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Matchwebs

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MATCHWEBS

A Thesis

Presented to

The Faculty of the Department of Mathematics

San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Katherine Shelley

August 2007

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ABSTRACT

MATCHWEBS

by Katherine Shelley

A matchweb is a minimal connected normalized matching bipartite graph $\Lambda(X, Y)$. The purpose of this paper is to define matchwebs and to reduce the problem of classifying matchwebs to the enumeration and classification of their underlying structures: core graphs and bases.

This paper begins with a classification of matchwebs with $|X| = 1$ and 2, and then generalizes to all normalized matching graphs with $|Y| = m|X|$ for $m \in \mathbb{N}$. Second, all bases up to order 8 will be systematically enumerated. Next, matchwebs with the basic n -pod, n -spider, and n -fence cores will be classified.

Finally it will be shown that there is at most one matchweb $\Lambda(X, Y)$ with $n = |X|$, $k = |Y|$, and a given core Λ^* ; and that matchweb solutions are unique “mod n ”, i.e., there are unique classes of matchwebs given $n = |X|$, a given core Λ^* , and value of $k \pmod{n}$.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

“Graph theory is a delightful playground for the exploration of proof techniques in discrete mathematics, and its results have applications in many areas of the computing, social, and natural sciences.”
D. West [Wes01]

Graphs use simple structures to model practical problems in the sciences. The most complex of problems can be easily modeled with graphs, abstracting the problem at hand, as well as creating an understandable arena for research. Among problems of science, economics, and operations, graphs can be used to model complex interactions on many levels. For example, the Marriage Problem credited to Philip Hall [Hal35] is a graphical representation of a problem that deals with optimal match-making between a group of n men and n women.

Graphs are also used to deal with problems in areas of mathematics different from graph theory, including combinatorics and set theory. Fundamental results in set theory can be represented easily in terms of graphs and matchings. Furthermore, proofs of those theorems follow from basic graph theory results. By attaching numerical characteristics to graphs, difficult theorems and proofs in combinatorics can be revisited in simpler terms and understood further than before. For example, Hall’s

Marriage Problem (refer to Section 2.2) is represented simply by a bipartite, or two level graph, in which solutions, if they exist, can be visualized clearly.

Though there is a wide gamut of graphs, in this paper we will mainly be considering a special type of graph called a matchweb. Matchwebs, under the name of *minimal LYM orders*, were first mentioned briefly by West, Harper, and Daykin [DHW83] as a graphical structure with special combinatorial properties. In particular, matchwebs can be used to investigate combinatorial problems such as *Anderson-Griggs posets*, partially ordered sets with symmetric and unimodal structures (see Section 4.4), as matchwebs are naturally embedded in the graphical representation of Anderson-Griggs posets. In this paper we will be using the term matchweb, and similar notation defined by Hsu, Logan, and Shahriari [HLS].

A matchweb is a connected normalized matching bipartite graph that is minimal in the sense that no edges can be removed to obtain a smaller normalized matching bipartite graph with the same vertex set. (See Chapter 5 for definitions.) The core of a matchweb $\Lambda(X, Y)$ (with $|X| \leq |Y|$) is a bipartite tree obtained by removing all leaves incident to X , and a core graph is a bipartite tree that has the structure of a core. The base of a matchweb is a graph obtained by replacing the vertices of Y in the matchweb core with a symbol of its valence. The purpose of this paper is to define and classify all matchwebs with a given size of X . Specifically, we wish to reduce the problem of classifying matchwebs to the enumeration and classification of core graphs and bases.

Beginning our classification, we consider all matchwebs with $|X| = 1$ or 2 , then generalize to all normalized matching graphs with $|Y| = m|X|$ for $m \in \mathbb{N}$. We

continue by defining a systematic method of enumerating all bases given $|X|$, and enumerate all bases up to order 8. Next, we classify all matchwebs with n -pod, n -spider, and n -fence core graphs, proving the existence and uniqueness of matchweb solutions for these core graphs given $n = |X|$ and $k = |Y|$.

As our final results, we show that given a core graph, $\Lambda^*(X, Y^*)$ and integers $n = |X|$ and k , there is at most one matchweb $\Lambda(X, Y)$ with $|X| = n$, $|Y| = k$, and core Λ^* . Finally, we prove uniqueness of matchweb solutions given $k = |Y|$ and uniqueness of solutions “mod n ”, i.e., unique classes of matchweb solutions given $n = |X|$ and a core graph, Λ^* .

1.2 Overview

From Chapter 2 through Chapter 4 we consider necessary background and motivational material for the understanding of matchwebs and their function. From Chapter 5 through Chapter 8 we discuss new results in the classification of matchwebs.

In Chapter 2 we discuss the foundation of matchwebs; we review graphs, trees, and matchings; and we investigate the properties, definitions, and theorems that will lead to the definition of matchweb. Specifically, we will consider the Hall Marriage Theorem and the Ford-Fulkerson Theorem, two results about matchings which inspire the definition of matchweb.

In Chapter 3 we continue to build our definition of matchwebs by looking at normalized matching graphs.

In Chapter 4 we consider the principle motivation behind matchweb study, posets (partially ordered sets). Posets are combinatorial structures that can be represented graphically and, in particular, with matchwebs.

In Chapter 5 we present the formal definition of matchweb and the first theorems regarding them. Continuing in Chapter 6, we study the construction of matchwebs, as well as their existence, by considering matchweb cores and bases and enumerating all matchweb bases up to order 8. We continue the classification of matchwebs by stating and proving specific theorems about three basic matchweb cores in Chapter 7.

In Chapter 8 we prove the uniqueness of matchwebs given their size and base. To conclude, in Chapter 9, we state open problems with matchwebs and conjectures left unproven by this paper.

1.3 Notation

We use the following terminology and notation.

Definition 1.3.1. The *natural numbers*, $\mathbb{N}$, are the set $\{1, 2, 3, 4, \dots\}$.

Definition 1.3.2. Let n be a non-negative integer. We define $[n] = \{1, 2, \dots, n\}$, with $[0] = \emptyset$.

Definition 1.3.3. Let X be a set. We denote the *k-subsets of X*, the set of all subsets of X of size k , by the set $\binom{X}{k}$.

Definition 1.3.4. Let A and B be sets. Then $A \setminus B = \{a \in A \mid a \notin B\}$.

Definition 1.3.5. Let $k, n \in \mathbb{N}$, *greatest integer less than $\frac{k}{n}$* , or *floor* of $\frac{k}{n}$, is denoted by $\left\lfloor \frac{k}{n} \right\rfloor$. The *least integer greater than $\frac{k}{n}$* , or *ceiling*, is denoted by $\left\lceil \frac{k}{n} \right\rceil$.

CHAPTER 2

GRAPHS AND MATCHINGS

To understand matchwebs we must first understand graphs and their properties. We begin with a background study of graph theory. In Section 2.1 we study the notion of graph in detail, and necessary terms and characteristics are defined. In Section 2.2 we define the idea of matching, furthering our setup of the matchweb problem, continuing in Section 2.3. Finally, in Section 2.4 we review material on trees that we will need for studying matchwebs.

2.1 Graphs

We begin our study of matchwebs and graphs with a few definitions. For the following standard material we follow West [Wes01].

Definition 2.1.1. A *graph* is a triple $G = (V, E, \partial)$ where

- (1) V is a set, called the *vertex set*;
- (2) E is a set, called the *edge set*; and
- (3) $\partial : E \rightarrow \mathcal{P}(V)$ (the power set of V), where $|\partial(e)| = 1$ or 2 for all $e \in E$.

Definition 2.1.2. Let Λ be a graph with vertex set V and edge set E . A *loop* is an $e \in E$ such that $|\partial(e)| = 1$. Λ has *multiple edges* if, for fixed $x, y \in V$, there exists more than one $e \in E$ such that $\partial(e) = \{x, y\}$. A graph $\Lambda = (V, E, \partial)$ with $\partial(E) \subseteq \binom{V}{2}$, such that for $e_1, e_2 \in E$, $\partial(e_1) = \partial(e_2)$ implies that $e_1 = e_2$, is called a *simple graph*, i.e., Λ has no loops or multiple edges.

In the case of a simple graph, we may consider E to be a subset of $\binom{V}{2}$ and ∂ to be the identity map on E , as edges in E are defined by the vertices they are adjacent to.

Definition 2.1.3. We say that $v \in V$ and $e \in E$ are *incident* if $v \in \partial(e)$.

Definition 2.1.4. Let G be a graph with vertex set V . The *valence* of $v \in V$ is

$$\text{valence}(v) = \text{the number of } e \in E \text{ incident with } v. \quad (2.1)$$

Definition 2.1.5. Two vertices v_1 and v_2 in V are *adjacent* if $\{v_1, v_2\} = \partial(e)$ for some $e \in E$. We denote this relation by $v_1 \sim v_2$.

Definition 2.1.6. Let $\Lambda = (V, E, \partial)$ be a graph. For $F \subseteq V$, the *neighbors of F* , or $\Gamma(F)$, are the set of all vertices $v \in V$ such that for some $w \in F$, w is adjacent to v .

Definition 2.1.7. A graph $H = (V', E', \partial')$ is a *subgraph* of a graph $G = (V, E, \partial)$ if H forms a graph under the rules of G , i.e.,

- (1) $V' \subseteq V$ is the vertex set of H .
- (2) $E' \subseteq E$ is the edge set of H .
- (3) $\partial'(e) = \partial(e) \subseteq V'$ for all $e \in E'$.

Definition 2.1.8. A simple graph $\Lambda = (V, E, \partial)$ is called a *bipartite graph* if $V = X \cup Y$, where X and Y are disjoint, and for every $e \in E$, $\partial(e) = \{x, y\}$ for some $x \in X$ and $y \in Y$. We denote such a graph by $\Lambda = \Lambda(X, Y)$.

Definition 2.1.9. A graph $\Lambda = (V, E, \partial)$ is called a *3-level graph* if

- (1) $V = X_1 \cup Y \cup X_2$, where all three sets are pairwise disjoint.
- (2) $E = E_1 \cup E_2$, with E_1 and E_2 disjoint.
- (3) For $i = 1, 2$ and every $e_i \in E_i$, $\partial(e_i) = \{x_i, y\}$ for some $x_i \in X_i$, $y \in Y$.

2.2 Matchings and the Hall Marriage Theorem

In this section we consider a graphical construction known as a matching, as well as a special problem in graph theory and combinatorics, the Hall Marriage Problem [Hal35], seen in Figure 2.1.

Figure 2.1: The Hall Marriage problem

The Hall Marriage Problem is an optimization problem from combinatorics. Consider a set of n men and n women. Assume that each of the women has a subset of men that she would be willing to marry and each of the men is indifferent. The question posed is: when does a solution exist? In other words, when can matches be made between all people such that each person is happy with their mate?

The solution to the Marriage Problem is given by the following more general definition and theorem. We follow Anderson [And02], using bipartite graphs instead of sets.

Definition 2.2.1. Let $\Lambda(X, Y)$ be a bipartite graph. A *matching* of X into Y is an injective function $f : X \rightarrow Y$ such that for all $x \in X$, $f(x) \in \Gamma(x)$. Note that we may also consider a matching to be a subgraph of Λ such that

- (1) $V = X \cup f(X)$ is the vertex set.
- (2) For each $x \in X$, there is an edge $e \in E$ such that $\partial(e) = \{x, f(x)\}$.

Note that if $|X| = |Y|$ then the matching f defined above is a bijection, or a *perfect matching*.

Theorem 2.2.2 (The Hall Marriage Theorem). *Let $\Lambda(X, Y)$ be a bipartite graph with $|X| \leq |Y|$. Then the following are equivalent:*

- (1) *There exists a matching $f : X \rightarrow Y$.*
- (2) *For all $F \subseteq X$, $|\Gamma(F)| \geq |F|$.*

Proof. Let $\Lambda(X, Y)$ be a bipartite graph.

$$(1) \Rightarrow (2)$$

First assume there exists a matching $f : X \rightarrow Y$. By the definition of matching, f is an injective function such that for all $x \in X$, $f(x) \in \Gamma(x)$. Also, for any $F \subseteq X$, $f(F) \subseteq \Gamma(F)$. Since f is injective, $|f(F)| = |F|$, so we have $|\Gamma(F)| \geq |f(F)| = |F|$.

$$(2) \Rightarrow (1)$$

Assume for any $F \subseteq X$, $|\Gamma(F)| \geq |F|$. We will show by induction on the size of X

that there exists $f : X \rightarrow Y$ such that f is injective and for all $x \in X$, $f(x) \in \Gamma(X)$.

For the base step, let $|X| = 1$. Since $|\Gamma(X)| \geq |X| = 1$, it is clear that there is at least one injective mapping from X to $\Gamma(X)$. Hence a matching exists from X into Y .

For the inductive step, assume $(2) \Rightarrow (1)$ for $|X| \leq k$. Suppose $|X| = k + 1$. We have two cases to consider: First, the case where every nonempty subset of X has “overkill,” i.e., each nonempty, proper subset of X has a strictly larger set of neighbors, $|\Gamma(F)| > |F|$; and second, the case where at least one nonempty proper subset of X has a set of neighbors of equal size, $|\Gamma(F)| = |F|$.

Case 1: Assume for all non-empty $F \subset X$, $|\Gamma(F)| \geq |F| + 1$. Choose $x_1 \in X$ to be “matched” to $y_1 \in \Gamma(\{x_1\})$. (See Figure 2.2.) Let $X_2 = X \setminus \{x_1\}$, $Y_2 = Y \setminus \{y_1\}$, and for $F \subseteq X_2$, let $\Gamma_2(F) = \Gamma(F) \setminus \{y_1\}$.

Figure 2.2: Case 1

Since $|X_2| \leq k$, by induction, to show that there is a matching $f^* : X_2 \rightarrow Y_2$, it suffices to show that $|\Gamma_2(F)| \geq |F|$ for every nonempty, proper $F \subseteq X_2$. Since $|\Gamma_2(F)| = |\Gamma(F) \setminus \{y_1\}| \geq |\Gamma(F)| - 1$, $|\Gamma(F)| \geq |F| + 1$ implies that

$$|\Gamma_2(F)| \geq |F| + 1 - 1 = |F|. \quad (2.2)$$

So there exists a matching $f^* : X_2 \rightarrow Y_2$. If we define $f : X \rightarrow Y$ such that $f(x_1) = y_1$ and $f(x) = f^*(x)$ for all $x \in X_2$, f is a matching from X to Y .

Case 2: Let X_1 be a nonempty, proper subset of X such that $|\Gamma(X_1)| = |X_1|$. Define $Y_1 = \Gamma(X_1)$, $X_2 = X \setminus X_1$, and $Y_2 = Y \setminus Y_1$. (See Figure 2.3.)

Figure 2.3: Case 2

We first show there exists a matching $f_1 : X_1 \rightarrow Y_1$. Let $F \subseteq X_1$, and let $\Gamma_1(F) = \Gamma(F) \cap Y_1$. Since F is a subset of X_1 , $\Gamma(F)$ is a subset of $\Gamma(X_1)$. Hence, $\Gamma(F) \cap Y_2 = \emptyset$. So $\Gamma_1(F) = \Gamma(F)$. Thus we have $|\Gamma_1(F)| = |\Gamma(F)| \geq |F|$ by assumption (2). So, by the induction hypothesis, a matching exists from X_1 into Y_1 .

Let $G \subseteq X_2$. Define $\Gamma_2(G) = \Gamma(G \cup X_1) \setminus Y_1 = \Gamma(G) \cap Y_2$. Then

$$|\Gamma_2(G)| = |\Gamma(G \cup X_1) \setminus Y_1| = |\Gamma(G \cup X_1) \setminus \Gamma(X_1)|. \quad (2.3)$$

Since $\Gamma(X_1)$ is a subset of $\Gamma(G \cup X_1)$,

$$|\Gamma(G \cup X_1) \setminus \Gamma(X_1)| = |\Gamma(G \cup X_1)| - |\Gamma(X_1)| \geq |G \cup X_1| - |\Gamma(X_1)|, \quad (2.4)$$

by assumption (2). Since we are in case 2, $|\Gamma(X_1)| = |X_1|$, so $|\Gamma_2(G)| \geq |G \cup X_1| - |X_1|$. Furthermore, $G \cap X_1 = \emptyset$; hence (2.3) and (2.4) give

$$|\Gamma_2(G)| \geq |G| + |X_1| - |X_1| = |G|. \quad (2.5)$$

So, by the induction hypothesis, there exists a matching $f_2 : X_2 \rightarrow Y_2$. Hence we have a matching $f : X \rightarrow Y$ defined by $f(x) = f_1(x)$ for all $x \in X_1$ and $f(x) = f_2(x)$ for all $x \in X_2$. $\square$

2.3 Ford-Fulkerson Theorem

We will now consider a 3-level version of Theorem 2.2.2 and its proof. The results found in this section will aid in our understanding of the true motivation behind matchwebs and will be referred to in Section 4.4. Throughout we follow Anderson [And02], using bipartite graphs. First, we must begin with a definition.

