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# ADDING A CLUB SUBSET OF $\omega_2$ WITHOUT COLLAPSING EITHER $\omega_1$ OR $\omega_2$

A Thesis

Presented to

The Faculty of the Department of Mathematics

San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Ivan G. Zaigralin

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ABSTRACT

ADDING A CLUB SUBSET OF $\omega_2$ WITHOUT COLLAPSING EITHER $\omega_1$ OR $\omega_2$

by Ivan Zaigralin

The goal is to provide a characterization of sets which have club subsets in cardinal preserving generic extensions. Several results are to be presented and compared, the major cases being $\omega_1$ and $\omega_2$ preserving extensions, as well as different approaches to forcing. Where possible, a constructible forcing will be described.

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## 1. Introduction

Here we will bring together and relate several results concerning the characterization of sets which have club subsets in some $\omega_1$ and $\omega_2$ preserving extension. These results are:

(1) If there exists a disjoint club sequence on $A \subseteq (\omega_2 \cap \operatorname{cof} \omega_1)$, then $A$ does not have a club subset in any $\omega_1$ and $\omega_2$ preserving outer model.

(2) If $A \subseteq \omega_1$ is stationary, then $A$ has a club subset in an $\omega_1$ preserving outer model with no new $\omega$-sequences of ordinals. If the CH holds in the inner model, then $\omega_2$ is preserved in the outer model.

(3) If $A \subseteq \omega_1$ is stationary, then $A$ has a club subset in an $\omega_1$ an $\omega_2$ preserving outer model (even if CH is false in the inner model). New $\omega$-sequences of ordinals are added.

(4) If $A \subseteq \kappa$ is fat stationary, then under suitable cardinal arithmetic hypotheses, there exists an outer model in which $A$ has a club subset and all cardinals are preserved.

(5) If $\omega_2$ is the $L$-successor of $\omega_1$ and $D \subseteq \omega_2$ is very stationary, with a witness in $L$, then there exists a constructible forcing that adds a club subset to $\omega_2$ and preserves all cardinals.

The two major cases we will have to deal with will be $\omega_1$ and $\omega_2$ case. As for the first one, our results will imply that the following statements are equivalent:

(1) $A \subseteq \omega_1$ has a club subset in an $\omega_1$-preserving outer model.

(2) $A \subseteq \omega_1$ has a club subset in a set forcing extension.

(3) $A \subseteq \omega_1$ is stationary.

The first statement is not first-order, but the other two are.

The $\omega_2$ case is different. The result due to Friedman (Theorem 6.1) is based on the following result.

**Definition 1.1.** Say that $A \subseteq \omega_2$ is *satiable* if $A$ has a club subset in some $\omega_1$ and $\omega_2$ preserving outer model.

**Theorem 1.2** (Anticharacterization).

(1) *Let $L_{min}$ be the minimum model of ZFC, that is, let $L_{min}$ be standard transitive universe satisfying ZFC $+ V = L +$ "there are no standard models of ZFC". Then*

$$\{S \in L_{min} \mid$$
$$S \subseteq \omega_2^{L_{min}} \text{ is satiable with respect to outer models of } L_{min}\} \notin L_{min}.$$

(2) *If $V$ is sufficiently non-minimal, then*

$$\{S \in V \mid$$
$$S \subseteq \omega_2 \text{ is satiable with respect to outer models of } V\} \notin V.$$

(3) *Given any countable standard transitive model $V$ of ZFC, there exists a definably class generic outer model $V'$ of $V$ such that*

$$\{S \in V' \mid$$
$$S \subseteq \omega_2^{V'} \text{ is satiable with respect to outer models of } V'\} \notin V'.$$

*Definably class generic* here means that there exists a $V$-definable class forcing property $P$ and a filter $G$ on $P$ meeting every dense class definable over $V$ and such that $V' = V[G]$. *Sufficiently non-minimal* means that there exists a $V$-inaccessible cardinal $\kappa$ and a $\Sigma_2$-elementary $j : \mathrm{Hyp}(V_\kappa) \to \mathrm{Hyp}(V)$ with critical point $\kappa$ and such that $j(V_\kappa) = V$. $\mathrm{Hyp}(X)$ is the smallest admissible set with $X$ as an element. The theorem appears in the work by M.C. Stanley [8].

This theorem implies that for models of ZFC, there cannot be a uniform, parameter free first-order equivalent to the second-order property "$A \subseteq \omega_2$ has a club subset in some $\omega_1$ and $\omega_2$ preserving outer model." Our motivation, then, is to say as much as we can about the satiable subsets of $\omega_2$, even though a first-order characterization relative to ZFC is impossible.

## 2. Forcing Conditions And Generic Sets

Here we will define the notion of a generic set and discuss how these can be added to transitive models to obtain generic extensions. Let us start by giving some definitions.

**Definition 2.1.** A set $X$ is *transitive* whenever

$$\forall x \in X \ \forall y \in x (y \in X)$$

**Definition 2.2.** A set $P$ together with a binary relation $\leqslant$ is *partially ordered* (or a *poset*) if the order relation $\leqslant$ is antisymmetric, reflexive, and transitive.

Let $\mathfrak{M}$ be an arbitrary transitive model of ZFC, which we will call the *ground model*, and consider a non-empty poset $(P, <) \in M$. We will call $P$ a *notion of forcing* and its elements—the *forcing conditions*. We will say that $p$ *is stronger than* $q$ iff $p < q$. We say that $p$ and $q$ are *compatible* whenever there exists $r$ such that $r \leqslant p$ and $r \leqslant q$, *incompatible* otherwise. A subset $W$ of $P$ is *incompatible* when its elements are pairwise incompatible. We say that a subset $D$ of $P$ is *dense* if $\forall p \in P \ \exists d \in D (d \leqslant p)$.

Moreover, a subset $F$ of $P$ is a *filter on* $P$ if the following conditions are satisfied:

(1) $F \neq \varnothing$
(2) $p \leqslant q \wedge p \in F \rightarrow q \in F$
(3) $p, q \in F \rightarrow \exists r \in F (r \leqslant p \wedge r \leqslant q)$

A subset $G$ of $P$ is *generic over* $\mathfrak{M}$ if

(1) $G$ is a filter on $P$
(2) If $D$ is dense in $P$ and $D \in \mathfrak{M}$, then $G \cap D \neq \varnothing$

A subset $D$ of $P$ is *dense below* $p$ if for every $p' \leqslant p$ there is $q \in D$ such that $q \leqslant p'$. A subset $U$ of $P$ is *open* if $p \in U \wedge q < p \rightarrow q \in U$. Finally, a subset $C$ of $P$ is *predense* if every $p \in P$ is compatible with some $q \in C$. A subset $W$ of $P$ is an *antichain* in $P$ if it is predense in $P$ and if any two distinct $p, q \in W$ are incompatible.

*Example* 2.3. This example, due to Jech [1], will make it clear how adding a set leads to collapsing of $\omega_1$. Let $\mathfrak{M}$ be the ground model. We want to add a set $G$ to $\mathfrak{M}$ to enlarge $\mathfrak{M}$ in such a way that there exists a mapping in the extension $\mathfrak{M}[G]$ of $\omega$ onto $\omega_1^{\mathfrak{M}}$. (When such a mapping exists, we say that $\omega_1$ has "collapsed".) Suppose that a mapping exists with the required property, call it $f$. For each $n < \omega$ the initial segment $f \restriction n$ is a finite sequence $(f(0), f(1), \ldots, f(n-1))$, and it lies in $\mathfrak{M}$. Let $P$ be the set of all finite sequences $(\alpha_0, \ldots, \alpha_n)$ of countable ordinals. The set $P$ will be our notion of forcing, and its elements—the forcing conditions.

