San Jose State University
SJSU ScholarWorks

Master's Theses | Master's Theses and Graduate Research

2008

The search of graphs equienergetic with edge deleted subgraph

Wilson A. Florero
San Jose State University

Follow this and additional works at: http://scholarworks.sjsu.edu/etd_theses

Recommended Citation
Florero, Wilson A., "The search of graphs equienergetic with edge deleted subgraph" (2008). *Master's Theses*. 3634.
http://scholarworks.sjsu.edu/etd_theses/3634

This Thesis is brought to you for free and open access by the Master's Theses and Graduate Research at SJSU ScholarWorks. It has been accepted for inclusion in Master's Theses by an authorized administrator of SJSU ScholarWorks. For more information, please contact scholarworks@sjsu.edu.

THE SEARCH OF GRAPHS EQUIENERGETIC WITH EDGE DELETED SUBGRAPH

A Thesis

Presented to

The Faculty of the Department of Mathematics

San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Wilson A. Florero

December 2008

UMI Number: 1463413

INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

UMI®

UMI Microform 1463413
Copyright 2009 by ProQuest LLC.
All rights reserved. This microform edition is protected against unauthorized copying under Title 17, United States Code.

ProQuest LLC
789 E. Eisenhower Parkway
PO Box 1346
Ann Arbor, MI 48106-1346

© 2008

Wilson A. Florero

ALL RIGHTS RESERVED

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Dr. Wasin So

Dr. Jane Day

Dr. Leslie Foster

APPROVED FOR THE UNIVERSITY

12/1/08

ABSTRACT

THE SEARCH OF GRAPHS EQUIENERGETIC WITH EDGE DELETED SUBGRAPH

by Wilson A. Florero

The *energy* of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix. Two graphs are *equienergetic* if they have the same energy. We are particularly interested in finding graphs which are equienergetic with a subgraph having one edge less. We begin our study of the equienergecity problem by using various MATLAB scripts to search for graphs with the described property. Using the acquired data as our foundation, we demonstrate how some of the examples found belong to an infinite family of graphs with equienergetic subgraphs having one edge less. Additionally, for those graphs whose generalization is unknown, we explain some of our findings and provide conjectures. We end our study by extending some of our theoretical results to equienergetic graphs with more than one edge removed, and provide more problems for further study.

Dedication

I dedicate this thesis to my grandparents Humberto Salinas Zamora and Maria Torrejón Aracena. They have been my inspiration in completing this project.

Acknowledgements

I would like to thank my family and friends for their patience, support, encouragement, and understanding during the development of this project. I thank my father, Wilson Florero, for all the sacrifices he has done to get me through school. I am grateful to my mother, Mirian Salinas, for giving me hope and cheering me up in hard times. I thank my sister, Lizeth Florero, for her support in the final stages of this project.

I would also like to express my gratitude to my thesis committee for guiding me in this mathematical journey. I thank my advisor, Dr. So, for keeping me on my feet since this project started. I owe special thanks to Dr. Day for the important suggestions she provided during the writing of this thesis. Finally, I thank Dr. Foster for challenging me during my master's level examinations.

Wilson A. Florero

Table of Contents

Chapter

List of Tables

Table

List of Figures

Figure

List of Notations

Page of First Appearance

Chapter 1

Introduction

Throughout, G will be a *simple graph*, namely, a graph with no loops or multiple edges. Denote $V(G)$, $E(G)$, and $A(G)$ the vertex set, edge set, and adjacency matrix of G, respectively. The characteristic polynomial and spectrum of a graph G are those of its corresponding adjacency matrix. Since G is a simple graph on n vertices, $A(G)$ will be an $n \times n$ real symmetric matrix, so that its eigenvalues will always be real numbers.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $A(G)$; then the *energy* of G is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$ [1]. Moreover, two graphs are *equienergetic* if they have the same energy. Obviously if two graphs are *cospectral*, i.e., they have the same spectrum, then they are equienergetic. So we will focus on non cospectral graphs. As it was shown in [2] the energy of a graph can either decrease, stay the same, or increase after an edge is deleted. *In this thesis we are interested in finding graphs whose energy remains the same after one edge has been deleted.* In Chapter 2 a MATLAB program (script) is presented that is used to find graphs with the required property. The results are reported in Chapter 3 and 4, where some of the graphs are shown to belong to an infinite family of graphs with a specific characteristic. In Chapter 5, a few graphs from the results of Chapter 2 are studied whose general characteristics are still unknown. Finally, Chapter 6 presents a small generalization of some of the results of Chapter 3 and also investigates further some graphs introduced in Chapter 5.

Chapter 2

MATLAB Search

We begin our analysis of the equienergetic property by searching all connected graphs on 1 through 11 vertices, trying to identify all those that are equienergetic with a subgraph having one fewer edge. An exhaustive search shows that, up to isomorphism, there is exactly one such graph with 6 vertices, two with 9 vertices, and six with 10 vertices; and there are (up to machine accuracy) at least 5 with 11 vertices. Because the number of graphs on n vertices grows exponentially, their adjacency matrices for each corresponding number of vertices are stored in separate files for easier accesibility. Our search was conducted by running variations of the MATLAB script **equiEnergetic** on files **CN6** through **CN9**, files **mat1** through **mat12**, and files **11vertices0** through **11vertices499**[1] . The results for **equiEnergetic** and its variations are presented in Table 2.1. This script is an alternate program to the one developed by Dr. Wasin So, who ran his for $n \leq 10$. His results, along with alternate proofs and examples found for $n = 11$ are presented in this thesis.

[1] See Appendix B for an explanation of the adjacency matrix files, the equiEnergetic script, its variations, and sample runs.

MATLAB script: **equiEnergetic**

```
% Searches for Equienergetic Graphs of one edge less.
% Final version
load CN9.mat
tic                         %Timer started.

SIZE = 9; %Matrix size
EPSILON = 0.0000000000001; %Error between energy of A and
                            %its submatrix before displayment.
L = 1;

fprintf('\n =================== \n');
fprintf(' | Beginning Search! |');
fprintf('\n =================== \n');
for k=1:261080
  A = yy(:,:,k);
  E1 = norm(eig(A),1);

%Edge matrix E_ij is created and subtracted from A.
 for m=1:(SIZE-1)
    column = m + 1;
        for n = column:SIZE
            if n~=m
                E = zeros(SIZE);E(m,n)=1;E(n,m)=1;
                   if A-E>=0
                     E2 = norm(eig(A-E),1);

                       if abs(E1-E2)<= EPSILON
                         while L<=1
                         fprintf('\n ***************************************** \n');
                         fprintf('\n Matrix %1.0f with energy E = %1.10f. \n',k,E1);
                         fprintf('\n has equienergetic submatrices given by: \n');
                         L = L + 1;
                         end
                         fprintf('\n After removing edge e=(%1.0f,%1.0f): \n',m,n);
                         display(A-E);
                         fprintf(' \n with energy E2 = %1.10f. \n',E2);
                       end

                   end


            end
         end
 end

L = 1;
end
fprintf('\n ================= \n');
fprintf(' | End of Search! |');
fprintf('\n ================= \n');
fprintf(' || Elapsed Time = %1.0f seconds|| \n',toc);
```

end of MATLAB script

The following table summarizes the results of **equiEnergetic** script and its variations. It provides the matrix number and files the matrices were obtained from along with their approximate energy.

MATLAB Results		
File	**6 vertex Matrix# and edges to be removed**	**≈Energy**
CN6	#59; {(1,6),(2,5)}	8.2925287399
File	**9 vertex Matrix# and edges to be removed**	**≈Energy**
CN9	#32042; {(1,9),(2,9)}	13.1231
	#50501=$KK(4,1,3)$; {(1,9),(3,9),(5,9)}	14.0 (Exact)
File	**10 vertex Matrix# and edges to be removed**	**≈Energy**
mat2	#608111; {(9,10)}	14.4556401300
	#783993; {(9,10)}	14.9485518944
	#811571; {(3,8),(5,9)}	13.4156343655
mat5	#798072; {(9,10)}	15.1428709008
mat8	#761124; {(7,10)}	15.1260746083
mat12	#661923=$KK(5,3)$; {(1,8),(2,9),(3,10)}	16.1904821077
File	**11 vertex Matrix# and edges to be removed**	**≈Energy**
11vertices19	#5464; {(2,8)}	17.5952415806
	#5465; {(2,8),(3,8)}	17.5952415806
11vertices173	#1982932; {(9,10)}	20.4852813742
11vertices211	#469937; {(7,9)}	18.2925287398
11vertices280	#1157259; {(1,6)}	16.2925287398

Table 2.1: Search results for the script **equiEnergetic** and its variations.

The notation $KK(\cdot,\cdot)$ and $KK(\cdot,\cdot,\cdot)$ is defined in the preceding List of Notation and will be discussed throughly in Chapter 3 and 4, respectively. The results of Table 2.1 in graphical form are shown next. An asterisk on an edge means that edge can be deleted without changing the energy.

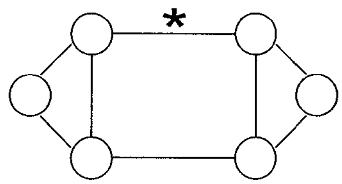


Figure 2.1: The graph #59 from file **CN6.mat**.

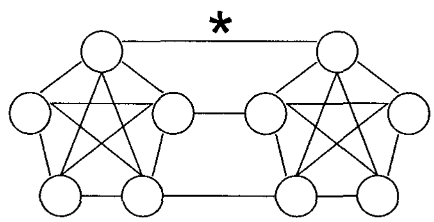


Figure 2.2: The graph #661923 from file **mat12**.

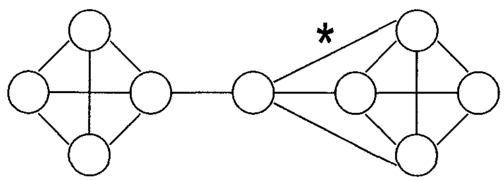


Figure 2.3: The graph #50501 from file **CN9**.

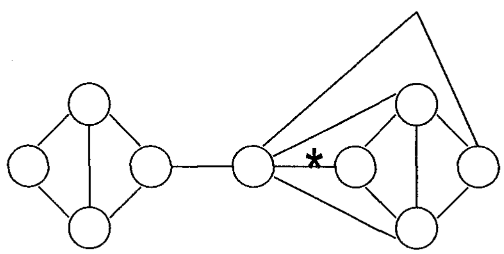


Figure 2.4: The graph #32042 from file **CN9**.

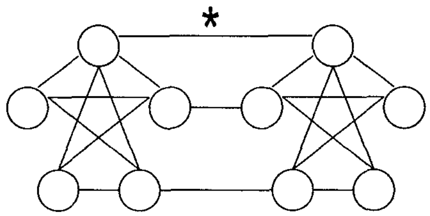


Figure 2.5: The graph #761124 from file **mat8**.

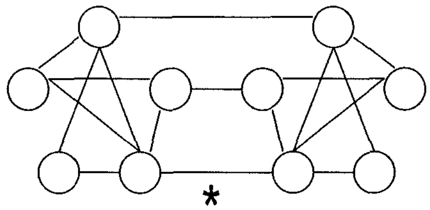


Figure 2.6: The graph #798072 from file **mat5**.

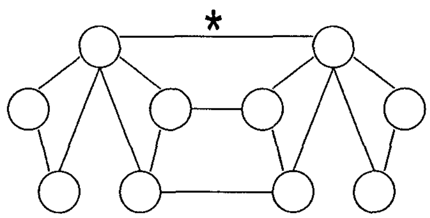


Figure 2.7: The graph #608111 from file **mat2**.

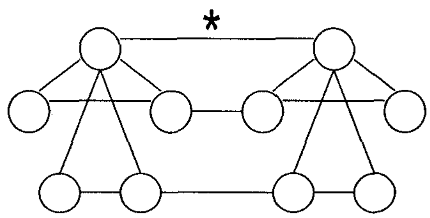


Figure 2.8: The graph #783993 from file **mat2**.

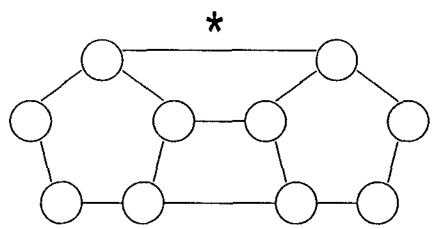


Figure 2.9: The graph #811571 from file **mat2**.

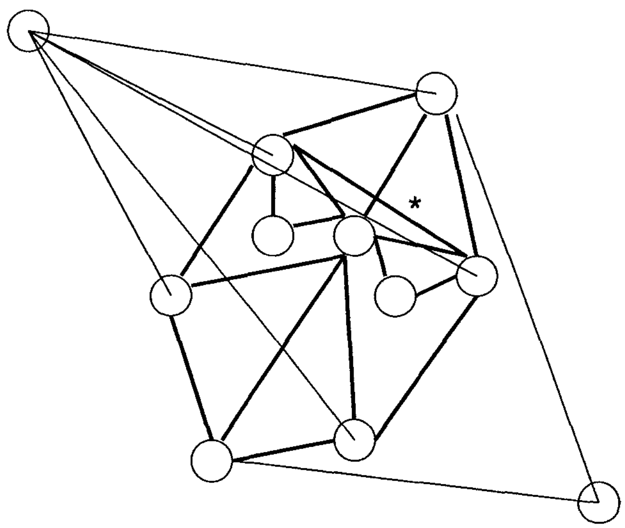
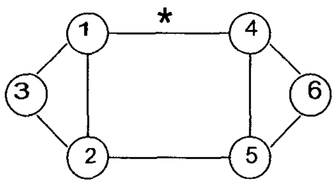


Figure 2.10: The graph #469937 from file **11vertices211**.

Now we begin our analysis of some of the preceding examples. The first graph we study is that of Figure 2.1, which is shown to be part of an infinite family of graphs that have a prescribed structure. In fact, there is such an infinite family which includes the graphs in both Figures 2.1 and 2.2. The detailed study of these two graphs follows next.

Chapter 3

First Infinite Family



In Chapter 2 we introduced a MATLAB script that obtained various graphs with the desired equienergetic properties. A pair of them are those from Figures 2.1 and 2.2. A pattern that can be observed from both of them is that they are graphs consisting of two copies of complete graphs joined by intermediate edges between them. We wondered if these graphs belong to a family of graphs with similar characteristics. Would the next graph be one composed of two complete graphs, with 7 vertices each, and joined by intermediate edges? The answer is yes, and that is what we show next.

The results in this chapter are from [2], but an alternate calculation is given here for the characteristic polynomial of this type of graph.

3.1 The $KK(m,r)$ graph

Notice that if we label the left hand side and right hand side complete graphs of Figure 2.1 first, then its corresponding adjacency matrix will have the form

$$A = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

One can check that the graph from Figure 2.2 can also be labeled in this way so that its adjacency matrix has the above similar block form. We can in fact use this same labeling process for the next matrices to obtain the following generalization. For $m \geq 3$ and $1 \leq r \leq m$, let $KK(m,r)$ denote the graph composed of two copies of the complete graph K_m joined by r parallel edges. Hence $KK(m,r)$ has $2m$ vertices and $m^2 - m + r$ edges. Let $A = A(KK(m,r))$ and $K = A(K_m)$ be the adjacency matrices of $KK(m,r)$ and K_m, respectively. Note $A \in M_n(\mathbb{R}), n = 2m$.

