2007

Contour integration with applications

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Poel, Misako van der, "Contour integration with applications" (2007). *Master's Theses*. 3389.
DOI: <https://doi.org/10.31979/etd.cfsm-nh3w>
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CONTOUR INTEGRATION

WITH

APPLICATIONS

A Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Misako van der Poel

May 2007

UMI Number: 1445273

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ABSTRACT

CONTOUR INTEGRATION WITH APPLICATIONS

by Misako van der Poel

Complex analysis is considered a powerful tool in solving problems in mathematics, physics, and engineering.

We use contour integration to evaluate real integrals on the intervals $[0, 2\pi]$ and $(-\infty, \infty)$. We also discuss research on how to avoid contour integration to calculate integrals. We evaluate the Fresnel integrals using two different contours and we find a

formula for evaluating $\int_0^{\infty} \left(\frac{\sin x}{x} \right)^n dx$ by using two methods.

One of Cauchy's great works is his contribution to finite sums and series. Here, it is shown how summation of series is computed using Lindelof's formulas. Also we derive the Hardy-Egorychev identity.

Cauchy's calculus of residues is extremely important. Using this method, we solve ordinary differential equations and derive Cauchy-Tanaka integral formula.

ACKNOWLEDGEMENTS

Having met my thesis advisor Dr. Samih Obaid, who is outstandingly nice and amazingly knowledgeable, is one of the luckiest happenings in my life. I will never forget that time when we laughed and laughed together.

Also, it is honorable to have respectable Dr. Hidefumi Katsuura, who is a man of strong faith and has his own outlook on the world, as one of my thesis committee members.

I chose my thesis topic from complex analysis because of Dr. Marilyn Blockus. I appreciate her impressive lectures and warm-hearted personality.

Also, I would like to say thank you to a mathematics instructor at Foothill College, Mr. Brian Stanley, who gave me a strong impact on choosing mathematics as my major.

Finally, I dedicate my thesis and gratitude to Karen and Michelle who chose to be born as my daughters.

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INTRODUCTION

Cauchy is considered as a principal founder of complex function theory. However, the brilliant Swiss mathematician Euler (1707-1783) also took an important role in the field of complex analysis.

The symbol i for $\sqrt{-1}$ was first used by Euler, who also introduced π as the ratio of the length of a circumference of a circle to its diameter and e as a base for the natural logarithms, respectively. He also developed one of the most useful formulas in mathematics. That is Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$, which can be derived from De Moivre's theorem. Letting $\theta = \pi$ in Euler's formula, we obtain the equation $e^{i\pi} + 1 = 0$. The equation, $e^{i\pi} + 1 = 0$, contains what some mathematicians believe as the five most significant numbers in all of mathematics, i.e. e, π, i and the two integers 0 and 1. He also gave the remarkable result $i^i = e^{\frac{-\pi}{2}}$, which means that an imaginary power of an imaginary number can be a real number, and he showed that the system of complex numbers is closed under the elementary transcendental operations.

Moreover, Euler was the first person to have used complex analysis methods for trying to prove Fermat's last theorem, and he actually proved the impossibility of integer solutions of $x^3 + y^3 = z^3$. Euler's contribution to complex analysis is significant. His work laid the foundation from which another mathematical giant, Cauchy, developed the full scale theory of complex analysis and its applications.

Complex analysis is essentially the study of complex numbers, their derivatives, and integrals with many other properties. On the other hand, number theory is concerned with the study of the properties of the natural numbers. These two fields of study seem to be unrelated to each other. However, in number theory, it is sometimes impossible to prove or solve some problems without the use complex analysis. The same is true in many problems in applied mathematics, physics and engineering.

In a calculus course, the exact value of some real-valued definite integrals cannot be computed by finding its antiderivative because some antiderivatives are impossible to find in terms of elementary functions. In these cases, the integrals are only calculated numerically. Amazingly, complex analysis enables us to evaluate many of these integrals. One important application of complex analysis is to find the values of real definite integrals by calculating contour integrals in the complex plane. A real definite

integral is an integral $\int_a^b f(x)dx$, where $f(x)$ is a real-valued function with real numbers

a and b . Here, a general definite integral is taken in the complex plane, and we have

$\int_C f(z)dz$ where C is a contour and z is a complex variable. Contour integration

provides a method for evaluating integrals by investigating singularities of the function in domains of the complex plane. The theory of functions of a real variable had been developed by Lagrange, but the theory of functions of a complex variable had been achieved by the efforts of Cauchy (see [8]).

The French mathematician Augustin-Louis Cauchy (1789 -1857) brought great contributions to mathematics, providing foundations for mathematical analysis,

establishing the limit concept and general theory of convergence, and defining the definite integral as the limit of a sum. Cauchy also devised the first systematic theory of complex numbers. Especially, his best-known work in complex function theory provides the Cauchy integral theorem as a powerful tool in analysis.

Moreover, his discovery of the calculus of residues is extremely valuable and its applications are marvelous because it can be applied to the evaluation of definite integrals, the summation of series, solving ordinary and partial differential equations, difference and algebraic equations, to the theory of symmetric functions, and moreover to mathematical physics.

The name of Cauchy appears today in connection with a number of theorems in many fields in mathematics. Cauchy published 789 papers and 7 books in mathematics. In his greatest contribution to mathematics, Cauchy provided foundations for mathematical analysis. In his first classic *Cours d'analyse de l'Ecole Royale Polytechnique* ("Course on Analysis for the Ecole Royal Polytechnic," 1821), he presented some of these foundations, including the establishment of precise definition of "the limit" for calculus.

In the *Resume*, Cauchy gave his definition of the derivative as a limit and defined the definite integral as the limit of a sum. As a consequence, Weierstrass and Riemann extended his work on complex function theory, and also Riemann and Lebesgue improved on his definition of the integral. In his book, he also retained the mean-value theorem much as Lagrange had derived it. Unlike Lagrange, Cauchy used tools from

integral calculus to obtain Taylor's theorem. Cauchy also did more with convergence properties for Taylor's series.

In the *Cours d'analyse*, he had formulated the Cauchy criterion independently of Bolzano early in 1817. In his writings, Cauchy proved and often utilized the ratio, root, and integral tests, thereby establishing the first general theory of convergence (see [8]).

Among those great works, the importance of Cauchy's residue theory cannot be underestimated. Let us talk here about many applications of contour integration, first in elasticity theory (see [3], [11], [28]), in hydrodynamics (see [20]), in special functions (see [24]), and in number theory and specifically the applications to Riemann zeta function (see [17], [13], [25],[32]).

In chapter 1, we will state some basic known results in complex analysis and chapter 2 will be devoted to the evaluation of some real-valued definite integrals including some recent research results on the subject. In chapter 3, we will explore the applications of Cauchy's residue theory in evaluating finite sums of series using Lindelof [19], Egorychev [12], and Henry [14] methods. In chapter 4, we introduce the works of Cauchy and Hermite in solving ordinary differential equations and give some examples as applications to the methods. In the end we discuss the results of Tanaka [31] on computing functions of matrices.

CHAPTER 1

Basic Results Related to Contour Integrals

We introduce some fundamental results in complex analysis. These are essential to understand the subsequent chapters which include applications of Cauchy's residue theory.

1.1 Basic Theorems in Complex Analysis

Analytic Functions

Definition. Let $z = x + iy$ be a complex variable. A function $f(z)$ is analytic at z_0 in a domain D if it is differentiable in a neighborhood of z_0 . It is analytic in D if it is differentiable at each point in D . If a function $f(z)$ fails to be analytic at some point z_0 but is analytic at some point in every neighborhood of z_0 , then we say z_0 is a singular point for $f(z)$.

There are three types of singularities, removable, poles, and essential singularities (see [27]). These singularities play an important role in contour integration. We shall assume that the reader is familiar with the notion of contour integrals and properties of integrals (see [10]).

Next we give a definition for a contour (see [27]). There are other equivalent definitions (see [10]).

Contour

For complex integration, contours are indispensable. Let us make it clear what a Contour is.

Definition. A contour Γ is either a single point z_0 or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, \dots, \gamma_n)$ such that the terminal point of γ_k coincides with the initial point of γ_{k+1} for each $k = 1, 2, \dots, n-1$. In this case one can write $\Gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$. For the definitions of a smooth curve and a directed smooth curve, see [27].

Cauchy's Integral Theorem

The amazing characteristic of Cauchy's integral theorem is that the value of the contour integral of $f(z)$ is always zero as long as the singularities of $f(z)$ are outside the closed contour C . In other words, if a simple closed contour C is drawn in a complex plane, and if a function $f(z)$ of the complex variable $z = x + iy$ is analytic at every point on C and within C , then the line integral of $f(z)$ taken along C is zero, *i.e.*

$$\oint_C f(z) dz = 0.$$

Cauchy's Integral Formula

Cauchy's integral theorem states that if $f(z)$ is analytic in a simply connected domain D , then $\oint_C f(z) dz = 0$, where C is any loop in D . Strangely, we have the

following. If a is inside C , then

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz = f(a).$$

It is considered one of the greatest achievements of Cauchy for its usefulness in solving boundary value problems. By extending this result, Cauchy obtained the following.

The Generalized Cauchy's Integral Formula

If f is analytic inside and on the simple closed positively oriented contour C and if a is any point inside C , then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad (n = 1, 2, 3, \dots).$$

The Cauchy integral formula and the Generalized Cauchy's integral formula can be used to prove the residue theorem.

1.2 Theorems Involving Cauchy's Residues

In this section, we discuss ideas related to the evaluation of complicated definite integrals of real valued functions simply by computing residues.

First, we focus on how analytic functions in an annulus can be expanded as a Laurent series.

Laurent Series

Since the functions defined by power series have derivatives of all orders and analytic functions also have derivatives of all orders, it seems that there may be some connections between analytic functions and power series. Roughly speaking, a Laurent series is an expanded form of a Taylor series employing both positive and negative powers of the complex variable z . Laurent series is a version of generalizing the Taylor series method to analytic functions in an annulus $r < |z - z_0| < R$.

Theorem [27]. Suppose $f(z)$ is analytic in the annulus $r < |z - z_0| < R$. Then

$f(z)$ can be expressed there as the sum of two series

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j},$$

both series converging in the annulus, and converging uniformly in any closed subannulus $r < \rho_1 \leq |z - z_0| \leq \rho_2 < R$. The coefficient a_j is given by

$$a_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} d\zeta \quad (j = 0, \pm 1, \pm 2, \dots),$$

where C is any positively oriented simple closed contour lying in the annulus and containing z_0 in its interior.

The computation of a Laurent series expansion is tedious in most cases. Since the residue of $f(z)$ at z_0 is defined to be the coefficient a_{-1} in the Laurent expansion, we want formulas that enable us to find a_{-1} easily, which lead to calculating the residue of $f(z)$.

Residue

Definition [27]. If $f(z)$ has an isolated singularity at the point z_0 , then the

coefficient a_{-1} of $\frac{1}{z - z_0}$ in the Laurent expansion for $f(z)$ around z_0 is called the residue of $f(z)$ at z_0 and is denoted by

$$\text{Res}[f(z); z_0].$$

Here are the methods for evaluating residues at poles. In general, when $f(z)$ has a

simple pole at $z = \alpha$, the residue is

$$a_{-1} = \lim_{z \rightarrow \alpha} [(z - \alpha)f(z)] = \text{Res}[f(z); z_0],$$

and when $f(z)$ has pole of order m at $z = \alpha$, then the residue of $f(z)$ at $z = \alpha$ is

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \rightarrow \alpha} \left[\frac{d^{m-1}}{dz^{m-1}} (z - \alpha)^m f(z) \right] = \text{Res}[f(z); z_0].$$

Cauchy's Residue Theorem. Let D be a simply connected domain and let C be a simple closed positively oriented contour that lies in D . If $f(z)$ is analytic inside and on C except at the points $z_1, z_2, \dots, z_n$ inside C , then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^n \text{Res}[f(z); z_j].$$

Residue at Infinity and Some Examples

Definition. We say $f(z)$ has a singular point at $z = \infty$ if $f\left(\frac{1}{w}\right)$ has singularity at

$w = 0$. Let $f(z)$ have an isolated singular point at $z = \infty$. Then the residue of $f(z)$ at infinity is defined by

$$\text{Res}[f(z); \infty] = -\frac{1}{2\pi i} \oint_C f(z) dz,$$

where C is a circle of big radius $|z| = R$ in the counterclockwise direction so that the only singularity of $f(z)$ in $|z| \geq R$ is the point $z = \infty$.

Note that the use of negative sign is different from the residue at finite isolated singularities. The sign effectively means that the residue is being computed at the point

that is exterior to the circle in the clockwise direction. It can be shown that residue of $f(z)$ at infinity is equal to negative the coefficient of w in the series representation of $f\left(\frac{1}{w}\right)$. The Cauchy residue theorem may be rewritten as follows.

$$\oint_C f(z) dz = -2\pi i \sum_{j=1}^{\infty} \text{Res}[f(z); b_j]$$

where the b_j 's are outside C , for $j = 1, 2, 3, \dots$.

Example 1. Let $f(z) = e^z = 1 + z + \frac{z^2}{2!} + \dots$. Then $f\left(\frac{1}{w}\right) = 1 + \frac{1}{w} + \frac{1}{2w^2} + \dots$.

Thus, $f(z)$ has an essential singularity at $z = \infty$. Since the coefficient of w in the expansion of $f\left(\frac{1}{w}\right)$ is zero, then $\text{Res}(e^z; \infty) = 0$.

Example 2. Let $g(z) = \frac{1}{1-z}$. Then $g\left(\frac{1}{w}\right) = \frac{-w}{1-w}$. Clearly $g\left(\frac{1}{w}\right)$ is analytic at $w = 0$.

Thus, $g(z)$ is analytic at $z = \infty$.

Notice that $g(w) = -w(1 + w + w^2 + \dots + \dots)$ for $|w| < 1$.

Hence, $\text{Res}[g(z); \infty] = 1$. This is a result of the definition of the residue at infinity which can give non zero residue at infinity even though the function is analytic at infinity.

1.3 Additional Important Theorems

In this section, three important theorems are introduced. Two of them are the Argument theorem and Rouché's theorem which are derived from Cauchy's residue theorem. They have important practical applications and are related to the functions whose isolated singularities are poles.

Argument Theorem. Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C . Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P$$

where N and P are, respectively, the number of zeros and poles of $f(z)$ inside C (multiplicity included) (see [29]).

Rouché's Theorem [33]. If $f(z)$ and $g(z)$ are analytic inside and on a simple closed contour C and if the strict inequality

$$|g(z)| < |f(z)|$$

holds at each points on C , then $f(z) + g(z)$ and $f(z)$ have the same total number of zeros (counting multiplicities) inside C .

The following theorem is important in finding special expansions of transcendental functions (see [15]).

Mittag-Leffler's Expansion Theorem. Let a function $f(z)$ be analytic in a domain D except at a finite number of poles. If

I) the only singularities of $f(z)$ in the finite z plane are the simple poles at

$z = a_1, z = a_2, \dots, z = a_n, \dots$, where $0 < |a_1| < |a_2| < \dots < |a_n| < \dots$,

II) $b_1, b_2, \dots, b_n, \dots$ are the residues of $f(z)$ at $z = a_1, a_2, \dots, a_n, \dots$, respectively, and

III) there is a sequence of closed contours C_N , either circles or squares as shown below,

such that none of the contours C_N contains any of the poles of $f(z)$.

Let C_N be a square and let the minimum distance R_N of C_N from the origin approach

infinity as $N \rightarrow \infty$, while the length L_N of C_N is $O(R_N)$ and such that $|f(z)| \leq M$ where

M is independent of N on C_N . Then

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\}$$

for all z except the poles.

CHAPTER 2

REAL-VALUED INTEGRALS

Here we present some examples of evaluating real-valued integrals by using several contours. The diverse applications of Cauchy's residue theorem will be apparent. We will present examples including the Fresnel integrals, integrals of logarithmic functions and we will show some results by Apostol [1], Arora [2], and Young [34].

2.1 Real-valued Integrals on the Interval $[0, 2\pi]$

Example 1. Let us evaluate the integral

$$\int_0^{2\pi} \frac{1}{2 + \sin \theta} d\theta.$$

Let $z = e^{i\theta}$. Then $\frac{1}{z} = e^{-i\theta}$. And $\sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$, $\frac{1}{iz} dz = d\theta$, and $|z| = 1$.

So we have $2 + \sin \theta = 2 + \frac{z - \frac{1}{z}}{2i} = 2 - \frac{i}{2} \left(z - \frac{1}{z} \right) = \frac{1}{2} \left(4 - iz + \frac{i}{z} \right)$. We then have

$$\frac{1}{2 + \sin \theta} d\theta = \frac{2}{\left(4 - iz + \frac{i}{z} \right)} \frac{1}{iz} dz = \frac{2}{4iz + z^2 - 1} dz = \frac{2}{z^2 + 4iz - 1} dz.$$

Solving $z^2 + 4iz - 1 = 0$ gives us that

$$z = \frac{-4i \pm \sqrt{(4i)^2 - 4(-1)}}{2} = \frac{-4i \pm \sqrt{-16 + 4}}{2} = \frac{-4i \pm \sqrt{-12}}{2} = \frac{-4i \pm 2\sqrt{3}i}{2} = i(-2 \pm \sqrt{3}),$$

which means that the roots are purely imaginary. Now we factor the polynomial as follows;

$$z^2 + 4iz - 1 = [z - i(-2 + \sqrt{3})][z - i(-2 - \sqrt{3})] = [z - i(\sqrt{3} - 2)][z + i(\sqrt{3} + 2)].$$

Let $p = i(\sqrt{3} - 2)$ and $q = -i(\sqrt{3} + 2)$, then p lies within the circle $|z| = 1$, while q

lies outside the circle $|z| = 1$. The function $(z - q)^{-1}$ is analytic in $|z| \leq 1$.

Thus, we have

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{1}{(z - q)(z - p)} dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{\frac{1}{z - q}}{z - p} dz.$$

Let $f(z) = \frac{1}{z - q}$. Then we obtain that

$$f(p) = \frac{1}{2\pi i} \int_{|z|=1} \frac{\frac{1}{z - q}}{z - p} dz = \frac{1}{i(\sqrt{3} - 2) - (-i(\sqrt{3} + 2))} = \frac{1}{i\sqrt{3} - i2 + i\sqrt{3} + i2} = \frac{1}{2i\sqrt{3}}$$

by Cauchy's integral formula. Then we can have

$$\oint \frac{1}{2 + \sin \theta} d\theta = \frac{4\pi i}{2i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

□

Example 2. We will evaluate the real value integral

$$I = \int_0^{2\pi} \frac{1}{1 - 2a \cos \theta + a^2} d\theta \text{ for } 0 < a < 1$$

by transforming it into a contour integral.