Now consider two elements of $P$, $p$ and $q$. If $p \subset q$, then, in a way, $q$ gives us more information about $f$, and so we are justified in saying that $q$ is *stronger* than $p$. By convention, we write it as $q < p$. That is,

$$q < p \iff p \subset q \tag{1}$$

We can see that this $<$ relation induces a partial ordering on $P$. Note also that both $P$ and $(P, <)$ are in $\mathfrak{M}$.

Next, consider $G = \{f \restriction n \mid n \in \omega\}$. $G \subset P$, and we can recover $f$ from $G$. Namely, $f = \bigcup G$, and therefore $G$ cannot be a set in $\mathfrak{M}$, on pain of having a surjective map from $\omega$ to $\omega_1$ in $\mathfrak{M}$. In order for $\bigcup G$ to be a surjective map from $\omega$ to $\omega_1^{\mathfrak{M}}$, $G$ must have the following properties:

(1) If $p, q \in G$, then $p$ and $q$ are compatible, i.e. $p(k) = q(k)$ for all $k \in \operatorname{dom} p \cap \operatorname{dom} q$.
(2) $\forall n \in \omega \; \exists p \in G \; (n \in \operatorname{dom} p)$
(3) $\forall \alpha \in \omega_1^{\mathfrak{M}} \; \exists p \in G \; (\alpha \in \operatorname{ran} p)$

We now forget about $f$ and let the set of conditions $G$ be generic over $\mathfrak{M}$. Since $G$ is a filter, $f = \bigcup G$ must be a function. We can also see that for all $n \in \omega$ the set

$$D_n = \{p \in P \mid n \in \operatorname{dom} p\} \tag{2}$$

is dense in $P$. Similarly, for all $\alpha \in \omega_1^{\mathfrak{M}}$ the set

$$E_\alpha = \{p \in P \mid \alpha \in \operatorname{ran} p\} \tag{3}$$

is dense in $P$. Therefore $G \cap D_n \neq \varnothing$ for all $n$ and $G \cap E_\alpha \neq \varnothing$ for all $\alpha$, and so $\operatorname{dom} f = \omega$ while $\operatorname{ran} f = \omega_1^{\mathfrak{M}}$. We can conclude that every transitive model $\mathfrak{N}$ extending $\mathfrak{M}$ and containing $G$ satisfies that $\omega_1^{\mathfrak{M}}$ is a countable ordinal.

To generalize the previous example, we can prove a lemma, also appearing in Jech's work [1], which gives us a simple way of collapsing an arbitrary cardinal.

**Definition 2.4.** If $\kappa$ is a regular cardinal, we will write $\lambda^{<\kappa}$ to denote $\sup\{\lambda^\alpha \mid \alpha < \kappa\}$

*Lemma* 2.5. Let $\kappa$ be a regular cardinal and let $\lambda > \kappa$ be an arbitrary cardinal. There exists a notion of forcing $(P, <)$ such that it collapses $\lambda$ onto $\kappa$, that is, $\lambda$ has cardinality $\kappa$ in the generic extension. Moreover,

(1) every cardinal $\alpha \leqslant \kappa$ in the ground model $\mathfrak{M}$ remains a cardinal in $\mathfrak{M}[G]$, and
(2) if $\lambda^{<\kappa} = \lambda$, then every cardinal $\alpha > \lambda$ remains a cardinal in the extension.

*Proof.* Let $P$ be the set of all functions $p$ such that

(1) $\operatorname{dom} p \subseteq \kappa$ and $|\operatorname{dom} p| < \kappa$,
(2) $\operatorname{ran} p \subseteq \lambda$,

and let $p < q$ iff $p \supset q$, as before. Let $G$ be a generic filter on $P$ and let $f = \bigcup G$. As in the example above, $f$ is a function, and it maps $\kappa$ onto $\lambda$. This is true because for all $\alpha < \kappa$ and all $\beta < \lambda$ the sets $D_\alpha = \{p \mid \alpha \in \operatorname{dom} p\}$ and $E_\beta = \{p \mid \beta \in \operatorname{ran} p\}$ are dense.

In order to show (1) and (2) we observe that $(P, <)$ is $\alpha$-closed for every $\alpha < \kappa$, and therefore all cardinals $\leqslant \kappa$ are preserved; and if $\lambda^{<\kappa} = \lambda$, then $|P| = \lambda^+$ and it follows that all cardinals above $\lambda$ are preserved. □

## 3. Forcing With Finite Conditions: $\omega_1$ Case

There are two theorems in this section. Theorem (3.4) adds a club subset to a stationary subset of $\omega_1$ using countable conditions. The proof is the precursor to theorems (4.3) and (4.4), which generalize it. This theorem is due to Baumgartner, Harrington, and Kleinberg [3] and independently to Jensen. The second theorem in this section, adding a club subset with finite conditions, is due to Baumgartner.

We start out by laying down some groundwork for discussing the preservation of $\omega_1$.

**Definition 3.1.** The set $D$ is said to be *predense* below $p$ if every condition below $p$ is compatible with an element of $D$.

*Lemma* 3.2. Suppose that for every $p$ and $D_i$, $i < \omega$, where each $D_i$ is predense below $p$, there are $q \leqslant p$ and countable $d_i \subseteq D_i$, $i < \omega$, where $d_i$ is predense below $q$ for each $i < \omega$. Then $\omega_1$ is preserved.

*Proof.* Assume that there is some $p$ which forces $\sigma$ to be a surjective map from $\omega$ to $\omega_1$. Then we can consider

$$D_i = \{q \mid \text{there exists } \alpha < \omega_1 \text{ such that } q \text{ forces } \sigma(i) = \alpha\} \tag{4}$$

$D_i$ is dense, and hence predense. Let $d_i \subseteq D_i$ and $q \leqslant p$ be such that $d_i$ is countable and predense below $p$. Let

$$\beta_i = \sup\{\alpha \mid \exists r \in d_i\ (r \Vdash \sigma(i) = \alpha)\}$$

Then $\beta_i < \omega_1$. Set $\beta = \sup_{i<\omega} \beta_i$. Then $\beta < \omega_1$ and $q \Vdash \sigma(i) < \beta$, for all $i < \omega$. It follows that there is $q \leqslant p$ and a countable $\beta$ such that $q$ forces $\sigma(i) < \beta$ for each $i < \omega$, that is, $q$ forces that $\sigma$ is bounded, which is a contradiction. □

Now consider a theorem which appears in Baumgartner [3].

**Definition 3.3.** A set $S \subseteq \mathscr{F}_{\omega_1}(X)$ is *stationary* if for each club $C \subseteq \mathscr{F}_{\omega_1}(X)$ the intersection $S \cap C$ is nonempty.

**Theorem 3.4.** *Let $X$ be a subset of $\omega_1$. Then the following conditions are equivalent:*

(1) *There exists an outer model which preserves $\omega_1$ and $X$ has a club subset in that model.*

(2) *$X$ is stationary.*

*Proof.* (1) implies (2) because any two club sets have a non-empty intersection. Conversely, consider the forcing $P$ with its conditions being closed, countable subsets of $X$, ordered by end-extension. Then $P$ adds a club subset to $X$. We must show that $\omega_1$ is preserved. To that end, suppose that $\langle D_i \mid i < \omega \rangle$ is a sequence of sets, each $D_i$ predense below $p$. Choose a continuous elementary chain $\langle M_j \mid j < \omega_1 \rangle$, where each $M_j$ is a countable elementary submodel of $H_\theta$, $\theta$ being sufficiently large and regular, so that $X$, $p$, and $\langle D_i \mid i < \omega \rangle$ belong to $M_0$ and $M_j \in M_{j+1}$ for each $j$. Since $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is club, we can choose $j$ in such a way that $\alpha = M_j \cap \omega_1 \in X$. Moreover, since each $D_i \cap M_j$ is predense below $p$ on $P \cap M_j$, we can choose $p = p_0 \geqslant p_1 \geqslant p_2 \geqslant \dots$ so that $p_{i+1} \in M_j$ and it extends some $r_i \in D_i$ for each $i < \omega$, while at the same time $\sup\left(\bigcup_i p_i\right) = \alpha$. Then $q = \bigcup_i p_i \cup \{\alpha\}$ is a condition which extends $p$, and for each $i$, $d_i = D_i \cap M_j$ is predense below $q$. That gives us the statement of lemma (3.2), which implies that $\omega_1$ is preserved. □

At that point, Baumgartner asked a question: if $X$ is a constructible, stationary subset of $\omega_1$, then is there a constructible forcing $P$ which preserves $\omega_1$ and adds a club subset to $X$? The following theorem, due to Baumgartner, answers this question. (The proof also appears in Friedman's paper [6].)

