9-1-2023

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Chaos, Solitons and Fractals 174 (2023) 113718

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Stochastic resonance in the recovery of signal from agent price expectations

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ARTICLE INFO

Keywords:
Signal detection in financial markets
Price expectations
Stochastic resonance
Multicomponent expectations
Networked agents

ABSTRACT

Contributions that noise can make to the objective of detecting signal in agent expectations for price in financial markets are examined. Although contrary to most assumptions on exogenous noise in financial markets as increasing both risk and uncertainty in the detection of signal, a basis for the contribution that noise can have to agent objectives in signal detection through stochastic resonance (SR) is well-documented across disciplines. After reviewing foundations for the micro-processing of expectations, a multi-component model of networked agents that includes a component of bounded rational processing and a component that has been cited as generating "herding" behavior in financial markets is offered. The signal-to-noise ratios in the proposed models provide a basis to investigate SR in an application to financial markets. Results with both deterministic and stochastic forms of the proposed model support SR as a process in which randomness can contribute to the recovery of signal in agent expectation. Additionally, predictive models that indicate the sensitivity of the occurrence of SR to the parameters of the models of agent expectations were estimated and cross-validated. The discriminative ability of the models is reported through Area Under the Receiver Operating Curve (AUROC) methodology. These results extend the cross-discipline demonstrations of SR to models of price in financial markets.

1. Introduction

Agent expectations have long been established as intermediaries in market behavior across a range of disciplinary contexts (e.g., [1–5]). In financial economics, an increasing number of applications continue to indicate the relationship of measured expectations to market behavior in price and returns (e.g., [6,7]). Accounts of the dynamics of market-related expectations, commonly assume at least bounded rationality (e.g., [8]) in the use of information to recover signal on price expectations. As now extensively addressed (e.g., [9]), traders can be categorized as either fundamentalists or "trend followers" ("noise") traders. Fundamentalists can be represented as at least bounded rational in the use of objective information in trading. So-called "noise" traders are commonly considered to be trend-followers or momentum traders. Seminal discourse that includes Grossman and Stiglitz [10] and Black [11] has established the importance of "noise" traders in the dynamics of market price. Also, see Peress and Schmidt [12]. In the absence of such traders, market price would reduce to private and public information.

In most models of expectations including those in which agents are categorized as either fundamentalists or "noise" traders, exogenous randomness is assumed to increase uncertainty and risk. Our interest is in extending the consideration of randomness in expectations as a price signal in financial markets. As will be reviewed, there are sizable demonstrations across disciplines of stochastic resonance (e.g., the compilation in McDonnell, Stocks, Pearce, and Abbott [13]) as a process through which randomness can increase efficiency in the detection of the signal. In the discourse to follow, the conditions of SR in models of agent price expectations for the market price will be directly investigated in deterministic and stochastic market models. Results that will be reported support the presence of SR in the signal-to-noise ratios (SNR) of the proposed models of price expectations. As will be reviewed, these results are among the few demonstrations of SR in economic processing and the first we can locate that investigate SR in comprehensive models of price expectations in financial markets. The next section will directly elaborate on the research problem and objectives.

1.1. Research problem and objectives

Agent expectations have been theorized to efficiently anticipate price in financial markets (e.g., [4,7]). The research objective of this

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https://doi.org/10.1016/j.chaos.2023.113718
Received 8 April 2023; Received in revised form 5 June 2023; Accepted 13 June 2023
Available online 12 July 2023

manuscript is to increase our understanding of dynamics in agent expectations for a market price through (1) elaborations of behavioral foundations of micro-processing in these dynamics, (2) investigation of conditions of SR under which the exogenous randomness that is endemic in financial markets can increase efficiency in the recovery of signal and (3) testing of the sensitivities of the occurrence of SR to model parameters. Our applications are in network models that represent an interaction between agents as addressed accounts that recognize "herding" in financial markets (e.g., [14]). Results for the research objectives in the estimation of what is the signature of SR in the signal-to-noise ratio (SNR) of multi-component models of expectations will be reported.

