cell Ω_k , let $u^k \equiv \sum_{i=1}^m b_i^k \Psi_i^k$ where b_i^k is a coefficient and m^k is a positive integer denoting the total number of basis elements for the k -th cell.

The moment collocation requirement for internal interfaces between cells Ω_i and Ω_j is that

$$\int_{\Gamma_{kj}} \gamma_{kj}(u^k) w_q^{kj} ds = \int_{\Gamma_{jk}} \gamma_{jk}(u^j) w_q^{kj} ds \text{ for } q = 1, \dots, n_{kj}. \quad (3.2)$$

Using the linearity of the trace map γ_{kj} and the integral, this requirement becomes

$$\sum_{i=1}^m b_i^k \int_{\Gamma_{kj}} \gamma_{kj}(\Psi_i^k) w_q^{kj} ds = \sum_{i=1}^m b_i^j \int_{\Gamma_{kj}} \gamma_{jk}(\Psi_i^j) w_q^{kj} ds \quad (3.3)$$

for $q = 1, 2, \dots, n_{kj}$.

$$\text{Let } d_i^{kj,q} = \int_{\Gamma_{kj}} \gamma_{kj}(\Psi_i^k) w_q^{kj} ds, \text{ for } i = 1, \dots, m^k.$$

Then the requirement is

$$\sum_{i=1}^m b_i^k d_i^{kj,q} - \sum_{i=1}^m b_i^j d_i^{jk,q} = 0, \text{ for } q = 1, 2, \dots, n_{kj}. \quad (3.4)$$

The coefficients $\{b_i^k\}$ for the representation on each of the NC cells can be concatenated to form a vector

$$\underline{b} \equiv (b_1^1, b_2^1, \dots, b_{m_1}^1, b_1^2, b_2^2, \dots, b_{m_2}^2, \dots) = (\underline{b}^1, \underline{b}^2, \dots, \underline{b}^k) \quad (3.5)$$

where $\underline{b}^j = (b_1^j, b_2^j, \dots, b_{m_j}^j)$.

The internal moment collocation can then be represented as

$$\underline{b} \cdot (\underline{0}, \dots, \underline{d}^{kj,q}, \underline{0}, \dots, -\underline{d}^{jk,q}, \underline{0}, \dots, \underline{0}) = 0, \quad (3.6)$$

where

$$\underline{d}^{kj,q} = \left(d_1^{kj,q}, d_2^{kj,q}, \dots, d_m^{kj,q} \right).$$

Note that $\underline{d}^{kj,q}$ is the same size, namely m^k , as $\underline{b}^k$.

If $\underline{d}_{kj,q} \equiv (\underline{0}, \dots, \underline{d}^{kj,q}, \underline{0}, \dots, -\underline{d}^{jk,q}, \underline{0}, \dots, \underline{0})$, then the set of $\underline{b}$ such that the resulting u will satisfy the moment collocations is $\{ \underline{b} : \underline{b} \cdot \underline{d}_{kj,q} = 0 \}$. A similar set $\underline{d}_{k0,q}$ exist such that $\{ \underline{b} : \underline{b} \cdot \underline{d}_{k0,q} = 0 \}$ will produce a u satisfying the weakly zero boundary condition. We denote the set that satisfies both of these conditions by $G = \{ \underline{d}_{kj,q} \}^\perp$ in $\mathbb{R}^N$ where $N = \sum_{k=1}^{NC} m^k$. (NC is the number of cells.) In fact, such $\{ \underline{b} \}$ comprise the null space of the linear operator on $\mathbb{R}^N$ given by the matrix D with rows $\{ \underline{d}_{ij,q} \}$. There will be $M \equiv \sum n_{kj}$ such rows; matrix D is the lower left-hand block of the matrix shown in figure 1b of section 1.

$\{ \underline{d}_{ij,q} \}^\perp$ is a subspace, so if we choose a basis $\{ \underline{c} \}$ for this subspace and express our approximations in terms of this basis, this will eliminate the necessity to enforce internal moment collocation with Lagrange multipliers as was done previously.

The following was pointed out by Dr. Les Foster. We can calculate the null space of D by first finding the QR factorization of D^T . D is a $M \times N$ matrix where $N > M$. Let $D^T = QR$ where Q is unitary and R is upper triangular. Then $D = R^T Q^T$. Since we are looking for $\underline{x}$ such that $D\underline{x} = 0$, an easy argument

shows that the last $N - M$ columns of Q will be an orthonormal basis for the null space of D . These columns can be obtained using linear algebra software packages such as LINPACK.

Suppose that last $N - M$ columns of Q are $(q_1^1, \dots, q_N^1), (q_1^2, \dots, q_N^2), \dots, (q_1^{N-M}, \dots, q_N^{N-M})$. If we enumerate $\{\Psi_i^j\}$ as

$$\left\{ \Psi_1^1, \Psi_2^1, \dots, \Psi_m^1, \Psi_1^2, \Psi_2^2, \dots, \Psi_m^2, \dots \right\},$$

then there will be N such Ψ_i^j . Denote the Ψ_k^i with this enumeration as $\left\{ \Phi^1, \Phi^2, \dots, \Phi^N \right\}$ and form $B_i^n \equiv \sum_{j=1}^N q_j^i \Phi^j$, where the superscript n on B indicates that $[n]$ collocations hold. These $\{B_i^n\}$ will be a basis for $G_0[n][m]$.

For simplicity of exposition, we will assume that the $\{B_i^n\}$ is an L_2 orthonormal set. For this to be the case, it is sufficient that the basis functions $\{\Psi_j^k\}$ defined on Ω_k be an $L_2(\Omega)$ orthonormal set, as the following argument shows: (In $\mathbb{R}^2$, if the cells Ω_k are parallelograms or triangles, then the $\{\Psi_j^k\}$ that we use in the our implementation of the algorithm are L_2 orthonormal on Ω_k .)

By defining Ψ_i^k to be zero outside of Ω_k , the $\{\Psi_i^k\}$ are an orthonormal set in $L_2(\Omega)$ in the subscripts, i and k i.e. $(\Psi_i^k, \Psi_j^l)_0 = \delta_k^l \delta_i^j$. The $\{\Phi^j\}$ will be orthonormal as well.

Recalling that $\left\{ (q_1^1, \dots, q_N^1), (q_1^2, \dots, q_N^2), \dots, (q_1^{N-M}, \dots, q_N^{N-M}) \right\}$

gives an orthonormal set in $\mathbb{R}^N$, we have

$$\begin{aligned}
\left(B_i^n, B_j^n \right)_0 &= \sum_{k,l=1}^N q_k^i q_l^j \left(\Phi^k, \Phi^l \right)_0 \\
&= \sum_{k,l=1}^N q_k^i q_l^j \delta_l^k = \sum_k q_k^i q_k^j = \delta_j^i.
\end{aligned} \tag{3.7}$$

Hence $\{B_i^n\}$ will be orthonormal also.

Although the argument above provides such a basis for fixed $[n]$ and $[m]$, we show in appendix B that a *complete* $L_2(\Omega)$ orthonormal basis $\left\{ B_i^n \right\}_{i=1}^\infty$ can be constructed for $G_0[n]$, with the following property: If P_n^m is the orthogonal projection of $G_0[n]$ onto the span of $\{ B_1^n, \dots, B_m^n \}$, then for any $u \in G_0[n]$,

$$\lim_{m \rightarrow \infty} \| u - P_n^m u \|_{H_{CD}} = 0.$$

A further discussion of the application of this reduction to elliptic problems is in appendix B.

We extend our definitions to appropriate domains for parabolic equations.

Let $\Omega \subset \mathbb{R}^K$ and define $\Omega^T \equiv \Omega \times [0, T]$.

$G_0([n], m)$ will represent $G_0[n][m]$ where now m is the total number of basis functions B_k^n .

Define $G^T([n], m) \equiv \{f: [0, T] \rightarrow G_0([n], m)\}$.

Members v of $G_0^T([n], m)$ can be represented as

$$v \equiv \sum_{i=1}^m b_i^n(t) B_i^n(x). \tag{3.8}$$

Let $H_0^{1,T}(\Omega) \equiv \{f: [0, T] \rightarrow H_0^1(\Omega)\}$.

Before we continue, we present a lemma will be required later.

Lemma 3.5.

Let $v_p \in G_0[n]$. Then

$$\begin{aligned} \int_{\Omega} \sum_p^K \frac{\partial}{\partial x^p} v_p dx &= \sum_{i,j} \int_{\Gamma_{ij}} \sum_p^K \left(\gamma_{ij}(v_p) - \gamma_{ji}(v_p) \right) n_p^{ij} ds \\ &\quad + \sum_{i,0} \int_{\Gamma_{i0}} \sum_p^K \left(\gamma_{i0}(v_p) \right) n_p^{i0} ds \end{aligned}$$

where n_p^{ij} is the p -th component of the unit normal vector to Γ_{ij} , outward with respect to the i -th cell Ω_i .

Proof:

Let $v_p \in G_0[n]$. Partitioning our domain Ω into subdomains Ω_i such that $\bar{\Omega} = \bigcup_i \bar{\Omega}_i$, we get

$$\int_{\Omega} \sum_p \frac{\partial}{\partial x^p} v_p dx = \sum_{i=1}^{NC} \left[\int_{\Omega_i} \left(\sum_p \frac{\partial}{\partial x^p} v_p \right) dx \right]. \quad (3.9)$$

Using the divergence theorem, this is

$$\sum_{i=1}^{NC} \left[\int_{\partial\Omega_i} \left(\sum_p \gamma_i(v_p) n_p^i \right) ds \right] \quad (3.10)$$

where γ_i is the trace operator on Ω_i and n_p^i is the p -th component of the unit outward normal from cell Ω_i . Since each interface lies between two adjacent cells, if we write our summation in terms of the Γ_{ij} , equation 3.10 becomes

$$\begin{aligned} \sum_{i,j} \left[\int_{\Gamma_{ij}} \left(\sum_p \gamma_{ij}(v_p) n_p^{ij} \right) ds + \int_{\Gamma_{ji}} \left(\sum_p \gamma_{ji}(v_p) n_p^{ji} \right) ds \right] \\ + \sum_i \int_{\Gamma_{i0}} \left(\sum_p \gamma_{i0}(v_p) n_p^{i0} \right) ds, \end{aligned} \quad (3.11)$$

where n_p^{ij} is the p -th component of the unit normal vector to Γ_{ij} , outward with respect to Ω_i . The last term represents the sides of cells that are part of $\partial\Omega$.

Since $n_p^{ij} = -n_p^{ji}$, equation 3.11 can be represented as

$$\begin{aligned} \sum_{i,j} \left[\int_{\Gamma_{ij}} \sum_p \left(\gamma_{ij}(v_p) - \gamma_{ji}(v_p) \right) n_p^{ij} ds \right] \\ + \sum_i \int_{\Gamma_{i0}} \left(\sum_p \gamma_{i0}(v_p) n_p^{i0} \right) ds \end{aligned} \quad \square$$

Section 4.