**Theorem 3.5.** *Let $X$ be a stationary subset of $\omega_1$. Then there exists a forcing $P$ which adds a club subset to $X$, such that $P$ preserves $\omega_1$ and $P$ belongs to $L[X]$.*

*Proof.* We can add a club subset to $X$ with finite conditions, using the technique described by Uri Abraham [5]. A condition is a finite set $p$ of disjoint closed intervals in $\omega_1$, whose left endpoints belong to $X$. Thus we allow the one-point intervals $[\alpha, \alpha]$, $\alpha \in X$. As we did before, we say that a condition $q$ *extends* $p$ iff $q \supseteq p$.

For a generic $G$, we say that $C_G$ is the set of all left endpoints of intervals in $\bigcup G$. $C_G$ must be an unbounded subset of $X$. Each countable ordinal either is a member of some interval in $G$ or fails to be a limit point in $X$. It follows that $C_G$ is closed in $X$. It remains to show that $\omega_1$ is preserved.

So suppose that $p$ is a condition and $D_i$, $i < \omega$ are predense below $p$. Choose a continuous elementary chain $\langle M_j \mid j < \omega_1 \rangle$ of countable elementary submodels of $H_\theta$, $\theta$ being large and regular, so that $X$, $p$, and $\langle D_i \mid i < \omega \rangle$ all belong to $M_0$ and $M_j \in M_{j+1}$ for all $j$. Since $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is club, we can choose $j$ in such a way that $\alpha = M_j \cap \omega_1$ is in $X$. Let $q$ be the condition $p \cup \{[\alpha, \alpha]\}$. If $r$ extends $q$ and $r_0 = r \restriction \alpha$ then every extension $s_0$ of $r_0$ in $P \cap M_j$ is compatible with $r$, since $[\alpha, \alpha] \in q$. It follows that $d_i = D_i \cap M_j$ is predense below $q$ for every $i$, because if $r < q$ then we can choose $s_0 \leqslant r_0$ such that it extends a condition in $d_i$. Now, since $s_0$ is compatible with $r$, $r$ must be compatible with some element of $d_i$. Hence $\omega_1$ is preserved. □

## 4. Forcing Club Sets

In this section we will consider some positive results due to Abraham and Shelah [5]. Specifically, we will see that given an arbitrary uncountable cardinal $\kappa$ and a stationary $S \subseteq \kappa$, we may be able to find a generic extension which adds a club subset to $S$ without collapsing any cardinals $\leqslant \kappa$. The sufficient condition for such an extension to be found is that $S$ has to be fat. Moreover, the proofs of the following two theorems only work if GCH is assumed; for if $2^{\aleph_0} > \aleph_1$, then our forcing posets will cause $\aleph_2$ to collapse. The following theorem, due to Abraham, establishes the first interesting result.

**Definition 4.1.** A stationary set $S \subseteq \kappa$ is *fat* if for every club $C \subseteq \kappa$, $S \cap C$ contains closed subsets with arbitrarily large order types less than $\kappa$.

**Definition 4.2.** A cardinal $\kappa$ is said to be *strongly inaccessible* if it is an uncountable, regular strong limit cardinal. We say that $\kappa$ is a *strong limit* cardinal if it cannot be obtained through the power-set operation, i.e., for all $\lambda < \kappa$, $2^\lambda < \kappa$.

**Theorem 4.3.** *Let $\kappa$ be either a strongly inaccessible cardinal or the successor of a regular cardinal $\mu$ such that $\mu = \mu^{<\mu}$, and let $S \subseteq \kappa$ be fat. Then there is a poset $P$ such that*

(1) *Forcing with $P$ adds a club $C \subseteq S$.*
(2) *Forcing with $P$ preserves all cardinals $\leqslant \kappa$.*
(3) *Cardinality of $P$ is $2^{<\kappa}$, and hence if $2^{<\kappa} = \kappa$, then all cardinals are preserved in the extension.*

*Proof.* For a given fat $S \subseteq \kappa$ we define $P = \{p \mid p \subseteq S$ is a closed and bounded set of ordinals$\}$. $P$ is partially ordered by end-extensions, namely,

$$p \leqslant p' \iff p = p' \cap (\sup(p) + 1)$$

The cardinality of $P$ is $2^{<\kappa}$, and if $G$ is a $V$-generic filter over $P$ then $C = \bigcup\{p \mid p \in G\}$ is a club subset of $S$. It remains to show that our forcing does not cause any new sets with cardinality $< \kappa$ to appear in $V[G]$. We must prove that, given a regular cardinal $\tau < \kappa$ and a sequence $D = \langle D_i \mid i \in \tau \rangle$ of dense open subsets of $P$, it holds that $\bigcap_{i<\tau} D_i$ is dense in $P$.

Fix $p$ and choose $\lambda$ so that $H(\lambda)$, which is the collection of all sets with hereditary cardinality $< \lambda$, contains $P$. Let $M = \langle H(\lambda), \in \rangle$. Define a sequence $\langle M_\alpha \mid \alpha < \kappa \rangle$ of elementary substructures of $M$ as follows:

(1) $P$, $p$, $D \in M_0$, and some fixed well-ordering of $|P|$ are in $M_0$. In other words, $M_0$ contains the universe of $P$.

(2) $\tau + 1 \in M_0$.

(3) $M_\alpha$ is of cardinality $< \kappa$, and if $\alpha < \beta$ then $M_\alpha \subset M_\beta$ and for limit $\delta$, $M_\delta = \bigcup_{\eta<\delta} M_\eta$.

(4) The intersection of the universe of $M_\alpha$ with $\kappa$, $c_\alpha = M_\alpha \cap \kappa$ is an ordinal and $\langle c_\alpha \mid \alpha < \kappa \rangle$ is a continuous increasing sequence, cofinal in $\kappa$.

To define $M_\alpha$, we recall that it was assumed that $\beta^{<\beta} < \kappa$ for all $\beta < \kappa$. Hence for $\beta < \alpha < \kappa$, $M_\alpha$ contains each subset of $\beta$ of cardinality $< |\beta|$.

Observe that $E = \{\alpha \mid \alpha = c_\alpha\}$ is a club subset of $\kappa$. Since $S$ is fat, $S \cap E$ contains a closed subset of order-type $\tau + 1$, call it $A$. Let $\alpha = \sup A$. It follows that $A \cap \xi \in M_\alpha$ for each $\xi < \alpha$. We can construct in $M_\alpha$ and increasing sequence in $P$, $\langle p_i \mid i < \tau \rangle$, of length $\tau$, such that $p_{i+1} \in D_i \cap M_\alpha$. We begin with $p = p_0$. If $p_i \in P \cap M_\alpha$ is defined then $p_{i+1}$ is the first member of $D_i$ such that the ordinal interval $(\sup p_i, \sup p_{i+1})$ has a non-empty intersection with $A$. For a limit $\delta < \tau$ we let

$$p_\delta = \bigcup_{i<\delta} p_i \cup \left(\sup\left(\bigcup_{i<\delta} p_i\right)\right)$$

Only a proper initial segment of $A$ is used in the definition of $p_\delta$, and hence $p_\delta \in M_\alpha$; and since $A \subseteq S$ is closed, $p_\delta \subseteq S$. Finally,

$$p_\tau = \bigcup_{i<\tau} p_i \cup \{\alpha\} \in \bigcap_{i<\tau} D_i$$

shows that $\bigcap_{i<\tau} D_i$ is dense in $P$. □

Difficulties will arise if we try to adapt the proof of Theorem (4.3) to the case where $\kappa = \mu^+$, $\mu$ singular. E.g., consider a fat set $S \subseteq \aleph_\omega^+$. If we follow the proof of the theorem above and define structures $M_\alpha$ of cardinality $\aleph_\omega$ then we cannot conclude that $M_\alpha$ are closed under countable unions of $p \in P$, since $\aleph_\omega^{\aleph_0} > \aleph_\omega$. We encounter the same problem in the case when $\tau^{<\tau} \geqslant \kappa$ for some $\tau < \kappa$. There is, however, an interesting result for $\kappa = \mu^+$, where $\mu$ is a singular strong limit. The following theorem, also due to Abraham [5], provides an example for $\kappa = \aleph_\omega^+$.