Definition 2.3.1. Let $\Lambda(X_1, Y, X_2)$ be a 3-level graph. We define the *N-graph* of Λ to be the bipartite graph $\Lambda_N = (V_N, E_N, \partial_N)$, where

- (1) $V_N = (X_1 \cup Y_2) \cup (X_2 \cup Y_1)$, where $Y_i = Y \times \{i\}$. (The Y_i are copies of Y .)
- (2) The edges in E_N are:
  - (a) For every $\{x_i, y\} \in \Lambda(X_1, Y, X_2)$, $x_i \in X_i$, $y \in Y$, and $i = 1, 2$, we have an edge $\{x_i, y_i\}$, where $y_i = (y, i)$.
  - (b) For every $y \in Y$, we have an edge $\{y_1, y_2\}$, where $y_i = (y, i)$ with $i = 1, 2$.

See Figure 2.4.

In order to prove the results of this section we must consider the following lemma, due to Perfect [Per69].

Lemma 2.3.2. *Let $\Lambda(X_1, Y, X_2)$ be a 3-level graph with $|X_1| = |X_2| \leq |Y|$. Then the following are equivalent:*

(1) *There is a matching $f_1 : X_1 \rightarrow Y$ and a matching $f_2 : X_2 \rightarrow Y$ in Λ such that $f_1(X_1) = f_2(X_2)$.*

(2) *There is a perfect matching $f : X_1 \cup Y_2 \rightarrow X_2 \cup Y_1$ in Λ_N , the N -graph of Λ .*

Proof. (1) $\Rightarrow$ (2)

Suppose there exist $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ such that f_1 and f_2 are one-to-one and $f_1(X_1) = f_2(X_2)$. Because f_2 is an injection, f_2^{-1} is well defined on $f_2(X_2)$, and we can define

$$g : X_1 \cup Y_2 \rightarrow X_2 \cup Y_1, \quad (2.6)$$

where

$$(1) \quad g = f_1 \text{ on } X_1,$$

$$(2) \quad g = f_2^{-1} \text{ on } f_2(X_2),$$

$$(3) \quad g(y, 2) = (y, 1) \text{ on } Y \setminus f_2(X_2).$$

(See Figure 2.4.)

We know that g is onto because

$$g(X_1 \cup Y_2) = g(X_1) \cup g(Y_2) = f_1(X_1) \cup \{g(y, 2)\} \cup f_2^{-1} \circ f_2(X_2), \quad (2.7)$$

where y ranges over all elements of $Y \setminus f_2(X_2)$. So we get

$$g(X_1 \cup Y_2) = f_1(X_1) \cup (Y_1 \setminus f_1(X_1)) \cup X_2 = Y_1 \cup X_2. \quad (2.8)$$

Figure 2.4: Perfect matching in Λ_N

We know that g is one-to-one because g is defined by the matchings f_1 and f_2 , which are injective, and a one-to-one map from $Y_2 \setminus f_2(X_2)$ to $Y_1 \setminus f_1(X_1)$. Hence g is a bijection. So we have a matching from $X_1 \cup Y_2$ to $Y_1 \cup X_2$.

(2) $\Rightarrow$ (1)

Suppose there exists a matching $g : X_1 \cup Y_2 \rightarrow X_2 \cup Y_1$. Since $|X_1 \cup Y_2| = |X_2 \cup Y_1|$, g is a perfect matching; in particular, g is invertible. We may define ϕ , the natural projection of Y_1 and Y_2 to Y , by $\phi(y, i) = y$ for all $y \in Y$. Applying ϕ to g , we get matchings $f_1 = \phi \circ g|_{X_1}$ and $f_2 = \phi \circ g^{-1}|_{X_2}$ such that $f_1 : X_1 \rightarrow Y$, $f_2 : X_2 \rightarrow Y$. Because g is a perfect matching, $\Gamma_N(X_1) \subseteq Y_1$, and $\Gamma_N(X_2) \subseteq Y_2$, we know that

$$f_1(X_1) = \phi \circ g|_{X_1}(X_1) = \{y \in Y \mid (y, 1) \text{ is not matched by } g \text{ to } (y, 2)\}, \quad (2.9)$$

and similarly,

$$f_2(X_2) = \phi \circ g^{-1}|_{X_2}(X_2) = \{y \in Y \mid (y, 2) \text{ is not matched by } g \text{ to } (y, 1)\}. \quad (2.10)$$

So $f_1(X_1) = f_2(X_2)$. (See Figure 2.4.) □

Theorem 2.3.3 (Ford-Fulkerson Theorem). *Let $\Lambda(X_1, Y, X_2)$ be a 3-level graph such that $m = |X_1| = |X_2| \leq |Y|$. Then the following are equivalent:*

(1) There exists a matching $f_1 : X_1 \rightarrow Y$ and a matching $f_2 : X_2 \rightarrow Y$ such that $f_1(X_1) = f_2(X_2)$.

(2) For all $F \subseteq X_1$ and $G \subseteq X_2$, $|\Gamma_1(F) \cap \Gamma_2(G)| \geq |F| + |G| - m$.

We define Γ_1 and Γ_2 as the neighbors of subsets of X_1 and X_2 respectively.

Figure 2.5: f_1 and f_2

Proof. (1) $\Rightarrow$ (2)

Let $f_1 : X_1 \rightarrow Y$ and $f_2 : X_2 \rightarrow Y$ be matchings such that $f_1(X_1) = f_2(X_2)$. The matchings f_i are injective, so an inverse mapping $f_2^{-1} : f_2(X_2) \rightarrow X_2$ is well defined, and $f = f_2^{-1} \circ f_1 : X_1 \rightarrow X_2$ is bijective. (See Figure 2.5.)

Let $F \subseteq X_1$ and $G \subseteq X_2$. We then have

$$|f(F) \cap G| = |f(F)| + |G| - |f(F) \cup G|. \quad (2.11)$$

It follows from $f(F) \subseteq X_2$ that $|f(F) \cup G| \leq m$. This gives

$$|f(F) \cap G| \geq |f(F)| + |G| - m. \quad (2.12)$$

Figure 2.6: f , Γ_1 , and Γ_2

Because f is a bijection we have that $|F| = |f(F)|$. Furthermore, $f_2(f(F) \cap G) = f_1(F) \cap f_2(G) \subseteq \Gamma_1(F) \cap \Gamma_2(G)$, because f_1 and f_2 are matchings, and f_2 is injective, so we have

$$|\Gamma_1(F) \cap \Gamma_2(G)| \geq |f(F) \cap G| \geq |F| + |G| - m. \quad (2.13)$$

(See Figure 2.6.)

(2) $\Rightarrow$ (1)

Assume for all $F \subseteq X_1$ and $G \subseteq X_2$, $|\Gamma(F) \cap \Gamma(G)| \geq |F| + |G| - m$. Consider Λ_N , the N-graph of Λ . (See Definition 2.3.1 and Figure 2.4.) By Lemma 2.3.2, to prove the existence of f_1 and f_2 , it suffices to show that there exists a matching in Λ_N ; i.e., in the notation of Definition 2.3.1, for all $F \subseteq X_1 \cup Y_2$, $|\Gamma(F)| \geq |F|$, where Γ is defined in Λ_N .

Let $F \subseteq X_1 \cup Y_2$, let $F_x = F \cap X_1$, and let $F_y = F \cap Y_2$. Thus we have

$$|\Gamma(F)| = |\Gamma(F_x \cup F_y)| = |\Gamma(F_x)| + |\Gamma(F_y)| - |\Gamma(F_x) \cap \Gamma(F_y)|. \quad (2.14)$$

Let $F'_y = \Gamma(F_y) \cap Y_1 \subseteq Y_1$, and note that F'_y is naturally perfectly matched with F_y .

(See Figure 2.7.) First we observe that

$$\Gamma(F_x) \setminus (\Gamma(F_x) \cap \Gamma(F_y)) = \Gamma(F_x) \cap (Y_1 \setminus F'_y), \quad (2.15)$$

since $\Gamma(F_x) \subseteq Y_1$. Also, since $\Gamma(F_y) \cap Y_1 = F'_y$,

$$\Gamma(F_y) = F'_y \cup (\Gamma(F_y) \cap X_2), \quad (2.16)$$

where $F'_y \cap (\Gamma(F_y) \cap X_2) = \emptyset$. Let $H = X_2 \setminus \Gamma(F_y)$, the elements of X_2 not adjacent to F_y . (See Figure 2.7.) Also, let $\phi : Y \times [i] \rightarrow Y$ be the natural projection $\phi((y, i)) = y$. So from (2.14) and (2.16) we get

$$|\Gamma(F)| = |\Gamma(F_x)| - |\Gamma(F_x) \cap \Gamma(F_y)| + |F'_y| + |\Gamma(F_y) \cap X_2|. \quad (2.17)$$

Figure 2.7: Λ_N

We know that $\Gamma(F_x) \cap \Gamma(F_y) \subseteq \Gamma(F_x)$ and $|\Gamma(F_y) \cap X_2| = |X_2| - |H|$, so

$$|\Gamma(F)| = |\Gamma(F_x) \setminus (\Gamma(F_x) \cap \Gamma(F_y))| + |X_2| - |H| + |F'_y| \quad (2.18)$$

Also, $|X_2| = m$, so with (2.15) we get

$$|\Gamma(F)| = |\Gamma(F_x) \cap (Y_1 \setminus F'_y)| + m - |H| + |F'_y| \quad (2.19)$$

Applying the projection ϕ to $\Gamma(F_x) \cap (Y_1 \setminus F'_y)$, we get

$$\phi(\Gamma(F_x) \cap (Y_1 \setminus F'_y)) = \Gamma_1(F_x) \cap (Y \setminus \phi(F_y)). \quad (2.20)$$

We defined H as the set of all elements of X_2 not adjacent to F_y , so $\Gamma_2(H) \subseteq Y \setminus \phi(F_y)$.

So $\Gamma_1(F_x) \cap (Y \setminus \phi(F_y))$ contains $\Gamma_1(F_x) \cap \Gamma_2(H)$. Hence (2.19) gives

$$|\Gamma(F)| \geq |\Gamma_1(F_x) \cap \Gamma_2(H)| + m - |H| + |F'_y|. \quad (2.21)$$

By assumption (1), $|\Gamma_1(F_x) \cap \Gamma_2(H)| \geq |F_x| + |H| - m$, so

$$|\Gamma(F)| \geq |F_x| + |H| - m + m - |H| + |F'_y| = |F_x| + |F'_y|. \quad (2.22)$$

We have defined F'_y to be perfectly matched with F_y , so (2.22) becomes

$$|\Gamma(F)| \geq |F_x| + |F_y| = |F|. \quad (2.23)$$

Hence, by the Hall Marriage Theorem, we know that there exists a matching $f : X_1 \cup Y_2 \rightarrow X_2 \cup Y_1$, and by Lemma 2.3.2 the theorem holds. $\square$

2.4 Trees

In this section we discuss a special type of graph called a tree. Trees are necessary to understanding matchwebs and their behavior because matchwebs are trees. We begin by defining trees and their characteristics.

Definition 2.4.1. A *path of length n* in a graph $\Lambda = (V, E, \partial)$, is a sequence of vertices $(v_0, \dots, v_n)$ such that $v_i \sim v_{i+1}$. A *circuit of length n* is a path with $v_0 = v_n$. A path has a *backtrack* if $v_i = v_{i+2}$ for some i . (See Figure 2.8.)

A path is *reducible* if it contains backtracks. A path of length 0 is called a *trivial path* (or trivial circuit). Note that if a path has a backtrack $v_i = v_{i+2}$, then $(v_0, \dots, v_i, v_{i+3}, \dots, v_n)$ is a shorter path with the same endpoints. A path with no backtracks is called a *reduced path*. So if there exists a path from x to y , we may assume that it is reduced.

Figure 2.8: Backtrack

Definition 2.4.2. A *cycle*, $C = (v_0, v_1, \dots, v_n = v_0)$, is a nontrivial reduced circuit such that there is no $0 < i, j < n$ such that $v_i = v_j$, $i \neq j$. In other words, the vertices of C can be rearranged to appear in a circle. Notice that if a graph contains a nontrivial reduced circuit, then a shortest possible portion of that circuit without internal repetition of vertices is a nontrivial reduced cycle. So if a graph contains a nontrivial reduced circuit, it must contain a nontrivial reduced cycle.

The following definition motivates both the topic of this section as well as the definition of matchwebs.

Definition 2.4.3. Let $x, y \in V$. We say that x is *connected* to y if there is a reduced path from x to y .

Theorem 2.4.4. *The relation “ x is connected to y ” is an equivalence relation.*

Proof. It suffices to show that “ x is connected to y ” is reflexive, symmetric, and transitive. Given any $x \in V$ we may find a reduced path from x to itself, namely, the trivial path (x) . So “connected” is reflexive.

Let $x, y \in V$ such that x is connected to y . Then we can find a reduced path from x to y , say $(x, v_1, \dots, v_{n-1}, y)$. Hence we can find a reduced path from y to x , namely, $(y, v_{n-1}, \dots, v_1, x)$. So y is connected to x , i.e. “connected” is symmetric.

Suppose $x, y, z \in V$ such that x is connected to y and y is connected to z . So we can find a reduced path $(x, v_1, \dots, v_{n-1}, y)$ from x to y , and, similarly, a reduced path $(y, w_1, \dots, w_{m-1}, z)$ from y to z . So there is a path $(x, v_1, \dots, v_{n-1}, y, w_1, \dots, w_{m-1}, z)$ which can be shortened to a reduced path from x to z . So x is connected to z , i.e., “connected” is transitive. $\square$

Definition 2.4.5. Let $\Lambda = (V, E, \partial)$ be a graph with $x \in V$. The *path component* of x is the equivalence class of x under the equivalence relation “ x is connected to y ”.

Definition 2.4.6. A graph $\Lambda = (V, E, \partial)$ is *connected* if each pair of vertices in V is connected by a path in Λ , i.e., V is the path component of every vertex in Λ . Otherwise Λ is *disconnected*.

We now come to the main definition of this section.

Definition 2.4.7. A *tree* is a connected graph that has no nontrivial reduced circuits.

Definition 2.4.8. A *leaf* of a graph is a vertex v with $\text{valence}(v) = 1$ and its incident edge.

We will now state and prove the theorems necessary for understanding the nature of trees.

Theorem 2.4.9. Let $T = (V, E, \partial)$ be a graph. Then the following are equivalent:

- (1) T is a tree.
- (2) For every $x, y \in V$ there exists a unique reduced path from x to y .

Proof. (1) $\Rightarrow$ (2)

We know because T is a tree that T is connected, i.e., for every $x, y \in V$ there is a path

connecting them. Suppose for $x, y \in V$ there are two distinct reduced paths connecting them. We may denote the paths by their vertices, $(x, v_1, v_2, \dots, v_k, \dots, v_m, \dots, y)$ and $(x, w_1, w_2, \dots, w_k, \dots, w_n, \dots, y)$. Let k be the smallest integer such that $v_{k+1} \neq w_{k+1}$, and pick m, n such that v_i is not in the second path for $k < i < m$ and w_j is not in the first path for $k < j < n$, and where $v_m = w_n$. The sequence of vertices $(v_k, \dots, v_m = w_n, w_{n-1}, \dots, w_k = v_k)$ is a reduced circuit because the two paths are reduced and no $v_i = w_j$, contradicting assumption (1). Hence there can only be one reduced path from x to y . (See Figure 2.9.)

Figure 2.9: Two distinct paths from x to y

(2) $\Rightarrow$ (1)

Let T be a graph such that for every $x, y \in V$ there exists a unique reduced path from x to y . We know T is connected because we can always find a path between any two vertices, so it suffices to show that T contains no nontrivial reduced circuits. Assume that T does contain a nontrivial reduced circuit, $(v_0, \dots, v_n = v_0)$. Thus we have v_0 adjacent to v_1 in two ways, $v_0 \sim v_1$ and $v_0 \sim v_{n-1} \sim v_{n-2} \sim \dots \sim v_1$. Both paths are reduced because the circuit $(v_0, \dots, v_n = v_0)$ is reduced. This is a contradiction however, because for each pair of vertices there is a unique reduced path connecting them. So T contains no nontrivial reduced circuits, and hence T is a tree. $\square$

Theorem 2.4.10. *Let T be a tree. Pick $e \in E$ such that $\partial(e) = \{x, y\}$. Let $T_x = (V_x, E_x, \partial)$ be the path component of x in $T \setminus \{e\}$, and $T_y = (V_y, E_y, \partial)$ be the path component of y in $T \setminus \{e\}$. Then we have disjoint unions $V = V_x \cup V_y$, $E = E_x \cup$*

$\{e\} \cup E_y$, and T_x and T_y are trees with fewer vertices than T .

Proof. Let T be a tree and let $e \in E$ such that $\partial(e) = \{x, y\}$. Since T is connected, if there is a vertex $v \in V$ such that v is not connected to x in T_x , then there is a unique reduced path from v to x through e , which means that $v \in V_y$. Similarly, if there is a vertex w such that w is not connected to y , then $w \in V_x$. Therefore, T_x and T_y are connected. Since T contains no nontrivial reduced circuits, no subgraph of T can contain a nontrivial reduced circuit, so we know that T_x and T_y , which are subgraphs of T , contain no nontrivial reduced circuits. Hence T_x and T_y are trees.

We now observe that every $v \in V$ is connected in T to x either through e , and hence $v \in V_y$, or v is connected in T to x not through e , so $v \in V_x$. So $V_x \cap V_y = \emptyset$ and $V = V_x \cup V_y$.

