Fall 2010

Applications of Boundary Value Problems

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APPLICATIONS OF BOUNDARY VALUE PROBLEMS

A Thesis

Presented to

The Faculty of the Department of Mathematics

San Jose State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

Annie Hien Nguyen

December 2010

The Designated Thesis Committee Approves the Thesis Titled

APPLICATIONS OF BOUNDARY VALUE PROBLEMS

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APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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December 2010

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ABSTRACT

APPLICATIONS OF BOUNDARY VALUE PROBLEMS

by Annie Hien Nguyen

In this thesis, we solved the Saint-Venant's torsion problem for beams with different cross sections bounded by simple closed curves using various methods. In addition, we solved the flexure problem of beams with certain curvilinear cross sections.

The first method was derived by Bassali and Obaid. We focused on cross sections bounded by hyperbolas, circular groves, lemniscate of Booth, and sectorial cross sections. The second method used Tchebycheff polynomials to solve the torsion problem corresponding to the circle and ellipse. The third method used conformal mapping to derive the solution of different cross sections bounded by curvilinear edges.

The flexure problem has been reduced to six boundary value problems; three are Dirichlet and three are Neumann problems. We derived the flexure functions corresponding to a certain boundary.

ACKNOWLEDGEMENTS

I would like to express my thanks to my committee members Dr. Roger Dodd and Dr. Mohammad Saleem for their suggestions and advice. I would also like to thank Dr. Richard Kubelka for his patience and assistance in helping me fulfill my responsibilities with the Graduate Studies & Research Office. A special thanks to my fiancé, friends, and family for being there for me through my rough times. Also, I would like to thank Shalini Reddy for always understanding me and helping me through my journey at San José State University.

But most of all, my deepest gratitude goes to my advisor, Dr. Samih Obaid, who has spent many hours working with me over the past year and a half. Without his diligence, patience, professionalism, and more importantly, his understanding, I would not have succeeded. Furthermore, thank you Dr. Obaid for teaching me mathematics and for modeling what kind of teacher I should be for my high school students.

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INTRODUCTION

The classical Saint-Venant's torsion problem has been extensively researched by mathematicians because of its important effects in the real world. The torsion problem deals with an elastic beam bounded by a cylinder. The cylinder's surface is free of external load; instead, the load is applied to the bases of the cylinder. When the beam is twisted, the stresses cause the generators of the cylinder to become helical curves and might eventually lead to the failure of the beam. Therefore, the solution of special boundary value problems connected with potential theory, which governs this situation, is crucial not just to mathematicians but to engineers and physicists as well.

When the beam is being twisted there are different cross sections that emerge from the beam; therefore, the torsion problem has many different types of problems. Many mathematicians have solved the torsion problem with different uniform, simply connected, cross sections of the cylinder bounded by closed curves such as a circle, ellipse, equilateral triangle, and cardioid. However, many others such as Bassali [1], [2], Bassali & Obaid [4], Sokolnikoff [13], Stevenson [14], Obaid and Rung [11], and Abassi [1] have developed different methods to solve a torsion problem.

Like the preceding authors, we start with a prismatic beam bounded by a cylindrical surface of finite length l and bases perpendicular to the generators of the cylinder where one of the bases lies in the x-y plane and the other lies in the plane $z = l$. Applying a pair of couples of equal magnitude and opposite directions to the bases of the beam results in a twist of the beam, which produces displacements and shearing stresses. We solve the problem for different cross sections bounded by simple closed curves.

Twisting the beam can result in beam breakage so a calculation of the torsional rigidity of the beam, which measures the strength of the beam is needed. In order to solve the torsion problem, a harmonic function $\psi(x, y)$ is needed, which is the harmonic conjugate of the torsion function $\phi(x, y)$ in the cross section S that satisfies the boundary condition $\psi(x, y) = \frac{1}{2}(x^2 + y^2)$ on the boundary of S.

In this thesis, we use four different methods to solve the two-dimensional Dirichlet problem. In the first chapter, we show two methods to solve the torsion problem. The first method was derived by Bassali and Obaid [4]. They use a complex analytic function $F(z)$ in the domain S and find a corresponding harmonic function $\psi(x, y)$ that satisfies the boundary condition $\psi(x, y) = \frac{1}{2}r^2$. We work with cross sections bounded by hyperbolas, circular groves, lemniscate of Booth, and more. The second method is to use Tchebycheff polynomials to solve the torsion problem for cross sections bounded by simple closed curves. In the third method, we use conformal mapping to derive the solutions to the torsion problem of different cross sections bounded by curvilinear edges. The last problem is an extension of the torsion function. Stevenson [14] has reduced the flexure problem to solving six boundary value problems; three are Dirichlet and three are Neumann problems, where one of the Dirichlet functions is a torsion function. We derive the flexure functions for a certain boundary.

CHAPTER 1

SOLUTIONS TO SAINT-VENANT'S TORSION PROBLEM USING BASSALI-OBAID METHOD AND TCHEBYCEFF POLYNOMIALS

In this chapter, we focus on two methods. First, we solve Saint-Venant's torsion problem using the Bassali-Obaid method [4]. Afterwards, we show the solution to the torsion problem of a cylinder using Tchebycheff polynomials.

Section 1.1 Essential Equations

Let S be a uniform cross section bounded by a simple closed curve Γ in the z-plane, where $z = x + iy = re^{i\theta}$. We set Z = 0 to be perpendicular to the generators of the cylinder, which is elastically isotropic. Twisting the cylinder along its axis at its two bases by two couples of equal magnitude and opposite directions results in displacement *(u, v, w)* and the non-vanishing stresses τ_{zx}, τ_{zy} as notated by Sokolnikoff [13] to be

$$u + iv = iTzZ,\ w = T\phi(x, y) = \tfrac{1}{2}T\left[\Omega(z) + \overline{\Omega}(\overline{z})\right] \qquad (1.1.1)$$

$$\tau_{zx} = \mu T\left(\frac{\partial \phi}{\partial x} - y\right) = \mu T\left(\frac{\partial \psi}{\partial y} - y\right) \qquad (1.1.2)$$

$$\tau_{zy} = \mu T\left(\frac{\partial \phi}{\partial y} + x\right) = \mu T\left(-\frac{\partial \psi}{\partial x} + x\right) \qquad (1.1.3)$$

$$\tau_{zx} - i\tau_{zy} = \mu T\left[\Omega'(z) - i\overline{z}\right], \qquad (1.1.4)$$

where T is the constant twist per unit length, μ is the rigidity of the material of the cylinder, $\phi(x,y)$ is the torsion function which is harmonic inside S, $\psi(x,y)$ is the harmonic function conjugate to $\phi(x,y)$ and

$$\Omega(z) = \phi(x,y) + i\psi(x,y) \tag{1.1.5}$$

is the complex torsion function. The function $\psi(x,y)$ must be harmonic in S and it must satisfy the boundary condition

$$\psi(x,y) = \frac{1}{2}\left(x^2 + y^2\right) \quad on\ \Gamma. \tag{1.1.6}$$

To solve the torsion problem for a cross section S, it is more convenient to find the stress function $\Psi(x,y)$

$$\Psi(x,y) = \psi(x,y) - \frac{1}{2}\left(x^2 + y^2\right), \tag{1.1.7}$$

where $\psi(x,y)$ and $\Psi(x,y)$ must be finite and continuous in S.

Once $\psi(x,y)$ is determined we can use it in relations (1.1.2) and (1.1.3) to compute the stresses. Since the material of the cylinder is isotropic as opposed to orthotropic material, we are left with only two elastic constants. Therefore, we only have two stress components given as

$$\tau_{zx} = \mu T \frac{\partial \Psi}{\partial y}, \quad \tau_{zy} = -\mu T \frac{\partial \Psi}{\partial x}. \tag{1.1.8}$$

The torsional rigidity determines the strength of the beam, which is represented by

$$D = 2\mu \iint_S \Psi(x,y)\,dxdy. \tag{1.1.9}$$

In polar coordinates (r,θ) the stresses are represented by

$$\tau_{zr} = \frac{\mu T}{r}\frac{\partial \Psi}{\partial \theta}, \quad \tau_{z\theta} = -\mu T \frac{\partial \Psi}{\partial r} \tag{1.1.10}$$

and the torsional rigidity is

$$D = 2\mu \iint_S \Psi(r,\theta)\, r\, dr\, d\theta \,. \tag{1.1.11}$$

The shearing stress is given by

$$\sigma(r,\theta) = \sqrt{\tau_{zr}{}^2 + \tau_{z\theta}{}^2}, \tag{1.1.12}$$

which provides the amount of force upon the cross section of the bar while being twisted.

Section 1.2 The Bassali-Obaid Method

The process of Bassali-Obaid's method [4] is to start with a complex analytic function $F(z)$ in the domain S, enclosed by Γ, and write the boundary Γ in terms of the analytic function. The function $F(z)$ would be used to find the stress function $\Psi(x,y)$ corresponding to the given cross section S.

Assume the general equation of a given simple closed boundary Γ has the form

$$\operatorname{Re} F(z) - \left(\alpha x^2 + \beta y^2 + \gamma\right) = 0\,, \tag{1.2.1}$$

where α, β, γ are given real constants with the condition that $\alpha + \beta \neq 0$. We assume that the harmonic function $\psi(x,y)$ takes the form

$$\psi(x,y) = A \operatorname{Re} F(z) + B\left(x^2 - y^2\right) + C, \tag{1.2.2}$$

where *A*,*B*, and *C* are constants to be determined. We can find *A*, *B*, and *C* by substituting the boundary condition (1.1.6) and the equation of the boundary (1.2.1) in (1.2.2):

$$A\left(\alpha x^2+\beta y^2+\gamma\right)+B\left(x^2-y^2\right)+C=\frac{1}{2}\left(x^2+y^2\right) \quad \text{on} \quad \Gamma$$

which provides the following linear equations in the unknowns *A*,*B*, and *C*:

$$\alpha A+B=\frac{1}{2}, \quad \beta A-B=\frac{1}{2}, \quad \gamma A+C=0.$$

Solving these linear equations will provide

$$A=\frac{1}{\alpha+\beta}, \quad B=\frac{\beta-\alpha}{2(\alpha+\beta)}, \quad C=-\frac{\gamma}{\alpha+\beta}. \tag{1.2.3}$$

Substituting (1.2.3) into (1.2.2), the harmonic function ψ becomes

$$\psi(x,y)=\frac{1}{\alpha+\beta}\left[\operatorname{Re}F(z)+\frac{1}{2}(\beta-\alpha)\left(x^2-y^2\right)-\gamma\right]. \tag{1.2.4}$$

Substituting ψ from (1.2.4) into (1.1.7), the stress function takes the general form

$$\Psi(x,y)=\frac{1}{\alpha+\beta}\left[\operatorname{Re}F(z)-\left(\alpha x^2+\beta y^2+\gamma\right)\right]. \tag{1.2.5}$$

We notice that the stress function in (1.2.5) is a multiple of the function that appears in (1.2.1). Also it is obvious that the stress function satisfies Poisson's equation $\nabla^2\Psi=-2$ in S and the boundary condition $\Psi=0$ on Γ.

Section 1.3 Cross Section Bounded by a Hyperbola and a Straight-Line

To solve the torsion problem for a cylinder whose cross section S is bounded by the closed curve Γ is to use the Bassali-Obaid method [4] as discussed in section 1.2. In order to rewrite the equation of the closed curve Γ in the form (1.2.1) we need to find a special analytic function $F(z)$ and numbers α, β, γ that corresponds to Γ. Once we are able to do this, we use (1.2.5) to find the stresses and the torsional rigidity.

As an example, consider the case $b > a \geq 0$ and assume

$$F(z) = z^3 - a^2 z, \quad \alpha = b, \quad \beta = -3b, \quad \gamma = -a^2 b \qquad (1.3.1)$$

to satisfy with the desired Γ. Since $z = x + iy$, $\operatorname{Re} F(z) = x^3 - 3xy^2 - a^2 x$. Substituting $F(z), \alpha, \beta, \gamma$ from (1.3.1) into (1.2.1), the Cartesian equation of Γ becomes

$$x^3 - 3xy^2 - a^2 x - bx^2 + 3by^2 + a^2 b = 0 \qquad (1.3.2)$$

or

$$\left(x^2 - 3y^2 - a^2\right)\left(x - b\right) = 0. \qquad (1.3.3)$$

The cross section is bounded by one branch of the hyperbola $x^2 - 3y^2 = a^2$ and a vertical line $x = b$ as shown in Fig. 1.

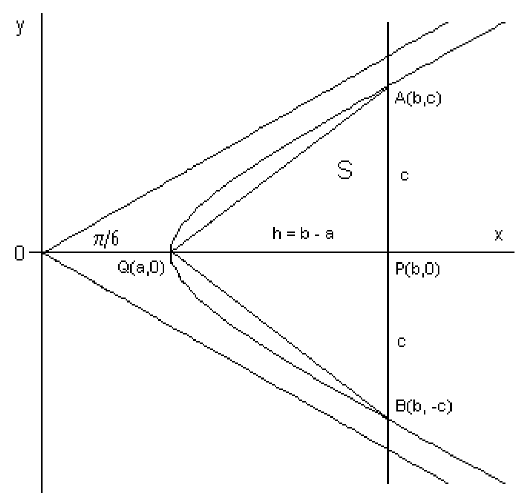

Fig. 1 Section bounded by a hyperbola and a straight line.

This hyperbola has an eccentricity of $\frac{2}{\sqrt{3}}$. Using (1.2.5), the stress function takes the form

$$\Psi(x,y) = -\frac{1}{2b}\left(x^3 - 3xy^2 - a^2x - bx^2 + 3by^2 + a^2b\right). \qquad (1.3.4)$$

To find the torsional rigidity D, we substitute (1.3.4) into (1.1.9):

$$D = \frac{\mu\sqrt{3}}{90b}\left[\left(2b^4 - 9a^2b^2 - 8a^4\right)\sqrt{b^2 - a^2} + 15a^4b\ln\frac{b + \sqrt{b^2 - a^2}}{a}\right].$$

Since the computation of the torsional rigidity D is time consuming and tedious, we refer the reader to see [4].

To find the stresses, we first have to find the partial differentiation of the stress function Ψ, which becomes

$$\frac{\partial\Psi}{\partial y} = \frac{1}{b}\left(3xy - 3by\right), \quad \frac{\partial\Psi}{\partial x} = -\frac{1}{2b}\left(3x^2 - 3y^2 - a^2 - 2bx\right).$$

Using the above partial derivatives, the stresses for this cross-section are determined by

$$\tau_{zx} = \frac{-3\mu Ty(b-x)}{b}, \quad \tau_{zy} = \frac{-\mu T\left(3y^2 - 3x^2 + 2bx + a^2\right)}{2b}. \quad (1.3.5)$$

On the straight segment APB, we have $x = b$, so the stresses on APB are

$$\tau_{zx} = 0, \quad \tau_{zy} = \frac{\mu T\left(b^2 - 3y^2 - a^2\right)}{2b}.$$

Substituting the above stresses into (1.1.11), the shearing stress function on the line segment is

$$\sigma(x,y) = \frac{\mu T\left(b^2 - 3y^2 - a^2\right)}{2b}. \qquad (1.3.6)$$

To determine the maximum value of shearing stress on the line segment, we differentiate (1.3.6) with respect to y, which is

$$\frac{d\sigma}{dy} = -\frac{3\mu Ty}{b} = 0$$

Using the first derivative test, the maximum y- value occurs at $y = 0$. The maximum value of the resultant shearing stress occurs at point P (b, 0) where

$$\sigma_p = \frac{\mu T\left(b^2 - a^2\right)}{2b}. \qquad (1.3.7)$$

On the hyperbolic boundary of AQB in Fig. 1, we have $y = \sqrt{\frac{x^2 - a^2}{3}}$, so the stresses on AQB are determined by

$$\tau_{zx} = \frac{-\mu T(b-x)\sqrt{3\left(x^2 - a^2\right)}}{b}, \quad \tau_{zy} = \frac{-\mu Tx(b-x)}{b}.$$

Substituting the above stresses into (1.1.11), the shearing stress function becomes

$$\sigma(x, y) = \frac{\mu T(b-x)}{b}\sqrt{4x^2 - 3a^2}. \qquad (1.3.8)$$

Differentiating (1.3.8) with respect to *x* gives

$$\frac{d\sigma}{dx} = -\frac{\mu T}{b}\left[\frac{8x^2 - 3a^2 - 4bx}{\sqrt{4x^2 - 3a^2}}\right] = 0,$$

so the maximum x-value occurs at $x = \frac{b + \sqrt{b^2 + 6a^2}}{4}$. The maximum value of the shearing stress on the hyperbolic boundary is

$$\sigma_{\max} = \frac{\mu T\sqrt{2}}{8b}\left(3b - \sqrt{b^2 + 6a^2}\right)\left(b^2 - 3a^2 + b\sqrt{b^2 + 6a^2}\right)^{1/2} \qquad (1.3.9)$$

and it occurs at two points with abscissa f where

$$f=\frac{b+\sqrt{b^2+6a^2}}{4}.$$

Section 1.4 Cross Sections Bounded by Circular Grooves

To solve the torsion problem for a cross section with a circular grove we choose

$$F(z)=z-\frac{b^2}{z},\quad \alpha=\beta=\frac{1}{2a},\quad \gamma=-\frac{b^2}{2a} \qquad (1.4.1)$$

with the condition $0\le b<2a$. Substituting $F(z),\ \alpha,\ \beta,\ \gamma$ from (1.4.1) into (1.2.1), the polar equation of Γ becomes

$$\left(r^2-b^2\right)\left(\frac{\cos\theta}{r}-\frac{1}{2a}\right)=0. \qquad (1.4.2)$$

The cross section is bounded by two circles $r=b$ and $r=2a\cos\theta$. Using (1.2.5), the stress function takes the polar form

$$\Psi(r,\theta)=a\left(r\cos\theta-\frac{b^2}{r}\cos\theta+\frac{b^2-r^2}{2a}\right). \qquad (1.4.3)$$

For the second cross section with a circular groove we choose in (1.2.1)

$$F(z)=z^2-\frac{b^4}{z^2},\quad \alpha=\beta=m,\quad \gamma=-ma^2. \qquad (1.4.4)$$

We investigate this case in more detail. The polar equation of Γ for the second case takes the form

$$\left(r^2-a^2\right)\left(\frac{r^2+a^2}{r^2}\cos 2\theta-m\right)=0. \qquad (1.4.5)$$

The cross section is bounded by the circle $r=a$ and the curve

$$r^2 = \frac{a^2 \cos 2\theta}{m - \cos 2\theta}. \tag{1.4.6}$$

Let $r = b$ when $\theta = 0$, which leads to $b^2 = \dfrac{a^2}{m-1}$. The condition $b > 0$ leads to $m - 1 > 0$.