**Theorem 4.4.** *Suppose that $\aleph_\omega$ is strong limit and $S \subseteq \aleph_\omega^+$ is fat. Then there is a forcing poset $P$ of cardinality $2^{\aleph_\omega}$ which adds a club subset to $S$ without collapsing any cardinals $\leqslant \aleph_\omega$.*

*Proof.* Let $S$ be fat, $S \subseteq \aleph_\omega^+$. Without loss of generality, assume that $S \cap \aleph_\omega = \varnothing$. As we did in the proof of Theorem (4.3), define $P$ to be the set of all bounded closed subsets of $S$. As before, we must prove that no new sets of size $\leqslant \aleph_\omega$ are added after forcing with $P$. It suffices to show that for all $n \leqslant \omega$ the intersection of $\aleph_n$ many dense open subsets in $P$ is dense. Fix a sequence of dense open sets $\langle D_i \mid i < \aleph_n \rangle$, as well as a condition $p \in P$. We will find an extension of $p$ in $\bigcap_{i<\aleph_n} D_i$. Let $F(x,y)$ be a function such that for $\aleph_\omega \leqslant \alpha < \aleph_\omega^+$ we have

$$K : \beta \mapsto F(\alpha, \beta),\ \beta < \alpha$$

is a bijection, $K(\alpha) = \aleph_\omega$. Next, pick a sequence $\langle M_\alpha \mid \alpha < \aleph_\omega^+ \rangle$ of structures of cardinality $\aleph_\omega$, just like in Theorem (4.3), but require also that $F \in M_0$ and $\aleph_\omega \subset M_0$. Let

$c_\alpha = M_\alpha \cap \aleph_\omega^+$ and let $C = \{\alpha \mid \alpha = c_\alpha\}$ ($C$ is club). Suppose that $2^{\aleph_{n-1}} = \lambda < \aleph_\omega$; then we can use the fact that $S$ is fat to obtain a closed $B \subset S \cap C$ of order-type $\lambda^+$. Define a function $h : [B]^2 \to \omega$ in the following way: for all $a, b \in B$, $a < b$, let $h(a,b) = k$ iff $k$ is the least integer such that $F(b,a) \in \aleph_k$. Using the partition relation (described in detail by Williams [9]),

$$(2^{\aleph_{n-1}})^+ \to (\aleph_n)^2_{\aleph_{n-1}}$$

find $A \subseteq B$ of order-type $\aleph_n$ which is homogeneous for some color $k$. Let $\alpha = \sup A$. Now, construct an increasing sequence $\langle p_i \mid i < \aleph_n \rangle$, just like in the proof of Theorem (4.3). Since every ordinal which is a limit in $A$ is a member of $S$, $p_\delta \in P$ for limit $\delta < \aleph_n$. To make sure that $p_\delta \in M_\alpha$, we will show that every bounded subset $X \subseteq A$ is in $M_\alpha$. Pick $b \in A$, $b > x$ for all $x \in X$. Then $F(b,x) < \aleph_k$ for all $x \in X$. The set $\{F(b,x) \mid x \in X,\ x < b\}$ is a subset of $\aleph_k$, and so it must be also in $M_\alpha$. It follows that $X \in M_\alpha$. The proof can now be completed just like that of Theorem (4.3). □

*Remark* 4.5. In respect to a forcing which preserves the cardinals and adds a club subset to a fat $S \in \kappa$, not much can be said if the GCH is not assumed. Baumgartner [3] provides a positive answer for $\kappa = \aleph_1$, but for $\kappa = \aleph_2$ the question is still unanswered.

## 5. Disjoint Club Sequences and Fat Stationary Sets

In this section we will explore a more refined result due to Friedman and Krueger [2], which is related to the existence of disjoint club sequences. First, we need more definitions. For a set $X$ which contains $\omega_1$, $\mathscr{P}_{\omega_1}(X)$ will denote the collection of countable subsets of $X$.

**Definition 5.1.** A subset $C$ is of $\mathscr{P}_{\omega_1}(X)$ is *cofinal* if, given any $x \in \mathscr{P}_{\omega_1}(X)$ there exists $z \in C$ such that $x \subseteq z$.

**Definition 5.2.** A set $C \subseteq \mathscr{P}_{\omega_1}(X)$ is *club* if it is closed under unions of countable increasing sequences and is cofinal in $\mathscr{P}_{\omega_1}(X)$.

**Definition 5.3.** If $\kappa$ is a regular cardinal let $\operatorname{cof}\kappa$ (and $\operatorname{cof}(<\kappa)$ respectively) denote the class of ordinals with cofinality $\kappa$ (and cofinality less than $\kappa$ respectively).

If $\kappa$ is a successor of a regular uncountable cardinal $\mu$, this is equivalent to the statement that for every club $C \subseteq \kappa$, $S \cap C$ contains a closed subset with order type $\mu + 1$. In particular, if $A \subseteq \kappa^+ \cap \operatorname{cof}\mu$ is stationary then $A \cup \operatorname{cof}(<\mu)$ is fat.

**Definition 5.4.** If $V$ is a transitive model of ZFC, we will say that $W$ is an *outer model* of $V$ if $W$ is a transitive model of ZFC such that $V \subseteq W$ and $W$ has the same ordinals as $V$. Moreover, $(W, V) \models$ ZFC, i.e. $W$ satisfies instances of replacement and separation that are formulated in a language with a predicate symbol in the inner model $V$.

**Definition 5.5.** Let $T$ be a cofinal subset of $\mathscr{P}_{\omega_1}(\omega_2)$. We say that $T$ is *thin* if for all $\beta < \omega_2$ the set $\{a \cap \beta \mid a \in T\}$ has size less than $\omega_2$.

In particular, if CH holds, then $\mathscr{P}_{\omega_1}(\omega_2)$ itself is thin.

**Definition 5.6** (Martin's Maximum). MM is the statement that whenever $P$ is a forcing poset which preserves stationary posets of $\omega_1$, then for any collection $D$ of dense subsets of $P$ with $|D| \leqslant \omega_1$, there is a filter $G \subset P$ which intersects each dense set in $D$.

We now introduce a combinatorial property of $\omega_2$ which implies that there does not exist a thin stationary subset of $\mathscr{F}_{\omega_1}(\omega_2)$. This property follows from Martin's Maximum and is equiconsistent with Mahlo cardinal. The property will imply that there exists a fat stationary subset of $\omega_2$ which cannot acquire a club subset by any forcing poset which preserves $\omega_1$ and $\omega_2$. To be sure, we will not give a proof of the consistency of the existence of a disjoint club sequence; rather, we are going to show that if there exists a disjoint club sequence on $A \subseteq \omega_2 \cap \operatorname{cof} \omega_1$, then $A$ does not have a club subset in any $\omega_1$ or $\omega_2$-preserving outer model.