Let $z = e^{i\theta}$ and $\frac{1}{z} = e^{-i\theta}$. Then $\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$ and $d\theta = \frac{dz}{iz}$.

Since $1 - 2a \cos \theta + a^2 = 1 - 2a \cdot \frac{1}{2} \left(z + \frac{1}{z} \right) + a^2 = 1 + a^2 - a \left(z + \frac{1}{z} \right)$, we have

$$\begin{aligned}\frac{d\theta}{1-2a\cos\theta+a^2} &= \frac{1}{1+a^2-a\left(z+\frac{1}{z}\right)} \cdot \frac{1}{iz} dz = \frac{1}{i\left(z(1+a^2)-az^2-a\right)} dz \\ &= \frac{1}{i(-az^2+(1+a^2)z-a)} dz.\end{aligned}$$

Now we know that

$$-az^2+(1+a^2)z-a = -a\left(z^2-\frac{z}{a}-az+1\right) = -a\left(z-\frac{1}{a}\right)(z-a).$$

$$\text{Hence, } \frac{1}{i} \int_{|z|=1} \frac{1}{-a\left(z-\frac{1}{a}\right)(z-a)} dz = \frac{1}{i} \int_{|z|=1} \frac{\frac{1}{-a\left(z-\frac{1}{a}\right)}}{z-a} dz.$$

Let $f(z) = \frac{1}{-a\left(z-\frac{1}{a}\right)i}$. Clearly $f(z)$ is analytic inside $|z| \leq 1$ because $z = \frac{1}{a}$ is outside

the circle $|z| = 1$. Then

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{\frac{1}{-a\left(z-\frac{1}{a}\right)i}}{z-a} dz = f(a) \text{ by Cauchy's integral formula.}$$

Thus, we conclude that

$$I = 2\pi i f(a) = 2\pi \frac{1}{-a\left(a-\frac{1}{a}\right)} = 2\pi \frac{1}{(-a^2+1)} = \frac{2\pi}{1-a^2}.$$

□

Example 3. We wish to evaluate the integral

$$I = \int_0^{2\pi} \cos^{2n+1} \theta d\theta \text{ where } n = 0, 1, 2, \dots$$

Let $z = e^{i\theta}$. We have as before, $\frac{1}{iz} dz = d\theta$ and $|z| = 1$. In addition,

$$\cos z = \frac{z + \frac{1}{z}}{2}, \quad \cos^{2n+1} z = \left(\frac{z + \frac{1}{z}}{2} \right)^{2n+1} = \frac{1}{2^{2n+1}} \left(z + \frac{1}{z} \right)^{2n+1}. \quad \text{Then}$$

$$\begin{aligned} I &= \int_{|z|=1} \frac{1}{2^{2n+1}} \left(z + \frac{1}{z} \right)^{2n+1} \frac{1}{iz} dz = \frac{1}{i2^{2n+1}} \int_{|z|=1} \left(z + \frac{1}{z} \right)^{2n+1} \frac{1}{z} dz = \frac{1}{i2^{2n+1}} \int_{|z|=1} \frac{(z^2 + 1)^{2n+1}}{z^{2n+1}} \frac{1}{z} dz \\ &= \frac{1}{i2^{2n+1}} \int_{|z|=1} \frac{(z^2 + 1)^{2n+1}}{z^{2n+2}} dz \end{aligned} \quad (1.1)$$

By the binomial theorem, we have

$$(1 + z^2)^{2n+1} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (z^2)^k.$$

So (1.1) can be written as

$$\begin{aligned} &\frac{1}{i2^{2n+1}} \int_{|z|=1} \frac{1}{z^{2n+2}} \left[1 + \binom{2n+1}{1} (z^2) + \binom{2n+1}{2} (z^2)^2 + \cdots + \binom{2n+1}{n-1} (z^2)^{n-1} \right. \\ &\quad \left. + \binom{2n+1}{n} (z^2)^n + \binom{2n+1}{n+1} (z^2)^{n+1} + \cdots + \binom{2n+1}{2n-1} (z^2)^{2n-1} + \binom{2n+1}{2n} (z^2)^{2n} + \binom{2n+1}{2n+1} (z^2)^{2n+1} \right] dz \\ &= \frac{1}{i2^{2n+1}} \int_{|z|=1} \left[\frac{1}{z^{2n+2}} + \binom{2n+1}{1} \frac{z^2}{z^{2n+2}} + \binom{2n+1}{2} \frac{z^4}{z^{2n+2}} + \cdots \right. \\ &\quad \left. + \binom{2n+1}{n-1} \frac{z^{2n-2}}{z^{2n+2}} + \binom{2n+1}{n} \frac{z^{2n}}{z^{2n+2}} + \binom{2n+1}{n+1} \frac{z^{2n+2}}{z^{2n+2}} + \cdots \right. \\ &\quad \left. + \binom{2n+1}{2n-1} \frac{z^{4n-2}}{z^{2n+2}} + \binom{2n+1}{2n} \frac{z^{4n}}{z^{2n+2}} + \binom{2n+1}{2n+1} \frac{z^{4n+2}}{z^{2n+2}} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{i2^{2n+1}} \int_{|z|=1} \left[\frac{1}{z^{2n+2}} + \binom{2n+1}{1} \frac{1}{z^{2n}} + \binom{2n+1}{2} \frac{1}{z^{2n-2}} + \cdots \right. \\
&\quad \left. + \binom{2n+1}{n-1} \frac{1}{z^4} + \binom{2n+1}{n} \frac{1}{z^2} + \binom{2n+1}{n+1} 1 + \cdots + \binom{2n+1}{2n-1} z^{2n-4} + \binom{2n+1}{2n} z^{2n-2} + z^{2n} \right] dz. \quad (1.2)
\end{aligned}$$

Since there is no $\frac{1}{z}$ term in (1.2), the coefficient of $\frac{1}{z}$ is zero. From (1.1), $z = 0$ is

inside $|z| = 1$, and $\frac{(z^2 + 1)^{2n+1}}{z^{2n+2}}$ has a pole at $z = 0$. By the definition of the residue of

$$\frac{(z^2 + 1)^{2n+1}}{z^{2n+2}} \text{ at } z = 0, \operatorname{Res}(0) = 0.$$

Thus, $I = \frac{1}{i2^{2n+1}} \cdot 2\pi i \cdot (0) = 0$ for all $n \in \mathbb{N}$. □

Example 4. Next we will evaluate the integral

$$I = \int_0^{2\pi} \cos^{2n} \theta \, d\theta, \text{ where } n = 0, 1, 2, 3, \dots.$$

We know that

$$\cos z = \frac{z + \frac{1}{z}}{2}, \quad \cos^{2n} z = \left(\frac{z + \frac{1}{z}}{2} \right)^{2n} = \frac{1}{2^{2n}} \left(z + \frac{1}{z} \right)^{2n}.$$

Thus,

$$\begin{aligned}
I &= \int_{|z|=1} \frac{1}{2^{2n}} \left(z + \frac{1}{z} \right)^{2n} \frac{1}{iz} dz = \frac{1}{i2^{2n}} \int_{|z|=1} \frac{1}{z} \left(\frac{z^2 + 1}{z} \right)^{2n} dz = \frac{1}{i2^{2n}} \int_{|z|=1} \frac{1}{z} \frac{(z^2 + 1)^{2n}}{z^{2n}} dz \\
&= \frac{1}{i2^{2n}} \int_{|z|=1} \frac{(z^2 + 1)^{2n}}{z^{2n+1}} dz. \quad (1.3)
\end{aligned}$$

By using the binomial theorem, $(z^2 + 1)^{2n}$ can be written as $\sum_{k=0}^{2n} \binom{2n}{k} (z^2)^k$.

Hence, (1.3) is written in the form

$$\begin{aligned}
I &= \frac{1}{i2^{2n}} \int_{|z|=1} \frac{1}{z^{2n+1}} \left[1 + \binom{2n}{1} (z^2)^1 + \binom{2n}{2} (z^2)^2 + \dots \right. \\
&\quad \left. \dots + \binom{2n}{n} (z^2)^n + \binom{2n}{2n-1} (z^2)^{2n-1} + \binom{2n}{2n} (z^2)^{2n} \right] dz \\
&= \frac{1}{i2^{2n}} \int_{|z|=1} \left[\frac{1}{z^{2n+1}} + \binom{2n}{1} \frac{1}{z^{2n-1}} + \binom{2n}{2} \frac{1}{z^{2n-3}} + \dots + \binom{2n}{n} \frac{1}{z} + \binom{2n}{n+1} z + \dots \right. \\
&\quad \left. \dots + \binom{2n}{2n-1} z^{2n-3} + \binom{2n}{2n} z^{2n-1} \right] dz. \tag{1.4}
\end{aligned}$$

Now the coefficient of $\frac{1}{z}$ is $\binom{2n}{n}$. From (1.3), $z = 0$ is inside $|z| = 1$, and $\frac{(z^2 + 1)^{2n}}{z^{2n+1}}$ has

a pole at $z = 0$. By the definition of the residue of $\frac{(z^2 + 1)^{2n}}{z^{2n+1}}$ at $z = 0$, $\text{Res}(0) = \binom{2n}{n}$.

So (1.4) can be rewritten as $I = \frac{1}{i2^{2n}} \cdot 2\pi i \binom{2n}{n} = \frac{\pi}{2^{2n-1}} \binom{2n}{n}$.

Hence, $I = \frac{\pi}{2^{2n-1}} \binom{2n}{n}$ for all $n \in \mathbb{N}$. □

2.2 Real-valued Integrals on the Interval $(-\infty, \infty)$

Example 1. Some integrals of the form $\int_{-\infty}^{\infty} f(x) dx$ may not be evaluated without

contour integration. Here we present the method for solving this type of integrals. Let us compute

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx,$$

where a and b are any positive real numbers.

Consider the integral $\int_C f(z) dz$, where $f(z) = \frac{z^2}{(z^2 + a^2)(z^2 + b^2)}$, and C is a closed

contour consisting of the upper half of a large circle $|z| = R$ and the real axis from $-R$ to R as shown below.

Since $(z^2 + a^2)(z^2 + b^2) = 0$ implies that $(z - ia)(z + ia)(z - ib)(z + ib) = 0$, $f(z)$ has simple poles at $z = \pm ia$, $z = \pm ib$ for $a, b > 0$. The poles that lie inside the contour are $z = ia$ and $z = ib$. So we have

$$\begin{aligned} \operatorname{Res}[f(z); ia] &= \lim_{z \rightarrow ia} (z - ia)f(z) = \lim_{z \rightarrow ia} \frac{(z - ia)z^2}{(z - ia)(z + ia)(z^2 + b^2)} = \frac{(ia)^2}{(2ia)((ia)^2 + b^2)} \\ &= \frac{-a^2}{2ia(-a^2 + b^2)} = \frac{ai}{-2(a^2 - b^2)} = \frac{ai}{2(b^2 - a^2)} \end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}[f(z); ib] &= \lim_{z \rightarrow ib} (z - ib)f(z) = \lim_{z \rightarrow ib} \frac{(z - ib)z^2}{(z - ib)(z + ib)(z^2 + a^2)} = \frac{(ib)^2}{(2ib)((ib)^2 + a^2)} \\ &= \frac{-b^2 i}{-2b(a^2 - b^2)} = \frac{bi}{2(a^2 - b^2)}.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{Res}[f(z); ia] + \operatorname{Res}[f(z); ib] &= \frac{ai}{-2(a^2 - b^2)} + \frac{bi}{2(a^2 - b^2)} = \frac{i}{2} \left(\frac{b - a}{a^2 - b^2} \right) \\ &= -\frac{i}{2} \left[\frac{a - b}{(a - b)(a + b)} \right] = \frac{-i}{2(a + b)}.\end{aligned}$$

By Cauchy's residue theorem, we have

$$\int_{\Gamma} f(z) dz = 2\pi i \left(\frac{-i}{2(a + b)} \right) = \frac{2\pi}{2(a + b)} = \frac{\pi}{a + b}.$$

That is

$$\int_{-R}^R f(x) dx + \int_{\Gamma_R} f(z) dz = \int_{-R}^R \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx + \int_{\Gamma_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} dz = \frac{\pi}{a + b}.$$

Now we show that $\int_{\Gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

$$\left| \int_{\Gamma_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} dz \right| \leq \int_{\Gamma_R} \frac{|z^2| |dz|}{|z^2 + a^2| |z^2 + b^2|} \leq \int_{\Gamma_R} \frac{|z|^2 |dz|}{(|z|^2 - a^2)(|z|^2 - b^2)}$$

$$\text{since } |z^2 + a^2| > |z|^2 - a^2 \text{ and } |z^2 + b^2| > |z|^2 - b^2$$

$$\leq \int_{\Gamma_R} \frac{R^2}{(R^2 - a^2)(R^2 - b^2)} R d\theta$$

since $z = Re^{i\theta}$, $dz = Rie^{i\theta}d\theta$, $|z| = R$, and $|dz| = Rd\theta$

$$= \frac{R^3\pi}{(R^2 - a^2)(R^2 - b^2)}.$$

Since the denominator of the integrand is of degree 4 and the numerator is of degree 3,

$$\frac{R^3\pi}{(R^2 - a^2)(R^2 - b^2)} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\text{Thus, } \int_{C_R} \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} dz = 0 \text{ as } R \rightarrow \infty.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a+b} \quad \square$$

Example 2. We now evaluate an integral related to the gamma function (see [23]).

We will show that

$$\int_{-\infty}^{\infty} \frac{e^{mx}}{1 + e^x} dx = \frac{\pi}{\sin \pi m}, \quad 0 < m < 1$$

by the contour integration around the rectangular path with fixed height $2\pi i$ along the line $y = 0$, $x = R$, $y = 2\pi$ and $x = -R$ and then letting R tend to infinity.

Since $e^{\pi i} = -1$, the function $f(z) = \frac{e^{mz}}{1 + e^z}$, $0 < m < 1$, has only one pole at $z = \pi i$ within

the given rectangular region.

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z) = e^{mz}$ and $q(z) = 1 + e^z$. Now we want to show that

$$\operatorname{Res}[f(z); \pi] = \frac{p(\pi i)}{q'(\pi i)} = -e^{m\pi i}.$$

Both of the functions $p(z)$ and $q(z)$ are analytic at $z = \pi i$, and q has a simple zero at $z = \pi i$ because $q'(\pi i) \neq 0$, while $p(\pi i) \neq 0$. So $f(z)$ has a simple pole at $z = \pi i$. Then we have

$$\begin{aligned} \operatorname{Res}[f(z); \pi] &= \lim_{z \rightarrow \pi i} (z - \pi i) f(z) = \lim_{z \rightarrow \pi i} (z - \pi i) \frac{p(z)}{q(z)} = \lim_{z \rightarrow \pi i} \frac{z - \pi i}{q(z)} \cdot p(z) \\ &= \lim_{z \rightarrow \pi i} \frac{1}{\frac{1}{(z - \pi i)}} \cdot \frac{p(z)}{q(z)} = \frac{1}{q'(\pi i)} p(\pi i) \end{aligned} \quad (2.1)$$

$$\text{But } \frac{p(\pi i)}{q'(\pi i)} = \frac{e^{m\pi i}}{e^{\pi i}} = \frac{e^{m\pi i}}{-1} = -e^{m\pi i}, \text{ since } q'(z) = e^z.$$

$$\text{Thus, } \operatorname{Res}[f(z); \pi] = -e^{m\pi i}.$$

By Cauchy's residue theorem,

$$\begin{aligned} \int_{-R}^R \frac{e^{mx}}{1 + e^x} dx + i \int_0^{2\pi} \frac{e^{m(R+iy)}}{1 + e^{R+iy}} dy + \int_R^0 \frac{e^{m(x+2\pi i)}}{1 + e^{x+2\pi i}} dx + i \int_{2\pi}^0 \frac{e^{m(-R+iy)}}{1 + e^{-R+iy}} dy &= 2\pi i (-e^{m\pi i}) \\ &= -2\pi i e^{m\pi i}. \end{aligned} \quad (2.2)$$

Since $|e^{iy}| = 1$ and $|e^z| = e^x$ ($z = x + iy$), we have

$$|e^{m(R+iy)}| = e^{mR} \text{ and } |e^{R+iy} + 1| \geq |e^{R+iy}| - 1 = e^R - 1. \text{ So we have}$$

$$\left| \int_0^{2\pi} \frac{e^{m(R+iy)i}}{1+e^{R+iy}} dy \right| \leq \frac{e^{mR}}{e^R - 1} \int_0^{2\pi} dy = \frac{2\pi e^{mR}}{e^R - 1}. \text{ Since } 0 < m < 1, \lim_{R \rightarrow \infty} \int_0^{2\pi} \frac{e^{m(R+iy)i}}{1+e^{R+iy}} dy = 0.$$

$$\text{Similarly, we have that } \lim_{R \rightarrow \infty} \int_{2\pi}^0 \frac{e^{m(-R+iy)}}{1+e^{-R+iy}} dy = 0.$$

Therefore, (2.2) can be written as

$$\begin{aligned} \int_{-R}^R \frac{e^{mx}}{1+e^x} dx + \int_R^R \frac{e^{m(x+2\pi)}}{1+e^{x+2\pi}} dx &= -2\pi i e^{m\pi i}, \text{ which implies that} \\ \int_{-R}^R \frac{e^{mx}}{1+e^x} dx - \int_R^R \frac{e^{mx} \cdot e^{2\pi m i}}{1+e^x \cdot e^{2\pi}} dx &= \int_{-R}^R \left[\frac{e^{mx}}{1+e^x} - \frac{e^{mx} \cdot e^{2\pi m i}}{1+e^x \cdot e^{2\pi}} \right] dx = \int_{-R}^R \frac{e^{mx} - e^{mx} \cdot e^{2\pi m i}}{1+e^x} dx \\ &= \int_{-R}^R \frac{e^{mx} (1 - e^{2\pi m i})}{1+e^x} dx \\ &= -2\pi i e^{m\pi i}. \end{aligned} \quad (2.3)$$