In addition, $a < b$ corresponds to $m < 2$. Therefore, we have $1 < m < 2$. If we put $m = 5/4$ in (1.4.6), we obtain Fig. 2 below.

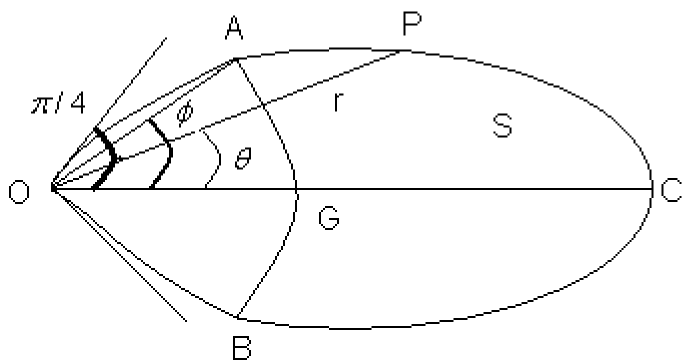

Fig. 2 Section bounded by a circular groove.

The curve (1.4.6) passes through the pole O. The circle and the closed curve intersects at $A(a, \phi)$ and $B(a, -\phi)$ where $0 < \phi = \dfrac{1}{2}\cos^{-1}\dfrac{m}{2} < \dfrac{\pi}{6}$. The stress function in polar form is

$$\Psi(r, \theta) = \frac{1}{2m}\left[\left(r^2 - \frac{a^4}{r^2}\right)\cos 2\theta + ma^2 - mr^2\right]. \tag{1.4.7}$$

Differentiating (1.4.7) with respect to r and θ, and substituting the results in (1.1.10), the stresses for this cross section are given by

$$\tau_{zr} = -\frac{\mu T}{m}\left(r - \frac{a^4}{r^3}\right)\sin 2\theta, \quad \tau_{z\theta} = -\frac{\mu T}{m}\left[\left(r + \frac{a^4}{r^3}\right)\cos 2\theta - mr\right]$$

respectively. Using the above stresses in (1.1.12), the shearing stress is

$$\sigma(r,\theta)=\frac{\mu T}{mr^3}\left[r^8\left(1-2m\cos\theta+m^2\right)+2a^4r^4\left(\cos 4\theta-m\cos 2\theta\right)+a^8\right]^{1/2}.$$

Letting $t=\cos 2\theta$ leads to $r^2=\dfrac{a^2t}{m-t}$. So the above shearing stress becomes

$$\sigma(\theta)=\mu Ta\left(\frac{2}{m}-\frac{1}{t}\right)\sqrt{\frac{m^2-2mt^3+t^4}{t(m-t)}}, \qquad (1.4.8)$$

where $-\phi\le\theta\le\phi$.

On the circle $r=a$, the shearing stress is

$$\sigma(\theta)=\mu Ta\left(\frac{2t}{m}-1\right),$$

where $t=\cos 2\theta$. At C and G in Fig. 2 we get t = 1, so the shearing stress becomes

$$\sigma_C=\mu Ta\left(\frac{2}{m}-1\right)\sqrt{m-1},\ \sigma_G=\mu Ta\left(\frac{2}{m}-1\right)$$

respectively where $\sigma_G>\sigma_C$. At A and B we get $t=\cos 2\phi={m}/{2}$, so the shearing stress becomes $\sigma_A=\sigma_B=0$. Refer to [4] for the torsional rigidity D for both cases of the cross sections with circular grooves.

Section 1.5 Cross Sections Bounded by Hyperbolic Arcs

To solve the classical torsion problem for the cross section bounded by hyperbolic arcs we choose

$$F(z)=z^4,\ \alpha=-4c^2,\quad \beta=\lambda\alpha,\quad \gamma=\tfrac{1}{2}\left(1+6\lambda+\lambda^2\right)c^4 \qquad (1.5.1)$$

where λ is a parameter. Substituting (1.5.1) into (1.2.1) leads to the Cartesian equation of the curve Γ as follows:

$$x^4 - 6x^2y^2 + y^4 + 4c^2x^2 + 4\lambda c^2y^2 - \tfrac{1}{2}c^4\left(1 + 6\lambda + \lambda^2\right) = 0\,. \qquad (1.5.2)$$

The Cartesian equation of Γ may be rewritten in the form

$$\left\{x^2 - \left(3 + 2\sqrt{2}\right)y^2 + \left(\tfrac{3+\lambda}{\sqrt{2}} + 2\right)c^2\right\}\left\{x^2 - \left(3 - 2\sqrt{2}\right)y^2 - \left(\tfrac{3+\lambda}{\sqrt{2}} - 2\right)c^2\right\} = 0\,. \quad (1.5.3)$$

So, the cross section S is bounded by two hyperbolas

$$\left(3 + 2\sqrt{2}\right)y^2 - x^2 = \left(\frac{3+\lambda}{\sqrt{2}} + 2\right)c^2\,, \qquad (1.5.4a)$$

$$x^2 - \left(3 - 2\sqrt{2}\right)y^2 = \left(\frac{3+\lambda}{\sqrt{2}} - 2\right)c^2\,. \qquad (1.5.4b)$$

Solving the systems of equations in (1.5.4a) and (1.5.4b) leads to

$$a = \tfrac{1}{2}c\sqrt{1 + 3\lambda}\,, \quad b = \tfrac{1}{2}c\sqrt{3 + \lambda}\,, \qquad (1.5.5)$$

where $(\pm a, \pm b)$ represents the intersection points of the two hyperbolas. To find the asymptotes of the hyperbolas we set the right side of (1.5.4a,b) equal to zero, which gives

$$y = \pm\left(\sqrt{2} - 1\right)x = \pm x \tan\tfrac{\pi}{8}\,, \qquad (1.5.6a)$$

$$y = \pm\left(\sqrt{2} + 1\right)x = \pm x \tan\tfrac{3\pi}{8} \qquad (1.5.6b)$$

respectively.

Using (1.2.5) the stress function in polar form corresponding to Γ becomes

$$\Psi(r,\theta) = \frac{\tfrac{1}{2}\left(1 + 6\lambda + \lambda^2\right)c^4 - 4c^2r^2\left(\cos^2\theta + \lambda\sin^2\theta\right) - r^4\left(\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta\right)}{4c^2(1+\lambda)}$$

or $$\Psi(r,\theta) = \frac{\left(1 + 6\lambda + \lambda^2\right)c^4 - 4c^2r^2\left\{1 + \lambda + (1-\lambda)\cos 2\theta\right\} - 2r^4\cos 4\theta}{8c^2(1+\lambda)}\,. \quad (1.5.7)$$

In addition, the polar forms of the hyperbolas in (1.5.4a,b) respectively become

$$r_1^2 = c^2 A \big/ \left(\sqrt{2} - 2\cos 2\theta\right), \tag{1.5.8a}$$

$$r_2^2 = c^2 B \big/ \left(\sqrt{2} + 2\cos 2\theta\right), \tag{1.5.8b}$$

where $A = \left(1+\sqrt{2}\right) + \lambda\left(\sqrt{2}-1\right) = \tan\frac{3\pi}{8} + \lambda\tan\frac{\pi}{8}$ and similarly $B = \tan\frac{\pi}{8} + \lambda\tan\frac{3\pi}{8}$. To simplify our functions, we introduce two new parameters, k and ϕ, where

$$k = \frac{a}{b} = \cot\phi = \sqrt{\frac{1+3\lambda}{3+\lambda}}\ . \tag{1.5.9}$$

For the boundary of the cross section to become a curvilinear rectangle, the right side of (1.5.4b) needs to be greater than zero, so the restriction on λ is

$$2\sqrt{2} - 3 < \lambda < \infty\ . \tag{1.5.10}$$

When $\lambda = 0$, $k = \dfrac{1}{\sqrt{3}}$ and $\phi = \dfrac{\pi}{3}$ the hyperbolas bounding the cross section are

$$r_1^2 = c^2\left(\sqrt{2}-1\right) \big/ \left(\sqrt{2} - 2\cos 2\theta\right),$$

$$r_2^2 = c^2\left(\sqrt{2}-2\right) \big/ \left(\sqrt{2} + 2\cos 2\theta\right),$$

which are shown in Fig. 3.

Let $(\pm a_1, \pm b_1)$ corresponds to (λ_1, c_1) and $(\pm a_2, \pm b_2)$ corresponds to (λ_2, c_2), so (1.5.5) can be rewritten as

$$a_v = \tfrac{1}{2} c_v \sqrt{1+3\lambda_v}\,, \quad b_v = \tfrac{1}{2} c_v \sqrt{3+\lambda_v} \tag{1.5.11}$$

for $v = 1, 2$. Setting $a_1 = b_2$ and $a_2 = b_1$ leads to

$$\frac{c_1}{c_2} = \sqrt{\frac{3+\lambda_2}{1+3\lambda_1}} = \sqrt{\frac{1+3\lambda_2}{3+\lambda_1}}\ . \tag{1.5.12}$$

Solving (1.5.12) gives

$$\lambda_1 \lambda_2 = 1. \tag{1.5.13}$$

Using (1.5.13) in (1.5.12), it reduces to

$$\frac{c_1}{c_2} = \sqrt{\lambda_2} = \frac{1}{\sqrt{\lambda_1}}. \tag{1.5.14}$$

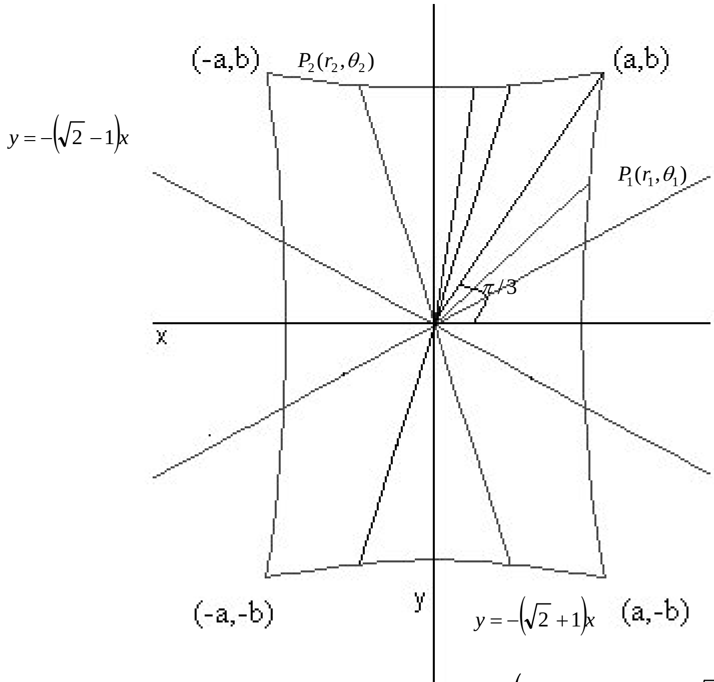

Fig. 3 A curvilinear rectangle $(\lambda = 0, k = 1/\sqrt{3})$

Due to (1.5.13) one of the λ's has to be less than or equal to 1, so the range of λ becomes $2\sqrt{2} - 3 < \lambda \le 1$. Substituting the new values of λ into (1.5.9), the range of k becomes $\sqrt{2} - 1 < k \le 1$ and $\pi/4 \le \phi < 3\pi/8$. For $\lambda = 1$, $a = b = c$, $k = 1$ and $\phi = \frac{\pi}{4}$, the curvilinear rectangle becomes a curvilinear square due to r_1^2 and r_2^2 intersecting at $(\pm c, \pm c)$. The corresponding hyperbolas have the forms $r = 2c^2 / (1 \pm \sqrt{2}\cos 2\theta)$. In the special case of $\lambda = 2\sqrt{2} - 3$, $k = \sqrt{2} - 1$ and $\phi = \frac{3\pi}{8}$, (1.5.4b) degenerates into two straight lines $y = \pm\left(\sqrt{2}+1\right)x$. The cross section in this case is bounded by these lines and the hyperbola $r_1^2 = 4c^2\left(\sqrt{2}-1\right)/\left(1-\sqrt{2}\cos 2\theta\right)$.

Substituting (1.5.7) into (1.1.11), the torsional rigidity for cross sections bounded by hyperbolic arcs is

$$D = \frac{\mu}{(1+\lambda)c^2}\left[\int_0^{\phi}\left\{\tfrac{1}{2}\left(1+6\lambda+\lambda^2\right)c^4 r_1^2 - \left(1+\lambda+\overline{1-\lambda}\cos 2\theta\right)c^4 r_1^4 - \tfrac{1}{3}r_1^6\cos 4\theta\right\}d\theta + \int_{\phi}^{\pi/2}\left\{\tfrac{1}{2}\left(1+6\lambda+\lambda^2\right)c^4 r_2^2 - \left(1+\lambda+\overline{1-\lambda}\cos 2\theta\right)c^4 r_2^4 - \tfrac{1}{3}r_2^6\cos 4\theta\right\}d\theta\right], \tag{1.5.15}$$

where r_1^2, r_2^2 are given by (1.5.8a,b) respectively. Setting $\lambda = 0$ into (1.5.15) the torsional rigidity can be shown to be

$$D(c,0) = \frac{1}{4}\mu c^4\left[6\ln\left(\sqrt{2}+\sqrt{3}\right) - 2\sqrt{2}\ln\left(2+\sqrt{3}\right) - \sqrt{6}\right]. \tag{1.5.16}$$

For $\lambda = 1$ and the special case $\lambda = 2\sqrt{2}-3$, the torsional rigidities are

$$D(c,1) = 4\mu c^4\left[\ln\left(\sqrt{2}+1\right) - \frac{\sqrt{2}}{3}\right], \tag{1.5.17}$$

$$D\left(c, 2\sqrt{2}-3\right) = \frac{4}{3}\mu b^4\left(3-2\sqrt{2}\right)\left(3\ln 2 - 2\right) \tag{1.5.18}$$

respectively. Refer to [4] for the computation of each torsional rigidity case.

For $\lambda = 0$, the stress function of (1.5.7) gives

$$\Psi(r,\theta) = \frac{1}{8c^2}\left[c^2 - 4c^2r^2 - 4c^2r^2\cos 2\theta - 2r^4\cos 4\theta\right]. \tag{1.5.19}$$

Differentiating (1.5.19) with respect to r and θ, and substituting the results into (1.1.10), the stresses are given by

$$\tau_{zr} = \mu Tr\left[\sin 2\theta + \tfrac{r^2}{c^2}\sin 4\theta\right], \quad \tau_{z\theta} = \mu Tr\left[1 + \cos 2\theta + \tfrac{r^2}{c^2}\cos 4\theta\right] \qquad (1.5.20)$$

respectively. The distribution of shearing stress results by replacing (1.5.20) into (1.1.12) leads to

$$\sigma(r,\theta) = 2\mu Tr\sqrt{\tfrac{r^4}{4c^4} + \cos^2\theta + \tfrac{r^2}{c^2}\left(\cos\theta\cos 3\theta\right)}. \qquad (1.5.21)$$

Similarly, the general formula for the stresses and the shearing stress are

$$\tau_{zr} = \frac{\mu Tr}{1+\lambda}\left[(1-\lambda)\sin 2\theta + \tfrac{r^2}{c^2}\sin 4\theta\right], \quad \tau_{z\theta} = \frac{\mu Tr}{1+\lambda}\left[1 + \lambda + (1-\lambda)\cos 2\theta + \tfrac{r^2}{c^2}\cos 4\theta\right],$$

$$\sigma(r,\theta) = \frac{2\mu Tr}{1+\lambda}\sqrt{\tfrac{r^4}{4c^4} + \cos^2\theta + \lambda^2\sin^2\theta + \tfrac{r^2}{c^2}\left(\cos\theta\cos 3\theta - \lambda\sin\theta\sin 3\theta\right)}$$

respectively.

Section 1.6 Cross Section with Two Linear Sides and Certain Curved Bases

In this section the boundary of the cross section consist of two straight lines $\theta = \pm\dfrac{\pi}{2n}$ and the curved base

$$\left(\frac{r}{c_0}\right)^{n-2} = \frac{\cos 2\theta - \cos\frac{\pi}{n}}{\cos n\theta\left(1 - \cos\frac{\pi}{n}\right)}, \ \left(1 < n < \infty, -\tfrac{\pi}{2n} \le \theta \le \tfrac{\pi}{2n}, n \ne 2\right). \qquad (1.6.1)$$

This curve is symmetric with respect to the x-axis and crosses the point $P_0(c_0, 0)$ at a 90° angle. Before we can begin using Bassali-Obaid's method [4] to solve this torsion problem, we need to find the Cartesian equation of Γ_n.

It is well-known that if *n* is an even integer *2m*, then we have the following identities

$$\cos 2m\theta = 2^{m-1} \prod_{v=1,3,\ldots}^{2m-1} \left(\cos 2\theta - \cos \frac{v\pi}{2m} \right), \quad (m = 2,3,\ldots) \qquad (1.6.2)$$

and

$$\prod_{v=1,3,\ldots}^{2m-1} \sin^2 \frac{v\pi}{4m} = 2^{1-2m}. \qquad (1.6.3)$$

Setting $n = 2m$ and substituting (1.6.2) and (1.6.3) into the curved base, (1.6.1) becomes

$$r^{2m} \left[\prod_{v=1,3,\ldots}^{2m-1} \left(\cos 2\theta - \cos \frac{v\pi}{2m} \right) \right] = \frac{c_0^{2m-2} r^2 \left(\cos 2\theta - \cos \frac{\pi}{2m} \right)}{2^m \sin^2 \frac{\pi}{4m}}. \qquad (1.6.4)$$

Note the left side of (1.6.4) becomes 'n' products of $r^2 \left(\cos 2\theta - \cos \frac{v\pi}{2m} \right)$, so (1.6.4) gives

$$\left[\prod_{v=1,3,\ldots}^{2m-1} \left(r^2 \cos 2\theta - r^2 \cos \frac{v\pi}{2m} \right) \right] = \frac{c_0^{2m-2} \left(r^2 \cos 2\theta - r^2 \cos \frac{\pi}{2m} \right)}{2^m \sin^2 \frac{\pi}{4m}}. \qquad (1.6.5)$$

Converting the polar coordinates into Cartesian coordinates in (1.6.5), we get

$$\prod_{v=1,3,\ldots}^{2m-1} \left[x^2 - y^2 \left(\frac{1 + \cos \frac{v\pi}{2m}}{1 - \cos \frac{v\pi}{2m}} \right) \right] = \frac{c_0^{2m-2} \left[x^2 \left(1 - \cos \frac{\pi}{2m} \right) - y^2 \left(1 + \cos \frac{\pi}{2m} \right) \right]}{2^m \sin^2 \frac{\pi}{4m} \prod_{v=1,3,\ldots}^{2m-1} \left(1 - \cos \frac{\pi}{2m} \right)}.$$

After a lot of meticulous substitutions, (1.6.1) becomes

$$\prod_{v=3,5,\ldots}^{2m-1} x^2 - y^2 \cot^2 \frac{v\pi}{4m} = c_0^{2m-2} \qquad (1.6.6)$$

for the case of n being an even integer.

