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Extensions of Morley's theorem

Richard Harvey Gordon
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EXTENSIONS OF MORLEY'S THEOREM

A Thesis

Presented to

the Faculty of the Department of Mathematics

San Jose State College

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

Richard Harvey Gordon

June 1970

APPROVED FOR THE DEPARTMENT OF MATHEMATICS

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Abstract: Extensions of Morley's Theorem by Richard Harvey Gordon

A review of the literature since 1899, when Morley offered his original hypothesis, is presented. Various authors have given proofs of the theorem, shown the existence of 27 such triangles, and established relationships to other mathematical concepts: isogonal conjugates, Lemoine point, Hessian points and axis, circles of Apollonius, Brianchon hexagons. The author explores the various intersections of the n -sectors of the angles of a general triangle, showing, incidentally, the uniqueness of the Morley equilateral triangle. Turning to the quadrilateral, the intersections of the tri-sectors of: the angles of a parallelogram, rectangle, trapezoid, and kite-shaped quadrilateral are examined, as well as those of the n -sectors. An important by-product of the author's investigation of the p -sided polygon confirms the uniqueness of the equilateral triangle of Morley's theorem. The author concludes with a discussion of the associated isogonal attributes and thoughts for further study.

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R.H.G.

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CHAPTER I

INTRODUCTION

Ironically, Frank Morley never published a formal proof for the theorem stated below. He discovered the identity as part of a more general analysis near the end of the nineteenth century and casually mentioned it to friends in Cambridge, England, around 1904. In later years he expressed mild surprise over the interest excited by this theorem. Figure 1-1 illustrates the theorem.

Morley's Theorem: If the angles of any triangle be trisected, the triangle formed by the meets of pairs of trisectors, each pair being adjacent to the same side of the given triangle, is equilateral.

Figure 1-1

Survey of the Literature

The works of Morley¹ no doubt contain the theory which led to the discovery of his trisection theorem. These works are included mainly for completeness, inasmuch as several authors have referenced them. H. F. Baker includes a note by Morley on "A Construction for an Equilateral Triangle from Any Given Triangle."² This note, however, was apparently written after several proofs were already in existence.

Coxeter³ and Taylor and Marr⁴ pinpoint the origin of Morley's theorem in the year 1899. The first published proofs of Morley's theorem appeared in 1909, some ten years after its discovery.⁵ Many proofs in the literature deal only with the equilateral triangle formed by intersections of adjacent interior trisectors. Taylor and Marr⁶ were

¹Listed in the Bibliography for general reference.

²H. F. Baker, An Introduction to Plane Geometry (London: Bentley House, 1943) p. 345.

³H. S. M. Coxeter, F. R. S., Introduction to Geometry Second Edition (New York: John Wiley and Sons, 1969) p. 23.

⁴F. Glanville Taylor and W. L. Marr, "The Six Trisectors of each of the Angles of a Triangle," Proceedings of the Edinburgh Mathematics Society Vol. 32 (1914) pp. 119-131.

⁵M. Salyanarayana, Mathematical Questions and their Solutions from the Educational Times Vol. 15 (1909) pp. 23-24; M. T. Naraniengar, Ditto, p. 47.

⁶Taylor and Marr, loc. cit.

among the first, if not the first, to show the existence of not one, but twenty-seven "Morley" triangles (i.e., triangles which may be formed from the intersections of adjacent trisectors). These triangles come about by considering three possible angles defined by two rays emanating from a point: A , $2\pi - A$, $2\pi + A$, where $A < \pi$. Figure 1-2 shows the six trisectors of the angle DOE. Each trisector is labeled with its angular measure from the line OE in a counterclockwise direction. The acute angle DOE has angular measure A . The lines $\frac{A}{3}$ and $\frac{2A}{3}$ are the trisectors of this acute angle. The trisectors of the external angle, $2\pi - A$, are $\frac{2\pi + 2A}{3}$ and $\frac{2\pi + A}{3}$. The trisectors of the third angle, $2\pi + A$, are $\frac{4\pi + A}{3}$ and $\frac{4\pi + 2A}{3}$.

Figure 1-2

Taylor and Marr introduced biangular coordinates to describe the adjacent trisectors of each angle. The adjacent trisectors are $\frac{A+2p\pi}{3}$, $\frac{B+2q\pi}{3}$, and $\frac{C+2r\pi}{3}$, where a positive direction for the angles is a rotation about a vertex toward the interior of the triangle and p, q, r , can take on any of the values 0, 1, 2. Thus a general Morley triangle is $D_{qr}E_{rp}F_{pq}$. Figure 1-3 illustrates this notation. The

Figure 1-3

figure contains twenty-seven Morley triangles of which just eighteen are equilateral triangles. The nine non-equilateral Morley triangles occur when $p+q+r \equiv 1 \pmod{3}$. In addition there are nine associated (non-Morley) equilateral triangles. Thus Figure 1-3 contains a total of twenty-seven equilateral triangles and nine non-equilateral triangles.

Figure 1-4 illustrates the three types of triangles found in Figure 1-3. Triangle $D_{10}E_{02}F_{21}$ is a Morley equilateral triangle. Triangle $D_{22}E_{20}F_{02}$ is also a Morley triangle, however, since $p=0$ and $q=r=2$ implies that $p+q+r \equiv 1 \pmod{3}$, it is not equilateral. Triangle $D_{10}D_{01}D_{22}$ is an equilateral triangle. It is not a Morley triangle since

Figure 1-4

each of its vertices is the intersection of certain trisectors of the angles at B and C only.

The authors show that the vertices of the twenty-seven equilateral triangles as well as the nine non-equilateral triangles lie on three sets of three parallel lines. They also develop the equations for the parallel lines in the biangular coordinate system.

The figure also shows twenty-seven points labeled L_{qr} , M_{rp} , and N_{pq} . The authors state that these points are the isogonal conjugates of D_{qr} , E_{rp} , and F_{pq} , respectively. Definition: Two lines are isogonal with respect to an angle if the bisector of the angle also bisects the angle formed by the two lines.

Definition: Two points are isogonal conjugates with respect to a triangle if the two lines from the points to each vertex of the triangle are isogonal lines.

The twenty-seven isogonal points lie six by six on nine conics through the triangle ABC. These nine conics meet in threes on the circumcircle. The authors prove that these three meeting points form an equilateral triangle whose sides are parallel to the sides of triangle DEF.

In a further extension, the authors note that the lines AL_{qr} , BM_{rp} , and CN_{pq} are concurrent and denote the point of concurrency as the pole of the triangle $D_{qr}E_{rp}F_{pq}$; they then demonstrate that there are six triads of

equilateral triangles and three triads of non-equilateral triangles such that each triad is two by two in perspective, and the poles of each triad are collinear. They further show that each line $D_{qr}D_{q+1,r+1}$, $E_{qr}E_{q+1,r+1}$, and $F_{qr}F_{q+1,r+1}$ contain three poles. Thus there are also nine triads of concurrent lines in the figure. The authors conclude their paper with a tabular presentation of the nine collinear triads and the nine concurrent triads.

In an independent article⁷ Taylor first shows that the nine triads of poles decompose into three triads of concurrent lines. The three points of concurrency, which are the Lemoine point, the positive Hessian point, and the negative Hessian point, are collinear on the Hessian axis. The Lemoine point is by definition the isogonal conjugate of the centroid of the triangle. This is also called the symmedian point. The two Hessian points are the points of intersection of the three circles of Apollonius, which are coaxal. The Hessian axis is the radical axis of these three circles. Figure 1-5 illustrates these points. By definition, a circle of Apollonius is a circle through a vertex of a triangle in which the other two vertices are

⁷F. Glanville Taylor, "The Relation of Morley's Theorem to the Hessian Axis and Circumcenter," Proceedings of the Edinburgh Mathematics Society, Vol. 32 (1914), pp. 132-135.

Figure 1-5

inverse points. Thus circle C_A in Figure 1-5 has B and C as inverse points.

Secondly, Taylor⁸ shows that the nine lines of the form $D_{qr}D_{q+1,r+1}$ pass by threes through three points. These points are collinear on a line through the circumcenter, O, of the triangle ABC, perpendicular to BC. Similarly the lines EE and FF yield lines through O and perpendicular to CA and AB, respectively.

⁸Ibid.

Marr⁹ demonstrates a construction method for recovering the twenty-seven Morley triangles given the three points on the Hessian axis. The Lemoine point yields the nine non-equilateral triangles and the two Hessian points each yield nine equilateral triangles. In the course of his construction, he shows and utilizes the fact that certain pairs of lines of poles of Morley triangles meet on one of the three circles of Apollonius.

In a trigonometric approach, Dobbs¹⁰ concerns himself with all twenty-seven Morley triangles. A by-product of this approach is the orientation of the Morley triangles with respect to the given triangle (i.e., the angle of inclination that the sides of the Morley triangles make with the sides of the given triangle). Referring to Figure 1-1, FE is inclined to BC at an angle of $\beta - \gamma$; DF is inclined to CA at an angle of $\gamma - \alpha$; ED is inclined to AB at an angle of $\alpha - \beta$. This result was also observed by Taylor and Marr.¹¹

⁹W. L. Marr, "Morley's Trisection Theorem: An Extension and its relation to the Circles of Apollonius," Proceedings of the Edinburgh Mathematics Society, Vol. 32 (1914), pp. 136-150.

¹⁰W. J. Dobbs, "Morley's Triangle," American Mathematics Monthly, Vol. 22 (Feb. 1938), pp. 50-57.

¹¹Taylor and Marr, op. cit., p. 123.

Loria¹² considers the larger question of equilateral triangles derived from a general triangle, of which the set of Morley triangles is a subset. The author makes an important contribution by finding the lengths of the sides of the twenty-seven equilateral triangles in Morley's figure in terms of the radius of the circumcircle and the three angles of the given triangle. It is this paper which discloses a letter from Morley, himself, in which he recalls having revealed the trisection theorem to friends in the year 1904.

Based on Dobbs' and Loria's work, Gibbons¹³ goes on to show the existence of three centroids, each of which is common to six scalene triangles, three of which are among the nine scalene Morley triangles. He also states that there are three coaxial systems of three circles each, passing through the vertices of the original triangle, ABC. The radical axis of each system is a median of ABC. Thus, the radical center of the nine circles is the centroid of ABC. The author also shows how to obtain the area of the nine scalene Morley triangles.

¹²Gino Loria, "Triangles Equilateraux Derives d'un Triangle Quelconque," Mathematics Gazette, Vol. 23 (1939), pp. 364-372.

¹³N. M. Gibbons, "The Non-Equilateral Morley Triangles," Mathematics Gazette, Vol. 26 (1942), pp. 81-86.

Of the many proofs considering just one of Morley's triangles, Lubin's¹⁴ proof is unique in that he defines the sides and angle trisectors of the given triangle as vector quantities and thus the sides of the Morley triangle are the results of certain vector operations. Of Bricard's¹⁵ work, Lubin says, "This author proved Morley's theorem and extensions by means of an ingenious use of sets of concurrent lines without finding their points of intersection."¹⁶

Smart¹⁷ extends the underlying theme of Morley's theorem to the square. He finds that intersections of adjacent trisectors form a square. He also finds that a square results when the restriction of "adjacent" is removed. By defining the n -sectors of an angle as those lines which partition the angle into n congruent parts, the author then goes on to consider the n -sectors of the angles of a square and shows that the intersections of adjacent n -sectors form a square.

¹⁴Clarence Lubin, "A Proof of Morley's Theorem," American Mathematics Monthly, Vol. 62 (1955), pp. 110-112.

¹⁵R. Bricard, "Sur les Droites Moyennes d'un Tri-angle," Nouvelles Annales de Mathematique, 5th Series, vol. ii lxxxi (1923), pp. 241-254.

¹⁶Lubin, op. cit., p. 112.

¹⁷James R. Smart, "The N -Sectors of the Angles of a Square," The Mathematics Teacher (May 1967), pp. 459-463.

Introduction to Chapters II-VI

The remaining chapters of this thesis will attempt to generalize Morley's theorem in two ways. First, the n -sectors of the angles of a triangle will be considered and the triangle generated by the various intersections of n -sectors examined for equilateral properties. Secondly, the triangle will be extended to a general quadrilateral and, further, to a general p -sided polygon. In each case, the figure generated by the various intersections of n -sectors will be studied to ascertain the existence of or conditions for existence of equilateral properties.

Also, this thesis will attempt to relate Morley's theorem and the Morley figure to other notions of higher geometry.

CHAPTER II

THE n -SECTORS OF A TRIANGLE

In this chapter the triangle formed by the intersections of n -sectors of the angles will be studied. First, the intersections of adjacent n -sectors for arbitrary n will be considered. Next, with the restriction that the n -sectors be adjacent removed, the intersections of non-adjacent n -sectors will be considered for arbitrary n . Lastly, the intersections of specific n -sectors for arbitrary n will be studied.

Adjacent n -Sectors

Let the interior of each of the angles of a triangle be partitioned into n congruent parts. The line, which together with a side of the triangle forms an angle with measure $1/n$ th of the angle of the triangle, is termed an adjacent n -sector. Figure 2-1 illustrates the situation. The vertices of triangle xyz are the intersections of adjacent n -sectors. Triangle xyz is to be examined for any equilateral properties.

A formal statement of the question is

Conjecture I: The triangle whose vertices are the three intersections of adjacent n -sectors of the angles of a triangle is equilateral for all n . ($n=3,4,\dots$)

Figure 2-1

A necessary, though not sufficient, condition for Conjecture I to be universally true is that it be true in the limit as n goes to infinity. Figure 2-2 illustrates this limiting case where x lies on AB , y lies on BC , and z lies on CA .

Figure 2-2

Figure 2-3

To determine where on AB the point x lies, consider Figure 2-3, which shows two angles, A and B, (The vertex designation and the magnitude of the angle have the same symbol.) and the intersection of their adjacent n-sectors.