**Definition 5.7.** A *disjoint club sequence* on $\omega_2$ is a sequence $\langle C_\alpha \mid \alpha \in A \rangle$ such that $A$ is a stationary subset of $\omega_2 \cap \operatorname{cof}(\omega_1)$, each $C_\alpha$ is a club subset of $\mathscr{F}_{\omega_1}(\alpha)$, and $C_\alpha \cap C_\beta$ is empty for all $\alpha < \beta$ in $A$.

**Definition 5.8.** A subset $S \subseteq \mathscr{F}_{\omega_1}(\omega_2)$ is *local club* if there is a club set $C \subseteq \omega_2$ such that for all uncountable $\alpha \in C$, $S \cap \mathscr{F}_{\omega_1}(\alpha)$ contains a club subset in $\mathscr{F}_{\omega_1}(\alpha)$.

Note that local club sets are also stationary. To see that, let $K$ be be a club set in $\mathscr{F}_{\omega_1}(\omega_2)$; then there exists a club $K' \subseteq \omega_2$ such that if $\alpha \in K'$ has cofinality $\omega_1$, then $K \cap \mathscr{F}_{\omega_1}(\alpha)$ is club in $\mathscr{F}_{\omega_1}(\alpha)$. Now suppose that $S \subseteq \mathscr{F}_{\omega_1}(\omega_2)$ is local club. Let $C$ be club in $\omega_2$ with the property that $S \cap \mathscr{F}_{\omega_1}(\alpha)$ is club, for all $\alpha \in C$. Let $\alpha \in C \cap K'$ have cofinality $\omega_1$. Then

$$(S \cap \mathscr{F}_{\omega_1}(\alpha)) \cap (K \cap \mathscr{F}_{\omega_1}(\alpha)) \neq \varnothing,$$

since both of these sets are club in $\mathscr{F}_{\omega_1}(\alpha)$.

*Lemma* 5.9. Suppose that $S \subseteq \mathscr{F}_{\omega_1}(\omega_2)$ is local club. Then $S$ is local club in any outer model $W$ with the same $\omega_1$ and $\omega_2$.

*Proof.* We let $C$ be local club subset of $\omega_2$, i.e. for every uncountable $\alpha \in C$, $S \cap \mathscr{F}_{\omega_1}(\alpha)$ contains a club subset in $\mathscr{F}_{\omega_1}(\alpha)$. Then $C$ is also club in $W$. For each uncountable $\alpha \in C$ we fix a bijective map $g_\alpha : \omega_1 \to \alpha$. Then $\{g_\alpha``i \mid i < \omega_1\}$ is a club subset of $\mathscr{F}_{\omega_1}(\alpha)$. If we intersect that set with $S$, we get a club subset of $S \cap \mathscr{F}_{\omega_1}(\alpha)$, namely, $\{a_i^\alpha \mid i < \omega_1\}$ which is increasing and continuous. This latter set remains a club subset of $\mathscr{F}_{\omega_1}(\alpha)$ in $W$. □

*Lemma* 5.10. Suppose there is a disjoint club sequence $\langle C_\alpha \mid \alpha \in A\rangle$ on $\omega_2$. Let $W$ be an outer model with the same $\omega_1$ and $\omega_2$ in which $A$ is still stationary. Then there is a disjoint club sequence $\langle D_\alpha \mid \alpha \in A\rangle$ in $W$.

*Proof.* By the proof of lemma (5.9), each $C_\alpha$ contains a club set $D_\alpha$ in $W$. Since $\omega_1$ is preserved, each $\alpha$ in $A$ still has $\operatorname{cof}\alpha = \omega_1$. □

**Theorem 5.11.** *Suppose that $\langle C_\alpha \mid \alpha \in A\rangle$ is a disjoint club sequence on $\omega_2$. Then $A \cup \operatorname{cof}\omega$ does not contain a club subset.*

*Proof.* For contradiction, suppose that $A \cup \operatorname{cof}\omega$ does indeed contain a club subset. Without loss of generality we can assume that $2^{\omega_1} = \omega_2$ (otherwise we can work in a generic extension $W$ by COLL $(\omega_2, 2^{\omega_1})$; in $W$ the set $A \cap \operatorname{cof}\omega$ contains a club subset and by lemma (5.10) there is a disjoint club sequence $\langle D_\alpha \mid \alpha \in A\rangle$).

Before we go on, we need some basic facts concerning the regressive functions. If $C \subseteq \mathscr{F}_{\omega_1}(X)$ is club then there exists a function $F : X^{<\omega} \to X$ such that every $a \in \mathscr{F}_{\omega_1}(X)$ which is closed under $F$ is in $C$. If $F : X^{<\omega} \to \mathscr{F}_{\omega_1}(X)$ is a function and $Y \subseteq X$, we say that $Y$ is *closed under* $F$ for all $\gamma \in Y^{<\omega}$, where $F(\gamma) \subseteq Y$. A partial function $H : \mathscr{F}_{\omega_1}(X) \to X$ is *regressive* if for all $a \in \operatorname{dom} H$ we have $H(a) \in a$.

Now, since $2^{\omega_1} = \omega_2$, $H(\omega_2)$ has the size $\omega_2$. Let $h$ be a bijective map: $h : H(\omega_2) \to \omega_2$, and let $\mathfrak{A}$ denote the structure $\langle H(\omega_2), \in, h\rangle$. Define $B$ to be a set of all $\alpha \in$

$\omega_2 \cap \mathrm{cof}(\omega_1)$ such that there exists an increasing and continuous sequence $\langle N_i \mid i < \omega_1 \rangle$ of countable elementary substructures of $\mathfrak{A}$ such that

(1) $N_i \in N_{i+1}$ for all $i < \omega_1$, and

(2) the set $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is club in $\mathscr{P}_{\omega_1}(\alpha)$.

We will now show that $B$ is stationary in $\omega_2$. Let $C \subseteq \omega_2$ be an arbitrary club set, and let $\mathfrak{B}$ be the expansion of $\mathfrak{A}$ by the function $\alpha \mapsto \min(C \setminus \alpha)$. Define by induction an increasing and continuous sequence $\langle N_i \mid i < \omega_1 \rangle$ of elementary substructures of $\mathfrak{B}$ such that $N_i \in N_{i+1}$ for all $i < \omega_1$. Let $N = \bigcup\{N_i \mid i < \omega_1\}$. Then $\omega_1 \subseteq N$ and so $N \cap \omega_2$ is an ordinal. If we let $\alpha = N \cap \omega_2$, then $\alpha \in C$ and $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is club in $\mathscr{P}_{\omega_1}(\alpha)$. It is clear that $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is closed. To see that it is unbounded, let $x = \{\xi_n \mid n < \omega\}$ be an element of $\mathscr{P}_{\omega_1}(\alpha)$. In general $x \notin N$. However, for each $n$ there exists $i_n < \omega_1$ such that $\xi_n \in N_{i_n}$. If $i = \sup_{n<\omega} i_n$, then $x \subseteq N_i$. So $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is club and hence $\alpha \in B \cap C$.

Since $A \cup \mathrm{cof}\,\omega$ contains a club subset, $A \cap B$ must be stationary. For each $\alpha \in A \cap B$ we fix a sequence $\langle N_i^\alpha \mid i < \omega_1 \rangle$, just as described in the definition of $B$ above. It follows that $\{N_i^\alpha \cap \omega_2 \mid i < \omega_1\} \cap C_\alpha$ is club in $\mathscr{P}_{\omega_1}(\alpha)$, and so there must exist a club set $c_\alpha \subseteq \omega_1$ such that $\{N_i^\alpha \cap \omega_2 \mid i \in c_\alpha\}$ is club and is a subset of $C_\alpha$. Let $i_\alpha = \min(c_\alpha)$ and let $d_\alpha = c_\alpha \setminus \{i_\alpha\}$.