Because of the symmetric nature of $KK(m,r)$, it is easy to label the vertices so that

$$A = \left[\begin{array}{c|c} K & D_r \\ \hline D_r & K \end{array}\right]$$

where $K, D_r \in M_m(\mathbb{R}), m = \frac{n}{2}, \quad K = [1]_m - I_m$ and $D_r = diag(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0)$. Using this new notation, we will denote the graphs from Figure 2.1 and Figure 2.2 by $KK(3,2)$ and $KK(5,3)$, respectively. In more detail, the graph and adjacency matrix for $KK(5,3)$ are as follows:

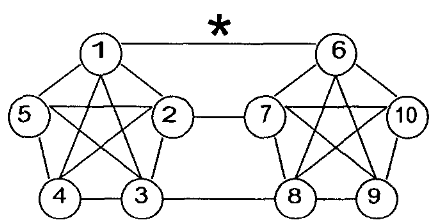


Figure 3.1: The graph $KK(5,3)$.

with adjacency matrix

$$A(KK(5,3)) = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

Remark 3.1. At this point it is clear what we mean by "parallel edges"; however, in order to stay consistent with the literature, and for the rest of this thesis, we will use the phrase "independent edges" instead.

3.2 Finding the characteristic polynomial of $KK(m,r)$

Continue to let $A = A(KK(m,r)) = \left[\begin{array}{c|c} K & D_r \\ \hline D_r & K \end{array}\right]$ as above. We are interested in $\mathcal{E}(KK(m,r))$, so we compute the characteristic polynomial, $p_A(z)$, of A. The following Lemma is a well known exercise that will be useful.

Lemma 3.1. *Let $B, C \in M_n(\mathbb{R})$, then*

$$det\left(\left[\begin{array}{c|c} B & C \\ \hline C & B \end{array}\right]\right) = det(B+C)det(B-C).$$

Proof: We perform a similarity on $\left[\begin{array}{c|c} B & C \\ \hline C & B \end{array}\right]$:

$$\left[\begin{array}{c|c} I & -I \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} B & C \\ \hline C & B \end{array}\right]\left[\begin{array}{c|c} I & I \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{c|c} B-C & 0 \\ \hline C & B+C \end{array}\right]$$

Then taking the determinant of both sides yields the desired result.□

Applying Lemma 3.1 to our matrix A gives:

$$\begin{aligned} p_A(z) &= det\left(\left[\begin{array}{c|c} K - zI_m & D_r \\ \hline D_r & K - zI_m \end{array}\right]\right) \\ &= det(K + D_r - zI_m)det(K - D_r - zI_m) \\ &= p_{K+D_r}(z)p_{K-D_r}(z) \end{aligned} \tag{3.1}$$

So the characteristic polynomial $p_A(z)$ depends on the characteristic polynomials $p_{K+D_r}(z)$ and $p_{K-D_r}(z)$. Next we calculate each of these.

3.2.1 Finding the polynomial $p_{K+D_r}(z)$

Notice the matrix $K + D_r \in M_m(\mathbb{R})$ may be rewritten as $K + D_r = (D_r - I_m) + [1]_m = diag(0, \ldots, 0, \underbrace{-1, \ldots, -1}_{m-r}) + ee^T$, where $e^T = [1 \ldots 1]$. Let $\alpha_i = \lambda_i(K + D_r)$; then it follows by the Rank One Perturbation Interlacing Theorem in Appendix A, that

$$-1 \le \underbrace{\alpha_1 \le \cdots \le \alpha_{m-r-1}}_{m-r-1} \le -1 \le \alpha_{m-r} \le 0 \le \underbrace{\alpha_{m-r+1} \le 0 \le \cdots \le \alpha_{m-1}}_{r-1} \le 0 \le \alpha_m$$

i.e., α_{m-r} and α_m are unknown but

$$\alpha_j = \begin{cases} -1, & \text{for } j = 1, 2, \ldots, m - r - 1 \\ 0, & \text{for } j = m - r + 1, \ldots, m - 1 \end{cases} \tag{3.2}$$

Then from (3.2), the characteristic polynomial of $K + D_r$ is:

$$\begin{aligned} p_{K+D_r}(z) &= z^{r-1}(z+1)^{m-r-1}(z - \alpha_{m-r})(z - \alpha_m) \\ &= z^{r-1}(z+1)^{m-r-1}[z^2 - (\alpha_{m-r} + \alpha_m)z + \alpha_{m-r}\alpha_m] \end{aligned} \tag{3.3}$$

Lemma 3.2. *The right hand side quadratic in (3.3) equals* $z^2 - (m-1)z - r$.

Proof: We know that $\sigma(K + D_r) = \{(-1)^{(m-r-1)}, \alpha_{m-r}, 0^{(r-1)}, \alpha_m\}$ and it is easy to check that $tr(K + D_r) = r$. But we also know that $tr(K + D_r) = \sum_{i=1}^{m} \lambda_i(K + D_r)$ so that $r = (-1)(m - r - 1) + \alpha_{m-r} + \alpha_m$. Thus we have

$$\boxed{\alpha_{m-r} + \alpha_m = m - 1} \tag{3.4}$$

By considering $(K + D_r)^2$ we have :

$$\begin{aligned}
(K + D_r)^2 &= \left[\begin{array}{c|c} [1]_r & [1]_{r\times(m-r)} \\ \hline [1]_{(m-r)\times r} & J_{(m-r)} \end{array}\right]^2 \text{,where } J_{(m-r)} = [1]_{(m-r)} - I_{(m-r)} \\
&= \left[\begin{array}{c|c} [1]_r + [1]_{r\times(m-r)}[1]_{(m-r)\times r} & * \\ \hline * & [1]_{(m-r)\times r}[1]_{r\times(m-r)} + J^2_{(m-r)} \end{array}\right] \\
&= \left[\begin{array}{c|c} [r]_r + [m-r]_r & * \\ \hline * & [r]_{(m-r)} + \begin{bmatrix} m-r-1 & & * \\ & \ddots & \\ * & & m-r-1 \end{bmatrix} \end{array}\right]
\end{aligned}$$

So

$$(K + D_r)^2 = \left[\begin{array}{c|c} [m]_r & * \\ \hline * & \begin{bmatrix} m-1 & & * \\ & \ddots & \\ * & & m-1 \end{bmatrix} \end{array}\right]$$

Now recall that for any matrix $C \in M_m(\mathbb{R})$, we also have $tr(C^k) = \sum_{i=1}^{m} \lambda_i(C)^k$.

Thus

$$\begin{aligned}
& tr((K + D_r)^2) = \sum_{i=1}^{m} \lambda_i(K + D_r)^2 \\
\Rightarrow\ & mr + (m-1)(m-r) = (m - r - 1) + \alpha_{m-r}^2 + \alpha_m^2 \\
\Rightarrow\ & 2\alpha_{m-r}\alpha_m + (m-1)^2 = -2r + (\alpha_{m-r} + \alpha_m)^2 \\
\Rightarrow\ & 2\alpha_{m-r}\alpha_m + (m-1)^2 = -2r + (m-1)^2, \text{ by (3.4)}
\end{aligned}$$

Hence we have

$$\boxed{\alpha_{m-r} \cdot \alpha_m = -r} \tag{3.5}$$

$\square$

This completes the proof of Lemma 3.2 and we now have

$$\boxed{p_{B+K_r}(z) = z^{r-1}(z+1)^{m-r-1}[z^2 - (m-1)z - r]} \tag{3.6}$$

3.2.2 Finding the polynomial $p_{K-D_r}(z)$

We use the same approach to find the characteristic polynomial of $K - D_r \in M_m(\mathbb{R})$. Note that

$$\begin{aligned} K - D_r &= [1]_m - I_m - diag(\underbrace{1,\dots,1}_{r},0,\dots,0) \\ &= diag(\underbrace{-2,\dots,-2}_{r},-1,\dots,-1) + ee^T \end{aligned}$$

so if we let $\beta_i = \lambda_i(K - D_r)$, then by Theorem A.1 we have:

$$-2 \le \underbrace{\beta_1 \le -2 \le \cdots \le \beta_{r-1}}_{r-1} \le -2 \le \beta_r \le -1 \le \underbrace{\beta_{r+1} \le -1 \le \cdots \le \beta_{m-1}}_{m-r-1} \le -1 \le \beta_m$$

Hence

$$\begin{aligned} p_{K-D_r}(z) &= (z+1)^{m-r-1}(z+2)^{r-1}(z-\beta_r)(z-\beta_m) \\ &= (z+1)^{m-r-1}(z+2)^{r-1}[z^2 - (\beta_r+\beta_m)z + \beta_r\beta_m] \end{aligned} \tag{3.7}$$

Lemma 3.3. *The right hand side quadratic in (3.7) equals* $z^2-(m-3)z-(2m-r-2)$.

Proof: We know that $\sigma(K - D_r) = \{(-2)^{(r-1)}, \beta_r, (-1)^{(m-r-1)}, \beta_m\}$ and it is easy to check that $tr(K - D_r) = -r$. But we also know that $tr(K + D_r) = \sum_{i=1}^{m} \beta_i$, hence $-r = -2\cdot(r-1) - (m-r-1) + \beta_r + \beta_m$. Thus we have

$$\boxed{\beta_r + \beta_m = m - 3} \tag{3.8}$$

Proceeding as in the proof of Lemma 3.2, we have:

$$
\begin{aligned}
(K-D_r)^2 &= \left[\begin{array}{c|c} ([1]-2I)_r & [1]_{r\times(m-r)} \\ \hline [1]_{(m-r\times r)} & \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix} \end{array}\right]^2 \\
&= \left[\begin{array}{c|c} ([1]_r^2-4[1]_r+4I_r)+[m-r] & * \\ \hline * & \begin{bmatrix} m-1 & & * \\ & \ddots & \\ * & & m-1 \end{bmatrix} \end{array}\right] \\
&= \left[\begin{array}{c|c} \begin{bmatrix} m & & * \\ & \ddots & \\ * & & m \end{bmatrix}_r & * \\ \hline * & \begin{bmatrix} m-1 & & * \\ & \ddots & \\ * & & m-1 \end{bmatrix} \end{array}\right]
\end{aligned}
$$

Thus

$$
\begin{aligned}
& tr((K-D_r)^2) = \sum_{i=1}^{m} \beta_i^2 \\
\Rightarrow\ & m\cdot r + (m-1)(m-r) = 4(r-1) + (m-r-1) + \beta_r^2 + \beta_m^2 \\
\Rightarrow\ & m^2 - 2m = 2r - 5 + \beta_r^2 + \beta_m^2 \\
\Rightarrow\ & (m-3)^2 + 2\beta_r\beta_m = 2r + 4 - 4m + (\beta_r+\beta_m)^2 \\
\Rightarrow\ & (\beta_r+\beta_m)^2 + 2\beta_r\beta_m = 2r + 4 - 4m + (\beta_r+\beta_m)^2, \text{ by (3.8)}
\end{aligned}
$$

Hence we have

$$
\boxed{\beta_r\cdot\beta_m = r + 2 - 2m} \tag{3.9}
$$

□

This completes the proof of Lemma 3.3. We now have by equations (3.7), (3.8), and (3.9) that

$$
\boxed{p_{K-D_r}(z) = (z+1)^{m-r-1}(z+2)^{r-1}[z^2-(m-3)z-(2m-r-2)]} \tag{3.10}
$$

3.2.3 Calculating $\mathcal{E}(KK(m,r))$

Now that we know $p_{K+D_r}(z)$ and $p_{K-D_r}(z)$ explicitly, we can easily compute $p_A(z)$. By equations (3.1), (3.6), and (3.10) the characteristic polynomial of A, $p_A(z)$, is given by:

$$\boxed{z^{r-1}(z+1)^{2(m-r-1)}(z+2)^{r-1}[z^2-(m-1)z-r][z^2-(m-3)z+(r+2-2m)]}$$

Hence $\sigma(A)$ is given by

$$\left\{0^{(r-1)}, (-1)^{(2m-2r-2)}, (-2)^{(r-1)}, \tfrac{(m-1)\pm\sqrt{m^2+1-2(m-2r)}}{2}, \tfrac{(m-3)\pm\sqrt{m^2+1+2(m-2r)}}{2}\right\}.$$

Thus by definition we have

$$\begin{aligned}
\mathcal{E}(KK(m,r)) &= \sum_{i=1}^{n}|\lambda_i| \\
&= (2m-2r-2)+2(r-1)+\tfrac{(m-1)+\sqrt{m^2+1-2(m-2r)}}{2} \\
&\quad +\left|\tfrac{(m-1)-\sqrt{m^2+1-2(m-2r)}}{2}\right| + \tfrac{(m-3)+\sqrt{m^2+1+2(m-2r)}}{2} \\
&\quad +\left|\tfrac{(m-3)-\sqrt{m^2+1+2(m-2r)}}{2}\right|.
\end{aligned}$$

But since $(m-1)\le\sqrt{m^2+1-2(m-2r)} \Leftrightarrow r\ge 0$ and that $(m-3)\le\sqrt{m^2+1+2(m-2r)} \Leftrightarrow m\ge\frac{r+2}{2}$ it easily follows that the energy of $KK(m,r)$ is given by

$$\boxed{\mathcal{E}(KK(m,r)) = 2m-4+\sqrt{m^2+1-2(m-2r)}+\sqrt{m^2+1+2(m-2r)}}$$

3.3 The main result

Now that we know an explicit formula for $\mathcal{E}(KK(m,r))$, we can say what are all values of m and r for which $\mathcal{E}(KK(m,r))$ and $\mathcal{E}(KK(m,r-1))$ have the same value. A compilation of all the information obtained of $KK(m,r)$ leads to the main result of this chapter.

Theorem 3.2. *For $m \geq 3$ and $1 \leq r \leq m$, let $KK(m,r)$ denote the graph composed of two copies of the complete graph K_m joined by r independent edges between them. Then $\mathcal{E}(KK(m,r)) = \mathcal{E}(KK(m,r-1))$ if and only if $m = 2r-1$.*

Proof: We have that

$$\begin{aligned}
& \mathcal{E}(KK(m,r)) = \mathcal{E}(KK(m,r-1)) \\
\Leftrightarrow\quad & 2m - 4 + \sqrt{m^2+1-2(m-2r)} + \sqrt{m^2+1+2(m-2r)} \\
& = 2m - 4 + \sqrt{m^2+1-2(m-2(r-1))} + \sqrt{m^2+1+2(m-2(r-1))} \\
\Leftrightarrow\quad & \sqrt{(m^2+1-2(m-2r)) \cdot (m^2+1+2(m-2r))} \\
& = \sqrt{(m^2+1-2(m-2r+2)) \cdot (m^2+1+2(m-2r+2))} \\
\Leftrightarrow\quad & (m^2+1)^2 - 4(m-2r)^2 = (m^2+1)^2 - 4(m-2r+2)^2 \\
\Leftrightarrow\quad & m = 2r-1.
\end{aligned}$$

□

In other words, if one of the r independent edges is removed, the new graph $KK(m,r-1)$ has the same energy if and only if m is odd and $r = \frac{m+1}{2}$. Thus there are infinitely many graphs with the desired equienergetic property.