Now suppose there exists an edge $e' \in E$, $e' \neq e$, such that $\partial(e') = \{x', y'\}$ where $x' \in V_x$ and $y' \in V_y$. Because T is a tree, for any pair of vertices in T a unique reduced path connecting them can always be found. So consider the paths in T between x' and y' . We have one path (x', y') passing through e' , and a second reduced path $(x', v_1, \dots, x, y, v_2, \dots, y')$, which passes through e , because x is connected to y uniquely through e . This is a contradiction, by Theorem 2.4.9. So there can be no edge $e' \in E$ such that $e' \neq e$ and e' connects T_x and T_y . So $E_x \cap E_y = \emptyset$ and $E = E_x \cup \{e\} \cup E_y$. $\square$

The following definitions are necessary for proving the final results of this section as well as many results on matchwebs.

Definition 2.4.11. Let $T = (V, E, \partial)$ be a tree. To *prune* T means to remove a leaf from T .

Definition 2.4.12. A graph T can be *pruned to a point* if after some sequence of repeated prunings, only a single vertex remains.

The following theorem is the main result of this section and its statement will appear and be utilized numerous times throughout this paper.

Theorem 2.4.13. *Let $T = (V, E, \partial)$ be a graph. Then the following are equivalent:*

- (1) T is a tree.
- (2) T is connected with $|V| = |E| + 1$. (T has $|V| - 1$ edges.)
- (3) T can be pruned to a point.

Proof. (1) $\Rightarrow$ (2)

Since T is a tree, by definition, T is connected. So we need only show that when T has $|V|$ vertices, T has $|V| - 1$ edges.

We proceed by induction on $|V|$. For the base step consider $|V| = 1$. We have a tree with only one vertex, and no edges, so $|E| = 0 = 1 - 1 = |V| - 1$.

For the inductive step, assume that for T a tree with vertex set V such that $|V| < n$, then we have $|E| = |V| - 1$. Consider a tree $T = (V, E, \partial)$ with $|V| = n$. From Theorem 2.4.10, we know that we can remove an edge e such that $\partial(e) = \{x, y\}$ from T to get two disjoint trees, $T_x = (V_x, E_x, \partial)$ and $T_y = (V_y, E_y, \partial)$. We know by the inductive hypothesis that $|E_x| = |V_x| - 1$ and $|E_y| = |V_y| - 1$. From Theorem 2.4.10 we have that

$$|E| = |E_x \cup \{e\} \cup E_y| = |E_x| + |\{e\}| + |E_y|. \quad (2.24)$$

So we get

$$|E| = (|V_x| - 1) + 1 + (|V_y| - 1) = |V_x| + |V_y| - 1 = |V| - 1. \quad (2.25)$$

$$(2) \Rightarrow (3)$$

Let T be connected with $|V| = |E| + 1$. We proceed by induction on $|V|$. For the base step let $|V| = 1$; then T is a single vertex and, clearly, can be pruned to a point.

Now suppose for all connected graphs $T = (V, E, \partial)$ such that $|V| < n$ and $|E| = |V| - 1$, T can be pruned to a point. Let T be a connected graph with $|V| = n > 1$ and $|V| - 1$ edges. Note that since T is connected and $|V| > 1$, every vertex has valence ≥ 1 .

Let $v \in V$. By definition, $\text{valence}(v)$ is the number of edges incident with v . Summing over V we get

$$\sum_{v \in V} \text{valence}(v) = 2(\text{number of edges}), \quad (2.26)$$

since every edge in T is incident with exactly two vertices. From this we get

$$\sum_{v \in V} \text{valence}(v) = 2(|V| - 1) = 2|V| - 2. \quad (2.27)$$

If for every $v \in V$ $\text{valence}(v) \geq 2$, then $\sum_{v \in V} \text{valence}(v) \geq 2|V|$. So there must exist at least two $v \in V$ such that $\text{valence}(v) = 1$, otherwise we contradict our assumption of $|V|$ vertices and $|V| - 1$ edges. Hence for $\text{valence}(v) = 1$, v and its incident edge can be pruned from T . This leaves a connected graph with $|V| - 1$ vertices and $|V| - 2$ edges, which, by the induction hypothesis, can be pruned to a

point. So we can prune T to a single point.

$$(3) \Rightarrow (1)$$

Let $T = (V, E, \partial)$ be a graph such that T can be pruned to a single vertex. We will prove by induction on the size of V that T is a tree.

For the base step let $|V| = 1$; then T is a tree. For the inductive step suppose all graphs with vertex set size $k < n$ that can be pruned to a point are trees. Let $T = (V, E, \partial)$ be a graph with vertex set $|V| = n$ such that T can be pruned to a point. We know that there is an $x \in V$ such that we may prune x and its incident edge $\{v, x\}$ from T to get a graph $T' = (V', E', \partial)$ that can be pruned to a point and hence, by the inductive hypothesis, is a tree. So for every $v, w \in V'$, v is connected to w through $\{v, x\}$. Furthermore, for every $w \in V'$ we know v is connected to w in T' and hence x is connected to w . So T is connected.

Since $\{v, x\}$ is the only edge incident to x , x is connected to T' through a single edge. If a nontrivial circuit in T passes through x , we may remove all backtracks from the circuit $(v_0, \dots, v, x, v, \dots, v_n = v_0)$, and hence, remove x . So if there is a nontrivial reduced circuit C in T , then C is in T' . However, T' is a tree and thus contains no nontrivial reduced circuits so T contains no nontrivial reduced circuits. Hence T is a tree. $\square$

The following results are given directly by Theorem 2.4.13 and its proof. Again, these results will become useful for many statements and proofs about matchwebs and properties of matchwebs.

Corollary 2.4.14. *Let T be a tree with $|V| \geq 2$. Then there exist distinct $x, y \in V$ such that $\text{valence}(x) = \text{valence}(y) = 1$.*

Corollary 2.4.15. *Pruning a leaf from a tree results in a smaller tree.*

CHAPTER 3

NORMALIZED MATCHING

In this chapter we continue our study of graphs by considering a special characteristic of certain bipartite graphs and matchings, normalized matching. The normalized matching condition is yet another necessary condition in the definition of matchweb.

In Section 3.1 we define normalized matching and properties of normalized matching graphs. In Section 3.2, we define and study regular flow solutions. Amongst other results of normalized matching, we end this chapter with an elegant result in Theorem 3.2.6.

3.1 Normalized matching bipartite graphs

As before, we begin our study of normalized matching bipartite graphs with necessary definitions and results, following Anderson [And02].

Definition 3.1.1. A bipartite graph $\Lambda(X, Y)$ is a *normalized matching bipartite graph* if for every $F \subseteq X$,

$$|\Gamma(F)| \geq \frac{|Y|}{|X|} |F|. \quad (3.1)$$

Example 3.1.2. The following bipartite graph shown in Figure 3.1 is an example of a normalized matching bipartite graph. Every vertex of X is adjacent to each vertex of Y , (refer to Definition 5.2.1) so for any $F \subseteq X$ we have

$$|\Gamma(F)| = |Y| \geq \frac{|Y|}{|X|} |F|. \quad (3.2)$$

Figure 3.1: Normalized matching bipartite graph

Theorem 3.1.3. *The normalized matching condition is symmetric in X and Y , i.e., if $\Lambda(X, Y)$ is a normalized matching bipartite graph, then for every $G \subseteq Y$,*

$$|\Gamma(G)| \geq \frac{|X|}{|Y|} |G|. \quad (3.3)$$

Proof. Suppose that for all $F \subseteq X$, (3.1) holds. Let G be a subset of Y , such that $\Gamma(G) = \{x \in X \mid x \sim y \in G\}$. Let $F = X \setminus \Gamma(G) \subseteq X$. Then $\Gamma(F) \cap G = \emptyset$, and

$$|Y| \geq |G| + |\Gamma(F)|, \quad (3.4)$$

$$|X| = |\Gamma(G)| + |F|. \quad (3.5)$$

From (3.1), (3.4) becomes

$$|Y| \geq |G| + \frac{|Y|}{|X|} |F|,$$

$$|Y||X| \geq |G||X| + |Y||F|,$$

$$|G||X| \leq |Y|(|X| - |F|).$$

From (3.5), $|\Gamma(G)| = |X| - |F|$, so we get $|Y||\Gamma(G)| \geq |G||X|$, or $|\Gamma(G)| \geq \frac{|X|}{|Y|} |G|$. $\square$

3.2 Regular flows

Although necessary, the property of normalized matching in a bipartite graph can be difficult to investigate. In this section we define a special construction on certain bipartite graphs called regular flows, and determine their relationship with normalized matching bipartite graphs. Our definitions follow Engel [Eng97].

Definition 3.2.1. Let $\Lambda(X, Y)$ be a bipartite graph with edge set E . A *flow on Λ* is a function $f : E \rightarrow \mathbb{N} \cup \{0\}$.

Definition 3.2.2. Let $\Lambda(X, Y)$ be a bipartite graph with edge set E . Then Λ *admits a regular flow* if there exists a flow f on Λ such that for every v in X ,

$$\sum_{v \in \partial(e)} f(e) = |Y|, \quad (3.6)$$

and for every w in Y ,

$$\sum_{w \in \partial(e)} f(e) = |X|. \quad (3.7)$$

Example 3.2.3. The example, shown in Figure 3.2, is a bipartite graph with a regular flow. A particular flow solution is represented on the edges. Here $|X| = 3$ and $|Y| = 4$.

Figure 3.2: Regular flow

Definition 3.2.4. Let $\Lambda(X, Y)$ be a bipartite graph with vertex set $V = X \cup Y$, and edge set E . The (k, n) -clone of $\Lambda(X, Y)$ is the graph $\tilde{\Lambda} = (V_c, E_c, \partial_c)$ where

- (1) $V_c = (X \times [k]) \cup (Y \times [n])$.
- (2) $E_c = E \times [k] \times [n]$.
- (3) If $\partial(e) = \{x, y\}$, then $\partial_c(e, i, j) = \{(x, i), (y, j)\}$.

See Figure 3.3.

Figure 3.3: The $(3, 2)$ -clone of $\Lambda(X, Y)$

The result of the following lemma will be utilized in the proof of Theorem 3.2.6, the main result of this chapter.

Lemma 3.2.5. *If $\Lambda(X, Y)$ is normalized matching and $\tilde{\Lambda}(X, Y)$ is the (k, n) -clone of Λ , then $\tilde{\Lambda}(X, Y)$ is also normalized matching.*

Proof. From Theorem 3.1.3, normalized matching is symmetric and cloning can be done on one side at a time, so it suffices to show that the $(1, n)$ -clone of $\Lambda(X, Y)$ is normalized matching.

Let $F \subseteq X$. Define $\Gamma_i(F) = \Gamma(F) \cap (Y, i)$. We know $|\Gamma_i(F)| \geq \frac{|(Y, i)|}{|X|} |F|$, so

$$\sum_{i=1}^n |\Gamma_i(F)| \geq \sum_{i=1}^n \frac{|(Y, i)|}{|X|} |F|. \quad (3.8)$$

Also, $(Y, i) \cap (Y, j) = \emptyset$ for all $i \neq j$, and $\Gamma_i(F) \subseteq (Y, i)$, so $\Gamma_i(F) \cap \Gamma_j(F) = \emptyset$ for all $i \neq j$. So $|\Gamma_i(F) \cup \Gamma_j(F)| = |\Gamma_i(F)| + |\Gamma_j(F)|$, i.e.,

$$|\Gamma(F)| = \sum_{i=1}^n |\Gamma_i(F)| \geq \sum_{i=1}^n \frac{|(Y, i)|}{|X|} |F| = \frac{|\cup_{i=1}^n (Y, i)|}{|X|} |F| = \frac{|Y \times [n]|}{|X|} |F|. \quad (3.9)$$

The lemma follows. $\square$

We now state and prove the final, elegant result of this chapter.

Theorem 3.2.6. *Let $\Lambda(X, Y)$ be a bipartite graph with edge set E . Then the following are equivalent:*

(1) Λ admits a regular flow.

(2) Λ is normalized matching.

Proof. (1) $\Rightarrow$ (2)

Suppose f is a flow such that for all $v \in X$, $\sum_{v \in \partial(e)} f(e) = |Y|$ and for all $w \in Y$, $\sum_{w \in \partial(e)} f(e) = |X|$. Let $F \subseteq X$. Consider the sum S of the flows on the edges incident to F in X . On the one hand,

$$S = \sum_{v \in F} \sum_{v \in \partial(e)} f(e) = \sum_{v \in F} |Y| = |F| |Y|, \quad (3.10)$$

and on the other hand,

$$S \leq \sum_{w \in \Gamma(F)} \sum_{w \in \partial(e)} f(e) = \sum_{w \in \Gamma(F)} |X| = |\Gamma(F)| |X|. \quad (3.11)$$

Comparing (3.10) and (3.11) gives $|F| |Y| \leq |\Gamma(F)| |X|$, which means that $|\Gamma(F)| \geq \frac{|Y|}{|X|} |F|$. So $\Lambda(X, Y)$ is normalized matching.

(2) $\Rightarrow$ (1)

Suppose $\Lambda(X, Y)$ is a normalized matching bipartite graph. Let $|X| = n$ and $|Y| = k$. Consider $\Lambda(X \times [k], Y \times [n]) = \tilde{\Lambda}$, the (k, n) -clone of $\Lambda(X, Y)$. By Lemma 3.2.5, this new bipartite graph is normalized matching, so for every $F \subseteq X \times [k]$,

$$|\Gamma(F)| \geq \frac{|Y \times [n]|}{|X \times [k]|} |F| = \frac{nk}{nk} |F| = |F|. \quad (3.12)$$

By the Hall Marriage Theorem there exists a matching $g : X \times [k] \rightarrow Y \times [n]$. We also know that g defines a subgraph H of $\tilde{\Lambda}$. Consider the natural projection $\phi : \tilde{\Lambda} \rightarrow \Lambda(X, Y)$ such that for all $x \in X$, $y \in Y$, $i \in [n]$ and $j \in [k]$, $\phi((x, j)) = x$ and $\phi((y, i)) = y$. Because $|X \times [k]| = |Y \times [n]| = nk$ the matching g is a bijection. Hence each element of $X \times [k]$ gets matched to exactly one element of $Y \times [n]$ and vice versa. So in the projection $\phi(\Lambda(X \times [k], Y \times [n])) = \Lambda(X, Y)$ each $x \in X$ is incident to k edges of $\phi(H)$ because there are k copies of x in $X \times [k]$, and each $y \in Y$ is incident to n edges of $\phi(H)$ because there are n copies of y in $Y \times [n]$. Hence $\Lambda(X, Y)$ admits a regular flow, namely,

$$f(e) = |\phi^{-1}(e)|, \quad (3.13)$$

where $\phi^{-1}(e) \in H$. See Figure 3.4. □

Figure 3.4: $\phi : \tilde{\Lambda}(X, Y) \rightarrow \Lambda(X, Y)$

CHAPTER 4

POSETS

In this chapter we investigate the true motivation behind studying matchwebs: posets. Posets are closely linked with multi-layered graphs, especially normalized matching graphs. In many interesting cases a normalized matching bipartite graph is embedded into the graphical structure of a poset.

In our study of posets we first, in Section 4.1, consider some necessary initial definitions of posets and their properties. We continue our investigation of posets in Section 4.2 with a study of a special kind of posets called normalized matching posets. In Section 4.3 we consider two natural examples of posets which turn out to be normalized matching posets. We conclude this chapter in Section 4.4 with the main result, the Anderson-Griggs Theorem. This final result is the background motivation of matchwebs. For the following notation and material we follow Anderson [And02].

4.1 Posets

In this section we will begin our study of posets with introductory definitions.

Definition 4.1.1. A *partially ordered set*, or poset, is a set P on which an order relation $\leq$ is defined satisfying:

- (1) For all $x \in P$, $x \leq x$.

(2) For all $x, y \in P$, if $x \leq y$ and $y \leq x$, then $x = y$.

(3) For all $x, y, z \in P$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

If $x \leq y$ such that $x \neq y$, then we say $x < y$. If there exists a $z \in P$ such that $z \leq x$ for all $x \in S$ then we say z is the *zero element*, or 0, of P . An element $x \in P$ is called a *minimal element* of P if there is no $y \in P$ such that $y < x$. Similarly we call $x \in P$ a *maximal element* of P if there is no $y \in P$ such that $x < y$. If $x, y \in P$ such that $x \leq y$ or $y \leq x$ then we say that x and y are *comparable*.

Definition 4.1.2. If $X \subseteq P$ such that $X = \{x_1, x_2, \dots, x_n\}$ and $x_1 < x_2 < \dots < x_n$, then X is called a *chain*.

Definition 4.1.3. If $x, y \in P$, $x < y$, and there is no $z \in P$ such that $x < z < y$, then we say y *covers* x .

Definition 4.1.4. A *saturated chain of length n* is a chain $x_0 < x_1 < \dots < x_n$ such that x_{i+1} covers x_i .

Definition 4.1.5. Let P be a finite poset. For $x \in P$, the *rank of x* , $r(x)$, is the length of all saturated chains from 0 to x . If r is well-defined then we say P is *ranked* and we define the rank of P to be the largest rank of P , denoted $r(P)$. Associated with the rank function of P are the *rank numbers*, N_i , defined by

$$N_i = \text{number of } x \in P \text{ of rank } i. \quad (4.1)$$

We also define the *levels* P_i of P by, $P_i = \{x \in P \mid r(x) = i\}$.