As $R \rightarrow \infty$, (2.3) becomes

$$\int_{-\infty}^{\infty} \frac{e^{mx} (1 - e^{2\pi m i})}{1+e^x} dx = -2\pi i e^{m\pi i}. \quad (2.4)$$

Note that $e^{2\pi m i} \neq 1$ for $0 < m < 1$. Thus (2.4) can be written as

$$(1 - e^{2\pi m i}) \int_{-\infty}^{\infty} \frac{e^{mx}}{1+e^x} dx = -2\pi i e^{m\pi i}, \text{ which implies that } \int_{-\infty}^{\infty} \frac{e^{mx}}{1+e^x} dx = -\frac{2\pi i e^{m\pi i}}{1 - e^{2\pi m i}}. \text{ But}$$

$$\frac{-2\pi i e^{m\pi i}}{1 - e^{2\pi m i}} = \frac{-2\pi i e^{m\pi i}}{-(e^{2\pi m i} - 1)} = \frac{2\pi i}{(e^{2\pi m i} - 1)e^{-m\pi i}} = \frac{2\pi i}{e^{m\pi i} - e^{-m\pi i}} = \frac{\pi}{\sin(m\pi)}.$$

$$\text{Hence, } \int_{-\infty}^{\infty} \frac{e^{mx}}{1+e^x} dx = \frac{\pi}{\sin(m\pi)}, \text{ for } 0 < m < 1.$$

It is well-known that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \text{ where } \Gamma(z) \text{ is the gamma function (see [24]).}$$

Thus, we can conclude that

$$\int_{-\infty}^{\infty} \frac{e^{mx}}{1+e^x} dx = \Gamma(m)\Gamma(1-m).$$

□

2.3 Integrals Involving Logarithms

Example 1. Here we discuss a real integral involving logarithms. In this example, we will show how to avoid contour integrals. Only recently Arora [2] discovered this real method for evaluating the integral:

$$I = \int_0^{\infty} \frac{\ln x}{(1+x^2)^2} dx = -\frac{\pi}{4}.$$

We use the substitution $x = \tan \theta$. Then $dx = \frac{1}{\cos^2 \theta} d\theta = \sec^2 \theta d\theta$. When $x = 0$,

$\theta = 0$ and when $x = \infty$, $\theta = \frac{\pi}{2}$, we have $(1+x^2)^2 = (1+\tan^2 \theta)^2 = \sec^4 \theta$. So we have

$$\begin{aligned} I &= \int_0^{\infty} \frac{\ln x}{(1+x^2)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\ln(\tan \theta)}{\sec^4 \theta} \sec^2 \theta d\theta = \int_0^{\frac{\pi}{2}} \frac{\ln(\tan \theta)}{\sec^2 \theta} d\theta = \int_0^{\frac{\pi}{2}} \cos^2 \theta \ln(\tan \theta) d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2} \right) \ln(\tan \theta) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1+\cos 2\theta) \ln(\tan \theta) d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\tan \theta) d\theta + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2\theta \ln(\tan \theta) d\theta. \end{aligned}$$

Also we have $\ln(\tan \theta) = \ln\left(\frac{\sin \theta}{\cos \theta}\right) = \ln(\sin \theta) - \ln(\cos \theta)$.

$$\text{Hence, } I = \left[\frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta \right] + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2\theta \ln(\tan \theta) d\theta = I_1 + I_2.$$

By evaluating the third integral by parts, letting $u = \ln(\tan \theta)$ and $dv = \cos 2\theta d\theta$, we'll

obtain

$$du = \frac{1}{\tan \theta} \frac{1}{\cos^2 \theta} d\theta = \frac{1}{\sin \theta \cos \theta} d\theta = \frac{2}{\sin 2\theta} d\theta \text{ and } v = \frac{1}{2} \sin 2\theta.$$

So we have

$$\begin{aligned} I_2 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2\theta \ln(\tan \theta) d\theta = \frac{1}{2} \left[\frac{1}{2} \sin 2\theta \ln(\tan \theta) \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \cdot 2 \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\sin 2\theta} d\theta \right] \\ &= \frac{1}{4} \sin 2\theta \ln(\tan \theta) \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \theta \Big|_0^{\frac{\pi}{2}} \\ &= \frac{1}{4} \lim_{\theta \rightarrow \frac{\pi}{2}^-} [\sin 2\theta \ln(\tan \theta)] - \frac{\pi}{4}. \end{aligned} \tag{3.1}$$

Using L'Hopital's rule, we can show that the value of the limit is zero as followings:

$$\lim_{\theta \rightarrow \frac{\pi}{2}^-} \sin 2\theta \ln(\tan \theta) = \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\ln(\tan \theta)}{\csc(2\theta)} = \lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{\frac{\sec^2 \theta}{\tan \theta}}{-2 \csc(2\theta) \cot(2\theta)} = 0.$$

So we have

$$\lim_{\theta \rightarrow \frac{\pi}{2}^-} \frac{1}{4} \sin 2\theta \ln(\tan \theta) - \frac{\pi}{4} = -\frac{\pi}{4}.$$

$$\text{Thus, } I_2 = -\frac{\pi}{4}.$$

$$\text{Now we evaluate the integral } I_1 = \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\sin \theta) d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta.$$

$$\text{Let } h(\theta) = \sin \theta. \text{ Then we have } h\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta.$$

If $g(\theta) = \ln(\sin \theta)$, then $g\left(\frac{\pi}{2} - \theta\right) = \ln\left(\sin\left(\frac{\pi}{2} - \theta\right)\right) = \ln(\cos \theta)$.

Hence, we have

$$I_1 = \frac{1}{2} \int_0^{\frac{\pi}{2}} g(\theta) d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} g\left(\frac{\pi}{2} - \theta\right) d\theta.$$

Since $\frac{1}{2} \int_0^{\frac{\pi}{2}} g(\theta) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} g\left(\frac{\pi}{2} - \theta\right) d\theta$ (see [2]), We have

$$\frac{1}{2} \int_0^{\frac{\pi}{2}} g(\theta) d\theta - \frac{1}{2} \int_0^{\frac{\pi}{2}} g\left(\frac{\pi}{2} - \theta\right) d\theta = 0.$$

Therefore, we have shown that $I = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos 2\theta \ln(\tan \theta) d\theta = -\frac{\pi}{4}$.

Thus, $I = \int_0^{\infty} \frac{\ln x}{(1+x^2)^2} dx = -\frac{\pi}{4}$. □

Example 2. We will evaluate another real integral involving logarithms. It is our objective to show that:

$$I = \int_0^{\infty} \frac{\log(1+x^2)}{1+x^2} dx = \pi \log 2.$$

In order to do this, we choose $f(z) = \frac{\text{Log}(z+i)}{z^2+1}$.

We will evaluate $\int_C f(z) dz$, where C is a contour consisting of real axis from $-\mathbf{R}$ to $\mathbf{R}$

and the upper half of the circle $|z| = \mathbf{R}$ as shown below.

Since the denominator of $f(z)$, $z^2 + 1$, is written as $(z+i)(z-i)$ and i is inside the upper half-plane, $f(z)$ has a pole of order 1 at $z = i$. Then the residue of $f(z)$ at $z = i$ is $\lim_{z \rightarrow i} (z-i)f(z) = \lim_{z \rightarrow i} (z-i) \frac{\text{Log}(z+i)}{(z-i)(z+i)} = \lim_{z \rightarrow i} \frac{\text{Log}(z+i)}{(z+i)} = \frac{\text{Log}(i+i)}{(i+i)} = \frac{\text{Log}(2i)}{2i}$.

Hence,

$$\begin{aligned} \oint_{C_R} f(z) dz &= 2\pi i \frac{\text{Log}(2i)}{2i} = \pi [\text{Log}(2) + \text{Log}(i)] = \pi \left[\text{Log}(2) + \text{Log}|i| + \frac{\pi}{2}i \right] \\ &= \pi \left[\text{Log}(2) + \text{Log}1 + \frac{\pi}{2}i \right] = \pi \left[\text{Log}(2) + \frac{\pi i}{2} \right]. \end{aligned} \quad (3.2)$$

$$\text{We now have } \oint_{C_R} f(z) dz = \int_{-R}^R \frac{\text{Log}(x+i)}{x^2+1} dx + \int_{C_R} f(z) dz = \pi \left[\text{Log}(2) + \frac{\pi i}{2} \right].$$

Notice that

$$\int_{-R}^R \frac{\text{Log}(x+i)}{x^2+1} dx = \int_{-R}^0 \frac{\text{Log}(x+i)}{x^2+1} dx + \int_0^R \frac{\text{Log}(x+i)}{x^2+1} dx.$$

Replacing x by $-x$ in the first integral on the right hand side, we have

$$\begin{aligned} &\int_0^R \frac{\text{Log}(i-x)}{x^2+1} dx + \int_0^R \frac{\text{Log}(x+i)}{x^2+1} dx \\ &= \int_0^R \left[\frac{\text{Log}(i-x)}{x^2+1} + \frac{\text{Log}(x+i)}{x^2+1} \right] dx = \int_0^R \frac{\text{Log}(i-x) + \text{Log}(x+i)}{x^2+1} dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\infty} \frac{\operatorname{Log}[(i-x)(x+i)]}{x^2+1} dx = \int_0^{\infty} \frac{\operatorname{Log}(-x^2-1)}{x^2+1} dx = \int_0^{\infty} \frac{\operatorname{Log}(-(x^2+1))}{x^2+1} dx \\
&= \int_0^{\infty} \frac{\operatorname{Log}(-1) + \operatorname{Log}(x^2+1)}{x^2+1} dx = \int_0^{\infty} \left(\frac{\pi i}{x^2+1} + \frac{\operatorname{Log}(x^2+1)}{x^2+1} \right) dx \\
&= \int_0^{\infty} \frac{\operatorname{Log}(x^2+1)}{x^2+1} dx + \int_0^{\infty} \frac{\pi i}{x^2+1} dx.
\end{aligned}$$

So we have

$$\int_C f(z) dz = \int_0^{\infty} \frac{\operatorname{Log}(x^2+1)}{x^2+1} dx + \int_0^{\infty} \frac{\pi i}{x^2+1} dx + \int_{C_R} \frac{\operatorname{Log}(z+i)}{z^2+1} dz. \quad (3.3)$$

Now consider the integral $\int_{C_R} \frac{\operatorname{Log}(z+i)}{z^2+1} dz$.

We want to show that the integral over $C_R \rightarrow 0$ as $R \rightarrow \infty$.

Notice that

$$\left| \int_{C_R} f(z) dz \right| = \left| \int_{C_R} \frac{\operatorname{Log}(z+i)}{z^2+1} dz \right| \leq \max_{z \in C_R} \left| \frac{\operatorname{Log}(z+i)}{z^2+1} \right| \int_{C_R} |dz|.$$

Since $R^2 - 1 = |z|^2 - 1 \leq |z|^2 + 1 \leq |z|^2 + 1$, $\frac{1}{|z|^2+1} \leq \frac{1}{|z|^2-1}$.

Now consider $|\operatorname{Log}(z+i)|$. Let $z^* = \operatorname{Re}^{i\theta} + i = R \cos \theta + i(R \sin \theta + 1)$ and let

$$R^* = |z^*|.$$

Let $|\operatorname{Log}(z+i)| = |\operatorname{Log}(z^*)| = |\operatorname{Log} R^* e^{i\theta^*}| = |\operatorname{Log} R^* + i\theta^*| \leq |\operatorname{Log} R^*| + \pi$ on C_R , where

$z^* = \operatorname{Re}^{i\theta} + i = R \cos \theta + i(R \sin \theta + 1)$. Then

$|\operatorname{Log}(z^*)| \leq \ln R^* + \pi$ on C_R . We also have

$$R^* = |z^*| = \sqrt{R^2 \cos^2 \theta + (1 + R \sin \theta)^2} = \sqrt{R^2 + 2R \sin \theta + 1} \leq \sqrt{(R+1)^2} = R+1.$$

Thus, we have

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{[\ln(R+1) + \pi] \cdot \pi R}{R^2 - 1}.$$

Using L'Hopital's rule, we have

$$\frac{[\ln(R+1) + \pi] \cdot 2\pi R}{R^2 - 1} \rightarrow 0 \text{ as } R \rightarrow \infty, \text{ so } \int_{C_R} f(z) dz \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Using (3.2), equation (3.3) becomes

$$\int_C f(z) dz = \int_0^R \frac{\log(x^2 + 1)}{x^2 + 1} dx + \int_0^R \frac{\pi i}{x^2 + 1} dx = \pi \left[\log 2 + \frac{\pi i}{2} \right].$$

Since

$$\int_0^\infty \frac{\log(x^2 + 1)}{x^2 + 1} dx + \pi i \arctan x \Big|_0^\infty = \int_0^\infty \frac{\log(x^2 + 1)}{x^2 + 1} dx + \pi i \frac{\pi}{2} = \pi \left[\log 2 + \frac{\pi i}{2} \right].$$

$$\text{Hence, } I = \int_0^\infty \frac{\log(x^2 + 1)}{x^2 + 1} dx = \pi \log 2.$$

□

Example 3. The following integral was first shown by Cauchy.

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = -\frac{1}{2} \pi \ln 2. \quad (3.4)$$

In order to do this, we use a result we prove earlier.

$$J = \int_0^\infty \frac{\ln(x^2 + 1)}{x^2 + 1} dx = \pi \ln 2. \quad (3.5)$$

Now we want to show that (3.5) can be transformed into (3.4).

Let $x = \tan \theta$ and $dx = \frac{1}{\cos^2 \theta} d\theta$. Then (3.5) reduces to

$$J = \int_0^{\frac{\pi}{2}} \frac{\ln(\tan^2 \theta + 1)}{(\tan^2 \theta + 1)} \cdot \frac{1}{\cos^2 \theta} d\theta = \pi \ln 2.$$

Since $\tan^2 \theta + 1 = \frac{1}{\cos^2 \theta}$, We have

$$\begin{aligned} J &= \int_0^{\frac{\pi}{2}} \frac{\ln\left(\frac{1}{\cos^2 \theta}\right)}{\frac{1}{\cos^2 \theta}} \cdot \frac{1}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{2}} [\ln 1 - \ln(\cos^2 \theta)] d\theta = \int_0^{\frac{\pi}{2}} -2 \ln(\cos \theta) d\theta \\ &= -2 \int_0^{\frac{\pi}{2}} \ln(\cos \theta) d\theta = \pi \ln 2. \end{aligned} \tag{3.6}$$

Replacing θ with x in (3.6), we have $-2 \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \pi \ln 2$.

We get $\int_0^{\frac{\pi}{2}} \ln(\cos x) dx = -\frac{1}{2} \pi \ln 2$.

$$\begin{aligned} \text{Since } \int_0^a f(x) dx &= \int_0^a f(a-x) dx, \text{ then } \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{\pi}{2} - x\right)\right) dx \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = -\frac{1}{2} \pi \ln 2. \end{aligned}$$

Thus we also conclude that

$$\int_0^{\frac{\pi}{2}} \ln(\sin x) dx - \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = 0, \text{ which implies that } \int_0^{\frac{\pi}{2}} \ln(\tan x) dx = 0$$

$$\text{Therefore, } \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = -\frac{1}{2} \pi \ln 2. \quad \square$$

2.4 Fresnel Integrals

It is our objective to show that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}$. These integrals are called the Fresnel integrals. We will evaluate these integrals over two different contours.

First we will evaluate the Fresnel Integrals

$$I_1 = \int_0^\infty \cos x^2 dx \text{ and } I_2 = \int_0^\infty \sin x^2 dx$$

by using the triangular contour C where $C = C_1 \oplus C_2 \oplus C_3$ shown below. The contour consists of three segments C_1, C_2 , and C_3 i.e. the line segment C_1 from $z = 0$ to $z = R$, C_2 the line segment from $z = R$ to $z = R + iR$, and C_3 is the line segment from $z = R + iR$ to $z = 0$.

Consider $\oint_{C_1} e^{-z^2} dz$, where C_1 is the boundary of the triangle. From $z = 0$ to $z = R$, we

have $z = x$ and $dz = dx$, so we have the integral reduces to

$$\int_{C_1} e^{-z^2} dz = \int_0^R e^{-x^2} dx.$$

Since on C_2 from $z = R$ to $z = R + iR$, we have $z = R + iy$ and $dz = i dy$, so

$$\int_{C_2} e^{-z^2} dz = i \int_0^R e^{-(R+iy)^2} dy.$$

On the line segment C_3 , we have $z = x + ix = (1+i)x$ and $dz = (1+i)dx$. So

$$\int_{C_3} e^{-z^2} dz = \int_R^0 e^{-(x(1+i))^2} (1+i) dx = \int_R^0 e^{-x^2(1+i)^2} (1+i) dx.$$

Let C be $C_1 \oplus C_2 \oplus C_3$. Then

$$\begin{aligned} \oint_C e^{-z^2} dz &= \int_0^R e^{-x^2} dx + i \int_0^R e^{-(R+iy)^2} dy + (1+i) \int_R^0 e^{-(1+i)^2 x^2} dx \\ &= \int_0^R e^{-x^2} dx + (1+i) \int_R^0 e^{-2ix^2} dx + i \int_0^R e^{-(R+iy)^2} dy. \end{aligned}$$

Since the function, e^{-z^2} has no singularities in the finite complex plane, we have

$$\int_0^R e^{-x^2} dx + (1+i) \int_R^0 e^{-2ix^2} dx + i \int_0^R e^{-(R+iy)^2} dy = 0$$

by Cauchy's integral theorem. Also, notice that

$$\left| i \int_0^R e^{-(R+iy)^2} dy \right| = \left| i \int_0^R e^{-(R^2+2iRy-y^2)} dy \right| = \left| i \int_0^R e^{-(R^2-y^2)} \cdot e^{-2iRy} dy \right| \leq \int_0^R e^{-(R^2-y^2)} dy$$

and $R^2 - y^2 \geq R(R-y)$ for $R \geq y \geq 0$. So we have

$$\int_0^R e^{-(R+iy)^2} dy \rightarrow 0 \text{ as } R \rightarrow \infty. \text{ This implies that } \int_0^\infty e^{-x^2} dx = (1+i) \int_0^\infty e^{-2ix^2} dx = \frac{\sqrt{\pi}}{2}.$$

Hence,

$$(1+i) \int_0^{\infty} (\cos(2x^2) - i \sin(2x^2)) dx = \frac{\sqrt{\pi}}{2}.$$

Equating the real and imaginary parts and making the substitution $x = \frac{\sqrt{2}}{2} t$

$$\int_0^{\infty} \cos x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^{\infty} \sin x^2 dx.$$

□

Next we will prove the above relation by using different contour C , the boundary of the sector of a circle of radius R and central angle $\frac{\pi}{4}$ as shown below.