Corollary 3.3. *For $k \in \mathbb{N}$, $\mathcal{E}(KK(2k+1,k+1)) = \mathcal{E}(KK(2k+1,k))$.*

Proof: Applying Theorem 3.2 to $r = k+1$, for $k \in \mathbb{N}$, gives the desired result.

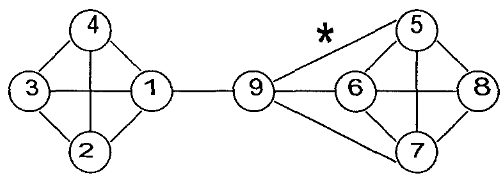
□

Remark 3.4. The results presented in this Chapter are from [2], with an alternate proof here for the characteristic polynomial of $KK(m, r)$. In [2], the authors make use of the identity: For $n \geq 2$, $det(D+J) = \prod_{i=1}^{n} d_i + \sum_{i=1}^{n} p_i$, where $D = diag(d_1, \ldots, d_n)$, $p_j = \prod_{\substack{i=1 \\ i \neq j}}^{n} d_i$, and J is the matrix consisting of all 1's and the same size as D. Using this and equation (3.1) we have $p_A(z) = det(B + D_r - zI_m)det(B - D_r - zI_m) = det((-D_r - (1+z)I_m) + J)det((D_r - (1+z)I_m) + J)$, where $(-D_r - (1+z)I_m)$ and $(D_r - (1+z)I_m)$ are diagonal matrices. Applying the above identity twice, the authors obtain the same characteristic equation $p_A(z)$ as obtained here in section 3.2.3, and the proof follows.

Now that we have demonstrated that the graphs from Figures 2.1 and 2.2 belong to an infinite family, we would like to see if this is also possible for the graph of Figure 2.3. The answer is in the affirmative, as we will show in the next chapter.

Chapter 4

Second Infinite Family



The next graph we investigate is that of Figure 2.3, shown labeled above. It is composed of 2 copies of K_4, i.e.,the complete graph on 4 vertices, a center vertex, and connections between the center vertex and the complete graphs. The question is: can we define a sequence of graphs which have the same characteristics. It is not easy to see how to do that, but the authors of [5] did and we will now study this new family.

4.1 The $KK(m, r, s)$ graph

By labeling the complete graphs first and the center vertex last, as shown above, the adjacency matrix takes the form of $A(K_4) \oplus A(K_4)$ bordered by two vectors of length 4, consisting of 1's and 0's. For example, labeling the graph in Figure 2.3 in this way gives its adjacency matrix

$$A = \left[\begin{array}{cccc|cccc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \tag{4.1}$$

Using this same labeling process with larger K_m helps find the following generalization. For $m \geq 1$ and $1 \leq r, s \leq m$, let $KK(m, r, s)$ denote the graph composed of two copies of the complete graph K_m with a center vertex, where the center vertex is connected to the left complete graph by r edges and to the right complete graph by s edges, respectively. Thus $KK(m, r, s)$ has $2m+1$ vertices and $m^2+r+s-2$ edges. Let $A = A(KK(m, r, s))$ and $K = A(K_m)$ be the adjacency matrix of $KK(m, r, s)$ and K_m, respectively. Then $A \in M_n(\mathbb{R}), n = 2m+1$, and by using the above labeling scheme, it is easy to show that A has the form

$$A = \left[\begin{array}{c|c|c} K & 0 & x_r \\ \hline 0 & K & x_s \\ \hline x_r & x_s & 0 \end{array}\right]$$

where $x_r, x_s \in \mathbb{R}^m$, and x_p is defined to be the column vector whose first p entries are 1 and the other $m-p$ entries are 0. For example, the matrix from (4.1) is the adjacency matrix of $KK(4, 1, 3)$. We now assume that $m \geq 4$ because one can easily check that no examples of $KK(m, r, s)$ with the equienergetic property exists for $mleq3$. Now with the notational foundation established, we proceed in solving for the characteristic equation of A.

4.2 Finding the eigenvalues of $KK(m, r, s)$

In this section we will apply results from [4] and [5]. Continue to let $A = A(KK(m, r, s))$ with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$, where $n = 2m + 1$. Let $A = A(K_m)$. Recall that $\sigma(K) = \{(-1)^{(m-1)}, m-1\}$ (See Appendix A). Let $H = K_m \oplus K_m$ denote the disjoint union of two copies of K_m, and set $B = A(H) = A(K_m) \oplus A(K_m)$, that is, $B = K \oplus K$ so that $\sigma(B) = \{(-1)^{(2m-2)}, (m-1)^{(2)}\}$; then by Theorem A.2 (Principal Submatrix Interlacing)

$$\lambda_1 \leq -1 \leq \underbrace{\lambda_2 \leq -1 \leq \cdots \leq -1 \leq \lambda_{n-3}}_{n-4} \leq -1 \leq \lambda_{n-2} \leq m-1 \leq \lambda_{n-1} \leq m-1 \leq \lambda_n$$

Thus $\lambda_2 = \cdots = \lambda_{n-3} = -1$ and $\lambda_{n-1} = m - 1$; however, λ_1, λ_{n-2}, and λ_n are still unknown. But

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$
$$\Rightarrow \quad 0 = (-1)(n-4) + (m-1) + \lambda_1 + \lambda_{n-2} + \lambda_n$$

so that

$$\boxed{\lambda_1 + \lambda_{n-2} + \lambda_n = m - 2.} \tag{4.2}$$

Identify the vertices of $K_m \oplus K_m$ with $\{1, \ldots, 2m\}$. Then $KK(m, r, s)$ is obtained by adding vertex $2m+1$ to the graph $K_m \oplus K_m$. Define $S = \{1, 2, \ldots, r, m+1, \ldots, m+s\}$. So S is all vertices joined to the new one. Note $\sum_{k \in S} e_k = \begin{bmatrix} x_r \\ x_s \end{bmatrix} \in \mathbb{R}^{2m}$. It is clear that the characteristic polynomial of B is given by $p_B(z) = (z+1)^{2m-2}[z-(m-1)]^2$, and -1 and $m-1$ are the only distinct eigenvalues of B.

Now we can apply Theorem A.3, which comes from [4], to say the characteristic polynomial of A is

$$p_A(z) = p_B(z)\left(z - \frac{\rho_1^2}{z-(m-1)} - \frac{\rho_2^2}{z+1}\right) \tag{4.3}$$

where $\rho_i = || \sum_{k\in S} P_i e_k|| = ||P_i \sum_{k\in S} e_k|| = \left\| P_i \begin{bmatrix} x_r \\ x_s \end{bmatrix} \right\|$.

Remark 4.1. Theorem A.3 relies on the spectral decomposition of B as $B = \gamma_1 P_1 + \gamma_2 P_2$ where $\gamma_1 = m-1$ and $\gamma_2 = -1$ and each P_i is the orthogonal projection onto the eigenspace E_{γ_i} for $i = 1,2$. Note that $m-1$ is an eigenvalue of B of multiplicity 2, so that

$$\{u_1, u_2\} := \left\{ \frac{1}{\sqrt{m}} \begin{bmatrix} x_m \\ 0 \end{bmatrix}, \frac{1}{\sqrt{m}} \begin{bmatrix} 0 \\ x_m \end{bmatrix} \right\}$$

can be taken as an orthonormal basis of E_{m-1}, hence $P_1 = u_1 u1^T + u_2 u_2^T$. Therefore $P_2 = (m-1)P_1 - B$, because $B = (m-1)P_1 + (-1)P_2$.

It remains to find the expressions for ρ_1^2 and ρ_2^2. This is done next.

4.2.1 Computing the value ρ_1^2:

We know $P_1 = u_1 u_1^T + u_2 u_2^T$ and $u_1 u_1^T = \frac{1}{\sqrt{m}} \begin{bmatrix} x_m \\ 0 \end{bmatrix} \frac{1}{\sqrt{m}} [x_m \; 0] = \frac{1}{m} \left[\begin{array}{c|c} [1]_m & 0 \\ \hline 0 & 0 \end{array}\right]$.

Similarly, $u_2 u_2^T = \frac{1}{m} \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & [1]_m \end{array}\right]$. Hence $P_1 = \frac{1}{m} \left[\begin{array}{c|c} [1]_m & 0 \\ \hline 0 & [1]_m \end{array}\right]$.

Then

$$\begin{aligned} P_1 \begin{bmatrix} x_r \\ x_s \end{bmatrix} &= \frac{1}{m} \left[\begin{array}{c|c} [1]_m & 0 \\ \hline 0 & [1]_m \end{array}\right] \begin{bmatrix} x_r \\ x_s \end{bmatrix} \\ &= \frac{1}{m} \begin{bmatrix} [1]_m x_s \\ [1]_m x_s \end{bmatrix} = \frac{1}{m} \begin{bmatrix} [r]_{m\times 1} \\ [s]_{m\times 1} \end{bmatrix} \end{aligned}$$

Thus $\rho_1^2 = \left\| P_1 \begin{bmatrix} x_r \\ x_s \end{bmatrix} \right\|^2 = \frac{1}{m^2}(r^2 m + s^2 m)$.

Hence

$$\boxed{\rho_1^2 = \frac{r^2 + s^2}{m}} \tag{4.4}$$

4.2.2 Computing the value ρ_2^2:

Notice $P_1 = \frac{1}{m}(B+I)$ so $P_2 = (m-1)P_1 - B = \frac{m-1}{m}(B+I) - B = \frac{-1}{m}B + \frac{m-1}{m}I$.
Then $P_2 \begin{bmatrix} x_r \\ x_s \end{bmatrix} = -\frac{1}{m} \begin{bmatrix} (K-(m-1)I)x_r \\ (K-(m-1)I)x_s \end{bmatrix}$.

But $[1]_m x_r = [r]_{m\times 1} \Rightarrow (K + I_m)x_r = [r]_{m\times 1} \Rightarrow Kx_r = [r]_{m\times 1} - x_r$.

Thus

$$\begin{aligned} P_2 \begin{bmatrix} x_r \\ x_s \end{bmatrix} &= -\frac{1}{m} \begin{bmatrix} [r]_{m\times 1} - x_r - (m-1)x_r \\ [s]_{m\times 1} - x_s - (m-1)x_s \end{bmatrix} \\ &= -\frac{1}{m} \begin{bmatrix} [r]_{m\times 1} - mx_r \\ [s]_{m\times 1} - mx_s \end{bmatrix}. \end{aligned}$$

In other words, we have

$$P_2 \begin{bmatrix} x_r \\ x_s \end{bmatrix} = -\frac{1}{m} \begin{bmatrix} r-m \\ \vdots \\ r-m \\ r \\ \vdots \\ r \\ s-m \\ \vdots \\ s-m \\ s \\ \vdots \\ s \end{bmatrix},$$

where there are $r, m-r, s$ and $m-s$ of each different variable, respectively. Thus $\rho_2^2 = \left\| P_2 \begin{bmatrix} x_r \\ x_s \end{bmatrix} \right\|^2 = \frac{1}{m^2}[(m-r)r^2 + (r-m)^2 r + (m-s)s^2 + (s-m)^2 s]$ or

$$\boxed{\rho_2^2 = r + s - \frac{r^2+s^2}{m}} \tag{4.5}$$

4.3 Calculating the zeros of $p_A(z)$

By equations (4.3), (4.4), (4.5), and some simplification, we find that the characteristic polynomial of A, $p_A(z)$ is:

$$\boxed{(z+1)^{2m-3}(z-m+1)[z^3 + (2-m)z^2 - (m+r+s-1)z + (m-1)(r+s) - (r^2+s^2)]}$$

Hence the unknown eigenvalues λ_1, λ_{n-2}, and λ_n must come from the cubic polynomial

$$\boxed{q_1(z) = z^3 + (2-m)z^2 - (m+r+s-1)z + (m-1)(r+s) - (r^2+s^2)}. \tag{4.6}$$

Moreover, recall that by the interlacing inequality at the beginning of this section, regardless what the values of m, r, and s are, we will always have that $\lambda_1 \leq -1 \leq \lambda_{n-2} \leq m-1 \leq \lambda_n$ so that $\lambda_1 < 0$, and $\lambda_n > 0$ always.

Let α_i be the eigenvalues of the graph $KK(m, r, s-1)$, then substituting $s-1$ for s on the above expressions yields formulas for the eigenvalues α_1, α_{n-2}, and α_n; They will be the zeros of

$$\boxed{q_2(z) = z^3 + (2-m)z^2 - (m+r+s-2)z + (m-1)(r+s-1) - (r^2+(s-1)^2)}, \tag{4.7}$$

which can be rewritten as $q_2(z) = q_1(z) + z + (2s-m)$. Additionally, the zeros of $q_2(z)$ will also have the expected properties: $\alpha_1 \leq -1 \leq \alpha_{n-2} \leq m-1 \leq \alpha_n$ and so $\alpha_1 < 0$, and $\alpha_n > 0$. Moreover, by equation (4.2), namely

$$\boxed{\alpha_1 + \alpha_{n-2} + \alpha_n = m-2} \tag{4.8}$$

Notice that if $q_1(z)$ and $q_2(z)$ have a common zero, say z_0, then $q_2(z_0) = q_1(z_0) + z_0 + (2s-m)$, which implies that $z_0 = m-2s$, i.e., if $q_1(z)$ and $q_2(z)$ have a common zero, then it must be $m-2s$. Finally, since λ_1, λ_{n-2}, and λ_n are zeros of $q_1(z) = z^3 + (2-m)z^2 - (m+r+s-1)z + (m-1)(r+s) - (r^2+s^2)$, we have that $q_1(z) = (z-\lambda_1)(z-\lambda_{n-2})(z-\lambda_n)$ implies that

$$\begin{aligned} \lambda_1 + \lambda_{n-2} + \lambda_n &= m-2 && (4.9)\\ \lambda_1\lambda_{n-2} + \lambda_1\lambda_n + \lambda_{n-2}\lambda_n &= -(m+r+s-1) && (4.10)\\ \lambda_1\lambda_{n-2}\lambda_n &= (r^2+s^2) - (m-1)(r+s). && (4.11) \end{aligned}$$

Similarly for $\alpha_1, \alpha_{n-2}, \alpha_n$, and $q_2(z)$, it follows from (4.7) that

$$\begin{aligned}
\alpha_1 + \alpha_{n-2} + \alpha_n &= m - 2 && (4.12)\\
\alpha_1\alpha_{n-2} + \alpha_1\alpha_n + \alpha_{n-2}\alpha_n &= -(m + r + s - 2) && (4.13)\\
\alpha_1\alpha_{n-2}\alpha_n &= (r^2 + s^2) - (m-1)(r+s) + (m - 2s). && (4.14)
\end{aligned}$$

Combining the above two set of equations we have

$$\begin{aligned}
\lambda_1 + \lambda_{n-2} + \lambda_n &= m - 2 = \alpha_1 + \alpha_{n-2} + \alpha_n && (4.15)\\
\lambda_1\lambda_{n-2} + \lambda_1\lambda_n + \lambda_{n-2}\lambda_n &= -(m + r + s - 1) &&\\
&= \alpha_1\alpha_{n-2} + \alpha_1\alpha_n + \alpha_{n-2}\alpha_n - 1 && (4.16)\\
\lambda_1\lambda_{n-2}\lambda_n &= (r^2 + s^2) - (m-1)(r+s) &&\\
&= \alpha_1\alpha_{n-2}\alpha_n + (2s - m) && (4.17)
\end{aligned}$$

4.4 Conditions for equienergicity

We would now like to find when $KK(m, r, s)$ and $KK(m, r, s-1)$ are equienergetic. Our first step is the following Lemma.