Definition 4.1.6. In a ranked poset P , a *maximal chain* is a chain that contains one element of each rank. Note that maximal chains are saturated.

Definition 4.1.7. An *antichain* in a poset P is a set of elements of P such that no two elements are comparable.

4.2 Normalized matching posets

In this section we will further our understanding of posets and their relationship with graphs and normalized matching. Again, we begin with important definitions.

Definition 4.2.1. Let P be a poset. The *Hasse diagram* of P is the graph $H = (P, E, \partial)$, where the vertex set of H is the set of all elements of P , and $E = \{\{x, y\} \mid x \text{ covers } y \text{ or } y \text{ covers } x\}$ is the edge set of H . The k^{th} level of P , or P_k , is the set $\{x \in P \mid r(x) = k\}$. It follows that $|P_k| = N_k$.

The following definitions motivate the main theorem of this chapter, Theorem 4.2.7.

Definition 4.2.2. Let P be a ranked poset with rank numbers N_i . We say that P has the *LYM property* if whenever A is an antichain in P with p_i elements of rank i , then

$$\sum_i \frac{p_i}{N_i} \leq 1. \quad (4.2)$$

Definition 4.2.3. Let P be a ranked poset with $B \subseteq P_k$. The *shade* of B is the set of all elements of P of rank $k + 1$ which are comparable with at least one element of B , and is denoted ∇B . Similarly, for $B \subseteq P_k$, the *shadow* of B is the set of all elements of P of rank $k - 1$ that are comparable with at least one element of B , and is denoted ΔB .

Definition 4.2.4. Let P be a poset of rank n with rank numbers N_i . We say that P has the *normalized matching property for posets* if for $0 \leq k \leq n - 1$ and $B \subseteq P_k$

$$\frac{|\nabla B|}{N_{k+1}} \geq \frac{|B|}{N_k}. \quad (4.3)$$

The following lemma links the ideas of normalized matching bipartite graphs with posets and leads to the motivation behind our study of matchwebs.

Lemma 4.2.5. *Let P be a poset of rank n with rank numbers N_i , and let H be the Hasse diagram of P . Then P has the normalized matching property for posets if and only if, for all $0 \leq k \leq n-1$, P_k and P_{k+1} form a normalized matching bipartite graph in H .*

Proof. Let $Y = P_{k+1}$ and let $X = P_k$. So we have that $|Y| = N_{k+1}$ and $|X| = N_k$. For $F \subseteq X$,

$$\nabla F = \{y \in Y \mid y \geq x\} = \{y \in Y \mid y \sim x\} = \Gamma(F). \quad (4.4)$$

So the normalized matching property for posets becomes

$$|\nabla(F)| \geq \frac{|P_{k+1}|}{|P_k|} |F|. \quad (4.5)$$

Similarly, the normalized matching property for bipartite graphs becomes

$$|\Gamma(F)| \geq \frac{|Y|}{|X|} |F|. \quad (4.6)$$

So P_k and P_{k+1} form a normalized matching bipartite graph for all k if and only if P has the normalized matching property for posets. $\square$

Definition 4.2.6. Let P be a ranked poset. We say that P has a *regular covering by chains* if there exists a non-empty collection of maximal chains C in P such that, for each k , every element of rank k occurs in the same number of chains in C .

The following theorem is the main result of this section.

Theorem 4.2.7. *Let P be a ranked poset with rank numbers N_i and Hasse diagram H , with P_k the k^{th} level of H . Then the following are equivalent:*

- (1) P has the LYM property.
- (2) P is normalized matching.

(3) P has a regular covering by chains.

Proof. Our proof follows Anderson [And02].

(1) $\Rightarrow$ (2)

Suppose that whenever A is an antichain of P with p_i elements of rank i , then

$$\sum_i \frac{p_i}{N_i} \leq 1 \quad (4.7)$$

Let $F \subseteq P_k$. Then

$$\nabla F = \{y \in P_{k+1} \mid y > x \text{ for some } x \in F\}. \quad (4.8)$$

Let $D = P_{k+1} \setminus \nabla F$; so we have that $F \cup D$ is an antichain in P . By the LYM property we get

$$\sum_{i=k}^{k+1} \frac{p_i}{N_i} = \frac{|F|}{N_k} + \frac{|D|}{N_{k+1}} \leq 1. \quad (4.9)$$

Then $D = P_{k+1} \setminus \nabla F$ gives

$$\frac{|F|}{N_k} + \frac{|P_{k+1}| - |\nabla F|}{N_{k+1}} = \frac{|F|}{N_k} - \frac{|\nabla F|}{N_{k+1}} + 1 \leq 1. \quad (4.10)$$

Hence we obtain the result

$$\frac{|F|}{N_k} \leq \frac{|\nabla F|}{N_{k+1}}. \quad (4.11)$$

So P is a normalized matching poset.

(2) $\Rightarrow$ (3)

Let P be normalized matching. Let $N = \prod_{i=0}^n N_i$. We know by Lemma 4.2.5, that $\Lambda(P_k, P_{k+1})$ is a normalized matching bipartite graph. Consider $\tilde{\Lambda}_k$, the $\left(\frac{N}{N_k}, \frac{N}{N_{k+1}}\right)$ -clone of Λ , and let $\tilde{H} = \cup_{k=0}^{n-1} \tilde{\Lambda}_k$. By Lemma 3.2.5, we know that $\tilde{\Lambda}_k$ is normalized matching, and $|\tilde{P}_k| = |\tilde{P}_{k+1}|$, so there exists a perfect matching, $f_k : \tilde{P}_k \rightarrow \tilde{P}_{k+1}$, for all $k = 0, \dots, n-1$. Each f_k defines a graph G_k .

Now consider $G = \cup_{k=0}^{n-1} G_k \subseteq \tilde{H}$. The components of G define maximal chains in $\tilde{P}$, so the natural projection $\phi : \tilde{P} \rightarrow P$ defines a collection C of maximal chains in P when applied to G . (See Figure 4.1.)

Figure 4.1: $\phi : \tilde{P} \rightarrow P$

Let $x \in P_k$. There are $\frac{N}{N_k}$ copies of x in $\tilde{P}_k$, so x is an element of $\frac{N}{N_k}$ chains in C . This holds for each $x \in P_k$. So there exists C , a collection of maximal chains in P , such that for every k , each element of rank k occurs in the same number of chains in C . So P has a regular covering by chains.

(3) $\Rightarrow$ (1)

Suppose P is a ranked poset with rank numbers N_i such that P has a regular covering by chains. In other words, there exists C , a nonempty collection of maximal chains in P , such that for every k , every element of rank k occurs in the same number of chains in C . In particular, every element of rank k occurs in $\frac{|C|}{N_k}$ chains in C .

Let A be an antichain with p_k elements of rank k . Let C_k be the set of chains of C containing an element of A of rank k . It follows that $|C_k| = \frac{p_k |C|}{N_k}$. Also, since two elements of an antichain cannot be in the same chain, $\sum_{i=1}^n |C_i| \leq |C|$. Hence we

have

$$\sum_{i=0}^n \frac{p_i}{N_i} = \sum_{i=0}^n \frac{p_i |C|}{N_i |C|} = \frac{1}{|C|} \sum_{i=0}^n \frac{p_i |C|}{N_i} = \frac{1}{|C|} \sum_{i=0}^n |C_i| \leq \frac{|C|}{|C|} = 1. \quad (4.12)$$

So P has the LYM property. $\square$

The following definition and corollaries follow directly from Theorem 4.2.7.

Definition 4.2.8. Let P be a poset of rank n with rank numbers N_i . We say that P has the *Sperner property* if

$$|A| \leq \max_{0 \leq i \leq n} \{N_i\} \quad (4.13)$$

for any antichain A of P . (Note that the largest level of P is an antichain, so this is the smallest possible maximum.)

Corollary 4.2.9 (Sperner's Theorem). *Let P be a ranked poset with rank numbers N_i . If P has any of the three properties from Theorem 4.2.7, then P has the Sperner property.*

Proof. Let P be an LYM poset with maximal rank number $N_k = N$. Let A be an antichain of P with p_i members of rank i . Certainly we have the identity $|A| = \sum_{i=0}^n p_i$. From the LYM property we get

$$|A| = \frac{N}{N} \sum_{i=0}^n p_i \leq N \sum_{i=0}^n \frac{p_i}{N_i} \leq N. \quad (4.14)$$

So we have that $|A| \leq N$, and the corollary holds. $\square$

Corollary 4.2.10 (Strong Sperner's Theorem). *If the largest level of the Hasse Diagram of P is unique, then the unique largest antichain of P is the largest level.*

Proof. Let P be defined as above with A an antichain of P . As above, we observe

$$|A| = \frac{N}{N} \sum_{i=0}^n p_i \leq N \sum_{i=1}^n \frac{p_i}{N_i} \leq N, \quad (4.15)$$

where the k^{th} level of P is its unique largest level. So if there exists any $p_i \neq 0$ for $i \neq k$ then since $\frac{N}{N_i} > 1$, $\frac{Np_i}{N} < \frac{Np_i}{N_i}$, and $|A| < N$. $\square$

4.3 Examples

We now consider two examples of posets, their structures, and normalized matching properties.

Example 4.3.1. Boolean Lattice; $\mathcal{P}([n])$.

The subsets of $[n]$ form a ranked poset $\mathcal{P}([n])$ where $\subseteq$ is set inclusion and $r(A) = |A|$ for all $A \subseteq [n]$. In $P = \mathcal{P}([n])$, $P_k = \binom{[n]}{k}$ and $|P_k| = N_k = \binom{n}{k}$. If $A \subseteq P_k$, then

$$\nabla A = \{B \subseteq [n] \mid |B| = k + 1 \text{ and } A \subset B\}, \quad (4.16)$$

and

$$\Delta A = \{B \subset [n] \mid |B| = k - 1 \text{ and } B \subset A\}. \quad (4.17)$$

Theorem 4.3.2. For $A_1, A_2 \in \binom{[n]}{k}$, $|\nabla A_1| = |\nabla A_2|$ and $|\Delta A_1| = |\Delta A_2|$.

Proof. Let $A_1, A_2 \in \binom{[n]}{k}$. Since $\Delta A_i = \{B \subseteq [n] \mid |B| = k - 1 \text{ and } B \subseteq A_i\}$, ΔA_i is the set of all $k - 1$ subsets of A_i . So since $|A_i| = k$, there are $\binom{k}{k-1}$ subsets of A_i , independent of i . So $\Delta A_1 = \Delta A_2$.

Conversely, $\nabla A_i = \{A_i \cup \{j\} \mid j \notin A_i\}$. So $|\nabla A_i| = |\{j \mid j \notin A_i\}| = n - k$. Again, this is independent of i , so $|\nabla A_1| = |\nabla A_2|$. $\square$

Example 4.3.3. Linear Lattice; $V(n, q)$ ($n \in \mathbb{N}, q = \text{a prime power}$)

The set of all subspaces of F_q^n , where F_q is a finite field and $\leq$ is subspace inclusion, forms a poset $V(n, q)$. To analyze this poset we must first recall two important lemmas from linear algebra. We follow Gallian [Gal06], Chapter 19.

Lemma 4.3.4. *Let V be a vector space over a field F with $W \subseteq V$ a subspace of V . If $\{w_1, \dots, w_k\}$ forms a basis for W and $v \in V$ such that $v \notin W$, then $\{v, w_1, \dots, w_k\}$ is linearly independent, and $\dim(\text{Span}\{v, w_1, \dots, w_k\}) = k + 1$.*

Lemma 4.3.5. *Let U be a vector space over a field F with $\dim(U) = k + 1$ and $\{u_1, \dots, u_k\}$ linearly independent in U . Then there exists a $u_{k+1} \in U$ such that $\{u_1, \dots, u_k, u_{k+1}\}$ is a basis for U .*

In the Hasse diagram of $V(n, q)$, P_k is the set of all k -dimensional subspaces of F_q^n , where $|P_k| = \begin{bmatrix} n \\ k \end{bmatrix}_q$, such that

$$\begin{aligned} [n]_q &= q^{n-1} + q^{n-2} + \dots + 1 \\ [n]!_q &= [n]_q [n-1]_q \cdots [1]_q \\ \begin{bmatrix} n \\ 0 \end{bmatrix}_q &= 1 = [1]_q \\ \begin{bmatrix} n \\ 1 \end{bmatrix}_q &= \frac{q^n - 1}{q - 1} = [n]_q \\ \begin{bmatrix} n \\ 2 \end{bmatrix}_q &= \frac{(q^n - 1)(q^n - q)}{(q^2 - 1)(q^2 - q)} = \frac{[n]_q [n-1]_q}{[1]_q [2]_q} \\ &\vdots \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i} = \frac{[n]!_q}{[k]!_q [n-k]!_q} \end{aligned}$$

(See Engel [Eng97].)

If $\dim(W) = k$, then

$$\nabla W = \{U \mid U \leq F_q^n, \dim(U) = k+1, \text{ and } W \leq U\} \quad (4.18)$$

$$\Delta W = \{U \mid U \leq F_q^n, \dim(U) = k-1, \text{ and } U \leq W\}. \quad (4.19)$$

Theorem 4.3.6. *For $W_1, W_2 \in P_k$, $|\nabla W_1| = |\nabla W_2|$ and $|\Delta W_1| = |\Delta W_2|$.*

Proof. Let $W_i \in P_k$ for $i = 1, 2$. From abstract linear algebra, (e.g. Gallian [Gal06]) we know that $W_1 \approx W_2$ by mapping a basis of W_1 to a basis of W_2 . So there exists a bijective correspondence between the subspaces of W_1 and the subspaces of W_2 . So $|\Delta W_1| = |\Delta W_2|$ for all $W_1, W_2 \in P_k$.

Let $W \in P_k$, $P_k \subseteq F_q^n$, and let $\{w_1, \dots, w_k\}$ be a basis for W . Then by Lemma 4.3.4, if $v \in V$, such that $v \notin W$, then $\{w_1, \dots, w_k, v\}$ is a linearly independent set and $\dim(\text{Span}\{w_1, \dots, w_k, v\}) = k+1$. Conversely, we also know, by Lemma 4.3.5, if $W \subseteq V$ and $\dim V = k+1$, then there exists a $v \in V$ such that $\{w_1, \dots, w_k, v\}$ is a basis for V . Since ∇W is the set of all spaces V such that $\dim V = k+1$ and $W \leq V$, we must count all vectors $v \in F_q^n$ such that $v \notin W$ that form a basis for a unique vector space of dimension $k+1$ in F_q^n . Since every V accounts for $|V \setminus W|$ such v , we have

$$|\nabla W| = \frac{|F_q^n \setminus W|}{|V \setminus W|} = \frac{q^n - q^k}{q^{k+1} - q^k} = [n - k]_q, \quad (4.20)$$

independent of the choice of W . □

Theorem 4.3.7. *Let P be a poset of rank n with H the Hasse diagram of P . Suppose, for all $0 \leq k \leq n$, and for all $x_1, x_2 \in P_k$, $|\nabla x_1| = |\nabla x_2|$ and $|\Delta x_1| = |\Delta x_2|$. Then P is normalized matching.*

Proof. Let $|\nabla x| = n$ and $|\Delta y| = m$ for each $x \in P_k$ and each $y \in P_{k+1}$. Consider the

number of edges in $\Lambda = \Lambda(P_k, P_{k+1})$. On the one hand, we have nN_k edges incident to P_k , on the other hand, we have mN_{k+1} edges incident to P_{k+1} . Hence we have

$$nN_k = mN_{k+1}. \quad (4.21)$$

In other words,

$$\frac{N_k}{m} = \frac{N_{k+1}}{n}. \quad (4.22)$$

We want to show that Λ admits a regular flow. Let $f(e) = \frac{N_{k+1}}{n} = \frac{N_k}{m}$ for all edges $e \in \Lambda$. Consider $\Lambda(P_k, P_{k+1})$. The flow into $x \in P_k$ is

$$\sum_{x \in \partial(e)} f(e) = f(e_1) + \dots + f(e_n) = n \frac{N_{k+1}}{n} = N_{k+1}. \quad (4.23)$$

Similarly we have that the flow into $y \in P_{k+1}$ is

$$\sum_{y \in \partial(e)} f(e) = f(e'_1) + \dots + f(e'_m) = m \frac{N_k}{m} = N_k. \quad (4.24)$$

So Λ admits a regular flow and hence is normalized matching. $\square$

We can now make the following statement about the Boolean Lattice and Linear Lattice.

Corollary 4.3.8. *For $n \in \mathbb{N}$ and q a prime power, $\mathcal{P}([n])$ and $V(n, q)$ are normalized matching posets.*

4.4 Anderson-Griggs Theorem

We conclude this chapter with the main result of our study of normalized matching posets, the Anderson-Griggs Theorem. The Anderson-Griggs Theorem regards the decomposition of posets, which links normalized matching bipartite graphs and

matchwebs to the graphical structure of posets.

In this section we set up the Anderson-Griggs Theorem with definitions about the structure of posets and their decomposition. We also recall Section 2.3 and the Ford-Fulkerson Theorem and relate that result to the study of poset structure in Lemma 4.4.4. We conclude with the Anderson-Griggs Theorem (Theorem 4.4.6), the final result of this chapter, as well as stating the connection between posets and matchwebs.