Proof of the convergence of approximations to the solution of parabolic equations.

Define $a(u,v) \equiv \int_{\Omega} \left(\sum_{i,j=1}^K A_{ij}(x) D_i u D_j v + A_0 uv \right) dx$
and $Au \equiv - \sum_{i,j} \frac{\partial}{\partial x^i} \left(A_{ij}(x) \frac{\partial u}{\partial x^j} \right) + A_0 u$. We assume that operator A is self-adjoint.

Suppose that we have a smooth $u(x,t)$ that solves our partial differential equation. So u satisfies

$$\frac{\partial u}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial u}{\partial x_j} \right) + A_0(x)u = f(x,t), \quad (4.1)$$

$$u(x,0) = u^0(x) \text{ and } u(x,t) \Big|_{x \in \partial\Omega} = 0 \text{ for } t \in [0,T].$$

Then for any $v \in H_0^1(\Omega)$,

$$(u' + Au, v)_0 = (f, v)_0$$

where $u' \equiv \frac{\partial u}{\partial t}$. Using linearity of the inner product, we get

$$(u', v)_0 + (Au, v)_0 = (f, v)_0. \quad (4.2)$$

Since $(Au, v)_0 = a(u, v)$ for $v \in H_0^1(\Omega)$, equation 4.2 is equivalent to

$$(u', v)_0 + a(u, v) = (f, v)_0. \quad (4.3)$$

We can find an approximate solution to equation 4.3 above by

finding some $u_{n,m} \equiv \sum_{i=1}^m b_i^n(t) B_i^n \in G_0^T([n], m)$ such that

$$(u'_{n,m}, B_k^n)_0 + a(u_{n,m}, B_k^n) = (f, B_k^n)_0 \text{ for } k = 1, \dots, m \text{ and} \quad (4.4)$$

$$u_{n,m} \Big|_{t=0} = u_{n,m}^0,$$

where we approximate the initial data by the L_2 orthogonal

projection of u^0 : $u_{n,m}^0 \equiv \sum_{i=1}^m d_i^n B_i^n$ where $d_i^n \equiv (u^0, B_i^n)_0$.

Substituting for $u_{n,m}$ in equation 4.4, we get

$$\left(\left(\sum_{i=1}^m b_i^n B_i^n \right)', B_k^n \right)_0 + a \left(\sum_{i=1}^m b_i^n B_i^n, B_k^n \right) = (f, B_k^n)_0, \quad k = 1, \dots, m,$$

and $b_i^n(0) = d_i^n$.

Using bilinearity of the inner product and of operator $a(.,.)$, and the fact that only b_i^n is a function of t , this equation is

$$\sum_{i=1}^m b_i^n(t) (B_i^n, B_k^n)_0 + \sum_{i=1}^m b_i^n(t) a(B_i^n, B_k^n) = (f, B_k^n)_0, \quad k = 1, \dots, m. \quad (4.5)$$

Now $(B_i^n, B_k^n)_0 = \delta_i^k$, so writing in vector notation, equation 4.5 becomes

$$\underline{b}' + \left(a(B_i^n, B_k^n) \right)^T \underline{b} = (f, B_k^n)$$

$\underline{b}(0) = \underline{d}.$ (4.6)

$\left(a(B_i^n, B_k^n) \right)$ is a symmetric positive definite matrix of moderate size, and this system of ordinary differential equations can be solved by conventional methods. (see appendix C)

If we multiply equation 4.4 by b_k^n for each k and sum over k , we get

$$\begin{aligned} \sum_k b_k^n (u'_{n,m}, B_k^n)_0 + \sum_k b_k^n a(u_{n,m}, B_k^n) &= \sum_k b_k^n (f, B_k^n)_0, \text{ or} \\ (u'_{n,m}, \sum_k b_k^n B_k^n)_0 + a(u_{n,m}, \sum_k b_k^n B_k^n) &= (f, \sum_k b_k^n B_k^n)_0, \text{ or} \\ (u'_{n,m}, u_{n,m})_0 + a(u_{n,m}, u_{n,m}) &= (f, u_{n,m})_0 \end{aligned} \quad (4.7)$$

A similar argument shows that

$$(u'_{n,m}, v)_0 + a(u_{n,m}, v) = (f, v)_0 \text{ for } v \in G_0([n], m) \text{ or } G_0^T([n], m).$$

With these preparations, we can now state the main theorem.

Theorem.

Assume that the $A_{ij} \in C^2(\Omega)$, $A_0 \in C^1(\Omega)$ and

$$f \in C([0, T]; C^1(\Omega)); f|_{\partial\Omega} = 0.$$

Assume that there exists a $c > 0$ such that

$$\sum_{i,j} A_{ij}(x) z_i z_j \geq c \sum_{i=1}^K z_i^2 \text{ for any } z_i \in \mathbb{R}$$

and $A_0(x) \geq c$ for $x \in \Omega$ (an ellipticity assumption)

and $A_{ij}(x) = A_{ji}(x)$ (self-adjointness.)

Let $[n]$ be a multi-index and m be a positive integer. Assume that our actual solution u of equation 4.1 satisfies the following conditions:

$$u \in C([0, T]; C^3(\Omega)), \frac{\partial u}{\partial t} \in C([0, T]; C_0(\Omega)),$$

$$u|_{\partial\Omega} = 0 \text{ and } u(0, x) = u_0(x) \in C^3(\Omega) \cap C_0(\Omega).$$

Then for any $\varepsilon > 0$, there exists $[N]$ such that for $[n] > [N]$, there exists an integer M such that for $m > M$,

$$\int_0^T \|u - u_{n,m}\|_{H_{CD}}^2 dt < \varepsilon$$

where $u_{n,m}$ is the approximation obtained from the Cell Discretization Algorithm.

Sufficient conditions for the required regularity for the solution $u(x, t)$ can be found in Lions and Magenes [8].

We will first reduce the problem and then prove a series of lemmas.

Recall that our problem is to find u such that

$$u' - \sum_{p,q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} \right) + A_0(x)u = f(x,t)$$

$$u(x,0) = u_0(x); u(x,t) \Big|_{x \in \partial\Omega} = 0, t \in [0, \tau]$$

where " u' " denotes the derivative of u with respect to t .

A solution will satisfy

$$\int_{\Omega} \left[u' - \sum_{p,q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} \right) + A_0(x)u - f(x,t) \right] v \, dx = 0$$

for any $v \in G_0^T([n],m)$.

Using the product rule for differentiation on the second integral term, we have

$$\begin{aligned} \int_{\Omega} [u'v] \, dx - \int_{\Omega} \left[\sum_{p,q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} v \right) \right] \, dx \\ + \int_{\Omega} \sum_{p,q} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} \frac{\partial v}{\partial x^p} \right) \, dx \\ + \int_{\Omega} \left[A_0(x)u - f(x,t) \right] v \, dx = 0. \end{aligned} \quad (4.8)$$

Rearranging terms,

$$\begin{aligned} \int_{\Omega} [u'v] \, dx + \int_{\Omega} \sum_{p,q} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} \frac{\partial v}{\partial x^p} \right) \, dx \\ + \int_{\Omega} \left[A_0(x)u - f(x,t) \right] v \, dx \\ - \int_{\Omega} \left[\sum_{p,q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} v \right) \right] \, dx = 0. \end{aligned} \quad (4.9)$$

Or writing in terms of inner products, this is

$$(u', v)_0 + a(u, v) - (f, v)_0 - \int_{\Omega} \left[\sum_{p, q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} v \right) \right] dx = 0 \quad (4.10)$$

Since our approximation $u_{n,m} \in G_0^T([n], m)$, we can substitute $v = u_{n,m}$ to get

$$(u', u_{n,m})_0 + a(u, u_{n,m}) - (f, u_{n,m})_0 - \int_{\Omega} \sum_{p, q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m} \right) dx = 0. \quad (4.11)$$

Using Lemma 3.5, the fourth term can be represented as

$$\begin{aligned} & - \int_{\Omega} \sum_{p, q} \frac{\partial}{\partial x^p} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m} \right) dx \\ &= - \sum_{ij} \int_{\Gamma_{ij}} \sum_{p, q} \left(\gamma_{ij} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m}^i \right) - \gamma_{ji} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m}^j \right) \right) n_p^{ij} dS \\ & \quad - \sum_{i0} \int_{\Gamma_{i0}} \sum_{p, q} \left(\gamma_{i0} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m}^i \right) \right) n_p^{ij} dS \end{aligned}$$

where Γ_{ij} is the interface between cell i and cell j and γ_{ij} is the trace map defined above, and $(n_1^{ij}, n_2^{ij}, \dots, n_k^{ij})$ is the unit normal on Γ_{ij} , outward with respect to Ω_i .

We then substitute this expression into equation 4.11 to get

$$\begin{aligned}
& (u', u_{n,m})_0 + a(u, u_{n,m}) - (f, u_{n,m})_0 \\
& - \sum_{\Gamma_{ij}} \int_{\Gamma_{ij}} \sum_{p,q} \left(\gamma_{ij} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m} \right) - \gamma_{ji} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m} \right) \right) n_p^{ij} dS \\
& - \sum_{\Gamma_{i0}} \int_{\Gamma_{i0}} \sum_{p,q} \left(\gamma_{i0} \left(A_{pq}(x) \frac{\partial u}{\partial y^q} u_{n,m} \right) \right) n_p^{i0} dS = 0. \tag{4.12}
\end{aligned}$$

Writing the last two integrals in terms of inner products, and recalling the definition of $D_n u$ [10], we get, for our smooth solution u and $u_{n,m} \in G_0^T([n], m)$,

$$\begin{aligned}
& (u', u_{n,m})_0 + a(u, u_{n,m}) - \sum_{\Gamma_{ij}} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} \\
& - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} = (f, u_{n,m})_0. \tag{4.13}
\end{aligned}$$

Rearranging terms,

$$\begin{aligned}
& (u', u_{n,m})_0 = (f, u_{n,m})_0 - a(u, u_{n,m}) \\
& + \sum_{\Gamma_{ij}} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} \\
& + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}. \tag{4.14}
\end{aligned}$$

Note that the first summation term should be small due to the moment collocations across the internal cell interfaces, while the last term should be small due to the collocation with the homogeneous boundary condition.

Now

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 &= \frac{1}{2} \frac{d}{dt} (u - u_{n,m}, u - u_{n,m})_0 \\
&= (u' - u'_{n,m}, u - u_{n,m})_0 \\
&= (u', u)_0 - (u'_{n,m}, u)_0 - (u', u_{n,m})_0 + (u'_{n,m}, u_{n,m})_0. \tag{4.15}
\end{aligned}$$

Let P_n^m be the H_{CD} orthogonal projection map of $H_0^1(\Omega) \rightarrow G_0([n], m)$. Then for all $w \in G_0([n], m)$,

$$(u - P_n^m u, w)_{H_{CD}} = 0.$$

We extend the above projection to the mapping:

$$H_0^{1,T}(\Omega) \rightarrow G_0^T([n], m).$$

Since $u \in H_0^1(\Omega)$, we have

$$(u', u)_0 = (f, u)_0 - a(u, u) \tag{4.16}$$

$$\text{and } (u_{n,m}', u)_0 = (u_{n,m}', u - P_n^m u)_0 + (u_{n,m}', P_n^m u)_0 \tag{4.17}$$

where P_n^m is the extension of the projection from $H_0^1(\Omega) \rightarrow G_0([n], m)$.