Let $C = C_1 \oplus C_2 \oplus C_3$. Since e^{iz^2} is entire, by Cauchy's theorem, we have

$\int_C e^{iz^2} dz = 0$. That is,

$$\int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz = 0. \quad (4.1)$$

On C_1 , let $z = x$ for $0 \leq x \leq R$, on C_2 , let $z = Re^{i\theta}$ for $0 \leq \theta \leq \frac{\pi}{4}$, and on C_3 , let

$$z = re^{i\frac{\pi}{4}} \text{ for } 0 \leq r \leq R.$$

Thus, we have

$$\oint_C e^{iz^2} dz = \int_0^R e^{ix^2} dx + \int_4^\pi e^{iR^2 e^{2i\theta}} i \operatorname{Re} e^{i\theta} d\theta + \int_R^0 e^{ir^2 e^{2i}} e^{\frac{\pi}{4}i} dr = 0$$

$$\Rightarrow \int_0^R (\cos x^2 + i \sin x^2) dx + \int_4^\pi e^{iR^2 (\cos 2\theta + i \sin 2\theta)} i \operatorname{Re} e^{i\theta} d\theta - e^{\frac{\pi i}{4}} \int_0^R e^{ir^2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)} dr = 0$$

$$\Rightarrow \int_0^R (\cos x^2 + i \sin x^2) dx = e^{\frac{\pi}{4}i} \int_0^R e^{ir^2(i)} dr - \int_4^\pi e^{iR^2 \cos 2\theta - R^2 \sin 2\theta} i \operatorname{Re} e^{i\theta} d\theta$$

$$\Rightarrow \int_0^R (\cos x^2 + i \sin x^2) dx = e^{\frac{\pi}{4}i} \int_0^R e^{-r^2} dr - \int_4^\pi e^{(iR^2 \cos 2\theta - R^2 \sin 2\theta)} i \operatorname{Re} e^{i\theta} d\theta. \quad (4.2)$$

$$\text{As } R \rightarrow \infty \text{ we have } e^{\frac{\pi}{4}i} \int_0^\infty e^{-r^2} dr = e^{\frac{\pi}{4}i} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{4} + i \frac{\sqrt{2\pi}}{4}.$$

Next, we will show the second integral on the right side of (4.2) tends to zero as

$$R \rightarrow \infty.$$

The absolute value of the integral is

$$\left| \int_4^\pi e^{(iR^2 \cos 2\theta - R^2 \sin 2\theta)} i \operatorname{Re} e^{i\theta} d\theta \right| \leq \int_4^\pi e^{-R^2 \sin 2\theta} R d\theta$$

$$(\text{let } \phi = 2\theta, \text{ then } d\phi = 2d\theta \Rightarrow \frac{1}{2}d\phi = d\theta)$$

$$= \frac{R}{2} \int_0^\pi e^{-R^2 \sin \phi} d\phi.$$

Since the slope of the dashed line in the next graph can be expressed as $m = \frac{2}{\pi}$ for

$0 \leq \theta \leq \frac{\pi}{2}$, and the equation of the line is $y = \frac{2}{\pi} \theta$ (refer to Jordan's Inequality).

We have $\sin \theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$. So we obtain

$$\frac{R}{2} \int_0^{\pi} e^{-R^2 \sin \phi} d\phi \leq \frac{R}{2} \int_0^{\pi} e^{-R^2 \frac{2\phi}{\pi}} d\phi = -\frac{R}{2} \cdot \frac{\pi}{2R^2} \left[e^{-\frac{2\phi}{\pi} R^2} \right]_{\phi=0}^{\phi=\frac{\pi}{2}} = -\frac{\pi}{4R} [e^{-R^2} - 1].$$

As $R \rightarrow \infty$, $-\frac{\pi}{4R} [e^{-R^2} - 1] \rightarrow 0$.

Hence, $\int_0^{\infty} (\cos x^2 + i \sin x^2) dx = \frac{\sqrt{2\pi}}{4} + i \frac{\sqrt{2\pi}}{4}$, and by equating real and imaginary parts,

we obtain

$$\int_0^{\infty} \cos x^2 dx = \int_0^{\infty} \sin x^2 dx = \frac{\sqrt{2\pi}}{4}.$$

□

2.5 Young's and Apostol's Formulas

Let

$$L(n) = \int_0^{\infty} \left(\frac{\sin x}{x} \right)^n dx$$

where n is some positive integer.

It is well-known that $L(1) = L(2) = \frac{\pi}{2}$. E.T.H.Wang asked whether it is possible to find

$L(3), L(4)$ and $L(5)$. In 1980, Young [34] and Apostol [1] found independently a formula for $L(n)$ using two different methods, the first by Young using contour integration and the second by Apostol using integration by parts. Here, we will find $L(2), L(5)$, and $L(n)$ using contour integration. Also we will find $L(n)$ by using integration by parts.

We will start by evaluating $L(2)$ where the power of $\frac{\sin x}{x}$ is even. Then we will find

$L(5)$ and conclude by finding a formula for $L(n)$.

$$\text{We know } \cos 2\theta = 1 - 2(\sin \theta)^2 \text{ and } \sin^2 \theta = \frac{1}{2}[1 - \cos 2\theta]. \quad (5.1)$$

Let $f_2(z) = e^{2iz} - 1$. Then $\frac{f_2(z)}{z^2} = \frac{e^{2iz} - 1}{z^2}$ is analytic in the whole plane except for a simple pole of order 2 at $z = 0$.

Now let C be the contour which consists of the semicircle C_R , the line segment from $-R$ to $-r$, the semicircle C_r , and the line segment from r to R as shown below in the

figure. Then, $\int_C \frac{f_2(z)}{z^2} dz = 0$ since $z = 0$ lies outside C .

So we have

$$\int_C \frac{f_2(z)}{z^2} dz = \int_{-R}^{-r} \frac{f_2(x)}{x^2} dx + \int_{C_r} \frac{f_2(z)}{z^2} dz + \int_r^R \frac{f_2(x)}{x^2} dx + \int_{C_R} \frac{f_2(z)}{z^2} dz = 0.$$

Consider the contour integral around the small semicircle C_r .

$$\begin{aligned} \int_{C_r} \frac{f_2(z)}{z^2} dz &= \int_{C_r} \frac{1}{z^2} (e^{2iz} - 1) dz = \int_{C_r} \left[\frac{1}{z^2} \left(\frac{2iz}{1!} + \frac{(2iz)^2}{2!} + \frac{(2iz)^3}{3!} + \dots \right) \right] dz \\ &= \int_{C_r} \left[\frac{1}{z} \frac{(2i)}{1!} + \frac{(2i)^2}{2!} + \frac{(2i)^3}{3!} z + \dots \right] dz \\ &= 2i \int_{C_r} \frac{1}{z} dz \text{ since } \int_{C_r} \left(\frac{(2i)^2}{2!} + \frac{(2i)^3}{3!} z + \dots \right) dz = 0 \text{ as } r \rightarrow 0. \end{aligned}$$

Let $z = re^{i\theta}$; $dz = ire^{i\theta} d\theta$. Then

$$2i \int_{C_r} \frac{1}{z} dz = 2i \int_{\pi}^0 \frac{1}{re^{i\theta}} ire^{i\theta} d\theta = -2 \int_{\pi}^0 d\theta = -2\theta \Big|_{\pi}^0 = -2(0 - \pi) = 2\pi. \quad (5.2)$$

Thus, $\lim_{r \rightarrow 0} \int_{C_r} \frac{1}{z} dz = -\pi i$.

We will show that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{f_2(z)}{z^2} dz = 0.$$

Let $z = Re^{i\theta}$; $dz = iRe^{i\theta} d\theta$. Then for sufficiently large R , we will get

$$\left| \int_{C_R} \frac{f_2(z)}{z^2} dz \right| \leq \int_{C_R} \frac{|e^{2iz} - 1|}{|z|^2} |dz| \leq \int_{C_R} \frac{|e^{2iz}| + 1}{|Re^{i\theta}|^2} |iRe^{i\theta} d\theta| = \int_0^\pi \frac{e^{-2y} + 1}{R^2} R d\theta = \frac{(e^{-2y} + 1)\pi}{R} \rightarrow 0$$

as $R \rightarrow \infty$.

Hence, $\lim_{R \rightarrow \infty} \int_{C_R} \frac{f_2(z)}{z^2} dz = 0$. So we have $\int_{-R}^r \frac{f_2(x)}{x^2} dx + \int_r^R \frac{f_2(x)}{x^2} dx + 2\pi = 0$.

Let $I = \int_{-R}^r \frac{f_2(x)}{x^2} dx$ and $J = \int_r^R \frac{f_2(x)}{x^2} dx$. Replacing x by $-x$ in J and letting

$R \rightarrow \infty$ and $r \rightarrow 0$, we have

$$I + J = \int_0^\infty \frac{f_2(x)}{x^2} dx + \int_0^\infty \frac{f_2(-x)}{x^2} dx + 2\pi = 0 = \int_0^\infty \left(\frac{e^{i2x} - 1}{x^2} + \frac{e^{-i2x} - 1}{x^2} \right) dx + 2\pi = 0 \text{ and}$$

$$\int_0^\infty \frac{2\cos(2x) - 2}{x^2} dx = -2\pi \Rightarrow \int_0^\infty \frac{\cos(2x) - 1}{x^2} dx = -\pi \Rightarrow \int_0^\infty \frac{1 - \cos(2x)}{x^2} dx = \pi$$

$$\Rightarrow \int_0^\infty \frac{2\sin^2 x}{x^2} dx = \pi \Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Before evaluating $L(n)$ for $n = 5$, we want to prove the identity

$$16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta$$

by using DeMoivre's formula

$$(\cos \theta + i \sin \theta)^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta), \text{ for } n = 1, 2, 3, 4, \dots$$

We first use the binomial theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n,$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$, $r = 1, 2, \dots, n$. Since $\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, we have

$$\begin{aligned} (\sin \theta)^5 &= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^5 = \frac{1}{2^5 i^5} (e^{i\theta} - e^{-i\theta})^5 \\ &= \frac{1}{32i} \left[(e^{i\theta})^5 - \binom{5}{1}(e^{i\theta})^4(e^{-i\theta}) + \binom{5}{2}(e^{i\theta})^3(e^{-i\theta})^2 \right. \\ &\quad \left. - \binom{5}{3}(e^{i\theta})^2(e^{-i\theta})^3 + \binom{5}{4}(e^{i\theta})^1(e^{-i\theta})^4 - \binom{5}{5}(e^{-i\theta})^5 \right] \\ &= \frac{1}{32i} [e^{i5\theta} - 5e^{i3\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-i3\theta} - e^{i5\theta}] \\ &= \frac{1}{32i} [(e^{i5\theta} - e^{-i5\theta}) - 5(e^{i3\theta} - e^{-i3\theta}) + 10(e^{i\theta} - e^{-i\theta})] \\ (\sin \theta)^5 &= \frac{1}{32i} [2i(\sin 5\theta) - 10i(\sin 3\theta) + 20i(\sin \theta)]. \end{aligned}$$

Multiplying both sides of the above equation by 16, we will get

$$16 \sin^5 \theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta. \quad (5.3)$$

The above identity indicates that we need to choose $F_5(z)$ in the form

$$F_5(z) = e^{5iz} - 5e^{3iz} + 10e^{iz}, \text{ where } z \text{ is a complex variable.}$$

Let $f_5(z) = F_5(z) + P_2(z)$ where $P_2(z) = -5z^2 - 6$, and the necessity of this choice of

$P_2(z)$ will be clear on the next page. Here we have

$$f_5(z) = e^{5iz} - 5e^{3iz} + 10e^{iz} - 5z^2 - 6, \text{ and}$$

$\frac{f_5(z)}{z^5} = \frac{e^{5iz} - 5e^{3iz} + 10e^{iz} - 5z^2 - 6}{z^5}$ is analytic in the whole complex plane except for a simple pole at $z = 0$.

Let C be the contour as shown in the previous figure. Then $\int_C \frac{f_5(z)}{z^5} dz = 0$ since

$z = 0$ is outside the contour. So we have

$$\oint_C \frac{f_5(z)}{z^5} dz = \int_{-R}^R \frac{f_5(x)}{x^5} dx + \int_{C_r} \frac{f_5(z)}{z^5} dz + \int_{-R}^R \frac{f_5(x)}{x^5} dx + \int_{C_R} \frac{f_5(z)}{z^5} dz = 0.$$

After replacing x by $-x$ in $\int_{-R}^R \frac{f_5(x)}{x^5} dx$, we have

$$\begin{aligned} \oint_C \frac{f_5(z)}{z^5} dz &= - \int_{x=-R}^{x=-r} \frac{f_5(-x)}{x^5} dx + \int_{-R}^R \frac{f_5(x)}{x^5} dx + \int_{C_r} \frac{f_5(z)}{z^5} dz + \int_{C_R} \frac{f_5(z)}{z^5} dz \\ &= - \int_{-R}^R \frac{f_5(-x)}{x^5} dx + \int_{-R}^R \frac{f_5(x)}{x^5} dx + \int_{C_r} \frac{f_5(z)}{z^5} dz + \int_{C_R} \frac{f_5(z)}{z^5} dz. \end{aligned} \quad (5.4)$$

Now Consider $\int_{C_r} \frac{f_5(z)}{z^5} dz$.

$$\begin{aligned} \int_{C_r} \frac{f_5(z)}{z^5} dz &= \int_{C_r} \frac{e^{5iz} - 5e^{3iz} + 10e^{iz} - 5z^2 - 6}{z^5} dz \\ &= \int_{C_r} \frac{1}{z^5} \left[\left(1 + \frac{5iz}{1!} + \frac{(5iz)^2}{2!} + \frac{(5iz)^3}{3!} + \dots \right) - 5 \left(1 + \frac{3iz}{1!} + \frac{(3iz)^2}{2!} + \dots \right) \right. \\ &\quad \left. + 10 \left(1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \dots \right) - 5z^2 - 6 \right] dz \\ &= \int_{C_r} \frac{1}{z^5} \left[\left(\frac{(5i)^4}{4!} - 5 \cdot \frac{(3i)^4}{4!} + 10 \cdot \frac{(i)^4}{4!} \right) z^4 + \dots \right] dz \end{aligned}$$

$$\begin{aligned}
&= \int \left[\frac{1}{z} \cdot \frac{1}{4!} (625 - 405 + 10) + \frac{1}{z^5} (0) + \frac{1}{z^4} (0) + \frac{1}{z^3} (0) \right. \\
&\quad \left. + \frac{1}{z^2} (0) + \frac{1}{z^0} (\dots) + z(\dots) + z^2(\dots) + \dots \right] dz \\
&= \left(\frac{625 - 405 + 10}{4!} \right) \int_{C_r} \frac{1}{z} dz \text{ since other integrals become zero as } r \rightarrow 0.
\end{aligned}$$

Notice the coefficients of $\frac{1}{z^3}$ and $\frac{1}{z^5}$ are zeros because of the special choice of $P_2(z)$ we made earlier.

$$\text{Hence, } \int_{C_r} \frac{f_5(z)}{z^5} dz = \frac{230}{24} \int_{C_r} \frac{1}{z} dz = \frac{115}{12} \int_{C_r} \frac{1}{z} dz.$$

Since $z = re^{i\theta}$ and $dz = ire^{i\theta} d\theta$, we have

$$\begin{aligned}
\frac{115}{12} \int_{C_r} \frac{1}{z} dz &= \frac{115}{12} \int_{\pi}^0 \frac{1}{re^{i\theta}} ire^{i\theta} d\theta = \frac{115}{12} i \int_{\pi}^0 d\theta = \frac{115i}{12} \theta \Big|_{\pi}^0 \\
&= \frac{115i}{12} (0 - \pi) = -\frac{115}{12} \pi i.
\end{aligned} \tag{5.5}$$

Next we will show that $\int_{C_R} \frac{f_5(z)}{z^5} dz$ becomes zero as $R \rightarrow \infty$.

Let $z = Re^{i\theta}$, then for sufficiently large R , we will get

$$\left| \int_{C_R} \frac{f_5(z)}{z^5} dz \right| \leq \int_{C_R} \frac{|e^{i5z} - 5e^{3iz} + 10e^{iz} - 5z^2 - 6|}{|z^5|} |dz|$$

$$\begin{aligned}
&\leq \int_R \frac{|e^{i5z}| + 5|e^{3iz}| + 10|e^{iz}| + 5|z|^2 + 6}{|z|^5} |dz| \\
&= \frac{e^{-5y} + 5e^{-3y} + 10e^{-y} + 5R^2 + 6}{R^5} R \int_0^\pi d\theta = \pi \frac{(e^{-5y} + 5e^{-3y} + 10e^{-y} + 5R^2 + 6)}{R^4},
\end{aligned}$$

which becomes zero as $R \rightarrow \infty$ because $y \geq 0$.

Hence, $\lim_{R \rightarrow \infty} \int_{C_R} \frac{f_5(z)}{z^5} dz = 0$.

We now consider the sum $I + J$ of the first two integrals on the right side of (5.4)

as $R \rightarrow \infty$ and $r \rightarrow 0$. Then

$$\begin{aligned}
I + J &= \int_0^\infty \frac{f_5(x)}{x^5} dx - \int_0^\infty \frac{f_5(-x)}{x^5} dx \\
&= \int_0^\infty \frac{e^{5ix} - 5e^{3ix} + 10e^{ix} - 5x^2 - 6 - (e^{-5ix} - 5e^{-3ix} + 10e^{-ix} - 5x^2 - 6)}{x^5} dx \\
&= \int_0^\infty \frac{1}{x^5} ((e^{5ix} - e^{-5ix}) - 5(e^{3ix} - e^{-3ix}) + 10(e^{ix} - e^{-ix})) dx \\
&= \int_0^\infty \frac{1}{x^5} ((2i)\sin(5x) - 5(2i)\sin(3x) + 10(2i)\sin(x)) dx \\
I + J &= 2i \int_0^\infty \frac{1}{x^5} (\sin 5x - 5\sin 3x + 10\sin x) dx. \tag{5.6}
\end{aligned}$$

Using the trigonometric identity $16\sin^5 x = \sin 5x - 5\sin 3x + 10\sin x$, we obtain

$$I + J = 2i \int_0^\infty \frac{1}{x^5} (16\sin^5 x) dx = 32i \int_0^\infty \left(\frac{\sin x}{x} \right)^5 dx.$$

Hence, $32i \int_0^\infty \left(\frac{\sin x}{x} \right)^5 dx - \frac{115}{12}\pi i = 0$, which implies that

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^5 dx = \frac{115}{12} \pi i \cdot \frac{1}{32i} = \frac{115\pi}{384}.$$

We want to show how to derive the formulas for $L(n)$ in two different ways.