Lemma 4.1. *Assume $m \geq 4$ and $1 \leq r, s \leq m$. Let λ_1, λ_{n-2} and λ_n be the roots of the cubic polynomial $q_1(z) = z^3 + (2-m)z^2 - (m+r+s-1)z + (m-1)(r+s) - (r^2+s^2)$ obtained from the characteristic polynomial of $KK(m, r, s)$. Then*

$$\begin{cases}
\text{(i) } r^2 + (1-2s)r + (2m^2s - 8ms^2 + 8s^3 - m^2 + 6ms - 9s^2 - m + 3s) = 0\\
\text{(ii) } m - 2s < 0\\
\text{(iii) } (m-1)(r+s) - (r^2+s^2) > 0.
\end{cases}$$

$$\text{if and only if} \begin{cases} \lambda_1 = m - 2s < 0 \\ \lambda_{n-2}, \lambda_n > 0 \end{cases}$$

Proof: Suppose conditions (i), (ii), and (iii) hold. Note that condition (i) is equivalent to saying $q_1(m-2s) = 0$. Since λ_1, λ_{n-2}, and λ_n are zeros of the cubic polynommial $q_1(z)$, we have that $q_1(z) = (z-\lambda_1)(z-\lambda_{n-2})(z-\lambda_n) \Rightarrow -\lambda_1\lambda_{n-2}\lambda_n = (m-1)(r+s) - (r^2+s^2)$. But by (iii), we have $\lambda_1\lambda_{n-2}\lambda_n < 0$. Moreover, since λ_1 and λ_n are always negative and positive, respectively, we must have that $\lambda_{n-2} > 0$. But by (i), $m-2s$ must be a zero of $q_1(z)$, and by (ii), it must be negative. Thus $\lambda_1 = m-2s$, and the result follows. As for the converse, $\lambda_1 = m-2s$ and negative, implies conditions (i) and (ii). Condition (iii) is evident by the information provided by all three eigenvalues. □

We wish to find conditions on m, r and s when $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$. Since $\sigma(KK(m,r,s))$ is now known, we easily find that $\mathcal{E}(KK(m,r,s)) = 3m - 4 + |\lambda_1| + |\lambda_{n-2}| + |\lambda_n|$ and $\mathcal{E}(KK(m,r,s-1)) = 3m-4+|\alpha_1|+|\alpha_{n-2}|+|\alpha_n|$. Thus $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$ if and only if $\sum_{i=1}^{n}|\lambda_i| = \sum_{i=1}^{n}|\alpha_i|$ if and only if $|\lambda_1| + |\lambda_{n-2}| + |\lambda_n| = |\alpha_1| + |\alpha_{n-2}| + |\alpha_n|$. But $\lambda_1, \alpha_1 < 0$ and $\lambda_n, \alpha_n > 0$ always, so $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$ if and only if

$$\boxed{-\lambda_1 + |\lambda_{n-2}| + \lambda_n = -\alpha_1 + |\alpha_{n-2}| + \alpha_n} \tag{4.18}$$

We continue our analysis with the next lemma, which is a slight modification of part of a theorem found in [5].

Lemma 4.2. *Suppose λ_i and α_i are the zeros for $q_1(z)$ and $q_2(z)$, respectively for $i = 1, n-2$, and n. If $\lambda_{n-2} \leq 0$ and $\alpha_{n-2} > 0$, then*

$$\lambda_n = (s-a-1) + \sqrt{(s-a-1)^2 + a(m-2)},$$

where $a = \lambda_1\lambda_{n-2}\lambda_n = (r^2+s^2) - (m-1)(r+s)$.

Proof: Since $\lambda_{n-2} \leq 0$ and $\alpha_{n-2} > 0$, it follows by (4.18) that $-\lambda_1 - \lambda_{n-2} + \lambda_n = -\alpha_1 + \alpha_{n-2} + \alpha_n$. Combining this with equation (4.15) implies that

$$\lambda_n = \alpha_{n-2} + \alpha_n \tag{4.19}$$

$$\alpha_1 = \lambda_1 + \lambda_{n-2} \tag{4.20}$$

Now by equation (4.16) we have that

$$\begin{aligned}
& \lambda_1\lambda_{n-2} + (\lambda_1 + \lambda_{n-2})\lambda_n = \alpha_1(\alpha_{n-2}+\alpha_n) + \alpha_{n-2}\alpha_n - 1 \\
\Rightarrow\ & \lambda_1\lambda_{n-2} + (\alpha_1)\lambda_n = \alpha_1(\lambda_n) + \alpha_{n-2}\alpha_n - 1, \text{ by (4.19) and (4.20)} \\
\Rightarrow\ & \lambda_1\lambda_{n-2} = \alpha_{n-2}\alpha_n - 1 \\
\Rightarrow\ & \frac{\lambda_1\lambda_{n-2}\lambda_n}{\lambda_n} = \frac{\alpha_1\alpha_{n-2}\alpha_n}{\alpha_1} - 1 \\
\Rightarrow\ & \alpha_1(\lambda_1\lambda_{n-2}\lambda_n) = (\alpha_1\alpha_{n-2}\alpha_n)\lambda_n - \alpha_1\lambda_n \\
\Rightarrow\ & \alpha_1(\lambda_1\lambda_{n-2}\lambda_n + \lambda_n) = (\alpha_1\alpha_{n-2}\alpha_n)\lambda_n \\
\Rightarrow\ & \alpha_1 = \left(\frac{\alpha_1\alpha_{n-2}\alpha_n}{\lambda_1\lambda_{n-2}\lambda_n + \lambda_n}\right)\lambda_n
\end{aligned}$$

Let $a = \lambda_1\lambda_{n-2}\lambda_n$ and $b = \alpha_1\alpha_{n-2}\alpha_n$, then by (4.17) it follows that $b = a + (m-2s)$. Thus

$$\alpha_1 = \left(\frac{b}{a+\lambda_n}\right)\lambda_n. \tag{4.21}$$

Now by (4.15), we have that

$$\begin{aligned}
& \alpha_1 + (\alpha_{n-2} + \alpha_n) = m - 2 \\
\Rightarrow\ & \left(\frac{b}{a+\lambda_n}\right)\lambda_n + \lambda_n = m-2, \text{ by (4.21) and (4.19)} \\
\Rightarrow\ & \lambda_n^2 + (a+b-m+2)\lambda_n - a(m-2) = 0 \\
\Rightarrow\ & \lambda_n^2 + 2(a-s+2)\lambda_n - a(m-2) = 0, \text{ since } b = a + (m-2s).
\end{aligned}$$

Finally, since $\lambda_n > 0$ always, we have that

$$\lambda_n = (s - a - 1) + \sqrt{(s - a - 1)^2 + a(m - 2)},$$

where $a = \lambda_1 \lambda_{n-2} \lambda_n = (r^2 + s^2) - (m - 1)(r + s)$. $\square$

Before we proceed to the main theorem, we have one more lemma.

Lemma 4.3. *For $m \geq 4$ and $1 \leq r, s \leq m$, we have that*

$$\mathcal{E}(KK(m, r, s)) = \mathcal{E}(KK(m, r, s - 1)) \Leftrightarrow \begin{cases} \lambda_1 = m - 2s < 0 \\ \lambda_{n-2}, \lambda_n > 0 \end{cases}$$

Proof($\Leftarrow$): Suppose that $\lambda_1 = m - 2s < 0$ and that $\lambda_{n-2}, \lambda_n > 0$. Recall that $q_2(z) = q_1(z) + z + (2s - m)$, where $q_1(z)$ and $q_2(z)$ are cubic polynomials coming from the characteristic polynomials of $KK(m, r, s)$ and $KK(m, r, s - 1)$, respectively. From this equation it is clear that if $q_1(z)$ and $q_2(z)$ have a common zero, then it must be given by $m - 2s$. In other words, if $m - 2s$ is a zero of $q_1(z)$, then it must also be a zero of $q_2(z)$. Now consider the constant term of $p_2(z)$; by Lemma 4.1 we have that

$$(m - 1)(r + s - 1) - (r^2 + s^2 - 2s + 1) = \underbrace{(m - 1)(r + s) - (r^2 + s^2)}_{>0} + \underbrace{2s - m}_{>0} > 0.$$

Then if $p_2(z) = (z - \alpha_1)(z - \alpha_{n-2})(z - \alpha_n)$, arguing as in the proof of Lemma 4.1, we obtain $\alpha_1 = m - 2s$ and $\alpha_{n-2}, \alpha_n > 0$. Thus $\lambda_1 = \alpha_1$ and $\lambda_{n-2}, \alpha_{n-2} > 0$ give:

$$\begin{aligned} -\lambda_1 + |\lambda_{n-2}| + \lambda_n &= -\lambda_1 + \lambda_{n-2} + \lambda_n \\ &= -2\lambda_1 + (\lambda_1 + \lambda_{n-2} + \lambda_n) \\ &= -2\alpha_1 + (\alpha_1 + \alpha_{n-2} + \alpha_n), \text{ by (4.2) and (4.8)} \\ &= -\alpha_1 + |\alpha_{n-2}| + \alpha_n \end{aligned}$$

Hence $-\lambda_1 + |\lambda_{n-2}| + \lambda_n = -\alpha_1 + |\alpha_{n-2}| + \alpha_n$, which implies that $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$, by the argument leading to equation (4.18). □

Proof(⇒): Suppose that $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$; then $-\lambda_1 + |\lambda_{n-2}| + \lambda_n = -\alpha_1 + |\alpha_{n-2}| + \alpha_n$. We will show that we must have $\alpha_{n-2}, \lambda_{n-2} > 0$.

Claim ($\boldsymbol{\alpha_{n-2} > 0}$): By way of contradiction, suppose $\alpha_{n-2} \leq 0$. If $\lambda_{n-2} > 0$, then (4.18) gives $-\lambda_1 + \lambda_{n-2} + \lambda_n = -\alpha_1 - \alpha_{n-2} + \alpha_n$, which implies that $\lambda_{n-2} + \lambda_n = \alpha_n$. But since $KK(m,r,s-1)$ is a subgraph of the connected graph $KK(m,r,s)$, and their adjacency matrices are irreducible and nonnegative, λ_n and α_n are their dominant eigenvalues; and B is a proper submatrix of A, we have $\lambda_n > \alpha_n$. This is by the Perron-Frobenius Theory [3]. Thus $\lambda_{n-2}+\lambda_n = \alpha_n < \lambda_n$ implies that $\lambda_{n-2} < 0$, a contradiction. On the other hand, if $\lambda_{n-2} \leq 0$, then (4.18) reduces to $-\lambda_1 - \lambda_{n-2} + \lambda_n = -\alpha_1 - \alpha_{n-2} + \alpha_n \Rightarrow -(\lambda_1 + \lambda_{n-2} + \lambda_n) + 2\lambda_n = -(\alpha_1 + \alpha_{n-2} + \alpha_n) + 2\alpha_n \Rightarrow \lambda_n = \alpha_n$. Thus λ_n is a common zero of $q_1(z)$ and $q_2(z)$. Then $q_2(z) = q_1(z) + z + (2s - m)$ implies that $\lambda_n = \alpha_n = m - 2s$. But $\lambda_n \geq m - 1 \Rightarrow m - 2s \geq m - 1 \Rightarrow s = 0$, which contradicts our assumption that $s \geq 1$. Therefore $\alpha_{n-2} \leq 0$ is not possible. □

With this established, (4.18) reduces to:

$$\boxed{-\lambda_1 + |\lambda_{n-2}| + \lambda_n = -\alpha_1 + \alpha_{n-2} + \alpha_n} \tag{4.22}$$

Claim ($\boldsymbol{\lambda_{n-2} > 0}$): By way of contradiction, assume $\lambda_{n-2} \leq 0$. Since $\lambda_1 < 0$ and $\lambda_n > 0$ always, it follows that $\lambda_1\lambda_{n-2}\lambda_n = (r^2 + s^2) - (m-1)(r+s) \geq 0$. We have the following cases:

Case 1 ($r = s = m$): In this case, we have $\alpha_1\alpha_{n-2}\alpha_n = (r^2 + s^2) - (m-1)(r+s) +$

$(m-2s) = m > 0$. But by the previous claim, $\alpha_{n-2} > 0$, and since $\alpha_1 < 0$ and $\alpha_n > 0$ always, we must have $\alpha_1\alpha_{n-2}\alpha_n < 0$, a contradiction.

Case 2 ($r = s = m-1$): Here we have that $\lambda_1\lambda_{n-2}\lambda_n = (r^2+s^2)-(m-1)(r+s) = 0$. Since λ_1 and λ_n are nonzero, we must have that $\lambda_{n-2} = 0$. Thus (4.22) reduces to $\lambda_n - \lambda_1 = -\alpha_1 + \alpha_{n-2} + \alpha_n$. This and the fact that $\lambda_1 + \lambda_{n-2} + \lambda_m = m - 2 = \alpha_1 + \alpha_{n-2} + \alpha_m$ implies that $\lambda_1 = \alpha_1$. Thus λ_1 is a common zero of $q_1(z)$ and $q_2(z)$. But $q_2(z) = q_1(z) + z + (2s - m)$ implies that $\lambda_1 = \alpha_1 = m - 2s = 2 - m$. Now since $\lambda_{n-2} = 0$, by (4.15) we have $\lambda_1\lambda_n = -(m+r+s-1) = 3-3m$. But (4.2) implies that $\lambda_n = m-2-\lambda_1 = m-2-(2-m) = 2m-4$. Therefore, $3-3m = \lambda_1\lambda_n = (2-m)(2m-4)$ or $2m^2 - 11m + 11 = 0 \Rightarrow m \notin \mathbb{N}$, a contradiction.

Case 3 ($r = m$ and $s \leq m-1$): Since $\alpha_1\alpha_{n-2}\alpha_n < 0$, we have that

$$\begin{aligned} & \alpha_1\alpha_{n-2}\alpha_n = (r^2+s^2) - (m-1)(r+s) + (m-2s) < 0, \text{ by (4.14)} \\ \Rightarrow\ & s^2 - (m+1)s + 2m < 0, \text{ since } r = m \\ \Rightarrow\ & \frac{(m+1) - \sqrt{m^2-6m+1}}{2} < s < \frac{(m+1) + \sqrt{m^2-6m+1}}{2}, \end{aligned}$$

for $m^2-6m+1 \geq 0$. However, $2 < \frac{(m+1)-\sqrt{m^2-6m+1}}{2} < 3$ and $m-2 < \frac{(m+1)-\sqrt{m^2-6m+1}}{2} < m-1$ for $m > 6$. Hence we must have that $3 \leq s \leq m-2$ for $m > 6$.
On the other hand, since $\lambda_{n-2} \leq 0$ by assumption, we have that $\lambda_1\lambda_{n-2}\lambda_n \geq 0$. Using this with (4.11) we have that

$$\begin{aligned} & \lambda_1\lambda_{n-2}\lambda_n = (r^2+s^2) - (m-1)(r+s) \geq 0 \\ \Rightarrow\ & s^2 - (m-1)s + m \geq 0, \text{ since } r = m \end{aligned}$$

If $m = 6$ this is true for all s. If $m \geq 6$, then it is true when

$$s \leq \frac{(m-1)-\sqrt{m^2-6m+1}}{2} \text{ or } s \geq \frac{(m-1)+\sqrt{m^2-6m+1}}{2},$$

because $m^2-6m+1 \geq 0$. But $1 < \frac{(m-1)-\sqrt{m^2-6m+1}}{2} < 2$ and $m-3 < \frac{(m-1)+\sqrt{m^2-6m+1}}{2} < m-2$ for $m > 6$ implies that $s \leq 1$ or $s \geq m-2$. This and the fact that $3 \leq s \leq m-2$ from our previous argument, implies that $s = m-2$ for $m > 6$.