Using equations 4.7, 4.14, 4.16 and 4.17, above, equation 4.15 becomes

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 &= [(f, u)_0 - a(u, u)] \\
&\quad - [(u_{n,m}', u - P_n^m u)_0 + (u_{n,m}', P_n^m u)_0] \\
&\quad - \left[(f, u_{n,m})_0 - a(u, u_{n,m}) + \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} \right. \\
&\quad \quad \left. + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} \right] \\
&\quad + [(f, u_{n,m})_0 - a(u_{n,m}, u_{n,m})]. \tag{4.18}
\end{aligned}$$

Rearranging 4.18, we get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 &= (f, u)_0 - (f, u_{n,m})_0 + (f, u_{n,m})_0 \\
&- a(u, u) + a(u, u_{n,m}) - a(u_{n,m}, u_{n,m}) \\
&- (u_{n,m}', u - P_n^m u)_0 - (u_{n,m}', P_n^m u) \\
&- \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}.
\end{aligned} \tag{4.19}$$

Due to the fact that $P_n^m u \in G_0([n], m)$,

$$(u_{n,m}', P_n^m u)_0 = (f, P_n^m u)_0 - a(u_{n,m}, P_n^m u). \tag{4.20}$$

Note that the second and the third terms of equation 4.19 sum to zero. With these results, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 &= (f, u)_0 + 0 \\
&- a(u, u) + a(u, u_{n,m}) - a(u_{n,m}, u_{n,m}) \\
&- (u_{n,m}', u - P_n^m u)_0 - (f, P_n^m u)_0 + a(u_{n,m}, P_n^m u) \\
&- \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}.
\end{aligned} \tag{4.21}$$

Since operator A is self adjoint, $a(u_{n,m}, u) = a(u, u_{n,m})$.

Adding and subtracting $a(u_{n,m}, u)$ on the second line, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 &= (f, u)_0 \\
&- a(u, u) + a(u, u_{n,m}) - a(u_{n,m}, u_{n,m}) + a(u_{n,m}, u) - a(u_{n,m}, u) \\
&- (u_{n,m}', u - P_n^m u)_0 - (f, P_n^m u)_0 + a(u_{n,m}, P_n^m u) \\
&- \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}.
\end{aligned} \tag{4.22}$$

Noting that

$$\begin{aligned}
& -a(u, u) + a(u, u_{n,m}) - a(u_{n,m}, u_{n,m}) + a(u_{n,m}, u) \\
& = - \left[a(u, u - u_{n,m}) - a(u_{n,m}, u - u_{n,m}) \right] \\
& = - a(u - u_{n,m}, u - u_{n,m}),
\end{aligned}$$

and substituting for the first four terms of the second line of equation 4.22, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 = (f, u)_0 \\
& - a(u - u_{n,m}, u - u_{n,m}) - a(u_{n,m}, u) \\
& - (u_{n,m}', u - P_n^m u)_0 - (f, P_n^m u)_0 + a(u_{n,m}, P_n^m u) \\
& - \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}. \tag{4.23}
\end{aligned}$$

Moving the first term on the second line in equation 4.23 to the left hand side and rearranging terms, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 + a(u - u_{n,m}, u - u_{n,m}) \\
& = (f, u)_0 - (f, P_n^m u)_0 \\
& - a(u_{n,m}, u) + a(u_{n,m}, P_n^m u) \\
& - (u_{n,m}', u - P_n^m u)_0 \\
& - \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}. \tag{4.24}
\end{aligned}$$

By using bilinearity, we have

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|u - u_{n,m}\|_0^2 + a(u - u_{n,m}, u - u_{n,m}) \\
&= (f, u - P_n^m u)_0 - a(u_{n,m}, u - P_n^m u) \\
&\quad - (u_{n,m}', u - P_n^m u)_0 \\
&- \sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} - \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega}.
\end{aligned} \tag{4.25}$$

Integrating with respect to t over $[0, \tau]$, we get

$$\begin{aligned}
& \frac{1}{2} \int_0^\tau \frac{d}{dt} \|u - u_{n,m}\|_0^2 dt + \int_0^\tau a(u - u_{n,m}, u - u_{n,m}) dt \\
&= \int_0^\tau (f, u - P_n^m u)_0 dt - \int_0^\tau a(u_{n,m}, u - P_n^m u) dt \\
&\quad - \int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt \\
&- \int_0^\tau \left[\sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} \right] dt.
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\text{Since } \frac{1}{2} \int_0^\tau \frac{d}{dt} \|u - u_{n,m}\|_0^2 dt &= \frac{1}{2} \|u - u_{n,m}\|_0^2 \Big|_0^\tau \\
&= \frac{1}{2} \|u(\tau) - u_{n,m}(\tau)\|_0^2 - \frac{1}{2} \|u^0 - u_{n,m}^0\|_0^2,
\end{aligned}$$

and since $c \|u - u_{n,m}\|_{H_{CD}}^2 \leq a(u - u_{n,m}, u - u_{n,m})$ by the coercivity of A , where c is some positive constant, we obtain

$$\begin{aligned}
& \frac{1}{2} \|u(\tau) - u_{n,m}(\tau)\|_0^2 + c \int_0^\tau \|u - u_{n,m}\|_{H_{CD}}^2 dt \\
& \leq \frac{1}{2} \|u^0 - u_{n,m}^0\|_0^2 + \int_0^\tau (f, u - P_n^m u)_0 dt \\
& \quad - \int_0^\tau a(u_{n,m}, u - P_n^m u) dt - \int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt \\
& \quad - \int_0^\tau \left[\sum_{\Gamma_{ij}} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} \right] dt.
\end{aligned} \tag{4.27}$$

We will show that the terms (1), (2) and (4) in equation 4.27 can be made small, and we will describe a method to handle terms (3) and (5).

Lemma 4.1.

For any $\varepsilon > 0$ and any multi-index $[N]$, there exists a positive integer $M = M([N], \varepsilon)$ such that for $m > M$,

$$\|u^0 - u_{n,m}^0\|_0 < \varepsilon.$$

Proof:

For any $[n]$ and smooth $u^0 \in G_0[n]$, the assumptions of lemma 3.4 hold and for any $[n]$ and $\varepsilon > 0$, there is some M such that $m > M \Rightarrow \|u^0 - P_n^m u^0\|_{H_{CD}} < \varepsilon$.

Now $\|u^0 - P_n^m u^0\|_{H_{CD}} \geq \|u^0 - P_n^m u^0\|_{L_2}$. But our choice of initial condition $u_{n,m}^0$ in terms of the L_2 orthonormal basis $\{B_i^n\}$ minimizes $\|u^0 - v\|_0$ over $v \in G_0([n], m)$.

Therefore

$$\|u^0 - P_n^m u^0\|_0 \geq \|u^0 - u_{n,m}^0\|_0. \quad \square$$

The following lemma will be used several times.

Lemma 4.2.

Suppose that $\{g_k(t)\}$ is a set of continuous functions on $[0, T]$ and suppose that, for each t , $\{g_k(t)\}$ is monotonically decreasing to zero as $k \rightarrow \infty$. Then for $\varepsilon > 0$, there exists a integer M such that for $k > M$,

$$\|g_k(t)\|_{C[0,T]} < \varepsilon.$$

Proof:

This is Dini's Theorem.

Lemma 4.3.

For any $\varepsilon > 0$, any multi-index $[N]$ and any $v \in C[0, \tau; H_0^1(\Omega)]$, there exists a positive integer $M = M([N], \varepsilon)$ such that for $m > M$,

$$\int_0^\tau \|v - P_n^m v\|_0 dt < \varepsilon.$$

Proof:

$(P_n^m v)(t) \in G_0([n], m)$ is the unique vector that minimizes the expression $\|v(t) - w\|_{H_{CD}}$ over $w \in G_0([n], m)$. If $m' > m$, then

$$G_0([n], m') \supset G_0([n], m).$$

So

$$\|v(t) - (P_n^{m'} v)(t)\|_{H_{CD}} \leq \|v(t) - (P_n^m v)(t)\|_{H_{CD}}.$$

$(P_n^m v)(t)$ can be shown to be in $C([0, \tau]; H_{CD})$.

Since $\lim_{m \rightarrow \infty} \|v(t) - (P_n^m v)(t)\|_{H_{CD}} = 0$ by lemma 3.4,

$$g^m(t) \equiv \|v(t) - (P_n^m v)(t)\|_{H_{CD}}$$

satisfies the conditions of lemma 4.2. Therefore, $g^m(t)$ can be made uniformly small on $[0, \tau]$. Lemma 4.3 follows, since

$$g^m(t) \geq \|v(t) - (P_n^m v)(t)\|_0. \quad \square$$

Lemma 4.4.

For $\varepsilon > 0$, for any multi-index $[n]$ there is an integer $M = M([n], \varepsilon)$ such that for $m > M$, $\int_0^\tau (f, u - P_n^m u)_0 dt < \varepsilon$.

Proof:

If $f \in L_\infty([0, \tau]; L_2)$, $(f, u - P_n^m u)_0 \leq \|f\|_0 \|u - P_n^m u\|_0$ by the Schwarz inequality. So

$$\left| \int_0^\tau (f, u - P_n^m u)_0 dt \right| \leq \|f\|_{0, \tau} \int_0^\tau \|u - P_n^m u\|_0 dt,$$

where $\|f\|_{0,\tau}$ is the L_∞ norm in term of t . Therefore, by selecting $\varepsilon_0 = \frac{\varepsilon}{\|f\|_{0,\tau}}$ and using lemma 4.3 above, there exists a suitable $M([n],\varepsilon)$. $\square$

Lemma 4.5. (Temam [11])

$\|u_{n,m}(\cdot, t)\|_0$ is uniformly bounded in t .