Method I. The contour integration method

The following trigonometric identities are well-known (see [1]).

$$(-1)^{\frac{n}{2}} 2^{n-1} \sin^n \theta = \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} \cos(n-2k)\theta + (-1)^{\frac{n}{2}} \frac{1}{2} \binom{n}{\frac{n}{2}} \text{ if } n \text{ is even.}$$

$$(-1)^{\frac{n-1}{2}} 2^{n-1} \sin^n \theta = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} \sin(n-2k)\theta \text{ if } n \text{ is odd.}$$

The following definition for $f_n(z)$ is the generalization of $f_5(z)$ that we discussed before.

$$f_n(z) = P_{n-3}(z) + \begin{cases} \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n}{k} e^{i(n-2k)z} & n \text{ even} \\ \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} e^{i(n-2k)z} & n \text{ odd} \end{cases}$$

in which $P_{n-3}(z)$, a polynomial of degree $n-3$, is determined so that $\frac{f_n(z)}{z^n}$ is analytic in

the complex plane except for a simple pole at $z=0$ and chosen so that the integral over

C_r exists. The residue of $\frac{f_n(z)}{z^n}$ at $z=0$ is $\frac{f_n^{(n-1)}(0)}{(n-1)!}$.

As before integrating over the contour C and taking limits as $r \rightarrow 0$ and $R \rightarrow \infty$, we will

have that $\int_{C_R} \frac{f_n(z)}{z^n} dz \rightarrow 0$ and $\int_{C_r} \frac{f_n(z)}{z^n} dz \rightarrow -\pi i \operatorname{Res} \left[\frac{f_n(z)}{z^n}; 0 \right]$.

This gives

$$L(n) = \begin{cases} \frac{(-1)^{\frac{n}{2}} \pi i^n}{2^n (n-1)!} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} (n-2k)^{n-1} & \text{if } n \text{ is even} \\ \frac{(-1)^{\frac{(n-1)}{2}} \pi i^{(n-1)}}{2^n (n-1)!} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} (n-2k)^{n-1} & \text{if } n \text{ is odd} \end{cases}.$$

Now we introduce Apostol's proof of the above identity using integration by parts.

Method II. Let $f(x) = \sin^m x$ with $m = 2, 3, \dots$. We wish to evaluate the integral

$$I_m = \int_0^\infty \frac{f(x)}{x^m} dx \text{ by parts.}$$

Let $u = f(x)$, $du = f'(x)dx$ and $dv = x^{-m}dx$, $v = \frac{1}{-m+1} x^{-m+1}$. Then

$$\begin{aligned} I_m &= -\frac{1}{m-1} x^{-m+1} f(x) \Big|_0^\infty + \frac{1}{m-1} \int_0^\infty x^{-(m-1)} f'(x) dx \\ &= \lim_{c \rightarrow \infty} -\frac{1}{m-1} (c^{-(m-1)} f(c)) + \frac{1}{m-1} \int_0^\infty \frac{f'(x)}{x^{m-1}} dx = \frac{1}{m-1} \int_0^\infty \frac{f'(x)}{x^{m-1}} dx, \text{ because } |\sin^m x| \leq 1. \end{aligned}$$

Using integration by parts again, we get

$$u = f'(x), du = f''(x)dx \text{ and } dv = x^{-(m-1)}dx, v = \frac{-1}{(m-2)} x^{-(m-2)}, \text{ and}$$

$$I_m = \int_0^\infty \frac{f(x)}{x^m} dx = \frac{1}{m-1} \cdot \frac{1}{m-2} \int_0^\infty \frac{f''(x)}{x^{(m-2)}} dx.$$

Repeating the process we obtain

$$\int_0^\infty \frac{f(x)}{x^m} dx = \frac{1}{(m-1)!} \int_0^\infty f^{(m-1)}(x) \cdot x^{-(m-(m-1))} dx = \frac{1}{(m-1)!} \int_0^\infty \frac{f^{(m-1)}(x)}{x} dx. \quad (5.7)$$

When m is an even number, i.e. $m = 2n$, $n = 1, 2, \dots$, equation (5.7) can be written as

$$\int_0^{\infty} \frac{f(x)}{x^m} dx = \frac{1}{(2n-1)!} \int_0^{\infty} \frac{f^{(2n-1)}(x)}{x} dx. \quad (5.8)$$

Now we use the identity [1]:

$$\sin^{2n} x = \frac{1}{2^{(2n-1)}} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{2n}{k} \cos[(2n-2k)x] + \frac{1}{2^{2n}} \binom{2n}{n}. \quad (5.9)$$

Now let $f(x) = \sin^{2n} x$ in (5.8).

Taking derivative of $f(x)$, $(2n-1)$ times using (5.9), we have

$$f^{(2n-1)}(x) = \frac{1}{2^{(2n-1)}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (2n-2k)^{(2n-1)} \sin[(2n-2k)x]. \quad (5.10)$$

Substituting (5.10) into (5.8), we obtain

$$\begin{aligned} \int_0^{\infty} \frac{f(x)}{x^m} dx &= \frac{1}{(2n-1)!} \int_0^{\infty} \frac{f^{(2n-1)}(x)}{x} dx \\ &= \frac{1}{(2n-1)!} \int_0^{\infty} \left[\frac{1}{x} \cdot \frac{1}{2^{(2n-1)}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (2n-2k)^{(2n-1)} \sin[(2n-2k)x] \right] dx \\ &= \frac{1}{(2n-1)!} \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (2n-2k)^{(2n-1)} \int_0^{\infty} \frac{1}{x} \sin[(2n-2k)x] dx. \end{aligned} \quad (5.11)$$

Now we consider the integral $\int_0^{\infty} \frac{\sin[(2n-2k)x]}{x} dx$.

Let $x = \frac{t}{2n-2k}$. Then $dx = \frac{1}{2n-2k} dt$. So

$$\int_0^{\infty} \frac{\sin[(2n-2k)x]}{x} dx = \frac{1}{2n-2k} \int_0^{\infty} \frac{\sin t}{t} dt = \int_0^{\infty} \frac{\sin t}{t} dt.$$

It is well-known that $\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$.

Hence (5.11) can be written as

$$\frac{1}{(2n-1)!} \frac{1}{2^{2n-1}} \cdot \frac{\pi}{2} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (2n-2k)^{(2n-1)} = \frac{\pi}{2^{2n} (2n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (2n-2k)^{2n-1}.$$

Thus,

$$L(2n) = \frac{\pi}{2^{2n} (2n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} (2n-2k)^{2n-1}. \quad (5.12)$$

Next, we consider (5.7) when m is an odd integer, i.e. $m = 2n + 1$, for $n = 0, 1, 2, 3, \dots$.

Let $h(x) = \sin^{(2n+1)}(x)$ with $n = 0, 1, 2, \dots$

Then we want to evaluate the integral

$$L(2n+1) = \int_0^\infty \frac{h(x)}{x^{2n+1}} dx. \quad (5.13)$$

Let $u = h(x)$, $du = h'(x)dx$ and $dv = x^{-(2n+1)}dx$, $v = -\frac{1}{2n} x^{-2n}$. Then (5.13) can be

rewritten as

$$L(2n+1) = -\frac{1}{2n} x^{-2n} h(x) \Big|_{x=0}^{x=\infty} + \frac{1}{2n} \int_0^\infty x^{-2n} h'(x) dx = \frac{1}{2n} \int_0^\infty \frac{h'(x)}{x^{2n}} dx, \text{ as before.}$$

Repeating integration by parts $(2n)$ times yields

$$L(2n+1) = \frac{1}{(2n)!} \int_0^\infty h^{(2n)}(x) \cdot x^{-((2n+1)-2n)} dx = \frac{1}{(2n)!} \int_0^\infty \frac{h^{(2n)}(x)}{x} dx. \quad (5.14)$$

Now we use the identity

$$\sin^{(2n+1)}(x) = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^{n-k} \binom{2n+1}{k} \sin[(2n-2k+1)x] \text{ and } h(x) = \sin^{(2n+1)}(x). \text{ Then}$$

$$h^{(2n)}(x) = \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n-2k+1)^{2n} \sin[(2n-2k+1)x]. \quad (5.15)$$

Substituting (5.15) into (5.14), we have

$$L(2n+1) = \frac{1}{(2n)!} \frac{1}{2^{2n}} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n-2k+1)^{2n} \int_0^\infty \frac{\sin[(2n-2k+1)x]}{x} dx. \quad (5.16)$$

Since $\int_0^\infty \frac{\sin[(2n-2k+1)x]}{x} dx = \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$, we have

$$L(2n+1) = \frac{\pi}{2^{2n+1} (2n)!} \sum_{k=0}^n (-1)^k \binom{2n+1}{k} (2n-2k+1)^{2n}. \quad (5.17)$$

So (5.12) and (5.17) can be written as

$$L(m) = \frac{\pi}{2^m (m-1)!} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \binom{m}{k} (m-2k)^{m-1} \quad (5.18)$$

where $m = 1, 2, 3, 4, \dots$ and $\lfloor x \rfloor$ is the greatest integer function. By using (5.18), we have

$$L(2) = \frac{2\pi}{2^2 (2-1)!} = \frac{\pi}{2}.$$

$$\begin{aligned} L(3) &= \frac{\pi}{2^3 (3-1)!} \left[(-1)^0 \binom{3}{0} (3-2(0))^{3-1} + (-1)^1 \binom{3}{1} (3-2(1))^{3-1} \right] = \frac{\pi}{8 \cdot 2!} (3^2 + (-1)(3)(1)^2) \\ &= \frac{\pi}{16} (9 - 3) = \frac{6\pi}{16} = \frac{3\pi}{8}. \end{aligned}$$

$$\begin{aligned} L(4) &= \frac{\pi}{2^4 (4-1)!} \left[(-1)^0 \binom{4}{0} (4-2(0))^{4-1} + (-1)^1 \binom{4}{1} (4-2(1))^{4-1} \right] = \frac{\pi}{16 \cdot 3!} (4^3 + (-1)(4)(2^3)) \\ &= \frac{\pi}{96} (64 - 32) = \frac{32\pi}{96} = \frac{\pi}{3}. \end{aligned}$$

$$L(5) = \frac{\pi}{2^5 (5-1)!} \left[(-1)^0 \binom{5}{0} (5-0)^{5-1} + (-1)^1 \binom{5}{1} (5-2)^{5-1} + (-1)^2 \binom{5}{2} (5-2(2))^{5-1} \right]$$

$$= \frac{\pi}{32 \cdot 4!} (5^4 - (5)(3)^4 + 10(1)^4) = \frac{\pi}{(32)(24)} (625 - 405 + 10)$$

$$= \frac{\pi(230)}{(32)(24)} = \frac{\pi(115)}{(16)(24)} = \frac{115\pi}{384}.$$

$$L(6) = \frac{\pi}{2^6(6-1)!} \left[(-1)^0 \binom{6}{0} (6-0)^{6-1} + (-1)^1 \binom{6}{1} (6-2)^{6-1} + (-1)^2 \binom{6}{2} (6-2(2))^{6-1} \right]$$

$$= \frac{\pi}{64 \cdot 5!} (6^5 - 6 \cdot (4)^5 + 15(2)^5) = \frac{\pi}{64 \cdot 120} (7776 - 6144 + 480) = \frac{\pi(2112)}{64 \cdot 120} = \frac{11\pi}{40}.$$

CHAPTER 3

SUMMATION OF SERIES

In 1831, Cauchy announced the theorem that an analytic function of a complex variable $w = f(z)$ can be expanded about a point $z = z_0$ in a power series that is convergent for all values of z within a circle having z_0 as center with a radius $r > 0$ (see [6]). From this time on, the use of infinite series became an essential part of the theory of functions of both real and complex variables.

In this chapter, two famous summation formulas and two identities are discussed.

3.1 Lindelof's Summation Formulas and Examples

Amazingly, the residue theorem enables us to figure out the values of some series. First, we will introduce Lindelof's summation formulas, *i.e.*

1. $\sum_{n=-\infty}^{\infty} f(n) = - \{ \text{sum of residues of } \pi f(z) \cot(\pi z) \text{ at all the poles of } f(z) \},$
2. $\sum_{n=-\infty}^{\infty} (-1)^n f(n) = - \{ \text{sum of residues of } \pi f(z) \csc(\pi z) \text{ at all the poles of } f(z) \}.$

Second, we will have an example as an application of formula 1. And third, we will discuss how to represent summation of a series in integral form and find a summation formula which was first derived by Hardy.

Let C_N be the contour shown below, which is a square.

Let $|f(z)| \leq \frac{M}{|z|^k}$ where $k > 1$ along the path C_N and M is a constant independent of N .

1. We want to show that under the above condition on $f(z)$,

$$\sum_{n=-\infty}^{\infty} f(n) = - \{ \text{sum of residues of } \pi f(z) \cot(\pi z) \text{ at all the poles of } f(z) \}.$$

Here we have two possible cases, *i.e.*

I) $f(z)$ has a finite number of poles, and

II) $f(z)$ has infinite number of poles.

We assume in addition that $f(z)$ is a meromorphic function and $f(z)$ has no singularities at $z = n$. That is, $f(z)$ is analytic at $z = \pm 1, \pm 2, \dots$, and all of its singularities are only poles.

Hence, the function $F(z) = \pi f(z) \cot(\pi z) = \frac{\pi f(z) \cos(\pi z)}{\sin(\pi z)}$ is a meromorphic function with

simple poles at $z = 0, \pm 1, \pm 2, \dots$.

We also assume that the only singularities of $f(z)$ occur at $z_1, z_2, \dots, z_m$.

We shall only consider the case where $f(z)$ has a finite number of poles.

Choose some large number N so that all poles of $F(z)$ lie inside C_N . Then

$$\operatorname{Res}[f(z)\cot(\pi z); n], \quad n = 0, \pm 1, \pm 2, \dots \quad (1.1)$$

$$= \lim_{z \rightarrow n} (z - n) \pi \frac{\cos(\pi z)}{\sin(\pi z)} f(z) = \lim_{z \rightarrow n} \frac{(z - n)}{\sin(\pi z)} \lim_{z \rightarrow n} \pi f(z) \cos(\pi z)$$

$$= \lim_{z \rightarrow n} \frac{1}{\pi \cos(\pi z)} \lim_{z \rightarrow n} \pi f(z) \cos(\pi z) = \frac{1}{\pi \cos(\pi n)} \cdot \pi f(n) \cos(n\pi) = f(n) \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Now let $F(z) = f(z)\cot(\pi z)$. Applying the residue theorem, we have

$$\begin{aligned} \int_{C_N} f(z)\cot(\pi z) dz &= 2\pi i \left[\sum_{n=-N}^N \operatorname{Res}(F(z); n) + \sum \operatorname{Res}(F(z); z_k) \right] \\ &= 2\pi i \left[\sum_{n=-N}^N f(n) + \sum \operatorname{Res}(F(z); z_k) \right]. \end{aligned}$$

Now we consider $\sum \operatorname{Res}[F(z); z_k]$.

As $N \rightarrow \infty$, the square C_N becomes large enough to contain all of the points

$z_1, z_2, \dots, z_m$ in the summation $\sum \operatorname{Res}[F(z); z_k]$ so that its summation index will be from

$k = 1$ to $k = m$.

Hence,

$$\lim_{N \rightarrow \infty} \int_{C_N} F(z) dz = 2\pi i \left[\sum_{n=-\infty}^{\infty} f(n) + \sum_{k=1}^m \operatorname{Res}(F(z); z_k) \right]. \quad (1.2)$$

Next we will show $\int_{C_N} F(z)dz \rightarrow 0$ as $N \rightarrow \infty$. Here we have to consider

$$\left| \int_{C_N} \pi f(z) \cot(\pi z) dz \right|.$$

First, we want to show that

$$|\cot(\pi z)| \leq 2 \text{ for all } z \text{ on } C_N. \quad (1.3)$$

Dealing with each side of the square C_N separately, we have $z = N + \frac{1}{2} + iy$ where

$-N - \frac{1}{2} \leq y \leq N + \frac{1}{2}$ on the vertical side. By using the formulas

$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$ and $|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$, we get

$$\begin{aligned} |\cot(\pi z)| &= \left| \frac{\cos(\pi z)}{\sin(\pi z)} \right| = \frac{\sqrt{\cos^2 \left(\pi \left(N + \frac{1}{2} \right) \right) + \sinh^2(\pi y)}}{\sqrt{\sin^2 \left(\pi \left(N + \frac{1}{2} \right) \right) + \sinh^2(\pi y)}} = \frac{|\sinh(\pi y)|}{\sqrt{1 + \sinh^2(\pi y)}} \\ &\leq \frac{|\sinh(\pi y)|}{|\sinh(\pi y)|} = 1 \leq 2. \end{aligned}$$

Since $\cot(\pi z)$ is odd function, we can deal with the left vertical side, and similarly the horizontal sides are taken care of.

Using the inequality (1.3), we see that

$$I = \left| \int_{C_N} \pi f(z) \cot(\pi z) dz \right| \leq \int_{C_N} |\pi \cot(\pi z)| |f(z)| |dz| \leq \pi \int_{C_N} |\cot(\pi z)| \frac{M}{|z|^k} |dz|. \quad (1.4)$$

Let $z = x + iy$. Then the right hand side of (1.4) can be written as

$I \leq 2\pi \frac{M}{N^k} (8N + 4)$. Taking the limit as $N \rightarrow \infty$, we have that $\lim_{N \rightarrow \infty} 2\pi \frac{M}{N^k} (8N + 4) = 0$

because $k > 1$.