The proposed models of the price expectations of agents are multi-component. In contrast to the definitions of different categories of agents, it is assumed that individual agents, process as fundamentalists under certain conditions and as the equivalent of momentum traders under other conditions. A form of the conditions in a transition function that weighs the relative weighting of components in terms of distances between market price and measures of fundamental or "intrinsic" value will be offered. In applications, fundamental or "intrinsic" values can be given a metric in price-earnings (PE) or price-earnings to growth (PEG) ratios.

1.2. Organization of the discourse

In the next section, background studies of price expectations in financial markets will be reviewed and the basis for a multi-component model and a transition function, that defines component dominance will be introduced. In the sections that follow, background studies of stochastic resonance will be reviewed and forms that integrate the presence of SR in multicomponent models of agent expectations will be offered. The signal-to-noise ratio that is the common base for the detection of SR will subsequently be defined for the proposed models of agent price expectations. Consistent with the methodology in SR [13], the presence of a maximum in the SNR as a function of an exogenous noise level in computational exercises that integrate market data will then be examined. Predictive models that relate model parameters in the forms of the SNR to the incidence of SR will also be reported. A final section will review the results on SR in the models of price expectations and consider directions for subsequent study.

2. Multicomponent agent-based models in financial markets

Multicomponent models of agents in financial markets typically represent agents who are at least bounded rational and those who are predominantly influenced by other agents. The dynamic process in the latter is commonly cited as "herding" (e.g., [14,15]. Harras and Sornette [16], for example, offer a multi-component model of adaptive agents for the growth of a "bubble" in a financial market. In their model, agents switch from buyers to sellers or vice-versa, through exogenously-driven, "news-based" adjustments to the weights of the model components. In exercises that these investigators report, news sources appear to increase the correlation between agents in their states and presage or are coordinate with the growth of a price "bubble".

In a similar direction, Eckrot, Jurczyk, and Morgenstern [17] investigate a multi-component agent-based model that allows for a case of multiple assets and information for which there is either public or private access. Their model is cited as an Ising analog and shows properties of fat tails and volatility clustering that have long been recognized in financial markets. Although, not strictly in a multi-component model, Li and Teo [18] have considered a combinatorial form for uncertainty and randomness in financial markets under an alternative definition of skewness in the randomness. The investigators integrate their definition of randomness in a modified mean-variance model.

While the present study agrees with previous investigators that market behavior in aggregates is likely to be generated by contrasts in agent processing that introduce different dynamics, the processing of multiple components in individual agents will be extended. Rather than grouping traders in categories of fundamentals and "noise" traders that have different dynamics, the proposed multi-component models of agent processing generate price expectations in financial markets from competition between components in bounded rational processing and the "copying", of other agents that underlies "herding" [e.g. Rangvid et al. [19]]. The non-linear dynamics of a transition function that define the relative weight of each component reflects agent behavior in response to the distance between market price and metrics of fundamental value. Behavioral foundations for the proposed micro-processing in expectations for price will be elaborated on in the next section.

2.1. Multicomponent model of agent heuristics in the formation of expectations

2.1.1. Bounded rational processing

In agreement with a range of specifications of expectations, it is recognized that agents do not generally process objective information on the market in a form that is strictly consistent with rationality (e.g., [7]). Theorists of bounded rationality [8,20,21] recognize that agents are limited in their information processing capabilities and commonly use approximation heuristics in this processing. In a representation of this component, it is assumed that agents use predictive information on fundamentals of value that underlie market price (e.g., PE or PEG ratios) in forming their expectations but adjust their expectations based on feedback from the past accuracy of comparable information in predicting price.

2.1.2. Imitative processing and herding

While there are contentions that even bounded rationality can maintain efficient markets (e.g., Lo [22]), behavior that exhibits so-called "excessive exuberance" (Shiller [23]) and agent copying of the expectations of others is widely recognized as departing from strict rationality and generating "herding" in the expectations of agents (e.g., [19]). Chen, Tan, and Zheng [14] have, for example, investigated an agent-based model in which "herding" exists at levels of individual securities, market sectors, and overall market levels. Consistent with these and other investigators, a form for imitative processing in a second component of agent expectations for price that directly follows from agent observations of the states or levels of referent individuals will be provided.