The law of sines yields

$$\frac{C_n x_n}{\sin \frac{A}{n}} = \frac{A x_n}{\sin \left(\frac{\pi}{2} - \frac{A}{n} \right)} \quad \text{and} \quad \frac{C_n x_n}{\sin \frac{B}{n}} = \frac{B x_n}{\sin \left(\frac{\pi}{2} - \frac{B}{n} \right)}$$

Thus

$$\frac{A x_n}{B x_n} = \frac{\sin \left(\frac{\pi}{2} - \frac{A}{n} \right)}{\sin \frac{A}{n}} \cdot \frac{\sin \frac{B}{n}}{\sin \left(\frac{\pi}{2} - \frac{B}{n} \right)}$$

Using a trigonometric identity

$$\frac{Ax_n}{Bx_n} = \frac{\cos \frac{A}{n}}{\sin \frac{A}{n}} \cdot \frac{\sin \frac{B}{n}}{\cos \frac{B}{n}}$$

Thus

$$\frac{Ax_n}{Bx_n} = \cot \frac{A}{n} \cdot \tan \frac{B}{n}$$

Now

$$\cot \frac{A}{n} = \frac{1}{\left(\frac{A}{n}\right)} - \frac{1}{3}\left(\frac{A}{n}\right) - \frac{1}{45}\left(\frac{A}{n}\right)^3 - \frac{2}{945}\left(\frac{A}{n}\right)^5 - \dots$$

$$\tan \frac{B}{n} = \left(\frac{B}{n}\right) + \frac{1}{3}\left(\frac{B}{n}\right)^3 + \frac{2}{15}\left(\frac{B}{n}\right)^5 + \dots$$

so that

$$\cot \frac{A}{n} \cdot \tan \frac{B}{n} = \frac{B}{A} + \frac{B^3}{An^2} - \frac{1}{3} \frac{AB}{n^2} + \frac{2}{15} \frac{B^5}{AB^4} - \dots$$

Thus it is easily seen that

$$\lim_{n \rightarrow \infty} \cot \frac{A}{n} \tan \frac{B}{n} = \frac{B}{A}$$

Therefore

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{Ax_n}{Bx_n} = \frac{B}{A}$$

Since the locus of the points, C_n , is not a continuous function of n , it must be shown that C_n can be made arbitrarily close to the limit point x on AB . In Figure 2-3 observe that

$$C_n x_n = AC_n \sin \frac{A}{n}$$

but

$$\sin \frac{A}{n} \leq \frac{A}{n} \text{ for } 0 \leq \frac{A}{n} \leq \frac{\pi}{2}$$

so that

$$C_n x_n \leq AC_n \cdot \frac{A}{n}$$

Now let any $\epsilon > 0$ be given. The Archimedean property¹ for real numbers guarantees that there exists an N_1 such that for $n > N_1$, $\frac{A}{n} < \frac{\pi}{2}$ (A is principal valued) and also that there exists an N_2 such that for $n > N_2$, $AC_n \cdot \frac{A}{n} < \epsilon$. Thus, for $N_0 = \max(N_1, N_2)$ and $n > N_0$, $C_n x_n < \epsilon$. So C_n will approach a limit point on AB . Thus the point x on AB is the limit point of the locus of points C_n . Therefore, the limiting case presented in Figure 2-2 is a valid necessary condition for Conjecture I. Now referring to Figure 2-2, it has been shown that $\frac{Ax}{Bx} = \frac{B}{A}$, $\frac{By}{Cy} = \frac{C}{B}$, and $\frac{Cz}{Az} = \frac{A}{C}$.

¹ Neal H. McCoy, Introduction to Modern Algebra (Boston: Allyn and Bacon, Inc., 1964), p. 85.

It can now be shown by example that the necessary condition for Conjecture I is not satisfied. Consider Figure 2-4 which depicts the limiting triangle xyz of Figure 2-2 for an isosceles triangle whose equal angles, A and B , are large.

Let AB denote the length of the line segment AB . This is not to be confused with the product of two angles A and B , which will be written as $A \cdot B$. Let C recede to infinity along the altitude so as to maintain the isosceles nature of ABC . By virtue of the ratios of the angles A , B , C i.e., $\frac{B}{A}$, $\frac{C}{B}$, and $\frac{A}{C}$ the point x will remain at the midpoint of AB while the points y and z will approach B and A , respectively. Thus yz will approach AB , while xy and xz will each approach $\frac{1}{2}AB$. Therefore, the triangle xyz is not equilateral.

Figure 2-4

So it must be concluded that Conjecture I is not true in general. Theorem 2.1 summarizes this result.

Theorem 2.1: The triangle whose vertices are the three intersections of adjacent n -sectors (for all $n=3,4,\dots$) of the angles of a triangle is not equilateral.

Non-Adjacent n -Sectors

Theorem 2.1 does not apply, however, to the case of a particular value of n , say $n=4$. This case, as well as many others, can be studied by assigning an index to the n -sectors, so that the intersection of non-adjacent n -sectors can be studied. Let the n -sectors of any two angles of a triangle be numbered consecutively starting with zero as their common side, as in Figure 2-5. Now the

Figure 2-5

intersection of adjacent n -sectors corresponds to the intersection of m th n -sectors for $m=1$; the intersection of the first pair of non-adjacent n -sectors corresponds to $m=2$, etc. Note that m cannot exceed $m=n-1$.

Observe that the case of $n=4$, $m=1$ is equivalent to the case of $n=12$, $m=3$. But this case can be studied along with the other cases of $n=12$ $m=1$, $n=12$ $m=2$, $n=12$ $m=4$, etc. Conjecture II generalizes this idea still further.

Conjecture II: There exists a positive integer, m (other than $\frac{n}{3}$) such that the three intersections of the m th n -sectors of the angles of a triangle are the vertices of an equilateral triangle.

The case of $m = \frac{n}{3}$, provided n is divisible by 3, is the case of trisectors which results in Morley's triangle. A more general possibility is suggested by Conjecture III.

Conjecture III: There exists positive integers, m and k , such that the three intersections, formed by intersecting the m th n -sector of one angle of a triangle with the k th n -sector of a second angle of the triangle, are the vertices of an equilateral triangle, where k is the index of the right-hand angle when the triangle is oriented as in Figure 2-5.

An analysis of the more general case of Conjecture III will now be developed. Conjecture II will then be the special case for $m=k$.

Figure 2-6 is a generalization to n -sectors of Lubin's work². The circumscribed circle is drawn for the given triangle, where r is its radius. The intersections

Figure 2-6

²Lubin, op. cit., pp. 110-112.

of the triangle with the circumcircle can be assigned the polar coordinates as shown.

Lubin states the following lemma;³ If two intersecting lines cut the circle $|z|=r$ in the points given by the complex numbers a, b and c, d , respectively, then the point of intersection of the lines is given by

$$(2.2) \quad z = \frac{a^{-1}b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

Application of this formula to Figure 2-6 will yield the polar coordinates of the points x, y , and z , namely Z_x, Z_y , and Z_z . These vector quantities are the vertices of the triangle generated in Conjecture III.

But first let

$$p = e^{\frac{iA}{n}} \quad q = e^{\frac{iB}{n}} \quad s = e^{\frac{i2\pi}{n}}$$

Then

$$(2.3a) \quad Z_x = r \frac{1+p^{-k}q^{-n+k}-p^{-m}q^{-n}}{p^{-k}q^{-n+k}-p^{-m}q^{-n}}$$

$$(2.3b) \quad Z_y = r \frac{p^{-n+q-k}s^k-p^{-n+m}q^{-m-1}}{p^{-n}q^{-k}s^k-p^{-n+m}q^{-m}}$$

³Ibid.

$$(2.3c) \quad Z_Z = r \frac{p^{-n+q-n+m} s^{-m} p^{-n+k} q^{-n}}{p^{-n} q^{-n+m} s^{-m} p^{-n+k} q^{-n}}$$

A better form is obtained by multiplying the numerator and denominator of Z_X by $p^m q^n$, of Z_Y by $p^n q^m$, and of Z_Z by $p^n q^n$.

Thus

$$(2.4a) \quad Z_X = r \frac{p^m q^n + p^{m-k} q^k - q^n - p^m}{p^{m-k} q^k - 1}$$

$$(2.4b) \quad Z_Y = r \frac{q^m + p^n q^{m-k} s^k - p^m - p^n q^m}{q^{m-k} s^k - p^m}$$

$$(2.4c) \quad Z_Z = r \frac{q^n + p^n q^m s^{-m} - p^k q^n - p^n}{q^m s^{-m} - p^k}$$

It is necessary to put equation (2.4) into a form that allows the differences of Z_X , Z_Y , and Z_Z to be taken. Synthetic division will result in a remainder in each of the three expressions; this is unacceptable. An alternate approach is to multiply the numerator and denominator of one expression by the denominators of the other two. Then Z_X , Z_Y , and Z_Z will each have the same denominator.

Thus

$$(2.5a) \quad Z_X = \frac{rN_X}{(p^{m-k} q^k - 1)(q^{m-k} s^k - p^m)(q^m s^{-m} - p^k)}$$

Where

$$\begin{aligned}
 N_X = & p^m q^{n+2m-k} s^k - m + p^{m-k} q^{2m} s^k - m - q^{n+2m-k} s^k - m \\
 & - p^m q^{2m-k} s^k - m - p^{2m} q^{n+m} s^{-m} - p^{2m-k} q^{m+k} s^{-m} \\
 & + p^m q^{n+m} s^{-m} + p^{2m} q^m s^{-m} - p^{m+k} q^{n+m-k} s^k - p^m q^m s^k \\
 & + p^k q^{n+m-k} s^k + p^{m+k} q^{m-k} s^k + p^{2m+k} q^n + p^{2m} q^k \\
 & - p^{m+k} q^n - p^{2m+k}
 \end{aligned}$$

And

$$(2.5b) \quad Z_Y = \frac{r N_Y}{(p^{m-k} q^k - 1) (q^{m-k} s^k - p^m) (q^m s^{-m} - p^k)}$$

Where

$$\begin{aligned}
 N_Y = & p^{m-k} q^{2m+k} s^{-m} + p^{n+m-k} q^{2m} s^k - m - p^{2m-k} q^{m+k} s^{-m} \\
 & - p^{n+m-k} q^{2m+k} s^{-m} - q^{2m} s^{-m} - p^n q^{2m-k} s^k - m \\
 & + p^m q^m s^{-m} + p^n q^{2m} s^{-m} - p^m q^{m+k} - p^{n+m} q^m s^k \\
 & + p^{2m} q^k + p^{n+m} q^{m+k} + p^k q^m + p^{n+k} q^{m-k} s^k \\
 & - p^{m+k} - p^{n+k} q^m
 \end{aligned}$$

And

$$(2.5c) \quad Z_Z = \frac{r N_Z}{(p^{m-k} q^k - 1) (q^{m-k} s^k - p^m) (q^m s^{-m} - p^k)}$$

Where

$$\begin{aligned}
 N_z = & p^{m-k} q^{n+m} s^k + p^{n+m-k} q^{2m} s^{k-m} - p^m q^{n+m} s^k \\
 & - p^{n+m-k} q^m s^k - q^{n+m-k} s^k - p^n q^{2m-k} s^{k-m} \\
 & + p^k q^{n+m-k} s^k + p^n q^{m-k} s^k - p^{2m-k} q^{n+k} \\
 & - p^{n+2m-k} q^{m+k} s^{-m} + p^{2m} q^{n+k} + p^{n+2m-k} q^k \\
 & + p^m q^n + p^{n+m} q^m s^{-m} - p^{m+k} q^n - p^{n+m}
 \end{aligned}$$

Now

$$\vec{xy} = z_y - z_x, \quad \vec{yz} = z_z - z_y, \quad \vec{zx} = z_x - z_z$$

So

$$(2.6a) \quad \vec{xy} = r \frac{N_y - N_x}{(p^{m-k} q^k - 1)(q^{m-k} s^k - p^m)(q^m s^{-m} - p^k)}$$

If $\vec{xy}$, $\vec{yz}$, and $\vec{zx}$ are to be the sides of an equilateral triangle, then these three vectors must be inclined to one another by 60° . Since the vectors 1 , $s^{\frac{n}{3}}$, and $s^{-\frac{n}{3}}$ are each inclined to one another by 60° , the sides $\vec{yz}$ and $\vec{zx}$ can be expressed as

$$(2.6b) \quad \vec{yz} = rs^{\frac{n}{3}} \frac{(N_z - N_y)s^{-\frac{n}{3}}}{(p^{m-k} q^k - 1)(q^{m-k} s^k - p^m)(q^m s^{-m} - p^k)}$$

And

$$(2.6c) \quad \vec{zx} = rs^{-\frac{n}{3}} \frac{(N_x - N_z) s^{\frac{n}{3}}}{(p^{m-k} q^k - 1)(q^{m-k} s^k - p^m)(q^m s^{-m} - p^k)}$$

Let

$$N_{yx} = N_y - N_x, \quad N_{zy} = (N_z - N_y) s^{-\frac{n}{3}}, \quad N_{xz} = (N_x - N_z) s^{\frac{n}{3}}$$

Then

$$(2.7a) \quad N_{yx} = q^{n+2m-k} s^{k-m} - p^m q^{n+2m-k} s^{k-m} - p^m q^{n+m} s^{-m} \\
- p^n q^{2m-k} s^{k-m} - p^k q^{n+m-k} s^k + p^{n+m} q^{m+k} \\
+ p^{2m} q^{n+m} s^{-m} - p^{n+m} q^m s^k - p^{n+k} q^m \\
+ p^{m+k} q^{n+m-k} s^k + p^{m+k} q^n + p^{n+k} q^{m-k} s^k \\
- p^m q^{m+k} - q^{2m} s^{-m} - p^{m-k} q^{2m} s^{k-m} + p^m q^m s^{-m} \\
+ p^k q^m + p^m q^{2m-k} s^{k-m} + p^m q^m s^k + p^{n+m-k} q^{2m} s^{k-m} \\
+ p^n q^{2m} s^{-m} - p^{2m} q^m s^{-m} - p^{m+k} - p^{m+k} q^{m-k} s^k \\
+ p^{2m+k} + p^{m-k} q^{2m+k} s^{-m} - p^{n+m-k} q^{2m+k} s^{-m} \\
- p^{2m+k} q^n$$

$$\begin{aligned}
 (2.7b) \quad N_{zy} = & p \frac{m-k}{q} \frac{n+m}{s} k - \frac{n}{3-p} \frac{m}{q} \frac{n+m}{s} k - \frac{n}{3-p} \frac{2m-k}{q} \frac{n+k}{s} - \frac{n}{3} \\
 & -p \frac{n+m-k}{q} \frac{m}{s} k - \frac{n}{3+p} \frac{k}{q} \frac{n+m-k}{s} k - \frac{n}{3+p} \frac{m}{q} \frac{n}{s} - \frac{n}{3} \\
 & -p \frac{n+2m-k}{q} \frac{m+k}{s} - \frac{n}{3-p} \frac{n+m}{q} \frac{m+k}{s} - \frac{n}{3+p} \frac{2m}{q} \frac{n+k}{s} - \frac{n}{3} \\
 & +p \frac{n+2m-k}{q} \frac{k}{s} - \frac{n}{3+p} \frac{n+m}{q} \frac{m}{s} - \frac{n}{3+p} \frac{n+m}{q} \frac{m}{s} k - \frac{n}{3} \\
 & +p \frac{n+k}{q} \frac{m}{s} - \frac{n}{3-p} \frac{m+k}{q} \frac{n}{s} - \frac{n}{3-p} \frac{n+m}{s} - \frac{n}{3} \\
 & -p \frac{n+k}{q} \frac{m-k}{s} k - \frac{n}{3+p} \frac{2m-k}{q} \frac{m+k}{s} - \frac{n}{3+p} \frac{m}{q} \frac{m+k}{s} - \frac{n}{3} \\
 & +q \frac{2m}{s} - \frac{n}{3-p} \frac{m}{q} \frac{m}{s} - \frac{n}{3-p} \frac{k}{q} \frac{m}{s} - \frac{n}{3-p} \frac{n}{q} \frac{2m}{s} - \frac{n}{3} \\
 & -p \frac{2m}{q} \frac{k}{s} - \frac{n}{3-p} \frac{m+k}{s} - \frac{n}{3+p} \frac{n}{q} \frac{m-k}{s} k - \frac{n}{3} \\
 & -q \frac{n+m-k}{s} k - \frac{n}{3-p} \frac{m-k}{q} \frac{2m+k}{s} - \frac{n}{3+p} \frac{n+m-k}{q} \frac{2m+k}{s} - \frac{n}{3}
 \end{aligned}$$

$$\begin{aligned}
 (2.7c) \quad N_{xz} = & -q \frac{n+2m-k}{s} k - \frac{n}{3-p} \frac{m-k}{q} \frac{n+m}{s} k + \frac{n}{3} \\
 & +p \frac{m}{q} \frac{n+2m-k}{s} k - \frac{n}{3+p} \frac{m}{q} \frac{n+m}{s} - \frac{n}{3+p} \frac{m}{q} \frac{n+m}{s} k + \frac{n}{3} \\
 & +p \frac{2m-k}{q} \frac{n+k}{s} - \frac{n}{3+p} \frac{n+m-k}{q} \frac{m}{s} k + \frac{n}{3+p} \frac{n}{q} \frac{2m-k}{s} k - \frac{n}{3} \\
 & -p \frac{m}{q} \frac{n}{s} - \frac{n}{3+p} \frac{n+2m-k}{q} \frac{m+k}{s} - \frac{n}{3-p} \frac{2m}{q} \frac{n+m}{s} - \frac{n}{3} \\
 & -p \frac{2m}{q} \frac{n+k}{s} - \frac{n}{3-p} \frac{n+2m-k}{q} \frac{k}{s} - \frac{n}{3-p} \frac{n+m}{q} \frac{m}{s} - \frac{n}{3} \\
 & -p \frac{m+k}{q} \frac{n+m-k}{s} k + \frac{n}{3+p} \frac{n+m}{s} - \frac{n}{3-p} \frac{2m-k}{q} \frac{m+k}{s} - \frac{n}{3} \\
 & +p \frac{m-k}{q} \frac{2m}{s} k - \frac{n}{3-p} \frac{m}{q} \frac{2m-k}{s} k - \frac{n}{3-p} \frac{m}{q} \frac{m}{s} k + \frac{n}{3}
 \end{aligned}$$

$$\begin{aligned}
& -p^{n+m-k} q^{2m} s^{k-m+\frac{n}{3}} p^{2m} q^m s^{-m+\frac{n}{3}} p^{2m} q^k s^{\frac{n}{3}} \\
& +p^{m+k} q^{m-k} s^{k+\frac{n}{3}} p^n q^{m-k} s^{k+\frac{n}{3}} p^{2m+k} s^{\frac{n}{3}} \\
& +q^{n+m-k} s^{k+\frac{n}{3}} p^{2m+k} q^n s^{\frac{n}{3}}
\end{aligned}$$

The foregoing results are summarized in the following theorem.