So from (1.2), we have

$$0 = 2\pi i \left[\sum_{n=-\infty}^{\infty} f(n) + \sum_{k=1}^m \text{Res}(F(z); z_k) \right], \text{ which implies that } \sum_{n=-\infty}^{\infty} f(n) = - \sum_{k=1}^m \text{Res}(F(z); z_k).$$

Hence, we conclude that

$$\sum_{n=-\infty}^{\infty} f(n) = - \{ \text{sum of residues of } \pi \cot \pi z f(z) \text{ at the poles of } f(z) \}. \quad \square$$

2. Lindelof's Alternating Summation Formula.

We will show that

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = - \{ \text{sum of residues of } \pi f(z) \csc(\pi z) \text{ at all the poles of } f(z) \}$$

using the contour C_N .

We know that $\csc \pi z = \frac{1}{\sin \pi z}$ has simple poles at $z = 0, \pm 1, \pm 2, \dots$. We assume that $f(z)$

has no poles at $z = 0, \pm 1, \pm 2, \dots$.

Let $F(z) = \pi f(z) \csc(\pi z)$. Then

$$\text{Res}[F(z); n], \quad n = 0, \pm 1, \pm 2, \dots$$

$$\begin{aligned} &= \lim_{z \rightarrow n} (z - n) \pi \frac{1}{\sin(\pi z)} f(z) = \lim_{z \rightarrow n} \frac{(z - n)}{\sin(\pi z)} \lim_{z \rightarrow n} \pi f(z) = \lim_{z \rightarrow n} \frac{1}{\pi \cos(\pi z)} \lim_{z \rightarrow n} \pi f(z) \\ &= \frac{1}{\pi \cos(n\pi)} \cdot \pi f(n) = (-1)^n f(n). \end{aligned} \tag{1.5}$$

By the residue theorem, we have

$$\int_{C_N} F(z) dz = 2\pi i \left[\sum_{n=-N}^N (-1)^n f(n) + \sum \operatorname{Res}[F(z); b_k] \right], \quad (1.6)$$

where b_k 's are the singularities of $f(z)$, for $k = 1, 2, \dots, m$, and $\csc \pi z$ has no poles at b_k .

Then as $N \rightarrow \infty$, we have

$$\lim_{N \rightarrow \infty} \int_{C_N} F(z) dz = 2\pi i \left[\sum_{n=-\infty}^{\infty} (-1)^n f(n) + \sum_{k=1}^m \operatorname{Res}[F(z); b_k] \right].$$

It can be shown that (see Spiegel [29]) $\lim_{N \rightarrow \infty} \int_{C_N} F(z) dz = 0$.

Thus, (1.6) can be rewritten as

$$0 = 2\pi i \left[\sum_{n=-N}^N (-1)^n f(n) + \sum_{k=1}^m \operatorname{Res}[F(z); b_k] \right], \text{ which implies}$$

$$\text{that } \sum_{n=-N}^N (-1)^n f(n) = - \sum_{k=1}^m \operatorname{Res}[F(z); b_k].$$

So we conclude that

$$\sum_{n=-N}^N (-1)^n f(n) = - \{ \text{sum of residues of } \pi \csc \pi z f(z) \text{ at the poles of } f(z) \}. \quad \square$$

Here we present an example to show how Lindelof's first formula is applied to evaluate the sum of a series.

We want to prove that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

$$\text{Let } f(z) = \frac{1}{(2z+1)^2}.$$

Since $f(z) = \frac{1}{\left[2\left(z + \frac{1}{2}\right)\right]^2} = \frac{1}{4\left(z + \frac{1}{2}\right)^2}$, $f(z)$ has a pole of order 2 at $z = -\frac{1}{2}$.

Then

$$\begin{aligned} \operatorname{Res}\left[\frac{\pi \cot(\pi z)}{4\left(z + \frac{1}{2}\right)^2}; -\frac{1}{2}\right] &= \lim_{z \rightarrow -\frac{1}{2}} \frac{1}{(2-1)!} \frac{d}{dz} \left[\left(z - \left(-\frac{1}{2}\right)\right)^2 \frac{\pi \cot(\pi z)}{4\left(z + \frac{1}{2}\right)^2} \right] \\ &= \lim_{z \rightarrow -\frac{1}{2}} \frac{d}{dz} \left[\frac{\pi}{4} \cot(\pi z) \right] = \lim_{z \rightarrow -\frac{1}{2}} \frac{\pi}{4} (-\pi \csc^2(\pi z)) = -\frac{\pi^2}{4}. \end{aligned}$$

Since $\sum_{n=-\infty}^{\infty} f(n) = -\operatorname{Res}\left[\pi \cot \pi z f(z); -\frac{1}{2}\right]$, we have $\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} = -\left(-\frac{\pi^2}{4}\right) = \frac{\pi^2}{4}$.

We know that $\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=-\infty}^{-1} \frac{1}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Let $n = -m$, $m \in \mathbb{N}$. Then

$$\sum_{n=-\infty}^{-1} \frac{1}{(2n+1)^2} = \sum_{m=1}^{\infty} \frac{1}{(-2m+1)^2} = \sum_{m=1}^{\infty} \frac{1}{(-(2m-1))^2} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Hence,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} &= \sum_{n=-\infty}^{-1} \frac{1}{(2n+1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots\right) \\ &= 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}. \end{aligned}$$

Hence,

$$\sum_0^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \quad \square$$

Lindelof's theorems can be used to evaluate the ζ -function $\zeta(2m)$ in terms of the Bernoulli numbers, where m is a positive integer (see [17], [25]). $\zeta(2m+1)$ is still unknown in a simple form. For more formulas on the summation of series, see [4].

3.2 Hardy-Egorychev Identity

We will compute a finite sum in the form

$$\sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^k \binom{m-k}{k} \frac{1}{m-k}, \quad m = 1, 2, \dots$$

This sum was first evaluated by Hardy. Here we will introduce Egorychev's elegant method for evaluating sums using contour integrals (See [12], [14]).

$$\text{Let } S_m = \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^k \binom{m-k}{k} \frac{1}{m-k}, \quad m = 1, 2, \dots$$

First, we want to show that

$$\binom{m-k}{k} \frac{1}{m-k} = \frac{1}{m} \left[\binom{m-k}{k} + \binom{m-k-1}{k-1} \right]. \quad \text{Then}$$

$$\text{L.H.S.} = \binom{m-k}{k} \frac{1}{m-k} = \frac{(m-k)!}{k!(m-2k)!} \cdot \frac{1}{m-k} = \frac{(m-k)(m-k-1)!}{k!(m-2k)!(m-k)} = \frac{(m-k-1)!}{k!(m-2k)!}$$

And

$$\text{R.H.S.} = \frac{1}{m} \left(\binom{m-k}{k} + \binom{m-k-1}{k-1} \right)$$

$$\begin{aligned}
&= \frac{1}{m} \left[\frac{(m-k)!}{k!(m-2k)!} + \frac{(m-k-1)!}{(k-1)!(m-2k)!} \right] = \frac{1}{m} \left[\frac{(m-k)! + k(m-k-1)!}{k(k-1)!(m-2k)!} \right] \\
&= \frac{1}{m} \left[\frac{(m-k)(m-k-1)! + k(m-k-1)!}{k(k-1)!(m-2k)!} \right] = \frac{1}{m} \left[\frac{m(m-k-1)!}{k!(m-2k)!} \right] = \frac{(m-k-1)!}{k!(m-2k)!}.
\end{aligned}$$

Thus, we have proved the combinatorial identity:

$$\binom{m-k}{k} \frac{1}{m-k} = \frac{1}{m} \left(\binom{m-k}{k} + \binom{m-k-1}{k-1} \right). \quad (2.1)$$

Now consider $\binom{m-k}{k}$ and $\binom{m-k-1}{k-1}$.

By the lemma [12], we obtain

$$\binom{\alpha}{n} = \operatorname{Re} s \left[\frac{(1+w)^\alpha}{w^{n+1}}; 0 \right] = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(1+w)^\alpha}{w^{n+1}} dw, \text{ for } \alpha \text{ is a positive integer and } 0 < \rho < 1,$$

which is the coefficient of $\frac{1}{w}$ in the expansion of $\frac{(1+w)^\alpha}{w^{n+1}}$. Because

$$\binom{m-k}{k} = \binom{m-k}{m-2k}, \text{ we can write}$$

$$\binom{m-k}{k} = \operatorname{Re} s \left[\frac{(1+w)^{m-k}}{w^{k+1}}; 0 \right] = \operatorname{Re} s \left[\frac{(1+w)^{m-k}}{w^{m-2k+1}}; 0 \right] = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(1+w)^{m-k}}{w^{m-2k+1}} dw, \quad (2.2)$$

where ρ is a small positive number.

$$\text{Since } \binom{m-k-1}{k-1} = \binom{m-k-1}{m-2k},$$

$$\binom{m-k-1}{k-1} = \operatorname{Re} s \left[\frac{(1+w)^{m-k-1}}{w^k}; 0 \right] = \operatorname{Re} s \left[\frac{(1+w)^{m-k-1}}{w^{m-2k+1}}; 0 \right] = \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(1+w)^{m-k-1}}{w^{m-2k+1}} dw. \quad (2.3)$$

Substituting (2.2) and (2.3) into (2.1), we have

$$\begin{aligned}
\binom{m-k}{k} \frac{1}{m-k} &= \frac{1}{m} \left[\frac{1}{2\pi i} \int_{|w|=\rho} \frac{(1+w)^{m-k}}{w^{m-2k+1}} dw + \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(1+w)^{m-k-1}}{w^{m-2k+1}} dw \right] \\
&= \frac{1}{m} \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(1+w)^{m-k} + (1+w)^{m-k-1}}{w^{m-2k+1}} dw.
\end{aligned} \tag{2.4}$$

The integrand $\frac{(1+w)^{m-k} + (1+w)^{m-k-1}}{w^{m-2k+1}}$ can be rewritten as

$$\begin{aligned}
\frac{(1+w)^{m-k} + (1+w)^{m-k-1}}{w^{m-2k+1}} &= \frac{(1+w)(1+w)^{m-k-1} + (1+w)^{m-k-1}}{w^{m-2k+1}} = \frac{((1+w)+1)(1+w)^{m-k-1}}{w^{m-2k+1}} \\
&= \frac{(2+w)(1+w)^{m-k-1}}{w^{m+1}w^{-2k}} = \frac{(2+w)(1+w)^{m-1}(1+w)^{-k}}{w^{m+1}w^{-2k}} \\
&= \frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \frac{w^{2k}}{(1+w)^k} = \frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \left(\frac{w^2}{1+w} \right)^k.
\end{aligned} \tag{2.5}$$

Replacing (2.5) with the integrand of (2.4), we get

$$\binom{m-k}{k} \frac{1}{m-k} = \frac{1}{m} \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \left(\frac{w^2}{1+w} \right)^k dw.$$

Hence,

$$\begin{aligned}
S_m &= \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \binom{m-k}{k} \frac{1}{m-k} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \frac{1}{m} \frac{1}{2\pi i} \int_{|w|=\rho} \frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \left(\frac{w^2}{1+w} \right)^k dw \\
&= \frac{1}{m} \frac{1}{2\pi i} \int_{|w|=\rho} \left[\frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k \left(\frac{w^2}{1+w} \right)^k \right] dw
\end{aligned}$$

$$= \frac{1}{2\pi im} \int_{|w|=\rho} \left[\frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left(\frac{-w^2}{1+w} \right)^k \right] dw. \quad (2.6)$$

Consider the geometric sum

$$\sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \left(\frac{-w^2}{1+w} \right)^k.$$

For $k > \left\lfloor \frac{m}{2} \right\rfloor$ and $|w| = \rho < 1$, we can choose $\rho = \frac{1}{2}$ so that $\sum_{k=0}^{\infty} \left(\frac{-w^2}{1+w} \right)^k$ converges

uniformly on the contour of integration $|w| = \rho = \frac{1}{2}$. Notice that the finite sum has been

replaced by a series because all the integrals are equal to zeros for $k > \left\lfloor \frac{m}{2} \right\rfloor$.

Hence, (2.6) can be written as

$$\begin{aligned} S_m &= \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} \left[\frac{(2+w)(1+w)^{m-1}}{w^{m+1}} \sum_{k=0}^{\infty} \left(\frac{-w^2}{1+w} \right)^k \right] dw \\ &= \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} \left[\frac{(2+w)(1+w)^m}{w^{m+1}(1+w)} \left(\frac{1}{1 - \left(\frac{-w^2}{1+w} \right)} \right) \right] dw \\ &= \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} \left[\frac{(2+w)(1+w)^m}{w^{m+1}(1+w)} \left(\frac{1}{1 + w - (-w^2)} \right) \right] dw \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} \left[\frac{(2+w)(1+w)^m}{w^{m+1}(1+w)} \left(\frac{1+w}{1+w+w^2} \right) \right] dw = \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} \left[\frac{(2+w)(1+w)^m}{w^{m+1}(1+w+w^2)} \right] dw \\
&= \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} \left[\frac{(w+2)(1+w)^m}{w^{m+1}(w^2+w+1)} \right] dw.
\end{aligned}$$

We will evaluate the above integral using Cauchy's residue theorem. We will compute the residues at the poles outside the circle of radius $1/2$ instead of computing the residue at $z = 0$ which is difficult.

$$\text{Let } h(z) = \frac{(z+2)(1+z)^m}{z^{m+1}(z^2+z+1)}, \quad z \in \mathbb{C}. \quad (2.7)$$

Then $h(z)$ has two simple poles at $z = \frac{-1 \pm \sqrt{3}i}{2}$ other than $z=0$.

Let $\alpha = \frac{-1 + \sqrt{3}i}{2}$ and $\beta = \frac{-1 - \sqrt{3}i}{2}$. Then we have $z^2 + z + 1 = (z - \alpha)(z - \beta) = 0$, which

implies $\alpha^2 + \alpha + 1 = 0$ and similarly for β . So (2.7) can be written as

$$h(z) = \frac{(z+2)(1+z)^m}{z^{m+1}(z-\alpha)(z-\beta)}.$$

Hence,

$$\text{Res}[h(z); \alpha] = \lim_{z \rightarrow \alpha} (z - \alpha) \frac{(z+2)(1+z)^m}{z^{m+1}(z-\alpha)(z-\beta)} = \lim_{z \rightarrow \alpha} \frac{(z+2)(1+z)^m}{z^{m+1}(z-\beta)} = \frac{(\alpha+2)(1+\alpha)^m}{\alpha^{m+1}(\alpha-\beta)}, \text{ and}$$

$$\text{Res}[h(z); \beta] = \lim_{z \rightarrow \beta} (z - \beta) \frac{(z+2)(1+z)^m}{z^{m+1}(z-\alpha)(z-\beta)} = \frac{(\beta+2)(1+\beta)^m}{\beta^{m+1}(\beta-\alpha)}.$$

So we have

$$\begin{aligned}
S_m &= \frac{1}{2\pi im} \int_{|w|=\frac{1}{2}} h(z) dz = -\frac{1}{m} \left[\frac{(\alpha+2)(\alpha+1)^m}{\alpha^{m+1}(\alpha-\beta)} + \frac{(\beta+2)(\beta+1)^m}{\beta^{m+1}(\beta-\alpha)} \right] \\
&= -\frac{1}{m} \left[\frac{\beta^{m+1}(\alpha+2)(\alpha+1)^m - \alpha^{m+1}(\beta+2)(\beta+1)^m}{(\alpha-\beta)} \right] \\
&\text{since } \alpha\beta = \frac{-1+\sqrt{3}i}{2} \cdot \frac{-1-\sqrt{3}i}{2} = \frac{1+3}{4} = 1 \\
&= -\frac{1}{m} \left[\frac{[\beta(\alpha+1)]^m \beta(\alpha+2) - [\alpha(\beta+1)]^m \alpha(\beta+2)}{(\alpha-\beta)} \right] \\
&= -\frac{1}{m} \left[\frac{(\alpha\beta + \beta)^m (\alpha\beta + 2\beta) - (\alpha\beta + \alpha)^m (\alpha\beta + 2\alpha)}{\alpha - \beta} \right] \\
&= -\frac{1}{m} \left[\frac{(1+\beta)^m (1+2\beta) - (1+\alpha)^m (1+2\alpha)}{\alpha - \beta} \right] \\
&= -\frac{1}{m} \left[\frac{\left(1 + \frac{-1-\sqrt{3}i}{2}\right)^m \left(1 + 2\left(\frac{-1-\sqrt{3}i}{2}\right)\right) - \left(1 + \frac{-1+\sqrt{3}i}{2}\right)^m \left(1 + 2\left(\frac{-1+\sqrt{3}i}{2}\right)\right)}{\frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}} \right] \\
&= -\frac{1}{m} \left[\frac{\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^m (-\sqrt{3}i) - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^m (\sqrt{3}i)}{\sqrt{3}i} \right] \\
&= -\frac{1}{m} \left[\frac{\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^m (-\sqrt{3}i)(\sqrt{3}i) - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^m (\sqrt{3}i)(\sqrt{3}i)}{(\sqrt{3}i)(\sqrt{3}i)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3m} \left[\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)^m (3) - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^m (-3) \right] = \frac{1}{m} \left[\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right)^m + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^m \right] \\
&= \frac{1}{m} \left[\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right)^m + \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)^m \right] \\
&= \frac{1}{m} \left[\left(\cos \frac{m\pi}{3} - i \sin \frac{m\pi}{3} \right) + \left(\cos \frac{m\pi}{3} + i \sin \frac{m\pi}{3} \right) \right] = \frac{1}{m} \left(2 \cos \frac{m\pi}{3} \right) \\
&= \frac{2}{m} \cos \frac{\pi}{3} m.
\end{aligned}$$

Hence, we have obtained Hardy's identity:

$$\sum_{k=0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-1)^k \binom{m-k}{k} \frac{1}{m-k} = \begin{cases} \frac{(-1)^n}{3n-1}, m = 3n-1 \\ \frac{2(-1)^n}{3n}, m = 3n \\ \frac{(-1)^n}{3n+1}, \text{otherwise} \end{cases}$$

□

CHAPTER 4

APPLICATIONS

4.1 General Solutions of Ordinary Differential Equations

Here we introduce Cauchy's methods for solving ordinary differential equations using residues. Specially, we will find the general solutions for linear homogeneous and non-homogeneous differential equations with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

and

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = F(x)$$

where $a_0 = 1$ and a_j 's, $j = 1, 2, 3, \dots$ are given constants (see [21]).