If $r = m$ and $s = m-2$, then (4.6) reduces to $q_1(z) = z^3 + (2-m)z^2 - 3(m-1)z - 2$, and by Lemma 4.2, $\lambda_n = (m-5) + \sqrt{m^2-8m+21}$. But λ_n is a zero of $q_1(z)$, so

$$\begin{aligned}
& q_1(\lambda_n) = 0 \\
\Rightarrow\ & (2m^3 - 35m^2 + 194m - 365) + (2m^2 - 27m + 79)\sqrt{m^2-8m+21} = 0 \\
\Rightarrow\ & (2m^3 - 35m^2 + 194m - 365)^2 - (2m^2 - 27m + 79)^2(m^2 - 8m + 21) = 0 \\
\Rightarrow\ & 8m^4 - 146m^3 + 872m^2 - 2106m + 2164 = 0,
\end{aligned}$$

and one can check numerically that the zeros of this equation are not integers, a contradiction.

Case 4 ($r \leq m-1$ and $s = m$): Since $\alpha_1\alpha_{n-2}\alpha_n < 0$, we have that

$$\begin{aligned}
& \alpha_1\alpha_{n-2}\alpha_n = (r^2+s^2) - (m-1)(r+s) + (m-2s) < 0, \text{ by (4.14)} \\
\Rightarrow\ & r^2 - (m-1)r < 0, \text{ since } s = m \\
\Rightarrow\ & 1 \leq r \leq m-2,
\end{aligned}$$

On the other hand, since $\lambda_{n-2} \leq 0$ by assumption, we have that $\lambda_1\lambda_{n-2}\lambda_n \geq 0$, so that

$$\begin{aligned}
& \lambda_1\lambda_{n-2}\lambda_n = (r^2+s^2) - (m-1)(r+s) \geq 0, \text{ by (4.11)} \\
\Rightarrow\ & r^2 - (m-1)r + m \geq 0, \text{ since } s = m \\
\Rightarrow\ & r \leq \frac{(m-1)-\sqrt{m^2-6m+1}}{2} \text{ or } r \geq \frac{(m-1)+\sqrt{m^2-6m+1}}{2},
\end{aligned}$$

for $m^2 - 6m + 1 \geq 0$. But $1 < \frac{(m-1)-\sqrt{m^2-6m+1}}{2} < 2$ and $m - 3 < \frac{(m-1)+\sqrt{m^2-6m+1}}{2} < m-2$ for $m > 6$ implies that $r \leq 1$ or $r \geq m-2$. This and the fact that $1 \leq r \leq m-2$ from our previous argument, implies that $r = 1$ or $r = m - 2$ for $m > 6$.

If $r = 1$ and $s = m$, then (4.6) reduces to $q_1(z) = z^3 + (2 - m)z^2 - 2mz - 2$, and by Lemma 4.2, $\lambda_n = (m - 3) + \sqrt{m^2 - 4m + 5}$. But λ_n is a zero of $q_1(z)$, so

$$\begin{aligned}
& q_1(\lambda_n) = 0 \\
\Rightarrow\quad & (2m^3 - 18m^2 + 50m - 46) + (2m^2 - 14m + 20)\sqrt{m^2 - 4m + 5} = 0 \\
\Rightarrow\quad & (2m^3 - 18m^2 + 50m - 46)^2 - (2m^2 - 14m + 20)^2(m^2 - 4m + 5) = 0 \\
\Rightarrow\quad & 4m^4 - 40m^3 + 136m^2 - 200m + 116 = 0,
\end{aligned}$$

and one can check numerically that $m \notin \mathbb{N}$, a contradiction.

Finally, if $r = m - 2$ and $s = m$, then (4.6) reduces to $q_1(z) = z^3 + (2 - m)z^2 - 3(m - 1)z - 2$, and by Lemma 4.2, $\lambda_n = (m - 3) + \sqrt{m^2 - 4m + 5}$. But λ_n is a zero of $q_1(z)$, so

$$\begin{aligned}
& q_1(\lambda_n) = 0 \\
\Rightarrow\quad & (2m^3 - 19m^2 + 56m - 55) + (2m^2 - 15m + 23)\sqrt{m^2 - 4m + 5} = 0 \\
\Rightarrow\quad & (2m^3 - 19m^2 + 56m - 55)^2 - (2m^2 - 15m + 23)^2(m^2 - 4m + 5) = 0 \\
\Rightarrow\quad & 8m^4 - 90m^3 + 352m^2 - 594m + 380 = 0,
\end{aligned}$$

and one can check numerically that $m \notin \mathbb{N}$, also a contradiction. □

Before we proceed to the main results, we have the following remark.

Remark 4.2. Note that if $s = m$, then by Lemma 4.3 $\mathcal{E}(KK(m, r, s)) = \mathcal{E}(KK(m, r, s-1))$ if and only if $r^2 + (1 - 2m)r + (2m^3 - 4m^2 + 2m) = 0$ if and only if $r =$

$\frac{(2m-1)+\sqrt{-8m^3+12m^2-12m+1}}{2}$ for $-8m^3+12m^2-12m+1 \geq 0$. But $-8m^3+12m^2-12m+1 \geq 0$ if and only if $m=1$. Therefore we can assume $s < m$.

4.5 The main result

Lemmas 4.1 and 4.3 in the previous section provide some conditions as to when $KK(m,r,s)$ and $KK(m,r,s-1)$ have the same energy. Merging both Lemmas 4.1 and 4.3 lead to the main result of this Chapter.

Theorem 4.3. *For $m \geq 4$ and $1 \leq r, s \leq m$, let $KK(m,r,s)$ denote the graph composed of two copies of the complete graph K_m with a center vertex, where the center vertex is connected to the left complete graph by r edges and to the right complete graph by s edges, respectively. Then*
$\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$ if and only if

$$\begin{cases} (i)\ r^2 + (1-2s)r + (2m^2s - 8ms^2 + 8s^3 - m^2 + 6ms - 9s^2 - m + 3s) = 0 \\ (ii)\ m - 2s < 0 \\ (iii)\ (m-1)(r+s) - (r^2+s^2) > 0. \end{cases}$$

i.e., if and only if $m-2s$ is the only negative root of $q_1(z)$ where $q_1(z)$ is the cubic factor of the characteristic polynomial of $KK(m,r,s)$ obtained above.

Although Theorem 4.3 provides conditions for the existence of the $KK(m,r,s)$ family of equienergetic graphs, it does not provide explicit values m, r, and s, like that of Theorem 3.2 in Chapter 3. This is because an explicit formula for $\mathcal{E}(KK(m,r,s))$ is unknown for arbitrary values m, r, and s. However, for values of m, r, and s that do satisfy the conditions of Theorem 4.3, we are able to explicitly provide the value $\mathcal{E}(KK(m,r,s))$, as shown in the next Corollary.

Corollary 4.4. *Suppose m, r, and s satisfy the conditions of Theorem 4.3, then* $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1)) = 2(m+2s-3)$.

Proof: We know that $\mathcal{E}(KK(m,r,s)) = 3m-4+|\lambda_1|+|\lambda_{n-2}|+|\lambda_n|$, where λ_1, λ_{n-2} and λ_n are the roots of the cubic polynomial $q_1(z) = z^3+(2-m)z^2-(m+r+s-1)z+(m-1)(r+s)-(r^2+s^2)$, obtained from the characteristic polynomial of $KK(m,r,s)$. Since $\lambda_1 = m-2s < 0$, we can factor $q_1(z)$ as $q_1(z) = [z-(m-2s)][z^2+2(1-s)z+c]$, where $c = \frac{(m-1)(r+s)-(r^2+s^2)}{2s-m} > 0$. Thus we have that $\lambda_1 = m-2s, \lambda_{n-2} = (s-1)-\sqrt{(s-1)^2-c}$, and $\lambda_n = (s-1)+\sqrt{(s-1)^2-c}$. Hence we have

$$\begin{aligned}
\mathcal{E}(KK(m,r,s)) &= 3m-4+|\lambda_1|+|\lambda_{n-2}|+|\lambda_n| \\
&= 3m-4+|m-2s| \\
&\quad +|(s-1)-\sqrt{(s-1)^2-c}|+|(s-1)+\sqrt{(s-1)^2-c}| \\
&= 3m-4-(m-2s) \\
&\quad +(s-1)-\sqrt{(s-1)^2-c}+(s-1)+\sqrt{(s-1)^2-c}
\end{aligned}$$

Therefore, by Theorem 4.3, we have

$$\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1)) = 2(m+2s-3). \quad \square$$

Theorem 4.3 provides conditions for the values m, r, and s for equienergicity; however, these conditions don't provide explicit formulas for the triplet (m,r,s). The following corollary shows that the graph $KK(m,r,s)$ is indeed part of an infinite family of graphs and also provides some explicit formulas the values m, r, and s can take.

Corollary 4.5. *There is an infinite family of graphs of the type $KK(m,r,s)$ each of which is equienergetic with a subgraph $KK(m,r,s-1)$.*

Proof: Let $k \in \mathbb{N}$, and suppose that m, r, and s satisfy any of the following conditions:

(i) For $k \geq 1$, $r = k^2$, $s = k^2 + k + 1$, and $m = 2k^2 + k + 1$.

(ii) For $k \geq 3$, $r = k^2$, $s = k^2 - k + 1$, and $m = 2k^2 - 3k + 2$.

(iii) For $k \geq 2$, $r = k$, $s = 2k^2 + 1$, and $m = 4k^2 - k + 1$.

(iv) For $k \geq 3$, $m = r = 4k^2 - 9k + 6$, and $s = 2k^2 - 4k + 3$.

Then it follows by Theorem 4.3 that $\mathcal{E}(KK(m, r, s)) = \mathcal{E}(KK(m, r, s - 1))$. □

The above Corollary provides some values that m, r, and s can take that satisfy the condition $\mathcal{E}(KK(m, r, s)) = \mathcal{E}(KK(m, r, s - 1))$; however, conditions (i)-(iv) do not entirely partition all possible values m, r, and s can take. Table B.1 in Appendix B shows the first few m, r, and s values and the corresponding condition they belong to.

In this chapter we have shown that the graph from Figure 2.3 belongs to an infinite family with the necessary equienergetic property. The next question, as expected, is whether or not the graph from Figure 2.4 also belongs to an infinite family with this property. Unfortunately the answer is still unknown, but we investigate this graph's properties next.

Chapter 5

Unknown Family

5.1 Unknown Family on 9 vertices

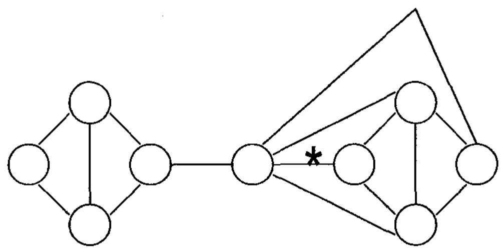


Figure 5.1: The graph #32042 from **CN9**.

Unlike Chapters 3 and 4, where $KK(3,2)$ and $KK(4,1,3)$ are shown to belong to an infinite family of graphs, it is still unknown whether the graph in Figure 5.1 is part of an infinite family of graphs or not. In Chapter 3 the graph $KK(m,r)$ was defined as consisting of two copies of K_m joined by r edges between them, where $m = 2r-1$ and $r \in \mathbb{N}$ with $r \geq 2$. In Chapter 4 the graph $KK(m,r,s)$ was introduced, this one consists of two copies of the graph K_m joined to an intermediate vertex by r and s edges to the left and right, respectively. In the case of Figure 5.1 notice that we have two copies of a graph U with 4 vertices but $U \neq K_4$, and U is joined to an intermediate vertex by 1 edge to the left and 4 edges to the right. We call this graph $UU(4,1,4)$. It is equienergetic with the subgraph obtained by removing the marked edge. However we do not yet know of a way to generalize this example to obtain larger graphs with this property. It would seem reasonable to try letting U be:

i *the complete graph* K_m *with an edge removed.*

ii *the complete graph* K_m *with* k *edges removed.*

iii *the cycle graph* C_m *with an additional edge.*

We tried to use the MATLAB scripts **GnrlztnOfScnd9VrtxGraphSearch-1edgeLess** and **GnrlztnOf10VrtxCycleGraphSearch** shown in Appendix B.1.4 and B.1.5, respectively, to investigate such questions; however, memory limitations prevented us from testing many graphs when $m \geq 10$.

Recall that Table B.1 in Appendix B provides values m, r, and s that satisfy the conditions of equienergeticity for the graph $K(m, r, s)$. In Table B.1 it is seen that the next smallest value of m is $m = 11$, and it is true that the graph $KK(11, 4, 7)$ is the next graph in the sequence of the infinite family. This fact with the similar structure of $UU(m, r, s)$ and $KK(m, r, s)$, gives us hope that $UU(4, 1, 4)$ might indeed belong to an infinite family of graphs where U belongs to one of the three categories outlined by the above three questions.

We end this section by analytically showing that $UU(4, 1, 4)$ and $UU(4, 1, 3)$ have the same energy. Labelling the two copies of graphs first and the middle vertex last in Figure 5.1 one can show that the adjacency matrices and characteristic polynomials of $UU(4, 1, 4)$ and $UU(4, 1, 3)$ are given by:

$$A(UU(4,1,4)) = \left[\begin{array}{cccc|cccc|c}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{array}\right]$$

with characteristic polynomial

$$p_{A(UU(4,1,4))}(z) = z(z+1)^3(z+2)(z^2-z-4)(z^2-4z+1), \text{ and also}$$

$$A(UU(4,1,3)) = \left[\begin{array}{cccc|cccc|c}
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{array}\right]$$

with characteristic polynomial given by

$$p_{A(UU(4,1,3))}(z) = z(z+1)^3(z+2)(z^2-z-4)(z^2-4z+2).$$

From here it is clear that $\mathcal{E}(UU(4,1,4)) = \mathcal{E}(UU(4,1,3)) = 9 + \sqrt{17}$.