Theorem 2.2: The sides of the triangle generated in Conjecture III are the vector quantities $\vec{xy}$, $\vec{yz}$, and $\vec{zx}$ given by equations (2.6) and (2.7).

For triangle xyz to be equilateral, it is required that

$$(2.8) \quad N_{yx} = N_{zy} = N_{xz}$$

in both magnitude and phase.

The expressions (2.7) are rather formidable. Even for the special case of $m=k$ (Conjecture II) a solution to (2.8) is not readily apparent. Perhaps some insight could be gained by considering the case of trisectors for which $n=3$ and $m=k=1$.

Denoting equations (2.7) by $N_{yx}(3,1,1)$, $N_{zy}(3,1,1)$, and $N_{xz}(3,1,1)$ for this case, (2.7) become

$$\begin{aligned}
 (2.9a) \quad N_{yx}(3,1,1) = & q^4 - pq^4(1+s^{-1}) - p^3q - pq^3s + p^4q^2 \\
 & + p^2q^4s^{-1} - p^4q(s+1) + p^2q^3(s+1) + p^4s \\
 & - pq^2 - q^2(s^{-1}+1) + pq(s^{-1}+2+s) \\
 & + p^3q^2(1+s^{-1}) - p^2qs^{-1} - p^2(1+s) \\
 & + p^3 + q^3s^{-1} - p^3q^3(s^{-1}+1)
 \end{aligned}$$

$$\begin{aligned}
 (2.9b) \quad N_{zy}(3,1,1) = & q^4 - pq^4(1+s^{-1}) - p^3q + pq^3(1+s^{-1}) \\
 & - p^4q^2(s^{-2}+s^{-1}) + p^2q^4s^{-1} \\
 & + p^4q(s^{-1}+s^{-2}+1+s^{-1}) - p^2q^3s^{-1} \\
 & - p^4(s^{-1}+1) + pq^2(s^{-2}+s^{-1}) + q^2s^{-2} \\
 & - pq(s^{-2}+s^{-1}) - p^3q^2s^{-2} - p^2qs^{-1} \\
 & + p^2s^{-1} + p^3 - q^3(1+s^{-2}) + p^3q^3s^{-2}
 \end{aligned}$$

$$\begin{aligned}
 (2.9c) \quad N_{xz}(3,1,1) = & -q^4(s+s^2) + pq^4(s+1+s^2+s) \\
 & + p^3q(s^2+s) - pq^3s + p^4q^2 - p^2q^4(1+s) \\
 & - p^4q(s+1) - p^2q^3s^2 + p^4s - pq^2 + q^2s \\
 & - pq(s+s^2) - p^3q^2s + p^2q(1+s) + p^2s^2 \\
 & - p^3(s^2+s) + q^3s^2 + p^3q^3s
 \end{aligned}$$

Recalling that $s = e^{i\frac{2\pi}{n}}$, one observes that for $n=3$, $s = e^{i\frac{2\pi}{3}}$ and thus

$$(2.10) \quad 1 + s + s^{-1} = 0$$

Also note

$$(2.11) \quad s^{-1} = s^2 \pmod{3}$$

By making the indicated substitutions, it can be shown that

$$(2.12) \quad N_{yx}(3, 1, 1) = N_{zy}(3, 1, 1) = N_{xz}(3, 1, 1) =$$

$$\begin{aligned} & q^4 + pq^4s - p^3q - pq^3s + p^4q^2 + p^2q^4s^{-1} + p^4qs^{-1} \\ & + p^2q^3s^{-1} + p^4s - pq^2 + q^2s + pq - p^3q^2s - p^2qs^{-1} \\ & + p^2s^{-1} + p^3 + q^3s^{-1} + p^3q^3s \end{aligned}$$

So for the case of trisectors, equations (2.7) are not termwise identical, but they are reducible to termwise identical polynomials.

Thus if an m exists such that Conjecture II is true, there must exist identities similar to (2.10) and (2.11) to aid in the reduction of (2.7).⁴ However, the last four terms of (2.7) present a basic limitation. The last four terms are

⁴It can be shown for the case $m=k$ that if $m = (3\ell + 1)\frac{n}{3}$, where ℓ is a non-negative integer, that the first twenty-four terms of equations (2.7a), (2.7b), and (2.7c) are reducible to three polynomials which are termwise identical.

for (2.7a)

$$p^{3m+q} s^{3m-m} - p^n q^{3m-m} - p^{3m} q^n$$

for (2.7b)

$$p^n s^{m-\frac{n}{3}-q} s^{m-\frac{n}{3}-q} s^{3m-m-\frac{n}{3}+p} q^n s^{3m-m-\frac{n}{3}}$$

for (2.7c)

$$-p^n s^{m+\frac{n}{3}-p} s^{3m-\frac{n}{3}+q} s^{m+\frac{n}{3}+p} q^{3m} s^{\frac{n}{3}}$$

It is apparent that for arbitrary p and q , these expressions are equal only for $n=3m$. Thus it must be concluded that Conjecture II is not generally true. This result is contained in the following theorem.

Theorem 2.3: There is no integer m , other than $m=\frac{n}{3}$ such that the three intersections of the m th n -sectors of the angles of a triangle are the vertices of an equilateral triangle.

In the light of the foregoing results, it seems hopeless to consider Conjecture III, where an added variable is introduced. However, there is some information to be gained. Again, first consider the limiting case of Conjecture III as a necessary condition for all n .

Referring to Figure 2-3, if C_n were the intersection of $\frac{A}{m}$ and $\frac{B}{kn}$, where m and k are positive integers, then clearly,

$$\lim_{n \rightarrow \infty} \frac{Ax_n - kB}{Bx_n - mA}$$

Let $h = \frac{k}{m}$. Thus in Figure 2-2, $\frac{Ax}{Bx} = \frac{hB}{A}$, $\frac{By}{Cy} = \frac{hC}{B}$, and $\frac{Cz}{Az} = \frac{hA}{C}$.

$$\text{Since } Ax+Bx=AB, \frac{Ax}{AB} = \frac{\frac{hB}{A}}{1+\frac{hB}{A}}, \text{ and } \frac{Bx}{AB} = \frac{1}{1+\frac{hB}{A}}.$$

$$\text{Similarly, } \frac{By}{BC} = \frac{\frac{hC}{B}}{1+\frac{hC}{B}} \text{ and } \frac{Cy}{BC} = \frac{1}{1+\frac{hC}{B}}.$$

$$\text{And } \frac{Cz}{AC} = \frac{\frac{hA}{C}}{1+\frac{hA}{C}} \text{ and } \frac{Az}{AC} = \frac{1}{1+\frac{hA}{C}}.$$

By the law of cosines

$$(xz)^2 = (Az)^2 + (Ax)^2 - 2(Az)(Ax)\cos A$$

$$= \left(\frac{AC}{1+\frac{hA}{C}} \right)^2 + \left(\frac{AB\frac{hB}{A}}{1+\frac{hB}{A}} \right)^2 - 2 \frac{(AC)(AB)\frac{hB}{A}}{\left(1+\frac{hA}{C}\right)\left(1+\frac{hB}{A}\right)} \cos A$$

Again the law of cosines yields

$$(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos A$$

which implies

$$\cos A = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)}$$

so

$$\begin{aligned}
 (xz)^2 &= \frac{(AC)^2}{\left(1+\frac{hA}{C}\right)^2} + \frac{(AB)^2 \left(\frac{hB}{A}\right)^2}{\left(1+\frac{hB}{A}\right)^2} - \frac{2(AC)(AB)\frac{hB}{A}}{\left(1+\frac{hA}{C}\right)\left(1+\frac{hB}{A}\right)} \cdot \frac{(AB)^2 + (AC)^2 - (BC)^2}{2AB \cdot AC} \\
 &= \frac{(AC)^2}{\left(1+\frac{hA}{C}\right)^2} + \frac{(AB)^2 \left(\frac{hB}{A}\right)^2}{\left(1+\frac{hB}{A}\right)^2} - \frac{\frac{hB}{A} \left((AB)^2 + (AC)^2 - (BC)^2 \right)}{\left(1+\frac{hA}{C}\right)\left(1+\frac{hB}{A}\right)}
 \end{aligned}$$

which reduces to

(2.13a)

$$(xz)^2 = (AB)^2 \frac{\frac{hB}{A} \left(\frac{h^2 B}{C} - 1 \right)}{\left(1+\frac{hB}{A}\right)^2 \left(1+\frac{hA}{C}\right)} + (BC)^2 \frac{\frac{hB}{A}}{\left(1+\frac{hA}{C}\right)\left(1+\frac{hB}{A}\right)} + (AC)^2 \frac{1 - \frac{h^2 B}{C}}{\left(1+\frac{hA}{C}\right)^2 \left(1+\frac{hB}{A}\right)}$$

Similarly

(2.13b)

$$(xy)^2 = (AB)^2 \frac{1 - \frac{h^2 C}{A}}{\left(1+\frac{hB}{A}\right)^2 \left(1+\frac{hC}{B}\right)} + (BC)^2 \frac{\frac{hC}{B} \left(\frac{h^2 C}{A} - 1 \right)}{\left(1+\frac{hC}{B}\right)^2 \left(1+\frac{hB}{A}\right)} + (AC)^2 \frac{\frac{hC}{B}}{\left(1+\frac{hB}{A}\right)\left(1+\frac{hC}{B}\right)}$$

and

(2.13c)

$$(yz)^2 = (AB)^2 \frac{\frac{hA}{C}}{\left(1+\frac{hC}{B}\right)\left(1+\frac{hA}{C}\right)} + (BC)^2 \frac{1 - \frac{h^2 A}{B}}{\left(1+\frac{hC}{B}\right)^2 \left(1+\frac{hA}{C}\right)} + (AC)^2 \frac{\frac{hA}{C} \left(\frac{h^2 A}{B} - 1 \right)}{\left(1+\frac{hA}{C}\right)^2 \left(1+\frac{hC}{B}\right)}$$

The question now is whether an h exists such that $xz=xy=yz$. The existence of such an h will present two possibilities. First, k and m may be finite, in which case the intersection of their n -sectors for large n will approach the limiting case, and the necessary condition for Conjecture III to be true will be satisfied. Second, k and m may become infinite (even though their ratio is finite) as n becomes infinite; hence, the limiting case is not a valid necessary condition for Conjecture III. It will now be shown that such an h does exist.

It is observed in equations (2.13) that the determinant of the matrix of coefficients of $(AB)^2$, $(BC)^2$, and $(AC)^2$ reduces to

$$h^9 + 3h^6 + 3h^3 + 1$$

and is equal to zero for only one real number, namely $h=-1$. Thus the inverse matrix exists for all $h > 0$ and represents a system of equations linear in $(xz)^2$, $(xy)^2$, $(yz)^2$. So, for $xz=xy=yz$ this system can be solved for h in terms of the angles (A , B , and C) and the sides (AB , BC and AC) of the given triangle.⁵ This expression for h will be quite unwieldy, and even so, will not yield any finite properties

⁵A non-trivial solution must exist, for in the general case triangle ABC is not equilateral and $h=0$ implies that triangle xyz coincides with ABC . Further, only rational solutions, h , would have any meaning.

of k and m . Thus it is indeterminate that the necessary condition for Conjecture III is satisfied. Hence, Conjecture III must remain unanswered for the general case. This case will be discussed again in Chapter IV.

It is interesting to note that, due to the symmetry of equations (2.13), if ABC is an equilateral triangle, then $xz=xy=yz$ for all values of h . Thus one is motivated to consider equations (2.7) for $p=s^{\frac{1}{3}}$ and $q=s^{\frac{2}{3}} \pmod{n}$. Not too surprisingly, it is found that

$$\begin{aligned}
 N_{yx}=N_{zx}=N_{xz}= & 1+2s^{\frac{1}{3}}(2n+m+k) -2s^{\frac{1}{3}}(2n+2m+k) -s^{\frac{1}{3}}(2n) \\
 & -s^{\frac{1}{3}}(n+m+k) -s^{\frac{1}{3}}(2n+2m+2k) +s^{\frac{1}{3}}(n+3m+2k) \\
 & +s^{\frac{1}{3}}(2n+m) -s^{\frac{1}{3}}(n+3m+3k) -2s^{\frac{1}{3}}(n+2m+k) \\
 & +s^{\frac{1}{3}}(2n+3m+2k) +2s^{\frac{1}{3}}(n+2m+2k) -2s^{\frac{1}{3}}(3m+2k) \\
 & -2s^{\frac{1}{3}m} -s^{\frac{1}{3}}(2m+2k) +4s^{\frac{1}{3}}(2m+k) +s^{\frac{1}{3}}(3m+3k) \\
 & +s^{\frac{1}{3}}(n+m) -s^{\frac{1}{3}}(m+k)
 \end{aligned}$$

Thus if triangle ABC is equilateral, so is triangle xyz for all values of n , m , and k . Thus the following is proved:

Theorem 2.4: The triangle whose vertices are the intersections of the m th n -sector of one angle with the k th n -sector of another angle of an equilateral triangle, is equilateral for all values of m , k , and n .

A Trigonometric Approach

It seems desirable at this point to introduce an alternate approach for the vector analysis used thus far. A trigonometric analysis will now be demonstrated for the case of adjacent 4-sectors. The lengths of the sides of the generated triangle will be found.