Theorem 1. Consider the differential equation with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_{n-1} y' + a_n y = 0. \quad (1.1)$$

Let f be an arbitrary analytic function of the complex variable z , whose zeros do not coincide with the zeros of the polynomial

$$g(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n.$$

Then the general solution of the equation (1.1) is given by

$$y(x) = \sum \operatorname{Res} \left[\frac{f(z) e^{zx}}{g(z)} \right], \quad (1.2)$$

where the summation is taken over all the singularities of the function

$$H(z) = \frac{f(z) e^{zx}}{g(z)},$$

i.e. the summation is taken over all the zeros of the polynomial $g(z)$.

We now show that (1.2) is a solution of the homogeneous differential equation (1.1). We assume that

$$y = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \right].$$

Then

$$y' = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \cdot z \right],$$

$$y'' = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \cdot z^2 \right],$$

$\vdots$

$$y^{(k)} = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \cdot z^k \right] \quad (k = 1, 2, \dots, n).$$

Hence,

$$\begin{aligned} & y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y \\ &= \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} z^n \right] + a_1 \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} z^{n-1} \right] + \dots \\ & \dots + a_{n-1} \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} z \right] + a_n \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \right] \\ &= \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} (z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n) \right] = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \cdot g(z) \right] \\ &= \sum \operatorname{Res} [f(z)e^{zx}] = 0 \text{ since } f(z) \text{ is analytic.} \end{aligned}$$

Thus, (1.2) is indeed a solution of (1.1). We will omit the proof that (1.2) is a general

solution of (1.1). For details, see [21] and [9].

Now we will show some examples.

Example 1. The case of distinct real roots. We want to find the general solution of the differential equation

$$y'' - 3y' + 2y = 0. \quad (1.3)$$

Let $f(z)$ be any arbitrary entire function whose zeros are not 1 and 2, and let

$g(z) = z^2 - 3z + 2$. Then $g(z) = (z-1)(z-2)$. Clearly the zeros of $f(z)$ do not coincide with the zeros of $g(z)$. We know the general solution is given by

$$y = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \right] \text{ by (1.2)}$$

where the summation is taken over $z = 1$ and $z = 2$. So we have

$$\begin{aligned} y &= \operatorname{Res} \left(\frac{f(z)e^{zx}}{g(z)}; 1 \right) + \operatorname{Res} \left(\frac{f(z)e^{zx}}{g(z)}; 2 \right) = \lim_{z \rightarrow 1} \frac{f(z)e^{zx}}{(z-2)} + \lim_{z \rightarrow 2} \frac{f(z)e^{zx}}{(z-1)} \\ &= -f(1)e^x + f(2)e^{2x}. \end{aligned}$$

Let $c_1 = -f(1)$ and $c_2 = f(2)$.

Hence, the general solution of the differential equation is $y = c_1 e^x + c_2 e^{2x}$. $\square$

Example 2. The case of repeated real roots. It is our objective to find the general solution of the differential equation

$$y'' - 4y' + 4y = 0. \quad (1.4)$$

Let $f(z)$ be an arbitrary entire function whose zeros do not include 2, and let

$g(z) = z^2 - 4z + 4$. Then $g(z) = (z-2)^2$. The general solution is given by

$$y = \sum \operatorname{Res} \left[\frac{f(z)e^{zx}}{g(z)} \right].$$

Then we have

$$\begin{aligned} y &= \operatorname{Res} \left[\frac{f(z)e^{zx}}{(z-2)^2}; 2 \right] = \lim_{z \rightarrow 2} \frac{1}{(2-1)!} \frac{d}{dz} \left[(z-2)^2 \frac{f(z)e^{zx}}{(z-2)^2} \right] = \lim_{z \rightarrow 2} [f(z)xe^{2x} + f'(z)e^{2x}] \\ &= f(2)xe^{2x} + f'(2)e^{2x}. \end{aligned}$$

Let $f'(2) = c_1$ and $f(2) = c_2$.

Thus, we have obtained the general solution, $y = c_1e^{2x} + c_2xe^{2x}$. □

Now we wish to find the general solution of a nonhomogeneous ordinary differential equation with constant coefficients. We start by introducing a lemma.

Lemma. Let $g(z)$ be any polynomial of degree n . Then

1. $\sum \operatorname{Res} \left[\frac{z^k}{g(z)} \right] = 0$, $0 < k < n-1$, where the sum is taken over the zeros of $g(z)$.
2. $\sum \operatorname{Res} \left[\frac{z^{(n-1)}}{g(z)} \right] = 1$, where the sum is taken over the zeros of $g(z)$.

The proof of both cases is not difficult if we think of the sum of the above residues to be equal to minus the residue of the function at infinity. So all that is needed to complete the proof is to let $H(z) = \left[\frac{z^k}{g(z)} \right]$ and find the coefficient of $-w$ in the series expansion of $H\left(\frac{1}{w}\right)$.

Theorem 2. The general solution of the differential equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_n = F(x) \tag{1.5}$$

is given by

$$y = \sum \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx} \right] + \sum \operatorname{Res} \left[\frac{e^{zx}}{g(z)} \int_{x_0}^{\infty} e^{-zt} F(t) dt \right] \quad (1.6)$$

where $f(z)$ is an arbitrary analytic function of the complex variable z , whose zeros do not coincide with zeros of the polynomial

$$g(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Proof. We want to find the solution of (1.5) in the form

$$y = \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} e^{zx} \right], \quad (1.7)$$

where the sum is taken over all the zeros of $g(z)$.

Now consider

$$Y = \sum \operatorname{Res} \left[\frac{h(z)}{g(z)} e^{zx} \right], \quad (1.8)$$

where h is a function of z only.

We now require that the first $n-1$ derivatives of y , defined by (1.7) to be the same as the first $n-1$ derivatives of Y defined by (1.8), i.e. $y^{(n-1)} = Y^{(n-1)}$.

The first derivatives of y and Y are

$$y' = \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{e^{zx}}{g(z)} + \frac{h(z, x)}{g(z)} z e^{zx} \right] \text{ and}$$

$$Y' = \sum \operatorname{Res} \left[\frac{h(z)}{g(z)} z e^{zx} \right], \text{ respectively.}$$

In order to have $y' = Y'$, we must have

$$\sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{e^{zx}}{g(z)} \right] = 0.$$

Hence,

$$y' = \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z e^{zx} \right]. \quad (1.9)$$

Next we find y'' and Y'' .

$$y'' = \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z}{g(z)} e^{zx} + \frac{h(z, x)}{g(z)} z^2 e^{zx} \right], \text{ and}$$

$$Y'' = \sum \operatorname{Res} \left[\frac{h(z)}{g(z)} z^2 e^{zx} \right].$$

$$\text{To make } y'' = Y'', \text{ we must have } \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z}{g(z)} e^{zx} \right] = 0.$$

Hence,

$$y'' = \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^2 e^{zx} \right]. \quad (1.10)$$

Repeating taking derivatives, we will have

$$y^{(n-1)} = \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-2}}{g(z)} e^{zx} + \frac{h(z, x)}{g(z)} z^{n-1} e^{zx} \right],$$

$$Y^{(n-1)} = \sum \operatorname{Res} \left[\frac{h(z)}{g(z)} z^{n-1} e^{zx} \right], \text{ and } \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-2}}{g(z)} e^{zx} \right] = 0.$$

So we have

$$y^{(n-1)} = \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^{n-1} e^{zx} \right].$$

Thus,

$$y^{(n)} = \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-1}}{g(z)} e^{zx} + \frac{h(z, x)}{g(z)} z^n e^{zx} \right]. \quad (1.11)$$

Substituting (1.9), (1.10), and (1.11) into (1.5), we have

$$\begin{aligned} & \left[\sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^n e^{zx} \right] + a_1 \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^{n-1} e^{zx} \right] + \dots \right. \\ & \dots + a_{n-2} \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^2 e^{zx} \right] + a_{n-1} \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^1 e^{zx} \right] \\ & \left. + a_n \sum \operatorname{Res} \left[\frac{h(z, x)}{g(z)} z^0 e^{zx} \right] \right] + \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-1}}{g(z)} e^{zx} \right] = F(x). \end{aligned}$$

Notice the sum of all the terms except for the last on the left side is zero because when we combine these terms $g(z)$ in the numerator will cancel $g(z)$ in the denominator and Sum of the residues of an analytic function is zero. Thus we must have

$$\sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-1}}{g(z)} e^{zx} \right] = F(x).$$

Hence, we must find a function $h(z, x)$ such that

$$\sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^k}{g(z)} e^{zx} \right] = 0, \quad (k = 0, 1, \dots, n-2) \text{ and } \sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-1}}{g(z)} e^{zx} \right] = F(x) \quad (1.12)$$

are satisfied. If we choose

$$\frac{\partial h}{\partial x} e^{zx} = F(x),$$

then the previous lemma implies that both of the above equations are satisfied.

This is true since $\frac{\partial h}{\partial x} \neq 0$ and $e^{zx} \neq 0$. In order to obtain $\sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^k}{g(z)} e^{zx} \right] = 0$,

we have to have

$$\sum \operatorname{Res} \left[\frac{z^k}{g(z)} \right] = 0 \text{ for } k = 0, \dots, n-2, \text{ which is true by lemma 1. And in order to obtain}$$

$$\sum \operatorname{Res} \left[\frac{\partial h}{\partial x} \frac{z^{n-1}}{g(z)} e^{zx} \right] = F(x), \text{ we have to have}$$

$$\sum \operatorname{Res} \frac{z^{n-1}}{g(z)} = 1 \text{ because } \frac{z^{n-1}}{g(z)} \text{ is not a function of } x.$$

Hence,

$$\frac{\partial h}{\partial x} e^{zx} = F(x).$$

Dividing both sides of the above equation by e^{zx} , we get

$$\frac{\partial h}{\partial x} = e^{-zx} F(x).$$

Taking the integral of both sides, we have $\int \frac{\partial h}{\partial x} dx = \int e^{-zx} F(x) dx$ and

$$h(z, x) + K(z) = \int e^{-zx} F(x) dx \text{ which can be rewritten as}$$

$$h(z, x) = f(z) + \int_{x_0}^x e^{-zt} F(t) dt \quad (1.13)$$

where $f(z) = -K(z)$ is an arbitrary analytic function. Substituting (1.13) into (1.7), we obtain

$$\begin{aligned} y &= \sum \operatorname{Res} \left[\left(f(z) + \int_{x_0}^x e^{-zt} F(t) dt \right) \frac{e^{zx}}{g(z)} \right] \\ &= \sum \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx} \right] + \sum \operatorname{Res} \left[\frac{e^{zx}}{g(z)} \int_{x_0}^x e^{-zt} F(t) dt \right], \text{ which is exactly the same as (1.6).} \end{aligned}$$

Example 3. Find the general solution of the equation

$$y'' + y' - 6y = F(x).$$

We have $g(z) = z^2 + z - 6 = (z+3)(z-2)$. By (1.6),

$$\begin{aligned} y &= \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx}; -3 \right] + \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx}; 2 \right] \\ &\quad + \operatorname{Res} \left[e^{zx} \int_{x_0}^{\infty} e^{-zt} \frac{F(t)}{g(z)} dt; -3 \right] + \operatorname{Res} \left[e^{zx} \int_{x_0}^{\infty} e^{-zt} \frac{F(t)}{g(z)} dt; 2 \right] \\ &= \lim_{z \rightarrow -3} (z+3) \frac{f(z)}{(z+3)(z-2)} e^{zx} + \lim_{z \rightarrow 2} (z-2) \frac{f(z)}{(z-2)(z+3)} e^{zx} \\ &\quad + \lim_{z \rightarrow -3} (z+3) \frac{e^{zx}}{(z+3)(z-2)} \int_{x_0}^{\infty} e^{-zt} F(t) dt + \lim_{z \rightarrow 2} (z-2) \frac{e^{zx}}{(z-2)(z+3)} \int_{x_0}^{\infty} e^{-zt} F(t) dt \\ &= \frac{f(-3)}{-5} e^{-3x} + \frac{f(2)}{5} e^{2x} + \frac{e^{-3x}}{-5} \int_{x_0}^{\infty} e^{3t} F(t) dt + \frac{e^{2x}}{5} \int_{x_0}^{\infty} e^{-2t} F(t) dt \\ &= C_1 e^{-3x} + C_2 e^{2x} + \left(-\frac{1}{5} \int_{x_0}^{\infty} e^{3t} F(t) dt \right) e^{-3x} + \left(\frac{1}{5} \int_{x_0}^{\infty} e^{-2t} F(t) dt \right) e^{2x}. \quad \square \end{aligned}$$

Here, we want to show a special case. Let $F(t) = 1$. Then

$$\begin{aligned} &\left(-\frac{1}{5} \int_{x_0}^{\infty} e^{3t} F(t) dt \right) e^{-3x} + \left(\frac{1}{5} \int_{x_0}^{\infty} e^{-2t} F(t) dt \right) e^{2x} = -\frac{1}{5 \cdot 3} [e^{3t}]_{x_0}^{\infty} e^{-3x} + \frac{1}{5(-2)} [e^{-2t}]_{x_0}^{\infty} \\ &= -\frac{1}{15} [e^{3x} - e^{3x_0}] e^{-3x} - \frac{1}{10} [e^{-2x} - e^{-2x_0}] e^{2x} = \left(\frac{1}{15} e^{3x_0} \right) e^{-3x} + \left(\frac{1}{10} e^{-2x_0} \right) e^{2x} - \frac{1}{6}. \end{aligned}$$

Hence, $y = C_1^* e^{-3x} + C_2^* e^{2x} - \frac{1}{6}$ as expected.

Example 4. Let us find the general solution of the differential equation

$$y'' + 4y' + 4 = F(x).$$

We have $g(z) = z^2 + 4z + 4 = (z+2)^2$. By using (1.6),

$$\begin{aligned}
y &= \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx}; -2 \right] + \operatorname{Res} \left[\frac{e^{zx}}{g(z)} \int_{x_0}^{\infty} e^{-zt} F(t) dt; -2 \right] \\
&= \operatorname{Res} \left[\frac{f(z)}{(z+2)^2} e^{zx}; -2 \right] + \operatorname{Res} \left[\frac{e^{zx}}{(z+2)^2} \int_{x_0}^{\infty} e^{-zt} F(t) dt; -2 \right] \\
&= \frac{1}{(2-1)!} \lim_{z \rightarrow -2} \frac{d}{dz} \left[(z+2)^2 \frac{f(z)}{(z+2)^2} e^{zx} \right] + \frac{1}{(2-1)!} \lim_{z \rightarrow -2} \frac{d}{dz} \left[(z+2)^2 \frac{e^{zx}}{(z+2)^2} \int_{x_0}^{\infty} e^{-zt} F(t) dt \right] \\
&= \lim_{z \rightarrow -2} (f'(z) e^{zx} + f(z) x e^{zx}) + \lim_{z \rightarrow -2} \frac{d}{dz} \left(\int_{x_0}^{\infty} e^{zx} \cdot e^{-zt} F(t) dt \right) \\
&= f'(-2) e^{-2x} + f(-2) x e^{-2x} + \lim_{z \rightarrow -2} \frac{d}{dz} \left(\int_{x_0}^{\infty} e^{z(x-t)} F(t) dt \right) \\
&= f'(-2) e^{-2x} + f(-2) x e^{-2x} + \lim_{z \rightarrow -2} \int_{x_0}^{\infty} (x-t) e^{z(x-t)} F(t) dt \\
y &= (C_1 + C_2 x) e^{-2x} + \int_{x_0}^{\infty} (x-t) e^{-2(x-t)} F(t) dt.
\end{aligned}$$

□

Example 5. Let us find the general solution of the differential equation

$$y'' + 4y = F(x).$$

We have $g(z) = z^2 + 4 = (z - 2i)(z + 2i)$. By (1.6),

$$\begin{aligned}
y &= \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx}; 2i \right] + \operatorname{Res} \left[\frac{f(z)}{g(z)} e^{zx}; -2i \right] \\
&\quad + \operatorname{Res} \left[\frac{e^{zx}}{g(z)} \int_{x_0}^{\infty} e^{-zt} F(t) dt; 2i \right] + \operatorname{Res} \left[\frac{e^{zx}}{g(z)} \int_{x_0}^{\infty} e^{-zt} F(t) dt; -2i \right] \\
&= \operatorname{Res} \left[\frac{f(z)}{(z-2i)(z+2i)} e^{zx}; 2i \right] + \operatorname{Res} \left[\frac{f(z)}{(z-2i)(z+2i)} e^{zx}; -2i \right] \\
&\quad + \operatorname{Res} \left[\frac{e^{zx}}{(z-2i)(z+2i)} \int_{x_0}^{\infty} e^{-zt} F(t) dt; 2i \right] + \operatorname{Res} \left[\frac{e^{zx}}{(z-2i)(z+2i)} \int_{x_0}^{\infty} e^{-zt} F(t) dt; -2i \right]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{z \rightarrow 2i} (z - 2i) \frac{f(z)}{(z - 2i)(z + 2i)} e^{zx} + \lim_{z \rightarrow -2i} (z + 2i) \frac{f(z)}{(z + 2i)(z - 2i)} e^{zx} \\
&+ \lim_{z \rightarrow 2i} (z - 2i) \frac{e^{zx}}{(z - 2i)(z + 2i)} \int_{x_0}^{\infty} e^{-zt} F(t) dt + \lim_{z \rightarrow -2i} (z + 2i) \frac{e^{zx}}{(z - 2i)(z + 2i)} \int_{x_0}^{\infty} e^{-zt} F(t) dt \\
&= \frac{f(2i)}{4i} e^{2ix} + \frac{f(-2i)}{-4i} e^{-2ix} + \frac{e^{2ix}}{4i} \int_{x_0}^{\infty} e^{-2it} F(t) dt + \frac{e^{-2ix}}{-4i} \int_{x_0}^{\infty} e^{2it} F(t) dt \\
&= C_1 e^{2ix} + C_2 e^{-2ix} - \left(\frac{i}{4} \int_{x_0}^{\infty} e^{-2it} F(t) dt \right) e^{2ix} + \left(\frac{i}{4} \int_{x_0}^{\infty} e^{2it} F(t) dt \right) e^{-2ix}.
\end{aligned}$$

It is not difficult to see that $C_1 e^{2ix} + C_2 e^{-2ix}$ can be rewritten in a real form using Euler's formula. It is also clear that the sum of both integrals that represent the particular solution of the differential equation is a real-valued function. $\square$

4.2 Cauchy-Tanaka Integral Formula

We will introduce a formula similar to Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - a} dz, \text{ which will be used in calculating functions of matrices, without the}$$

usual diagonalization process. Lemma 1 and 2 will enable us to rewrite Lagrange's interpolation formula as a contour integral and similarly we will show that a function of a matrix A can be written as a contour integral.