If n is an odd integer $2m+1$, then we can use the following identities

$$\cos(2m+1)\theta = 2^m \cos\theta \prod_{v=1,3,\ldots}^{2m-1} \left(\cos 2\theta - \cos \frac{v\pi}{2m+1} \right), \quad (m = 1,2,3,\ldots)$$

and

$$\prod_{v=1,3,\ldots}^{2m-1} \sin^2 \frac{v\pi}{4m} = 2^{-2m}.$$

Similarly, the curved base for the odd integer n becomes

$$x\prod_{v=3,5,\ldots}^{2m-1}[x^2-y^2\cot^2\frac{v\pi}{4m+2}]=c_0^{2m-1}. \qquad (1.6.7)$$

For $n=3$, (1.6.1) reduces to a vertical line $x=c_0$ and Γ_3 is the equilateral triangle $OP_3P_3^{'}$ in Fig. 4 below. In addition, Fig. 4 displays the Cartesian equation of the curved base as the hyperbola $x^2-y^2\tan^2\frac{\pi}{8}=c_0^2$ when $n=4$. The Cartesian equation of Γ_5 becomes the cubic curve $x\left(x^2-y^2\tan^2\frac{\pi}{5}\right)=c_0^3$. The curved base simplifies to the quartic curve $\left(x^2-y^2\right)\left(x^2-y^2\tan^2\frac{\pi}{12}\right)=c_0^4$ for $n=6$.

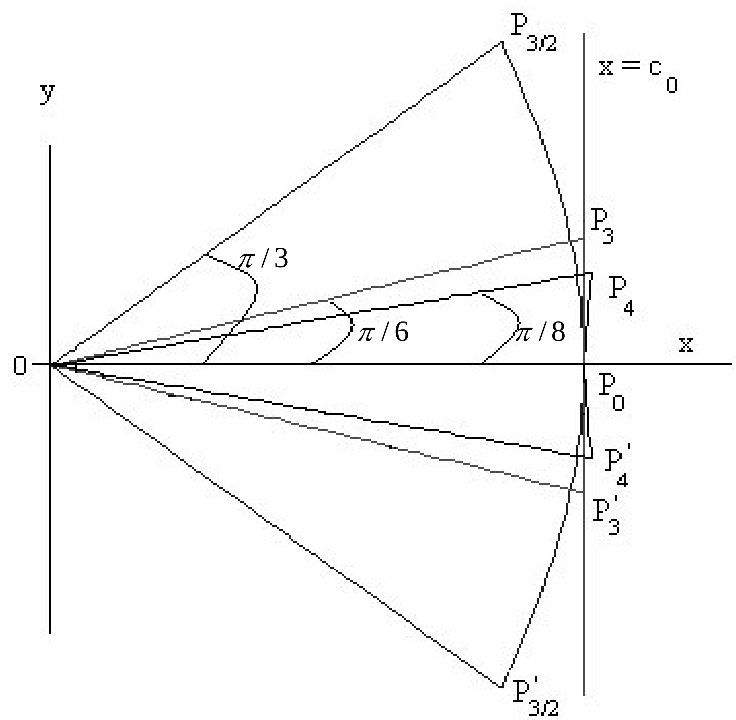

Fig. 4 Sectorial cross sections corresponding to *n* = 3/2, 3, and 4.

From Fig. 4, it can be proven that when $n>3$ the curved base extends to the right of the vertical line $x=c_0$ whereas for $n<3$ the curved base lies on the left of the line.

To continue with the Bassali-Obaid method [4], we need to find the stress function by choosing the right $F(z)$. For $n \neq 2$, let

$$\alpha = -\frac{t^2}{1-t^2}, \quad \beta = \frac{1}{1-t^2}, \quad \gamma = 0, \quad F(z) = C_n z^n \qquad (1.6.8)$$

where C_n is a constant. Replacing $t = \tan\frac{\pi}{2n}$ into (1.6.8) and using the new values for (1.2.5), the stress function becomes

$$\Psi_n(x,y) = C_n r^n \cos n\theta + \frac{x^2 \tan^2 \frac{\pi}{2n} - y^2}{2 - \sec^2 \frac{\pi}{2n}}. \qquad (1.6.9)$$

Using a few trigonometric substitutions, the stress function in polar form is

$$\Psi_n(r,\theta) = \left(\cos 2\theta - \cos\frac{\pi}{n}\right)\left[\frac{C_n r^n \cos n\theta}{\cos 2\theta - \cos\frac{\pi}{n}} + \frac{r^2 \sec\frac{\pi}{n}}{2}\right] \qquad (1.6.10)$$

with condition of $1 < n < \infty, n \neq 2$.

Using the boundary condition, we found $\theta = \pm\frac{\pi}{2n}$ and

$$r^{n-2} = -\frac{\left(\cos 2\theta - \cos\frac{\pi}{n}\right)\sec\frac{\pi}{n}}{2C_n \cos n\theta}. \qquad (1.6.11)$$

To keep (1.6.11) consistent with (1.6.1) we set $C_n = \frac{1}{2} c_0^{2-n}\left(1 - \sec\frac{\pi}{n}\right)$, which leads to

$$F(z) = \frac{\left(1 - \sec\frac{\pi}{n}\right) z^n}{2c_0^{2-n}}. \qquad (1.6.12)$$

Substituting C_n into (1.6.10) we obtain

$$\Psi_n(r,\theta) = \frac{r^2}{2}\left[\left(\frac{r}{c_o}\right)^{n-2} \cos n\theta + \frac{\cos 2\theta - \left(\frac{r}{c_O}\right)^{n-2} \cos n\theta}{\cos\frac{\pi}{n}} - 1\right]. \qquad (1.6.13)$$

Substituting the partial derivatives of (1.6.13) with respect to r and θ into (1.1.10), the stress components at any point (r,θ) are given by

$$\tau_{zr} = \mu Tr\left[\frac{n}{2}\left(\frac{r}{c_0}\right)^{n-2} \sin n\theta\left(\sec\frac{\pi}{n}-1\right)-\sin 2\theta\sec\frac{\pi}{n}\right],$$

$$\tau_{z\theta} = \mu Tr\left[1-\cos 2\theta\sec\frac{\pi}{n}+\frac{n}{2}\left(\frac{r}{c_0}\right)^{n-2} \cos n\theta\left(1-\sec\frac{\pi}{n}\right)\right] \qquad (1.6.14)$$

respectively. Using (1.6.14) in (1.1.12), the shearing stress is

$$\begin{aligned}\sigma_n(r,\theta) = \mu Tr\Big[1+\sec^2\tfrac{\pi}{n}-2\sec\tfrac{\pi}{n}\cos 2\theta+\tfrac{n^2}{4}\left(\sec\tfrac{\pi}{n}-1\right)^2\left(\tfrac{r}{c_0}\right)^{2n-4} \\ n\left(\sec\tfrac{\pi}{n}-1\right)\left(\tfrac{r}{c_0}\right)^{n-2}\left\{\cos n\theta-\sec\tfrac{\pi}{n}\cos(n-2)\theta\right\}\Big]^{1/2}.\end{aligned} \qquad (1.6.15)$$

The amount of force upon the cross section of the bar on the x-axis while being twisted is given by

$$\sigma_n(r,0) = \left|1-\sec\tfrac{\pi}{n}\right|\cdot\left|1-\tfrac{n}{2}\left(\tfrac{r}{c_0}\right)^{n-2}\right|. \qquad (1.6.16)$$

The shearing stress at any point $\left(r,\frac{\pi}{2n}\right)$ of OP_n is

$$\sigma_n\left(r,\tfrac{\pi}{2n}\right) = \mu Tr\left|\tan\tfrac{\pi}{n}+\tfrac{n}{2}\left(1-\sec\tfrac{\pi}{n}\right)\left(\tfrac{r}{c_0}\right)^{n-2}\right|. \qquad (1.6.17)$$

Substituting (1.6.1) into (1.6.15) gives the shearing stress at any point (r,θ) of P_0P_n as

$$\sigma_n(\theta) = \frac{\mu Tr}{2\left|\cos\frac{\pi}{n}\right|}\left[(n-2)^2\left(\cos 2\theta-\cos\tfrac{\pi}{n}\right)^2+\left\{n\tan n\theta\left(\cos 2\theta-\cos\tfrac{\pi}{n}\right)-2\sin 2\theta\right\}^2\right]^{1/2}.$$

To find the torsional rigidity, we substitute (1.6.13) into (1.1.11) to obtain

$$D_n = 4\mu\int_0^{\pi/2n} r^4\left[\left(\frac{r}{c_0}\right)^{n-2}\left\{\frac{\cos n\theta\left(\cos\frac{\pi}{n}-1\right)}{2(n+2)\cos\frac{\pi}{n}}\right\}-\frac{\cos\frac{\pi}{n}-\cos 2\theta}{8\cos\frac{\pi}{n}}\right]d\theta. \qquad (1.6.18)$$

Further simplifying D_n by using (1.6.1) and $r^4 = c_0^4\left[\dfrac{\cos 2\theta - \cos\frac{\pi}{n}}{\cos n\theta\left(1-\cos\frac{\pi}{n}\right)}\right]^{\frac{4}{n-2}}$ leads to

$$D_n = \frac{\mu c_0^4 (n-2)\left(2\sin^2\frac{\pi}{2n}\right)^{\frac{4}{2-n}}}{(n+2)\cos\frac{\pi}{n}}\int_0^{\pi/2n}\left(\cos 2\theta - \cos\frac{\pi}{n}\right)^{\frac{n+2}{n-2}}(\cos n\theta)^{\frac{4}{2-n}}\,d\theta\,. \quad (1.6.19)$$

To find the torsional rigidity for $n = 4$ we start with (1.6.18), which gives

$$D_4 = \frac{\mu c_0^4\left(3+2\sqrt{2}\right)}{6}\int_0^{\pi/8}\frac{\left(\sqrt{2}\cos 2\theta - 1\right)^3}{\cos^2 4\theta}\,d\theta\,. \quad (1.6.20)$$

Using the substitution rule where $x = 2\theta$ and $dx = 2d\theta$, (1.6.20) becomes

$$\begin{aligned} D_4 &= \frac{\mu c_0^4\left(3+2\sqrt{2}\right)}{6}\left\{\frac{1}{2}\int_0^{\pi/4}\frac{\sqrt{2}\cos x - 1}{\left(\sqrt{2}\cos x + 1\right)^2}\,dx\right\} \\ &= \frac{\mu c_0^4\left(3+2\sqrt{2}\right)}{6}(J-K), \end{aligned} \quad (1.6.21)$$

where $J = \frac{1}{2}\int_0^{\pi/4}\dfrac{1}{\sqrt{2}\cos x + 1}dx$ and $K = \int_0^{\pi/4}\dfrac{1}{\left(\sqrt{2}\cos x + 1\right)^2}dx$. To solve for the integral J,

we need to apply Weierstrass' method where $u = \tan\frac{x}{2}$, which leads to $\cos x = \frac{1-u^2}{1+u^2}$. So,

$$J = 2\int_0^{\tan\frac{\pi}{8}}\frac{du}{\left(\sqrt{2}+1\right)-\left(\sqrt{2}-1\right)u^2}\,. \quad (1.6.22)$$

Factoring out $\sqrt{2}-1$ allows us to use a well-known Calculus formula, $\int\frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$,

which (1.6.22) becomes

$$J = \frac{1}{2}\ln 2\,. \quad (1.6.23)$$

Similarly, we have

$$K = \frac{2}{\left(\sqrt{2}-1\right)^2}\int_0^{\tan\frac{\pi}{8}} \frac{\left(1+u^2\right)du}{\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}-u^2\right)^2}.$$

Letting $b^2 = \frac{\sqrt{2}+1}{\sqrt{2}-1}$ and using the substitution rule for $u = b\sin\phi$ for K leads to

$$\begin{aligned} K &= \tfrac{1}{b^3}\int_0^{\phi_1}\left(1+b^2\right)\sec^3\phi - b^2\sec\phi\, d\phi \\ &= \tfrac{1}{b^3}\left[\tfrac{1}{2}\left(1+b^2\right)\left(\sec\phi\tan\phi + \ln\left|\sec\phi+\tan\phi\right|\right) - b^2\ln\left|\sec\phi+\tan\phi\right|\right]_0^{\phi_1} \end{aligned}$$

where $\phi_1 = \sin^{-1}\left(\frac{1}{b}\tan\frac{\pi}{8}\right)$. Evaluating K gives

$$K = \frac{1}{2}\left(1-\ln 2\right). \tag{1.6.24}$$

Substituting (1.6.23) and (1.6.24) into (1.6.21), the torsion rigidity at $n = 4$ is denoted as

$$D_4 = \frac{1}{24}\left(3+2\sqrt{2}\right)\mu c_0^4\left(3\ln 2 - 2\right). \tag{1.6.25}$$

Refer to [6] for the proof of other torsion rigidities.

Section 1.7 Cross Section Bounded by the Lemniscate of Booth

In the previous sections, we guessed $F(z)$ to obtain solutions to the torsion problem. In this section, we find $F(z)$ that corresponds to a cross section of the lemniscate of Booth Γ. This is the opposite of what we have done before and it is more difficult. To do so, we need to derive the equation of the lemniscate of Booth.

Given the ellipse $\Gamma' : x'^2/a'^2 + y'^2/b'^2 = 1$ with $a' > b'$, its polar form is

$$\frac{\cos^2\theta}{a'^2} + \frac{\sin^2\theta}{b'^2} = \frac{1}{r'^2},$$

where its center O is set as the pole and (r',θ) is a polar coordinate of any point P' on the ellipse. Let $P(r,\theta)$ be the inverse of P' with respect to the circle $|z| = f$. Therefore, the inverse of the ellipse Γ' is the lemniscate of Booth Γ with polar equation

$$r^2 = f^4\left(\frac{\cos^2\theta}{a'^2} + \frac{\sin^2\theta}{b'^2}\right)$$

or

$$r^2 = a^2 - b^2\cos 2\theta \tag{1.7.1}$$

where

$$a^2 = \tfrac{1}{2} f^4\left(\frac{1}{b'^2} + \frac{1}{a'^2}\right), \quad b^2 = \tfrac{1}{2} f^4\left(\frac{1}{b'^2} - \frac{1}{a'^2}\right).$$

From the ellipse Γ', the foci are $\left(\pm\sqrt{a'^2 - b'^2}, 0\right)$ where $c'^2 = a'^2 - b'^2$. The inverted foci of Γ is $(\pm c, 0)$. By inversion, $cc' = f^2$ implies $c^2 = f^4/c'^2$, so we must have

$$c^2 = \frac{f^4}{a'^2 - b'^2} = \frac{a^4 - b^4}{2b^2}. \tag{1.7.2}$$

Next we need to develop a relationship between the polar coordinates of P to the center O and the two foci O_1, O_2 of Γ as poles as shown in Fig. 5 on the following page.

Based on law of cosines, we have

$$r_1^2 = r^2 + c^2 - 2cr\cos\theta, \quad r_2^2 = r^2 + c^2 + 2cr\cos\theta. \tag{1.7.3}$$

Furthermore, we have

$$r_1 r_2 = \sqrt{r^4 + c^4 - 2c^2 r^2\cos 2\theta}. \tag{1.7.4}$$

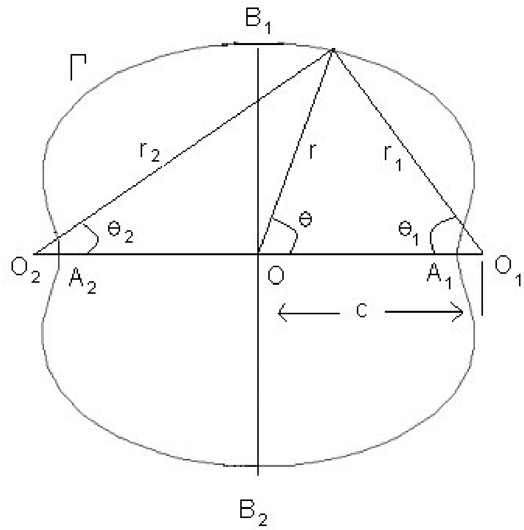

Fig. 5 The lemniscate of Booth section

Substituting $\cos 2\theta = {(a^2 - r^2)}/{b^2}$ from (1.7.1) and ${a^4}/{b^4} = 1 + {2c^2}/{b^2}$ from (1.7.2) into (1.7.4) we get

$$r_1 r_2 = \frac{a^2}{b^2} r^2 - c^2 = b^2 + c^2 - a^2 \cos 2\theta . \qquad (1.7.5)$$

Using (1.7.3) we establish further properties:

$$r_2 - r_1 = 2\cos\theta\sqrt{a^2 - b^2}, \quad r_2 + r_1 = r\sqrt{2\left(1 + {a^2}/{b^2}\right)} . \qquad (1.7.6)$$

The law of sines, $\dfrac{\sin\theta_1}{r_2} = \dfrac{\sin\theta_2}{r_1}$, leads to $\tan\left(\dfrac{\theta_1 - \theta_2}{2}\right) = \dfrac{r_2 - r_1}{r_2 + r_1}\tan\left(\dfrac{\theta_1 + \theta_2}{2}\right)$. Setting

$\phi = \pi - (\theta_1 + \theta_2)$ gives $\tan\left(\dfrac{\theta_1 - \theta_2}{2}\right) = \dfrac{r_2 - r_1}{r_2 + r_1}\cot\left(\dfrac{\phi}{2}\right)$.