The case of adjacent 4-sectors is represented in Figure 2-7. By elementary considerations, it is clear that

Figure 2-7

$$\frac{AD}{BD} = \cot \alpha \tan \beta$$

which implies that

$$\frac{AD}{AB} = \frac{\cot \alpha \tan \beta}{1 + \cot \alpha \tan \beta}$$

Now

$$Ax = \frac{AD}{\cos \alpha}$$

so

$$Ax = \frac{AB}{\cos \alpha} \cdot \frac{\cot \alpha \tan \beta}{1 + \cot \alpha \tan \beta}$$

or

$$(2.14a) \quad Ax = AB \frac{\sin \beta}{\sin(\alpha + \beta)}$$

Similarly it is found that

$$(2.14b) \quad Az = AC \frac{\sin \gamma}{\sin(\alpha + \gamma)}$$

The law of cosines for triangle Axz yields

$$(2.15) \quad (xz)^2 = (Ax)^2 + (Az)^2 - 2(Ax)(Az) \cos 2\alpha$$

Substituting (2.14) into (2.15) gives the square of the side xz of the triangle xyz:

(2.16a)

$$(xz)^2 = (AB)^2 \frac{\sin^2 \beta}{\sin^2(\alpha + \beta)} + (AC)^2 \frac{\sin^2 \gamma}{\sin^2(\alpha + \gamma)} - 2(AB)(AC) \frac{\sin \beta \sin \gamma \cos 2\alpha}{\sin(\alpha + \beta) \sin(\alpha + \gamma)}$$

Similarly, or by permuting the symbols ABC and $\alpha\beta\gamma$, the other two sides can be found.

(2.16b)

$$(xy)^2 = (BC)^2 \frac{\sin^2 \gamma}{\sin^2(\beta + \gamma)} + (AB)^2 \frac{\sin^2 \alpha}{\sin^2(\alpha + \beta)} - 2(BC)(AB) \frac{\sin \gamma \sin \alpha \cos 2\beta}{\sin(\beta + \gamma) \sin(\alpha + \beta)}$$

and

(2.16c)

$$(yz)^2 = (AC)^2 \frac{\sin^2 \alpha}{\sin^2(\alpha + \gamma)} + (BC)^2 \frac{\sin^2 \beta}{\sin^2(\beta + \gamma)} - 2(AC)(BC) \frac{\sin \alpha \sin \beta \cos 2\gamma}{\sin(\alpha + \gamma) \sin(\beta + \gamma)}$$

The two equalities,

$$(2.17a) \quad (xz)^2 = (xy)^2$$

$$(2.17b) \quad (xy)^2 = (yz)^2$$

suffice as conditions that triangle xyz be equilateral.

These equations can be simplified to an expression involving only sines and cosines of the angles. If this resulting expression is a trigonometric identity, then triangle xyz is equilateral for any triangle ABC. If the expression is not a trigonometric identity, then it represents a restriction of the triangle ABC so that triangle xyz is equilateral. This particular case of 4-sectors will be considered again in Chapter IV.

Summary

This chapter has dealt with the many intersections of the n-sectors of the angles of a triangle. The intersections

of certain n -sectors which result in an equilateral triangle have been studied.

An analysis has been developed to study the general case of a m th n -sector intersecting with a k th n -sector. Expressions for the sides of the resulting triangle have been derived.

It has been shown that the case of adjacent ($m=k=1$) n -sectors and arbitrary n ($n > 3$) does not yield an equilateral triangle. Further, the case $m=k$ yields an equilateral triangle only for $m=\frac{n}{3}$. The more general case for $m \neq k$ and arbitrary n proved inconclusive. For the special case of the given triangle being equilateral, the triangle formed by intersecting n -sectors is equilateral for all values of m , k , and n .

A case of intersecting adjacent 4-sectors was discussed, primarily, to demonstrate an alternate analysis to the vector analysis used previously.

CHAPTER III

THE QUADRILATERAL

It has been shown in the literature¹ that the intersections of the trisectors (indeed, n -sectors) of the angles of a square are the vertices of a square. It is the purpose of this chapter to consider the more general quadrilateral.

A general quadrilateral, together with the trisectors of the angles, is shown in Figure 3-1. Let a coordinate

Figure 3-1

¹ Smart, op. cit., pp. 459-463.

system be chosen with A as the origin and B lying on the positive x axis. The direction perpendicular to AB toward the interior of the quadrilateral is the positive y axis. Denote the abscissa by subscript "a" and the ordinate by subscript "o".

Thus the coordinates of the points w, x, y, and z are

$$(3.1a) \quad w_a = AB \frac{\cos \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$(3.1b) \quad w_o = AB \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$(3.1c) \quad x_a = AB - BC \frac{\sin \gamma \cos 2\beta}{\sin(\beta + \gamma)}$$

$$(3.1d) \quad x_o = BC \frac{\sin \gamma \sin 2\beta}{\sin(\beta + \gamma)}$$

$$(3.1e) \quad y_a = AD \cos 3\alpha - CD \frac{\sin \gamma \cos(3\alpha + 2\delta)}{\sin(\gamma + \delta)}$$

$$(3.1f) \quad y_o = AD \sin 3\alpha - CD \frac{\sin \gamma \sin(3\alpha + 2\delta)}{\sin(\gamma + \delta)}$$

$$(3.1g) \quad z_a = AD \frac{\cos 2\alpha \sin \delta}{\sin(\alpha + \delta)}$$

$$(3.1h) \quad z_o = AD \frac{\sin 2\alpha \sin \delta}{\sin(\alpha + \delta)}$$

Expressions can be written for the lengths of the sides or the slopes of the sides of the figure wxyz.

However, these expressions are unwieldy and inconclusive.

Simplification is achieved by further restricting the given quadrilateral ABCD.

The Parallelogram

First, consider the case when ABCD is a parallelogram.

Now

$$\alpha = \gamma \quad \text{and} \quad \beta = \delta$$

Also

$$AB=DC \quad \text{and} \quad AD=BC$$

Note also that

$$\alpha + \beta = \frac{\pi}{3}$$

Hence equations (3.1) reduce to

$$(3.2a) \quad w_a = AB \frac{\cos \alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

$$(3.2b) \quad w_o = AB \frac{\sin \alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

$$(3.2c) \quad x_a = AB - AD \frac{\sin \alpha \cos(\frac{2\pi}{3} - 2\alpha)}{\sin \frac{\pi}{3}}$$

$$(3.2d) \quad x_o = AD \frac{\sin \alpha \sin(\frac{2\pi}{3} - 2\alpha)}{\sin \frac{\pi}{3}}$$

$$(3.2e) \quad y_a = AD \cos 3\alpha - AB \frac{\sin \alpha \cos(\alpha + \frac{2\pi}{3})}{\sin \frac{\pi}{3}}$$

$$(3.2f) \quad y_O = AD \sin 3\alpha - AB \frac{\sin \alpha \sin(\alpha + \frac{2\pi}{3})}{\sin \frac{\pi}{3}}$$

$$(3.2g) \quad z_a = AD \frac{\cos 2\alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

$$(3.2h) \quad z_O = AD \frac{\sin 2\alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

Now the slope of the line xw can be found.

Since (3.3a) $x_O - w_O =$

$$AD \frac{\sin \alpha \sin(\frac{2\pi}{3} - 2\alpha)}{\sin \frac{\pi}{3}} - AB \frac{\sin \alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

and (3.3b) $x_a - w_a =$

$$AB \frac{\sin \frac{\pi}{3} - \cos \alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}} - AD \frac{\sin \alpha \cos(\frac{2\pi}{3} - 2\alpha)}{\sin \frac{\pi}{3}}$$

The slope of xw is

$$\frac{x_O - w_O}{x_a - w_a} =$$

$$\frac{-AB \sin \alpha \sin(\frac{\pi}{3} - \alpha) + AD \sin \alpha \sin(\frac{2\pi}{3} - 2\alpha)}{AB \left[\sin \frac{\pi}{3} - \cos \alpha \sin(\frac{\pi}{3} - \alpha) \right] - AD \sin \alpha \cos(\frac{2\pi}{3} - 2\alpha)}$$

Similarly the slope of yz is found.

$$(3.4a) \quad y_o - z_o =$$

$$AD \left[\sin 3\alpha - \frac{\sin 2\alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}} \right] - AB \frac{\sin \alpha \sin(\alpha + \frac{2\pi}{3})}{\sin \frac{\pi}{3}}$$

$$(3.4b) \quad y_a - z_a =$$

$$AD \left[\cos 3\alpha - \frac{\cos 2\alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}} \right] - AB \frac{\sin \alpha \cos(\alpha + \frac{2\pi}{3})}{\sin \frac{\pi}{3}}$$

so that the slope of yz is

$$\frac{y_o - z_o}{y_a - z_a} =$$

$$\frac{AD \left[\sin 3\alpha \sin \frac{\pi}{3} - \sin 2\alpha \sin(\frac{\pi}{3} - \alpha) \right] - AB \sin \alpha \sin(\alpha + \frac{2\pi}{3})}{AD \left[\cos 3\alpha \sin \frac{\pi}{3} - \cos 2\alpha \sin(\frac{\pi}{3} - \alpha) \right] - AB \sin \alpha \cos(\alpha + \frac{2\pi}{3})}$$

However, the slopes of xw and yz are equal since the coefficients of AB and AD in both the numerators and denominators of the slope expressions are equal. That is

$$(3.5a) \quad \sin \alpha \sin(\frac{\pi}{3} - \alpha) = \sin \alpha \sin(\alpha + \frac{2\pi}{3})$$

$$(3.5b) \quad \sin \alpha \sin(\frac{2\pi}{3} - 2\alpha) = \sin 3\alpha \sin \frac{\pi}{3} - \sin 2\alpha \sin(\frac{\pi}{3} - \alpha)$$

$$(3.5c) \quad \sin \frac{\pi}{3} - \cos \alpha \sin(\frac{\pi}{3} - \alpha) = -\sin \alpha \cos(\alpha + \frac{2\pi}{3})$$

$$(3.5d) \quad -\sin \alpha \cos(\frac{2\pi}{3} - 2\alpha) = \cos 3\alpha \sin \frac{\pi}{3} - \cos 2\alpha \sin(\frac{\pi}{3} - \alpha)$$

The validity of (3.5) must be shown. (3.5a) is true since, for any θ , $\sin \theta = \sin(\pi - \theta)$.

Thus

$$\sin(\alpha + \frac{2\pi}{3}) = \sin(\pi - \alpha - \frac{2\pi}{3}) = \sin(\frac{\pi}{3} - \alpha)$$

(3.5b) is true, since the three terms can be expanded using a trigonometric identity.

Thus

$$\sin \alpha \sin(\frac{2\pi}{3} - 2\alpha) = \frac{1}{2} \left[\cos(3\alpha - \frac{2\pi}{3}) - \cos(\frac{2\pi}{3} - \alpha) \right]$$

$$\sin 3\alpha \sin \frac{\pi}{3} = \frac{1}{2} \left[\cos(3\alpha - \frac{\pi}{3}) - \cos(3\alpha + \frac{\pi}{3}) \right]$$

$$\sin 2\alpha \sin(\frac{\pi}{3} - \alpha) = \frac{1}{2} \left[\cos(3\alpha - \frac{\pi}{3}) - \cos(\alpha + \frac{\pi}{3}) \right]$$

Thus (3.5b) implies

$$\cos(3\alpha - \frac{2\pi}{3}) - \cos(\frac{2\pi}{3} - \alpha) =$$

$$\cos(3\alpha - \frac{\pi}{3}) - \cos(3\alpha + \frac{\pi}{3}) - \cos(3\alpha - \frac{\pi}{3}) + \cos(\alpha + \frac{\pi}{3})$$

But $\cos \theta = -\cos(\pi + \theta) = -\cos(\pi - \theta)$ for any θ . So, the left-hand side becomes

$$\cos(3\alpha - \frac{2\pi}{3}) - \cos(\frac{2\pi}{3} - \alpha) = -\cos(3\alpha + \frac{\pi}{3}) + \cos(\frac{\pi}{3} + \alpha)$$

(3.5c) is true since rearrangement of the terms yields

$$-\sin \frac{\pi}{3} = -\cos \alpha \sin(\frac{2\pi}{3} + \alpha) + \sin \alpha \cos(\frac{2\pi}{3} + \alpha)$$

So

$$\begin{aligned}
 -\sin\frac{\pi}{3} &= \sin(\alpha - \frac{2\pi}{3} - \alpha) \\
 &= -\sin\frac{2\pi}{3} \\
 &= -\sin(\pi - \frac{2\pi}{3}) \\
 &= -\sin\frac{\pi}{3}
 \end{aligned}$$

(3.5d) is true, since the three terms can be expanded by a trigonometric identity.

Thus

$$\begin{aligned}
 \sin\alpha \cos(\frac{2\pi}{3} - 2\alpha) &= \frac{1}{2} \left[\sin(3\alpha - \frac{2\pi}{3}) + \sin(\frac{2\pi}{3} - \alpha) \right] \\
 \cos 3\alpha \sin\frac{\pi}{3} &= \frac{1}{2} \left[\sin(\frac{\pi}{3} - 3\alpha) + \sin(\frac{\pi}{3} + 3\alpha) \right] \\
 \cos 2\alpha \sin(\frac{\pi}{3} - \alpha) &= \frac{1}{2} \left[\sin(\frac{\pi}{3} - 3\alpha) + \sin(\frac{\pi}{3} + \alpha) \right]
 \end{aligned}$$

So (3.5d) implies

$$\begin{aligned}
 &-\sin(3\alpha - \frac{2\pi}{3}) - \sin(\frac{2\pi}{3} - \alpha) = \\
 &\sin(\frac{\pi}{3} - 3\alpha) + \sin(\frac{\pi}{3} + 3\alpha) - \sin(\frac{\pi}{3} - 3\alpha) - \sin(\frac{\pi}{3} + \alpha)
 \end{aligned}$$

Since $\sin\theta = \sin(\pi - \theta) = -\sin(\pi + \theta)$, the left-hand side becomes

$$-\sin(3\alpha - \frac{2\pi}{3}) - \sin(\frac{2\pi}{3} - \alpha) = \sin(\frac{\pi}{3} + 3\alpha) - \sin(\frac{\pi}{3} + \alpha)$$

Thus, xw and yz are parallel.