Lemma 1. Let $f(z)$ be a polynomial of degree n so that

$$f(z) = (z - a_1)(z - a_2) \cdots (z - a_n) \text{ has roots at } z = a_k, a_k \in \mathbf{C}, k = 1, 2, \dots, n,$$

$(a_i \neq a_j, i \neq j)$. Let $F(z)$ be any function of a complex variable such that $F(z)$ is analytic in the domain D enclosed by the closed contour C which contains all the points $a_1, a_2, \dots, a_n$.

Then we can find a polynomial $p(x)$ of at most $(n-1)^{th}$ degree such that

$$\begin{aligned} p(a_1) &= F(a_1), \\ p(a_2) &= F(a_2), \\ &\vdots \\ p(a_n) &= F(a_n), \end{aligned}$$

where $p(x)$ is given by the contour integral

$$p(x) = \frac{1}{2\pi i} \oint_C \frac{f(z) - f(x)}{z - x} \cdot \frac{F(z)}{f(z)} dz.$$

Proof. We know $p(z)$ has simple poles. Using the well-known Lagrange's interpolation formula of the $(n-1)^{th}$ degree (see [7]), we find the polynomial $p(x)$ in the form

$$p(x) = \sum_{k=1}^n \frac{f(x)}{(x - a_k) f'(a_k)} \cdot F(a_k), \quad (2.1)$$

which satisfies all the above conditions.

Next we will show that (2.1) can be rewritten as follows:

$$p(x) = \frac{1}{2\pi i} \oint_C \frac{f(z) - f(x)}{z - x} \cdot \frac{F(z)}{f(z)} dz \quad (2.2)$$

Since

$$\begin{aligned} \frac{f(z) - f(x)}{z - x} &= \frac{(z - a_1)(z - a_2) \cdots (z - a_n) - (x - a_1)(x - a_2) \cdots (x - a_n)}{z - x} \\ &= \frac{(z^n - x^n) + \alpha_1(z^{n-1} - x^{n-1}) + \cdots + \alpha_{n-2}(z - x)}{z - x}, \quad (\alpha_1, \dots, \alpha_n \in \mathbf{R}), \end{aligned}$$

$\frac{f(z) - f(x)}{z - x}$ has a removable singularity $z = x$.

Hence, $\frac{f(z) - f(x)}{z - x} \cdot \frac{F(z)}{f(z)}$ has simple poles at $z = a_1, z = a_2, \dots$, and $z = a_n$.

So

$$\begin{aligned} &\sum_{k=1}^n \operatorname{Res} \left[\frac{f(z) - f(x)}{z - x} \cdot \frac{F(z)}{f(z)}; a_k \right] \\ &= \sum_{k=1}^n \lim_{z \rightarrow a_k} (z - a_k) \frac{f(z) - f(x)}{z - x} \cdot \frac{F(z)}{f(z)} = \sum_{k=1}^n \left[\lim_{z \rightarrow a_k} \frac{z - a_k}{f(z)} \lim_{z \rightarrow a_k} \frac{f(z) - f(x)}{z - x} \cdot F(z) \right] \\ &= \sum_{k=1}^n \left[\lim_{z \rightarrow a_k} \frac{1}{f'(z)} \lim_{z \rightarrow a_k} \frac{f(z) - f(x)}{z - x} F(z) \right], \text{ by using L'Hopital's rule} \\ &= \sum_{k=1}^n \frac{1}{f'(a_k)} \cdot \frac{-f(x)}{a_k - x} F(a_k) = \sum_{k=1}^n \frac{f(x)}{(x - a_k) f'(a_k)} \cdot F(a_k), \text{ which coincides with (2.1)} \end{aligned}$$

Therefore,

$$p(x) = \frac{1}{2\pi i} \oint_C \frac{f(z) - f(x)}{z - x} \cdot \frac{F(z)}{f(z)} dz. \quad (2.3)$$

Lemma 2. Let A be any $n \times n$ matrix and let $f(z)$ be its characteristic polynomial so that $\det(zI - A) = f(z)$, where $f(z) = (z - a_1)(z - a_2) \cdots (z - a_n)$ for $a_i \in \mathbb{R}$, $i = 1, 2, 3, \dots, n$, and I is the identity matrix. We assume the eigenvalues are distinct.

Let $H(z)$ be any polynomial in z . Then we have

$$H(A) = \frac{1}{2\pi i} \oint_C (zI - A)^{-1} H(z) dz,$$

where C is the contour containing $a_1, a_2, \dots, a_n$.

Proof. Since $H(z)$ is a polynomial and analytic inside C , by Lemma I, we can find another polynomial $p(x)$ such that

$$\begin{aligned} p(a_1) &= H(a_1), \\ p(a_2) &= H(a_2), \\ &\vdots \\ p(a_n) &= H(a_n), \end{aligned}$$

and

$$p(x) = \frac{1}{2\pi i} \oint_C \frac{f(z) - f(x)}{z - x} \cdot \frac{H(z)}{f(z)} dz \quad (2.4)$$

Now consider $\frac{f(z) - f(x)}{z - x}$.

Since $f(z)$ is the polynomial of n^{th} degree and

$$\frac{f(z) - f(x)}{z - x} = \frac{1}{z - x} [(z^n - x^n) + \alpha_1(z^{n-1} - x^{n-1}) + \alpha_2(z^{n-2} - x^{n-2}) + \cdots + \alpha_{n-1}(1 - 1)]$$

for $\alpha_i \in \mathbb{R}$, $i = 1, 2, \dots, n-1$, $\frac{f(z) - f(x)}{z - x}$ is a polynomial of $(n-1)^{\text{th}}$ degree.

So $\frac{f(z)-f(x)}{z-x}$ can be written as $\sum_{k=0}^{n-1} x^k C_k(z)$, where $C_k(z)$ are polynomials of z .

Then $\frac{f(z)-f(x)}{z-x} = \sum_{k=0}^{n-1} x^k C_k(z)$ implies $f(z)-f(x) = (z-x) \sum_{k=0}^{n-1} x^k C_k(z)$.

By replacing x with A , we have

$$f(z)I - f(A) = (zI - A) \sum_{k=0}^{n-1} A^k C_k(z).$$

By the Cayley-Hamilton theorem, $f(A) = 0$, and this implies that

$$f(z)I = (zI - A) \sum_{k=0}^{n-1} A^k C_k(z).$$

Multiplying $(zI - A)^{-1}$ on both sides, we have $(zI - A)^{-1} f(z) = \sum_{k=0}^{n-1} A^k C_k(z)$.

The above result implies that we can rewrite (2.4) as

$$H(A) = p(A) = \frac{1}{2\pi i} \oint_C [zI - A]^{-1} H(z) dz.$$

This completes the proof. It is important to know that $H(z)$ can be replaced by any analytic function inside C .

Next we will give the general form of Tanaka's theorem and use it to compute some matrix functions.

Theorem. If f is an analytic function defined on the spectrum of a square matrix A , then

$$f(A) = \frac{1}{2\pi i} \oint_C f(z) (zI - A)^{-1} dz \quad (2.5)$$

where I is the identity matrix and C is a closed contour which encircles the spectrum of A .

Example 1. Consider a 2×2 matrix A so that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbb{C}$.

We want to determine A^m , $m \in \mathbb{N}$. If $ad - bc \neq 0$, then A is invertible.

So we have

$$(zI - A) = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} z-a & -b \\ -c & z-d \end{bmatrix},$$

and

$$(zI - A)^{-1} = \frac{1}{(z-a)(z-d) - (-b)(-c)} \begin{bmatrix} z-d & b \\ c & z-a \end{bmatrix} \text{ if } (z-a)(z-d) - bc \neq 0.$$

$$\text{Let } p(z) = \det(zI - A) = (z-a)(z-d) - (-b)(-c) = z^2 - (a+d)z + ad - bc$$

$$= (z-\alpha)(z-\beta) \tag{2.6}$$

Since

$$z^2 - (a+d)z + (ad - bc) = z^2 - (\alpha + \beta)z + \alpha\beta, \text{ then } \alpha + \beta = a + d \text{ and } \alpha\beta = ad - bc.$$

Now we have

$$(zI - A)^{-1} = \frac{1}{p(z)} \begin{bmatrix} z-d & b \\ c & z-a \end{bmatrix}. \text{ Then}$$

$$f(A) = \frac{1}{2\pi i} \oint_{\gamma} f(z) \frac{1}{p(z)} \begin{bmatrix} z-d & b \\ c & z-a \end{bmatrix} dz \tag{2.7}$$

Let $f(A) = A^m$. Then

$$A^m = \frac{1}{2\pi i} \oint_{\gamma} \frac{z^m}{p(z)} \begin{bmatrix} z-d & b \\ c & z-a \end{bmatrix} dz, \text{ with } f(z) = z^m$$

$$= \begin{bmatrix} \frac{1}{2\pi i} \oint_C \frac{(z-d)z^m}{p(z)} dz & \frac{1}{2\pi i} \oint_C \frac{bz^m}{p(z)} dz \\ \frac{1}{2\pi i} \oint_C \frac{cz^m}{p(z)} dz & \frac{1}{2\pi i} \oint_C \frac{(z-a)z^m}{p(z)} dz \end{bmatrix} \quad (2.8)$$

Consider two cases, *i.e.* when $\alpha \neq \beta$ and $\alpha = \beta$.

Case 1. If $\alpha \neq \beta$, applying Cauchy's theorem on each component of the matrix (2.8), we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{(z-d)z^m}{p(z)} dz &= \frac{1}{2\pi i} \oint_C \frac{(z-d)z^m}{(z-\alpha)(z-\beta)} dz \\ &= \operatorname{Res} \left[\frac{(z-d)z^m}{(z-\alpha)(z-\beta)}; \alpha \right] + \operatorname{Res} \left[\frac{(z-d)z^m}{(z-\alpha)(z-\beta)}; \beta \right] \\ &= \lim_{z \rightarrow \alpha} (z-\alpha) \frac{(z-d)z^m}{(z-\alpha)(z-\beta)} + \lim_{z \rightarrow \beta} (z-\beta) \frac{(z-d)z^m}{(z-\beta)(z-\alpha)} \\ &= \lim_{z \rightarrow \alpha} \frac{(z-d)z^m}{z-\beta} + \lim_{z \rightarrow \beta} \frac{(z-d)z^m}{z-\alpha} = \frac{(\alpha-d)\alpha^m}{\alpha-\beta} + \frac{(\beta-d)\beta^m}{\beta-\alpha} \\ &= \frac{(\alpha-d)\alpha^m - (\beta-d)\beta^m}{\alpha-\beta} \end{aligned} \quad (2.9)$$

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{bz^m}{(z-\alpha)(z-\beta)} dz &= \operatorname{Res} \left[\frac{bz^m}{(z-\alpha)(z-\beta)}; \alpha \right] + \operatorname{Res} \left[\frac{bz^m}{(z-\alpha)(z-\beta)}; \beta \right] \\ &= \lim_{z \rightarrow \alpha} (z-\alpha) \frac{bz^m}{(z-\alpha)(z-\beta)} + \lim_{z \rightarrow \beta} (z-\beta) \frac{bz^m}{(z-\beta)(z-\alpha)} \\ &= \frac{b\alpha^m}{\alpha-\beta} + \frac{b\beta^m}{\beta-\alpha} = \frac{b\alpha^m - b\beta^m}{\alpha-\beta} \end{aligned} \quad (2.10)$$

Similarly, we have

$$\begin{aligned}
\frac{1}{2\pi i} \oint_C \frac{cz^m}{p(z)} dz &= \operatorname{Res} \left[\frac{cz^m}{(z-\alpha)(z-\beta)}; \alpha \right] + \operatorname{Res} \left[\frac{cz^m}{(z-\alpha)(z-\beta)}; \beta \right] \\
&= \lim_{z \rightarrow \alpha} (z-\alpha) \frac{cz^m}{(z-\alpha)(z-\beta)} + \lim_{z \rightarrow \beta} (z-\beta) \frac{cz^m}{(z-\beta)(z-\alpha)} \\
&= \frac{c\alpha^m - c\beta^m}{\alpha - \beta}
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\frac{1}{2\pi i} \oint_C \frac{(z-a)z^m}{p(z)} dz &= \operatorname{Res} \left[\frac{(z-a)z^m}{(z-\alpha)(z-\beta)}; \alpha \right] + \operatorname{Res} \left[\frac{(z-a)z^m}{(z-\alpha)(z-\beta)}; \beta \right] \\
&= \lim_{z \rightarrow \alpha} (z-\alpha) \frac{(z-a)z^m}{(z-\alpha)(z-\beta)} + \lim_{z \rightarrow \beta} (z-\beta) \frac{(z-a)z^m}{(z-\beta)(z-\alpha)} \\
&= \frac{(\alpha-a)\alpha^m - (\beta-a)\beta^m}{\alpha - \beta}
\end{aligned} \tag{2.12}$$

Hence,

$$A^m = \frac{1}{\alpha - \beta} \begin{bmatrix} (\alpha-d)\alpha^m - (\beta-d)\beta^m & b(\alpha^m - \beta^m) \\ c(\alpha^m - \beta^m) & (\alpha-a)\alpha^m - (\beta-a)\beta^m \end{bmatrix} \tag{2.13}$$

We can easily show that (2.13) can be rewritten as

$$A^m = \frac{\alpha^m - \beta^m}{\alpha - \beta} A - \alpha\beta \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} I.$$

Case 2. If $\alpha = \beta$, then we have $(z-\alpha)(z-\beta) = (z-\alpha)^2$.

Each element of the matrix (2.8) are given below:

$$\begin{aligned}
\frac{1}{2\pi i} \oint_C \frac{(z-d)z^m}{(z-\alpha)^2} dz &= \operatorname{Res} \left[\frac{(z-d)z^m}{(z-\alpha)^2}; \alpha \right] = \lim_{z \rightarrow \alpha} \frac{d}{dz} \left[(z-\alpha)^2 \frac{(z-d)z^m}{(z-\alpha)^2} \right] \\
&= \lim_{z \rightarrow \alpha} [z^m + m(z-d)z^{m-1}] = \alpha^m + m(\alpha-d)\alpha^{m-1}
\end{aligned}$$

$$= \alpha^m + m\alpha^m - m d \alpha^{m-1} = (m+1)\alpha^m - m d \alpha^{m-1}.$$

$$\frac{1}{2\pi i} \oint_C \frac{bz^m}{p(z)} dz = \operatorname{Res} \left[\frac{bz^m}{(z-\alpha)^2}; \alpha \right] = \lim_{z \rightarrow \alpha} \frac{d}{dz} \left[(z-\alpha)^2 \frac{bz^m}{(z-\alpha)^2} \right] = \lim_{z \rightarrow \alpha} [mbz^{m-1}] = mb\alpha^{m-1}.$$

$$\frac{1}{2\pi i} \oint_C \frac{cz^m}{p(z)} dz = \operatorname{Res} \left[\frac{cz^m}{(z-\alpha)^2}; \alpha \right] = \lim_{z \rightarrow \alpha} \frac{d}{dz} \left[(z-\alpha)^2 \frac{cz^m}{(z-\alpha)^2} \right] = mc\alpha^{m-1}.$$

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{(z-a)z^m}{(z-\alpha)^2} dz &= \operatorname{Res} \left[\frac{(z-a)z^m}{(z-\alpha)^2}; \alpha \right] = \lim_{z \rightarrow \alpha} \frac{d}{dz} \left[(z-\alpha)^2 \frac{(z-a)z^m}{(z-\alpha)^2} \right] \\ &= \alpha^m + (\alpha-a)m\alpha^{m-1} = (m+1)\alpha^m - ma\alpha^{m-1}. \end{aligned}$$

Hence,

$$A^m = \begin{bmatrix} (m+1)\alpha^m - m d \alpha^{m-1} & mb\alpha^{m-1} \\ mc\alpha^{m-1} & (m+1)\alpha^m - ma\alpha^{m-1} \end{bmatrix} \quad (2.14)$$

Example 2. Let us compute the matrix function $f(A) = A^m$, which yields a formula for the Fibonacci sequence.

Let $a = 1$, $b = 1$, $c = 1$, and $d = 0$. Then

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } [zI - A] = \begin{bmatrix} z-1 & -1 \\ -1 & z \end{bmatrix}.$$

Let $p(z) = \det[zI - A]$. Then

$$p(z) = (z-1)z - 1 = z^2 - z - 1 = \left(z - \frac{1+\sqrt{5}}{2} \right) \left(z - \frac{1-\sqrt{5}}{2} \right).$$

Now let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then we have $p(z) = (z-\alpha)(z-\beta)$.

By using the formula (2.13), we have

$$A^2 = \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^2 & \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2 \\ \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2 & \left(\frac{1+\sqrt{5}}{2} - 1\right)\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2} - 1\right)\left(\frac{1-\sqrt{5}}{2}\right)^2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 2\sqrt{5} & \sqrt{5} \\ \sqrt{5} & \sqrt{5} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

$\vdots$

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix},$$

where F_n is the well-known Fibonacci sequence which is given by

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}.$$

Now let $a = 2z$, $b = 1$, $c = 1$, and $d = 0$. Then the matrix Q below generalizes the previous matrix. It is given by

$$Q(z) = \begin{bmatrix} 2z & 1 \\ 1 & 0 \end{bmatrix} \text{ and } Q^n(z) = \begin{bmatrix} F_n(z) & F_{n-1}(z) \\ F_{n-1}(z) & F_{n-2}(z) \end{bmatrix}, \text{ where } F_n(z) \text{ are the Fibonacci}$$

polynomials. The Fibonacci polynomials are given by

$$F_n(z) = \frac{1}{2\sqrt{z^2 + 1}} \left[\left(z + \sqrt{z^2 + 1} \right)^{n+1} - \left(z - \sqrt{z^2 + 1} \right)^{n+1} \right].$$

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