Using trigonometric identities, it can be shown that

$$\tan\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{r_2 - r_1}{r_2 + r_1}\sqrt{\frac{(r_2 + r_1)^2 - 4c^2}{4c^2 - (r_2 - r_1)^2}} \qquad (1.7.7)$$

at any point P. If P lies on Γ we have

$$\tan\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{b^2 \sin 2\theta}{r^2}$$

by substituting (1.7.6) into (1.7.7). The above relations imply $\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{r^2}{b\sqrt{2r_1 r_2}}$.

Therefore, we can deduce $\sqrt{r_1 r_2}\cos\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{r^2}{b\sqrt{2}}$, which leads to

$$\mathrm{Re}\sqrt{c^2 - z^2} = \frac{r^2}{b\sqrt{2}} = \frac{x^2 + y^2}{b\sqrt{2}}. \tag{1.7.8}$$

Comparing (1.7.8) with (1.2.1), we find

$$F(z) = \sqrt{c^2 - z^2}, \alpha = \beta = \frac{1}{b\sqrt{2}}, \gamma = 0. \tag{1.7.9}$$

Substituting these values into (1.2.5), the stress function in polar form becomes

$$\Psi(r, \theta) = \frac{b}{\sqrt{2}}\mathrm{Re}\sqrt{c^2 - z^2} - \frac{1}{2}r^2. \tag{1.7.10}$$

It is well-known that $\mathrm{Re}\sqrt{u + iv} = \frac{1}{\sqrt{2}}\sqrt{u + \left(u^2 + v^2\right)^{1/2}}$. We will set $u = c^2 - x^2 + y^2$ and $v = -2xy$. Using the given formula and replacing u, v into (1.7.10) we have

$$\Psi(r, \theta) = \frac{1}{2}\left[b\sqrt{c^2 - r^2\cos 2\theta + \left(c^4 - 2c^2 r^2 \cos 2\theta + r^4\right)^{1/2}} - r^2\right]. \tag{1.7.11}$$

Differentiating (1.7.11) with respect to θ and r, and substituting the results in (1.1.10), the stresses for this cross section are given by

$$\tau_{zr} = -\frac{\mu T a^2 b^2 r \sin 2\theta}{a^4 + b^4 - 2a^2 b^2 \cos 2\theta}, \quad \tau_{z\theta} = -\frac{\mu T a^2 r^3}{a^4 + b^4 - 2a^2 b^2 \cos 2\theta} \tag{1.7.12}$$

respectively. Substituting (1.7.12) into (1.1.11), the resultant tangential stress is

$$\sigma(r,\theta) = \frac{\mu T a^2 r}{\sqrt{b^4 - a^4 + 2a^2 r^2}}. \tag{1.7.13}$$

To find the extreme values of the shearing stress on Γ, we use (1.7.1) in (1.7.13); then we solve $\frac{\partial}{\partial\theta}\sigma(\theta) = 0$ for θ. The critical points occur at $\theta = 0, \pm\frac{\pi}{2}, \pi$. The maximum value is $\sigma_{A_1} = \sigma_{A_2} = \frac{\mu T a^2}{\sqrt{a^2 - b^2}}$ at $\theta = 0, \pi$ whereas the minimum value is $\sigma_{B_1} = \sigma_{B_2} = \frac{\mu T a^2}{\sqrt{a^2 + b^2}}$ at $\theta = \pm\frac{\pi}{2}$.

Substituting (1.7.10) into (1.1.11) where $z = re^{i\theta}$, the torsional rigidity becomes

$$D = 8\mu\int_0^{\pi/2}\int_0^{\sqrt{a^2 - b^2\cos 2\theta}}\left(\frac{b}{\sqrt{2}}\operatorname{Re}\sqrt{c^2 - r^2 e^{2i\theta}} - \frac{1}{2}r^2\right) r\,dr\,d\theta. \tag{1.7.14}$$

Simplifying (1.7.14) and using the substitution rule, the torsional rigidity becomes

$$D = 4\mu(I_1 - I_2) \tag{1.7.15}$$

where

$$I_2 = \int_0^{\pi/2}\int_0^{a^2 - b^2\cos 2\theta} r^3\,dr\,d\theta = \frac{\pi}{16}\left(2a^4 + b^4\right), \tag{1.7.16}$$

$$\begin{aligned} I_1 &= \frac{b}{\sqrt{2}}\operatorname{Re}\int_0^{\pi/2}\int_0^{a^2 - b^2\cos 2\theta}\sqrt{c^2 - ue^{2i\theta}}\,du\,d\theta \\ &= -\frac{b\sqrt{2}}{3}\operatorname{Re}\int_0^{\pi/2} e^{-2i\theta}\left[c^2 - e^{2i\theta}\left(a^2 - b^2\cos 2\theta\right)\right]^{3/2} d\theta. \end{aligned}$$

Substituting c^2 from (1.7.2) and $\cos 2\theta = \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)$ into I_1 we have

$$I_1 = -\frac{1}{6b^2}\operatorname{Re}\int_0^{\pi/2}(a^6 e^{-2i\theta} - 3a^4 b^2 + 3a^2 b^4 e^{2i\theta} - b^6 e^{4i\theta})\,d\theta.$$

Since we only want the real value of I_1 we get

$$I_1 = \frac{1}{6b^2}\int_0^{\pi/2} 3a^4b^2 d\theta = \frac{\pi a^4}{4}. \qquad (1.7.17)$$

Substituting (1.7.16) and (1.7.17) into (1.7.15) the resultant torsional rigidity is

$$D = \tfrac{1}{4}\mu\pi\left(2a^4 - b^4\right).$$

Section 1.8 Tchebycheff Polynomials and their Applications

Another method to solve Saint-Venant Torsion Problem is through the use of Tchebycheff polynomials developed by Mohammed M. Abbassi [1]. This method offers, in many cases, a simpler and direct approach to the solution of the torsion equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = -2\mu\alpha \qquad (1.8.1)$$

where Ψ represents the stress function, μ is the modulus of rigidity of the material of the cylinder, and α is the angle of twist per unit length of the bar. The solution of the torsion problem Ψ must satisfy the boundary condition that Ψ is a constant on the boundary of the cross section of the cylinder.

The torsional rigidity is defined a bit differently. The reason is that Abbasi's Ψ is $\mu\alpha$ times the $\mathrm{Bassali-Obaid}\ \Psi$. We decided to keep Abbasi's notation, so it is defined as the magnitude of the torque, M_t, of any cross section given as

$$M_t = 2\iint \Psi\, dxdy. \qquad (1.8.2)$$

The non-vanishing stresses are determined by

$$\tau_{zx} = \frac{\partial \Psi}{\partial y}, \quad \tau_{zy} = -\frac{\partial \Psi}{\partial x} \tag{1.8.3}$$

using Sokolnikoff's notation in [13]. The solution to (1.8.1) is denoted as

$$\Psi = -2\mu\alpha\left[\tfrac{1}{4}\left(x^2 + y^2\right) + \phi\right], \tag{1.8.4}$$

where ϕ satisfies Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{1.8.5}$$

Twice differentiating $\phi(r(x,y), \theta(x,y))$ with respect to *x* and *y* then substituting these values into (1.8.5), we obtain the well-known Laplace's equation in polar form as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{1.8.6}$$

A solution of (1.8.6) using separation of variables is given by

$$\phi = F_1(\theta) F_2(r) \tag{1.8.7}$$

with conditions of $\phi(r,\pi) = \phi(r,-\pi)$ and $\phi_\theta(r,\pi) = \phi_\theta(r,-\pi)$. Hence

$$\phi_{rr} = F_1 F_2^{''}, \quad \phi_r = F_1 F_2^{'}, \quad \phi_{\theta\theta} = F_1^{''} F_2. \tag{1.8.8}$$

Substituting (1.8.8) into (1.8.6) and dividing both sides by $F_1 F_2$ leads to

$$r^2 \frac{F_2^{''}}{F_2} + r\frac{F_2^{'}}{F_2} = -\frac{F_1^{''}}{F_1} = \lambda^2. \tag{1.8.9}$$

From the case for F_1 we determine the parameter λ^2 and substitute the boundary conditions into the solution of the differential equation $F_1^{''} + \lambda^2 F_1 = 0$. The case of $\lambda^2 < 0$ gives a contradiction to the solution of the differential equation; therefore, we have the parameter to be $\lambda^2 \geq 0$. Hence, we have

$$F_1(\theta) = E_n \cos n\theta + H_n \sin n\theta, \qquad (1.8.10)$$

where E_n and H_n are arbitrary constants and $n \neq 0$. For the case of F_2, we let $F_2(r) = r^\gamma$, which gives $F_2^{'}(r) = \gamma r^{\gamma-1}$ and $F_2^{''}(r) = \gamma(\gamma-1)r^{\gamma-2}$. Substituting these derivatives into $r^2F_2^{''} + rF_2^{'} - \lambda^2 F_2 = 0$ leads to $(\gamma^2 - \lambda^2)r^\gamma = 0$. Since $r \neq 0$, we have $\gamma = \pm\lambda$. Therefore, we obtain

$$F_2(r) = K_n r^\lambda + L_n r^{-\lambda}, \qquad (1.8.11)$$

where K_n and L_n are arbitrary constants and $n \neq 0$. For $n = 0$, (1.8.10) and (1.8.11) becomes

$$F_1 = a_0 + b_0\theta, \quad F_2 = c_o \log r + d_0 \qquad (1.8.12)$$

respectively with a_0, b_0, c_0, and d_0 as arbitrary constants. Substituting (1.8.10), (1.8.11), and (1.8.12) into (1.8.7) the solution to (1.8.6) is given by

$$\phi = A_0 + B_0\theta + (C_o + D_0\theta)\log r + \sum_{n=1}^{\infty}\left[\left(A_n r^n + B_n r^{-n}\right)\cos n\theta + \left(C_n r^n + D_n r^{-n}\right)\sin n\theta\right].$$

Replacing the above function into (1.8.4), the stress function is determined by

$$\begin{aligned}\Psi = -2\mu\alpha \Big\{ &\tfrac{1}{4}r^2 + A_0 + B_0\theta + (C_o + D_0\theta)\log r + \\ &\sum_{n=1}^{\infty}\left[\left(A_n r^n + B_n r^{-n}\right)\cos n\theta + \left(C_n r^n + D_n r^{-n}\right)\sin n\theta\right] \Big\}.\end{aligned} \qquad (1.8.13)$$

(1.8.13) can be re-written in terms of Tchebycheff polynomials. Before we do so, let's introduce a few properties. The first type of Tchebycheff polynomials is denoted by

$$T_n(\cos\theta) = \cos n\theta. \qquad (1.8.14)$$

Since $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta \cos\theta$ and using (1.8.14), we have proven the following recurrence relation

$$T_{n+1}(\cos\theta) - 2\cos\theta T_n(\cos\theta) + T_{n-1}(\cos\theta) = 0. \qquad (1.8.15)$$

The second type of Tchebycheff polynomials is given by

$$U_n(\cos\theta) = \sin n\theta. \qquad (1.8.16)$$

Since $\sin(n+1)\theta + \sin(n-1)\theta = 2\sin n\theta \cos\theta$, the recurrence relation becomes

$$U_{n+1}(\cos\theta) - 2\cos\theta U_n(\cos\theta) + U_{n-1}(\cos\theta) = 0. \qquad (1.8.17)$$

Using (1.8.14) and (1.8.16), the stress function in (1.8.13) leads to

$$\begin{aligned}\Psi = -2\mu\alpha \Big\{ &\tfrac{1}{4}r^2 + A_0 + B_0\theta + (C_o + D_0\theta)\log r + \\ &\sum_{n=1}^{\infty}\left[\left(A_n r^n + B_n r^{-n}\right)T_n(\cos\theta) + \left(C_n r^n + D_n r^{-n}\right)U_n(\sin\theta)\right] \Big\},\end{aligned} \qquad (1.8.18)$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1} y/x$. Using Abbassi's method, the solutions with $B_o\theta$, $(C_o + D_0)\log r$, and $\sum_{n=1}^{\infty}\left(C_n r^n + D_n r^{-n}\right)U_n(\sin\theta)$ are usually not needed, so we will set $B_0 = C_0 = D_0 = C_n = D_n = 0$. With these new conditions, (1.8.18) becomes

$$\Psi = -2\mu\alpha \left\{ \tfrac{1}{4}r^2 + \sum_{n=0}^{\infty}\left[\left(A_n r^n + B_n r^{-n}\right)T_n(\cos\theta)\right] \right\}. \qquad (1.8.19)$$

Therefore, if the equation of the cross section can be rewritten as

$$\tfrac{1}{4}r^2 + \sum_{n=0}^{\infty}\left[\left(A_n r^n\right)T_n(\cos\theta)\right] = 0 \qquad (1.8.20)$$

or

$$\tfrac{1}{4}r^2 + \sum_{n=0}^{\infty}\left[\left(A_n r^n + B_n r^{-n}\right)T_n(\cos\theta)\right] = 0, \qquad (1.8.21)$$

then the stress function of (1.8.19) is the solution for the torsion problem.

1.9 Cross Section Bounded by a Circle, or Ellipse, or Equilateral Triangle

In this section, we use Abbassi's method for three different cross sections: circle, ellipse, and equilateral triangle. First we need to convert the Cartesian equation of Γ into the form of (1.8.20). Once this is verified, we are able to use the stress function formula indicated in (1.8.19).

Given the cross section of any given circle, $x^2 + y^2 = a^2$, with radius *a*, the polar form becomes $r^2 - a^2 = 0$. To keep the cross section consistent with (1.8.20), we multiply both sides by $\frac{1}{4}$. Thus, letting $A_0 = -\frac{1}{4}a^2$ and the other constants as zero. So the stress function Ψ becomes

$$\Psi = -\tfrac{1}{2}\mu\alpha\left(r^2 - a^2\right)$$

or

$$\Psi = -\tfrac{1}{2}\mu\alpha\left(x^2 + y^2 - a^2\right).$$

In an elliptic cross section, converting the Cartesian equation, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, of Γ into the form of (1.8.20) requires the use of Tchebycheff polynomials. From (1.8.14), we can deduce $T_2(\cos\theta) = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$. Converting the Cartesian equation into polar coordinates and using Tchebycheff 2^{nd} degree polynomial gives

$$\frac{T_2(\cos\theta)+1}{2a^2} + \frac{1-T_2(\cos\theta)}{2b^2} = \frac{1}{r^2}.$$

Simplifying the above equation and keeping it consistent with (1.8.20), we need to divide both sides by $4\left(a^2 + b^2\right)$ leading to

$$\tfrac{1}{4}r^2 - \frac{a^2b^2}{2\left(a^2+b^2\right)} - \frac{r^2\left(a^2-b^2\right)T_2(\cos\theta)}{4\left(a^2+b^2\right)} = 0.$$

The above equation matches with (1.8.20) where $A_0 = -\frac{a^2b^2}{2(a^2+b^2)}$, $A_2 = -\frac{(a^2-b^2)}{4(a^2+b^2)}$,

and all the other constants are zero; therefore, the stress function Ψ is

$$\Psi = -\tfrac{1}{2}\mu\alpha\left[r^2 - \frac{2a^2b^2}{(a^2+b^2)} - \frac{(a^2-b^2)}{(a^2+b^2)}r^2T_2(\cos\theta)\right].$$

To convert the stress function Ψ back to Cartesian coordinates, we substitute $r^2 = x^2 + y^2$

and $T_2(\cos\theta) = \frac{x^2 - y^2}{x^2 + y^2}$, which leads to

$$\Psi = -\mu\alpha\frac{a^2b^2}{a^2+b^2}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right].$$

To find the stress function Ψ for a specific equilateral triangle cross section we need to use the 3rd degree Tchebycheff polynomial, $T_3(\cos\theta) = 4\cos^3\theta - 3\cos\theta$. The equation of the boundary of the triangular section we are solving is given by

$$\left(x + \tfrac{1}{3}a\right)\left(x - \sqrt{3}y - \tfrac{2}{3}a\right)\left(x + \sqrt{3}y - \tfrac{2}{3}a\right) = 0$$

or

$$\tfrac{1}{4}(x^2 + y^2) - \tfrac{1}{27}a^2 - \tfrac{1}{4a}(x^3 - 3xy^2) = 0.$$

Converting the above equation into polar coordinates gives

$$\tfrac{1}{4}r^2 - \tfrac{1}{27}a^2 - \tfrac{1}{4a}r^3T_3(\cos\theta) = 0,$$

which leads to $A_0 = -\frac{a^2}{27}$, $A_3 = -\frac{1}{4a}$, and the rest of the constants as zero. Therefore, we are able to use the stress function Ψ formula in (1.8.19), which is denoted by

$$\Psi = -\tfrac{1}{2}\mu\alpha\left[r^2 - \tfrac{4}{27}a^2 - \tfrac{1}{a}r^3T_3(\cos\theta)\right]$$

or

$$\Psi = -\tfrac{1}{2}\mu\alpha\left[r^2 - \tfrac{4}{27}a^2 - \tfrac{1}{a}(x^3 - 3xy^2)\right].$$

The stress function in all three cases is consistent with the Bassali-Obaid method in [4].

1.10 Cross Section Bounded by Special Curves

In this section, we follow Abbassi's method derived in Section 1.8 for two cases. The first case is a cross section bounded by two intersecting circles and the second case is a cardioid. In both cases, we rewrite the Cartesian equation of Γ to the form in (1.8.20) or (1.8.21) so that we can apply the stress function in (1.8.19).

The first special curve case gives $r = b$ to have a center on the circumference of the circle $r = 2a\cos\theta$, where b < a. Substituting $\cos\theta = \frac{x}{r}$ into $r = 2a\cos\theta$ and completing the square leads to $(x-a)^2 + y^2 = a^2$, which is shown in Fig. 6 below.