The economic importance that an equivalent of imitative processing can have to expectations has been cited in different disciplines [24–26]. In an extreme, qualitative text on expectations report a range of instances in which agents distinctly depart from rational or bounded rational processing (e.g., [27,28]). Such a so-called "mercurial" sensitivity with which "herding" can build has been observed across many applications in financial markets (e.g., [15,29,30]).

2.1.3. Transition function

A behavioral basis for dynamic variation in the relative weights of bounded rational and herding components of agent expectations will next be elaborated. Component dominance in the transition function to be proposed is defined by a distance of market price from metrics of fundamental value (e.g., historical PE or PEG ratios The transition function allows for what can be considered to be the more realistic agent behavior of one component dominating with continuing processing occurring in the other component. This transition function further introduces non-linear dynamics in price expectations since extreme departures of market price from metrics of fundamental value (as can be driven by the equivalent of "herding") are commonly "corrected" by what can be an abrupt transition to a level that is closer to fundamental value metrics.

2.1.4. Non-linear dynamics in the transition function

In the proposed agent processing, component dominance that is dominated by imitative processing can revert to bounded rational processing to correct for a price level of an asset relative to its fundamental value that an agent judges to be "excessive". A basis for the non-linear dynamics introduced in the transition function is implied by a number of psychological theorists who are sometimes labeled as "incongruity theorists" (e.g., [31,32]). These theorists maintain that while small violations of in-place expectations generate interest and attempts to explain and reconcile the violations in what we designate as bounded rational processing, larger violations can increasingly generate fear and "trepidation", and, in turn, "irrational" (e.g., imitative) processing that results in "herding" (e.g., Foster and Keane [33]).[1]

The stylized micro-processing in dynamics of bounded rational and imitative components followed can be qualitatively described in the graphic in Fig. 1. This graphic exemplifies the proposed account of component dominance in terms of the distance between price and fundamental value metrics under a consistent measure of dispersion.

Fig. 1 assumes a symmetrical positive and negative range of expectations for the price of an asset for which there is a market large enough to be efficient in matching buyers and sellers. At a mid-point of the range (ε_o), expectations are assumed to be approximately equal to metrics of fundamental value, i.e., $\varepsilon_0 = \overline{\nu}_0$. Even in what was designated as bounded rational processing, there can be more or less random variation around fundamental value as in the bounded region of *A* to *-A*. The levels of $\varepsilon_{\max}$ (or $-\varepsilon_{max}$) are denoted as the boundary of region *A* and can be considered as the boundary of a confidence interval in which a bounded rational component dominates.

Exogenous events such as the arrival of new technology or natural disasters can generate an increase in the distance of market price from fundamentals of intrinsic value that are still processed in a bounded rational component. In such cases, these events can be assumed by agents to presage an increase or decrease in the earnings stream from the asset. However, if the expectations of others (or even inference on these from the momentum of observed market prices) are of sufficient magnitude relative to volatility for a definable duration, agent expectations can be driven to a level that exceeds what has been designated as confidence intervals around the mid-point of fundamental value. When this occurs, there is commonly a transition to the dominance of the imitative processing that underlies "herding". A transition from bounded rational to imitative processing ($A \rightarrow B$ in the Figure) can be operationally defined to occur when the distance of market price from fundamental value is a multiple of standard deviations from its zero-point equivalence to fundamental value.

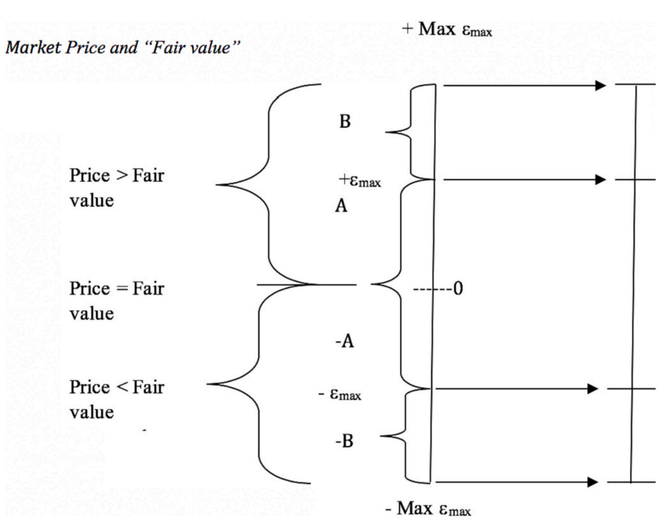

Fig. 1. Dynamics of multicomponent processing in expectations.