Definition 4.4.1. Let P be a ranked poset with rank function r . The elements $x_0, \dots, x_\ell$ of P form a *symmetric chain* if

- (1) x_{i+1} covers x_i for each $i < \ell$, and
- (2) $r(x_0) + r(x_\ell) = r(P)$,

where $r(P)$ is the largest rank in P .

In the case of the Boolean Lattice, $\mathcal{P}([n])$, $P_0, \dots, P_\ell$ form a symmetric chain if

- (1) $P_i \subset P_{i+1}$ and $|P_{i+1}| = |P_i| + 1$ for each $i < \ell$.
- (2) $|P_0| + |P_\ell| = n$.

In the case of the Linear Lattice, $V(n, q)$, $W_0, \dots, W_\ell$ form a symmetric chain if

- (1) $W_i \leq W_{i+1}$ and $\dim(W_{i+1}) = \dim(W_i) + 1$ for each $i < \ell$.
- (2) $\dim(W_0) + \dim(W_\ell) = n$.

Definition 4.4.2. A poset P of rank n that can be decomposed into symmetric chains is a *symmetric chain order*. We also say that P has a *symmetric chain decomposition*.

Definition 4.4.3. A ranked poset P is said to have *unimodal rank numbers* if $N_0 \leq \dots \leq N_{k-1} \leq N_k \geq N_{k+1} \geq \dots \geq N_n$ for some integer k .

The following lemma will be used in the proof of Theorem 4.4.6.

Lemma 4.4.4. Let $\Lambda(X_1, Y, X_2)$ be a three-level graph where each pair of consecutive levels is normalized matching and $|Y| \geq |X_1| = |X_2| = m$. Then the Ford-Fulkerson condition holds, i.e., for every $F \subseteq X_1$ and $G \subseteq X_2$, $|\Gamma(F) \cap \Gamma(G)| \geq |F| + |G| - m$.

Proof. Let $\Lambda(X_1, Y, X_2)$ be a three-level normalized matching graph with $|X_1| = |X_2| = m \leq |Y|$. Let $F \subseteq X_1$ and $G \subseteq X_2$. We know that

$$|Y| \geq |\Gamma(F) \cup \Gamma(G)| = |\Gamma(F)| + |\Gamma(G)| - |\Gamma(F) \cap \Gamma(G)|. \quad (4.25)$$

Normalized matching gives

$$|Y| \geq \frac{|Y|}{|X_1|} |F| + \frac{|Y|}{|X_2|} |G| - |\Gamma(F) \cap \Gamma(G)|. \quad (4.26)$$

$$|Y| \geq \frac{|Y|}{m} |F| + \frac{|Y|}{m} |G| - |\Gamma(F) \cap \Gamma(G)|. \quad (4.27)$$

$$\frac{m}{|Y|} |\Gamma(F) \cap \Gamma(G)| \geq |F| + |G| - m. \quad (4.28)$$

We know by assumption that $|Y| \geq m$ so we get

$$|\Gamma(F) \cap \Gamma(G)| \geq \frac{m}{|Y|} |\Gamma(F) \cap \Gamma(G)| \geq |F| + |G| - m. \quad (4.29)$$

The lemma follows. $\square$

Definition 4.4.5. We say that a rank n , normalized matching poset P with with unimodal rank numbers satisfying $N_i = N_{n-i}$ is an *Anderson-Griggs poset*.

Theorem 4.4.6 (Anderson-Griggs). *Every Anderson-Griggs poset is a symmetric chain order.*

Proof. Let P be a rank n , normalized matching poset with unimodal rank numbers such that $N_i = N_{n-i}$. We may assume, without loss of generality, that n is odd, for if it is even then the middle two rank numbers are $N_k = N_{k+1}$, where $n = 2k$. Then, by the Hall Marriage Theorem, we know there exists a perfect matching between P_k and P_{k+1} , and we may thus consider $\Lambda(P_k, P_{k+1})$ to be a single level in the Hasse diagram. (See Figure 4.2.)

Figure 4.2: $N_k = N_{k+1}$

Assume n is odd. Consider the three-level normalized matching graph $\Lambda(P_{k-1}, P_k, P_{k+1})$. By Lemma 4.4.4 and the Ford-Fulkerson Theorem, we know that there are matchings $f_1 : P_{k-1} \rightarrow P_k$ and $f_2 : P_{k+1} \rightarrow P_k$ such that $f_1(P_{k-1}) = f_2(P_{k+1})$. Hence we may consider $\Lambda(P_{k-1}, P_k, P_{k+1})$ to be a single level P^* in the Hasse diagram where $|P^*| = |P_{k-1}| = |P_{k+1}|$. (See Figure 4.2.) So we may again apply Lemma 4.4.4 and the Ford Fulkerson Theorem to $\Lambda(P_{k-2}, P^*, P_{k+2})$. Continuing to “grow chains” in this manner we obtain Figure 4.3.

Figure 4.3: A symmetric chain decomposition

Let C be a chain obtained by the process defined above, then $x_1 \in P_i$ and

$x_\ell \in P_{n-i}$ by unimodality. So $r(x_1) + r(x_\ell) = i + n - i = n$.

We can continue iterating until we reach P_0 and P_n . So all of P is included in the chain decomposition. Also, the chains at each level are disjoint and symmetric. Hence P has a symmetric chain decomposition. $\square$

We can conclude that Anderson-Griggs posets, posets with a symmetric chain decomposition, have a graphical structure given by their decompositions, and that graphical representation is in fact given by multiple layers of matchwebs of different sizes.

We are now prepared to define and understand matchwebs.

CHAPTER 5

MATCHWEBS

Matchwebs first appeared in West, Harper, and Daykin [DHW83], represented as two levels of the Hasse diagram of a minimal LYM order. In other words, a matchweb is a minimal normalized matching bipartite graph embedded in a level of an Anderson-Griggs poset.

We begin this chapter in Section 5.1 with the final necessary definitions and results in order to formally define a matchweb. We utilize the background material discussed in all previous sections. In Section 5.2 we begin our classification of all matchwebs by considering small ranks, i.e., small values of $|X| = n$.

5.1 Matchwebs

In this section we will formally define a matchweb, and state some initial properties of matchwebs.

Theorem 5.1.1. *Let $\Lambda(X, Y)$ be a disconnected, normalized matching bipartite graph. Then Λ is proportional; i.e., for $X = B \cup D$ and $Y = A \cup C$, both disjoint, such that $\Gamma(B) \subseteq A$ and $\Gamma(D) \subseteq C$, we have*

$$\frac{|A|}{|B|} = \frac{|C|}{|D|}. \quad (5.1)$$

Proof. Suppose $\Lambda(B \cup D, A \cup C)$ is normalized matching, so for every $F \subseteq B \cup D$, $|\Gamma(F)| \geq |F| \frac{|A \cup C|}{|B \cup D|}$. On one hand we have $B \subseteq B \cup D$ which gives

$$|A| \geq |\Gamma(B)| \geq |B| \frac{|A \cup C|}{|B \cup D|} = |B| \frac{|A| + |C|}{|B| + |D|}. \quad (5.2)$$

This gives

$$|A| |B| + |A| |D| \geq |A| |B| + |B| |C| \quad (5.3)$$

$$|A| |D| \geq |B| |C| \quad (5.4)$$

$$\frac{|A|}{|B|} \geq \frac{|C|}{|D|}. \quad (5.5)$$

On the other hand, $D \subseteq B \cup D$, so similarly,

$$\frac{|C|}{|D|} \geq \frac{|A|}{|B|}. \quad (5.6)$$

So from (5.5) and (5.6) we have that $\frac{|A|}{|B|} = \frac{|C|}{|D|}$. $\square$

The following definition will be used as part of the method for creating and enumerating matchwebs.

Definition 5.1.2. Given a bipartite graph $\Lambda(X, Y)$ with edge set E , a regular flow $f : E \rightarrow \mathbb{N} \cup \{0\}$, and a cycle $\{e_1, \dots, e_n\}$, we may remove circuits by *cyclic reduction* to obtain a new graph Λ_c , as follows:

- (1) Denote edges of the cycle as $e_1, \dots, e_{2k}$, where $n = 2k$. (Note that cycles are of even length.)
- (2) Without loss of generality let $f(e_1) = f_1$ be a smallest flow in the cycle. (I.e., $f_1 \leq f(e)$ for all edges e in the cycle.)

(3) For $1 \leq m \leq k$, let

$$f_c(e_{2m}) = f(e_{2m}) + f_1,$$

$$f_c(e_{2m+1}) = f(e_{2m+1}) - f_1, \text{ and}$$

$$f_c(e) = f(e)$$

for all edges e not in the cycle.

(4) Remove edges with flow equal to zero.

See Figure 5.1.

Figure 5.1: Cyclic reduction

Lemma 5.1.3. *Let $\Lambda(X, Y)$ be a normalized matching bipartite graph with edge set E , let $f : E \rightarrow \mathbb{N} \cup \{0\}$ be a regular flow on Λ , and let $\{e_1, \dots, e_{2k}\}$ be a cycle. Then $\Lambda_c(X, Y)$, obtained by cyclic reduction on Λ , is normalized matching.*

Proof. Let $C = \{v_0, \dots, v_{2k} = v_0\}$ be a cycle in Λ such that $\partial(e_i) = \{v_{i-1}, v_i\}$, $1 \leq i \leq 2k$, and let $f : E \rightarrow \mathbb{N} \cup \{0\}$ be a regular flow on Λ . We know by the normalized matching condition that for all $x \in X$,

$$\sum_{x \in \partial(e)} f(e) = |Y|, \quad (5.7)$$

and for all $y \in Y$,

$$\sum_{y \in \partial(e)} f(e) = |X|. \quad (5.8)$$

Without loss of generality, $v_0, v_2, \dots, v_{2k-2} \in X$ and $v_1, v_3, \dots, v_{2k-1} \in Y$. Let $f(e_1) = f_1$ be the minimum flow on the cycle C . Then, by cyclic reduction, for $1 \leq m \leq k$,

$$\begin{aligned} f_c(e_{2m}) &= f(e_{2m}) + f_1, \\ f_c(e_{2m-1}) &= f(e_{2m-1}) - f_1, \\ f_c(e) &= f(e), \end{aligned}$$

for all edges $e \notin C$. Then, because $x \notin \{v_0, v_2, \dots, v_{2k-1}\}$ is unaffected by cyclic reduction, we only need to check the vertices $x = v_{2m}$, for which the flow solution becomes

$$\sum_{x \in \partial(e), e \notin C} f(e) + (f(e_{2m}) + f_1) + (f(e_{2m+1}) - f_1) = \sum_{x \in \partial(e)} f(e) = |Y|. \quad (5.9)$$

Similarly, because $y \notin \{v_1, v_3, \dots, v_{2k-1}\}$ is unaffected by cyclic reduction, we need only check all vertices $y = v_{2m-1}$, for which the flow solution becomes

$$\sum_{y \in \partial(e), e \notin C} f(e) + (f(e_{2m-1}) - f_1) + (f(e_{2m}) + f_1) = \sum_{y \in \partial(e)} f(e) = |X|. \quad (5.10)$$

Hence Λ_c is normalized matching. $\square$

Definition 5.1.4. A *minimal normalized matching bipartite graph* is a normalized matching bipartite graph such that no edges can be removed to obtain a normalized matching bipartite graph with the same vertex set.

Theorem 5.1.5 follows naturally.

Theorem 5.1.5. *Any minimal normalized matching bipartite graph has no nontrivial reduced circuits.*

Proof. Let $\Lambda(X, Y)$ be a minimal normalized matching bipartite graph. Suppose Λ contains a nontrivial reduced circuit. Then by Lemma 5.1.3 we may obtain $\Lambda_c(X, Y)$ through cyclic reduction such that Λ_c is normalized matching with the same vertex set and fewer edges. This is a contradiction of the assumption that Λ is minimal normalized matching, so Λ contains no nontrivial reduced circuits. $\square$

We now approach the heart of this paper with our main definition.

Definition 5.1.6. A *matchweb* is a connected minimal normalized matching bipartite graph.

The following results from Theorem 5.1.5 and Section 2.4.

Corollary 5.1.7. *Every matchweb is a tree.* $\square$

Theorem 5.1.8. *Let $\Lambda(X, Y)$ be a matchweb with edge set E , and a regular flow f . Then for $e \in E$, $f(e) > 0$.*

Proof. Suppose $\Lambda(X, Y)$ is a matchweb with an edge e such that $f(e) = 0$. Then since the flow on e does not affect the regular flow solution of Λ , after removing e , Λ is still a normalized matching bipartite graph. This is a contradiction however, because Λ is a minimal normalized matching bipartite graph and edges cannot be removed to obtain a smaller normalized matching bipartite graph. $\square$

Theorem 5.1.9. *If $\Lambda(X, Y)$ is a connected normalized matching bipartite graph on which every regular flow is positive, then Λ is a matchweb.*

Proof. Suppose a connected normalized matching bipartite graph $\Lambda(X, Y)$ has all regular flows, $f > 0$. Then no edges can be removed without affecting the regular flow solution. Hence $\Lambda(X, Y)$ is a minimal normalized matching bipartite graph, in other words, Λ is a matchweb. $\square$

Example 5.1.10. In Figure 5.2 we see an example of a normalized matching bipartite graph with a regular flow that is not a matchweb.

Figure 5.2: A normalized matching bipartite tree is not a matchweb

5.2 Small ranks

In this section we begin our enumeration and classification of matchwebs with two specific examples. Namely, we will consider all matchwebs with $|X| = 1$ and $|X| = 2$.

Definition 5.2.1. A normalized matching bipartite graph $\Lambda(X, Y)$ is called a *complete bipartite graph* if X is adjacent to every element of Y . If $|X| = n$ and $|Y| = k$, then we denote $\Lambda = K_{n,k}$.

Definition 5.2.2. If $\Lambda(X, Y)$ is a complete bipartite graph such that $|X| = 1$ and $|Y| = k$, then we call Λ the *fan graph*, denoted $K_{1,k}$.

The first main matchweb classification, $|X| = 1$, follows as Theorem 5.2.3.

Theorem 5.2.3. *If $\Lambda(X, Y)$ is normalized matching, and $|X| = 1$, then Λ is a fan graph.*

Proof. We know that $|\Gamma(X)| \geq \frac{|Y|}{|X|} |X|$, so $|\Gamma(X)| \geq |Y|$. Since $\Gamma(X) \subseteq Y$, $|\Gamma(X)| = |Y|$. So $\Gamma(X) = Y$, everything in Y is adjacent to X and hence Λ is a fan. $\square$

Our second matchweb classification, $|X| = 2$, follows as Theorem 5.2.4.

Theorem 5.2.4. *Let $\Lambda(X, Y)$ be minimal normalized matching, with $|X| = 2$. If $|Y| = 2m$ then Λ is a disjoint union of two fan graphs, $K_{1,m}$. If $|Y| = 2m + 1$ then Λ is a union of two fan graphs, joined at a single vertex of Y .*

Proof. Let $\Lambda(X, Y)$ be a normalized matching bipartite graph with $X = \{x_1, x_2\}$. Consider $F \subset X$, with $|F| = 1$. Since Λ is normalized matching we have that

$$|\Gamma(F)| \geq \frac{|Y|}{|X|} |F| = \frac{|Y|}{2} \quad (5.11)$$

We have two cases to consider: $|Y| = 2m + 1$ and $|Y| = 2m$ for $m \in \mathbb{N}$.

Case 1: Let $|Y| = 2m$. With normalized matching we have

$$|\Gamma(x_1)| \geq \frac{2m}{2} = m. \quad (5.12)$$

Figure 5.3: $\Gamma(x_1) \cap \Gamma(x_2)$

Suppose $|\Gamma(x_1)| \geq m + 1$. We also know that $|\Gamma(x_2)| \geq m$, so $|\Gamma(x_1) \cap \Gamma(x_2)| \geq$

1. If $|\Gamma(x_1) \cap \Gamma(x_2)| \geq 2$, then Λ contains a cycle. (See Figure 5.3.) So we can assume, without loss of generality, that $|\Gamma(x_1) \cap \Gamma(x_2)| = 1$. Let $y \in Y$ such that y

is adjacent to x_1 and x_2 . We know the flow into y must be 2. So, since by Theorem 5.1.8, every regular flow is positive, if there exists $e_1, e_2 \in E$ such that $\partial(e_1) = \{x_1, y\}$ and $\partial(e_2) = \{x_2, y\}$, then $f(e_1) = f(e_2) = 1$.

Consider the flow solution at x_1 :

$$2m = \sum_{x_1 \in \partial(e)} f(e) = \sum_{x_1 \in \partial(e); e \neq e_1} f(e) + 1. \quad (5.13)$$

So we have that

$$\sum_{x_1 \in \partial(e); e \neq e_1} f(e) = 2m - 1. \quad (5.14)$$

This is a contradiction because, since f is regular, for each edge $e \in E$ such that e is not incident to y , $f(e) = 2$. So the sum could not be odd. The result is symmetric for the flow solution of x_2 . Hence $\Gamma(x_1) \cap \Gamma(x_2) = \emptyset$, and Λ is a disjoint union of two m -fans, $K_{1,m}$. (See Figure 5.4.)