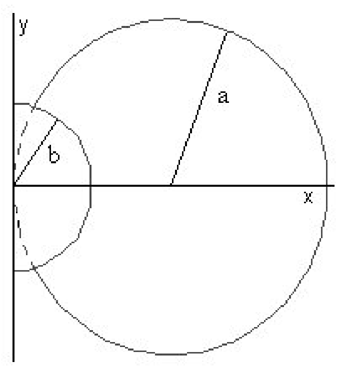

Fig. 6 A circle of radius a with a notch whose boundary is a circle of radius b

The cross section is given by

$$\left(r^2 - b^2\right)\left(1 - \frac{2a}{r}\cos\theta\right) = 0$$

or

$$\frac{1}{4}r^2 - \frac{1}{4}b^2 - \frac{1}{2}arT_1(\cos\theta) + \frac{1}{2}\frac{ab^2}{r}T_1(\cos\theta) = 0.$$

So, in this case (1.8.21) has values of $A_0 = -\frac{1}{4}b^2$, $A_1 = -\frac{1}{2}a$, and $B_1 = \frac{1}{2}ab^2$, which leads (1.8.19) to become

$$\Psi = -\tfrac{1}{2}\mu\alpha\left(r^2 - b^2\right)\left[1 - \tfrac{2a}{r}T_1(\cos\theta)\right].$$

In the second case, the cross section is bounded by a cardioid of the form

$$r = \frac{1+\cos\theta}{2a^2}$$

or
$$\left(2a^2 r - \cos\theta\right)^2 = 1.$$

Multiplying the above equation by a^2r^2 we have

$$a^2r^2\left(2a^2r - \cos\theta\right) = \tfrac{1}{2}r(1+\cos\theta) = r\cos^2\frac{\theta}{2}$$

or
$$2a^3r^2 - ar\cos\theta - r^{1/2}\cos\frac{\theta}{2} = 0.$$

To keep the above equation consistent with (1.8.20), we divide both sides by $\dfrac{1}{8a^3}$ leading to

$$\tfrac{1}{4}r^2 - \tfrac{1}{8a^2}rT_1(\cos\theta) - \tfrac{1}{8a^3}r^{1/2}T_{1/2}(\cos\theta) = 0.$$

Since the equation has the form in (1.8.20) where $A_{1/2} = -\frac{1}{8a^3}$, $A_1 = -\frac{1}{8a^2}$, and the rest of the constants are zeros, the stress function is denoted by

$$\Psi = -\tfrac{1}{2}\mu\alpha\left[r^2 - \tfrac{1}{2a^3}r^{1/2}T_{1/2}(\cos\theta) - \tfrac{1}{2a^2}rT_1(\cos\theta)\right].$$

Note that when we used separation of variables in Section 1.8 we defined $n \in \mathbf{Z}$ based on a circle. Since the above case is a cardioid, we are allowed to let $n = \frac{1}{2}$ where $T_{1/2}$ is considered a Tchebycheff function not a polynomial.

CHAPTER 2

SOLUTIONS TO SAINT-VENANT'S TORSION PROBLEM USING CONFORMAL MAPPING

In this chapter, we solve the Saint-Venant's torsion problem for two different cross sections in which the boundary is a regular curvilinear polygon of *n* sides and *n* rounded vertices using conformal mapping. To do so, we will map the boundary curves in the *z*-plane to the unit circle γ in the $\zeta -$plane under the conformal transformation $z = \omega(\zeta)$. To solve the torsion problem for γ, we will need to derive and use Schwarz and Poisson's formulas. Refer to [8] and [13].

2.1 Fundamental Equations

The complex function

$$\Omega(z) = \phi(x, y) + i\psi(x, y) \qquad (2.1.1)$$

is analytic if it has a unique derivative at every point in the given region S and it must satisfy the conditions of the Cauchy-Riemann equations

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}, \qquad (2.1.2)$$

where the partial derivatives are continuous functions of *x* and *y*. If $\Omega(z)$ does not have a derivative at some point z_0, then we call it a singular point of the analytic function. If $\Omega(z)$ is analytic, then we know (2.1.2) exists in S and the higher order derivatives as well. If all of these conditions are met, then ϕ and ψ satisfy the Laplace's equation

$$\nabla^2\phi = 0, \quad \nabla^2\psi = 0 \tag{2.1.3}$$

respectively. If the function $\Omega(z)$ is analytic and continuous in S bounded by a simple closed contour Γ, then Cauchy's Integral Theorem is applicable, which is

$$\int_{\Gamma} \Omega(z)dz = 0. \tag{2.1.4}$$

We can further generalize Cauchy's Integral Theorem to a closed, multiply connected region S. If $\Omega(z)$ is analytic and continuous in S bounded by the exterior simple contour Γ_0 and by the interior simple contours $\Gamma_1, \Gamma_2, ..., \Gamma_n$, then

$$\int_{\Gamma_0} \Omega(z)dz = \int_{\Gamma_1} \Omega(z)dz + \int_{\Gamma_2} \Omega(z)dz + \Lambda + \int_{\Gamma_n} \Omega(z)dz. \tag{2.1.5}$$

If $z = a$ is an interior point of S bounded by Γ, $\Omega(z)$ is continuous in a closed region S, and $\Omega(z)$ is analytic at any interior point in S, then Cauchy's Integral Formula is

$$\Omega(a) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\Omega(z)}{z-a}dz. \tag{2.1.6}$$

Cauchy's Integral Formula is differentiable given z to be any interior point in S. To keep the formula consistent we will change the variable of integration in (2.1.6) to ζ, which leads to

$$\Omega^{(n)}(z) = \frac{n!}{2\pi i}\int_{\Gamma} \frac{\Omega(\zeta)}{(\zeta - z)^{n+1}}d\zeta. \tag{2.1.7}$$

With the use of Cauchy's Integral formula, we can expand the analytic function $\Omega(z)$ as a Taylor's series such as

$$\Omega(z) = \Omega(a) + \Omega'(a)(z-a) + \cdots + \frac{\Omega^{(n)}(a)}{n!}(z-a)^n + \cdots. \tag{2.1.8}$$

The function $\Omega(z)$ can also be rewritten as a Laurent series. To do so, $\Omega(z)$ has to be continuous in the closed annular region formed by two concentric circles Γ_1 and Γ_2, where Γ_1 and Γ_2 also bounds the region S. In addition, $\Omega(z)$ has to be analytic at every interior point between the circles. If all these conditions are satisfied, then we have

$$\Omega(z) = \sum_{k=-\infty}^{\infty} b_k (z-a)^k , \tag{2.1.9}$$

where $b_k = \frac{1}{2\pi i}\int_{\Gamma} \frac{\Omega(z)}{(z-a)^{k+1}} dz$, $k = 0, \pm 1, \pm 2, \ldots$, and $z = a$ is the center of Γ_1 and Γ_2.

There are two cases for $\Omega(z)$ in (2.1.9). The first case is there exist a finite number of the coefficients with negative subscripts that are not zero, but in the second case there exist an infinite number of them. Given the first case, $\Omega(z)$ has a pole of order *n* at $z = a$, then (2.1.9) becomes

$$\Omega(z) = \frac{b_{-n}}{(z-a)^n} + \cdots + \frac{b_{-2}}{(z-a)^2} + \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots . \tag{2.1.10}$$

Substituting $z - a = \zeta$ into (2.1.10) and integrating around Γ enclosed by $z = a$ with no other singularity of leads to

$$\int_{\Gamma} \Omega(z) dzz = 2\pi i b_{-1} . \tag{2.1.11}$$

The quantity b_{-1} is known as the residue of $\Omega(z)$ at the pole $z = a$. To evaluate the residue of $\Omega(z)$ at a pole of order *n* we need to use the following formula

$$b_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left\{(z-a)^n \Omega(z)\right\}. \tag{2.1.12}$$

At a simple pole, $n = 1$, (2.1.12) becomes

$$b_{-1} = \lim_{z \to a} (z-a)\ \Omega(z). \tag{2.1.13}$$

If Γ encompasses n poles at singularities $a_1, a_2, ..., a_n$ with residues $c_{-1}, d_{-1}, ..., q_{-1}$, (2.1.11) leads to

$$\int_\Gamma \Omega(z) dz = 2\pi i \times (c_{-1} + d_{-1} + \cdots + q_{-1}). \tag{2.1.14}$$

If each pole is known in the Laurent expansion of $\Omega(z)$, then another way to evaluate (2.1.14) is to multiply $2\pi i$ with the sum of the coefficients of $\frac{1}{z-a_1}, \frac{1}{z-a_2}, \dots$.

The last fundamental equation needed is the Theorem of Harnack. This theorem uses functions of a complex variable that is applicable to elasticity. The region bounded by the unit circle $|z| \le 1$ transformed into the complex ζ - plane, where $\zeta = \xi + i\eta$ and $\xi, \eta \in \mathrm{R}$. So, $|\zeta| \le 1$ are points inside the unit circle, γ, and $\sigma = e^{i\theta}$ are points on the boundary of γ. All functions with θ are to be periodic. So Theorem of Harnack states that if $f(\theta)$ and $\varphi(\theta)$ are continuous real functions, all values of ζ are inside γ and

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\theta)}{\sigma - \zeta} d\sigma = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\theta)}{\sigma - \zeta} d\sigma \tag{2.1.15}$$

are satisfied , then $f(\theta) \equiv \varphi(\theta)$. If the points ζ are outside of γ instead, but the rest are the same, then $f(\theta) = \varphi(\theta) +$ constant. For proofs of all of these known theorems and formulas refer to [13].

2.2 Schwarz's and Poisson Formulas

In this section, we find the torsion function $\phi(x, y)$ and its conjugate $\psi(x, y)$, the problems of Dirichlet and Neumann, based on a circular region. These types of problems usually involve lots of difficulties, so we can give some formulas to help with finding the solution. In other words, we use the formulas of Schwarz and Poisson to solve the torsion problem with a circular cross section.

Let S be the region bounded by a unit circle as previously described in Section 2.1. Let the harmonic function $\phi(\xi, \eta)$ be determined when the boundary condition on the circle γ satisfies

$$\phi\big|_{\gamma} = f(\theta), \tag{2.2.1}$$

where $f(\theta)$ is a continuous real function of θ. The conjugate of the harmonic function $\phi(\xi, \eta)$ is $\psi(\xi, \eta)$ where $\psi(\xi, \eta)$ is found to be within an arbitrary constant from $\phi(\xi, \eta)$. So we have

$$\Omega(\zeta) = \phi(\xi, \eta) + i\psi(\xi, \eta) \tag{2.2.2}$$

is an analytic function of the complex variable $\zeta = \xi + i\eta$ for all values of $|\zeta| \leq 1$. Let us assume $\Omega(\zeta)$ is continuous in the closed region $|\zeta| \leq 1$, then (2.2.1) becomes

$$\Omega(\sigma) + \overline{\Omega}(\overline{\sigma}) = 2f(\theta) \tag{2.2.3}$$

on γ where $\Omega(\sigma) = f(\theta) + i\psi$ and $\overline{\Omega}(\overline{\sigma}) = f(\theta) - i\psi$. We define $\overline{\Omega}(\sigma) = \overline{\Omega(\overline{\sigma})}$ and $\overline{\Omega}(\overline{\sigma}) = \overline{\Omega(\sigma)}$ based on Sokolnikoff's notation [13]. So when we multiply $\dfrac{1}{2\pi i}\dfrac{d\sigma}{\sigma - \zeta}$ by both sides of (2.2.3) and integrate upon γ we have

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\Omega(\sigma)}{\sigma - \zeta}d\sigma + \frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Omega(\sigma)}}{\sigma - \zeta}d\sigma = \frac{1}{\pi i}\int_{\gamma}\frac{f(\theta)}{\sigma - \zeta}d\sigma. \tag{2.2.4}$$

Based on Theorem of Harnack, (2.2.4) is equivalent to (2.2.3). Furthermore, based on Cauchy's Integral Formula the first integral becomes

$$\Omega(\zeta) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\Omega(\sigma)}{\sigma - \zeta}d\sigma. \tag{2.2.5}$$

The second integral in (2.2.4) requires a little more calculation. Using (2.1.8) with the origin as the center of the expansion give us

$$\Omega(\zeta) = \Omega(0) + \Omega'(0)(\zeta) + \cdots + \frac{\Omega^{(n)}(0)}{n!}(\zeta)^n + \cdots. \tag{2.2.6}$$

Since $|\zeta| = 1$ and the points on the boundary of the unit circle are $\zeta = e^{i\theta} = \sigma$, we have $\overline{\zeta} = e^{-i\theta} = \dfrac{1}{\sigma}$. Multiplying (2.2.6) with $\dfrac{1}{2\pi i}\dfrac{d\sigma}{\sigma - \zeta}$ and integrating it upon σ leads to

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{\Omega(\sigma)}}{\sigma-\zeta}d\sigma = \frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{\Omega}(0)}{\sigma-\zeta}d\sigma + \frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{\Omega}'(0)}{\sigma(\sigma-\zeta)}d\sigma + \cdots + \frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{\Omega}^{(n)}(0)}{\sigma^n n!(\sigma-\zeta)}d\sigma + \cdots.$$

The first integral on the right side above is $\overline{\Omega}(0)$. By the Residue theorem, the rest of the integrals on the right side are zero. Using these findings, (2.2.4) becomes

$$\Omega(\zeta) + \overline{\Omega}(0) = \frac{1}{\pi i}\int_{\gamma}\frac{f(\theta)}{\sigma - \zeta}d\sigma. \tag{2.2.7}$$

Let $\overline{\Omega}(0) = a_0 - ib_0$. Rewriting (2.2.7) leads to

$$\Omega(\zeta) = \frac{1}{\pi i} \int_\gamma \frac{f(\theta)}{\sigma - \zeta} d\sigma - a_0 + ib_0 . \qquad (2.2.8)$$

If we set $\zeta = 0$, then $\Omega(0) = a_0 + ib_0$, which leads to

$$a_0 = \frac{1}{2\pi i} \int_\gamma \frac{f(\theta)}{\sigma} d\sigma . \qquad (2.2.9)$$

Replacing (2.2.8) into (2.2.9) gives

$$\Omega(\zeta) = \frac{1}{\pi i} \int_\gamma \frac{\sigma + \zeta}{\sigma - \zeta} f(\theta) d\sigma - \frac{1}{2\pi i} \int_\gamma \frac{f(\theta)}{\sigma} d\sigma + ib_0 . \qquad (2.2.10)$$

Combining the two integrals in (2.2.10) provides Schwarz's formula as

$$\Omega(\zeta) = \frac{1}{2\pi i} \int_\gamma \frac{\sigma + \zeta}{\sigma - \zeta} f(\theta) d\sigma + ib_0 . \qquad (2.2.11)$$

Note b_0 is undetermined because $\psi(\xi, \eta)$ is found to be within an arbitrary constant of $\phi(\xi, \eta)$. Substituting $\zeta = \rho e^{i\Theta} = \rho(\cos\Theta + i\sin\Theta)$ and $\sigma = e^{i\theta} = \cos\theta + i\sin\theta$ into (2.2.11) gives

$$\Omega(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\theta)\left[1 - \rho^2 + 2i\rho\sin(\Theta - \theta)\right]}{1 - 2\rho\cos(\theta - \Theta) + \rho^2} d\theta + ib_0 . \qquad (2.2.12)$$

To obtain the Poisson's formula, we will get the real part of (2.2.12) to be

$$\operatorname{Re}\Omega(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(\theta)\left[1 - \rho^2\right]}{1 - 2\rho\cos(\theta - \Theta) + \rho^2} d\theta . \qquad (2.2.13)$$

Poisson's formula determines the solution of the Dirichlet problem where the region is bounded by a circle. These formulas are restricted to solving a boundary which is a circle. In the next section, we expand the formulas to solve potential theory problems for different boundaries.

2.3 The Conformal Mapping Method

In this section, we generalize the formulas of Schwarz and Poisson to solve boundary value problems to any simply connected region by using conformal mapping. The mapping function $z = \omega(\zeta)$ corresponds to points $\zeta = \xi + i\eta$ of the complex ζ-plane to points $z = x + iy$ of the complex z - plane. $\omega(\zeta)$ is a conformal transformation if $\omega(\zeta)$ is analytic in the circle γ in the ζ - plane and $\omega'(\zeta) \neq 0$. If $\omega(\zeta)$ is a conformal mapping, then it will preserve the angle measures. Using conformal mapping, we can solve a torsion problem, which is also explained in this section.

As stated previously, the complex torsion function is

$$\Omega(z) = \phi(x, y) + i\psi(x, y), \qquad (2.3.1)$$

where $z = x + iy$, $\phi(x, y)$ is torsion function, and $\psi(x, y)$ is the conjugate torsion function of $\phi(x, y)$. The function $\Omega(z)$ is analytic in the cross section region of the beam denoted by S. Let S be a simply connected region, so we assume we have the function

$$z = \omega(\zeta) \qquad (2.3.2)$$

conformally maps S onto the unit circle $|\zeta| \leq 1$. Using (2.3.2) we can rewrite $\Omega(z)$ in terms of $\zeta = \xi + i\eta$ as

$$\Omega[\omega(\zeta)] = \phi(\xi, \eta) + i\psi(\xi, \eta) = f(\zeta), \qquad (2.3.3)$$

where $f(\zeta)$ is analytic inside the unit circle, γ.

As stated previously, the conjugate torsion functions ψ satisfies the boundary condition

$$\psi(x,y)=\frac{1}{2}\left(x^2+y^2\right)=\frac{z\bar{z}}{2}. \tag{2.3.4}$$

Substituting (2.3.2) into (2.3.4) leads to

$$\psi=\frac{\omega(\zeta)\bar{\omega}\left(\bar{\zeta}\right)}{2}, \tag{2.3.5}$$

which is the boundary condition on γ that needs to be satisfied by the imaginary part of $f(\zeta)$. Solving the torsion problem requires finding the real part of the analytic function

$$\frac{1}{i}f(\zeta)=\psi-i\phi, \tag{2.3.6}$$

while still satisfying the boundary condition in (2.3.5). Putting $f(\theta)=\frac{1}{2}\left\{\omega(\sigma)\bar{\omega}\left(\bar{\sigma}\right)\right\}$ and $\Omega(\zeta)=\frac{1}{i}f(\zeta)$ into (2.2.8) leads to

$$\frac{1}{i}f(\zeta)=\frac{1}{2\pi i}\int_{\gamma}\frac{\omega(\sigma)\bar{\omega}\left(\bar{\sigma}\right)}{\sigma-\zeta}d\sigma-a_0+ib_0. \tag{2.3.7}$$

Since $\sigma=e^{i\theta}$ and $\bar{\sigma}=e^{-i\theta}=1/\sigma$, (2.3.7) can be written as

$$f(\zeta)=\frac{1}{2\pi}\int_{\gamma}\frac{\omega(\sigma)\bar{\omega}(1/\sigma)}{\sigma-\zeta}d\sigma-ia_0-b_0, \tag{2.3.8}$$

which is the solution to the torsion problem. The torsional rigidity can be expressed in terms of $f(\zeta)$:

$$\begin{aligned}D&=\mu\iint_S\left(x^2+y^2\right)dxdy+\mu\iint_S\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)dxdy\\&=\mu I_0+\mu D_0,\end{aligned} \tag{2.3.9}$$

where

$$I_0 = -\frac{i}{4}\int_\gamma \left[\overline{\omega}(\overline{\sigma})\right]^2 \omega(\sigma)d\omega(\sigma) \tag{2.3.10}$$

and

$$D_0 = -\frac{1}{4}\int_\gamma \left[f(\sigma)+\overline{f}(\overline{\sigma})\right]d\left[\omega(\sigma)\overline{\omega}(\overline{\sigma})\right]. \tag{2.3.11}$$

Refer to [13] for proofs of the above known theorems and formulas.