In a similar manner, the slopes of the lines xy and wz will now be found and shown to be equal. From (3.2)

$$(3.6a) \quad y_O - x_O =$$

$$AD \left[\sin 3\alpha - \frac{\sin \alpha \sin(\frac{2\pi}{3} - 2\alpha)}{\sin \frac{\pi}{3}} \right] - AB \frac{\sin \alpha \sin(\alpha + \frac{2\pi}{3})}{\sin \frac{\pi}{3}}$$

and

$$(3.6b) \quad y_a - x_a =$$

$$AD \left[\cos 3\alpha + \frac{\sin \alpha \sin(\frac{2\pi}{3} - 2\alpha)}{\sin \frac{\pi}{3}} \right] - AB \left[1 + \frac{\sin \alpha \cos(\alpha + \frac{2\pi}{3})}{\sin \frac{\pi}{3}} \right]$$

So the slope of xy is

$$\frac{y_O - x_O}{y_a - x_a} =$$

$$\frac{AD \left[\sin 3\alpha \sin \frac{\pi}{3} - \sin \alpha \sin(\frac{2\pi}{3} - 2\alpha) \right] - AB \sin \alpha \sin(\alpha + \frac{2\pi}{3})}{AD \left[\cos 3\alpha \sin \frac{\pi}{3} + \sin \alpha \cos(\frac{2\pi}{3} - 2\alpha) \right] - AB \left[\sin \frac{\pi}{3} + \sin \alpha \cos(\alpha + \frac{2\pi}{3}) \right]}$$

For the line wz

$$(3.7a) \quad z_O - w_O = AD \frac{\sin 2\alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}} - AB \frac{\sin \alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

and

$$(3.7b) \quad z_a - w_a = AD \frac{\cos 2\alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}} - AB \frac{\cos \alpha \sin(\frac{\pi}{3} - \alpha)}{\sin \frac{\pi}{3}}$$

so the slope of wz is

$$\frac{z_o - w_o}{z_a - w_a} = \frac{AD \sin 2\alpha \sin(\frac{\pi}{3} - \alpha) - AB \sin \alpha \sin(\frac{\pi}{3} - \alpha)}{AD \cos 2\alpha \sin(\frac{\pi}{3} - \alpha) - AB \cos \alpha \sin(\frac{\pi}{3} - \alpha)}$$

Equating the coefficients of AD and AB of the slope expressions yields

$$(3.8a) \quad \sin 2\alpha \sin(\frac{\pi}{3} - \alpha) = \sin 3\alpha \sin \frac{\pi}{3} - \sin \alpha \sin(\frac{2\pi}{3} - 2\alpha)$$

$$(3.8b) \quad \sin \alpha \sin(\frac{\pi}{3} - \alpha) = \sin \alpha \sin(\alpha + \frac{2\pi}{3})$$

$$(3.8c) \quad \cos 2\alpha \sin(\frac{\pi}{3} - \alpha) = \cos 3\alpha \sin \frac{\pi}{3} + \sin \alpha \cos(\frac{2\pi}{3} - 2\alpha)$$

$$(3.8d) \quad \cos \alpha \sin(\frac{\pi}{3} - \alpha) = \sin \frac{\pi}{3} + \sin \alpha \cos(\alpha + \frac{2\pi}{3})$$

But these expressions are the same as (3.5) which have already been validated. So xy and wz are parallel also.

Thus the following theorem is true:

Theorem 3.1: The intersections of the adjacent trisectors of the angles of a parallelogram are the vertices of a parallelogram.

In general, the parallelogram generated in Theorem 3.1 is not a rhombus, as can be seen by considering the following example.

Figure 3-2a represents the results of Theorem 3.1. As the given parallelogram ABCD degenerates into the straight line AB by allowing the sides AD and BC to approach zero length, angle $\angle zxy$ approaches α , while angle $\angle zxw$ approaches β . The limiting case is shown in Figure 3-2b. Thus, a parallelogram $wxyz$ is generated whose diagonal xz does not bisect the angle wxy , since $\alpha \neq \beta$. Therefore, $wxyz$ is not a rhombus.

The Rectangle

Consider, now, the case when ABCD in Figure 3-1 is a rectangle. It will be shown that the lengths of the two adjacent sides of $wxyz$ are equal so that $wxyz$ is a rhombus. Thus the following theorem will be proved.

Theorem 3.2: The intersections of the adjacent trisectors of the angles of a rectangle are the vertices of a rhombus.

Since ABCD is a rectangle, it is a parallelogram with $\alpha = \beta = \frac{\pi}{6}$. Then the length of the line xw is

$$xw = \left[(x_a - w_a)^2 + (x_o - w_o)^2 \right]^{\frac{1}{2}}$$

Utilizing (3.5), equations (3.3) for this case become

$$x_o - w_o = \frac{-AB \sin^2 \frac{\pi}{6} + AD \sin \frac{\pi}{6} \sin \frac{\pi}{3}}{\sin \frac{\pi}{3}}$$

and

$$x_a - w_a = \frac{AB \sin \frac{\pi}{6} \cos \frac{\pi}{6} - AD \sin \frac{\pi}{6} \cos \frac{\pi}{3}}{\sin \frac{\pi}{3}}$$

so

$$\begin{aligned} xw &= \frac{1}{\sin \frac{\pi}{3}} \left[(AB)^2 \left(\sin^2 \frac{\pi}{6} \cos^2 \frac{\pi}{6} + \sin^4 \frac{\pi}{6} \right) \right. \\ &\quad \left. + (AD)^2 \left(\sin^2 \frac{\pi}{6} \cos^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{6} \sin^2 \frac{\pi}{3} \right) \right. \\ &\quad \left. - 2(AB)(AD) \left(\sin^2 \frac{\pi}{6} \cos \frac{\pi}{6} \cos \frac{\pi}{3} + \sin^3 \frac{\pi}{6} \sin \frac{\pi}{3} \right) \right]^{\frac{1}{2}} \\ xw &= \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} \left[(AB)^2 \left(\cos^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{6} \right) + (AD)^2 \left(\cos^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{6} \right) \right. \\ &\quad \left. - 2(AB)(AD) \left(\cos \frac{\pi}{6} \cos \frac{\pi}{3} + \sin \frac{\pi}{6} \sin \frac{\pi}{3} \right) \right]^{\frac{1}{2}} \\ xw &= \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} \left[(AB)^2 + (AD)^2 - 2(AB)(AD) \cos \frac{\pi}{6} \right]^{\frac{1}{2}} \end{aligned}$$

Similarly zw is found. (3.7) becomes

$$z_o - w_o = \frac{AD \sin \frac{\pi}{3} \sin \frac{\pi}{6} - AB \sin \frac{\pi}{6} \sin \frac{\pi}{6}}{\sin \frac{\pi}{3}}$$

and

$$z_a - w_a = \frac{AD \cos \frac{\pi}{3} \sin \frac{\pi}{6} - AB \cos \frac{\pi}{6} \sin \frac{\pi}{6}}{\sin \frac{\pi}{3}}$$

so that

$$zw = \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} \left[(AB)^2 (\cos^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{6}) + (AD)^2 (\cos^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{3}) - (AB)(AD) (\cos \frac{\pi}{3} \cos \frac{\pi}{6} + \sin \frac{\pi}{3} \sin \frac{\pi}{6}) \right]^{\frac{1}{2}}$$

$$zw = \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} \left[(AB)^2 + (AD)^2 - 2(AB)(AD) \cos \frac{\pi}{6} \right]^{\frac{1}{2}}$$

Thus

$$xw = zw$$

It is known that when the rectangle of Theorem 3.2 becomes a square, the rhombus becomes a square also.² The length of the square is easily found since now $AB=AD$.

²Smart, loc. cit.

Then

$$xw = \sqrt{2} \ AB \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} \left[1 - \cos \frac{\pi}{6} \right]^{\frac{1}{2}}$$

$$= 2 \ AB \frac{\sin \frac{\pi}{6}}{\sin \frac{\pi}{3}} \sin \frac{\pi}{12}$$

But recognize that

$$2 \sin \frac{\pi}{6} = 1$$

so that

$$xw = AB \frac{\sin \frac{\pi}{12}}{\sin \frac{\pi}{3}}$$

A Trapezoid

There is an interesting special case in which the intersections of the angle trisectors of a quadrilateral are the vertices of a rhombus which, itself, is composed of two Morley triangles with a common side. This case is best approached by starting with the Morley triangle of Figure 3-3a and constructing the quadrilateral and rhombus of Figure 3-3b.

In Figure 3-3a, triangle ABC is given along with its angle trisectors. Then triangle wxy is one of the eighteen

equilateral Morley triangles. On side wy construct an equilateral triangle, wyz. Also on wy construct an isosceles triangle, wyM, with equal angles θ , $\theta = \frac{\pi}{3} - \alpha$. Extend sides My and Mw to meet the lines AB and AC, respectively, in points E and D. Draw the lines Dz and Ez and let the points N and L be produced.

Now angle CDw can be found. First, by elementary considerations the following relations can be shown.

$$\angle DCw = \angle wCx = \angle xCB = \frac{\pi}{3} - \gamma$$

$$\angle EBy = \angle yBw = \angle wBC = \frac{\pi}{3} - \beta$$

$$\angle BwC = \frac{2\pi}{3} - \alpha$$

The law of sines applied to triangle AwC yields

$$wC = AC \frac{\sin \alpha}{\sin \beta}$$

The law of sines applied to triangle BxC yields

$$xC = BC \frac{\sin(\frac{\pi}{3} - \beta)}{\sin(\frac{2\pi}{3} - \alpha)}$$

If d is the diameter of the circumcircle of triangle ABC, note that

$$AC = d \sin 3\beta$$

and

$$BC = d \sin 3\alpha$$

Thus

$$\frac{wC}{xC} = \frac{\sin 3\beta \sin \alpha \sin(\frac{2\pi}{3} - \alpha)}{\sin 3\alpha \sin(\frac{\pi}{3} - \beta) \sin \beta}$$

Utilizing the identity³

$$\sin 3\phi = 4 \sin \phi \sin(\frac{\pi}{3} + \phi) \sin(\frac{\pi}{3} - \phi)$$

this becomes

$$\frac{wC}{xC} = \frac{\sin \beta \sin(\frac{\pi}{3} + \beta) \sin(\frac{\pi}{3} - \beta) \sin \alpha \sin(\frac{\pi}{3} + \alpha)}{\sin \alpha \sin(\frac{\pi}{3} + \alpha) \sin(\frac{\pi}{3} - \alpha) \sin(\frac{\pi}{3} - \beta) \sin \beta}$$

so that

$$\frac{wC}{xC} = \frac{\sin(\frac{\pi}{3} + \beta)}{\sin(\frac{\pi}{3} - \alpha)}$$

This expression is recognized as the law of sines for triangle xCw. Thus

$$\angle xwC = \frac{\pi}{3} - \alpha$$

Since

$$\angle ywx = \frac{\pi}{3}$$

³Dobbs, op. cit., p. 50.

it follows that

$$\angle FWC = \frac{\pi}{3} + \alpha$$

and

$$\angle DFW = \frac{2\pi}{3} + \alpha - \gamma$$

Also note that

$$\angle DWF = \theta = \frac{\pi}{3} - \alpha$$

Thus

$$\angle CDW = \gamma$$

Note that

$$\angle CWA = \beta$$

so that

$$\angle AWX = \frac{\pi}{3} - \alpha - \beta$$

Since

$$\angle xwz = \frac{2\pi}{3} \text{ and } \alpha + \beta + \gamma = \frac{\pi}{3}$$

it follows that

$$\angle Lwz = \frac{\pi}{3} - \gamma$$

Similarly, it is found that

$$\angle BEy = \beta$$

and

$$\angle Nyz = \frac{\pi}{3} - \beta$$

Now Coxeter⁴ has shown that, given an equilateral triangle, a triangle can be constructed for which the equilateral triangle is a Morley triangle. This construction is shown in Figure 3-4.

Figure 3-4

⁴Coxeter, op. cit., p. 23-25.

By constructing isosceles triangles on the sides of the given equilateral triangle such that

$$\alpha + \beta + \gamma = \frac{\pi}{3}$$

triangle ABC is generated where Ax and Az, Bx and By, and Cy and Cz are trisectors of the angles at A, B, and C, respectively. Further, Coxeter⁵ shows that

$$\alpha = \frac{\pi}{3} - \frac{A}{3}, \quad \beta = \frac{\pi}{3} - \frac{B}{3}, \quad \gamma = \frac{\pi}{3} - \frac{C}{3}.$$

Now in Figure 3-3b, since $\angle Lwz = \frac{\pi}{3} - \gamma$, a point L' on Aw can be chosen such that $\angle L'zw = \frac{\pi}{3} - \gamma$. L'z will meet My in a point E'. Similarly, a point N' on Ay such that $\angle N'zy = \frac{\pi}{3} - \beta$ results in a point D' on Mw. Now D'z and D'w must trisect the angle AD'E' and $\frac{D'}{3} = \gamma$. Also E'z and E'y must trisect the angle AE'D and $\frac{E'}{3} = \beta$. But it has been shown that $\angle CDw = \gamma$ and $\angle BEy = \beta$. Therefore, D and D' must coincide as also must E and E'. The conclusion is that triangle wyz is the Morley triangle of triangle ADE. Further, $\angle CDE = 3\gamma$ and $\angle BED = 3\beta$. Thus the intersections of the adjacent trisectors of the trapezoid BCDE are the vertices of the rhombus wxyz.

It must be realized that every trapezoid will not yield such a rhombus. A condition that a trapezoid does

⁵ Ibid.

generate such a rhombus can be derived. The requirement can be stated as follows: The intersection of the adjacent trisectors at B and E of the quadrilateral BCDE in Figure 3-3 must coincide with the intersection of the adjacent trisectors at A and E of the triangle ADE. A similar requirement must be placed on the trisectors at A, D, and C.

Figure 3-5 shows the necessary portions of trapezoid BCDE for this analysis, where the adjacent trisectors of the angles at A, B, and E are not concurrent at y. The above requirements are equivalent to

$$(3.9a) \quad Gy = G'y'$$

Figure 3-5

and

$$(3.9b) \quad Hw = H'w'.$$

Now

$$Gy = AE \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

But

$$\alpha = \frac{\pi}{3} - \beta - \gamma$$

By similar triangles,

$$AE = DE \frac{AB}{BC}$$

and by the law of sines

$$AB = BC \frac{\sin 3\gamma}{\sin 3\alpha}$$

so that

$$Gy = DE \frac{\sin(\frac{\pi}{3} - \beta - \gamma) \sin \beta \sin 3\gamma}{\sin(\frac{\pi}{3} - \gamma) \sin(\pi - 3\beta - 3\gamma)}$$

while

$$G'y' = BE \frac{\sin \beta \sin(\frac{\pi}{3} - \beta)}{\sin \frac{\pi}{3}}$$

Thus (3.9a) yields

$$(3.10a) \quad DE \frac{\sin(\frac{\pi}{3} - \beta - \gamma) \sin 3\gamma}{\sin(\frac{\pi}{3} - \gamma) \sin(\pi - 3\beta - 3\gamma)} = BE \frac{\sin(\frac{\pi}{3} - \beta)}{\sin \frac{\pi}{3}}$$

In a similar manner (3.9b) yields

$$(3.10b) \quad DE \frac{\sin(\frac{\pi}{3} - \beta - \gamma) \sin 3\gamma}{\sin(\frac{\pi}{3} - \gamma) \sin(\pi - 3\beta - 3\gamma)} = CD \frac{\sin(\frac{\pi}{3} - \gamma)}{\sin \frac{\pi}{3}}$$

The required condition for the trapezoid BCDE is found by taking the ratio of (3.10a) and (3.10b), which is

$$\frac{BE}{CD} = \frac{\sin 3\gamma}{\sin 3\beta}$$

The results of this section may be summarized in the following theorem.

Theorem 3.3: Let a trapezoid ABCD be given with parallel sides AB and CD and let the measure of each vertex angle have the same symbol as the vertex. If $\frac{AD}{BC} = \frac{\sin B}{\sin A}$, then the intersections of the angle trisectors are the vertices of a rhombus.

Corollary: The rhombus generated in Theorem 3.3 is composed of two Morley triangles with one side in common.

The Kite-Shaped Quadrilateral

There is yet another fruitful case for the quadrilateral. In Figure 3-1, let $AB=AD$ and $BC=CD$. It follows that triangles ABC and ADC are similar and that $\delta = \beta$.

Thus AC bisects the angles at A and C and the figure ABCD, with the angle trisectors, is symmetrical with respect

to AC. It follows that wz and xy are perpendicular to AC and each are bisected by AC.