Define $S = \{N_i^\alpha \mid \alpha \in A \cap B \wedge i \in d_\alpha\}$. If $N \in S$ then there is a unique pair $\alpha \in A \cap B$ and $i \in d_\alpha$ such that $N = N_i^\alpha$. To see that, assume that $N = N_i^\alpha = N_j^\beta$. It follows that $N \cap \omega_2 \in C_\alpha \cap C_\beta$, and so $\alpha = \beta$, which implies that $i = j$. Note that if $N_i^\alpha \in S$, then $N_{i_\alpha}^\alpha \in N_i^\alpha$. So the function $H : S \to H(\omega_2)$ defined by $H(N_i^\alpha) = N_{i_\alpha}^\alpha$ is well-defined and regressive.

We will prove that $S$ is stationary in $\mathscr{P}_{\omega_1}(H(\omega_2))$. Let $F : H(\omega_2)^{<\omega} \to H(\omega_2)$ be a function. Recall that $h$ is a bijection between $H(\omega_2)$ and $\omega_2$ and define $G : \omega_2^{<\omega} \to \omega_2$

by letting

$$G(\alpha_0,\ldots,\alpha_n)=h(F(h^{-1}(\alpha_0),\ldots,h^{-1}(\alpha_n))) \tag{5}$$

Let $E$ be a club subset of $\alpha$ in $\omega_2$ close under $G$. Fix $\alpha$ in $E\cap A\cap B$. It follows that there exists $i\in d_\alpha$ such that $N_i^\alpha\cap\omega_2$ is closed under $G$. We claim that $N_i^\alpha$ is closed under $F$. Given $a_0,\ldots,a_n\in N_i^\alpha$, the ordinals $h(a_0),\ldots,h(a_n)$ are in $N_i^\alpha\cap\omega_2$. Define $\gamma$ by letting

$$\gamma=G(h(a_0),\ldots,h(a_n))=h(F(a_0,\ldots,a_n)) \tag{6}$$

We can see that $\gamma\in N_i^\alpha\cap\omega_2$, and therefore $h^{-1}(\gamma)=F(a_0,\ldots,a_n)\in N_i^\alpha$.

Since $S$ is stationary and $H:S\to H(\omega_2)$ is regressive, there exists a stationary set $S^*\subseteq S$ and a fixed $N$ such that for all $N_i^\alpha\in S^*$ we have $H(N_i^\alpha)=N$. The set $S^*$ is stationary, and hence its size must be $\omega_2$. It implies that there are distinct $\alpha$ and $\beta$ such that for some $i\in d_\alpha$ and $j\in d_\beta$ both $N_i^\alpha$ and $N_j^\beta$ are the members of $S^*$. Then $N=N_{i_\alpha}^\alpha=N_{i_\beta}^\beta$, and hence $N\cap\omega_2$ is in $C_\alpha\cap C_\beta$, which is a contradiction. □

*Lemma* 5.12. Let $\langle C_\alpha\mid\alpha\in A\rangle$ be a disjoint club sequence and let $W$ be an outer model of $V$ with the same $\omega_1$ and $\omega_2$. Then in $W$, $A\cup\operatorname{cof}\omega$ does not contain a club subset.

*Proof.* If $A$ remains stationary in $W$, then by lemma (5.10) there is a disjoint club sequence $\langle D_\alpha\mid\alpha\in A\rangle$ in $W$. By Theorem (5.11) $A\cap\operatorname{cof}\omega$ does not contain a club subset in $W$. □

Since it is our goal to characterize the cases when we can (or cannot) add a club subset without it resulting in a collapse, it makes sense to restate this result as follows:

*Corollary* 5.13. If $\langle C_\alpha\mid\alpha\in A\rangle$ is a disjoint club sequence in the inner model $V$ of $W$, and $A\cap\operatorname{cof}\omega$ contains a club subset in $W$, then any forcing of $V$ which yields $W$ will collapse either $\omega_1$ or $\omega_2$.

## 6. Forcing With Finite Conditions: $\omega_2$ Case

Unfortunately, there is no analogous result for $\omega_2$. The following theorem is a direct consequence of the theorem due to Friedman [4].

**Theorem 6.1.** *Suppose that* $0^\sharp$ *exists. Then the set*

$$\{X \subseteq \omega_2^L \mid X \in L \text{ and } X \text{ contains a club subset in an inner model where } \omega_2 = \omega_2^L\}$$

*is not constructible, and has L-degree* $0^\sharp$.

E.g., there are some $X$ which belong to the set above but have no club subset in any set-generic extension of $L$ which preserves $\omega_2$.

There is, however, a sufficient condition (also due to Friedman [6]) for a subset of $\omega_2$ to contain a club subset in an extension which is cardinal-preserving.

**Definition 6.2.** $X \subseteq \omega_2$ is *very stationary* if for all $\alpha$ in some stationary $X \subseteq X \cap \operatorname{cof} \omega_1$, $X \cap \alpha$ contains a club subset of $\alpha$.

With a variant of $\Diamond$ at $\omega_2$ in $L$ we can construct disjoint very stationary subsets of $\omega_2$ in $L$. In general, $\Diamond_\kappa(Q)$, where $\kappa$ is a cardinal and $Q \subseteq \kappa$ is stationary, is the statement that there exists a sequence $\langle A_\alpha \mid \alpha \in Q\rangle$ such that

(1) $A_\alpha \subseteq \alpha$ for each $\alpha \in Q$.
(2) For each $A \subseteq \kappa$, the set $\{\alpha \in Q \mid A \cap \alpha = A_\alpha\}$ is stationary in $\kappa$.

The sufficient condition we have mentioned above is this:

**Theorem 6.3.** *If* $X \subseteq \omega_2$ *is very stationary then there is a set-forcing extension which preserves both* $\omega_1$ *and* $\omega_2$*, and* $X$ *contains a club subset in that extension.*

*Proof.* The proof is analogous to that of Theorem 3.4. We force with closed subsets of $X$ of order type less than $\omega_2$, ordered by end-extension. We can use the fact that $X$ is very stationary to show that if $p$ is a condition and $D_i$, $i < \omega_1$, are predense below $p$ then there exists $q \leqslant p$ which extends an element of $D_i$ for each $i$ (see the proof above for details). It follows that no new $\omega_1$-sequences are added by the forcing, which implies the preservation of $\omega_1$ and $\omega_2$. □

The most interesting question, however, is this: in the case of $\omega_2$, can we find a forcing similar to that of Baumgartner, which would add a club set with finite conditions? The following theorem will provide an answer.

**Definition 6.4.** $D_0$ *witnesses* that $D \subseteq \omega_2$ is very stationary if $D_0 \subseteq D \cap \operatorname{cof} \omega_1$ is stationary and $D \cap \alpha$ contains a club subset of $\alpha$ for each $\alpha \in D_0$.

**Theorem 6.5.** *Suppose that $\omega_2$ is the L-successor to $\omega_1$, that $D \subseteq \omega_2$ is constructible, and that there exists a constructible witness that $D$ is very stationary. Then there is a constructible forcing $P$ which preserves cofinalities and adds a club subset of $D$.*

*Proof.* Let $D_0 \in L$ witness that $D$ is very stationary. We will assume that successor elements of $D$ all have cofinality $\omega$, by replacing $D$ with $(D \cap \operatorname{Lim} D) \cup \{\alpha + \omega \mid \alpha \in D\}$. This suffices because the set of limit points of a club set contained in this set is a club set contained in $D$.

Our condition will be a pair $(A, S)$, where

(1) $A$ is a finite set of disjoint closed intervals, their left endpoints being elements of $D$, and their left endpoints of $\operatorname{cof} \omega_1$ being elements of $D_0$. Let $L$ denote the set of all left endpoints of intervals in $A$.

(2) $S$ is a finite set of countable, constructible $\Sigma_1$ elementary submodels $x$ of some $L_\beta$, $\beta$ limit, $\beta < \omega_2$, such that

  (a) $x \cap \alpha$ is unbounded in $\alpha \in x$ whenever $\operatorname{cof} \alpha = \omega$.