Even when imitative processing dominates, it is assumed here that there is a continuing presence of bounded rational processing. As imitative processing increasingly drives expectations further from what is fundamental value, bounded rational processing by an agent is assumed to continuously track the discrepancy. When the discrepancy is large enough, the bounded rational and imitative components can once again have equal weights in the expectations of an agent. In the Figure, this occurs at $Max\ \varepsilon_{max}$ or $-Max\ \varepsilon_{max}$. The difference between price and fundamental value can then fluctuate in a relatively small range around ε_0 for some time until the equivalent of another disturbance is large enough to drive it beyond the bounds of the confidence interval *(+A, −A* in the figure) and the subsequent dynamics repeat in an opposite direction.

A form of a transition function that represents these dynamics in component weights can be given as.

$$\alpha(t) = \begin{cases} 0.9, if & |V_t - P_t| \le \zeta_t \ or \ |V_t - P_t| \ge 2\zeta_t \\ 0.1, & otherwise \end{cases} \quad (1)$$

where ζ_t is a bound on the interval of difference between price and fundamental value as indicated in Fig. 1.

3. Stochastic resonance (SR)

The indirect and subtle ways that randomness can contribute to functional objectives have been engagingly recognized in qualitative discourse by authors that include Gladwell [35] and Taleb [36]. In applications to financial markets, a basis for a well-defined account in which exogenous randomness can contribute to the detection of a signal can be through stochastic resonance ([13]). In SR, a level of exogenous randomness has been shown to have the capability to make a sub-threshold signal detectable. The generality of SR is now supported by extensive numerical and empirical applications across disciplines that include Hopfield neural networks (e.g., [37–39]), predator signals in crayfish (e.g., [40]), and perceptual interpretation of ambiguous figures (e.g., [41,42]) as well as binary states of financial markets in "bubbles and crashes" (e.g., [43,44]).

Available applications of SR in financial markets have not addressed price expectations of agents and the multiple components of bounded rationality (e.g., Conlisk, [8]) and "herding" in referent agents (e.g., [30,45,46]) in their processing. Since expectations are precursors of market behavior (e.g., [7]), there is an opportunity to contribute to our understanding of dynamics in financial markets by extending the condition of exogenous randomness to the detection of signals in expectations.

Specification of agent expectations provided offers a more complete representation of continuous agent micro-processing in financial markets than has been introduced in previous demonstrations of SR in binary states of "bubbles" and "crashes" in market price (e.g., [43,44]). The results reported by investigators indicate the functionality that exogenous randomness can have through SR in extreme cases in "bubbles" and "crashes" of financial markets. We propose to elaborate on these models by addressing SR in the continuous processing of multi-component models of agent expectations for price. The range of applications across disciplines that have demonstrated the case of efficiency-increasing randomness does not ensure a comparable effect in the detection of signal for market price in agent expectations. However, the

[1] Tan [34] documents one recent example of emotion-based momentum and subsequent abrupt corrections toward fundamental value in the security markets of Shanghai and Shenzhen. Over this author's reference period, indexes for these markets more than doubled in the face of a decline in the growth of the Chinese economy from about 12 % to 79 % and no increase in earnings or real value in the underlying assets of listed companies. The subsequent crash reduced price indexes to a level that was actually less than many fundamental value metrics would assign to assess in the index.

results of the background studies that have been cited encourage further investigation of the case in agent expectations for market price.

3.1. Randomness as increasing efficiency in the detection of signal through SR

The information-theoretic tradition that follows from Shannon (e.g., Shannon and Weaver [47]; also see Cover and Thomas [48]) investigates the objective of recovering signal in the presence of noise. This is clearly an objective of agents in financial markets ([49,50]). This objective is commonly given a form in the signal-to-noise ratio (SNR). A range of investigations have elaborated and tested an account of stochastic resonance in which noise of a certain intensity can contribute to rather than detract from the detection of signal in the SNR (e.g., McDonnell, Stock, Pearce, and Abbott, [13]).