Figure 5.4: $K_{1,4} \cup K_{1,4}$

Case 2: Let $|Y| = 2m + 1$. With normalized matching we have

$$|\Gamma(x_1)| \geq \frac{2m+1}{2} > m. \quad (5.15)$$

We have a symmetric result for x_2 , so $|\Gamma(x_i)| \geq m + 1$ for $i = 1, 2$. By the Pigeonhole Principle we know that $|\Gamma(x_1) \cap \Gamma(x_2)| \geq 1$. Again, if $|\Gamma(x_1) \cap \Gamma(x_2)| = 2$ then we have a subgraph with an equal number of vertices and edges, hence we have a cycle. So we can eliminate the possibility that $|\Gamma(x_1) \cap \Gamma(x_2)| \geq 2$. So there is only

Figure 5.5: $|X| = 2, |Y| = 7$

one vertex $y \in Y$ such that both x_1 and x_2 are adjacent to y . So for every other $v \in Y$, v is adjacent to only one x_i . Hence Λ is a union of two $m + 1$ -fans, $K_{1,m+1}$ joined at a single vertex of Y . (See Figure 5.5.) $\square$

So the cases of matchwebs with one or two vertices in X have been fully classified. We have found that all minimal normalized matching bipartite graphs with $|X| = 1$ are matchwebs and a matchweb $\Lambda(X, Y)$ has $|X| = 2$ if and only if $|Y| = 2m + 1$, for a positive integer m .

We now consider a more general class of minimal normalized matching bipartite graphs, the case $|Y| = m|X|$. Theorem 5.2.5 is from Hsu, Logan, and Shahriari [HLS], but is represented here for completeness.

Theorem 5.2.5. *Let $\Lambda(X, Y)$ be a minimal normalized matching bipartite graph with $|X| = n$ and $|Y| = mn$, for $m \in \mathbb{Z}$. Then Λ is the union of m -fans, $K_{1,m}$.*

Proof. Let $\Lambda(X, Y)$ be a minimal normalized matching bipartite graph with $|Y| = m|X|$. By Lemma 3.2.5 we know that because $\Lambda(X, Y)$ is normalized matching, the $(m, 1)$ -clone of $\Lambda(X, Y)$, $\tilde{\Lambda}$, is normalized matching. We know that $|Y \times [1]| = |X \times [m]| = mn$. So for all $F \in X \times [m]$,

$$|\Gamma(F)| \geq \frac{|Y \times [1]|}{|X \times [m]|} |F| = |F|. \quad (5.16)$$

So by the Hall Marriage Theorem there exists a matching $g : X \times [m] \rightarrow Y \times [1]$. Since $|Y \times [1]| = |X \times [m]|$, g is a perfect matching, i.e., for each $(x, i) \in X \times [m]$, $\text{valence}((x, i)) = 1$.

Figure 5.6: $|Y| = 9$, $|X| = 3$

Consider the natural projection $\phi : \tilde{\Lambda} \rightarrow \Lambda(X, Y)$, such that for all $x \in X$ and $i \in [m]$, $\phi((x, i)) = x$. Because $\tilde{\Lambda}$ is the $(m, 1)$ -clone of Λ , $y \in Y$ is not affected. Since g is a perfect matching on $\tilde{\Lambda}$, the projection of g maps each element of X to m elements of Y , i.e., for all $x \in X$,

$$\text{valence}(\phi((x, i))) = m \text{ valence}((x, i)) = m. \quad (5.17)$$

So for all $x \in X$, $\text{valence}(x) = m$. Thus $\Lambda(X, Y)$ contains the disjoint union of m -fans $K_{1,m}$ as a subgraph. See Figure 5.6. However, since $\Lambda(X, Y)$ is minimal, $\Lambda(X, Y)$ is equal to the disjoint union of n m -fans. $\square$

We now move on to a general matchweb classification by building the basic matchweb structure.

CHAPTER 6

BUILDING MATCHWEBS

In this chapter we will begin to build the general matchweb structure. To do this we will consider the matchweb more precisely, considering the underlying structure, or core, of each matchweb.

In Section 6.1 we consider cores and bases, the fundamental structure of matchwebs, as well as their properties. In Section 6.2 we formally define a process of enumerating all possible matchweb bases, given $|X| = n$. Finally, in Section 6.3, we enumerate all matchweb bases up to rank 8, i.e., $|X| \leq 8$.

Note that for a matchweb $\Lambda(X, Y)$, without loss of generality, we assume $|X| \leq |Y|$, and by Theorem 5.2.5, if $|X| > 1$, $|X| < |Y|$.

6.1 Core graphs

In this section we will define the core graph and its relation to the basic structure of matchwebs, cores and bases.

Theorem 6.1.1. *Let $\Lambda(X, Y)$ be a matchweb with $|X| < |Y|$. Then for all $x \in X$, $\text{valence}(x) > 1$.*

Proof. Suppose $\Lambda(X, Y)$ is a matchweb such that there is an $x \in X$ with $\text{valence}(x) = 1$. Then x and its incident edge e_x form a leaf. Since $\Lambda(X, Y)$ is normalized matching, Λ admits a regular flow; in particular, given x ,

$$|Y| = \sum_{x \in \partial(e)} f(e) = f(e_x). \quad (6.1)$$

Suppose $y \in Y$ such that $x \sim y$. Then the flow into y is given by

$$|X| = \sum_{y \in \partial(e)} f(e) = \sum_{y \in \partial(e), e \neq e_x} f(e) + f(e_x). \quad (6.2)$$

Considering (6.1) we obtain

$$|X| = \sum_{y \in \partial(e), e \neq e_x} f(e) + |Y| > |X|. \quad (6.3)$$

This is a clear contradiction. $\square$

Definition 6.1.2. Let $\Lambda(X, Y)$ be a matchweb, then Λ has a *core*, $\Lambda^*(X, Y^*)$, obtained by removing all of the leaves incident to vertices of X . (See Figure 6.1.)

Figure 6.1: Matchweb core

Definition 6.1.3. A *core graph* is a connected bipartite tree $\Lambda^*(X, Y^*)$ such that, for every $y \in Y$, $\text{valence}(y) > 1$.

Matchweb cores are bipartite trees with a smaller Y vertex set and the same X vertex set. More precisely:

Theorem 6.1.4. A matchweb core is a core graph.

Proof. Let $\Lambda(X, Y)$ be a matchweb. We know that for all $x \in X$, $\text{valence}(x) > 1$. To obtain the core of Λ we must remove all leaves from Λ incident to $x \in X$. Since all $y \in Y$ with $\text{valence}(y) = 1$ have been removed, we are left with a bipartite graph $\Lambda^*(X, Y^*)$ such that for all $y \in Y^*$, $\text{valence}(y) > 1$. By Corollary 2.4.15, Λ^* is a tree. Hence the core Λ^* is a core graph. $\square$

Figure 6.2: Core graph to base; 2-pod base

Definition 6.1.5. The *base* of a core graph $\Lambda^*(X, Y^*)$ is a marked graph obtained by replacing each vertex of Y^* in the core graph by a symbol of its valence. The symbol for valence two is omitted for notation and replaced with an edge. A *pod* is a symbol used to denote the valence and adjacent vertices of a single vertex in Y^* . (See Figure 6.2.) The *order* of a base is defined to be $|X|$.

A base is the underlying skeletal structure of a core graph. The following definition motivates the systematic process of enumerating bases we utilize in our matchweb classification.

Definition 6.1.6. Given bases V and W , we define a *base isomorphism* to be a bijection $\varphi : V \rightarrow W$, such that

(1) If $v \sim w$ in V , if and only if $\varphi(v) \sim \varphi(w)$ in W .

(2) $\varphi(X_v) = X_w$ and $\varphi(Y_v) = Y_w$.

If $V = W$, then we say φ is a *base automorphism* of V .

Definition 6.1.7. Two vertices of a base are *symmetric* if they are equivalent under a base automorphism φ .

6.2 Core graph and base enumeration

We continue with our classification of all matchwebs by defining a systematic method for enumerating all matchweb bases. The following theorem motivates the rest of this chapter and its results.

Theorem 6.2.1. *If $\Lambda^*(X, Y^*)$ is a core graph, then there exists a $y \in Y^*$ such that there is at most one $x \sim y$ with $\text{valence}(x) > 1$.*

Proof. If for all $y \in Y^*$ there exist at least two $x \sim y$ such that $\text{valence}(x) > 1$, after pruning all the leaves from Λ^* we are left with a graph $\Lambda_r^*(X_r, Y^*)$ with all vertices of Y^* ; since in a core graph for every $y \in Y^*$, $\text{valence}(y) > 1$; and all vertices of X with valence greater than one and $|X_r| \geq 2$. So every vertex of Λ_r^* would have valence greater than one. But by Corollary 2.4.14 there must be at least two vertices in $V_r = X_r \cup Y^*$ such that $\text{valence}(v) = 1$. Hence there must exist one $y \in Y^*$ such that there is at most one $x \sim y$ with $\text{valence}(x) > 1$. $\square$

So every core graph contains a prunable pod that shares at most one vertex in X . See Figure 6.3.

We now consider the following method for enumerating all matchwebs. In order to understand matchwebs we must first understand their cores; and to enumerate all

can remove that pod with pruning to obtain a core graph with $|X| < n$, with a corresponding base. By the inductive hypothesis this new base can be pruned to either a pod or a point. Hence all bases can be obtained by the above algorithm. $\square$

6.3 Base enumeration up to order 8

We can now systematically enumerate all bases with the algorithm described in Section 6.2. In Figures 6.4 through 6.17 we have enumerated all bases up to order 8 up to isomorphism.

Figure 6.4: Matchweb base enumeration up to order 4

Figure 6.5: Matchweb base enumeration of order 5 and 6

Figure 6.6: Matchweb base enumeration of order 7

7) continued

Figure 6.7: Order 7 enumeration continued

7) continued

Figure 6.8: Order 7 enumeration continued

Figure 6.9: Matchweb base enumeration of order 8

8) continued

Figure 6.10: Order 8 enumeration continued

8) *continued*

Figure 6.11: Order 8 enumeration continued

8) continued

Figure 6.12: Order 8 enumeration continued

8) *continued*

Figure 6.13: Order 8 enumeration continued

8) continued

Figure 6.14: Order 8 enumeration continued

8) continued

Figure 6.15: Order 8 enumeration continued

8) continued

Figure 6.16: Order 8 enumeration continued

8) *continued*

Figure 6.17: Order 8 enumeration continued

CHAPTER 7

EXAMPLES

In the following chapter we consider specific examples of core graphs and their corresponding matchwebs. Using properties of regular flows, normalized matching, and modular arithmetic we prove matchweb solutions exist for three examples of core graphs.

We begin in Section 7.1 with a short background result. In Section 7.2, we define and discuss the n -pod. In Section 7.3, we define and discuss the n -spider. Finally, in Section 7.4 we define and discuss the n -fence.

7.1 From core graphs to cores

Lemma 7.1.1. *Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$ and a core $\Lambda^*(X, Y^*)$ with edge set E^* , and regular flow f on Λ . Then for $e \in E^*$, $0 < f(e) < n$.*

Proof. We know by Theorem 5.1.8 for all $e \in E^*$, $f(e) > 0$. Since f is a regular flow, for all $y \in Y$,

$$\sum_{y \in \partial(e)} f(e) = n. \quad (7.1)$$

If there is an edge $e \in E^*$ such that $f(e) > n$, then, because $\text{valence}(y) > 1$ for all

$y \in Y^*$, there must also be an edge e' such that $f(e') \leq 0$, which is not possible. So, for all $e \in E^*$, $0 < f(e) < n$. $\square$

We will now consider three basic core types and prove existence of their solutions as matchwebs. Uniqueness of solutions will be further discussed in Chapter 8.

7.2 The n -pod core

In this section we will discuss the basic n -pod core, the existence of its matchweb solution and uniqueness mod k , where $k = |Y|$ in the matchweb, $\Lambda(X, Y)$.

Definition 7.2.1. Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$. The core $\Lambda^*(X, Y^*)$ of Λ is called an n -pod if Λ^* contains only one vertex of Y^* to which every vertex in X is adjacent. (See Figure 7.1.)

Figure 7.1: The 5-pod

Theorem 7.2.2. Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$, $|Y| = k$, and an n -pod core. Then $|Y| \equiv 1 \pmod{n}$ and each vertex in X must be incident to the same number of leaves. Conversely, if $k, n, \ell \in \mathbb{N}$ such that $k = n\ell + 1$, then $\Lambda(X, Y)$ is a matchweb with $|X| = n$, $|Y| = k$, and an n -pod core such that each $x \in X$ is incident to ℓ leaves.

Proof. Consider the general n -pod core. In order to build a matchweb we must fill in leaves until we have $|Y| \geq |X|$ and a regular flow, f . The flow into each vertex in Y

must be n so we know that the value of f on each edge, e_i , in the core must be equal to 1 since all flows are positive. Also, since the added leaves are incident to only one vertex in Y , the flow on each leaf must be n . Let ℓ_i be the number of leaves adjacent to $x_i \in X$. (See Figure 7.1.) A regular flow gives

$$\begin{aligned} k &= n\ell_1 + 1 \\ k &= n\ell_2 + 1 \\ &\vdots \\ k &= n\ell_n + 1. \end{aligned} \tag{7.2}$$

So $|Y| = k \equiv 1 \pmod{n}$ and $\ell_i = \ell_j$ for all $1 \leq i, j \leq n$, by the Division Algorithm. In other words, each vertex of X is incident to the same number of leaves.

Conversely, let $k, n, \ell \in \mathbb{N}$ such that $k = n\ell + 1$. Then if we attach ℓ leaves to each $x_i \in X$ of the n -pod core $\Lambda^*(X, Y^*)$ and choose the other edge flows to be 1, (7.2) gives a regular flow solution on $\Lambda(X, Y)$. Since, this is the only possible solution, all flows are positive, no edges can be removed, so $\Lambda(X, Y)$ is a matchweb with $|X| = n$, $|Y| = k$, and ℓ leaves incident to each vertex of X . $\square$

7.3 The n -spider core

In this section we will discuss the basic n -spider core and existence of its matchweb solution. Uniqueness of its solution will be discussed further in Chapter 8.

Definition 7.3.1. Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$. The core of Λ is called an n -spider if Λ^* contains 1 vertex of X to which each vertex of Y^* is adjacent, and every other vertex of X is adjacent to a unique vertex of Y^* , as seen in Figure 7.2.

Figure 7.2: The 5-spider

Definition 7.3.2. Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$, $|Y| = k$, and an n -spider core Λ^* . Then we define the following notation,

x_i = vertices of X , $0 \leq i \leq n - 1$

y_j = vertices of Y^* , $1 \leq j \leq n - 1$

f_i = flow on edge $e_i = \{x_i, y_j\}$, $i \neq 0$

g_j = flow on edge $e_j = \{x_0, y_j\}$

ℓ_i = number of leaves incident to x_i .

Here x_0 is the special node of X that is adjacent to all y in Λ^* . (See Figure 7.2.)

Theorem 7.3.3. *If $\Lambda(X, Y)$ is a matchweb with $|X| = n$, $|Y| = k = nq + r$ ($0 \leq r < n$) by the Division Algorithm, and an n -spider core $\Lambda^*(X, Y^*)$, then $k \not\equiv 0 \pmod{n}$, and $q - n + r + 1 \geq 0$.*

Proof. We know from Theorem 5.2.5 that if $k = mn$ for $m \in \mathbb{N}$, $n > 1$, $\Lambda(X, Y)$ is not a matchweb. So $k \not\equiv 0 \pmod{n}$.

Let $k = nq + r$, $0 < r < n$. Let the flows on Λ be defined as in Definition 7.3.2.

Since Λ admits a regular flow, for all x_i , $1 \leq i \leq n - 1$, we have

$$k = n\ell_i + f_i \tag{7.3}$$

with $0 < f_i < n$, by Lemma 7.1.1. In other words, $\ell_i = q$ and $f_i = r$ for $1 \leq i \leq n-1$.

Also, we have

$$|Y| = k = \sum_{i=0}^{n-1} \ell_i + n - 1 = (n-1)q + \ell_0 + n - 1, \quad (7.4)$$

since we add ℓ_i leaves to each $x_i \in X$. So we have

$$k = nq + r = (n-1)q + \ell_0 + n - 1. \quad (7.5)$$

Simplifying, we get

$$\ell_0 = nq + r - (n-1)q - n + 1 = q - n + r + 1. \quad (7.6)$$

So in order for $\Lambda(X, Y)$ to be a matchweb with an n -spider core and $k = nq + r$, we must have $q - n + r + 1 \geq 0$. $\square$

Theorem 7.3.4. *Let $k, n \in \mathbb{N}$ such that $k = nq + r$, $0 < r < n$, by the Division Algorithm, where $q - n + r + 1 \geq 0$. Let*

$$\begin{aligned} f_i &= r \\ g_i &= n - r \\ \ell_i &= \begin{cases} q & \text{when } i \neq 0, \\ q - n + r + 1 & \text{when } i = 0. \end{cases} \end{aligned} \quad (7.7)$$

Then choosing flows f_i and g_i as above and a flow of n on leaves defines a positive regular flow on the graph $\Lambda(X, Y)$, which is an n -spider $\Lambda^*(X, Y^*)$ with ℓ_i leaves attached to each x_i , and $|Y| = k$.