Proof:

First, for any $c > 0$ we note that

$$2\alpha\beta \leq \frac{\alpha^2}{c} + c\beta^2, \quad (4.28)$$

$$\text{since } 0 \leq \left(\frac{1}{\sqrt{c}} \alpha - \sqrt{c} \beta \right)^2 = \alpha^2/c - 2\alpha\beta + c\beta^2.$$

$$\begin{aligned} \text{Now, } \frac{1}{2} \frac{d}{dt} \|u_{n,m}\|_0^2 &= (u_{n,m}', u_{n,m})_0 \\ &= -a(u_{n,m}, u_{n,m}) + (f, u_{n,m})_0. \end{aligned}$$

Rearranging terms,

$$\frac{1}{2} \frac{d}{dt} \|u_{n,m}\|_0^2 + a(u_{n,m}, u_{n,m}) = (f, u_{n,m})_0.$$

$$\text{Since } c \|u_{n,m}\|_{H_{CD}}^2 \leq a(u_{n,m}, u_{n,m}),$$

$$\frac{1}{2} \frac{d}{dt} \|u_{n,m}\|_0^2 + c \|u_{n,m}\|_{H_{CD}}^2 \leq |(f, u_{n,m})_0|.$$

$$\begin{aligned} \text{So } \frac{d}{dt} \|u_{n,m}\|_0^2 + 2c \|u_{n,m}\|_{H_{CD}}^2 &\leq 2|(f, u_{n,m})_0| \\ &\leq 2 \|f\|_0 \|u_{n,m}\|_0. \end{aligned}$$

By equation 4.28, we can majorize this last expression by

$$\begin{aligned} &\|f\|_0^2 / c + c \|u_{n,m}\|_0^2 \\ &\leq \|f\|_0^2 / c + c \|u_{n,m}\|_{H_{CD}}^2. \end{aligned} \quad (4.29)$$

Hence,

$$\begin{aligned} -\frac{d}{dt} \|u_{n,m}\|_0^2 + 2c \|u_{n,m}\|_{H_{CD}}^2 &\leq \|f\|_0^2 / c + c \|u_{n,m}\|_{H_{CD}}^2 \text{ or} \\ -\frac{d}{dt} \|u_{n,m}\|_0^2 + c \|u_{n,m}\|_{H_{CD}}^2 &\leq \|f\|_0^2 / c. \end{aligned}$$

$$\text{So} \quad -\frac{d}{dt} \|u_{n,m}\|_0^2 \leq \|f\|_0^2 / c. \quad (4.30)$$

Integrating both sides of the equation above with respect to variable t , we get

$$\int_0^\tau -\frac{d}{dt} \|u_{n,m}\|_0^2 dt \leq \int_0^\tau \frac{\|f\|_0^2}{c} dt. \quad (4.31)$$

$$\begin{aligned} \text{So} \quad \|u_{n,m}\|_0^2(\tau) - \|u_{n,m}\|_0^2(0) &\leq \int_0^\tau \frac{\|f\|_0^2}{c} dt \text{ or} \\ \|u_{n,m}\|_0^2(\tau) &\leq \int_0^\tau \frac{\|f\|_0^2}{c} dt + \|u_{n,m}\|_0^2(0). \end{aligned} \quad (4.32)$$

Now $\|u_{n,m}\|_0^2(0)$ is majorized by $\|u^0\|_0^2$ and

$$\int_0^\tau \frac{\|f\|_0^2}{c} dt \leq \int_0^T \frac{\|f\|_0^2}{c} dt$$

is independent of n and m , so the result follows. $\square$

Lemma 4.6.

For $\varepsilon > 0$, for any $[n]$, there exists an integer $M = M([n], \varepsilon)$ such that for $m > M$, $\int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt < \varepsilon$.

Proof:

$$\begin{aligned} \text{By integration by parts, } \int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt \\ = \int_0^\tau \frac{d}{dt} \left(u_{n,m}, u - P_n^m u \right)_0 dt - \int_0^\tau \left(u_{n,m}, \frac{\partial}{\partial t} (u - P_n^m u) \right)_0 dt \end{aligned}$$

$$= \left(u_{n,m}(\tau), u(\tau) - P_n^m u(\tau) \right)_0 - \left(u_{n,m}^0, u^0 - P_n^m u^0 \right)_0 \\ - \int_0^\tau \left(u_{n,m}, \frac{\partial}{\partial t} (u - P_n^m u) \right)_0 dt.$$

Then

$$\left| \int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt \right| \\ \leq \| u_{n,m}(\tau) \|_0 \| u(\tau) - P_n^m u(\tau) \|_0 + \| u_{n,m}^0 \|_0 \| u^0 - P_n^m u^0 \|_0 \\ + \| u_{n,m} \|_{0,\infty} \int_0^\tau \left\| \frac{\partial}{\partial t} (u - P_n^m u) \right\|_0 dt.$$

Now $\frac{\partial}{\partial t} (u - P_n^m u) = \frac{\partial u}{\partial t} - P_n^m \left(\frac{\partial u}{\partial t} \right)$ since we can show that

$\frac{\partial P_n^m u}{\partial t} = P_n^m \left(\frac{\partial u}{\partial t} \right)$ by considering the representation of u in terms of a finite H_{CD} -orthonormal basis for $G_0([n],m)$.

$$\text{Therefore, we have } \left| \int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt \right| \\ \leq \| u_{n,m}(\tau) \|_0 \| u(\tau) - P_n^m u(\tau) \|_0 + \| u_{n,m}^0 \|_0 \| u^0 - P_n^m u^0 \|_0 \\ + \| u_{n,m} \|_{0,\infty} \int_0^\tau \left\| \left(\frac{\partial}{\partial t} u - P_n^m \frac{\partial}{\partial t} u \right) \right\|_0 dt$$

Since $\| u_{n,m} \|$ is bounded in t by lemma 4.5 and since we have assumed that $\frac{\partial u}{\partial t}$ is in $C([0,\tau]; C_0(\Omega))$, we use lemma 4.3 to establish the result. $\square$

Note that requiring that $\frac{\partial u}{\partial t} \Big|_{\partial\Omega} = 0$ is a natural consequence of the fact that $u \Big|_{\partial\Omega} = 0$ for $t \in [0, T]$.

Lemma 4.7.

For $\varepsilon > 0$, there exists a multi-index $[n_0]$ such that for $[n] > [n_0]$,

$$\left| \int_0^\tau \left[\sum_{\Gamma_{ij}} \langle D_n u, g_{ij}(u_{n,m}) - g_{ji}(u_{n,m}) \rangle_{ij} + \langle D_n u, g(u_{n,m}) \rangle_{\partial\Omega} \right] dt \right| \leq 2 K \varepsilon \int_0^\tau \| u_{n,m} - u \|_{H_{CD}} dt.$$

where K is some constant independent of n and m , and m is sufficiently large so that $G_0([n], m)$ is non-trivial.

Proof:

We first note that

$$\sum_{\Gamma_{ij}} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} \quad (4.33)$$

$$= \sum_{\Gamma_{ij}} \langle D_n u, \gamma_{ij}(u_{n,m} - u) - \gamma_{ji}(u_{n,m} - u) \rangle_{ij} + \langle D_n u, \gamma(u_{n,m} - u) \rangle_{\partial\Omega} \quad (4.34)$$

since $\gamma_{ij}(u) = \gamma_{ji}(u)$ and $\gamma(u) = 0$.

Without loss of generality, we can assume that our moment collocation weights $\{ w_k^{ij} \}$ are orthonormal in $L_2(\Gamma_{ij})$, since the Gram-Schmidt process could be used to produce such a set from our original $\{ w_k^{ij} \}$; the space $G_0([n], m)$ would remain the same. The A_{ij} and $D_n u$ are sufficiently smooth so that we can write $D_n(u)$

$$= \sum_{k=1}^{\infty} d_k^{ij}(t) w_k^{ij} \text{ where } d_k^{ij}(t) \equiv \int_{\Gamma_{ij}} (D_n u) w_k^{ij} ds.$$

If $\Delta \equiv u_{n,m} - u$, then

$$\begin{aligned} & \sum_{\Gamma_{ij}} \langle D_n u, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij} \\ &= \sum_{\Gamma_{ij}} \langle \sum_{k=1}^{\infty} d_k^{ij}(t) w_k^{ij}, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij} \end{aligned}$$

$$= \sum_{\Gamma_{ij}} \left(\sum_{k=1}^{\infty} d_k^{ij}(t) \langle w_k^{ij}, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij} \right). \quad (4.35)$$

Since for each interface side, for $k \leq n_{ij}$,

$$\langle w_k^{ij}, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij} = 0,$$

equation 4.35 becomes

$$\sum_{\Gamma_{ij}} \left(\sum_{k=n_{ij}+1}^{\infty} d_k^{ij}(t) \langle w_k^{ij}, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij} \right).$$

Using the Schwarz inequality, this expression is bounded by

$$\sum_{\Gamma_{ij}} \left[\left(\sum_{k=n_{ij}+1}^{\infty} d_k^{ij}(t)^2 \right)^{1/2} \left(\sum_{k=n_{ij}+1}^{\infty} \langle w_k^{ij}, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij}^2 \right)^{1/2} \right]. \quad (4.36)$$

Since $\langle w_k^{ij}, \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \rangle_{ij}$ is the k -th Fourier coefficient of $[\gamma_{ij}(\Delta) - \gamma_{ji}(\Delta)]$ relative to the assumed orthonormal basis $\{w_k^{ij}\}$ of $L_2(\Gamma_{ij})$, we now have

$$(4.36) \leq \sum_{\Gamma_{ij}} \left[\left(\sum_{k=n_{ij}+1}^{\infty} d_k^{ij}(t)^2 \right)^{1/2} \| \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \|_{ij} \right].$$

By the trace theorem, there are constants c_i and c_j such that

$$\begin{aligned} \| \gamma_{ij}(\Delta) - \gamma_{ji}(\Delta) \|_{ij} &\leq \| \gamma_{ij}(\Delta) \|_{ij} + \| \gamma_{ji}(\Delta) \|_{ij} \\ &\leq c_i \| \Delta \|_{1,i} + c_j \| \Delta \|_{1,j}. \end{aligned}$$

$\| \cdot \|_{1,i}$ is the H^1 norm on cell Ω_i .

We now have

$$(4.36) \leq \sum_{\Gamma_{ij}} \left[\sum_{k=n_{ij}+1}^{\infty} \left(d_k^{ij}(t) \right)^2 \right]^{1/2} \left(c_i \| \Delta \|_{1,i} + c_j \| \Delta \|_{1,j} \right).$$

Letting $K = \max\{c_i\}$, and noting that

$$\|\Delta\|_{H_{CD}} \geq \|\Delta\|_{1,i} \text{ and } \|\Delta\|_{1,j},$$

we get

$$(4.36) \leq \sum_{\Gamma_{ij}} \left[\sum_{k=n_{ij}+1}^{\infty} \left(d_k^{ij}(t) \right)^2 \right]^{1/2} \left(2 K \|\Delta\|_{H_{CD}} \right). \quad (4.37)$$

$$\text{Recall that } D_n u = \sum_p \gamma_{ij} \left(\sum_q A_{pq} \frac{\partial u}{\partial x^q} \right) \cdot n_p^{ij}.$$

$$\text{Let } \varepsilon_2 \equiv \sum_{\Gamma_{ij}} \left[\sum_{k=n_{ij}+1}^{\infty} \left(d_k^{ij}(t) \right)^2 \right]^{1/2} \text{ and } d_k^{ij} \equiv \int_{\Gamma_{ij}} D_n(u(t)) w_k^{ij} ds.$$

From Parseval's equality we have the fact that

$$\sum_{k=1}^{\infty} (d_k^{ij})^2 = \|D_n u(t)\|_{ij}^2.$$

Hence, for a fixed t and any $\varepsilon > 0$, there exists an integer n such that $\sum_{k=n}^{\infty} (d_k^{ij})^2 < \varepsilon$. So since $\sum_{k=n}^{\infty} (d_k^{ij})^2 \geq \sum_{k=m}^{\infty} (d_k^{ij})^2$ for $m > n$, $\sum_{k=n}^{\infty} (d_k^{ij})^2$ is monotonically decreasing as $n \rightarrow \infty$.