3.2. Dynamics in stochastic resonance

Accounts of SR introduce a basis to expect that even the "weak periodic variation" that environments impose in price expectations can be made more detectable by randomness. Stochastic resonance is perhaps most easily understood in a case where the system's variation is not directly observable except in terms of "switching" events (i.e., a change of states as in "on-off" states). In a simple case of demonstrated SR, deterministic variation is often given weak periodic (e.g., sinusoidal) variation as signal. Here, random variation, as in Gaussian "noise" can be added to each agent to represent agent idiosyncrasy in signal. Noise can increase the impulse of the periodic signal and "switch" the state. Thus, the combination of a certain level of noise and signal can be more informative on the state of the system than the signal is in the absence of noise. This is schematically represented in Fig. 2.

In Fig. 2, signal by itself does not attain the minimum magnitude at which an agent can detect it as shown in the "no noise" and "weak noise" panels. The addition of an optimal Gaussian noise level that is coordinated with a magnitude of the signal can make the signal detectable as shown in the "optimal noise" panel. At too high a noise intensity, the "high noise" panel indicates that only noise can be detected.

The signal-to-noise ratio remains the common metric for detecting efficiency in the recovery of signal in information theory (e.g., [48]). In applications to SR, the presence of SR is typically indicated by (1) an identifiable maximum in the signal-to-noise ratio (SNR) as a function of noise intensity in the time domain and (2) a corresponding maximum for the Fourier Transform in the frequency domain. The explicit form given the SNR is dependent on the application and differs across applications that range from ocean engineering (e.g., [51]) to opinion dynamics (e.g., [52]) and financial and energy markets (e.g., [43,44,53]).

As noted, the direct demonstration of noise-induced enhancement of signal in nonlinear systems (e.g., [13,54]) does not ensure commensurate effects in agent expectations for market price. However, available demonstrations of SR across contextual representations do encourage explicit applications of the process to economic agents. In the section that follows, applications of SR in financial markets will be reviewed in further detail. Following this review, a form for the SNR in agent expectations that follow from the foundations of microprocessing and the transition function introduced in an earlier section will be proposed. We will then investigate whether SR can be demonstrated in the proposed forms for the SNR.

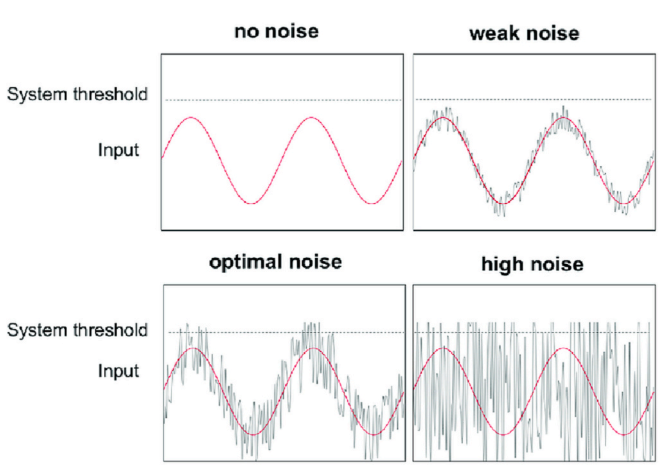

Fig. 2. Schematic of signal and noise in stochastic resonance.

3.3. Applications of stochastic resonance in financial markets

Xiao-Ming, Kai, and Qi [55] have examined SR in a model of the binary states of "bubbles" and "crashes" that follows Kiselev, Phillips, and Gabitov [56] where the interest rate is considered as an external signal and the randomness in traders' behavior is noise. Their results indicate the presence of SR that increases with noise intensity to a maximum. Krawiecki and Holyst [43] investigated a bi-stable model of a financial market designed to represent "bubbles and crashes" as subject to a weak external information-carrying signal and noise. These authors show that an information-carrying signal assumed to be too weak to induce the equivalent of "magnetization jumps" can be enhanced by external noise as in standard SR accounts. Both Helbing and Platkowski [57] and Krawiecki and Holyst [43] implement a form in which the arrival of information considered to have relatively weak importance to an objective can be enhanced by external noise as in stochastic resonance and result in the end to a "bubble" and a market "crash".