5.2 The graph $CC(m,r)$.

One of the results from Table 2.1 is the graph shown in Figure 5.2. This graph consists of two copies of C_5 (the cycle graph on 5 vertices) with 3 edges between them. Notice that this graph is very identical in structure to the graph $KK(m,r)$ defined

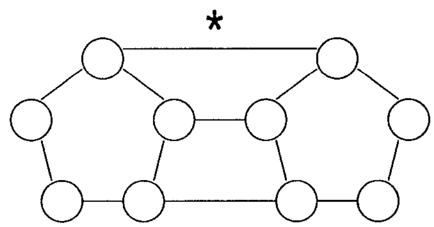


Figure 5.2: The graph #811571 from file **mat2**.

in Chapter 3, and for this reason we will denote the graph in Figure 5.2 by $CC(5,3)$, and in general $CC(m,r)$. Using the same labeling scheme as in the graphs of Chapter 3, we have

$$A(CC(5,3)) = \left[\begin{array}{ccccc|ccccc} 0&1&0&0&1&1&0&0&0&0\\ 1&0&1&0&0&0&1&0&0&0\\ 0&1&0&1&0&0&0&1&0&0\\ 0&0&1&0&1&0&0&0&0&0\\ 1&0&0&1&0&0&0&0&0&0\\ \hline 1&0&0&0&0&0&1&0&0&1\\ 0&1&0&0&0&1&0&1&0&0\\ 0&0&1&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0&0&1&0 \end{array}\right]$$

with characteristic polynomial

$$p_{CC(5,3)}(z) = (z-1)(z+2)z^2(z^2-2)(z^2-2z-2)(z^2+z-4)$$

After removing edge (1,6), we have

$$p_{CC(5,2)}(z) = z(z+1)(z-1)^2(z^2-3)(z^2-z-4)(z^2+2z-1)$$

From here it is clear that $\mathcal{E}(CC(5,3)) = \mathcal{E}(CC(5,2)) = 3+2(\sqrt{2}+\sqrt{3})+\sqrt{17}$.

At this point one might expect there to be more graphs of the type $CC(m,r)$, but a preliminary search using the script

GnrlztnOf10VrtxCycleGraphSearch found in Appendix B.1.5 provides evidence to believe that $CC(5,3)$ does not belong to an infinite family of the prescribed structure. Our data leads us to the following conjecture:

Conjecture 5.1. *Denote* $CC(m,r)$ *to be the graph composed of two copies of* C_m*, the cycle on* m *vertices, joined by* r *independent edges between them. Then* $\mathcal{E}(CC(m,r)) = \mathcal{E}(CC(m,r-1))$ *if and only if* $m=3$ *and* $r=2$ *or* $m=5$ *and* $r=3$.

5.3 Unknown Family on 10 vertices

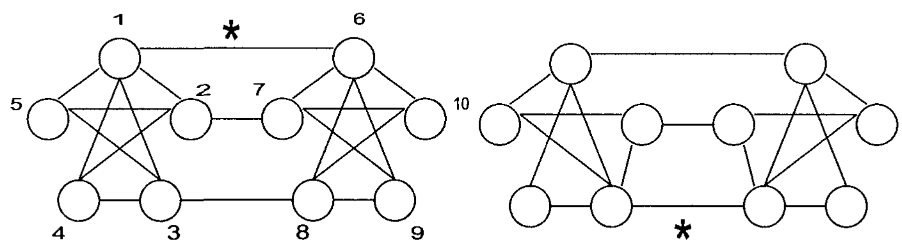


Figure 5.3: Graphs #761124 and #798072, from file **mat8** and **mat5**, respectively. Denote these graphs by G_1 and G_2, respectively.

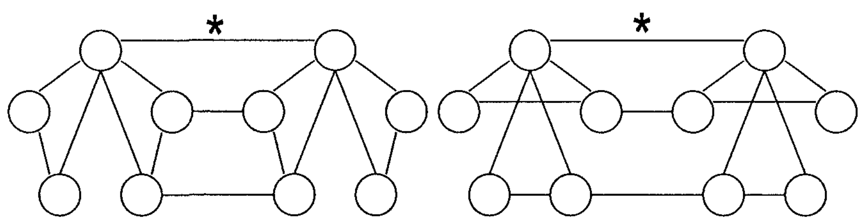


Figure 5.4: Graphs #608111 and #783993 from file **mat12** and **mat2**, respectively. Denote these graphs by G_3 and G_4, respectively.

The other four graphs obtained in Table 2.1 are those of Figures 5.3 and 5.4. Notice that all of these graphs constituted of two copies of a subgraph G of K_5 joined by 3 edges between them. For the same reasons explained in Section 5.1, it is also unclear whether these graphs belong to an infinite family of graphs or not. However in the next chapter we discuss the pattern they exhibit and the further study this might lead to. We end this section by analytically verifying the equienergetic property of each of these graphs. Denote G_i^* to be G_i minus an edge, for $i = 1, \ldots, 4$. Using the labeling scheme of graph #761124 of Figure 5.3 for the rest of the graphs we have:

Claim: $\mathcal{E}(G_1) = \mathcal{E}(G_1^*)$.

Proof: For G_1,

$$A(G_1) = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

has characteristic polynomial

$$p_{G_1}(z) = z^2(z-1)(z+1)(z+2)(z^2-7)(z^3-2z^2-7z-2)$$

After removing edge $(1,6)$, we have

$$p_{G_1^*}(z) = z^2(z-1)(z+1)^2(z^2-2z-6)(z^3+z^2-8z-10).$$

Let $f(z) = z^3 - 2z^2 - 7z - 2$ and assume $f(a_i) = 0$, for $i = 1, 2, 3$. Arranging the roots such that $a_1 \leq a_2 \leq a_3$, it is not hard to show that $a_1 < -1 < a_2 < 0 < a_3$. So we have

$\sigma(G_1) = \{0^{(2)}, -1, 1, 2, -\sqrt{7}, \sqrt{7}, a_1, a_2, a_3\}$. Thus $\mathcal{E}(G_1) = 4+2\sqrt{7}+|a_1|+|a_2|+|a_3| = 4 + 2\sqrt{7} - a_1 - a_2 + a_3$. Now, let $g(z) = z^3 + z^2 - 8z - 10$ and set $b_i = a_i - 1$, for $i = 1, 2, 3$. Then $g(b_i) = g(a_i - 1) = a_i^3 - 2a_i^2 - 7a_i - 2 = f(a_i) = 0$. Hence $\sigma(G_1^*) = \{0^{(2)}, -1, 1, 1 - \sqrt{7}, 1 + \sqrt{7}, b_1, b_2, b_3\}$. Thus we have

$$\begin{aligned} \mathcal{E}(G_1^*) &= 3 + |1 - \sqrt{7}| + |1 + \sqrt{7}| + |a_1 - 1| + |a_2 - 1| + |a_3 - 1| \\ &= 3 + (\sqrt{7} - 1) + (1 + \sqrt{7}) + (1 - a_1) + (1 - a_2) + (a_3 - 1) \\ &= 4 + 2\sqrt{7} - a_1 - a_2 + a_3 = \mathcal{E}(G_1). \end{aligned}$$

□

Claim: $\mathcal{E}(G_2) = \mathcal{E}(G_2^*)$.

Proof: The adjacency matrix for G_2 is given by

$$A(G_2) = \left[\begin{array}{ccccc|ccccc} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

has characteristic polynomial

$$p_{G_2}(z) = z(z + 2)(z^4 - 3z^3 - 4z^2 + 5z + 4)(z^4 + z^3 - 6z^2 - 5z + 2)$$

After removing edge $(3, 8)$, we have

$$p_{G_2^*}(z) = (z + 1)^2(z^4 + z^3 - 7z^2 - 8z + 3)(z^4 - 3z^3 - 3z^2 + 6z + 1)$$

Let $f_1(z) = z^4-3z^3-4z^2+5z+4$ and $f_2(z) = z^4+z^3-6z^2-5z+2$, and assume $f_1(a_i) = 0$ and $f_2(b_i) = 0$, for $i = 1, 2, 3, 4$. Arranging the roots such that $a_1 \leq \cdots \leq a_4$ and

$b_1 \leq \cdots \leq b_4$, it is not hard to show that $a_1 < -1 < a_2 < 0 < 1 < a_3 < a_4$ and $b_1 < -1 < b_2 < 0 < 1 < b_3 < 1 < b_4$. So we have $\sigma(G_2) = \{0, -2, a_1, \ldots, a_4, b_1, \ldots, b_4\}$. Thus $\mathcal{E}(G_2) = 2-a_1-a_2+a_3+a_4-b_1-b_2+b_3+b_4$. Now, let $g_1(z) = z^4+z^3-7z^2-8z+3$ and $g_2(z) = z^4-3z^3-3z^2+6z+1$, and set $a_i^{'} = a_i-1$ and $b_i^{'} = b_i+1$, for $i = 1, 2, 3,$and 4. Then $g_1(a_i^{'}) = g_1(a_i-1) = a_i^4 - 3a_i^3 - 4a_i^2 + 5a_i + 4 = f_1(a_i) = 0$. Similarly, $g_2(b_i^{'}) = g_2(b_i+1) = f_2(b_i) = 0$. Hence $\sigma(G_2^*) = \{(-1)^{(2)}, a_1^{'}, \ldots, a_4^{'}, b_1^{'}, \ldots, b_4^{'}\}$. Thus we have

$$\begin{aligned}
\mathcal{E}(G_2^*) &= 2 + \sum_{i=1}^{4} |a_i^{'}| + \sum_{i=1}^{4} |b_i^{'}| \\
&= 2 + \sum_{i=1}^{4} |a_i - 1| + \sum_{i=1}^{4} |b_i + 1| \\
&= 2 + (1 - a_1) + (1 - a_2) + (a_3 - 1) + (a_4 - 1) \\
&= +(-1 - b_1) + (-1 - b_2) + (b_3 + 1) + (b_4 + 1) \\
&= 2 - a_1 - a_2 + a_3 + a_4 - b_1 - b_2 + b_3 + b_4 = \mathcal{E}(G_2)
\end{aligned}$$

□

Claim: $\mathcal{E}(G_3) = \mathcal{E}(G_3^*)$.

Proof: For G_3 we have

$$A(G_3) = \left[\begin{array}{ccccc|ccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}\right]$$

has characteristic polynomial

$$p_{G_3}(z) = z(z-2)(z+2)(z+1)^2(z^2+2z-1)(z^3-4z^2+z+4)$$

After removing edge $(1,6)$, we have

$$p_{G_3^*}(z) = z(z-3)(z+2)(z+1)^2(z^2-2)(z^3-z^2-4z+2).$$

Let $f(z) = z^3 - 4z^2 + z + 4$ and assume $f(a_i) = 0$, for $i = 1,2,3$. Arranging the roots such that $a_1 \le a_2 \le a_3$, it is not hard to show that $a_1 < 0 < 1 < a_2 < a_3$. So we have $\sigma(G_3) = \{0, 2, -2, (-1)^{(2)}, -1-\sqrt{2}, -1+\sqrt{2}, a_1, a_2, a_3\}$. Thus $\mathcal{E}(G_3) = 6+(1+\sqrt{2})+(-1+\sqrt{2})+|a_1|+|a_2|+|a_3| = 6+2\sqrt{2}-a_1+a_2+a_3$. Now, let $g(z) = z^3-z^2-4z+2$ and set $b_i = a_i - 1$, for $i = 1,2,3$. Then $g(b_i) = g(a_i - 1) = a_i^3 - 4a_i^2 + a_i + 4 = f(a_i) = 0$. Hence $\sigma(G_3^*) = \{0, 3, -2, (-1)^{(2)}, -\sqrt{2}, \sqrt{2}, b_1, b_2, b_3\}$. Thus we have

$$\begin{aligned}\mathcal{E}(G_3^*) &= 7 + 2\sqrt{2} + |b_1| + |b_2| + |b_3| \\ &= 7 + 2\sqrt{2} + (1 - a_1) + (a_2 - 1) + (a_3 - 1) \\ &= 6 + 2\sqrt{2} - a_1 + a_2 + a_3 = \mathcal{E}(G_3)\end{aligned}$$

□

Finally,

Claim: $\mathcal{E}(G_4) = \mathcal{E}(G_4^*)$.

Proof: The adjacency matrix for G_4 is given by:

$$A(G_4) = \left[\begin{array}{ccccc|ccccc} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

has characteristic polynomial

$$p_{G_4}(z) = (z+1)(z^2+z-1)(z^2-z-1)(z^2-3z-1)(z^3+2z^2-4z-7)$$

After removing edge $(1,6)$, we have

$$p_{G_4^*}(z) = (z+2)(z^2+z-1)(z^2-z-1)(z^2-z-3)(z^3-z^2-5z-2)$$

Let $f(z) = z^3+2z^2-4z-7$ and assume $f(a_i) = 0$, for $i = 1,2,3$. Arranging the roots such that $a_1 \leq a_2 \leq a_3$, it is not hard to show that $a_1 < a_2 < -1 < 0 < a_3$. So we have $\sigma(G_4) = \{-1, \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2}, a_1, a_2, a_3\}$. Thus $\mathcal{E}(G_4) = 1 + \sqrt{5} + \sqrt{13} - a_1 - a_2 + a_3$. Now, let $g(z) = z^3 - z^2 - 5z - 2$ and set $b_i = a_i + 1$, for $i = 1,2,3$. Then $g(b_i) = g(a_i+1) = a_i^3 + 2a_i^2 - 4a_i - 7 = f(a_i) = 0$. Hence $\sigma(G_4^*) = \{-2, \frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{13}}{2}, \frac{1+\sqrt{13}}{2}, b_1, b_2, b_3\}$. Thus we have

$$\begin{aligned}
\mathcal{E}(G_4^*) &= 2+\sqrt{5}+\sqrt{13}+|b_1|+|b_2|+|b_3| \\
&= 2+\sqrt{5}+\sqrt{13}+(-a_1-1)+(-a_2-1)+(a_3+1) \\
&= 1+\sqrt{5}+\sqrt{13}-a_1-a_2+a_3 = \mathcal{E}(G_4)
\end{aligned}$$

□

5.4 Results on 11 vertices

The MATLAB script **equiEnergeticSize11** in Appendix B.1.7 allowed us to find at least 5 equienergetic matrices on 11 vertices. Because of the graphical complexity of the graphs corresponding to these matrices, it is difficult to draw these graphs in a useful way and hence determine an infinite family of graphs for them, if they exist. We end this chapter by simply presenting these matrices and analytically showing that they indeed have the equienergetic property.

Claim: Let $A(G)$ be the matrix #5464 from file **11vertices19** with corresponding graph G. Denote G^* to be G minus an edge. Then $\mathcal{E}(G) = \mathcal{E}(G^*) = 9 + 2\sqrt{5} + \sqrt{17}$.

Proof: The adjacency matrix for G

$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

has characteristic polynomial

$$p_G(z) = z(z+1)(z+2)(z^2-6z-1)(z^2+z-4)(z^2+z-1)^2.$$

After removing edge $(2, 8)$, we have

$$p_{G^*}(z) = z(z+1)(z+2)(z^2-6z+2)(z^2+z-4)(z^2+z-1)^2.$$

From here it is easy to show that $\mathcal{E}(G) = \mathcal{E}(G^*) = 9 + 2\sqrt{5} + \sqrt{17}$. $\square$

Claim: Let $A(G)$ be the matrix #5465 from file **11vertices19** with corresponding graph G. Denote G^* to be G minus an edge. Then $\mathcal{E}(G) = \mathcal{E}(G^*) = 9 + 2\sqrt{5} + \sqrt{17}$.