$\sum_{k=n}^{\infty} (d_k^{ij})^2$ is continuous in t since

$$\sum_{k=n+1}^{\infty} (d_k^{ij})^2 = \|D_n(u(t)) - \sum_{k=1}^n d_k^{ij} w_k^{ij}\|_{ij}^2.$$

$D_n u \in C(\Gamma_{ij} \times [0, T])$ and d_k^{ij} is continuous in t ,

Therefore, by applying Dini's Theorem (4.2), we can make $\sum_{k=n+1}^{\infty} (d_k^{ij})^2$ uniformly small in $t \in [0, T]$ for $[n]$ sufficiently large. Thus ε_2 can be made small, and

$$\int_0^{\tau} \left[\sum_{S_{ij}} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} \right] dt$$

$$\leq 2 K \varepsilon_2 \int_0^{\tau} \|\Delta\|_{H_{CD}} dt.$$

□

Lemma 4.8.

$\int_0^\tau a(u_{n,m}, u - P_n^m u) dt$ can be assumed to be zero.

Proof:

Since $P_n^m u - u$ is perpendicular to $u_{n,m}$ in the H_{CD} inner product,

$$(u_{n,m}, u - P_n^m u)_{H_{CD}} = 0.$$

If $Au = -\Delta u + u$, then $a(u_{n,m}, u - P_n^m u) = (u_{n,m}, u - P_n^m u)_{H_{CD}} = 0$.

For the general case, $a(v, w)$ is an inner product. $P_n^m u$ could be defined to be the projection induced by $a(u, v)$ and the entire argument could be made using $a(u, v)$ instead of $(u, v)_{H_{CD}}$. $\square$

With these lemmata established, we can prove the main theorem. We show the following:

For any $\varepsilon > 0$, there exists $[N]$ such that for $[n] > [N]$ there exists an integer M such that for $m > M$, $\int_0^\tau \|u - u_{n,m}\|_{H_{CD}}^2 dt < \varepsilon$.

Choose $[N]$ such that for $[n] > [N]$ Lemma 4.7 holds for any chosen ε_2 . We have established the following inequality:

$$\begin{aligned} & \frac{1}{2} \|u(\tau) - u_{n,m}(\tau)\|_0^2 + c \int_0^\tau \|u - u_{n,m}\|_{H_{CD}}^2 dt \\ & \leq \frac{1}{2} \|u^0 - u_{n,m}^0\|_0^2 + \int_0^\tau (f, u - P_n^m u)_0 dt \\ & \quad (1) \qquad (2) \\ & - \int_0^\tau a(u_{n,m}, u - P_n^m u) dt - \int_0^\tau (u_{n,m}', u - P_n^m u)_0 dt \\ & \quad (3) \qquad (4) \\ & - \int_0^\tau \left[\sum_{ij} \langle D_n u, \gamma_{ij}(u_{n,m}) - \gamma_{ji}(u_{n,m}) \rangle_{ij} + \langle D_n u, \gamma(u_{n,m}) \rangle_{\partial\Omega} \right] dt. \\ & \qquad (5) \end{aligned}$$

Letting $\Delta = u_{n,m} - u$, $D^1 = P_n^m u - u$, $D^2 = P_n^m u_t - u_t$, we use the previous lemmas to majorize this inequality to get

$$\begin{aligned}
\frac{1}{2} \|\Delta(\tau)\|_0^2 + c \int_0^\tau \|\Delta\|_{H_{CD}}^2 dt &\leq \varepsilon_1 + C^1 \int_0^\tau \|D^1\|_0 dt \\
&+ C^2 \|D^1\|_0 \Big|_{t=\tau} + C^3 \|D^1\|_0 \Big|_{t=0} \\
&+ C^4 \int_0^\tau \|D^2\|_0 dt + 2 K \varepsilon_2 \int_0^\tau \|\Delta\|_{H_{CD}} dt
\end{aligned}$$

where c and C^i , $i = 1, \dots, 4$ denote numbers independent of $[n]$ and m .

The lemmas allow us to write this as

$$\begin{aligned}
&\frac{1}{2} \|\Delta(\tau)\|_0^2 + c \int_0^\tau \|\Delta\|_{H_{CD}}^2 dt \\
&\leq \varepsilon_1 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + 2 K \varepsilon_2 \int_0^\tau \|\Delta\|_{H_{CD}} dt
\end{aligned}$$

where $\varepsilon_3 = C^1 \int_0^\tau \|D^1\|_0 dt$, $\varepsilon_4 = C^2 \|D^1\|_0 \Big|_{t=\tau}$, $\varepsilon_5 = C^3 \|D^1\|_0 \Big|_{t=0}$

and $\varepsilon_6 = C^4 \int_0^\tau \|D^2\|_0 dt$; the ε_i can be made arbitrarily small for sufficiently large m once some $[n]$ is chosen to satisfy lemma 4.7.

Letting $K_1 c = K$, and rearranging several terms, we get

$$\begin{aligned}
&c \int_0^\tau \|\Delta\|_{H_{cd}}^2 dt - 2 K_1 c \varepsilon_2 \int_0^\tau \|\Delta\|_{H_{cd}} dt \\
&\leq \varepsilon_1 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - \frac{1}{2} \|\Delta\|_0^2
\end{aligned}$$

By adding the term $c \int_0^\tau K_1^2 \varepsilon_2^2 dt$ on both sides,

$$c \int_0^\tau \left(\|\Delta\|_{H_{CD}} - K_1 \varepsilon_2 \right)^2 dt$$

$$\begin{aligned}
&= c \int_0^\tau \left(\|\Delta\|_{H_{CD}}^2 - 2 K_1 \epsilon_2 \|\Delta\|_{H_{CD}} + K_1^2 \epsilon_2^2 \right) dt \\
&\leq \epsilon_1 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 - \frac{1}{2} \|\Delta\|_0^2 + c \int_0^\tau K_1^2 \epsilon_2^2 dt \\
&\leq \epsilon' + c \int_0^\tau K_1^2 \epsilon_2^2 dt
\end{aligned}$$

where $\epsilon' = \epsilon_1 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6$.

Now we use the following lemma to conclude the proof.

Lemma 4.9.

If for all τ : $0 < \tau < T$, $\int_0^\tau (z - \sigma)^2 dt \leq \int_0^\tau \sigma^2 dt + \delta$ for σ, z and δ greater than or equal to 0, then

$$\int_0^\tau z^2 dt \longrightarrow 0 \text{ as } \sigma \longrightarrow 0 \text{ and } \delta \longrightarrow 0.$$

Proof:

First, note that if $\int_0^\tau z dt \longrightarrow \infty$ as $\sigma \longrightarrow 0$ and $\delta \longrightarrow 0$, then $\int_0^\tau (z - \sigma) dt \longrightarrow \infty$.

$$\begin{aligned}
\text{Now } \left| \int_0^\tau (z - \sigma) \cdot 1 dt \right| &\leq \sqrt{\int_0^\tau (z - \sigma)^2 dt} \sqrt{\int_0^\tau 1^2 dt} \\
&= \sqrt{\int_0^\tau (z - \sigma)^2 dt} \sqrt{\tau},
\end{aligned}$$

which implies that $\int_0^\tau (z - \sigma)^2 dt \longrightarrow \infty$.

However, the above statement contradicts our hypothesis. Hence

there is some B so that $\int_0^\tau z dt \leq B$.

$$\begin{aligned}
\text{So } \int_0^\tau z^2 dt &= \int_0^\tau (z^2 - 2\sigma z + \sigma^2) dt + \int_0^\tau (2z\sigma - \sigma^2) dt \\
&= \int_0^\tau (z - \sigma)^2 dt + \int_0^\tau (2z\sigma - \sigma^2) dt \\
&\leq \int_0^\tau \sigma^2 dt + \delta + \int_0^\tau (2z\sigma - \sigma^2) dt
\end{aligned}$$

by our hypothesis.

We have just argued that $\delta + 2\sigma \int_0^\tau z dt \leq \delta + 2\sigma B$ for some B .

Therefore, as $\sigma \rightarrow 0$ and $\delta \rightarrow 0$, $\int_0^\tau z^2 dt \rightarrow 0$. $\square$

We conclude the proof of convergence.

We have

$$c \int_0^\tau \left(\| \Delta \|_{H_{CD}} - K_1 \epsilon_2 \right)^2 dt \leq \epsilon' + c \int_0^\tau K_1^2 \epsilon_2^2 dt.$$

Dividing by c on both sides,

$$\int_0^\tau \left(\| \Delta \|_{H_{CD}} - K_1 \epsilon_2 \right)^2 dt \leq \frac{\epsilon'}{c} + \int_0^\tau K_1^2 \epsilon_2^2 dt$$

Letting $\delta = \frac{\epsilon'}{c}$, $\sigma = K_1 \epsilon_2$ and $z = \| \Delta \|_{H_{CD}}$, we have

$$\int_0^\tau (z - \sigma)^2 dt \leq \int_0^\tau \sigma^2 dt + \delta.$$

Since the previous lemmas have shown that δ and σ can be made small for sufficiently large $[n]$ and m , we can use lemma 4.9 to conclude that we can make

$$\int_0^\tau \| u - u_{n,m} \|_{H_{CD}}^2 dt < \epsilon. \quad \square$$

Appendix A.

We present here an informal description of the variational technique used to convert an elliptic problem to one of minimizing a functional.

The boundary value problem is to find u satisfying

$$-\left(\frac{\partial}{\partial x} \left(A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left(A_{21} \frac{\partial u}{\partial x} + A_{22} \frac{\partial u}{\partial y} \right) \right) + A_0 u - f = 0$$

on some domain Ω , where A_{ij} could be functions in terms of x and y , and u must satisfy certain boundary conditions. There are three types of boundary conditions. First, there is the Dirichlet

boundary condition where the value of our unknown u on the boundary Γ is given by some function. Second is the Neumann boundary condition, where a kind of a derivative of the unknown is given by some function. This is of form $\frac{\partial u}{\partial n} \Big|_{\Gamma} = g(x,y)$

$$\text{where } \frac{\partial u}{\partial n} \equiv \left(A_{11} \frac{\partial u}{\partial x} + A_{12} \frac{\partial u}{\partial y} \right) n_1 + \left(A_{21} \frac{\partial u}{\partial x} + A_{22} \frac{\partial u}{\partial y} \right) n_2$$

with the unit outward normal $\underline{n} = (n_1, n_2)$. The last is the mixed boundary condition where the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ and has Dirichlet and Neumann conditions on portions of the boundary:

$$u|_{\Gamma_1} = g_1(x,y) \text{ and } \frac{\partial u}{\partial n} \Big|_{\Gamma_2} = g_2(x,y).$$

An typical example is the equation $-\Delta u + A_0 u - f = 0$ defined over some domain Ω with boundary conditions:

$$u|_{\Gamma_1} = g_1 \text{ and } \frac{\partial u}{\partial n} \Big|_{\Gamma_2} = g_2. \quad (1)$$

We will proceed by using the example instead of the general partial differential equation given in the beginning.