2.4 Cross Sections Γ_n Defined by $z = \frac{c\zeta}{\left(1+m\zeta^n+p\zeta^{2n}\right)}$

Stevenson [14] used conformal mapping to solve the Dirichlet problem for cross sections in which the boundary is a regular curvilinear polygon of *n* sides and *n* rounded vertices. He mapped the boundary curves in the *z*-plane to the unit circle γ in the ζ – plane under the conformal transformation $z = \omega(\zeta) = c\zeta\left(1+m\zeta^n\right)$ with the conditions $c > 0$ and $0 \le |m|(n+1) \le 1$. In this section, we use Cauchy integral methods described in Section 2.3 to solve the torsion problem for the cross sections bounded by Γ_n in the z - plane which are mapped to the unit circle γ in the ζ – plane under a different mapping function from that of Stevenson. Let the conformal transformation be

$$z = \frac{c\zeta}{\left(1+m\zeta^n+p\zeta^{2n}\right)}, \quad c>0,\ \zeta = \xi + i\eta = \rho e^{i\theta}, \tag{2.4.1}$$

where $n \in \mathbf{N}^+$ and m, p are real parameters. To ensure the mapping will be conformal everywhere inside the curves, m and p will be chosen so that $z'(\zeta)$ will not be zero or infinite within Γ_n. See Bassali [2] for the details of the parameters.

To determine if Γ_n is a regular curvilinear polygons of *n* sides and *n* vertices, we need to obtain the parametric equations of the curves Γ_n. Multiplying (2.4.1) with its conjugate gives us

$$r^2 = z\bar{z} = \frac{c^2}{1+m^2+p^2+2m\cos n\theta+2mp\cos n\theta+2p\cos 2n\theta},$$

which gives the polar equation of Γ_n.

Using $z+\bar{z}=2x$ and dividing both sides of the above equation by r^2 leads to

$$\begin{aligned}\frac{2xc}{r^2} &= 2\cos\theta + m[\cos(n-1)\theta - i\sin(n-1)\theta + \cos(n-1)\theta + i\sin(n-1)\theta] + \\ &\quad p[\cos(2n-1)\theta - i\sin(2n-1)\theta + \cos(2n-1)\theta + i\sin(2n-1)\theta].\end{aligned}$$

Let $n_1 = n-1$ and $n_2 = 2n-1$. Then simplifying the above equation to

$$\frac{xc}{r^2} = \cos\theta + m\cos n_1\theta + p\cos n_2\theta\,. \qquad (2.4.2)$$

Similarly, we have

$$\frac{yc}{r^2} = \sin\theta - m\sin_1\theta - p\sin n_2\theta\,. \qquad (2.4.3)$$

The parametric equations of Γ_n from (2.4.2) and (2.4.3) can be shown to be regular curvilinear polygons of *n* sides and *n* vertices, so we can continue to find the complex torsion function as shown in Fig. 7 in the following page.

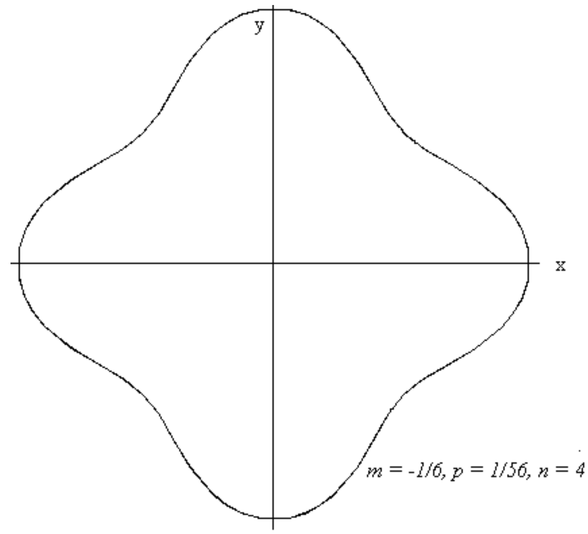

Fig. 7 Cross Section bounded by four arcs and approximately semi-circular arcs

The complex torsion function is

$$\Omega(z) = f(\zeta) = \phi(x,y) + i\psi(x,y), \tag{2.4.4}$$

where $f(\zeta)$ is the function that solves the torsion problem in the circle and satisfies the boundary condition in (2.3.4). Let σ be a point on the boundary of the circle. Thus $\overline{\sigma} = 1/\sigma$. Substituting $\omega(\sigma) = \dfrac{c\sigma}{1 + m\sigma^n + p\sigma^{2n}}$ and $\overline{\omega}(1/\sigma) = \dfrac{c/\sigma}{1 + m(1/\sigma)^n + p(1/\sigma)^{2n}}$

into (2.3.8) leads to $f(\zeta)$

$$f(\zeta) = \frac{c^2}{2\pi}\int_\gamma \frac{\sigma^{2n}}{(p\sigma^{2n} + m\sigma^n + 1)(\sigma^{2n} + m\sigma^n + p)(\sigma - \zeta)} d\sigma \text{ + constant.} \tag{2.4.5}$$

For *z* to be conformal everywhere inside the circle z' cannot be infinite in the interior of z, in other words, $1 + m\zeta^n + p\zeta^{2n} \neq 0$. Therefore,

$$p\sigma^{2n} + m\sigma^n + 1 = 0 \tag{2.4.6}$$

gives points outside the circle γ. Let λ' and λ'' be the two roots of the quadratic equation

$$pt^2 + mt + 1 = 0, \tag{2.4.7}$$

where $t = \sigma^n$. Using special formulas for the sum and the product of roots, we have

$$\lambda' + \lambda'' = -\tfrac{m}{p}, \quad \lambda'\lambda'' = \tfrac{1}{p}. \tag{2.4.8}$$

The $2n$ roots of (2.4.6) are λ_s', λ_s'' where $s = 1, 2, \ldots n$, $\lambda_s'^n = \lambda'$ and $\lambda_s''^n = \lambda''$. Since λ' is a root of (2.4.7), its factor will be $t - \lambda'$ or $\sigma^n - \lambda'$. So, we have

$$\sigma^n - \lambda' = (\sigma - \lambda_1')(\sigma - \lambda_2')\cdots(\sigma - \lambda_n'). \tag{2.4.9}$$

Taking the logarithm of both sides of (2.4.9) and then differentiating both sides with respect to σ leads to

$$\frac{n\sigma^{n-1}}{\sigma^n - \lambda'} = \sum_{s=1}^{n} \frac{1}{\sigma - \lambda_s'} = \frac{n}{\sigma} + \frac{1}{\sigma}\sum_{s=1}^{n} \frac{\lambda_s'}{\sigma - \lambda_s'}. \tag{2.4.10}$$

The last equality above is obtained by a simple algebraic trick. Rewriting (2.4.10) we have $\frac{1}{\sigma}\sum_{s=1}^{n} \frac{\lambda_s'}{\sigma - \lambda_s'} = \frac{n\sigma^{n-1}}{\sigma^n - \lambda'} - \frac{n}{\sigma}$. Multiplying both sides of this equality by σ and combining the two fractions gives the following property:

$$\sum_{s=1}^{n} \frac{\lambda_s'}{\sigma - \lambda_s'} = \frac{n\lambda'}{\sigma^n - \lambda'}. \tag{2.4.11}$$

To evaluate (2.4.5) we add the residues outside the circles with the residue at infinity. However, it can be shown that the residue at infinity is equal to zero since the denominator of the integrand in (2.4.5) has a high power; therefore, we only need the sum of the residues at the simple poles outside of the circle. In the case of $m^2 \neq 4p$, the simple poles are outside the circle. We shall use an extension of the residue theorem to evaluate the contour integral (2.4.5). Thus we have

$$f(\zeta) = -c^2 i \sum_{s=1}^{n} \left(R_s^{'} + R_s^{''} \right) + \text{constant}, \qquad (2.4.12)$$

where $R_s^{'}, R_s^{''}$ are the residues of the integrand in (2.4.5) at $\lambda_s^{'}, \lambda_s^{''}$ respectively. Using the formula for the residue at a simple pole we have

$$R_s^{'} = \lim_{\sigma \to \lambda_s^{'}} \frac{\sigma - \lambda_s^{'}}{1 + m\sigma^n + p\sigma^{2n}} \cdot \lim_{\sigma \to \lambda_s^{'}} \frac{\sigma^{2n}}{\left(p + m\sigma^n + \sigma^{2n}\right)\left(\sigma - \zeta\right)}. \qquad (2.4.13)$$

Since $\lambda_s^{'}$ is a root of (2.4.6), applying L' Hopital's Rule to the first limit in (2.4.13) and simplifying the limits gives

$$R_s^{'} = \frac{\lambda_s^{'}}{m\lambda_s^{'\,n} + 2np\lambda_s^{'\,2n}} \cdot \frac{\lambda_s^{'\,2n}}{\left(p + m\lambda_s^{'\,n} + \lambda_s^{'\,2n}\right)\left(\lambda_s^{'} - \zeta\right)}.$$

Simplifying the above equation with the use of these properties $m = -p\left(\lambda^{'} + \lambda^{''}\right)$ and $\lambda_s^{'n} = \lambda^{'}$ leads to

$$R_s^{'} = \frac{\lambda^{'}\lambda_s^{'}}{np\left(\lambda^{''} - \lambda^{'}\right)\left(p + m\lambda^{'} + \lambda^{'2}\right)\left(\zeta - \lambda_s^{'}\right)}. \qquad (2.4.14)$$

Similarly, we also get

$$R_s^{''} = \frac{\lambda^{''}\lambda_s^{''}}{np\left(\lambda^{'} - \lambda^{''}\right)\left(p + m\lambda^{''} + \lambda^{''2}\right)\left(\zeta - \lambda_s^{''}\right)}. \qquad (2.4.15)$$

Taking the summation of both sides of (2.4.14) gives

$$\sum_{s=1}^{n} R_s^{'} = \frac{\lambda^{'}}{np\left(\lambda^{''} - \lambda^{'}\right)\left(p + m\lambda^{'} + \lambda^{'2}\right)} \sum_{s=1}^{n} \frac{\lambda_s^{'}}{\left(\zeta - \lambda_s^{'}\right)}.$$

Using (2.4.11) into the above equation leads to

$$\sum_{s=1}^{n} R_s^{'} = \frac{\lambda^{'2}}{p\left(\lambda^{''} - \lambda^{'}\right)\left(p + m\lambda^{'} + \lambda^{'2}\right)\left(\zeta^{n} - \lambda^{'}\right)}. \qquad (2.4.16)$$

Similarly,

$$\sum_{s=1}^{n} R_s^{''} = \frac{\lambda^{''2}}{p\left(\lambda^{'} - \lambda^{''}\right)\left(p + m\lambda^{''} + \lambda^{''2}\right)\left(\zeta^{n} - \lambda^{''}\right)}. \qquad (2.4.17)$$

Substituting (2.4.16) and (2.4.17) into (2.4.12) and simplifying it leads to

$$f(\zeta) = \frac{c^2 i\left(mp\zeta^{n} + p + 1\right)}{(1-p)\left(p\zeta^{2n} + m\zeta^{n} + 1\right)\left\{(1+p)^2 - m^2\right\}} + \text{constant} \qquad (2.4.18)$$

for the case $m^2 \neq 4p$. In the case of $m^2 = 4p$, it turns out that it coincides with (2.4.18) because even though we have $\lambda^{'} = \lambda^{''}$ we still have $\lambda^{'} + \lambda^{''} = -\frac{m}{p}$ and $\lambda^{'}\lambda^{''} = \frac{1}{p}$. In addition, there are n double poles that lie outside of the circle, so we would still need to find $R_s^{'}$ and $R_s^{''}$. When finding $f(\zeta)$ all the $\lambda^{'}, \lambda^{''}$ ends up canceling each other anyway; therefore, the solution of $f(\zeta)$ becomes the same formula as in (2.4.18). The constant in (2.4.22) is given as

$$\text{constant} = \frac{-c^2 i(1+p)}{2(1-p)\left\{(1+p)^2 - m^2\right\}} \qquad (2.4.19)$$

by using the formula in (2.2.9). Substituting (2.4.19) into (2.4.18) gives the complex torsion function as follows:

$$f(\zeta) = \frac{c^2 i\left(mp\zeta^{n} + p + 1\right)}{2(1-p)\left\{(1+p)^2 - m^2\right\}}\left[\frac{(1+p) - m\zeta^{n}(1-p) - p\zeta^{2n}(1+p)}{p\zeta^{2n} + m\zeta^{n} + 1}\right]. \qquad (2.4.20)$$

Let $q = (1+p)/(1-p)$ and $k = (1+p)^2 - m^2$. Replacing the new values of q and k into (2.4.20) leads to

$$f(\zeta) = \frac{c^2 i}{2k}\left[\frac{q - m\zeta^n - pq\zeta^{2n}}{p\zeta^{2n} + m\zeta^n + 1}\right]. \tag{2.4.21}$$

Let $\rho = 1$, so we have $\zeta = e^{i\theta}$, which sets (2.4.21) as

$$f\left(e^{i\theta}\right) = \frac{c^2 i}{2k}\left[\frac{q - me^{in\theta} - pqe^{2in\theta}}{pe^{2in\theta} + me^{in\theta} + 1}\right]. \tag{2.4.22}$$

Multiplying the numerator and denominator of (2.4.26) by $1 + me^{-in\theta} + pe^{-2in\theta}$ gives

$$f\left(e^{i\theta}\right) = \frac{c^2 i}{2k}\left[\frac{q - p^2 q - m^2 - 2pqi\sin 2\theta + mi\sin n\theta(p - q - pq - 1)}{1 + m^2 + p^2 + 2m(1+p)\cos n\theta + 2p\cos 2n\theta}\right].$$

The imaginary part of $f\left(e^{i\theta}\right)$ is the solution to the torsion problem. So, the imaginary part of the above equation is

$$\psi = \frac{c^2}{2k}\left[\frac{q - p^2 q - m^2}{1 + m^2 + p^2 + 2m(1+p)\cos n\theta + 2p\cos 2n\theta}\right]. \tag{2.4.23}$$

Substituting the value of k and r from the polar equation for the curve into (2.4.23) gives

$$\psi = \operatorname{Im}\left\{f\left(e^{i\theta}\right)\right\} = \frac{1}{2}r^2. \tag{2.4.24}$$

as expected.

2.5 Cross Sections Γ_n Defined by $z = \frac{c\zeta}{\left(1 + m\zeta^n\right)}$

In the previous section, we found that the solution to the torsion problem by using Cauchy integral methods. To find the solution to the torsion problem using different conformal transformations is a similar process to Section 2.4. Since it is repetitive, we are not discussing the details of finding the solution again. Instead, we are deciphering the torsional rigidity for a cross section corresponding to the following conformal mapping

$$z = \frac{c\zeta}{\left(1 + m\zeta^n\right)}, \tag{2.5.1}$$

where $n = 2, 3, 4...$ and $\left|m(n-1)\right| \leq 1$.

Following the same procedures we used in Section 2.4 for the conformal transformation in (2.5.1), the function $f(\zeta)$ becomes

$$f(\zeta) = \frac{c^2}{2\pi m}\int_\gamma \frac{\sigma^n}{\left(\sigma^n + m\right)\left(\sigma^n + m^{-1}\right)\left(\sigma - \zeta\right)} d\sigma + \text{ constant.} \tag{2.5.2}$$

Integrating (2.5.2) as in the methods in Section 2.4, we get

$$f(\zeta) = \frac{c^2 i}{2\left(1 - m^2\right)} \frac{1 - m\zeta^n}{1 + m\zeta^n}. \tag{2.5.3}$$

Then the imaginary part of the complex torsion function in (2.5.3) is

$$\psi = \frac{c^2\left(1 - m^2\rho^{2n}\right)}{2\left(1 - m^2\right)\left(1 + 2m\rho^n \cos n\theta + m^2\rho^{2n}\right)}. \tag{2.5.4}$$

As in the earlier section, (2.5.4) equals to $\frac{1}{2}r^2$ on the boundary Γ_n of the cross section.

Refer to Bassali [3] for the details of the solution to the torsion problem.