(b) $\sup(x \cap \alpha) \in D$ whenever $\alpha \in (x \cap D_0); \omega_2$.

(c) For $\alpha, \beta \in D$, if $\alpha < \beta$ are adjacent, $\alpha \in x$, $\beta < \sup(x \cap \mathrm{Ord})$, then $\beta \in x$.

(3) Given any interval $I = [\alpha, \beta] \in A$ and any $x \in S$ we have:

(a) If $I \cap x \neq \varnothing$ then $I \in x$.

(b) If $I \cap x = \varnothing$ and $\alpha < \sup(x \cap \mathrm{Ord})$ then $\alpha_x \in L$, where $\alpha_x$ is the least ordinal $\geqslant \alpha$ in $x$.

(4) Let $F$ be the set of all elements of $L$ of cofinality $\omega_1$, together with $\omega_2$. For nice $x$, we say that the *$F$-height of* $x$ is the least element of $F$ which is greater than $\sup(x \cap \mathrm{Ord})$. As for our condition $p$, the following conditions must be met:

(a) If $x \in S$ and $\alpha \in F$ then $x \cap L_\alpha \in S$.

(b) If $x, y \in S$ have the same $F$-height, then $x = y$, $x \in y$, or $y \in x$.

From now on we will write $(A_p, S_p)$ and $F_p$ to designate the sets having to do with a condition $p$. We say that $q$ extends $p$ iff $A_q \supseteq A_p$ and $S_q \supseteq S_p$. Friedman breaks down the proof into three claims.

*Claim* 6.5.1. Fix $p \in P$.

(1) For any club $C \subseteq \omega_2$ there is $\alpha \in C$, $\operatorname{cof} \alpha = \omega_1$, such that $p \in L_\alpha$ and $p^*$, obtained by adding $[\alpha, \alpha]$ to $A_p$, is a condition extending $p$.

(2) Let $\alpha$ and $p^*$ be defined as above. If $q^* < p^*$ then there is $q < p$ in $L_\alpha$ such that every extension of $q$ in $L_\alpha$ is compatible with $q^*$.

*Claim* 6.5.2. Fix $p \in P$.

(1) For any club $C \subseteq \mathscr{P}_{\omega_1}(\omega_2)$ there is a constructible $x \in C$ such that $p \in x$ and $p^*$, defined by adding $x \cap L_\alpha$ to $S_p$ for all $\alpha \in F_p$ is a condition extending $p$.

(2) With $q^*$ and $p^*$ defined as above, if $q^* < p^*$ then there exists $q \in x$ which extends $p$, and every extension of $q$ in $x$ is compatible with $q^*$.

Claim 6.5.1 implies that $\omega_2$ is preserved, while claim 6.5.2 implies that $\omega_1$ is preserved. Since the size of $P$ is $\omega_2$, all cofinalities are preserved. To complete the proof, we also need

*Claim* 6.5.3. Let $G$ be $P$-generic and define

$$C_G = \{\gamma \mid \gamma \text{ is a left endpoint of some interval in } \cup \{A_p \mid p \in G\}\}$$

Then $C_G$ is a club subset of $D$.

*Proof of claim 6.5.1.*

*Step 1.* Choose $\alpha$ in the intersection of $C$ and $D_0$ such that $p \in L_\alpha$. We must show that the properties (1) through (4) are satisfied. $p^*$ satisfies (1) since $\alpha$ is greater than the right endpoint of any interval in $A_p$. Property (2) is the same for $P^*$ as it is for $p$. Property (3a) is also the same for $p^*$, since $\alpha$ is not a member of any element of $S_p$. Ditto for (3b), since $\alpha > \sup x$ for all $x \in S_p$. Finally, property (4) is satisfied because $\alpha \notin x$ for all $x \in S_p$.

*Step 2.* Let $q^*$ extend $p^*$ and define $q$ as follows:

$$A_q \text{ is } A_{q^*} \cap L_\alpha,$$
$$S_q \text{ is } S_{q^*} \cap L_\alpha$$

*Step 2a.* First of all, we will show that $q$ is a condition. It suffices to verify properties (3b) and (4).

For property (3b), assume that $I \cap x$ is empty and the left endpoint $\beta$ of $I = [\beta, \gamma]$ is $< \sup x$, where $I \in A_{q^*} \cap L_\alpha$ and $x \in S_{q^*} \cap L_\alpha$. Since $q^*$ is a condition, $\beta_x$ is the left endpoint of some interval $J$ in $S_{q^*}$. But since $[\alpha, \alpha] \in A_{q^*}$, the right endpoint of $J$ is $< \alpha$, and hence $J \in S_{q^*} \cap L_\alpha = S_q$.

For property (4), observe that $F_q = F_{q^*} \cap \alpha$, together with $\omega_2$. If $x \in S_q$ and $\beta \in F_q$ then $x \cap L_\beta \in S_{q^*}$, and therefore also in $S_q = S_{q^*} \cap L_\alpha$, because $x \cap L_\beta \in L_\alpha$. This gives us (4a). Now, if $x, y \in S_q$ have the same $F_q$-height then they also have the same $F_{q^*}$-height, since $x, y \in L_\alpha$. Moreover, $q^*$ is a condition, and $x, y \in S_{q^*}$, which gives us (4b), and we can conclude that $q$ is a condition.

Since $q^*$ extends $p^*$, it also extends $p$. But $p \in L_\alpha$, and so $q$ is a condition which extends $p$.

*Step 2b.* To complete the proof of this claim, we must also show that for any extension $r$ of $q$, $r \in L_\alpha$, there exists a common extension $t$ of $r$ and $q^*$. Define $t$ as follows:

$$
\begin{aligned}
A_t &= A_r \cup A_{q^*}, \\
S_t &= S_r \cup S_{q^*}
\end{aligned}
$$

It suffices to show that $t$ is a condition, for in that case it will be clear that it extends both $r$ and $q^*$. To finish the proof, we will verify that properties (1) through (4) hold for $t$.

For property (1), we observe that $r$ is a condition extending $q$, each interval in $A_r$ has its right endpoint $< \alpha$, and each interval $I \in A_{q^*} \setminus A_q$ has its left endpoint $\geqslant \alpha$. It follows that the intervals in $A_t$ are disjoint.

Property (2) is evident; seeing (2c) uses the fact that $D$ is unbounded in $\alpha$, since $\alpha \in D_0$.

For property (3), we fix $I \in A_t \setminus A_r$ and $x \in S_r$. It follows that $\sup(x \cap \mathrm{Ord}) < \alpha$ and the left endpoint of $I$ is $\geqslant \alpha$. That is, property (3) is vacuously true. So fix $I \in A_r$ and $x \in S_t \setminus S_r$. Then $x \cap L_\alpha \in S_q \subseteq S_r$, implying that property (3) holds for $I$ and $x \cap L_\alpha$. Since $I \cap x \neq \varnothing$ implies that $I \cap x \cap L_\alpha \neq \varnothing$, property (3a) holds for $I$ and $x$. And so does (3b). Really, if $I \cap x = \varnothing$ and the left endpoint $\beta$ of $I$ is $< \sup(x \cap \mathrm{Ord})$ then $I$ is disjoint from $x \cap L_\alpha$ and one of the following cases applies:

(i) If $\beta < \sup(x \cap \alpha)$, then $\beta_x = \beta_{x\cap\alpha}$ and therefore (3b) follows, since $r$ is a condition.

(ii) If $\beta_x = \alpha$, then (3b) follows because $[\alpha, \alpha] \in A_{q^*}$.

(iii) If $\beta_x = \alpha_x$, then we get (3b) because $q^*$ is a condition.

The remaining cases, where $I \in A_r$ and $x \in S_r$, or $I \in A_t \setminus A_r$ and $x \in S_t \setminus S_r$, follow immediately from $r$ and $q^*$ being conditions.