We first convert the problem of solving (1) to an equivalent variational problem.

To do this, we first need Green's Theorem (which can be readily obtained from Stoke's Theorem).

Green's Theorem.

Let Ω be a domain in the X-Y plane with boundary curve Γ oriented counterclockwise. Then

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = \int_{\Gamma} (P dx + Q dy).$$

Consider the functional expression

$$\iint_{\Omega} \left[-\Delta u + A_0 u \right] v dx dy. \quad (2)$$

Expanding (2), we get

$$\begin{aligned} & \iint_{\Omega} \left[-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + A_0 u \right] v dx dy \text{ or} \\ & \iint_{\Omega} \left[-\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) v - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) v + A_0 uv \right] dx dy. \end{aligned}$$

From the product rule: $\frac{\partial}{\partial x}(WV) = \frac{\partial W}{\partial x} V + W \frac{\partial V}{\partial x}$, we have the fact that

$$\frac{\partial W}{\partial x} V = -W \frac{\partial V}{\partial x} + \frac{\partial}{\partial x}(WV).$$

Then (2) is

$$\iint_{\Omega} \left\{ \left[\left(\frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial x} - \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} v \right) + \left(\frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} - \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} v \right) \right] + A_0 uv \right\} dx dy.$$

Now the bracketed terms can be re-grouped as

$$\begin{aligned} & \iint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} + A_0 uv \right] dx dy \\ & - \iint_{\Omega} \left[\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} v \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} v \right) \right] dx dy. \end{aligned} \quad (3)$$

The second integral term in (3) can be written as

$$- \iint_{\Omega} \left[\frac{\partial}{\partial x} (Q) - \frac{\partial}{\partial y} (P) \right] dx dy \quad (4)$$

where $Q = \frac{\partial u}{\partial x} v$ and $P = - \frac{\partial u}{\partial y} v$.

Using Green's Theorem, (4) is

$$\begin{aligned} & - \int_{\Gamma} \left[P dx + Q dy \right] ds \\ &= - \int_{\Gamma} \left[- \frac{\partial u}{\partial y} v dx + \frac{\partial u}{\partial x} v dy \right] ds. \end{aligned} \quad (5)$$

If the curve Γ is parametrically written as $(x(t), y(t))$, then (5) is

$$\begin{aligned} & - \int_{\Gamma} \left[- \frac{\partial u}{\partial y} v x'(t) dt + \frac{\partial u}{\partial x} v y'(t) dt \right] \\ &= - \int_{\Gamma} \left[\frac{\partial u}{\partial x} v y'(t) dt + \frac{\partial u}{\partial y} v (-x'(t)) dt \right] \\ &= - \int_{\Gamma} v \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right] \frac{(y'(t), -x'(t))}{\sqrt{y'(t)^2 + x'(t)^2}} \sqrt{y'(t)^2 + x'(t)^2} dt. \end{aligned}$$

The unit tangent vector to Γ in the counter-clockwise direction is $(x'(t), y'(t)) / \sqrt{y'(t)^2 + x'(t)^2}$, so the outward unit normal vector is $(y'(t), -x'(t)) / \sqrt{y'(t)^2 + x'(t)^2}$. For our example,

$$\frac{\partial u}{\partial n} \equiv \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right] \cdot \frac{(y'(t), -x'(t))}{\sqrt{y'(t)^2 + x'(t)^2}}, \text{ and}$$

$$\text{also } ds \equiv \sqrt{y'(t)^2 + x'(t)^2} dt,$$

so (5) can be written as

$$- \int_{\Gamma} v \frac{\partial u}{\partial n} ds.$$

Summarizing to this point, for any smooth v , (2) is

$$\iint_{\Omega} \left[\left(\frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} + A_0 uv \right] dx dy - \int_{\Gamma} v \frac{\partial u}{\partial n} ds.$$

If u is the solution, then $-\Delta u + A_0 u - f = 0$. So

$$\iint_{\Omega} [-\Delta u + A_0 u - f] v \, dx \, dy = \iint_{\Omega} [0] v \, dx \, dy = 0.$$

$$\begin{aligned} \text{Then } 0 &= \iint_{\Omega} [-\Delta u + A_0 u - f] v \, dx \, dy \\ &= \iint_{\Omega} [-\Delta u + A_0 u] v \, dx \, dy - \iint_{\Omega} f v \, dx \, dy \\ &= \iint_{\Omega} \left[\left(-\frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial x} + \left(-\frac{\partial u}{\partial y} \right) \frac{\partial v}{\partial y} + A_0 u v \right] dx dy - \int_{\Gamma} v \frac{\partial u}{\partial n} ds - \iint_{\Omega} f v \, dx dy. \end{aligned} \quad (6)$$

Define $(h, g)_0 \equiv \iint_{\Omega} h \cdot g \, dx \, dy$ and let

$$a(u, v) \equiv \iint_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + A_0 u v \right) dx \, dy.$$

Then (6) can be written as

$$\begin{aligned} a(u, v) - (f, v)_0 - \int_{\Gamma} v \frac{\partial u}{\partial n} ds &= 0 \text{ or} \\ a(u, v) - (f, v)_0 - \int_{\Gamma_1} v \frac{\partial u}{\partial n} ds - \int_{\Gamma_2} v \frac{\partial u}{\partial n} ds &= 0. \end{aligned} \quad (7)$$

Γ_1 is the portion of the boundary curve with a Dirichlet condition, and Γ_2 is the portion with the Neumann condition i.e. on Γ_2 , $\partial u / \partial n = g_2$.

Therefore, we have

$$a(u, v) - (f, v)_0 - \int_{\Gamma_1} v (\partial u / \partial n) ds - \int_{\Gamma_2} g_2 v ds = 0.$$

for any smooth v . If $v|_{\Gamma_2} = 0$, then

$$a(u, v) - (f, v)_0 - \int_{\Gamma_2} g_2 v ds = 0.$$

The following theorem is a converse of the above result.

Theorem.

Suppose that for $u \in C^2(\Omega)$ with $u|_{\Gamma_1} = g_1$ and $f \in C(\Omega)$,

$$a(u, v) - (f, v)_0 - \int_{\Gamma_2} g_2 v ds = 0 \text{ for all } v \in H_{\Gamma_1}^1. \quad (8)$$

Then u is a solution of (1).

Here, $H_{\Gamma_1}^1$ is a space of test-functions which we define here to be functions that have continuous derivatives on $\bar{\Omega}$ and are equal to zero on Γ_1 .

Proof:

It is given that $a(u,v) = (f,v)_0 - \int_{\Gamma_2} g_2 v \, ds = 0$ for all $v \in$

$$H_{\Gamma_1}^1. \text{ Let } Lu \equiv -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + A_0 u.$$

Since $(Lu,v) = a(u,v) - \int_{\Gamma} v \frac{\partial u}{\partial n} \, ds$ (by previous work),

$$a(u,v) = (Lu,v) + \int_{\Gamma} v \frac{\partial u}{\partial n} \, ds = (Lu,v) + \int_{\Gamma_1} v \frac{\partial u}{\partial n} \, ds + \int_{\Gamma_2} v \frac{\partial u}{\partial n} \, ds.$$

Substituting for $a(u,v)$ in equation 8, we get

$$\iint_{\Omega} Lu \cdot v \, dx dy + \int_{\Gamma_1} \frac{\partial u}{\partial n} v \, ds + \int_{\Gamma_2} \frac{\partial u}{\partial n} v \, ds - \iint_{\Omega} f v \, dx dy - \int_{\Gamma_2} g_2 v \, ds = 0$$

Rearranging terms, we obtain

$$\iint_{\Omega} Lu \cdot v \, dx dy - \iint_{\Omega} f v \, dx dy = - \int_{\Gamma_1} \frac{\partial u}{\partial n} v \, ds - \int_{\Gamma_2} \frac{\partial u}{\partial n} v \, ds + \int_{\Gamma_2} g_2 v \, ds$$

Now $v \in H_{\Gamma_1}^1$, so $v|_{\Gamma_1} = 0$. Then

$$\iint_{\Omega} Lu \cdot v \, dx dy - \iint_{\Omega} f v \, dx dy = - \int_{\Gamma_2} \frac{\partial u}{\partial n} v \, ds + \int_{\Gamma_2} g_2 v \, ds \text{ or}$$

$$\iint_{\Omega} (Lu - f) \cdot v \, dx dy = \int_{\Gamma_2} \left(g_2 - \frac{\partial u}{\partial n} \right) v \, ds \text{ for all } v \in H_{\Gamma_1}^1. \quad (9)$$

We first show that $Lu - f = 0$. Assume that $Lu - f \neq 0$ at some point $(x_0, y_0) \in \Omega$. Suppose $[Lu - f](x_0, y_0) > 0$. Since we have assumed that $u \in C^2(\bar{\Omega})$, $Lu - f$ is continuous at $(x_0, y_0) \in \Omega$. So for some $R > 0$, there exists a ball $B((x_0, y_0), R) \subset \Omega$ such that $[Lu - f](x,y) > 0$ for all points $(x, y) \in B((x_0, y_0), R)$. We

can construct a $v \in H_{\Gamma_1}^1$ such that $v > 0$ on $B((x_0, y_0), R)$ and $v \equiv 0$ outside $B((x_0, y_0), R)$. Since v is zero on Γ_2 , the right hand side of (9) will be zero. Then $0 = \iint_{\Omega} (Lu - f) v \, dx \, dy = \iint_{B((x, y), R)} (Lu - f) v \, dx \, dy$. But this cannot be true, for both continuous v and $(Lu - f)$ are positive on $B((x_0, y_0), R)$. Hence there can be no points (x_0, y_0) in Ω where $Lu - f > 0$. A similar argument holds if $Lu - f < 0$ at some point in Ω . Hence $Lu - f = 0$ in Ω .

Now (9) shows that $0 = \int_{\Gamma_2} \left(g_2 - \frac{\partial u}{\partial n} \right) v \, ds$ for all $v \in H_{\Gamma_1}^1$.