Proof: The adjacency matrix for G

$$
A(G) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
$$

has characteristic polynomial

$$p_G(z) = z(z+1)(z+2)(z^2-6z-1)(z^2+z-4)(z^2+z-1)^2.$$

After removing edge $(2, 8)$, we have

$$p_{G^*}(z) = z(z+1)(z+2)(z^2-6z+2)(z^2+z-4)(z^2+z-1)^2.$$

From here it is easy to show that $\mathcal{E}(G) = \mathcal{E}(G^*) = 9 + 2\sqrt{5} + \sqrt{17}$. $\square$

Claim: Let $A(G)$ be the matrix #1982932 from file **11vertices173** with corresponding graph G. Denote G^* to be G minus an edge. Then $\mathcal{E}(G) = \mathcal{E}(G^*) = 12 + 6\sqrt{2}$.

Proof: The adjacency matrix for G

$$
A(G) = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
$$

has characteristic polynomial

$$p_G(z) = (z-1)(z-5)(z+2)^3(z^2-2)^3.$$

After removing edge $(9, 10)$, we have

$$p_{G^*}(z) = (z^2-6z+6)(z+2)^3(z^2-2)^3.$$

From here it is easy to show that $\mathcal{E}(G) = \mathcal{E}(G^*) = 12 + 6\sqrt{2}$. □

Claim: Let $A(G)$ be the matrix #469937 from file **11vertices211** with corresponding graph G. Denote G^* to be G minus an edge. Then $\mathcal{E}(G) = \mathcal{E}(G^*) = 12 + 2(\sqrt{2} + \sqrt{3})$.

Proof: The adjacency matrix for G

$$A(G) = \begin{bmatrix} 0&0&0&0&0&1&1&0&1&1&1 \\ 0&0&0&0&0&0&1&1&0&1&1 \\ 0&0&0&0&0&0&1&0&0&0&1 \\ 0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&0&1 \\ 1&0&0&0&0&0&0&1&0&0&0 \\ 1&1&1&0&0&0&0&0&0&1&1 \\ 0&1&0&1&0&1&0&0&0&0&1 \\ 1&0&0&1&1&0&0&0&0&1&1 \\ 1&1&0&1&0&0&1&0&1&0&0 \\ 1&1&1&1&1&0&1&1&1&0&0 \end{bmatrix}$$

has characteristic polynomial

$$p_G(z) = z(z+1)(z-5)(z-1)^2(z^2-2)(z+2)^2(z^2+2z-2).$$

After removing edge $(7, 9)$, we have

$$p_{G^*}(z) = z(z-1)(z+1)(z^2-2)(z+2)^2(z^2+2z-2)(z^2-6z+6).$$

From here it is easy to show that $\mathcal{E}(G) = \mathcal{E}(G^*) = 12 + 2(\sqrt{2} + \sqrt{3})$. □

Claim: Let $A(G)$ be the matrix #469937 from file **11vertices280** with corresponding graph G. Denote G^* to be G minus an edge. Then $\mathcal{E}(G) = \mathcal{E}(G^*) = 10 + 2(\sqrt{2} + \sqrt{3})$.

Proof: The adjacency matrix for G

$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

has characteristic polynomial

$$p_G(z) = z^3(z^2 + 2z - 2)(z^2 - 2)(z^2 - 6z + 7)(z + 2)^2.$$

After removing edge $(1, 6)$, we have

$$p_{G^*}(z) = z^3(z + 3)(z + 2)(z^2 - 3)(z^2 - 2)(z^2 - 5z + 2).$$

From here it is easy to show that $\mathcal{E}(G) = \mathcal{E}(G^*) = 10 + 2(\sqrt{2} + \sqrt{3})$. $\square$

Chapter 6

Miscellaneous

6.1 Graphs equienergetic to subgraphs with more than one edge removed, part I

The focus of this thesis was the search of infinite families of graphs having the equienergetic property. Throughout, our main attention was devoted to graphs that are equienergetic to subgraphs with one fewer edge. A natural question to ask is what about graphs equienergetic to subgraphs with j deleted edges? Is the study of these types of graphs viable? The answer is yes, and, in fact, the following corollary to theorem 3.2 from Chapter 3 provides infinite families with these properties.

Corollary 6.1. *For $m \geq 3$, and $1 \leq r \leq m$, let $KK(m,r)$ denote the graph composed of two copies of the complete graph K_m joined by r independent edges between them. Then for $k \in \mathbb{N}$ and $1 \leq j < r$, $\mathcal{E}(KK(m,r)) = \mathcal{E}(KK(m,r-j))$ if and only if $m = 2r - j$.*

Proof: Replace 1 by j in the proof of Theorem 3.2. $\square$

Remark 6.2. Corollary 6.1 actually is valid for $0 \leq j \leq r$. For $j = 0$, the proof is trivial. For $j = r$, $A(KK(m,r)) = K \oplus K$, and the result follows; however, we avoid this case because $KK(m,r)$ becomes disconnected.

6.2 Graphs equienergetic to subgraphs with more than one edge removed part II

In Chapter 5, it was mentioned that it is not clear if each of the graphs #761124, #798072, #608111, and #783993 belong to an infinite family of graphs; however, considering these graphs together with $KK(5,3)$ and $CC(5,3)$ as a group, it is not hard to see the traits they have in common: Each of these graphs is a subgraph of the graph $KK(5,3)$ in a special way. For example, the graph #761124 consists of two copies of a subgraph of K_5 with 2 edges removed, joined by 3 independent edges between them. The graph $CC(5,3)$ contains two copies of a subgraph of K_5 with 5 edges removed.

Using this information we devise the following notation: define $K_iK_i(5,3)$ to be the graph consisting of two copies of a subgraph of K_5 minus i edges, connected by 3 independet edges. For example, the graph #761124 is $K_2K_2(5,3)$. Moreover, using this notation, $CC(5,3)$ is the graph $K_5K_5(5,3)$. Also, notice that both #60811 and #783993 are represented by $K_4K_4(5,3)$. This means that $K_4K_4(5,3)$ does not represent a graph, but rather a set of graphs. Hence, modifying our notation we denote $K_iK_i(5,3)$ to be the set of graphs consisting of two copies of a subgraph of K_5 minus i edges, connected by 3 independent edges between them.

At this point one might ask: Does this phenomena also occur in graphs with higher number of vertices? As expected, the answer is in the affirmative, and this is investigated further by the MATLAB script **ConjectureSetOfGraphs** presented in Section B.1.6 of Appendix B. Preliminary data shows that the set of graphs occur for $m = 5, 7$, and 9. We end this Chapter with the following conjecture:

Conjecture 6.1. *Let $KK(m,r)$ be the graph as defined in Chapter 3. Let $m = 2r-1$ and let K_i denote the subgraphs of K_m with i edges removed. Then the set of graphs*

$K_iK_i(m,r)$ i.e., the set of graphs consisting of two copies of a subgraph of K_m joined by r independent edges between them, gives rise to an infinite family of sets of graphs with the equienergetic property.

Chapter 7

Conclusion

We began our study of the equienergetic problem by using MATLAB scripts to search for graphs with the described property. Using the results of Chapter 2, we continued our study in Chapters 3 and 4 by demonstrating that some of these graphs belong to an infinite family. Unfortunately, we were not able to generalize all of our results; however, we presented some conjectures and findings that might shed some light on these graphs. Finally, Chapter 6 provided a slight generalization of the main results from Chapter 3. Our study ends with a conjecture of an additional extension of the main result of Chapter 3 to infinite family of sets of graphs equienergetic to subgraphs with more than one edge removed.

Bibliography

[1] I. Gutman, The energy of a graph, *Ber. Math-Stat. Sekt. Forschungszent. Graz* 103 (1978) 1-22.

[2] J. Day and W. So, Graph energy change due to edge deletion, *Linear Algebra and its Applications* 428 (2008) 2070-2078.

[3] R. Horn and C. Johnson, *Matrix Analysis,* Cambridge University Press, 1989.

[4] P. Rowlinson, The Spectrum of a Graph modified by the addition of a vertex, *Univ. Beograd. Publ Electrotehn. Fak* Ser. Mat. 3 (1992), 67-70.

[5] C. Li and W. So, Graphs equienergetic with edge-deleted subgraphs, *Linear and Multilinear Algebra* accepted.

Appendix A

Theorems

The following results are quoted from other sources for easy reference here.

Theorem A.1 (Rank One Perturbation Interlacing). *Let $A \in M_n(\mathbb{C})$ be any Hermitian matrix and suppose $B = xx^*$ where $x \in \mathbb{C}^n$. Suppose the eigenvalues of A and $A+B$, denoted by $\lambda_i(A)$ and $\lambda_i(A+B)$, respectively, are ordered in increasing order. Then the eigenvalues of A and $A+B$ have the following interlacing property:*

$$\lambda_1(A) \leq \lambda_1(A+B) \leq \lambda_2(A) \leq \lambda_2(A+B) \leq \cdots \leq \lambda_n(A) \leq \lambda_n(A+B).$$

Proof: See pg 182-183 in [3]. □

Theorem A.2 (Principal Submatrix Interlacing). *Let $B \in M_n(\mathbb{C})$ be a given Hermitian matrix, let $y \in \mathbb{C}^n$ be a give vector, and let $a \in \mathbb{R}$ be given. Let $A \in M_{n+1}(\mathbb{R})$ be the Hermitian matrix obtained by bordering B with y and a as follows:*

$$A = \left[\begin{array}{c|c} B & y \\ \hline y^* & a \end{array}\right]$$

Let $\mu_i = \mu_i(B)$ and $\lambda_i = \lambda_i(A)$ and assume they are ordered in increasing order. Then $\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \mu_n \leq \lambda_{n+1}$.

Proof: See pg 185 in [3]. □

Proposition: Suppose K_m is the complete graph with m vertices and let B be its corresponding adjacency matrix. Then the spectrum of B is given by, $\sigma(B) = \{(-1)^{(m-1)}, m-1\}$.

Proof: Let $I \in M_m(\mathbb{R})$, then it is clear that $\sigma(-I) = \{(-1)^{(m)}\}$. Moreover, we have that $B = [1]_m - I = (-I) + ee^T$, where $e^T = [1 \cdots 1]$. Thus by the rank 1 perturbation theorem, it follows that $\lambda_1(-I) \leq \lambda_1(B) \leq \cdots \leq \lambda_n(-I) \leq \lambda_n(B)$. But $\lambda_i(-I) = -1, \forall i$. So we have $\lambda_1(B) = \cdots = \lambda_{n-1}(B) = -1$. However, $tr(B) = \sum_{i=1}^{m} \lambda_i(B) = 0 \Rightarrow (-1)(m-1) + \lambda_n(B) = 0$ or $\lambda_n(B) = m-1$. Therefore, $\sigma(B) = \{(-1)^{(m-1)}, m-1\}$. □

The above are classic Theorems. One more recent result due to Rowlinson [4] is essential in Section 4.2 of this thesis. It is based on the following classical theory. An $n \times n$ real symmetric matrix A has pairwise orthogonal eigenspaces $E_1, \ldots, E_p$, one for each distinct eigenvalue $\gamma_1, \ldots, \gamma_p$, and $\mathbb{R}^n = E_1 \oplus \cdots \oplus E_p$. Then if $P_i : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection of $\mathbb{R}^n$ onto E_i, it is true that $A = \sum_{i=1}^{p} \gamma_i P_i$ (where we identify a matrix M with the map $x \to M_x$). This is called the Spectral Decomposition of A. Details can be found in [3].

With this notation, Rowlinson's Theorem can be stated:

Theorem A.3. *Let G be a finite graph whose adjacency matrix A has spectral decomposition $A = \sum_{i=1}^{p} \gamma_i P_i$, where $\gamma_1, \ldots, \gamma_p$ are the distinct eigenvalues of A. Let $\emptyset \neq S \subset V(G) = \{1, 2, \ldots, n\}$ and let G^* be the graph obtained from G by adding one* *new vertex whose neighbors are the vertices in S. Then*

$$p_{G^*}(z) = p_G(z)\left(z - \sum_{i=1}^{p} \frac{\rho_i^2}{z-\gamma_i}\right) \textit{ where } \rho_i = \left|\left|\textstyle\sum_{k\in S} P_i e_k\right|\right|.$$

Appendix B

Equienergetic Search Algorithms

B.1 MATLAB Scripts and functions

The research in this Thesis would not have been possible if the data for the adjacency matrices for all possible connected graphs with a prescribed number of vertices were available. For the connected graphs up to $m = 10$ vertices, the data is available to the public at `http://cs.anu.edu.au/~bdm/data/graphs.html`. For $m = 11$ vertices, the data must be generated using the program **NAUTY**. Detailed information and documentation, as well as the program itself can be found at `http://cs.anu.edu.au/~bdm/nauty/`

In MATLAB, programs are called scripts. Because of the format of the file **CN9**, we must use functions to extract the required matrices and calculate their energy. We introducue the functions and MATLAB scripts used in this Thesis next.

B.1.1 Functions

The following are MATLAB functions used within the main MATLAB scripts used in this Thesis.

OneEdgeLessUnkwnFmly function

The **OneEdgeLessUnkwnFmly** function is the core of the **GnrlztnOfScnd9VrtxGraphSearch-1edgeLess** script found in Section B.1.4. Its main purpose is to report the value m such that $\mathcal{E}(UU(m,r,s)) = \mathcal{E}(UU(m,r,s-1))$. This function takes matrices of the form

$$A = \left[\begin{array}{c|c|c} B & 0 & x_r \\ \hline 0 & B & x_s \\ \hline x_r & x_s & 0 \end{array}\right]$$

where B is the adjacency matrix of a subgraph of K_m.

MATLAB function: **OneEdgeLessUnkwnFmly**

```
function OneEdgeLessUnkwnFmly(A,SIZE)

% OneEdgeLessUnkwnFmly subtracts one edge from adjacency matrix depending
% on the parameters A (the matrix) and its corresponding SIZE, which is
% assumed to be odd.
EPSILON = 0.0000000000001;
L = 1;
E1 = norm(eig(A),1);
%Edge matrix E_ij is created and subtracted from A.
for m=1:SIZE
    if SIZE~=m
       E = zeros(SIZE);E(m,SIZE)=1;E(SIZE,m)=1;
         if A-E>=0
            E2 = norm(eig(A-E),1);

            if abs(E1-E2)<= EPSILON
             while L<=1
            fprintf('\n ***************************************** \n');
            fprintf('\n Matrix with energy E = %1.12f. \n',E1);
            fprintf('\n has equienergetic submatrices given by: \n');
            L = L + 1;
            fprintf('\n Look at value m=%1.0f. \n',(SIZE-1)/2);
             end
            fprintf('\n After removing edge e=(%1.0f,%1.0f): \n',m,SIZE);
            %display(A-E);
            fprintf(' \n with energy E2 = %1.12f. \n',E2);
            end
          end

     end
end
```

end of MATLAB function.

OneEdgeLessUnkwnFmlyDrDr

This function is a modification of the **OneEdgeLessUnkwnFmly** function. It takes matrices of the form

$$A = \left[\begin{array}{c|c} B & D_r \\ \hline D_r & B \end{array}\right]$$

where $B \in M_m(\mathbb{N})$. In the script **GnrlztnOf10VrtxCycleGraphSearch** found in Section B.1.5, $B = A(C_m)$.