Now in equations (3.1) let $\delta = \beta$, $AD=AB$, and $CD=BC$. Note also that $\alpha + 2\beta + \gamma = \frac{2\pi}{3}$, so that the coordinates for the points w, x, y, and z become

$$(3.11a) \quad w_a = AB \frac{\cos \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$(3.11b) \quad w_o = AB \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$(3.11c) \quad x_a = AB - BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \cos 2\beta}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

$$(3.11d) \quad x_o = BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \sin 2\beta}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

$$(3.11e) \quad y_a = AB \cos 3\alpha - BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \cos(3\alpha + 2\beta)}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

$$(3.11f) \quad y_o = AB \sin 3\alpha - BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \sin(3\alpha + 2\beta)}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

$$(3.11g) \quad z_a = AB \frac{\cos 2\alpha \sin \beta}{\sin(\alpha + \beta)}$$

$$(3.11h) \quad z_o = AB \frac{\sin 2\alpha \sin \beta}{\sin(\alpha + \beta)}$$

Expressions for the lengths of the diagonals zx and yw can be derived.

$$(zx)^2 = (z_a - x_a)^2 + (z_o - x_o)^2$$

where

$$z_a - x_a = AB \frac{\cos 2\alpha \sin \beta - \sin(\alpha + \beta)}{\sin(\alpha + \beta)} + BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \cos 2\beta}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

and

$$z_o - x_o = AB \frac{\sin 2\alpha \sin \beta}{\sin(\alpha + \beta)} - BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \sin 2\beta}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

so that

$$\begin{aligned} (zx)^2 = & (AB)^2 \left[\frac{\cos^2 2\alpha \sin^2 \beta + \sin^2(\alpha + \beta) + \sin^2 2\alpha \sin^2 \beta - 2\cos 2\alpha \sin \beta \sin(\alpha + \beta)}{\sin(\alpha + \beta)} \right] \\ & + (BC)^2 \left[\frac{\sin^2(\frac{\pi}{3} + \alpha + 2\beta)}{\sin^2(\frac{\pi}{3} + \alpha + \beta)} \right] \\ & + 2(AB)(BC) \left[\frac{\cos 2\alpha \sin \beta \sin(\frac{\pi}{3} + \alpha + 2\beta) \cos 2\beta - \sin(\alpha + \beta) \sin(\frac{\pi}{3} + \alpha + 2\beta) \cos 2\beta}{\sin(\alpha + \beta) \sin(\frac{\pi}{3} + \alpha + \beta)} \right. \\ & \left. - \frac{\sin 2\alpha \sin \beta \sin(\frac{\pi}{3} + \alpha + 2\beta) \sin 2\beta}{\sin(\alpha + \beta) \sin(\frac{\pi}{3} + \alpha + \beta)} \right] \end{aligned}$$

This reduces to

$$\begin{aligned}
 (zx)^2 = & \\
 & (AB)^2 \left[\frac{\sin^2 \beta + \sin^2(\alpha + \beta) - 2 \cos 2\alpha \sin \beta \sin(\alpha + \beta)}{\sin(\alpha + \beta)} \right] \\
 & + (BC)^2 \left[\frac{\sin^2(\frac{\pi}{3} + \alpha + \beta)}{\sin^2(\frac{\pi}{3} + \alpha + 2\beta)} \right] \\
 & + 2(AB)(BC) \left[\frac{\cos(2\alpha + 2\beta) \sin \beta \sin(\frac{\pi}{3} + \alpha + 2\beta) - \sin(\alpha + \beta) \sin(\frac{\pi}{3} + \alpha + 2\beta) \cos 2\beta}{\sin(\alpha + \beta) \sin(\frac{\pi}{3} + \alpha + \beta)} \right]
 \end{aligned}$$

Similarly for w_y

$$y_a - w_a =$$

$$AB \frac{\cos 3\alpha \sin(\alpha + \beta) - \cos \alpha \sin \beta}{\sin(\alpha + \beta)} - BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \cos(3\alpha + 2\beta)}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

and

$$y_o - w_o =$$

$$AB \frac{\sin 3\alpha \sin(\alpha + \beta) - \sin \alpha \sin \beta}{\sin(\alpha + \beta)} - BC \frac{\sin(\frac{\pi}{3} + \alpha + 2\beta) \sin(3\alpha + 2\beta)}{\sin(\frac{\pi}{3} + \alpha + \beta)}$$

Thus $(wy)^2$ reduces to

$$\begin{aligned}
 (wy)^2 = & \\
 (AB)^2 & \left[\frac{\sin^2(\alpha + \beta) + \sin^2\beta - 2\sin(\alpha + \beta)\sin\beta\cos 2\alpha}{\sin^2(\alpha + \beta)} \right] \\
 + (BC)^2 & \left[\frac{\sin^2(\frac{\pi}{3} + \alpha + 2\beta)}{\sin^2(\frac{\pi}{3} + \alpha + \beta)} \right] \\
 - 2(AB)(BC) & \left[\frac{\sin(\alpha + \beta)\sin(\frac{\pi}{3} + \alpha + 2\beta)\cos 2\beta - \sin\sin(\frac{\pi}{3} + \alpha + 2\beta)\cos(2\alpha + 2\beta)}{\sin(\alpha + \beta)\sin(\frac{\pi}{3} + \alpha + \beta)} \right]
 \end{aligned}$$

Note that the expressions for wy and zx are identical. Thus figure $wxyz$ is a square. Theorem 3.4 summarizes these results.

Theorem 3.4: The intersections of the angle trisectors of a kite-shaped quadrilateral are the vertices of a square.

The length of a side of this square is easily computed, since

$$zw = \left[(z_a - w_a)^2 + (z_o - w_o)^2 \right]^{\frac{1}{2}}$$

where

$$z_a - w_a = AB \frac{\cos 2\alpha \sin \beta - \cos \alpha \sin \beta}{\sin(\alpha + \beta)}$$

and

$$z_0 - w_0 = AB \frac{\sin 2\alpha \sin \beta - \sin \alpha \sin \beta}{\sin(\alpha + \beta)}$$

so that

$$\begin{aligned} zw &= \frac{AB}{\sin(\alpha + \beta)} \left[2\sin^2 \beta - 2\cos \alpha \cos 2\alpha \sin^2 \beta - 2\sin \alpha \sin 2\alpha \sin^2 \beta \right]^{\frac{1}{2}} \\ &= \sqrt{2} \ AB \frac{\sin \beta}{\sin(\alpha + \beta)} \left[1 - (\cos \alpha \cos 2\alpha + \sin \alpha \sin 2\alpha) \right]^{\frac{1}{2}} \\ &= \sqrt{2} \ AB \frac{\sin \beta}{\sin(\alpha + \beta)} \left[1 - \cos \alpha \right]^{\frac{1}{2}} \end{aligned}$$

Thus

$$zw = 2AB \frac{\sin \frac{\alpha}{2} \sin \beta}{\sin(\alpha + \beta)}$$

The n-Sectors of a Quadrilateral

The problem of angle n-sectors of a quadrilateral is considerably more complex than that of trisectors. The approach here will be to develop expressions which will suitably analyze any quadrilateral. The complexity of the expressions will forbid an easy analysis of any but the simplest of cases.

In developing coordinates for the intersection of n -sectors of a quadrilateral, it is desirable to do so in such a way that intersections of non-adjacent n -sectors as well as adjacent n -sectors may be considered. Figure 3-6 illustrates the situation for the intersections of arbitrary n -sectors of a general quadrilateral. Again the origin of the coordinate system is A , the positive x axis is AB , and the positive y coordinate is the distance perpendicular to AB toward the interior of the quadrilateral. The coordinates of w , x , y , and z can now be written.

Figure 3-6

$$(3.12a) \quad w_a = AB \frac{\cos \frac{mA}{n} \sin \frac{kB}{n}}{\sin(\frac{mA+kB}{n})}$$

$$(3.12b) \quad w_o = AB \frac{\sin \frac{mA}{n} \sin \frac{kB}{n}}{\sin(\frac{mA+kB}{n})}$$

$$(3.12c) \quad x_a = AB - BC \frac{\sin \frac{kC}{n} \cos \frac{(n-m)B}{n}}{\sin(\frac{mB+kC}{n})}$$

$$(3.12d) \quad x_o = BC \frac{\sin \frac{kC}{n} \sin \frac{(n-m)B}{n}}{\sin(\frac{mB+kC}{n})}$$

$$(3.12e) \quad y_a = AD \cos A - CD \frac{\sin \frac{mC}{n} \cos(A+D - \frac{kD}{n})}{\sin(\frac{mC+kD}{n})}$$

$$(3.12f) \quad y_o = AD \sin A - CD \frac{\sin \frac{mC}{n} \sin(A+D - \frac{kD}{n})}{\sin(\frac{mC+kD}{n})}$$

$$(3.12g) \quad z_a = AD \frac{\cos \frac{(n-k)A}{n} \sin \frac{mD}{n}}{\sin(\frac{kA+mD}{n})}$$

$$(3.12h) \quad z_C = AD \frac{\sin \frac{(n-k)A}{n} \sin \frac{mD}{n}}{\sin(\frac{kA+mD}{n})}$$

It seems obvious that, except for some of the simplest quadrilaterals, utilization of (3.12) would be tedious and perhaps unrewarding. Even for the simpler quadrilaterals the expressions are awkward. Considerations of symmetry will reduce the need for formal expressions of length or slope.

Special Cases

The case of the rectangle is shown in Figure 3-7. The symmetry of Figure 3-7 is such that if the portion

Figure 3-7

to the left of the line ℓ_2 is reflected in the line ℓ_1 and the result is reflected in the line ℓ_2 , then y and z are seen to coincide with w and x , respectively.

Thus

$$yz=wx$$

Similarly the portion of the figure above ℓ_1 first can be reflected in ℓ_2 and then reflected in ℓ_1 and it is found that $xy=wz$. Thus for all m , k , and n , $wxyz$ is a parallelogram. It is clear that for $m=k$, x and z lie on ℓ_1 , and w and y lie on ℓ_2 , so that $wxyz$ is a rhombus. Thus Theorem 3.5 can be postulated.

Theorem 3.5: The intersections of the m th n -sector of one angle with the k th n -sector of an adjacent angle of a rectangle are the vertices of a parallelogram for all values of m , k , and n . If $m=k$, the parallelogram is a rhombus.

If the given rectangle is a square, there is rotational symmetry. That is, let $ABCD$ be rotated by $\frac{\pi}{2}$ about the intersection of ℓ_1 and ℓ_2 . Then it is found that xy rotates into yz ; so that $\angle xyz$ must be $\frac{\pi}{2}$.

Similarly the other angles are found to be $\frac{\pi}{2}$. Thus $wxyz$ is a square for all m , k , and n , so that Theorem 3.5 has the following corollary.

Corollary: If the given rectangle is a square, the parallelogram is a square for all values of m , k , and n .

Figure 3-8 shows the case of the given quadrilateral being a parallelogram. There are no axes of symmetry here. However, choose two lines, l_1 and l_2 , concurrent with the diagonals, AC and BD, such that l_1 is parallel to AB and l_2 is perpendicular to l_1 . Then, identical arguments made for the rectangle apply here. That is, reflect in l_1 the portion of the figure to the left of l_2 and reflect this result in l_2 . Observe that $wx=yz$, and similarly that $xy=wz$, so that $wxyz$ is a parallelogram for all m , k , and n . Thus Theorem 3.6 can be stated.

Figure 3-8

Theorem 3.6: The intersections of the m th n -sector of one angle with the k th n -sector of an adjacent angle of a parallelogram are the vertices of a parallelogram.

Figure 3-9 shows the generalized n -sectors of the kite-shaped figure ABCD described in Theorem 3.4.

Figure 3-9

With $m \neq k$ there are no axes of symmetry. Thus the generalized coordinates (3.12) would have to be used to prove any conjecture about $wxyz$. However, with $m=k$, the

line, ℓ , is an axis of symmetry for the figure. Thus wz and xy are perpendicular to ℓ . Also by symmetry $yz=wx$.

In general, however, $wz \neq xy$. This may be shown through the use of (3.12) or by considering the following limiting case for $m=k=1$ and $n \rightarrow \infty$. In the limit, z goes to a point z' on AD given by

$$Az' = AD \frac{2B}{A+2B}$$

so that

$$\frac{w'z'}{2} = AD \frac{2B}{A+2B} \sin \frac{A}{2}$$

On the other hand, y goes to a point y' on CD given by

$$Cy' = CD \frac{2B}{C+2B}$$

so

$$\frac{x'y'}{2} = CD \frac{2B}{C+2B} \sin \frac{C}{2}$$

and in general

$$x'y' \neq w'z'$$

Thus for $m=k$ in Figure 3-9, $wxyz$ is an isosceles trapezoid.

So Theorem 3.7 can be stated.

Theorem 3.7: The intersections of the m th n -sector of one angle with m th n -sector of an adjacent angle of a kite-shaped quadrilateral are the vertices of an isosceles trapezoid.

Summary

A summary of this chapter shows that a set of coordinates was derived for the intersections of the angle trisectors of a general quadrilateral. These intersections were considered as the vertices of a second quadrilateral and this quadrilateral was examined for any special properties. Due to the complexity of the resulting expressions for the length or slope of the sides of the second quadrilateral, the investigation was limited to the special cases of the given quadrilateral including a parallelogram, a rectangle, and a kite-shaped quadrilateral. Further, a special trapezoid was found to generate a unique rhombus.

Finally, a set of coordinates was derived for the intersections of arbitrary n -sectors of a quadrilateral. These provided a means of analyzing any quadrilateral. However, these coordinates were not required to achieve adequate results for the rectangle and parallelogram since the symmetry of these figures proved sufficient. Symmetry, also, provided partial results for the kite-shaped quadrilateral.

CHAPTER IV

THE p -SIDED POLYGON

The p -sided polygon for p an arbitrary integer is difficult to analyze, since a specific, closed figure cannot be drawn as for the triangle and quadrilateral. However, if the polygon is described iteratively, useful information can be obtained concerning the lengths of each of the sides of the polygon whose vertices are the intersections of the angle n -sectors of the given p -sided polygon. The analysis which follows will be for the general case of intersections of arbitrary angle n -sectors from which the case of tri-sectors is an easy consequence.

Figure 4-1 shows two sides and three vertices of a p -sided polygon. Let A_i denote an arbitrary vertex of the p -sided polygon as well as the magnitude of the interior angle at the i th vertex. Let x_i denote the point of intersection of the m th n -sector of A_i with the k th n -sector of A_{i+1} . The index, i , goes from 1 to p . Figure 4-1 clearly is that portion of the p -sided polygon for which $i=1, 2, 3$.