Since we assume that Γ_2 is smooth, if there is some $(x(t_0), y(t_0)) \in \Gamma_2$ where $\left(g_2 - \frac{\partial u}{\partial n} \right)(x(t_0), y(t_0)) > 0$, then for $r > 0$ there will be some $B(t_0, r)$ where $\left(g_2 - \frac{\partial u}{\partial n} \right)(x(t), y(t)) > 0$. We can construct some $v \in H_{\Gamma_1}^1$ such that $v(x(t), y(t)) > 0$ for $t \in B(t_0, r)$, but $v(x(t), y(t)) = 0$ for t outside $B(t_0, r)$. Then again we would have

$$\begin{aligned} 0 &= \int_{\Gamma_2} \left(g_2 - \frac{\partial u}{\partial n} \right) v \, ds \\ &= \int_{\Gamma_2} \left(g_2 - \frac{\partial u}{\partial n} \right)(x(t), y(t)) v(x(t), y(t)) \sqrt{y'(t)^2 + x'(t)^2} \, dt \\ &= \int_{t_0-r}^{t_0+r} \left(g_2 - \frac{\partial u}{\partial n} \right)(x(t), y(t)) v(x(t), y(t)) \sqrt{y'(t)^2 + x'(t)^2} \, dt. \end{aligned}$$

But this cannot be true, for all terms in the integral are positive. Hence $g_2 - \frac{\partial u}{\partial n} \leq 0$ on Γ_2 . A similar contradiction occurs if $g_2 - \frac{\partial u}{\partial n} < 0$ at any point of Γ_2 . Hence $\frac{\partial u}{\partial n} = g_2$ on Γ_2 . $\square$

This brings us to the minimization problem that CDA actually solves.

Theorem.

If function u minimizes $\Phi(u) = a(u,u) - 2(f,u)_0 - 2\int_{\Gamma_2} g_2 u \, ds$ over all u such that $u|_{\Gamma_1} = g_1$, then u solves $Lu = f$, and $\frac{\partial u}{\partial n} = g_2$ on Γ_2 .

Proof:

Say that u_0 minimizes the expression. For any $v \in H_{\Gamma_1}^1$ and any $\varepsilon \in \mathbb{R}$, $(u_0 + \varepsilon v)|_{\Gamma_1} = g_1$, since $v|_{\Gamma_1} = 0$.

Then

$$\Phi(u_0) \leq \Phi(u_0 + \varepsilon v) = a(u_0 + \varepsilon v, u_0 + \varepsilon v) - 2(f, u_0 + \varepsilon v)_0 - 2\int_{\Gamma_2} g_2 (u_0 + \varepsilon v) \, ds.$$

$a(u,v)$ is bilinear, so

$$\begin{aligned} \Phi(u_0 + \varepsilon v) &= a(u_0, u_0) + \varepsilon a(u_0, v) + \varepsilon a(v, u_0) + \varepsilon^2 a(v, v) \\ &\quad - 2(f, u_0)_0 - 2\varepsilon(f, v)_0 - 2\int_{\Gamma_2} g_2 u_0 \, ds - 2\varepsilon\int_{\Gamma_2} g_2 v \, ds. \end{aligned}$$

Rearranging terms, we get

$$\begin{aligned} \Phi(u_0 + \varepsilon v) &= a(u_0, u_0) - 2(f, u_0)_0 - 2\int_{\Gamma_2} g_2 u_0 \, ds \\ &\quad + \varepsilon a(u_0, v) + \varepsilon a(v, u_0) - 2\varepsilon(f, v)_0 - 2\varepsilon\int_{\Gamma_2} g_2 v \, ds + \varepsilon^2 a(v, v), \text{ or} \\ \Phi(u_0 + \varepsilon v) &= \Phi(u_0) + \varepsilon a(u_0, v) + \varepsilon a(v, u_0) \\ &\quad - 2\varepsilon(f, v)_0 - 2\varepsilon\int_{\Gamma_2} g_2 v \, ds + \varepsilon^2 a(v, v) \end{aligned}$$

Operator L is self-adjoint, so $a(u_0, v) = a(v, u_0)$. Then this can be written as

$$\begin{aligned} \Phi(u_0) &\leq \Phi(u_0) + 2\varepsilon \left[a(u_0, v) - (f, v)_0 - \int_{\Gamma_2} g_2 v \, ds \right] + \varepsilon^2 a(v, v) \\ &\equiv \alpha + \beta\varepsilon + \gamma\varepsilon^2. \end{aligned}$$

The minimum for this expression exists when $\epsilon = 0$. The equation above is a simple parabola in terms of ϵ . Now from elementary calculus, the minimum exists when the derivative of the function is equal zero i.e. if $g(\epsilon) = \alpha + \beta\epsilon + \gamma\epsilon^2$, then there exists a minimum when $g'(\epsilon) = 0$. We know that the minimum occurs when $\epsilon = 0$. So

$$g'(\epsilon = 0) = 2 \left[a(u_0, v) - (f, v)_0 - \int_{\Gamma_2} g_2 v \, ds \right] + 0 = 0.$$

Thus $a(u_0, v) - (f, v) - \int_{\Gamma_2} g_2 v \, ds = 0$ for all $v \in H_{\Gamma_1}^1$.

Using the previous theorem, u solves $Lu = f$, with $\frac{\partial u_0}{\partial n} = g_2$ on Γ_2 , and $u_0|_{\Gamma_1} = g_1$ by assumption. $\square$

For completeness, we verify the bilinearity of $a(u, v)$:

We use the general form for $a(u, v)$.

$$\begin{aligned} a(u + \lambda w, v) &= \iint_{\Omega} \left[\sum_{i,j} A_{ij} D_j(u + \lambda w) D_i v + A_0(u + \lambda w)v \right] dx \, dy \\ &= \sum_{i,j} \iint_{\Omega} \left[A_{ij} \left[D_j(u) + \lambda D_j(w) \right] D_i v + A_0 u v + \lambda A_0 w v \right] dx \, dy \\ &= \sum_{i,j} \iint_{\Omega} \left[A_{ij} D_j(u) D_i v + A_0 u v \right] dx \, dy \\ &\quad + \lambda \sum_{i,j} \iint_{\Omega} \left[A_{ij} D_j(w) D_i v + A_0 w v \right] dx \, dy \\ &= a(u, v) + \lambda a(w, v). \end{aligned} \quad \square$$

Appendix B.

In this section we will further explore the method described in section 3 for constructing a basis that will automatically satisfy the moment collocation constraints. We will apply the results of using this basis to approximate the solution of elliptic problems. We will also show that the basis necessary to prove convergence for solutions to parabolic equations exists.

In section 3 we constructed a basis whose members are of the form $B_i^n \equiv \sum_{j=1}^N q_j^i \Phi^j$ where the $\{ q_j^i \}$ are orthonormal in $\mathbb{R}^N$ and the $\{\Phi^j\}$ is a suitable enumeration of the members of the Shauder basis for each individual cell. This is a basis for $G_0([n],m)$; the set of functions that are weakly continuous across internal interfaces and satisfy a weak zero boundary condition.

Therefore, any approximation satisfying the boundary condition weakly will have the form $v = \sum_i^m b_i B_i^n$. We first use this new basis to find an approximation to elliptic problems.

$$\text{Let } a(u,v) \equiv \iint_{\Omega} \left[\sum_{i,j} A_{ij} D_i u D_j v + A_0 uv \right] dx.$$

To approximate the solution to elliptic problems (with zero Dirichlet boundary data), appendix A requires that we find the function $v \in G_0([n],m)$ that minimizes

$$a(v, v) - 2(f, v) \text{ or } \sum_{i,j} b_i b_j a(B_i^n, B_j^n) - 2 \sum_i b_i (f, B_i^n).$$

This is equivalent of solving the linear system

$$\sum_i b_i a(B_i^n, B_j^n) - 2 (f, B_j^n) \text{ for } j = 1, \dots, m.$$

Now $a(B_i^n, B_j^n) = a\left(\sum_k^m q_k^i \Phi^k, \sum_q^m q_q^j \Phi^q\right) = \sum_{k,m} q_k^i q_q^j a(\Phi^k, \Phi^q)$ by the bilinearity of $a(.,.)$. These are the entries of CVC^T , where $V = a(\Phi^k, \Phi^q)$ is the upper left corner of the matrix in figure 1b of section 1:

$$\left(\begin{array}{c|c} V & U^T \\ \hline U & 0 \end{array} \right) = \left(\begin{array}{c|c} \begin{array}{ccc} \square & & \\ & \square & \\ & & \square \end{array} & U^T \\ \hline U & 0 \end{array} \right).$$

C is the matrix with orthonormal rows q_j^i that span the null space of matrix U .

Since V is symmetric and positive definite, $(V\underline{x}, \underline{x})_{\mathbb{R}^m} \geq 0$ and $(V\underline{x}, \underline{x})_{\mathbb{R}^m} = 0 \Leftrightarrow \underline{x} = 0$. So $(CVC^T\underline{y}, \underline{y})_{\mathbb{R}^k} = (V(C^T\underline{y}), C^T\underline{y})_{\mathbb{R}^m} \geq 0$. This will be 0 $\Leftrightarrow C^T\underline{y} = 0 \Leftrightarrow \underline{y} = 0$, since C consists of independent rows. So matrix CVC^T will be symmetric and positive definite also.

The size of the matrix CVC^T will be k by k where $k = (\text{size of matrix } V) - (\text{total number of collocations enforced})$. This will be reasonably small; about 140×140 for a domain composed of 10 triangular cells with 4 moment collocations enforced and 9 extra degrees of freedom (so that $3 \times 4 + 9$ basis functions are used on each cell. In our two dimensional implementation, this will give a full fifth order polynomial basis.)

This will also be the matrix that will appear in any parabolic equation with the same cell structure; its size allows the discovery of the necessary eigenvalues and vectors.

For any $G_0([n],m)$, an efficient way to construct a basis is the method suggested by Dr. Les Foster of San Jose State University using the QR factorization of U^T described in section 3. We need a complete basis for $G_0[n]$ with similar properties to prove the convergence of the approximations. The following argument shows that such a basis exists.

The rows of the matrix U were denoted in section 3 by $\{\underline{d}_{k,j}\}$. They are assumed to be independent. List these as $\{\underline{d}^1, \underline{d}^2, \dots, \underline{d}^P\}$ and form the matrix, $\begin{pmatrix} m_{ij} \end{pmatrix} = \begin{pmatrix} \underline{d}^i \cdot \underline{d}^j \end{pmatrix}$. It will be non-singular since the $\underline{d}^i$ are assumed to be independent.

Enumerate a spanning set $\{e^i\}$ for $\mathbb{R}^N$ as follows:

Let $e^1 = (1,0,0,0,\dots)$.

Let $e^2 = (0,\dots,0,1,0,\dots)$ where the one is in the (m^1+1) -th place, i.e. the place corresponding to the place of the first basis function for the second cell.

Let $e^3 = (0,\dots,0,1,0,\dots)$ where the 1 is in the location corresponding to the first basis function for the third cell, i.e. $m^1 + m^2 + 1$ place.

We continue this enumeration until we have enumerated all locations corresponding to the first basis function for each cell.

For e^{nc+1} we come back to the second function of the first cell. i.e. $e^{nc+1} = (0,1,0,\dots)$. The next term will be $e^{nc+2} = (0,\dots,0,1,0,\dots)$ where the 1 is in the location corresponding to the second basis function of the second cell.