Zhou, Zhong, Li, Li, and He [44] have simulated a modified Heston model of market price and directly demonstrated SR in the periodic volatility of two financial markets. Their results demonstrate that (1) optimal levels of volatility parameters can maximize the effects of both systematic and non-systematic randomness on market periodicity and (2) different correlation magnitudes between systematic and non-systematic noise can result in different critical points that can induce single or multiple instances of SR.

Although not directly evidenced in stochastic resonance, the importance of price that noise and drift (e.g., Black and Scholes [58]) can have in evolutionary games has also been recognized (Vega-Redondo [59]). Vukov, Szabó, and Szolnoki [60] have directly identified SR as generating cooperation in a Prisoner's Dilemma game. Less frequently, SR has been applied to the study of opinion formation (Kuperman and Zanette [61]; Tessone and Toral [52]) which the authors note as having analogs to binary states in market behavior.

As has been proposed, available studies of SR in financial markets can be extended by addressing agents' expectations for price that are precursors of market behavior (e.g., [6,7]). As indicated, the form and dynamics of the proposed model of multicomponent processing and a transition function for component dominance give a finer behavioral representation of the dynamics of expectations for price in financial markets than has been available in previous models. In the sections to follow, an integration of signal and noise in the SNR for multi-component models of price expectations will be introduced.

4. Investigating stochastic resonance in the dynamics of multicomponent expectations

A component of a multi-component model of "herding" in a model of expectations is most informedly investigated in network models since agents at collective levels exhibit structural configurations. Models of agent expectations when agents are integrated in small-world networks (SWN: Watts and Strogatz [62]) will be studied. For the integration of SWNs in the economic analysis also see Jackson [63,64]). SWNs parametrize structure in a network as a proportion (ρ) of the total vertices in the network that are disconnected from neighbors or next neighbors in a regular network and re-connected to a randomly selected vertex. This structural parameter has been related to path length (mean number of transfers across vertices to connect any two vertices in the network) and clustering in states or levels of variables in the network (e.g., Barrat and Weight [65]).

Path length and clustering have direct interpretability in economic analyses. On path length and economic efficiency in networks, see Cowan and Jonard [66], and Wen, Yang, and Zhou [67]. On clustering in networks and economic inequality, see D'Angelo and Lilla [68], Tsvetkova, Wagner, and Mao [69], and Brida, Carrera, and Segarra [70].

$$\varepsilon_{i,t+1} = \alpha_t(V_t + (V_{t-1} - P_{t-1})) + (1-\alpha_t)\sum_j M_{ij}(\varepsilon_{j,t} - \varepsilon_{i,t}) + A\,sin\,(\Omega_t) + \phi_{i,t} \tag{2}$$

$$\alpha_t = \begin{cases} \alpha \; if \; |P_t - V_t| \le \zeta_t \; or \; |P_t - V_t| \ge 2\,\zeta_t \\ 1-\alpha, \text{otherwise} \end{cases} \tag{3}$$

and

$$M_{ij} = \begin{cases} M \; if \; i \; and \; j \; are \; neighbors \; or \; remote \; connections \; in \; the \; small-world \; network \; (i.e., cohorts) \\ 0, otherwise \end{cases}$$

Relatively small proportions of total connections in the SWN that are to a remote vertex (e.g., $\rho \sim 0.001$) have been shown to have highly significant effects on path length and clustering in the network ([65]). In the exercises to follow, we directly investigate the relationship of the ρ parameter in SWNs in conjunction with other parameters to define efficiency and structure in the detection of signal in agent expectations.

Definitions of the SNR and SR in deterministic and stochastic models that operationalize the dynamics of agent expectations in networked agents will be implemented in deterministic and stochastic models. A deterministic model integrates empirics from market data in market price, earnings, and PE ratios of the S&P 500. A stochastic model follows historical traditions in representing market price as Brownian motion with drift and integrates market data on earnings and PE ratios of the S&P 500.