Proof. Let $k = nq + r$, and (7.7) hold. Choosing flows f_i and g_i as in Definition 7.3.2. Consider the flow into the vertices of Y . We define the flow on the leaves incident to X to be n , and for vertices of Y not incident to leaves, the flow is:

$$f_i + g_i = r + n - r = n. \quad (7.8)$$

So the flow solutions into Y are n . (See Figure 7.2.)

Consider the flow solutions into the nodes of X . We have two cases to consider:
 $i = 0$ and $i \neq 0$.

For the flow into x_0 , we have

$$n\ell_0 + \sum_{i=1}^{n-1} g_i = n(q - n + r + 1) + (n - 1)(n - r) = nq + r = k. \quad (7.9)$$

For the flow into x_i , $i \neq 0$, we have

$$n\ell_i + f_i = nq + r = k. \quad (7.10)$$

Furthermore, we have added $\sum_{i=0}^n \ell_i$ vertices to Y^* ; in other words,

$$|Y| = n - 1 + q(n - 1) + (q - n + r + 1) = nq + r = k. \quad (7.11)$$

Hence (7.7) defines a positive regular flow on $\Lambda(X, Y)$. It follows from Theorem 5.1.9 that $\Lambda(X, Y)$ is a matchweb, since this flow is unique.

□

7.4 The n -fence core

In this section we discuss the basic n -fence core, existence of its matchweb solution in Theorem 7.4.3, and uniqueness in Theorem 7.4.4.

Definition 7.4.1. Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$. The core graph of Λ is called an n -fence if Λ^* contains $n - 1$ vertices of Y each adjacent to exactly two vertices in X , as shown in Figure 7.3.

Figure 7.3: The 5-fence

Definition 7.4.2. Let $\Lambda(X, Y)$ be a matchweb with $|X| = n$, $|Y| = k$, and an n -fence core. We define the following notation on Λ^* :

x_i = vertices of X , $1 \leq i \leq n$

y_j = vertices of Y^* , $1 \leq j \leq n - 1$

f_i = flow on edge $\{x_i, y_i\}$

g_i = flow on edge $\{x_{i+1}, y_i\}$

ℓ_i = number of leaves incident to x_i .

(See Figure 7.3.)

The following theorem shows the uniqueness of solutions, i.e., the uniqueness of the matchweb with an n -fence core.

Theorem 7.4.3. Let $\Lambda(X, Y)$ be a matchweb with an n -fence core, $|X| = n$, $|Y| = k$, and let $k = nq + r$, $0 < r \leq n$, by the Division Algorithm. Then for $0 < i < n$,

$$f_i = ir \pmod{n}, \quad 0 < f_i < n, \quad (7.12)$$

$$g_i = n - f_i, \quad (7.13)$$

$$\ell_i = \begin{cases} q & \text{when } f_{i-1} > n - r \text{ (case 1)} \\ q - 1 & \text{when } f_{i-1} < n - r \text{ (case 2)}. \end{cases} \quad (7.14)$$

Furthermore, k and n are relatively prime.

The first case of ℓ_i is when there is “wrap-around” in the flow solutions, i.e., $f_i = f_{i-1} + r - n$ and in the second case there is no wrap-around, $f_i = f_{i-1} + r$.

Proof. We first show that the first property, $f_i = ir \pmod{n}$, holds.

We proceed by induction on i . For the base step let $i = 1$. Consider $x_1 \in X$. We know that the flows into x_1 must sum to k , and that the flow on each leaf incident to x_1 is n . So we have

$$n\ell_1 + f_1 = k. \quad (7.15)$$

So $f_1 = r$, i.e., $f_1 = 1r \pmod{n}$.

Suppose for all $i < m$, $f_i = ir \pmod{n}$. Consider $x_m \in X$. We know that the flow solution into x_m must sum to k , and the flow on each leaf incident to x_m is n . So the flow becomes

$$n\ell_m + f_m + g_{m-1} = k. \quad (7.16)$$

We know that $f_{m-1} + g_{m-1} = n$, i.e., $g_{m-1} = n - f_{m-1}$. By the inductive hypothesis $f_{m-1} = (m-1)r \pmod{n}$. So (7.16) becomes

$$n\ell_m + f_m + n - (m-1)r = k \pmod{n}. \quad (7.17)$$

If we consider the flow solution $\pmod{n}$ we get

$$f_m = (m-1)r + k = (m-1)r + r = mr \pmod{n}. \quad (7.18)$$

So for all $1 \leq i \leq n$, $f_i = ir \pmod{n}$.

For a regular flow, the sum into each vertex in Y must sum to n , so for all $y_i \in \Lambda^*$,

$$n = f_i + g_i. \quad (7.19)$$

So $g_i = n - f_i$ for all $1 \leq i \leq n - 1$.

Similarly, suppose for each $x_i \in X$ has ℓ_i leaves attached in $\Lambda(X, Y)$, then regular flows give

$$k = n\ell_i + f_i + g_{i-1}. \quad (7.20)$$

If $f_i = f_{i-1} + r - n$ then (7.20) becomes

$$k = n\ell_i + f_{i-1} + r - n + g_{i-1} = n\ell_i + (i-1)r + r - n + n - (i-1)r = n\ell_i + r. \quad (7.21)$$

So $\ell_i = q$.

If $f_i = f_{i-1} + r$, then (7.20) becomes

$$k = n\ell_i + f_{i-1} + r + g_{i-1} = n\ell_i + (i-1)r + r + n - (i-1)r = n(\ell_i + 1) + r. \quad (7.22)$$

So $\ell_i = q - 1$.

Finally, we prove $(k, n) = 1$. We know that $f_i = ir \pmod{n}$ and $g_i = n - f_i = -ir \pmod{n}$. If $(k, n) = d$ for some positive integer d , then there exist integers s and t such that $s = \frac{n}{d}$ and $t = \frac{k}{d}$. If $s < n$, then we have flow solutions

$$f_s = sr \equiv sk = tn = 0 \pmod{n} \quad (7.23)$$

$$g_s = -sr \equiv -sk = -tn = 0 \pmod{n}. \quad (7.24)$$

Since $0 < f_i, g_i < n$, it must be the case that $s \geq n$, so $s = n$ i.e., $d = 1$. Hence k and n are relatively prime. $\square$

We now consider the existence of matchweb solutions given an n -fence core.

Theorem 7.4.4. *Let k and n be relatively prime, and let (7.12) through (7.14) hold. Then this defines a matchweb $\Lambda(X, Y)$ with $|Y| = k$ and an n -fence core.*

Proof. We will prove by showing (7.12) through (7.14) defines a positive regular flow on a bipartite tree $\Lambda(X, Y)$.

Let $\Lambda(X, Y)$ be a connected bipartite tree with an n -fence core and ℓ_i leaves attached to each $x_i \in X$ as defined above. Consider the flow into the vertices in Y . We can define the flow on the leaves incident to X to be n . For vertices of Y not incident to leaves, the flow is

$$f_i + g_i = f_i + n - f_i = n. \quad (7.25)$$

So the flow into vertices of Y is n .

Now consider the flow into vertices of X . We will consider the two cases, one, where Λ has “wrap-around”, and two, where Λ has no “wrap-around”.

Case 1:

Suppose the flow solutions of Λ have wrap-around, i.e.,

$$\begin{aligned} f_i &= f_{i-1} + r - n \\ g_i &= n - f_i \\ \ell_i &= q. \end{aligned} \quad (7.26)$$

Then, by (7.26), the flow into vertex x_i is

$$n\ell_i + g_{i-1} + f_i = n(q) + n - f_{i-1} + f_{i-1} + r - n = nq + r = k. \quad (7.27)$$

Hence the flow is regular at x_i .

Case 2:

Suppose the flow solutions of Λ do not have wrap-around, i.e.,

$$\begin{aligned} f_i &= f_{i-1} + r \\ g_i &= n - f_i \\ \ell_i &= q - 1 \end{aligned} \tag{7.28}$$

Then the flow into vertex x_i is

$$n\ell_i + g_{i-1} + f_i = n(q-1) + n - f_{i-1} + f_{i-1} + r = nq + r = k \tag{7.29}$$

from (7.28). The flow is regular at x_i , hence $\Lambda(X, Y)$ admits a regular flow. Finally, all the flows are positive and unique, so $\Lambda(X, Y)$ is a matchweb by Theorem 5.1.9. $\square$

CHAPTER 8

UNIQUENESS OF SOLUTIONS OF CORES

In this chapter we consider the final main results of this paper. We begin with a formal statement of the final results of matchwebs in Section 8.1, we continue in Sections 8.2 and 8.3 by building the necessary tools to prove the first main result, and in Section 8.4, we finish with a discussion of algebraic topology and build tools to prove the final result.

8.1 Main results

We begin this section with a statement of the main results of this report, and continue with the definitions necessary to prove those results.

The goal is to classify all matchwebs. After the results of Chapter 6, we reduced this problem to the enumeration and classification of core graphs and bases, and found the existence of solutions in three specific core examples in Chapter 7. We now must ask the following:

Question 8.1.1. If a given core graph $\Lambda^*(X, Y^*)$ has a matchweb solution, i.e., there exists a matchweb with a core Λ^* , is that solution unique?

The answer, and two main results of the chapter, are:

Theorem 8.1.2. *Any given core graph, $\Lambda^*(X, Y^*)$, is the core of at most one matchweb, $\Lambda(X, Y)$, with $|Y| = k$.*

Theorem 8.1.3. *Any core graph has at most one matchweb solution “mod n ”.*

Theorem 8.1.2 will be proven as Corollary 8.3.3, and Theorem 8.1.3 will be stated and proved precisely in Theorem 8.4.6. First, we need to make a few definitions and statements.

8.2 Quasi-cores

In order to prove uniqueness of matchwebs it is natural to investigate all possible solutions obtained from matchweb cores. However, our proof requires a more general structure, that of the quasi-core.

Definition 8.2.1. An (n, k) -quasi-core (Λ^*, S) is defined to be:

- (1) A core graph $\Lambda^*(X, Y^*)$ with vertex set $V = X \cup Y^*$.
- (2) Two positive integers n and k .
- (3) A function $S : V \rightarrow \mathbb{N}$ such that
  - (a) for all $x \in X$, $S(x) \leq k$,
  - (b) for all $y \in Y^*$, $S(y) = n$.

Our next step in the proof of matchweb uniqueness is to consider properties of quasi-cores and some results of those properties.

Definition 8.2.2. Given an (n, k) -quasi-core, we define an (n, k) -quasi-core solution of $\Lambda^*(X, Y^*)$ to be a pair of functions $f : E \rightarrow \{1, \dots, n-1\}$ and $\ell : V \rightarrow \mathbb{N} \cup \{0\}$

such that

$$n\ell(x) + \sum_{x \in \partial(e)} f(e) = S(x), \quad (8.1)$$

$$\sum_{y \in \partial(e)} f(e) = S(y). \quad (8.2)$$

Definition 8.2.3. For fixed $k \in \mathbb{N}$, we define the k -matchweb quasi-core of a core graph $\Lambda^*(X, Y^*)$ by taking $n = |X|$ and $S(x) = k$ for all $x \in X$.

The following theorem relates quasi-cores to matchweb cores.

Theorem 8.2.4. *Let $\Lambda^*(X, Y^*)$ be the core of a matchweb, $\Lambda(X, Y)$ with $|Y| = k$. Then the k -matchweb quasi-core of Λ^* has a solution.*

Proof. Let $\Lambda^*(X, Y^*)$ be the core of a matchweb $\Lambda(X, Y)$ with $|Y| = k$, and let E^* be the set of all edges of Λ^* . Then by Lemma 7.1.1, we know there exists a flow $f^* : E^* \rightarrow \{1, \dots, n-1\}$ such that for all $x \in X$,

$$n\ell(x) + \sum_{x \in \partial(e), e \in E^*} f^*(e) = k, \quad (8.3)$$

where ℓ is the number of leaves attached to x . Similarly, for all $y \in Y$,

$$\sum_{y \in \partial(e)} f^*(e) = n. \quad (8.4)$$

Considering $\Lambda^*(X, Y^*)$ as a core graph, we have a (n, k) -quasi-core solution with $|X| = n$ and $S(x) = k$, where $f(e) = f^*(e)$, the flow on edge e , and $\ell(x)$ is the number of leaves incident to vertex x . In other words $\Lambda^*(X, Y^*)$ has a k -matchweb quasi-core solution. $\square$

We prove the converse of Theorem 8.2.4 in Corollary 8.3.5.

8.3 Uniqueness

In this section we prove the uniqueness of matchweb solutions of core graphs using results of quasi-core graphs and solutions. We begin with a technical lemma, which we use for an inductive proof of Theorem 8.1.2.

Lemma 8.3.1. *Let $(\Lambda^*(X, Y^*), S)$ be a (n, k) -quasi-core with vertex set $V = X \cup Y^*$ and edge set E . Suppose Λ^* contains a $y^* \in Y^*$ such that $\Gamma(y^*) = \{x_1, x_2, \dots, x_m\}$ ($m \geq 2$) where $\text{valence}(x_m) \geq 1$ and $\text{valence}(x_i) = 1$. Let $e_i = \{x_i, y^*\}$. (See Figure 8.1.)*

Figure 8.1: The m -pod

For $1 \leq i \leq m-1$, we define f_i and ℓ_i uniquely by

$$S(x_i) = n\ell_i + f_i, \quad (8.5)$$

with $0 \leq f_i < n$. We also define

$$f_m = S(y^*) - \sum_{i=1}^{m-1} f_i = n - \sum_{i=1}^{m-1} f_i. \quad (8.6)$$

Let $(\Lambda_r^*(X_r, Y_r^*), S_r)$ be the (n, k) -quasi-core obtained by removing the m -pod from Λ^* (i.e., remove $x_1, \dots, x_{m-1}, y^*$ and their edges, leaving x_m), and defining S_r

$$S_r(x) = \begin{cases} S(x) & \text{when } x \neq x_m \\ S(x) - f_m & \text{when } x = x_m \end{cases} \quad (8.7)$$

and $S_r(y) = S(y)$ for all $y \in Y_r^*$. Let E_r be the edge set of $\Lambda_r^*(X_r, Y_r^*)$.

For any $f_r : E_r \rightarrow \{1, \dots, n-1\}$, we define $\widehat{f}_r : E \rightarrow \{1, \dots, n-1\}$ by

$$\widehat{f}_r = \begin{cases} f_r(e) & \text{when } e \in E_r, \\ f_i & \text{when } e = e_i, 1 \leq i \leq m, \end{cases}$$

and for any $\ell_r : X_r \rightarrow \mathbb{N} \cup \{0\}$, we define $\widehat{\ell}_r : X \rightarrow \mathbb{N} \cup \{0\}$ by,

$$\widehat{\ell}_r = \begin{cases} \ell_r(x) & \text{when } x \in X_r \\ \ell_i & \text{when } x = x_i, 1 \leq i \leq m-1, \end{cases}$$

since $x_m \in X_r$. Then we have the following:

(1) If the m -pod is unsolvable, i.e. $\ell_i < 0$, $f_i = 0$ ($1 \leq i \leq m-1$), $f_m \leq 0$, or $S(x_m) - f_m \leq 0$, then (Λ^*, S) has no solution.

(2) Suppose $0 < f_i < n$ for all $1 \leq i \leq m-1$ and $S(x) - f_m > 0$. Then:

(a) If (f, ℓ) is a solution to (Λ^*, S) , then $(f|_{E_r}, \ell|_{X_r})$ is a solution to (Λ_r^*, S_r) .

(b) If (f_r, ℓ_r) is a solution to (Λ_r^*, S_r) , then $(\widehat{f}_r, \widehat{\ell}_r)$ is a solution to (Λ^*, S) .

In particular, if (Λ^*, S) has a solution, then (Λ^*, S) and (Λ_r^*, S_r) have the same number of solutions.

Proof. (1)

Suppose the pod is unsolvable, i.e., there is an $\ell_i < 0$, $f_i = 0$ ($1 \leq i \leq m-1$), $f_m \leq 0$, or $S(x_m) - f_m \leq 0$. Also, suppose Λ^* has a solution, i.e. there are functions $f : E \rightarrow \{1, \dots, n-1\}$ and $\ell : V \rightarrow \mathbb{N} \cup \{0\}$ such that for all $x \in X$,

$$n\ell(x) + \sum_{x \in \partial(e)} f(e) = S(x), \quad (8.8)$$

and for all $y \in Y^*$,

$$\sum_{y \in \partial(e)} f(e) = S(y). \quad (8.9)$$

So we must have, for $1 \leq i \leq m-1$,

$$n\ell(x_i) + f(e_i) = S(x_i), \quad (8.10)$$

with $0 < f(e_i) < n$ and

$$f(e_1) + f(e_2) + \dots + f(e_m) = S(y^*), \quad (8.11)$$

with $0 < f(e_m) < n$. So $f(e_i) = f_i$, and $\ell(x_i) = \ell_i$ by the Division Algorithm, and $f(e_m) = f_m$, by (8.6) and (8.11). Hence $\ell_i \geq 0$, and $f_i > 0$, $1 \leq i \leq m$, by the definition of f and ℓ ; contradiction.