We can continue this enumeration until we exhaust all selected basis functions.

Let $G \equiv$ the orthogonal complement in $\mathbb{R}^N$ to $\{\underline{d}_{kj}\}$.

The orthogonal projection of $\underline{e}^1$ on G can be found by solving

$$\begin{pmatrix} m_{ij} \end{pmatrix} \underline{x}^1 = \begin{pmatrix} \underline{e}^1 \cdot \underline{d}^1 \\ \underline{e}^1 \cdot \underline{d}^2 \\ \vdots \end{pmatrix}$$

So $\sum_i x_i \underline{d}^i \cdot \underline{d}^k = \underline{e}^1 \cdot \underline{d}^k$, for $k = 1, \dots, P$. Then $\underline{c}^1 \equiv \underline{e}^1 - \sum_i x_i \underline{d}^i \in G$, since $\underline{c}^1 \cdot \underline{d}^k = \underline{e}^1 \cdot \underline{d}^k - \sum_i x_i \underline{d}^i \cdot \underline{d}^k = 0$.

We can normalize $\underline{c}^1$, if $\underline{c}^1 \neq \underline{0}$. To obtain the next basis, we add a row and a column to (m_{ij}) to get

$$M^1 = \begin{pmatrix} (m_{ij}) & (\underline{e}^1 \cdot \underline{d}^1) \\ (\underline{e}^1 \cdot \underline{d}^1) \dots & 1 \end{pmatrix}.$$

So we solve

$$M^1 \underline{x}^2 = \begin{pmatrix} \underline{e}^2 \cdot \underline{d}^1 \\ \vdots \\ \underline{e}^2 \cdot \underline{d}^k \\ 0 \end{pmatrix}.$$

Hence, $\sum_i x_i^2 \underline{d}^i \cdot \underline{d}^k + x_{p+1}^2 (\underline{e}^1 \cdot \underline{d}^k) = \underline{e}^2 \cdot \underline{d}^k$ (if $k \leq p$)

and $\sum_i x_i^2 (\underline{e}^1 \cdot \underline{d}^i) + x_{p+1}^2 = 0$.

Form $\underline{c}^2 = \underline{e}^2 - \sum_i x_i^2 \underline{d}^i - x_{p+1}^2 \underline{e}^1$. Then

$\underline{c}^2 \cdot \underline{d}^k = \underline{e}^2 \cdot \underline{d}^k - \sum_i x_i^2 \underline{d}^i \cdot \underline{d}^k - x_{p+1}^2 \underline{e}^1 \cdot \underline{d}^k = 0$, if $k \leq p$

and $\underline{c}^2 \cdot \underline{e}^1 = \underline{e}^2 \cdot \underline{e}^1 - \sum_i x_i^2 \underline{d}^i \cdot \underline{e}^1 - x_{p+1}^2 = 0$.

Since $\underline{c}^2 \cdot \underline{e}^1 = 0$ and $\underline{c}^2 \cdot \underline{d}^k = 0$, and due to the fact that $\underline{c}^1$ is in the span of $\{ \underline{e}^1, \underline{d}_x \}$, we have $\underline{c}^2 \cdot \underline{c}^1 = 0$.

If we continue adding a further symmetric column and row to M^1 when $\underline{c}^2 \neq 0$, we will get an orthonormal set $\{\underline{c}^k\}$ of dimension $N-P$, spanning G ; exactly P of the computed projections should be 0 by a consideration of the dimensions involved.

Let $B_k^n \equiv \sum_i c_i^k \Phi^i$ where Φ^i is the enumeration of $\{\Psi_j^k\}$; $\{\Phi^1, \Phi^2, \dots\} \equiv \{\Psi_1^1, \Psi_2^1, \Psi_3^1, \dots, \Psi_m^1, \Psi_1^2, \Psi_2^2, \Psi_3^2, \dots\}$. Then any v satisfying the internal moment collocation constraints and matching the weakly zero boundary data will have form $v = \sum b_i B_i^n$.

This process can be (theoretically) be continued indefinitely for increasingly larger values of m to obtain a basis for $G_0[n]$. This basis will have the required property needed in section 3. By lemma 3.3, if P_n^m is the orthogonal H_{CD} projection of $G_0[n]$ onto the span of $\{ B_1^n, \dots, B_m^n \}$, then for any $u \in G_0[n]$,

$$\lim_{m \rightarrow \infty} \| u - P_n^m u \|_{H_{CD}} = 0.$$

Appendix C.

We present here the method to solve the problem: $\underline{u}' + A\underline{u} = \underline{f}$ where $\underline{u}|_{t=0} = \underline{u}^0$ (initial condition), $\underline{u}(t) \in \mathbb{R}^n$. We come upon this problem when we use the Cell Discretization Algorithm to solve parabolic partial differential equations. The following is abstracted from Boyce and DePrima [2].

In the equation $\underline{u}' + A\underline{u} = \underline{f}(t)$, $\underline{u}$ is the solution vector, $\underline{f}$ is a given vector and A is a matrix. " u " denotes the derivative of u with respect to time. Recall that $\underline{f}$ has been assumed to be continuous on interval $0 < t < \tau$.

We first look at the problem in $\mathbb{R}^1$: $u' + au = f$ with the initial condition $u|_{t=0} = u^0$. If we multiply both sides of our differential equation by e^{at} , we have

$$\begin{aligned} e^{at} \frac{du}{dt} + ae^{at}u &= e^{at} f, \text{ or} \\ \frac{d}{dt} \left(ue^{at} \right) &= e^{at} f. \end{aligned} \quad (1)$$

Integrating equation (1) over $[0, t]$ on both sides, we obtain

$$ue^{at} = \int_0^t e^{as} f \, ds + c, \quad (2)$$

where c is a constant that is determined by the initial condition: $u(0) e^0 = u(0) = 0 + c$, so $c = u^0$.

Therefore the solution is

$$u = e^{-at} \int_0^t e^{as} f \, ds + u^0 e^{-at}. \quad (3)$$

Let $\underline{u} = (u_1, u_2, \dots, u_n)$ and $\underline{f} = (f_1, f_2, \dots, f_n)$.

We first solve the homogeneous problem

$$\underline{u}' + A\underline{u} = \underline{0}.$$

We seek solutions of the form $\underline{u} = \underline{K}e^{-\lambda t}$ where we need to determine

$\underline{K}$ and r . Substituting into the differential equation gives

$$r\underline{K}e^{-rt} + A\underline{K}e^{-rt} = \underline{0}.$$

Since e^{-rt} is non-zero we obtain

$$\left(-r\underline{I} + A \right) \underline{K} = \underline{0}$$

where $\underline{I}$ is the $n \times n$ identity matrix. The problem above is same as the problem of determining the eigenvalues and eigenvectors for the matrix A .

Since matrix A is symmetric and all of the entries are real, it is Hermitian. Therefore, the eigenvalues of A are real and there is a full set of n linearly independent eigenvectors. Denoting the eigenvalues of the matrix A as r_i and the corresponding eigenvectors by $\underline{\xi}^i$, the eigenvalues r satisfy the equation $A\underline{\xi} = r\underline{\xi}$. (Strang [9])

The general solution of the homogeneous n dimensional problem is

$$\underline{u} = \sum_{i=1}^n c^i \underline{\xi}^i e^{-r_i t}$$

where c^i are constants.

For the inhomogeneous case we have the equation

$$\underline{u}' + A\underline{u} = \underline{f}. \quad (4)$$

with the initial condition $\underline{u}|_{t=0} = \underline{u}^0$.

Let T be the matrix whose columns are the eigenvectors $\underline{\xi}^1, \dots, \underline{\xi}^n$ of A , and define a new dependent variable $\underline{v}$ by $\underline{u} \equiv T\underline{v}$, where $\underline{v} = (v_1, v_2, \dots, v_n)$. Substituting into equation 4, we obtain

$$T\underline{v}' + AT\underline{v} = \underline{f}.$$

Multiplying by T^{-1} , we have

$$\begin{aligned}\underline{v}' + (T^{-1}AT)\underline{v} &= T^{-1}\underline{f}, \text{ or} \\ \underline{v}' + D\underline{v} &= \underline{g}\end{aligned}\quad (5)$$

where $\underline{g} \equiv T^{-1}\underline{f}$ and D is the diagonal matrix whose diagonal entries are the eigenvalues, $-r_1, \dots, -r_n$ of A arranged in the same order as the corresponding eigenvectors $\underline{\xi}^1, \dots, \underline{\xi}^n$. Since equation 5 is a system of n -uncoupled equations in terms of v_i , $i = 1, \dots, n$, we can solve each equation separately. Each equation has the form,

$$v_i'(t) + r_i v_i(t) = g_i(t), \quad j = 1, \dots, n \quad (6)$$

where $g_i(t)$ is a linear combination of $f_1(t), \dots, f_n(t)$. Note that equation 6 consists of n one-dimensional problems.

Hence

$$v_i(t) = e^{-r_i t} \int_0^t e^{r_i s} g_i(s) ds + c_i e^{-r_i t}, \quad i = 1, \dots, n.$$

The initial condition gives us that $\underline{u}^0 = \underline{u}(0) = T\underline{v}(0) = T\underline{c}$. Hence $\underline{c} = T^{-1}\underline{u}^0$.

With this $\underline{c}$, the solution is

$$u(t) = \sum_{j=1}^n \underline{\xi}^j \left(e^{-r_j t} \int_0^t e^{r_j s} g_j(s) ds + c_j e^{-r_j t} \right).$$

Bibliography

1. Aubin, J.P. **Approximation of Elliptic Boundary Value Problems.** Wiley, 1972.
2. Boyce, W. and R. DiPrima. **Elementary Differential Equations and Boundary Value Problems.** Wiley, 1977.
3. Coffey, Mark. **Some Aspects of Treating Parabolic Differential Equations with the Cell Discretization Algorithm.** IBM Palo Alto Scientific Center Report No. G230-3466, 1984.
4. Fleming, W. **Functions of Several Variables.** Springer-Verlag, 1977.
5. Greenstatdt, John. **Cell Discretization III - Treatment of Discrete Equations; (Hyperbolic and Parabolic Cases).** IBM New York Scientific Center Report No. 320-2914, July 1967.
6. Grivard, Pierre. **Elliptic Problems in Non-smooth Domains.** Pitman, 1985.
7. Kreyzig, E. **Introductory Functional Analysis with Applications,** Wiley, 1978.
8. Lion, J.L. and E. Magenes. **Non-Homogeneous Boundary Value Problems and Applications, Volume I.** Springer-Verlag, 1972.
9. Strang, G. **Linear Algebra with its Applications.** Academic Press, 1980.
10. Swann, Howard and M. Nishimura. **Implementation of the Cell Discretization Algorithm for Solving Elliptic Partial Differential Equations, Fall 1988, Final Report.** San Jose State University Center for Applied Mathematics and Computer Science, 1988.
11. Teman, R. **Navier-Stokes Equations.** Revised Edition. North-Holland, 1984.