A coordinate system is adopted similar to that used for the quadrilateral, where A_i is the origin, $A_i A_{i+1}$ the positive x axis, and the direction perpendicular to $A_i A_{i+1}$ toward the interior of the given polygon is the positive

Figure 4-1

y axis. Then the coordinates of the points x_1 and x_2 in Figure 4-1 can be derived. They are found to be

$$(4.1a) \quad x_{1a} = A_1 A_2 \frac{\cos \frac{mA_1}{n} \sin \frac{kA_2}{n}}{\sin \frac{mA_1 + kA_2}{n}}$$

$$(4.1b) \quad x_{1o} = A_1 A_2 \frac{\sin \frac{mA_1}{n} \sin \frac{kA_2}{n}}{\sin \frac{mA_1 + kA_2}{n}}$$

$$(4.1c) \quad x_{2a} = (A_1 A_2)(A_2 A_3) \frac{\cos \frac{(n-m)A_2}{n} \sin \frac{kA_3}{n}}{\sin \frac{mA_2 + kA_3}{n}}$$

$$(4.1d) \quad x_{2o} = A_2 A_3 \frac{\sin \frac{(n-m)A_2}{n} \sin \frac{kA_3}{n}}{\sin \frac{mA_2 + kA_3}{n}}$$

Now the length of the line, x_1x_2 , is

$$(x_1x_2)^2 = (x_{2a} - x_{1a})^2 + (x_{2o} - x_{1o})^2$$

Some of the work is reduced if the above expression is expanded.

$$(x_1x_2)^2 = x_{1a}^2 + x_{1o}^2 + x_{2a}^2 + x_{2o}^2 - 2x_{1a}x_{2a} - 2x_{1o}x_{2o}$$

Substituting the coordinates yields

$$(x_1x_2)^2 = (A_1A_2)^2 \frac{\sin^2 \frac{kA_2}{n}}{\sin^2 \frac{mA_1 + kA_2}{n}} + (A_1A_2)^2 + (A_2A_3)^2 \frac{\sin^2 \frac{kA_3}{n}}{\sin^2 \frac{mA_2 + kA_3}{n}}$$

$$-2(A_1A_2)(A_2A_3) \frac{\cos \frac{(n-m)A_2}{n} \sin \frac{kA_3}{n}}{\sin \frac{mA_2 + kA_3}{n}}$$

$$-2(A_1A_2)^2 \frac{\cos \frac{mA_1}{n} \sin \frac{kA_2}{n}}{\sin \frac{mA_1 + kA_2}{n}}$$

$$+2(A_1A_2)(A_2A_3) \frac{\cos \frac{mA_1}{n} \sin \frac{kA_2}{n} \cos \frac{(n-m)A_2}{n} \sin \frac{kA_3}{n}}{\sin \frac{mA_1 + kA_2}{n} \sin \frac{mA_2 + kA_3}{n}}$$

$$-2(A_1A_2)(A_2A_3) \frac{\sin \frac{mA_1}{n} \sin \frac{kA_2}{n} \sin \frac{(n-m)A_2}{n} \sin \frac{kA_3}{n}}{\sin \frac{mA_1 + kA_2}{n} \sin \frac{mA_2 + kA_3}{n}}$$

This can be reduced to

$$\begin{aligned}
 (4.2) \quad (x_1 x_2)^2 = & (A_1 A_2)^2 \left[\frac{\sin^2 \frac{mA_1}{n}}{\sin^2 \frac{mA_1 + kA_2}{n}} \right] \\
 & + (A_2 A_3)^2 \left[\frac{\sin^2 \frac{kA_3}{n}}{\sin^2 \frac{mA_2 + kA_3}{n}} \right] \\
 & - 2(A_1 A_2)(A_2 A_3) \left[\frac{\sin \frac{mA_1}{n} \cos \frac{(n-m-k)A_2}{n} \sin \frac{kA_3}{n}}{\sin \frac{mA_1 + kA_2}{n} \sin \frac{mA_2 + kA_3}{n}} \right]
 \end{aligned}$$

By advancing all subscripts by one, the length $x_2 x_3$ is obtained. In a similar manner all $x_i x_{i+1}$ is obtained. For the polygon $x_1 x_2 \dots x_p$ to be regular, it is required that $x_i x_{i+1} = x_{i+1} x_{i+2}$ for all i . This leads to a system of $p-1$ independent equations in three unknowns (m , k , and n).

Note that only for $p=4$ (a quadrilateral) is there a unique solution to this system. Even so, there is no guarantee that m , k , and n will be integers, as they must be for the solution to be meaningful.

The Triangle

If the restriction, $m=k$, is invoked, then the system of equations for a triangle ($p=3$) has a unique solution,

since the system is just two equations in two unknowns, m and n . Since Morley's triangle is a solution, it must be the only solution. Thus the following is true.

Theorem 4.1: Morley's triangle is unique in the sense that it is the only equilateral triangle which has as vertices the intersections of the m th n -sectors of the angles of a triangle.

Theorem 4.1 corroborates the results in Chapter II, wherein Theorem 2.3 was shown to be true.

Now, the special case of adjacent 4-sectors of a triangle, discussed in Chapter II, can be analyzed with positive results. For this case, $m=k=1$ and $n=4$. This is identical to the case of $m=k=3$ and $n=12$. But the Morley triangle corresponds to $m=4$ and $n=12$. By Theorem 4.1, this is the only solution when $n=12$. Thus $m=3$, $n=12$ is not a solution. Then $m=1$, $n=4$ is not a solution also. Thus the intersections of adjacent 4-sectors of a triangle are not the vertices of an equilateral triangle.

In a similar manner, it can be shown that if n is not divisible by three, there are no m th n -sectors ($m=k$) of a triangle whose intersections are the vertices of an equilateral triangle. Let a case be represented by $m=n_1$ and $n=n_1$, where $m=1, 2, \dots, n_1-1$ and $\frac{n_1}{3}$ is not an integer. This

case is identical to the case of $m=3m_1$ and $n=3n_1$. But the Morley triangle corresponds to $m=n_1$ and $n=3n_1$. Since $\frac{n_1}{3}$ is not an integer, $n_1 \neq 3m_1$. Thus by Theorem 4.1, the case $m=3m_1$, $n=3n_1$ is not a solution. Then the equivalent case of $m=m_1$ and $n=n_1$ is not a solution. This result is summarized in the following theorem.

Theorem 4.2: If n is not divisible by three, then the triangle whose vertices are the intersections of arbitrary m th n -sectors of a triangle is not equilateral.

This is a sharper form of Theorem 2.3.

Rather than letting $m=k$ to reduce the number of unknowns, the variable, n , may be fixed at some arbitrary integer, and for a triangle there will exist a unique real solution for m and k such that $x_1x_2x_3$ is equilateral. Here it is observed that if n is divisible by three, then necessarily $m=k=\frac{n}{3}$, which results in the Morley triangle. If n is not divisible by three, there is still a unique real solution for m and k , but it may not be an integral solution. This result shows that Conjecture III in Chapter II is or is not true accordingly as m and k are or are not integers.

The Polygon

An examination of (4.2) reveals that if the polygon $A_1A_2\dots A_p$ is regular, then so is $x_1x_2\dots x_p$ for all values

of m , k , and n . For in this case, $A_1 A_2 = A_i A_{i+1}$ for all i , and $A_1 = A_i$ for all i ,

so that

$$(4.3) \quad x_1 x_2 = \sqrt{2} \quad A_1 A_2 \frac{\sin \frac{mA_1}{n}}{\sin \frac{(m-k)A_1}{n}} \left[1 - \cos \frac{(n-m-k)A_1}{n} \right]^{\frac{1}{2}}$$

Thus the following theorem is proved.

Theorem 4.3: The intersections of the m th n -sector of each angle of a regular polygon with the k th n -sector of an adjacent angle are the vertices of a regular polygon for all m , k , and n .

For the case of trisectors, $m=k=1$ and $n=3$, so that (4.3) reduces to

$$(4.4) \quad x_1 x_2 = 2A_1 A_2 \frac{\sin \frac{A_1}{3} \sin \frac{A_1}{6}}{\sin \frac{2A_1}{3}}$$

Summary

A summary of Chapter IV shows that three vertices and two angles of a polygon are sufficient to define a side of a polygon generated by intersections of angle n -sectors. If the given polygon is defined iteratively, then the generated

polygon is completely defined. An expression for the length of an arbitrary side of the generated polygon was derived. Of significance, Theorem 4.1 points out the uniqueness of Morley's triangle. Theorem 4.3 shows that a regular polygon can generate a large number of regular polygons.

CHAPTER V

MORLEY'S FIGURE AND HIGHER GEOMETRY

As mentioned in Chapter I, Morley¹ discovered this theorem as a relatively minor result of his theory on n -lines in a plane. Coxeter² and others³ have demonstrated the relation of Morley's theorem to more abstract notions of geometry. This chapter will further pursue this concept.

It can be demonstrated easily that Morley's theorem is not projective. That is, the Morley figure as shown in Figure 1-1 is not invariant under projection from one plane to another. For if it were, the equilateral triangle would project into an equilateral triangle, which is tantamount to having similarity a projective property. This is a contradiction.

Isogonal Elements

It is advantageous at this point to pursue the concept of isogonality. A few properties of isogonal elements will be shown so that Morley's theorem may be stated in terms of isogonal lines.

¹Morley, op. cit., Transactions of the American Mathematical Society, Vol. I, pp. 97-115.

²Coxeter, op. cit., p. 115.

³Marr, op. cit., p. 136; Taylor, op. cit., p. 132.

Recall that two lines are an isogonal pair with respect to an angle if the bisector of the angle also bisects the angle of the two lines. Also recall that two points are isogonal conjugates with respect to a triangle if the two lines from the points to each vertex of the triangle are isogonal lines. The fundamental theorem of isogonals is the following:⁴

Theorem 5.1: If three lines from the vertices of a triangle are concurrent, their isogonals are concurrent also.

The following lemma will prove useful.

Lemma: Given four concurrent lines--a, b, c, d--the angle formed by a and b is equal to that of c and d if, and only if, the ratio of division of a and d by b [denoted as $(a,d)/b$] is equal to $\frac{1}{(a,d)/c}$.

Proof of Lemma: Figure 5-1 shows the conditions of the lemma, where θ and ϕ are the angles of interest. It is easy to see that

$$Hx = Gx \sin \theta$$

and

$$Jx = Gx \sin(\gamma + \phi)$$

⁴ Roger A. Johnson, Modern Geometry (Boston: Houghton Mifflin Co., 1929) p. 154.

Figure 5-1

so that

$$(5.1a) \quad (a, d)/b = - \frac{\sin \theta}{\sin(\gamma + \phi)}$$

The minus sign is necessary to show that the directed distances from x to H and J are in the opposite sense. Similarly it is found that

$$(5.1b) \quad (a, d)/c = - \frac{\sin(\gamma + \theta)}{\sin \phi}$$

The necessary portion of the lemma is obvious. To show the sufficiency of the lemma, observe that

$$(a, d)/b = \frac{1}{(a, d)/c}$$

implies

$$\sin \theta \sin(\gamma + \theta) = \sin \phi \sin(\gamma + \phi)$$

Applying a trigonometric identity yields

$$\frac{1}{2} \cos(\gamma) - \frac{1}{2} \cos(\gamma + 2\theta) = \frac{1}{2} (\cos \gamma) - \frac{1}{2} \cos(\gamma + 2\phi)$$

or

$$\cos(\gamma + 2\theta) = \cos(\gamma + 2\phi)$$

But this implies

$$\theta = \phi \pm n\pi$$

where n is an integer. Thus the lemma is proved.

Now isogonal lines can be defined in terms of line division.

Definition: Two lines are isogonal lines with respect to two given lines if the ratios in which each line divides the two given lines are the reciprocals of each other.

In particular, trisectors of an angle are isogonal lines. A ratio of division of two lines by a trisector is found by equating $\theta = \gamma = \phi$ in (5.1b). So that if c is a trisector in Figure 5-1, then

$$(a, d)/c = - \frac{\sin 2\theta}{\sin \theta}$$

which can be simplified to

$$(a, d)/c = -2 \cos \theta$$

The cross ratio, μ , for four lines is defined as

$$\mu = \frac{(a, d)/c}{(a, d)/b}$$

Thus for trisectors, the cross ratio is

$$\mu = 4 \cos^2 \theta$$

Now Morley's theorem takes the following form.

Theorem 5.2: Let three pencils of 4 lines each, with non-collinear vertices, be given such that each pencil has one line in common with each of the other two pencils. Let the two non-common lines of each pencil be an isogonal pair such that the cross ratio, μ , of the four lines is $\mu = 4 \cos^2 \psi$, where ψ is $\frac{1}{3}$ of the vertex angle of the common lines. Then the intersections of the non-common lines adjacent to each of the common lines are the vertices of an equilateral triangle.

Three Complete Quadrilaterals

Morley's figure exhibits three complete quadrilaterals. When the figure is carefully drawn as in Figure 5-2, it appears that certain diagonals of the complete quadrilaterals are concurrent in threes at four points. The quadrilaterals are $ABQ_2Q_5Q_8Q_{11}$, $BCQ_3Q_6Q_9Q_{12}$, and $ACQ_1Q_4Q_7Q_{10}$. The points of apparent concurrency are L, M, N, and O. It has been shown that $Q_2Q_4Q_6Q_8Q_{10}Q_{12}$

Figure 5-2

is a Brianchon hexagon.⁵ As such, the diagonals Q_2Q_8 , Q_4Q_{10} , and Q_6Q_{12} are concurrent, so that the point, O , is a valid point of concurrency. Thus let the following theorem be stated.

Theorem 5.3: Let three complete quadrilaterals be given such that each shares a vertex with the other two and the four lines of each quadrilateral are two pairs of isogonal lines with respect to the diagonals through the common

⁵Taylor and Marr, op. cit., p. 123

points. Then the three pairs of diagonals, not containing the common points, meet by threes in four points.

The proof of this theorem will require the following theorem⁶, which is illustrated by Figure 5-3.

Theorem 5.4: From each vertex of a triangle let a pair of isogonal rays be drawn. If no three of these are concurrent they intersect at twelve points besides the vertices. These twelve points are isogonally conjugate in pairs; each vertex of the given triangle may be connected with two pairs of them by new lines. These new lines intersect by threes in eight other points, which are four pairs of isogonal conjugates.

Figure 5-3 requires some explanation. It is drawn for arbitrary isogonal pairs, rather than the special case of trisectors. $Q_1, Q_2, \dots, Q_{12}$ are the intersections of the original isogonal rays from each vertex. Thus AQ_1 and AQ_{11} are the original isogonal rays drawn from vertex A. Similarly BQ_3, BQ_5 , and CQ_7, CQ_9 , are the original isogonal rays drawn from B and C, respectively. From the definition, Q_1 and Q_7 are isogonal conjugates. Similarly paired are Q_2 and Q_8, Q_6 and Q_{12} , etc. $P_1, P_2, \dots, P_8$ are the eight other

⁶Johnson, op. cit., p. 224.

Figure 5-3

points referred to in the second part of the theorem. P_2 , for instance, is found by drawing the new lines AQ_3 and BQ_1 . P_2 is the intersection of these two lines. The theorem states that CQ_8 also passes through P_2 . P_5 is similarly found and is seen to be the isogonal conjugate of P_2 .

Now the proof of Theorem 5.3 can be developed.

Figure 5-4 combines the essential parts of Figures 5-2 and 5-3 to prove that the point N in Figure 5-2 is a point of concurrency for the lines Q_1Q_7 , Q_2Q_8 , and Q_3Q_9 . Observe that the two quadrangles, $P_5Q_7Q_8Q_9$ and $P_2Q_1Q_2Q_3$ have two pairs of corresponding sides meeting at A and two other pairs meeting at B . A fifth pair of corresponding sides is Q_8P_5 and Q_2P_2 , meeting at a point G . However, since

Figure 5-4

P_2 and P_5 are points of concurrency according to Theorem 5.4, C is a point of perspectivity for triangles $Q_7Q_8P_5$ and $Q_2Q_3P_2$; so that corresponding sides of these two triangles intersect in collinear points. But Q_3P_2 and Q_7Q_8 intersect at A ; Q_2Q_3 and Q_7P_5 intersect at B . Thus G , the intersection of Q_8P_5 and Q_2P_2 , must lie on the line AB .

The intersections of five pairs of corresponding sides of the two quadrangles are collinear on AB . This guarantees that the sixth pair also lies on the line AB .⁷ Thus, the intersection of Q_7Q_9 and Q_1Q_3 lies on AB . But Q_7Q_8 and Q_1Q_2 intersect at A ; Q_8Q_9 and Q_2Q_3 intersect at B . Hence triangles $Q_7Q_8Q_9$ and $Q_1Q_2Q_3$ are perspective so that the lines joining corresponding vertices are concurrent. Thus N is a point of concurrency of Q_1Q_7 , Q_2Q_8 , and Q_3Q_9 .