Suppose $S(x_m) - f_m \leq 0$. Then we have

$$S(x_m) = n\ell(x_m) + \sum_{x_m \in \partial(e)} f(e) = n\ell(x_m) + \sum_{x_m \in \partial(e), e \neq e_m} f(e) + f_m. \quad (8.12)$$

So we have

$$n\ell(x_m) + \sum_{x_m \in \partial(e), e \neq e_m} f(e) \leq 0. \quad (8.13)$$

So either $\ell(x_m) < 0$ or $f(e) \leq 0$ for some $e \in E$. For a solution to exist, $f : E \rightarrow \{1, \dots, n-1\}$ and $\ell : V \rightarrow \mathbb{N} \cup \{0\}$, so a solution does not exist on Λ^* .

So if the pod is unsolvable, the quasi-core $(\Lambda^*(X, Y^*), S)$ has no solution.

(2a)

Suppose Λ^* has a solution given by (f, ℓ) such that for all $x \in X$ and $y \in Y$,

$$n\ell(x) + \sum_{x \in \partial(e)} f(e) = S(x), \quad (8.14)$$

$$\sum_{y \in \partial(e)} f(e) = S(y). \quad (8.15)$$

Consider $(f|_{E_r}, \ell|_{X_r})$, i.e., (f, ℓ) restricted to Λ_r^* . Condition (8.15) is unchanged so we need only check (8.14). For $x \in X_r$ such that $x \neq x_m$, we have

$$n\ell|_{X_r}(x) + \sum_{x \in \partial(e)} f|_{E_r}(e) = n\ell(x) + \sum_{x \in \partial(e)} f(e) = S(x) = S_r(x), \quad (8.16)$$

by the definition of S_r .

For x_m we have

$$n\ell|_{X_r}(x_m) + \sum_{x_m \in \partial(e)} f|_{E_r}(e) = n\ell(x_m) + \sum_{x_m \in \partial(e), e \neq e_m} f(e) = S(x_m) - f_m = S_r(x_m). \quad (8.17)$$

So $(f|_{E_r}, \ell|_{X_r})$ is a solution on (Λ_r^*, S_r) .

(2b)

Suppose (Λ_r^*, S_r) has a solution given by (f_r, ℓ_r) . Then for all $x \in X_r$ and $y \in Y_r^*$,

$$n\ell_r(x) + \sum_{x \in \partial(e)} f_r(e) = S_r(x) \quad (8.18)$$

$$\sum_{y \in \partial(e)} f_r(e) = S_r(y). \quad (8.19)$$

Consider $(\widehat{f}, \widehat{\ell})$, i.e., (f_r, ℓ_r) extended to Λ^* . Since (8.18) and (8.19) give a solution for $x \notin \{x_1, \dots, x_m\}$, by (8.6), we only need to consider $x = x_i$, $1 \leq i \leq m$. For $1 \leq i \leq m-1$,

$$n\widehat{\ell}(x_i) + \sum_{x_i \in \partial(e)} \widehat{f}(e) = n\ell_i + f_i = S(x_i). \quad (8.20)$$

For x_m , we have

$$n\widehat{\ell}(x_m) + \sum_{x_m \in \partial(e)} \widehat{f}(e) = n\ell_r(x_m) + \sum_{x_m \in \partial(e), e \neq e_m} f_r(e) + f_m = S_r(x_m) + f_m = S(x_m). \quad (8.21)$$

So $(\widehat{f}, \widehat{\ell})$ gives a solution on (Λ^*, S) .

It follows that (Λ^*, S) and (Λ_r^*, S_r) have the same number of solutions. $\square$

We now can state and prove the main result of this section.

Theorem 8.3.2. *Let (Λ^*, S) be a (n, k) -quasi-core. Then S gives at most one solution.*

Proof. We will proceed by induction on $|X|$. For the base step consider $X = \{x\}$, $Y^* = \emptyset$. We know $S(x) \leq k$. If (Λ^*, S) has a solution, then there exist functions $f : E \rightarrow \{1, \dots, n-1\}$ and $\ell : X \rightarrow \mathbb{N} \cup \{0\}$ such that

$$n\ell(x) + \sum_{x \in \partial(e)} f(e) = S(x). \quad (8.22)$$

Since Λ^* has no edges, we have

$$n\ell(x) = S(x). \quad (8.23)$$

Since $\ell(x) \in \mathbb{N}$ we know that $n|S(x)$. Hence $\ell(x) = \frac{S(x)}{n}$ is the only possible solution.

For the inductive step, assume that for all (n, k) -quasi-cores $(\Lambda^*(X, Y^*), S)$ with $|X| < M$, (Λ^*, S) gives at most one solution. Let Λ^* be a (n, k) -quasi-core defined by a function S with $|X| = M$. We know by Theorem 6.2.1 that there exists a $y^* \in Y^*$ such that there is at most one $x \sim y$ such that $\text{valence } y > 1$. Let $\Gamma(y^*) = \{x_1, \dots, x_m\}$; then we may obtain a core Λ_r^* from $\Lambda^*(X, Y^*)$ that satisfies the hypotheses of Lemma 8.3.1. By the inductive hypothesis, (Λ_r^*, S_r) has at most one solution. By Lemma 8.3.1 either (Λ^*, S) has no solutions or (Λ^*, S) and (Λ_r^*, S_r) have the same number of solutions; hence (Λ^*, S) has at most one solution. $\square$

The following corollaries result from Theorem 8.3.2, Theorem 5.1.9, and Theorem 8.2.4.

Corollary 8.3.3 (Uniqueness.). *Two matchwebs $\Lambda_1(X, Y)$ and $\Lambda_2(X, Y)$ with the same core $\Lambda^*(X, Y^*)$ are isomorphic graphs.*

Proof. We know $\Lambda_1(X, Y)$ and $\Lambda_2(X, Y)$ both define a $|Y|$ -matchweb quasi-core solution of Λ^* . Since, by Theorem 8.3.2, the quasi-core solution of Λ^* is unique, specifically $\ell(x)$, the number of leaves at each $x \in X$, determines Λ_1 and Λ_2 . Hence $\Lambda_1(X, Y) = \Lambda_2(X, Y)$. $\square$

Corollary 8.3.4. *$\Lambda(X, Y)$ is a matchweb if and only if Λ is a bipartite tree with a positive regular flow.*

Proof. If $\Lambda(X, Y)$ is a connected normalized matching bipartite graph with a positive regular flow then, by Theorem 8.3.2, that regular flow solution is unique and, by Theorem 5.1.9, Λ is a matchweb. The secondary implication is given by Theorem 5.1.8. $\square$

Corollary 8.3.5. *[Converse of Theorem 8.2.4.] If $\Lambda^*(X, Y^*)$ has a k -matchweb quasi-core solution, then Λ^* is the core of a matchweb $\Lambda(X, Y)$ with $|Y| = k$.*

Proof. Suppose the k -matchweb quasi-core of $\Lambda^*(X, Y^*)$, with edge set E^* , has a solution, i.e., there exist functions $f : E^* \rightarrow \{1, \dots, n-1\}$ and $\ell : X \rightarrow \mathbb{N} \cup \{0\}$ such that for all $x \in X$

$$n\ell(x) + \sum_{x \in \partial(e)} f(e) = k, \quad (8.24)$$

and for all $y \in Y^*$

$$\sum_{y \in \partial(e)} f(e) = |X| = n. \quad (8.25)$$

Let $\Lambda(X, Y)$ be a bipartite tree obtained by attaching $\ell(x)$ leaves to each x in $\Lambda^*(X, Y^*)$. For a regular flow solution, the flow on the leaves must be n . So, if the flow on each edge is given by f , with 8.24, the flow sum into each $x \in X$ is k . Similarly, because the flow on each leaf is n , and by 8.25, the flow sum into each $y \in Y$ is n . Hence $\Lambda(X, Y)$ has a positive regular flow. So by Corollary 8.3.4, Λ is a matchweb.

So $\Lambda(X, Y)$ is a matchweb, and $\Lambda^*(X, Y^*)$ is the core of Λ . □

8.4 Algebraic topology and uniqueness mod n

In order to understand and prove uniqueness of matchweb solutions “mod n ” we must recall elementary theorems in algebraic topology. For the following material we follow Hsu [Hsu00]. We also use the following notation: $\Lambda(X, Y)$ is a directed bipartite tree (Figure 8.2) with vertex set $V = X \cup Y$, edge set E , and X to Y the positive orientation. We also let R be a commutative ring with unity.

Definition 8.4.1. A R -valued 0-chain is a function $S : V \rightarrow R$. We can take R -linear combinations of 0-chains, so the R -module of all 0-chains on Λ is denoted by $C_0(\Lambda)$.

Definition 8.4.2. A R -valued 1-chain is a function $f : E \rightarrow R$. The R -module of all

Figure 8.2: Directed bipartite tree

1-chains on Λ is denoted by $C_1(\Lambda)$. For $v \in V$, we define $S_v : V \rightarrow R$ by

$$S_v(w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w. \end{cases}$$

Definition 8.4.3. We define an R -module homomorphism

$$\partial : C_1(\Lambda) \rightarrow C_0(\Lambda) \tag{8.26}$$

such that for $\{x, y\} \in E$, $x \in X$, $y \in Y$,

$$\partial f_{\{x, y\}} = S_y - S_x. \tag{8.27}$$

(This is where “orientation” appears.)

Definition 8.4.4. A 1-cycle on Λ is an element of $\ker \partial$, i.e., $f \in C_1(\Lambda)$ such that $\partial f = 0$.

Definition 8.4.5. Given a directed bipartite tree Λ , and a 1-chain, f , we define the *flow sum* of Λ to be

$$\sum_{e \in E} f(e) \tag{8.28}$$

If f is a 1-cycle, then the flow sum of Λ is 0.

Theorem 8.4.6. Let $\Lambda(X, Y)$ be a directed bipartite tree with at least one edge. Then the only 1-cycle with coefficients in R is 0.

Figure 8.3: Directed edge e

Proof. We proceed by induction on $|V|$. For the base step, suppose $\Lambda(X, Y)$ has two vertices, x and y . (See Figure 8.3.)

Let f be a 1-cycle on Λ , that is,

$$\partial f(x) = 0 \tag{8.29}$$

$$\partial f(y) = 0. \tag{8.30}$$

But, for $e = \{x, y\}$, $\partial f(x) = -f(e) = 0$. So $f(e) = 0$, and the only 1-cycle is 0.

For the inductive step assume that for all directed bipartite trees with $|V| < m$, the only 1-cycle is 0. Let $\Lambda(X, Y)$ be a directed bipartite tree with $|V| = m$. Consider a leaf of Λ , given by $e = \{x, y\}$. We can assume without loss of generality that $\text{valence}(x) = 1$. (See Figure 8.4.) Suppose f is a 1-cycle on Λ . Then $\partial f(x) = 0$, but we know that $\partial f(x) = -f(e) = 0$. So $f(e)$ does not affect the flow sum of Λ , and we need only consider $f|_{\Lambda_p}$, the 1-cycle defined on Λ after pruning e .

By the inductive hypothesis, we know that any 1-cycle on Λ_p is 0, so $f|_{\Lambda_p} = 0$. Hence $f = 0$ on Λ , and the only 1-cycle is 0. $\square$

We are now able to give a precise statement of Theorem 8.1.3.

Theorem 8.4.7. *Suppose $\Lambda_1(X, Y_1)$ and $\Lambda_2(X, Y_2)$ are matchwebs with flows f_1 and*

Figure 8.4: $\{x, y\}$ and Λ_p

f_2 respectively, and both with the core $\Lambda^*(X, Y^*)$ ($Y^* \subseteq Y_1$ and $Y^* \subseteq Y_2$) and suppose $|Y_1| = |Y_2| = k \pmod{n}$. Then $f_1|_{Y^*} \equiv f_2|_{Y^*} \pmod{n}$.

Proof. Without loss of generality, we can assume $k \geq |Y_1|$ and $k \geq |Y_2|$. Let $(\Lambda^*(X, Y^*), S)$ be the k -matchweb quasi-core of $\Lambda^*(X, Y^*)$. Since Λ_1 and Λ_2 are matchwebs, let (f_1, ℓ_1) be a solution for the $|Y_1|$ -matchweb quasi-core of $(\Lambda^*(X, Y^*), S)$ and (f_2, ℓ_2) be a solution for the $|Y_2|$ -matchweb quasi-core of $(\Lambda^*(X, Y^*), S)$. Then, for all $x \in X$,

$$n\ell_1(x) + \sum_{x \in \partial(e)} f_1(e) = S(x) = n, \quad (8.31)$$

$$n\ell_2(x) + \sum_{x \in \partial(e)} f_2(e) = S(x) = n. \quad (8.32)$$

Also, for all $y \in Y^*$,

$$\sum_{y \in \partial(e)} f_1(e) = |Y_1| \equiv k \pmod{n} \quad (8.33)$$

$$\sum_{y \in \partial(e)} f_2(e) = |Y_2| \equiv k \pmod{n}. \quad (8.34)$$

So for all $x \in X$ and $y \in Y^*$, we get

$$\sum_{x \in \partial(e)} f_1(e) - f_2(e) \equiv 0 \pmod{n} \quad (8.35)$$

$$\sum_{y \in \partial(e)} f_1(e) - f_2(e) \equiv 0 \pmod{n}. \quad (8.36)$$

So, $f_1 - f_2$ is a $\mathbb{Z}_n$ -valued 1-cycle. By Theorem 8.4.6 we know $f_1 - f_2 \equiv 0 \pmod{n}$, so $f_1|_{Y^*} \equiv f_2|_{Y^*} \pmod{n}$. $\square$

CHAPTER 9

CONCLUSIONS AND OPEN PROBLEMS

9.1 Summary

Matchwebs are connected minimal normalized matching bipartite graphs. In this paper we have defined matchwebs based on definitions and theorems from graph theory, abstract algebra, set theory, and combinatorics. Matchwebs occur naturally, embedded in the graphical representation of normalized matching posets. We discussed the background and motivations behind matchweb study, building our formal definition of a matchweb.

In our rigorous investigation of the structure and behavior of matchwebs we determined that matchwebs are unique given $n = |X|$, $k = |Y|$, and a core, $\Lambda^*(X, Y^*)$, obtained by removing leaves incident to X from $\Lambda(X, Y)$. We began our classification by investigating all matchwebs with $|X| = 1$ and $|X| = 2$. In the case of $n = 1$, we found the only matchweb is the fan graph, $K_{1,k}$. In the case of $n = 2$ we found that a normalized matching bipartite graph with $|X| = 2$ is a matchweb if and only if $|Y| = 2m + 1$ for $m \in \mathbb{Z}$; in the case that $|Y| = 2m$, $\Lambda(X, Y)$ is made up of two fan graphs, $K_{1,m}$. We also found that if $\Lambda(X, Y)$ is a minimal normalized matching bipartite graph with $|Y| = m|X|$, for $m \in \mathbb{Z}$, then Λ is the disjoint union of m -fans. Though in this case Λ is not a matchweb, it is a disjoint union of matchwebs.

We continued our classification of all matchwebs given $|X|$, by defining the underlying structure of matchwebs, cores and bases. Since classifying core graphs amounts to the classification of bases, we were able to reduce the problem of core graph classification to the enumeration of all bases given $|X|$. We then defined the systematic process for enumerating all bases, and listed the bases up to order 8.

Using our understanding of normalized matching bipartite graphs, we began our general matchweb classification by classifying all matchwebs with the n -pod, n -spider, and n -fence cores. We found that $\Lambda(X, Y)$, with $|X| = n$ and $|Y| = k$, is a matchweb with an n -pod core if and only if, $k \equiv 1 \pmod{n}$, and the number of leaves incident to each $x \in X$ is the same. Next, we found that $\Lambda(X, Y)$ is a matchweb with a n -spider core with any value of k given that the number of leaves incident to X is either q , the quotient found by the Division Algorithm on k and n , or $q - n + r + 1$, where r is the remainder found by the Division Algorithm. Finally, we found that $\Lambda(X, Y)$ is a matchweb with an n -fence core if and only if $(k, n) = 1$, meaning k and n are relatively prime, and the number of leaves incident to each $x \in X$ is q or $q - 1$, given by the Division Algorithm on k and n .

Finally, we stated our two main results of this paper: uniqueness of matchweb solutions given $n = |X|$, $k = |Y|$, and a core graph $\Lambda^*(X, Y^*)$, and uniqueness of matchweb solutions “mod n ”, or unique classes of matchwebs, given $n = |X|$, $|Y| \pmod{n}$, and a matchweb core. Using the concept of a quasi-core and properties from algebraic topology, we proved our final results of uniqueness.

9.2 Conjectures and open problems

We will close this thesis with some open problems with matchwebs and further areas of research.

Question 9.2.1. For fixed n , for which classes of $k \pmod n$ is there a matchweb $\Lambda(X, Y)$ with $|X| = n$, $|Y| = k$, and a given core, $\Lambda^*(X, Y^*)$?

Conjecture 9.2.2. *Any matchweb can be obtained by cyclic reduction from a complete bipartite graph $\Lambda(X, Y) = K_{k,n}$.*

Question 9.2.3 (Surgery Theorem). Are there similar patterns $\pmod n$, $(n, k) = 1$, etc.) for matchweb solutions with different cores? In other words, cores that are the union of different sized pods, spiders, and fences.

Conjecture 9.2.4 (Existence Theorem). *Given a core graph $\Lambda^*(X, Y^*)$, there exist $n, k \in \mathbb{N}$, such that Λ has a k -matchweb solution.*

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