For property (4a), it suffices to show that if $x \in S_t$ and $\beta \in F_t$ then $x \cap L_\beta \in S_t$. If $x \in S_r$ then either $\beta \in F_r$ or $\beta \geqslant \alpha$. If $\beta \in F_r$ then $x \cap L_\beta \in S_r \subseteq S_t$, since $r$ is a condition. If, on the other hand, $\beta \geqslant \alpha$, then $x \cap L_\beta = x \in S_r \subseteq S_t$. On the other hand, if $x \in S_{q^*}$ then either $\beta \in F_{q^*}$ or $\beta \in F_r$. If $\beta \in F_{q^*}$ then (4a) follows since $q^*$ is a condition. If $\beta \in F_r$ then $x \cap L_\beta = (x \cap L_\alpha) \cap L_\beta \in S_q \subseteq S_r$.

For property (4b), we must prove that if $x, y \in S_t$ have the same $F_t$-height, then $x \in y$, $y \in x$, or $x = y$. If $x \in S_r$ then $F_t$-height of $x$ is at most $\alpha$, implying that $y \in S_r$. Hence $x$ and $y$ have the same $F_r$-height and (4b) follows since $r$ is a condition. If $x \in S_{q^*} \setminus S_r$ then the $F_t$-height of $x$ is $> \alpha$, and hence $y \in S_{q^*}$. It follows that $x$ and $y$ have the same $F_{q^*}$-height, and (4b) follows since $q^*$ is a condition.

This concludes the proof of claim 6.5.1.

*The proof of the claim 6.5.2* follows the basic pattern of that of claim 6.5.1, but is somewhat more involved. We omit it here, but an interested reader may examine it in full in Friedman's paper [6].

*Proof of the claim 6.5.3.*

Here we must show that $C_G$ is club. It is clearly unbounded, so it remains to show that it is also closed.

Let $p$ be a condition. For contradiction, assume that

$$p \Vdash (\alpha \in \operatorname{Lim} C_G \text{ and } \alpha \notin C_G)$$

We assume that for all $y \in S_p$, $\alpha_y$ is either a left endpoint of some interval in $A_p$, or it is forced by $p$ that $\alpha_y \notin C_G$. For all $q < p$, $\alpha \notin I$ for all intervals $I \in A_q$, because in that case $q$ forces either that $\alpha \in C_G$ or that $\alpha$ is not the limit of elements of $C_G$.

Suppose that $y \in S_p$, $\alpha > \sup(y \cap \alpha)$, and $\alpha < \sup(y \cap \mathrm{Ord})$. Then $\alpha_y$ has to be a left endpoint of some $I \in A_p$, or else $p$ forces that $\alpha$ is not a limit point of $C_G$.

Consider $\beta$, the least element of $F_p$ which is $> \alpha$, and let

$$S = \{y \in S_p \mid \alpha \leqslant \sup(y \cap \mathrm{Ord}) < \beta\}$$

It follows that the elements of $S$ form an $\in$-chain.

**Case 1. $y \cap \alpha$ is cofinal in $\alpha$, for some $y \in S$.** Assume that $y \cap \alpha$ is cofinal in $\alpha$ for some $y \in S$, and let $y_0$ be the $\in$-least element with that property. If $\alpha_{y_0}$ is a left endpoint of some interval $I \in A_p$, then $\alpha \in D$, as is required by the condition (2b) in the beginning of the proof. We will see that it is possible to extend $p$ to force either that $\alpha \in C_G$ or that $\alpha$ is not a limit point of $C_G$. Before we do that, observe that $D \cap y_0 \cap \alpha$ is cofinal in $\alpha$, since there are cofinally many $\gamma < \alpha$ which are forced by extensions of $p$ into $C_G$, and for all such $\gamma \notin y_0$, $\gamma_{y_0} \in D_0 \subseteq D$. It follows from the property (2c) for $y_0$ that $D \cap y_0 \cap \alpha \cap \operatorname{cof} \omega$ is also cofinal in $\alpha$.

**Subcase 1.1. $\alpha_{y_0}$ is not the left endpoint of some interval in $A_p$.** If $\alpha_{y_0}$ is not the left endpoint of any interval in $A_p$, then we fix $\gamma \in D \cap y_0 \cap \alpha \cap \operatorname{cof} \omega$ such that $\gamma > \delta$ for all $\delta$, right endpoints of each interval in $A_p$ such that its left endpoint is less than $\alpha$ and at the same time larger than $\sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$. We will show that a condition results when the interval $I = [\gamma, \alpha_{y_0}]$ is added to $p$. This condition will then force that $\alpha \notin \lim C_G$ in contradiction to our initial hypothesis. By our choice of $\gamma$, $I$ is disjoint from each interval in $A_p$ with its left endpoint $< \alpha$. And since we assumed that $\alpha_{y_0}$ is not the left endpoint of an interval in $A_p$, and so neither is any ordinal between $\alpha$ and $\alpha_{y_0}$, $I$ is also disjoint from each interval in $A_p$ with its left endpoint $> \alpha$. By our choice of $\gamma$, $I$ also has an empty intersection with all $y \in S_p$ such that $\sup(y \cap \beta) < \alpha$. Moreover, $I$ is disjoint from every $y \in S_p$ with

$$\sup(y \cap \alpha) < \alpha \leqslant \sup(y \cap \beta),$$

since for such $y$ we can write $\alpha_y > \alpha_{y_0}$, because $\alpha_y$ is the left endpoint of some interval in $A_p$ while $\alpha_{y_0}$ is not. Every other $y \in S_p$ contains $y_0$, and hence also $I$ as its element. If $y \in S_p$ is disjoint from $I$ with $\gamma < \sup(y \cap \mathrm{Ord})$, then $\gamma_y = \alpha_y$ is a left endpoint of some interval in $A_p$.

**Subcase 1.2. $\alpha_{y_0}$ is the left endpoint of some interval in $A_p$.** If $\alpha_{y_0}$ is the left endpoint of some interval in $A_p$, then we set $I = [\alpha, \alpha]$. We will show that a condition results if we add $I$ to $p$ which will force that $\alpha \in C_G$, in contradiction to our initial hypothesis. Clearly, $I$ is disjoint from each interval in $A_p$, since $\alpha$ is not in any such interval. If $I$ intersects $y \in S_p$, then $I \in y$. If $I$ is disjoint from $y \in S_p$ and $\alpha < \sup(y \cap \mathrm{Ord})$, then $\alpha_y \geqslant \alpha_{y_0}$, on pain of having $y \cap L_\beta \in S$, $y_0 \in y \cap L_\beta$, and hence $\alpha = \sup(y_0 \cap \alpha_{y_0}) \in y$, which is against our hypothesis. So again, $\alpha_y$ must be the left endpoint of some interval in $A_p$, or else $\alpha_{y_0}$ cannot be such.

**Case 2. $y \cap \alpha$ is not cofinal in $\alpha$, for all $y \in S$.** For this last case, choose $I = [\gamma, \alpha]$, $\gamma \in D \cap \alpha \cap \operatorname{cof} \omega$ just as we did before, i.e.

(1) $\gamma > \delta$ for all $\delta$, right endpoints of each interval in $A_p$ such that its left endpoint is $< \alpha$, and
(2) $\gamma > \sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$

We will show that a condition results when we add $I$ to $p$. By our choice of $\gamma$, $I$ is disjoint from all intervals in $A_p$. $I$ is also disjoint from each $y \in S_p$, since $\sup(y \cap \alpha) < \gamma$

by our choice of $\gamma$ and the hypothesis for this case. (If $\alpha \in \gamma$, then $\alpha = \gamma_y$ and hence $p$ forces that $\alpha$ is not a limit of elements of $C_G$, since $\alpha$ is not the left endpoint of any interval in $A_q$ for each $q \leqslant p$.) If $y \in S_p$ and $\gamma < \sup(y \cap Ord)$, then $\alpha_y$, yet again, is the left endpoint of some interval in $A_p$.

The proof of this claim also concludes the proof of the theorem. □

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