MATLAB function: **OneEdgeLessUnkwnFmlyDrDr**

```
function OneEdgeLessUnkwnFmlyDrDr(A,SIZE)

OneEdgeLessDrDr subtracts one edge from adjacency matrix depending
% on the parameters A (the matrix) and its corresponding SIZE,
% which is assumed to be even. This function takes matrices of the
% form A=[B Dr;Dr B].

EPSILON = 0.000000000001;
L = 1;
E1 = norm(eig(A),1);
%Edge matrix E_ij is created and subtracted from A.
for m=1:(SIZE/2)
       E = zeros(SIZE);E(m,m+(SIZE/2))=1;E(m+(SIZE/2),m)=1;
         if A-E>=0
            E2 = norm(eig(A-E),1);
            %-----------------------------------------
            %display(A);
            if abs(E1-E2)<= EPSILON
             while L<=1
            fprintf('\n ****************************************** \n');
            fprintf('\n Matrix with energy E = %1.12f. \n',E1);
            display(A(1:SIZE/2,1:SIZE/2))
            fprintf('\n has equienergetic submatrices given by: \n');
            L = L + 1;
            fprintf('\n Look at value m=%1.0f. \n',SIZE/2);
             end
            fprintf('\n After removing edge e=(%1.0f,%1.0f): \n',m,m+(SIZE/2));
            %display(A-E);
            fprintf(' \n with energy E2 = %1.12f. \n',E2);
            end
```

```
        end
end
```

end of MATLAB function.

g62adj function

The **g62adj** function is the core of **equiEnergetic** script and its variations. Its main purpose it to convert the data contained in **CN6, ..., mat1, ..., mat12** into usable matrices.

MATLAB function: **g62adj**

```
function A=g62adj(v)

% convert a graph v in g6 format of a graph on 10 vertices
% to its adjacency matrix A

x=dec2bin(v-63);
[a,b]=size(x);
for i=1:a
    for j=1:b
        nx(i,j)=str2num(x(i,j));
    end
end
nx=[zeros(8,6-b) nx];

B=reshape(nx',1,48);

t=zeros(10);

for i=2:10
    for j=1:i-1
        t(i,j)=B((i-1)*(i-2)/2+j);
    end
end

A=t+t';
```

end of MATLAB function.

g62adj function

The **g62adj11** function is a modification of **g62adj** function used for the **equiEnergeticSize11** script found in Section B.1.7.

MATLAB function: **g62adj11**

```
function A=g62adj11(v)

% convert a graph v in g6 format of a graph on 11 vertices
% to its adjacency matrix A

x=dec2bin(v-63);
[a,b]=size(x);
for i=1:a
    for j=1:b
        nx(i,j)=str2num(x(i,j));
    end
end
nx=[zeros(10,6-b) nx];

B=reshape(nx',1,60);

t=zeros(11);

for i=2:11
    for j=1:i-1
        t(i,j)=B((i-1)*(i-2)/2+j);
    end
end

A=t+t';
```

end of MATLAB function.

B.1.2 equiEnergetic

The script **equiEnergetic** begins by initializing A and then computes its energergy E1. The following two *for-loops* and *if statement* create the edge matrix $E_{i,j}$, which is later subtracted from A. After ensuring that the diagonal entries are left untouched, it is checked that all the entries of the new matrix $A - E_{i,j}$ are nonnegative. If so, then the program continues by calculating its energy, and comparing it the energy of the matrix A. If the energy difference falls withing some prescribed error, the program continues and prints out the corresponding equienergetic matrices corresponding to the matrix A. A sample run is presented next.

Sample run of **equiEnergetic**

```
 ====================
 | Beginning Search! |
 ====================

 ****************************************

 Matrix 32042 with energy E = 13.1231056256.

 has equienergetic submatrices given by:

 After removing edge e=(1,9):

ans =

     0     0     0     0     1     0     1     0     0
     0     0     0     0     1     0     1     0     1
     0     0     0     0     0     1     0     1     1
     0     0     0     0     0     1     0     1     0
     1     1     0     0     0     0     1     0     1
     0     0     1     1     0     0     0     1     0
     1     1     0     0     1     0     0     0     1
     0     0     1     1     0     1     0     0     0
     0     1     1     0     1     0     1     0     0


 with energy E2 = 13.1231056256.

 After removing edge e=(2,9):

ans =

     0     0     0     0     1     0     1     0     1
     0     0     0     0     1     0     1     0     0
     0     0     0     0     0     1     0     1     1
     0     0     0     0     0     1     0     1     0
     1     1     0     0     0     0     1     0     1
     0     0     1     1     0     0     0     1     0
```

```
     1     1     0     0     1     0     0     0     1
     0     0     1     1     0     1     0     0     0
     1     0     1     0     1     0     1     0     0
```

 with energy E2 = 13.1231056256.

 ****

 Matrix 50501 with energy E = 14.0000000000.

 has equienergetic submatrices given by:

 After removing edge e=(1,9):

```
ans =

     0     0     1     0     1     0     1     0     0
     0     0     0     1     0     1     0     1     1
     1     0     0     0     1     0     1     0     1
     0     1     0     0     0     1     0     1     0
     1     0     1     0     0     0     1     0     1
     0     1     0     1     0     0     0     1     0
     1     0     1     0     1     0     0     0     0
     0     1     0     1     0     1     0     0     0
     0     1     1     0     1     0     0     0     0
```

 with energy E2 = 14.0000000000.

 After removing edge e=(3,9):

```
ans =

     0     0     1     0     1     0     1     0     1
     0     0     0     1     0     1     0     1     1
     1     0     0     0     1     0     1     0     0
     0     1     0     0     0     1     0     1     0
     1     0     1     0     0     0     1     0     1
     0     1     0     1     0     0     0     1     0
     1     0     1     0     1     0     0     0     0
     0     1     0     1     0     1     0     0     0
     1     1     0     0     1     0     0     0     0
```

 with energy E2 = 14.0000000000.

 After removing edge e=(5,9):

```
ans =
```

```
0    0    1    0    1    0    1    0    1
0    0    0    1    0    1    0    1    1
1    0    0    0    1    0    1    0    1
0    1    0    0    0    1    0    1    0
1    0    1    0    0    0    1    0    0
0    1    0    1    0    0    0    1    0
1    0    1    0    1    0    0    0    0
0    1    0    1    0    1    0    0    0
1    1    1    0    0    0    0    0    0

with energy E2 = 14.0000000000.

==================
| End of Search! |
==================
|| Elapsed Time = 234 seconds||
```

end of sample run.

B.1.3 PolySearcherCndtnChkr

The program **PolySearcherCndtnChkr** searches for values (m, r, s) such that $\mathcal{E}(KK(m, r, s)) = \mathcal{E}(KK(m, r, s-1))$. It takes advantage of the conditions outlined in Lemma 4.1.

MATLAB script: **PolySearcherCndtnChkr**

```
% Searches for a pattern in 3rd charac poly of KK(m,r,s) Graph.
% Final Version.
fprintf('\n =================== \n');
fprintf(' | Beginning Search! |');
fprintf('\n =================== \n');
for m=301:370
  for r=1:m
    for s=1:m
      if m-2*s < 0
        if (m+1)*(r+s)-(r^2+s^2)>0
          if r^2+(1-2*s)*r+(2*(m^2)*s-8*m*(s^2)+8*s^3-m^2+6*m*s-9*s^2-m+3*s)== 0
            fprintf('\n Look at values (m,r,s)=(%1.0f,%1.0f,%1.0f). \n',m,r,s);
          end

        end

      end

    end
  end
end

fprintf('\n ================= \n');
fprintf(' | End of Search! |');
fprintf('\n ================= \n');
```

end of MATLAB script.

The following table shows values obtained by **PolySearcherCndtnChkr**. The **Formula** column indicates what equation in Corollary 4.5 the corresponding values satisfy. Some values, like (26,6,15), don't fall in any of the four categories (i)-(iv).

(m, r, s) **such that** $\mathcal{E}(KK(m,r,s)) = \mathcal{E}(KK(m,r,s-1))$				
Formula	$m - 2s$	m	r	s
(i)	2	4	1	3
(i)	3	11	4	7
(ii)		11	9	7
(iii)		15	2	9
(iv)		15	15	9
(i)	4	22	9	13
(ii)		22	16	13
		26	6	15
		26	23	15
(iii)		34	3	19
(iv)		34	34	19
(i)	5	37	16	21
(ii)		37	25	21
(iii)		61	4	33
(iv)		61	61	33
(i)	6	56	25	31
(ii)		56	36	31
(iii)		96	5	51
(iv)		96	96	51
(i)	7	79	36	43
(ii)		79	49	43
(i)	8	106	49	57
(ii)		106	64	57
		118	34	63
		118	91	63

Table B.1: Shows search results for PolySearcherCndtnChkr.

B.1.4 GnrlztnOfScnd9VrtxGraphSearch-1edgeLess

This script attempts to find the next graph $UU(m, r, s)$, if it exists, such that $\mathcal{E}(UU(m, r, s)) = \mathcal{E}(UU(m, r, s - 1)$. It assumes that U is a subgraph of K_m minus an edge.

MATLAB script: **GnrlztnOfScnd9VrtxGraphSearch-1edgeLess**

```
% GnrlztnOfScnd9VrtxGraphSearch-1edgeLess
% Searches for Equienergetic Graphs of one edge less for UU(m,r,s).
% Conjecture is that the two copies are Km - edge.
% ERROR: m=10 OUT OF MEMORY
% Finished version

tic                         %Timer started.

for m=10:10; %m=10 OUT OF MEMORY.
SIZE = 2*m + 1;
Z = zeros(m);
%Constructing the matrix Km, complete for m edges.
Kme = ones(m)-eye(m);
Kme(1,2)=0;
Kme(2,1)=0;
%End of construction

x = zeros(m,1);
y = zeros(m,1);

%%%%%%%%%%%%%%%%%

fprintf('\n ==================== \n');
fprintf(' | Beginning Search! |');
fprintf('\n ==================== \n');

for k = 1:m
    x(k,1)=1;
    X1 = unique(perms(x), 'rows');
    SIZEX1 = size(X1,1); %# of rows.
    for i = 1:SIZEX1
       x1 = X1(i,:);
         for j = 1:m
           y(j,1)=1;
           X2 = unique(perms(y), 'rows');
           SIZEX2 = size(X2,1); %# of rows
```

```
            for p = 1:SIZEX2
            x2 = X2(p,:);
            A = [Kme Z x1';Z Kme x2'; x1 x2 0];
            E = norm(eig(A),1);
            OneEdgeLessUnkwnFmly(A,SIZE);
            end
        end
    end
end


fprintf('\n ================== \n');
fprintf(' | End of Search! |');
fprintf('\n ================== \n');
fprintf(' || Elapsed Time = %1.0f seconds|| \n',toc);
end

fprintf('\n ******************* \n');
fprintf(' | THE REAL END! |');
fprintf('\n ******************* \n');
```

end of MATLAB script.

B.1.5 GnrlztnOf10VrtxCycleGraphSearch

This script attemps to find graphs with the same structure as the graph in Section 5.2.

MATLAB script: **GnrlztnOf10VrtxCycleGraphSearch**

```
%GnrlztnOf10VrtxCycleGraphSearch
% Searches for Equienergetic Graphs of the graph CC(m,r),
% two cycles on m vertices connected by r edges between them.
% Final Version.
tic                        %Timer started.

fprintf('\n =================== \n');
fprintf(' | Beginning Search! |');
fprintf('\n =================== \n');
for m = 101:150  %Number of vertices for Cm.
SIZE = 2*m;
Dr = zeros(m);
Dr(1,1)=1;
%Constructing the matrix Cm, cycle of m vertices.
Cm = zeros(m);
Cm(1,m)=1;
for k = 1:(m-1)
    Cm(k,k+1)=1;
end
Cm=Cm+Cm';
%End of construction

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%Creating the Matrix A.
for r = 2:m
    Dr(r,r)=1;
    A = [Cm Dr; Dr Cm];
    OneEdgeLessUnkwnFmlyDrDr(A,SIZE)
end

end
fprintf('\n ================= \n');
fprintf(' | End of Search! | ' );
fprintf('\n ================= \n');
fprintf(' || Elapsed Time = %1.0f seconds|| \n',toc);
```

end of MATLAB script.

B.1.6 ConjectureSetOfGraphs

This script is used to find data to backup Conjecture 6.1 in Chapter 6. Its main purpose is to find equienergetic matrices of the form $A = \left[\begin{array}{c|c} B & D_r \\ \hline D_r & B \end{array}\right]$ where B is the adjacency matrix of a subgraph of the complete graph K_m. The script presented next uses the **CN9** file, but other files may be used as well.

MATLAB script: **ConjectureKiKi(m,r)**

```
% Searches for examples of the conjecture K_iK_i(m,r) graphs.
% Finished version
load CN9.mat

m = 9;
Dr = eye(m);
for p=(m+3)/2:m
    Dr(p,p)=0;
end
fprintf('\n ==================== \n');
fprintf(' | Beginning Search! |');
fprintf('\n ==================== \n');
for k=1:size(yy,3)
    B = yy(:,:,k);
    A = [B Dr; Dr B];
    OneEdgeLessUnkwnFmlyDrDr(A,2*m)
end
fprintf('\n ==================== \n');
fprintf(' | End of Search!    |');
fprintf('\n ==================== \n');
```

end of MATLAB script.

B.1.7 equiEnergeticSize11

The **equiEnergeticSize11** script is a variation of the **equiEnergetic** script found in Chapter 2. It works with the files **11vertices0**, ..., **11vertices499**.

MATLAB script: **equiEnergeticSize11**

```
% Searches for Equienergetic Graphs of one edge less.
% Finished version

for w = 3:5

  SIZE = 11; %Matrix size
  EPSILON = 0.0000000000001; %Error between energy of A
                             %and its submatrix before displayment.
  L = 1;

  filename = strcat('11vertices',int2str(w),'.g6');
  fid = fopen(strcat('11vertices',int2str(w),'.g6'));
  z = fread(fid);
  mat = reshape(z',12,length(z)/12);
  fclose(fid)

fileSIZE = size(mat,2);
tic                         %Timer started.

diary(strcat('11vertices',int2str(w)))


fprintf('\n =================== \n');
fprintf(' | Beginning Search! |');
fprintf('\n =================== \n');
fprintf('\n searh for file %s \n', filename);

 for k=1:fileSIZE
  A = g62adj11(mat(2:11,k));
  E1 = norm(eig(A),1);

  %Position matrix E^ij is created and subtracted from A.
  for m=1:SIZE
   column = m + 1;
    for n = column:SIZE
     if n~=m
       E = zeros(SIZE);E(m,n)=1;E(n,m)=1;
        if A-E>=0
```

```
        E2 = norm(eig(A-E),1);

          if abs(E1-E2)<= EPSILON
            while L<=1
            fprintf('\n ***************************************** \n');
            fprintf('\n Matrix %1.0f with energy E = %1.10f. \n',k,E1);
            fprintf('\n has equienergetic submatrices given by: \n');
            L = L + 1;
            end
            fprintf('\n After removing edge e=(%1.0f,%1.0f): \n',m,n);
            display(A-E);
            fprintf(' \n with energy E2 = %1.10f. \n',E2);
          end

        end


     end
    end
  end

 L = 1;
 end
 fprintf('\n ================== \n');
 fprintf(' | End of Search! |');
 fprintf('\n ================== \n');
 fprintf(' || Elapsed Time = %1.0f seconds|| \n',toc);

diary off

clear
end
```

end of MATLAB script.