As stated previously, the torsional rigidity is found by the following formulas

$$D = \mu I_0 + \mu D_0, \tag{2.5.5}$$

where

$$I_0 = -\frac{i}{4}\int_\gamma \left[\overline{\omega}\left(\overline{\sigma}\right)\right]^2 \omega(\sigma) d\omega(\sigma) \tag{2.5.6}$$

and

$$D_0 = -\frac{1}{4}\int_\gamma \left[f(\sigma) + \overline{f}\left(\overline{\sigma}\right)\right] d\left[\omega(\sigma)\overline{\omega}\left(\overline{\sigma}\right)\right]. \tag{2.5.7}$$

Let σ be a point on the boundary of the circle where $\overline{\sigma} = 1/\sigma$. On the boundary of the circle γ, (2.5.3) becomes

$$f(\sigma) = \frac{c^2 i}{2\left(1-m^2\right)} \frac{1-m\sigma^n}{1+m\sigma^n}. \tag{2.5.8}$$

Substituting $\overline{\sigma} = 1/\sigma$ into (2.5.8) and combining it with (2.5.8) leads to

$$f(\sigma) + f\left(\overline{\sigma}\right) = \frac{c^2 i\left(1-\sigma^{2n}\right)}{\left(1-m^2\right)\left(\sigma^n + m\right)\left(\sigma^n + m^{-1}\right)}. \tag{2.5.9}$$

Since $z = \omega(\zeta)$, on the boundary of the circle we have

$$\omega(\sigma) = \frac{c\sigma}{1+m\sigma^n} \text{ and } \overline{\omega}\left(\overline{\sigma}\right) = \frac{c\sigma^{n-1}}{\sigma^n + m}. \tag{2.5.10}$$

Given the product of the two equations in (2.5.10), then using the product rule gives

$$d\left[\omega(\sigma)\overline{\omega}\left(\overline{\sigma}\right)\right] = \frac{c^2 n\left(1-\sigma^{2n}\right)\sigma^{n-1}}{m\left(\sigma^n + m\right)\left(\sigma^n + m^{-1}\right)^2}. \tag{2.5.11}$$

Substituting (2.5.9) and (2.5.11) into (2.5.7) leads to

$$D_0 = \frac{ic^4 n}{4m\left(m^2-1\right)}\int_\gamma \frac{\left(1-\sigma^{2n}\right)^2 \sigma^{n-1}}{\left(\sigma^n + m\right)^3\left(\sigma^n + m^{-1}\right)^3} d\sigma. \tag{2.5.12}$$

Set $H(\sigma)$ to be the integrand of (2.5.12). Let λ_s be a root of $\sigma^n + m = 0$, which implies $\lambda_s^n = -m$. Similar to the previous section, integrating (2.5.12) using the residue theorem gives us

$$D_0 = \frac{\pi c^4 n}{2m(1-m^2)} \sum_{s=1}^{n} Q_s \,, \tag{2.5.13}$$

where Q_s is the residue of $H(\sigma)$ at λ_s. Expanding $H(\lambda_s + t)$ in powers of t helps in finding the residue as explained below.

Finding the coefficient of t^{-1} gives the desired result Q_s. So, we have

$$H(\lambda_s + t) = \frac{(\lambda_s + t)^{n-1} - 2(\lambda_s + t)^{3n-1} + (\lambda_s + t)^{5n-1}}{\left[(\lambda_s + t)^n + m\right]^3 \left[(\lambda_s + t)^n + \frac{1}{m}\right]^3}. \tag{2.5.14}$$

To find the coefficient of t^{-1}, we consider the case $n = 2$. Finding the result requires lots of calculations, so we expand it as a sum of three parts. The first fraction in (2.5.14) is

$$I = \frac{\lambda_s + t}{\left[(\lambda_s + t)^2 + m\right]^3 \left[(\lambda_s + t)^2 + \frac{1}{m}\right]^3}. \tag{2.5.15}$$

Using $\lambda_s^2 = -m$ to expand (2.5.15) leads to

$$I = \frac{\lambda_s + t}{t^3 (t + 2\lambda_s)^3 \left(t^2 + 2\lambda_s t + \frac{1}{m} - m\right)^3}.$$

Factoring out $2\lambda_s$ from the first parenthesis and $r = \frac{1}{m} - m$ from the second parenthesis in the denominator from the above equation gives us

$$I = \frac{\lambda_s + t}{8r^3 \lambda_s^3 t^3 \left(1 + \frac{t}{2\lambda_s}\right)^3 \left(1 + \frac{t^2 + 2\lambda_s t}{r}\right)^3}.$$

Raising the quantities of the denominator to the numerator gives each parenthesis a power of -3. Expanding each parenthesis with the use of Binomial theorem $(1 + y)^n = 1 + ny + \frac{n(n-1)}{2!} y^2 + \cdots$ and pulling only the coefficients of t^{-1} which is given by

$$\text{Coefficient of } (1/t) \text{ in } I = \frac{1}{8r^3\lambda_s^3}\left[\frac{24\lambda_s^3}{r^2}\right] = \frac{3}{r^5}. \qquad (2.5.16)$$

Following the same procedure to simplify the denominator, the second fraction in (2.5.14) is

$$II = \frac{-2(\lambda_s + t)^5}{8r^3\lambda_s^3 t^3\left(1+\frac{t}{2\lambda_s}\right)^3\left(1+\frac{t^2+2\lambda_s t}{r}\right)^3}. \qquad (2.5.17)$$

Repeating the same process as earlier, the coefficients of t^{-1} of (2.5.7) is

$$\frac{-1}{4r^3\lambda_s^3}\left(4\lambda_s^3 - \frac{24\lambda_s^5}{r} + \frac{24\lambda_s^7}{r^2}\right) = \frac{-1}{r^3} + \frac{6\lambda_s^2}{r^4} - \frac{6\lambda_s^4}{r^5}.$$

Simplifying the above equation further, we use $\lambda_s^2 = -m$ and $r = \frac{1-m^2}{m}$ giving us

$$\text{Coefficient of } (1/t) \text{ in } II = -\frac{1}{r^3} - \frac{6}{r^5}. \qquad (2.5.18)$$

The third fraction of (2.5.14) is

$$III = \frac{(\lambda_s + t)^9}{8r^3\lambda_s^3 t^3\left(1+\frac{t}{2\lambda_s}\right)^3\left(1+\frac{t^2+2\lambda_s t}{r}\right)^3}. \qquad (2.5.19)$$

Similarly, (2.5.19) is reduced to

$$\text{Coefficient of } (1/t) \text{ in } III = \frac{3}{r^5}. \qquad (2.5.20)$$

The sum of (2.5.16), (2.5.18), and (2.5.20) for the case n = 2 is

$$Q_s = -\frac{1}{r^3} = -\frac{m^3}{(1-m^2)^3}. \qquad (2.5.21)$$

Thus, D_0 for the case n = 2, is given by

$$D_0 = -\frac{2\pi c^4 m^2}{\left(1-m^2\right)^4}.$$

Similarly, we can use (2.5.6) to find I_0 for the case n = 2, which is

$$I_0 = \frac{\pi c^4\left[m^4 + 4m^2 + 1\right]}{2\left(1-m^2\right)^4}.$$

Referring to Bassali [3] for the computation of the general result with any n = 2, 3, 4…

leads to

$$D_0 = -\frac{\pi c^4 n m^2}{\left(1-m^2\right)^4} \tag{2.5.22}$$

and

$$I_0 = \frac{\pi c^4\left[1 - m^4 + m^2\left(m^2+2\right)n\right]}{2\left(1-m^2\right)^4}. \tag{2.5.23}$$

Adding (2.5.22) and (2.5.23) gives the torsional rigidity for any value n as follows:

$$D = \frac{\mu\pi c^4\left(1-m^4+m^4 n\right)}{2\left(1-m^2\right)^4}. \tag{2.5.24}$$

2.6 Cross Section Γ_n Defined by $r = a\cos^n \dfrac{\theta}{n}$, $\left(-\dfrac{n\pi}{2} \le \theta \le \dfrac{n\pi}{2}\right)$

In this section, we use conformal mapping and Fourier series to solve a torsion problem for special curves where $0 < n \le 2$. To derive the polar equation, we let the conformal transformation be

$$z = \omega(\zeta) = c(1+\zeta)^n \tag{2.6.1}$$

where $c = \dfrac{a}{2^n}$. On the boundary of the unit circle $\sigma = e^{i\psi}$, so we have

$\omega(\sigma) = c(1 + \cos\psi + i\sin\psi)^n$. Since $z = re^{i\theta}$ is a point on the contours C_n, (2.6.1)

becomes $re^{i\theta} = c\left(2\cos\frac{\psi}{2}\right)^n e^{in\psi/2}$. Therefore, $\theta = \frac{n\psi}{2}$ and

$$r = c\left(2\cos\frac{\psi}{2}\right)^n. \qquad (2.6.2)$$

Note we change the contours Γ_n as C_n to avoid confusion of the gamma function $\Gamma(x)$.

Substituting $\psi = \frac{2\theta}{n}$ into (2.6.2) gives

$$r = a\left(\cos\frac{\theta}{n}\right)^n, \qquad (2.6.3)$$

where $a = 2^n c$. Notice when $n = 1/2$, equation (2.6.3) gives the lemniscate of Bernoulli and when $n = 2$ we have a cardioid as shown in Fig. 8 below.

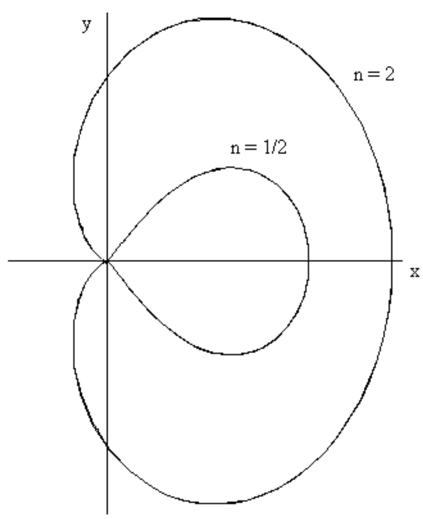

Fig. 8 Cross Sections corresponding to n = ½ and n = 2

To derive the complex torsion function for (2.6.3), we use the Fourier expansion of the function

$$f(\theta) = \cos^{2n}\frac{\theta}{n}, \quad (-L \le \theta \le L), \tag{2.6.4}$$

where $L = \frac{n\pi}{2}$. Since $f(\theta)$ is even, we know $b_k = 0$ and $f(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi\theta}{L}$.

Substituting (2.6.4) and $L = \frac{n\pi}{2}$ into the formula $a_k = \frac{2}{L}\int_0^L f(\theta)\cos\frac{k\pi}{L}d\theta$ gives

$$a_k = \frac{4}{n\pi}\int_0^{\frac{n\pi}{2}} \cos^{2n}\frac{\theta}{n}\cos\frac{2k\theta}{n}d\theta. \tag{2.6.5}$$

Using the substitution $\phi = \dfrac{\theta}{n}$, (2.6.5) becomes

$$a_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n}\phi\cos(2k\phi)d\phi. \tag{2.6.6}$$

To evaluate (2.6.6), we need Cauchy's formula which involves the gamma function $\Gamma(x)$:

$$\int_0^{\frac{\pi}{2}} (\cos t)^{\alpha}\cos(\beta t)dt = \frac{\pi\Gamma(1+\alpha)2^{-\alpha-1}}{\Gamma\left(1+\frac{1}{2}\alpha+\frac{1}{2}\beta\right)\Gamma\left(1+\frac{1}{2}\alpha-\frac{1}{2}\beta\right)}, \tag{2.6.7}$$

where $\operatorname{Re}\alpha > -1$. Using (2.6.7) with $\alpha = 2n$ and $\beta = 2k$, then (2.6.6) is simplified to

$$a_k = \frac{n}{2^{2n-2}} \cdot \frac{\Gamma(2n)}{\Gamma(1+n+k)\Gamma(1+n-k)} \tag{2.6.8}$$

for $k = 0, 1, 2, \ldots$. Using the properties of the gamma function

$$\Gamma(x+1) = x\Gamma(x) \text{ or } \Gamma^2(x+1) = x^2\Gamma^2(x)$$

and substituting (2.6.8) into the Fourier series expansion leads to

$$\cos^{2n}\frac{\theta}{n} = \frac{n\Gamma(2n)}{2^{2n-1}}\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\cos\frac{2k\theta}{n}}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]. \tag{2.6.9}$$

Squaring (2.6.3) and substituting the equivalent expression into (2.6.9) yields

$$\frac{1}{2}r^2 = nc^2\Gamma(2n)\mathrm{Re}\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\sigma^k}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]. \quad (2.6.10)$$

The complex torsion function $f(\zeta)$ at any point inside or on the unit circle γ is

$$f(\zeta) = nic^2\Gamma(2n)\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\zeta^k}{\Gamma(1+n+k)\Gamma(1+n-k)}\right], \quad (2.6.11)$$

where its imaginary part also satisfies the boundary condition of $\frac{1}{2}r^2$. Isolating ζ from (2.6.1) gives $\zeta = \left(z/c\right)^{\frac{1}{n}} - 1$. Substituting it into (2.6.11) gives

$$\Omega(z) = nic^2\Gamma(2n)\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\left\{\left(z/c\right)^{\frac{1}{n}} - 1\right\}^k}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]. \quad (2.6.12)$$

For $0 < n \le 1$, we have $\Omega'(0) = 0$; therefore, $\Omega(z)$ is analytic. For $1 < n \le 2$, we have $\Omega'(0) = \infty$ where the point $z = 0$ lies on the contour C_n. However, the point $z = 0$ lies inside the contour C_n and $\Omega'(0) = \infty$ for $n > 2$; therefore, $\Omega(z)$ is not analytic. So, we only use the range $0 < n \le 2$.

The torsion rigidity is defined as

$$D_n = \mu(I + J), \quad (2.6.13)$$

where $I = 2\iint_S \Psi(x,y)dxdy$ and $J = \frac{1}{2}\int_\gamma \omega(\sigma)\overline{\omega}(\sigma^{-1})f'(\sigma)d\sigma$.

Evaluating the first integral in (2.6.13) gives

$$I = 2\int_0^{\frac{n\pi}{2}}\int_0^{a\cos^n\frac{\theta}{2}} r^3 dr d\theta = \frac{a^4}{2}\int_0^{\frac{n\pi}{2}}\cos^{4n}\frac{\theta}{n}d\theta. \quad (2.6.14)$$

Using substitution $\phi = \frac{\theta}{n}$ into (2.6.14) and (2.6.7), I becomes

$$I = \frac{na^4}{2}\int_0^{\frac{\pi}{2}} \cos^{4n}\phi \, d\phi = \frac{\pi c^4 \Gamma(4n)}{4\Gamma^2(2n)}. \qquad (2.6.15)$$

Replacing ζ *by* σ in (2.6.1) and (2.6.11) and evaluating the second integral of (2.6.13) gives

$$J = nc^4\Gamma(2n)\sum_{k=1}^{\infty}\frac{kL_k}{\Gamma(1+n+k)\Gamma(1+n-k)}, \qquad (2.6.16)$$

where $L_k = i\int_\gamma (1+\sigma)^{2n}\sigma^{k-n-1}d\sigma$. Using the substitution rule for $\sigma = e^{i\psi}$, L_k is given by

$$L_k = -2^{2n}\int_0^{2\pi}\left(\cos\frac{\psi}{2}\right)^{2n} e^{ik\psi}\,d\psi = -2^{2n+1}\int_0^{\pi}\left(\cos\frac{\psi}{2}\right)^{2n}\cos k\psi\,d\psi .$$

Using substitution again for $\phi = \psi/2$ and applying (2.6.7), the above equation becomes

$$L_k = -2^{2n+2}\int_0^{\frac{\pi}{2}}(\cos\phi)^{2n}\cos 2k\phi\,d\phi = \frac{-4\pi n\Gamma(2n)}{\Gamma(1+n+k)\Gamma(1+n-k)}.$$

Replacing L_k into (2.6.16) leads to

$$J = -4\pi n^2c^4\Gamma^2(2n)\sum_{k=1}^{\infty}\frac{k}{\Gamma^2(1+n+k)\Gamma^2(1+n-k)}. \qquad (2.6.17)$$

Let H be the summation in (2.6.17). To evaluate (2.6.17), we need to use Pochhammer's formula for the hypergeometric functions

$${}_3F_2\left[\begin{matrix}a_1, a_2, a_3\\ b_1, b_2\end{matrix}; z\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{n!(b_1)_n(b_2)_n}z^n \qquad (2.6.18)$$

and Dixon's Forumla

$${}_3F_2\left[\begin{matrix}\alpha, \beta, \gamma\\ 1+\alpha-\beta, 1+\alpha-\gamma\end{matrix}; 1\right] = \frac{\Gamma(1+\frac{1}{2}\alpha)\Gamma(1+\frac{1}{2}\alpha-\beta-\gamma)\Gamma(1+\alpha-\beta)\Gamma(1+\alpha-\gamma)}{\Gamma(1+\alpha)\Gamma(1+\alpha-\beta-\gamma)\Gamma(1+\frac{1}{2}\alpha-\beta)\Gamma(1+\frac{1}{2}\alpha-\gamma)}.$$

First we need to introduce Pochhammer's symbol

$$(\alpha)_m = \alpha(\alpha+1)\Lambda\ (\alpha+m-1) = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}. \qquad (2.6.19)$$

Using Pochhammer's Symbol, it is obvious that we have $k = k!/(k-1)! = (2)_{k-1}/(k-1)!$,

$\Gamma(1+n+k) = (1+n)(2+n)_{k-1}\Gamma(1+n)$, and $\Gamma(1+n-k) = (-1)^k\Gamma(1+n)/\{(-n)(1-n)_{k-1}\}$.

Applying these three expressions, we have

$$H = \frac{1}{n^2(1+n)^2\Gamma^4(n)}\sum_{k=0}^{\infty}\frac{(2)_k}{k!}\left[\frac{(1-n)_k}{(2+n)_k}\right]^2. \qquad (2.6.20)$$

Let $\lambda = \frac{1}{n^2(1+n)^2\Gamma^4(n)}$. Applying (2.6.18) to the above summation, (2.6.20) becomes

$$H = \lambda\,{}_3F_2\left[\begin{matrix}2,1-n,1-n\\ 2+n,2+n\end{matrix};1\right]. \qquad (2.6.21)$$

Using Dixon's Formula in (2.6.21), we have

$$H = \frac{\lambda\Gamma(2)\Gamma(2n)\Gamma^2(2+n)}{\Gamma(3)\Gamma(1+2n)\Gamma^2(1+n)}. \qquad (2.6.21)$$

Simplifying (2.6.21) we get

$$H = \frac{1}{4n^3\Gamma^2(n)}. \qquad (2.6.22)$$

Substituting (2.6.22) into (2.6.17) leads to

$$J = \frac{-\pi c^4\Gamma^2(2n)}{n\Gamma^4(n)}. \qquad (2.6.23)$$

Replacing (2.6.15) and (2.6.23) into (2.6.13) gives the torsional rigidity to be

$$D_n = \frac{\mu\pi a^4}{2^{4n}}\left[\frac{\Gamma(4n)}{4\Gamma^2(2n)} - \frac{\Gamma^2(2n)}{n\Gamma^4(n)}\right]. \qquad (2.6.24)$$

CHAPTER 3

SOLUTIONS TO THE FLEXURE PROBLEM

Stevenson [14] was able to simplify the flexure problem of an isotropic beam to six functions noted as the three Dirichlet boundary value problems and three Neumann boundary value problems. One of the three Dirichlet problems turns out to be a torsion function. In this chapter, we solve the flexure problem for cross sections bounded by $r = 2^4 b \sin^4(\theta/4)$, $(-\pi, \pi]$, and $b > 0$. The process we use to solve for this specific boundary can be applied to the general form when the number 4 in the equation is replaced by any positive integer *n*.