After formally defining the SNR and the condition for the occurrence of SR, results that demonstrate that the implemented models of price expectations can evidence SR will be reported. In the application, note that as Fakir [71] has indicated, the demonstration of SR does not require stationary time series. Accordingly, background studies that include Zhou, Zhong, Li, Li, and He [44] have investigated SR in the occurrence of "bubbles and crashes", in non-stationary time series. Park and Phillips [72] have also demonstrated SR in non-stationarity in models of binary choice. Following the demonstration of SR in the proposed models, we will investigate parameters in levels, and combinations of models that discriminate cases of SR in the models will be investigated. These prediction exercises will implement Area Under the Receiver Operator Characteristics (AUROC) models that have been reported in economic applications including [72–75].

4.1. Deterministic model of agent multi-component expectation

For the exercise with a deterministic multi-component system, S&P market data were integrated into an estimation of the model. Price data and a historical PE-based multiple from S&P 500 market data were used to define a PE-based measure of what has been denoted as a fundamental value. Eq. (2) represents the dynamics of the multi-component model to be estimated with a sinusoidal forcing term for periodic signal.

Table 1
Parameter distributions for the deterministic model.

Variable	Range
ρ	(0,1)
M_{ij}	(0,0.2)
α	(0,1)
Ω	(0,5)
$\boldsymbol{A}$	(0,1000)

$$V_t = (0.2E_{t-1} + 0.8E_t)n(t) \tag{4}$$

$$\zeta_t = \left(\frac{\sum_{k=1}^{5}(V_{t-k} - P_{t-k} - \mu_t)^2}{5}\right)^{0.5} \tag{5}$$

$$\mu_t = \frac{\sum_{k=1}^{5}(V_{t-k} - P_{t-k})}{5} \tag{6}$$

In Eq. (2), ε_i is the expectation of the i^{th} agent for market price

V_t is a measure of "fundamental value" in the S&P 500 (the product of real S&P earnings and the mean historical PE ratio for the S&P index).

P_t is the observed monthly S&P 500 price index.

α_t is the relative weight of a component

M_{ij} is a magnitude parameter for weights of differences between the expectation of agents *i* and *j*

$\beta \ge 1$ parametrizes an exponentially increasing sensitivity of agent i to the distance between the expectations of the i^{th} and j^{th} agents. In numerical exercises, condition $\beta = 1$ was set.

Ω is a position in the cycle defined in radians

A is the magnitude of the cycle

ϕ_i is a sequence of independent N(0,κ) random variables to represent the arrival of exogenous randomness in agent expectations where κ is a small positive number.

Eq. (3) is a form for the transition function of the deterministic model.

In Eq. (4), E is a time series of inflation-adjusted earnings in the S&P 500

n(t) is a 5-year moving average of the S&P 500 PE ratio

In Eq. (5), ζ_t is a bound on the interval of difference between price and fundamental value as indicated in Fig. 1.

To parameterize the dynamics of the transition function in numerical exercises, it is proposed that the component in bounded rational processing will dominate when the difference between fundamental value and the observed price is within one standard deviation of the predicted price or when the difference is more than two standard deviations from predicted price. Processing in the imitative component of expectations that generates "herding" will dominate in the interval where the difference of observed from the predicted price is greater than one SD but less than two SDs.[2]

[2] As indicated previously, w a monotonic relationship of component dominance to the distance of market price to a price based on fundamental value is not assumed. Here, it is assume that as in observable market dynamics, market-oriented agents can respond to too large a discrepancy between fundamental value metrics and market price by a return to bounded rational processing of the discrepancy. This is because to most agents there are discrepancies so large that they are no longer credible in even bounded rational processing.

Table 1 summarizes model parameters and their ranges in the deterministic model.

In Table 1, ρ is the parameter of a small world network that defines the proportion of vertices that are connected to a vertex that is not a neighbor or a next-neighbor, M_{ij} is the weight of the interpersonal influence of agent j on agent i, α is the relative weight of a model component, Ω is the angle of the sinusoidal signal in radians, and A is the magnitude of the signal.