In a similar manner the points L and M can be shown to be points of concurrency. Thus Theorem 5.3 is proved. There are two immediate corollaries to this theorem.

Corollary 1: The points P_2 , N , and P_5 in Figure 5-4 are collinear. Similarly P_1 , L , and P_4 are collinear; P_3 , P_6 , and M are collinear; P_7 , O , and P_8 are collinear.

This result is immediate since triangles $Q_7Q_9P_5$ and $Q_1Q_3P_2$

⁷ William C. Graustein, Introduction to Higher Geometry (New York: MacMillan Company, 1930), p. 25.

are also in perspective, so that P_2P_5 is concurrent at N. Similarly, P_1P_4 and P_3P_6 are concurrent at L and M, respectively.

Corollary 2: There are four Brianchon hexagons in Figure 5-2, namely $Q_2Q_4Q_6Q_8Q_{10}Q_{12}$, $Q_1Q_5Q_6Q_7Q_{11}Q_{12}$, $Q_3Q_4Q_5Q_9Q_{10}Q_{11}$, and $Q_1Q_2Q_3Q_7Q_8Q_9$.

This is an obvious result since L, M, N, and O are the points of concurrency of the diagonals of the hexagons.

Another Complete Quadrilateral

Another interesting observation of Figure 5-3 is presented in the following theorem.

Theorem 5.5: There are eight triangles in Figure 5-3 individually in perspective with triangle ABC. The corresponding sides of these triangles intersect the sides of ABC in six points which lie in threes on four lines. These six points are then the vertices of a complete quadrilateral for which triangle ABC is the diagonal triangle.

Proof: Theorem 5.3 states that each of the points, $P_1, P_2, \dots, P_8$, are points of concurrency of three lines from the vertices of ABC. These eight points lie in twos on six lines. Thus from each of the eight points, there is a triangle whose vertices are three other of the eight points, in perspective with ABC. The eight triangles with their corresponding sides are presented in Table 5-1.

Point of Perspectivity	Perspective Triangle with ABC	Corresponding Sides with ABC		
		AB	BC	AC
P_1	$P_8P_2P_6$	P_8P_2	P_2P_6	P_8P_6
P_2	$P_3P_1P_7$	P_3P_1	P_1P_7	P_3P_7
P_3	$P_2P_8P_4$	P_2P_8	P_8P_4	P_2P_4
P_4	$P_7P_5P_3$	P_7P_5	P_5P_3	P_7P_3
P_5	$P_6P_4P_8$	P_6P_4	P_4P_8	P_6P_8
P_6	$P_5P_7P_1$	P_5P_7	P_7P_1	P_5P_1
P_7	$P_4P_6P_2$	P_4P_6	P_6P_2	P_4P_2
P_8	$P_1P_3P_5$	P_1P_3	P_3P_5	P_1P_5

Table 5-1

At this point the format of this proof becomes identical to that of Theorem 5.3. Observe that C is a point of perspectivity for triangles $P_4P_5P_7$ and $P_3P_8P_2$, so that the intersection of P_8P_2 and P_7P_5 , D, lies on AB. Thus quadrangles $P_4P_5P_6P_7$ and $P_1P_2P_3P_8$ have five pairs of corresponding sides on AB. Hence the intersection, E, of the sixth pair, P_6P_4 and P_3P_1 , is also on AB. In a similar manner, it can be shown that K, the intersection of P_8P_4 and P_1P_7 , and G, the intersection of P_2P_6 and P_5P_3 , lie on BC. Also, H, the intersection of P_3P_7 and P_8P_6 , and F, the intersection of P_5P_1 and P_2P_4 , lie on AC. Now D, K, and F are collinear

Corollary: The six points of intersection in Theorem 5.5, in pairs, divide the sides of triangle ABC harmonically with respect to the vertices of the triangle.

It should be noted that Theorems 5.3 and 5.5 are a result of the isogonal nature of the lines dividing the angles of the triangle, ABC. Therefore, Theorems 5.3 and 5.5 are not restricted to the trisecting lines of Morley's figure, but rather hold for the more general case of isogonal division of the angles of the triangle, ABC.

The Incenter of Morley's Triangle

An intriguing aspect of the Morley figure is the relation of the incenters of the given triangle and the Morley equilateral triangle. In Figure 5-6, the point, O, is located as the intersection of the diagonals of the Brianchon hexagon, $Q_2Q_4Q_6Q_8Q_{10}Q_{12}$. It has been shown that triangle $Q_6Q_8Q_{10}$ is an isosceles triangle; so that the diagonal, Q_2Q_8 , must bisect angle $Q_6Q_2Q_{10}$. Similarly the other diagonals of the hexagon bisect the angles, $Q_2Q_6Q_{10}$ and $Q_6Q_{10}Q_2$. Thus the point, O, is also the incenter of Morley's equilateral triangle. Recall that Corollary 1 of Theorem 5.3 shows that O lies on the line P_7P_8 .

Figure 5-6

The lines joining the isogonal conjugate pairs of the points $P_1, P_2, \dots, P_8$ are concurrent.⁸ Label this point E. Obviously, E lies on P_7P_8 .

Since an isogonal pair of lines is symmetric about the bisector of an angle, the incenter of triangle ABC must

⁸Johnson, op. cit., p. 224

lie within the boundary formed by the six isogonal lines, AP_7 , AP_8 , BP_7 , BP_8 , CP_7 , and CP_8 .

Now an impasse is reached, for the position of the incenter of triangle ABC seemingly cannot be more accurately located with the information thus far introduced. However, it appears that neither E nor O is the incenter of triangle ABC. It also appears that E and O bracket this incenter. It seems reasonable, therefore, to conjecture the following.
Conjecture: The incenter, O, of Morley's equilateral triangle is the isogonal conjugate of the point, E, as defined above.

Inversion in a Circle

An inversive transformation with respect to a circle preserves the angle measure of intersecting curves. Thus Morley's theorem will remain valid under inversion.

Definition: The angle of intersection of two circles is the angle formed by the tangents to the circles at the point of intersection. There is a small ambiguity, since the two intersecting tangents will form two angles. This ambiguity can be clarified as necessary.

Now the inverse of Morley's theorem can be stated.

Theorem 5.6: Let nine circles intersect in a point called the center, such that the sum of the angles of intersection of any three of them is π . Denote these three circles as

primary circles, and the angles as primary angles. Let the remaining six circles, in pairs, trisect the primary angles. Then the three circles, through the points of intersections of the adjacent trisecting circles and through the center, intersect at equal angles of $\frac{\pi}{3}$. "Adjacent" is defined as nearest to a primary circle.

The proof of this theorem merely requires verification that the inverse is Morley's result. There will be no loss in generality by letting two of the primary circles be degenerate (i.e., straight lines). Then the center of the nine circles can be placed on a vertex of a triangle and can become the center of inversion. Figure 5-7 illustrates the inversion.

The elementary rules governing an inversion are the following: A circle through the center of inversion transforms into a line not through the center. A circle not through the center transforms into a circle not through the center. A line not through the center transforms into a circle through the center. A line through the center is invariant. The circle of inversion is invariant.

The information in the previous paragraph makes Figure 5-7 fairly self-explanatory. The circle, R, is an arbitrary circle of inversion. The primary circles are AC'B', AB'B, and AC'C, the latter two being degenerate.

Figure 5-7

The figure demonstrates that Theorem 5.6 is the inverse of Morley's theorem, which is sufficient to complete the proof.

Summary

A summary of this chapter shows that, although Morley's theorem is not projective, it is inversive. The inverse theorem was stated. The Morley figure was studied from the point of view of angle trisectors being isogonal lines. Some interesting results were obtained concerning various perspectivities, which were not limited to the case of trisectors, but were valid for the more general case of isogonal lines. A quadrilateral was found for which the given triangle is the diagonal triangle. The relation between incenters of Morley's triangle and the given triangle was discussed. Although a firm relation was not established, the distance between them was shown to be bounded.

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

This thesis has been a study of Morley's theorem and its extensions. A survey of the literature yielded many proofs of the theorem as well as some extensions to more abstract notions of geometry. Most of the proofs of the theorem dealt merely with the internal angles of the triangle. However, some writers recognized that the external angles may also be trisected, and that there were twenty-seven Morley triangles of which just eighteen were equilateral. The extensions of the theorem in the literature encompassed such concepts as the Hessian axis and Hessian points, the Lemoine point, the circles of Apollonius, isogonal conjugate points, coaxal systems of circles, n -sectors of a square, Simpson lines, and Brianchon hexagons.

Some of the articles found in the literature were in foreign publications. Further insight into Morley's theorem might be gained in pursuing these articles. For instance, it could be very rewarding to learn what such a prominent mathematician as Lebesgue¹ has to say about

¹M. Henri Lebesgue, "Sur les n -Sectrices d'un Triangle," L'Enseignement Mathématique, Vol. 38 (1939-1940), pp. 39-58.

n-sectors of a triangle. Also, Bricard's² article may prove illuminating in view of Lubin's³ description.

This author has extended Morley's theorem, basically in two ways. First, sectors of a triangle other than trisectors were considered. Second, the triangle was extended to the general polygon where both trisectors and n-sectors were considered. Both cases were searched for the existence of results similar to Morley's (i.e., equilateral figures). Where these results were non-existent or indeterminate for the general case, special cases or conditions for existence were examined.

In Chapter II the lengths of the sides of the triangle, generated by intersections of angle n-sectors, were expressed as vector quantities. It was shown that for arbitrary n ($n > 3$) the intersections of adjacent n-sectors of a triangle do not yield an equilateral triangle. It was also shown that if $m=k$, then only for $m=\frac{n}{3}$ (trisectors) does an equilateral triangle result from the intersections of n-sectors.

Lubin's⁴ formula, expressed in equation (2.2), for the point of intersection of two vectors is limited to

²Bricard, op. cit., pp. 241-254.

³Lubin, op. cit., p. 112.

⁴Ibid.

vectors defined on a circle. Since a conic can always be drawn through four given points, an equivalent expression for vectors defined on a conic would give an added degree of freedom. This would allow a vector analysis to be developed for the general quadrilateral, similar to that developed for the triangle. Also four angles and three sides of a p -sided polygon are sufficient to describe two sides of the generated polygon; so that such an expression would allow the determination of the lengths of two sides and their included angle.

In Chapter III, the general quadrilateral was studied with its angles being both trisected and then n -sected. In each case, a set of trigonometric coordinates was developed for the points of intersection of adjacent trisectors (n -sectors). For trisectors, expressions for the lengths of the sides of the quadrilateral, formed by considering these points of intersection as vertices, were developed for the special cases of the given quadrilateral being a parallelogram, a rectangle, and a kite-shaped figure. For n -sectors, these same special cases were examined from the point of view of symmetry to obtain good results. Also a particular trapezoid was found to generate a rhombus composed of two Morley triangles.

In both Chapters II and III, only the internal angles of the polygon were n -sected ($n \geq 3$). Remaining for future

study are the other two angles at each vertex, $2\pi + A$ and $4\pi + A$. Also the generated polygon can be examined for similarity to the given polygon, in addition to being an equilateral figure. For example, Figure 3-9 was drawn for the case $n=5$, $m=1$, and $k=2$. The generated polygon appears to be similar to the given polygon.

A general p -sided polygon, defined iteratively, was discussed in Chapter IV. A condition for the generated polygon to be equilateral was stated as a system of $n-1$ equations in three unknowns. This allowed the deduction that Morley's triangle is unique in a certain sense and provided the sufficient condition that arbitrary n -sectors of a triangle generate an equilateral triangle. The uniqueness of Morley's triangle led to the conclusion that for $m \neq k$ n must be divisible by three for an equilateral triangle, formed by intersecting n -sectors, to exist at all. These results confirmed the conclusions of Chapter II. Also it was shown that a unique, real solution (perhaps non-integral) exists for a general quadrilateral to generate a rhombus from the intersections of arbitrary n -sectors.

In Chapter V, the three quadrilaterals in Morley's figure were found to have six of their diagonals intersecting by threes in four points. This was found to be a result of the isogonal properties of trisectors and thus holds true for the more general case. Various perspectivities were

shown to exist in the figure of intersecting isogonal lines. These led to the discovery of a complete quadrilateral associated with one particular set of perspectivities for which the given triangle is its diagonal triangle.

There are other perspectivities in Figure 5-3 which can be explored in future study (i.e., the twelve points $Q_1, Q_2, \dots, Q_{12}$ are centers of perspectivity). Further, repeated application of Theorem 5.4 may yield other centers of perspectivity.

An isogonal coordinate system for the triangle could reduce the complexity of the equations for the lengths of the sides of the generated triangle over those found with a vector or a trigonometric coordinate system as in Chapter II. In particular, Figure 6-1 shows one way a triangle can be partitioned so that the isogonal conjugates of every point in the shaded area lie in the unshaded area. If (m_1, m_2, m_3) is the isogonal coordinate of a point, its isogonal conjugate is $(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3})$.⁵ Thus by specifying the points in the shaded area, every point in the triangle is known. This could lead to a simple relation between m , k , and n , shown to exist in Chapter IV for intersecting arbitrary n -sectors of a triangle.

⁵Graustein, op. cit., p. 172.

Figure 6-1

The area of isogonal elements could be a rich subject for future study. Whereas this author has not specifically analyzed the literature on this subject, he has found relatively few references to it. Johnson⁶ is the best text found, and even so, contains just a few pages on isogonality. Taylor and Marr⁷ utilize some of the concepts of isogonality, with no explanation; perhaps, something can be gleaned by implication from their article. Some of the

⁶Johnson, op. cit., pp. 153-157.

⁷Taylor and Marr, op. cit., pp. 126-127.

questions that can be asked in this area are: Can an "isogonal dual" be defined so that the several theorems that do exist can be dualized? In particular, is a dual theorem of the fundamental theorem for isogonal lines true? Is there an underlying theory as to why the lines joining the four pairs of isogonal conjugate points, $P_1, P_2, \dots, P_8$ in Figure 5-3 are concurrent? What can be said about the cross-joins of a set of pairs of isogonal points? What form does the distance formula take in isogonal coordinates? Can the isogonal coordinates of the point of intersection of the joins of four points be determined from the isogonal coordinates of the four points? Can a sufficient condition that two points are isogonal conjugates be developed that lends itself to easy application (e.g., show that the points E and O in Figure 5-6 are isogonal conjugates)? These questions apparently have not been answered.

Even though Morley's figure is not projective, concurrency of lines and collineations of points are projective. The projective properties of Figure 5-3 could be examined in future study. All that would be lost is the isogonal properties of the lines and points.

A theorem stating the inverse of Morley's theorem was shown to be valid in Chapter V. It seems that the

theorem is still true if the condition that the sum of the angles of intersection of the primary circles be equal to π is removed. It remains to be verified that, in this case, the system of nine circles would invert into one of the "external" Morley triangles. There are other points which may be considered as centers of inversion, such as orthocenter, median, symmedian, and center of the nine point circle.

The extension of Morley's theorem to three-dimensional space is a reasonable area of future study. In particular, the extension to the tetrahedron should prove to be straightforward. Certainly inversion in space of any three-dimensional results would be valid. It also seems reasonable to expect the isogonal concept to extend to three-dimensional space as well.

In addition to accomplishing the original goals, this thesis uncovered several other interesting aspects of Morley's theorem and its extensions. Some of these were explored. Perhaps this thesis can serve as a point of departure for future study.

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