3.1 Essential Equations

In the flexure problem for a cylindrical elastic beam, we set the *z*-axis to be along the beam parallel to the generators of the cylinder and the *x*-axis is along the axis of symmetry of the cross section *S*. If the external forces are equivalent to loads $(W_x, W_y, 0)$ acting at the centroid $x = h,\ y = 0$ of the end of the beam, then Stevenson [14] proved the flexure problem is solved by finding six analytic functions corresponding to each cross section:

$$\omega_r = \phi_r + i\psi_r, \quad \Omega_r = \Phi_r + i\Psi_r \cdot (r = 1, 2, 3). \tag{3.1.1}$$

To determine the harmonic functions ψ_r and Φ_r we find the solutions to Laplace's equations with the boundary conditions

$$\psi_r = F_r, \quad \frac{\partial}{\partial n}(\Phi_r - F_r) = 0, \tag{3.1.2}$$

where

$$F_1 = \frac{x^3}{3},\ F_2 = \frac{y^3}{3},\ F_3 = \frac{x^2 + y^2}{2}, \qquad (3.1.3)$$

and n is normal to the boundary G of the cross-section S.

Here we consider the flexure problem corresponding to the boundaries G_n which are denoted as $r = c\left|\sin\left(\theta/_n\right)/\sin\left(\pi/_n\right)\right|^n$, $-\pi < \theta \le \pi$, and $n = 2, 3, 4...$ as shown in the Fig. 9 below.

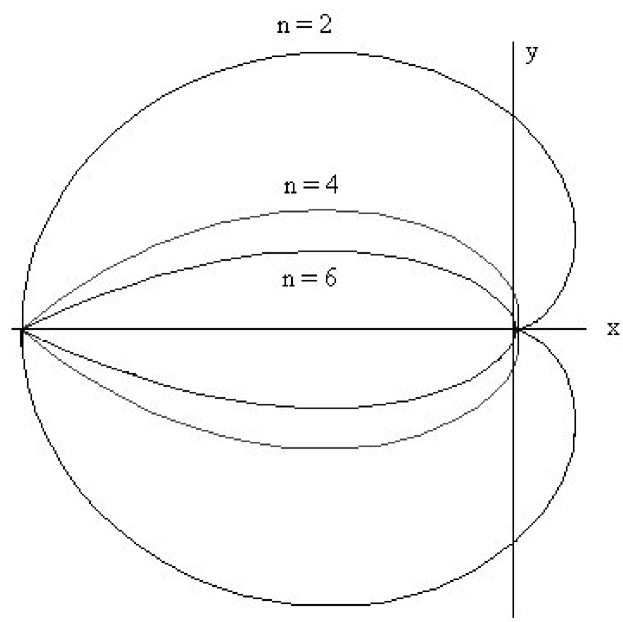

Fig. 9 Cross sections corresponding to n = 2, 4, and 6

The boundary conditions on ψ_1 and ψ_2 can be written in complex form as

$$\psi_1 + i\psi_2 = \frac{x^3}{3} + i\frac{y^3}{3} = \frac{z^{*3}}{12} + \frac{z^2 z^*}{4}. \qquad (3.1.4)$$

Note z^* is the conjugate of z. We can rewrite (3.1.4) as

$$\psi_1 + i\psi_2 = f(z) + F(z^*) \qquad (3.1.5)$$

such that

$$\psi_1 = \mathrm{Re}\left[f(z) + F(z^*)\right], \tag{3.1.6}$$

$$\psi_2 = \mathrm{Im}\left[f(z) - F(z^*)\right], \tag{3.1.7}$$

where $[f(z)]^* = f^*(z^*)$. From (3.1.1), we can rewrite the complex flexure functions as

$$\omega_1 = i\left[f(z) + F^*(z)\right], \tag{3.1.8}$$

and
$$\omega_2 = \left[f(z) - F^*(z)\right]. \tag{3.1.9}$$

By (3.1.2) and (3.1.3), we have $\psi_3 = F_3 = \frac{1}{2}r^2$, which is the same boundary condition of a torsion function. As stated previously, the solution to the torsion function is given by

$$\frac{1}{\alpha+\beta}\left[\mathrm{Re}\,h(z) + \frac{1}{2}(\beta-\alpha)\left(x^2 - y^2\right) - \gamma\right]. \tag{3.1.10}$$

Stevenson [14] has found that the third complex flexure function is a special case of (3.1.10) where $\alpha = \beta = 1$ and $\gamma = 0$. Therefore, the complex flexure function ω_3 is also a complex torsion function denoted as

$$\omega_3 = i\,\frac{h(z)}{2}. \tag{3.1.11}$$

The two boundary conditions on Φ_1 and Φ_2 for the flexure functions Ω_1 and Ω_2 are given by

$$\frac{\partial}{\partial n}\left[\left(\Phi_1 + i\Phi_2\right) - \left(\tfrac{1}{12}z^{*3} + \tfrac{1}{4}z^2 z^*\right)\right] = 0. \tag{3.1.12}$$

Since the equation of the boundary is

$$\mathrm{Im}\,\zeta = \mathrm{Im}(\xi + i\eta) = \mathrm{Im}\,H(z) = \text{constant}, \tag{3.1.13}$$

the boundary conditions in (3.1.12) is reduced to

$$\frac{\partial}{\partial \eta}(\Phi_1 + i\Phi_2) = \frac{i}{2} zz^* \frac{dz}{d\zeta} - \frac{i}{4}\left(z^{*2} + z^2\right)\frac{dz^*}{d\zeta^*}. \qquad (3.1.14)$$

Similarly, the boundary conditions of the flexure functions can be rewritten in a separable form of given functions $g_1(z)$ and $G_1(z^*)$ denoted as

$$\frac{\partial}{\partial \eta}(\Phi_1 + i\Phi_2) = g_1(z) + G_1(z^*). \qquad (3.1.15)$$

Let Φ_1 and Φ_2 be determined by

$$\Phi_1 + i\Phi_2 = t(z) + T(z^*), \qquad (3.1.16)$$

where $t(z)$ and $T(z^*)$ are found through the boundary condition of (3.1.15).

Substituting (3.1.16) into (3.1.15) leads to

$$i\frac{dt}{dz}\frac{dz}{d\zeta} = g_1(z) - C_1, \qquad (3.1.17)$$

and

$$-i\frac{dT}{dz^*}\frac{dz^*}{d\zeta^*} = G_1(z^*) + C_1, \qquad (3.1.18)$$

where C_1 is a constant. Once $t(z)$ and $T(z^*)$ are determined from (3.1.17) and (3.1.18), the flexure functions are given by

$$\Omega_1 = t(z) + T^*(z), \qquad (3.1.19)$$

and

$$\Omega_2 = -i[t(z) - T^*(z)]. \qquad (3.1.20)$$

Finding the flexure function Ω_3 is a similar process. The boundary condition on Φ_3 is

$$\frac{\partial}{\partial \eta}\Phi_3 = \frac{\partial}{\partial \eta}\left(\frac{1}{2}zz^*\right) = \mathrm{Im}\left(z^{5/4}z^*\right), \qquad (3.1.21)$$

because $\frac{\partial}{\partial \eta} = i\,(\frac{\partial}{\zeta} - \frac{\partial}{\partial \zeta^*})$ and (3.1.13).

Using $\Phi_3 = \frac{1}{2}\left[\Omega_3 + \Omega_3^*\right]$ and the boundary equation, we can obtain (3.1.21) to be

$$\frac{\partial}{\partial \eta}\Phi_3 = \operatorname{Im}\left[\Omega_3'(z) z^{5/4}\right]. \tag{3.1.22}$$

Rewriting the right side of (3.1.21) in a separable form of z and z^* gives us the right side of (3.1.22). From there, we need to integrate $\Omega_3'(z)$ in terms of z to get the third flexure function. For all the details see Obaid [10].

To find the torsional rigidity D of the beam we use

$$D = \mu \tau M_3, \tag{3.1.23}$$

where μ is the rigidity, τ is the twist , and $M_3 = M_{33} + \iint_S \left(x^2 + y^2\right) dS$ with

M_{33} = real part of $\iint_S \mathrm{iz}\frac{\mathrm{d}\omega_3}{\mathrm{dz}}\, dS$. Refer to [10] in order to decipher the twist, τ .

3.2 Cross Section G_4 Defined by $r = 2^4 b \sin^4 \frac{\theta}{4}$

In this section, we find the flexure functions for the cross section bounded by the closed curve G_4 with the polar equation

$$r = 2^4 b \sin^4 \frac{\theta}{4} \tag{3.2.1}$$

with the conditions of $-\pi < \theta \le \pi$ and $b > 0$. Given $z = x + iy$ and $\zeta = \xi + i\eta$, let

$$\zeta == \xi + i\eta = z^{-1/4}. \tag{3.2.2}$$

If $z = re^{i\theta}$, then the above relation gives the curve $\eta = -\frac{1}{2}b^{-1/4}$ as (3.2.1), i.e

$\eta =$ constant is the same boundary (3.2.1). To find the complex torsion function, ω_3, we need to first find the boundary condition $\psi_3 = \frac{1}{2}zz*$. The boundary equation (3.2.1) can be rewritten in the complex separable form as

$$z^{-1/4} - z*^{-1/4} = ib^{-1/4} \tag{3.2.3a}$$

or

$$(zz*)^{1/4} = -ib^{1/4}\left(z^{1/4} - z*^{1/4}\right). \tag{3.2.3b}$$

Simplifying (3.2.3b) leads to

$$zz* = b\left[(z + z*) - 4(zz*)^{1/4}\left(z^{1/2} + z*^{1/2}\right) + 6(zz*)^{1/2}\right]. \tag{3.2.4}$$

To evaluate (3.2.4), we need to break down each of the terms in it. So, we have

$$(zz*)^{1/2} = -2b^{1/2}\cos\tfrac{\theta}{2} + 4b^{3/4}\sin\tfrac{\theta}{4} = \mathrm{Re}\left[-2b^{1/2}z^{1/2} - 4ib^{3/4}z^{1/4}\right], \tag{3.2.5}$$

and

$$(zz*)^{1/4}\left(z^{1/2} + z*^{1/2}\right) = \mathrm{Re}\left[-2ib^{1/4}z^{3/4} + 2b^{1/2}z^{1/2} + 4ib^{3/4}z^{1/4}\right]. \tag{3.2.6}$$

Replacing (3.2.3b), (3.2.5), and (3.2.6) into (3.2.4) leads to

$$\mathrm{Re}\left[2bz + 8ib^{5/4}z^{3/4} - 20b^{3/2}z^{1/2} - 40ib^{7/4}z^{1/4}\right] - \left(x^2 + y^2\right) = 0. \tag{3.2.7}$$

Since (3.2.7) is consistent with (3.1.10) for $\alpha = \beta = 1$ and $\gamma = 0$, (3.1.11) applies to give the complex torsion function to be

$$\omega_3 = ibz - 4b^{5/4}z^{3/4} - 10ib^{3/2}z^{1/2} + 20b^{7/4}z^{1/4}. \tag{3.2.8}$$

To find (3.1.14), we multiply z to (3.2.4), so we get

$$\begin{aligned}\psi_1 + i\psi_2 = \frac{z*^3}{12} + b\left(z^2 + zz*\right) + 4ib^{5/4}\left(z^{7/4} - zz*^{3/4}\right)\\ -10b^{3/2}\left(z^{3/2} + zz*^{1/2}\right) - 20ib^{7/4}\left(z^{5/4} - zz*^{1/4}\right).\end{aligned} \tag{3.2.9}$$

Similarly, we have

$$zz^{*3/4} = ib^{3/4}z - 3bz^{3/4} + bz^{*3/4} - 6ib^{5/4}z^{1/2} - 4ib^{5/4}z^{*1/2} + 10b^{3/2}\left(z^{1/4} - z^{*1/4}\right),$$

$$zz^{*1/2} = -b^{1/2}z + 3bz^{1/2} + bz^{*1/2} - 2ib^{3/4}z^{3/4} + 4ib^{5/4}\left(z^{1/4} - z^{*1/4}\right),$$

and
$$zz^{*1/4} = -ib^{1/4}z + b^{1/2}z^{3/4} + ib^{3/4}z^{1/2} - b\left(z^{1/4} - z^{*1/4}\right).$$

Substituting the above 3 equations into (3.2.9) gives us

$$\begin{aligned} f(z) = {} & \tfrac{1}{4}bz^2 - 3bz^{3/4} + ib^{5/4}z^{7/4} - \tfrac{5}{2}b^{3/2}z^{3/2} - 5ib^{7/4}z^{5/4} \\ & + \tfrac{35}{4}b^2 z + 14ib^{9/4}z^{3/4} - 21b^{5/2}z^{1/2} \end{aligned} \tag{3.2.10}$$

and
$$F(z^*) = \tfrac{1}{12}z^{*3} + \tfrac{1}{4}b^2 z^* - 2ib^{9/4}z^{*3/4} - 9b^{5/2}z^{*1/2} + 30ib^{11/4}z^{*1/4}. \tag{3.2.11}$$

Replacing (3.2.10) and (3.2.11) into (3.1.8) and (3.1.9) gives the complex functions to be

$$\omega_1 = i\left[\begin{aligned} & \tfrac{1}{12}z^3 + \tfrac{1}{4}ibz^2 - b^{5/4}z^{7/4} - \tfrac{5}{2}ib^{3/2}z^{3/2} + 5b^{7/4}z^{5/4} \\ & \quad + 9ib^2 z - 16b^{9/4}z^{3/4} - 30ib^{5/2}z^{1/2} + 60b^{11/4}z^{1/4} \end{aligned}\right], \tag{3.2.12}$$

and
$$\omega_2 = \left[\begin{aligned} & \tfrac{-1}{12}z^3 + \tfrac{1}{4}bz^2 + ib^{5/4}z^{7/4} - \tfrac{5}{2}b^{3/2}z^{3/2} - 5ib^{7/4}z^{5/4} \\ & \quad + \tfrac{17}{2}b^2 z + 12ib^{9/4}z^{3/4} - 12b^{5/2}z^{1/2} \end{aligned}\right]. \tag{3.2.13}$$

Substituting (3.2.2) into the boundary condition in (3.1.14) gives us

$$\frac{\partial}{\partial\eta}\left(\Phi_1 + i\Phi_2\right) = i\left[z^{*13/4} + z^2 z^{*5/4} - 2z^{9/4}z^*\right]. \tag{3.2.14}$$

Following the procedures to find (3.1.19), (3.1.20), and (3.1.22) as described in Section 3.1, we obtain the following

$$\begin{aligned} \Omega_1 = {} & \tfrac{1}{12}z^3 + \tfrac{1}{4}bz^2 + \tfrac{9}{7}ib^{5/4}z^{7/4} - \tfrac{25}{6}b^{3/2}z^{3/2} - 11ib^{7/4}z^{5/4} + \tfrac{106}{4}b^2 z \\ & + 64ib^{9/4}z^{3/4} - 174b^{5/2}z^{1/2} - 660ib^{11/4}z^{1/4} + 330b^3 \ln z, \end{aligned} \tag{3.2.15}$$

$$\begin{aligned} \Omega_2 = {} & i[\tfrac{1}{12}z^3 + \tfrac{1}{4}bz^2 + \tfrac{9}{7}ib^{5/4}z^{7/4} - \tfrac{25}{6}b^{3/2}z^{3/2} - 11ib^{7/4}z^{5/4} + \tfrac{106}{4}b^2 z \\ & + 64ib^{9/4}z^{3/4} - 174b^{5/2}z^{1/2} - 660ib^{11/4}z^{1/4} + 330b^3 \ln z], \end{aligned} \tag{3.2.16}$$

and
$$\Omega_3 = bz + \tfrac{20}{3}ib^{5/4}z^{3/4} - 30b^{3/2}z^{1/2} - 140ib^{7/4}z^{1/4} + 70b^2 \ln z. \tag{3.2.17}$$

Since the process of finding the three Neumann flexure functions is tedious and similar to the earlier process of finding the three Dirichlet complex flexure functions, refer to [10] for all details.

To solve the flexure problem for cross sections bounded by any curve of the family of the form

$$\{\Gamma_n\}: r = a\cos^n \frac{\theta}{n}, \quad a > 0, \ -\pi < \theta \le \pi, \quad n = 2, 3, 4, ...$$

and

$$\{C_n\}: r = a\left|\sin\frac{\theta}{n}\right|^n, \quad a > 0, \ -\pi < \theta \le \pi, \quad n = 2, 3, 4, ...,$$

we need to apply the following two identities:

$$2^{n+k}(\cos\theta)^{n+k}\cos(n-k)\phi = \sum_{v=1}^{n} 2^v\left[\binom{n+k-v-1}{k-1} + \binom{n+k-v-1}{n-1}\right](\cos\phi)^v \cos v\phi$$

and

$$2^{n+k}(\cos\theta)^{n+k}\sin(n-k)\phi = \sum_{v=1}^{n} 2^v\left[\binom{n+k-v-1}{k-1} - \binom{n+k-v-1}{n-1}\right](\cos\phi)^v \sin v\phi\,.$$

Rung and Obaid have proven the above identities in two different ways. Refer to [11] and [12] for the proof of the identities and the complete solution of these flexure problems.

CONCLUSION

We have solved a flexure problem with a particular boundary equation and found the solution to the torsion problem for various cross sections by different methods. However, we touched only the "tip of the iceberg" in the world of potential theory. We know that, given a closed curve, where $\alpha + \beta = 0$ cannot be solved using the Bassali-Obaid method. However, there is a solution to the torsion problem where the closed curve satisfies $\alpha + \beta = 0$. We wonder if it is possible to find a simple solution to the torsion problem for the case of $\alpha + \beta = 0$. In addition, it is desirable to discover if there is an easy way to solve the torsion problem for a simple closed boundary of the form $\operatorname{Re} F(z) - \left(\alpha x^n + \beta y^n + \gamma\right) = 0$, where $\alpha + \beta \neq 0$, and $n = 3, 4, 5, \ldots$. We can also try to solve the torsion problem for other forms of the closed boundary curve. These are all areas of interest for future research.

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