4.2. Stochastic model of agent multicomponent expectations

A range of specifications (e.g., [58,76,77]) have represented realized market price as a stochastic process and is next given an explicit form. The form in SR for the model of Eqs. (3) and (4) with a stochastic market price will next be given. When the asset is an indexed bundle of equities, prices (P_t) have historically grown linearly with local perturbations some of which attain a magnitude to be classified as a cycle. Following a range of specifications, (see Paul and Baschnagel [77]), this dynamic can be given a simplified representation in Eq. (7) as Brownian (B) motion with drift:

$$P(t) = \gamma t + \delta B(t) \tag{7}$$

where γ is a growth parameter, B is a Weiner process, and δ is the magnitude of Brownian motion.

From the Law of the Iterated Logarithm for Brownian Motion (e.g., Khoshnevisan and Lewis [78]), it follows that:

$$lim_{t\to\infty}(2tloglogt)^{-\frac{1}{2}}B(t) = 1 \text{ with probability one}$$

If we assume that all agents can rationally predict the stock price perfectly in the long run, the Law of the Iterated Logarithm allows the further inference that in Eq. (2) and thereafter:

$$V(t) = \gamma \text{t}$$

Since the Brownian fluctuations become negligible in the limit, the ratio between the observed price and a metric of fundamental value in price can be assumed to get closer to 1 as time increases. However, local perturbations remain of major interest since they are persistent at any finite time.

Using basic properties of Brownian motion in Eq. (7), expectations can be written as:

$$\varepsilon_{i,t+1} = \alpha_t(V_t + (V_{t-1} - \xi_{t-1})) + (1-\alpha_t)\sum_j M_{ij}(\varepsilon_{j,t} - \varepsilon_{i,t}) + Asin(\Omega_t) \tag{8}$$

where $\alpha_t = \begin{cases} \alpha \text{ if } |P_t - V_t| \le (Var(P_t))^{\frac{1}{2}} \text{or } |P_t - V_t| \ge 2(Var(P_t))^{\frac{1}{2}} \\ 1-\alpha, \text{otherwise} \end{cases}$ and

$$M_{ij} = \begin{cases} M \text{ if } i \text{ and } j \text{ are neighbors or remote connections in the small} - \text{world network} \\ 0, \text{otherwise} \end{cases}$$

with $0 \le \alpha \le 1$, $P_t = \gamma t + \delta B_t$, $V_t = \gamma t$ and B_t is a standard Brownian motion

$$\alpha(t) = \begin{cases} \alpha_1, if\ |B(t)| \le \sqrt{t} \text{ or if } |B(t)| \ge 2\sqrt{t} \\ \alpha_2, otherwise \end{cases} \tag{9}$$

The stochastic model can be estimated in exercises with ξ in Eqs. (8) and (9) substituted for the price and in the transition function (α_1 and α_2), respectively.

Table 2 summarizes parameters and their ranges in the stochastic model.

The parameter ranges that were investigated in Table 2 correspond to those defined in Table 1 with the exception of the absence of a parameter of a forcing term for signal and the addition of γ as a time-delimited growth parameter in the function for stochastic price.

Table 2
Parameter distribution for the stochastic model.

Variable	Range
ρ	(0,1)
M_{ij}	(0,0.2)
α	(0,1)
γ	(0,5)

4.3. Definition of the SNR and SR

A range of small-world networks over the parameter (ρ) that represents an underlying topology for the model was initially defined for computational exercises. Each network is defined over 500 × 500 = 250,000 agents. In a deterministic model of agent expectations, the noise level will be defined by a parameter (κ) of the variance of an additive random variable (ϕ) to represent exogenous randomness in agent expectations. In the stochastic model, the noise level will be defined through δ which is in itself a parameter of the Brownian motion with drift that classical investigators have defined in generating market price.

In the definition of the SNR, let the noise level χ be represented by κ in a deterministic model and δ in a stochastic model. Let $\varepsilon_i^{\chi}(t)$ represent the time series of expectations of agent i that will be defined by the time series of predictor variables in Eqs. (2) and (3). Following investigators that include Krawiecki and Holyst [43], $f_i^{\chi}(t)$ was defined as a power spectral density of $\varepsilon_i^{\chi}(t)$. Let $t^* = \underset{t\in 12\ldots256}{argmax} f_i^{\chi}(t)$ stand for the location of the maximum of f_i^{χ}(t). The SNR can then be defined as

$$\text{SNR}(\chi) = E\left(\frac{f_i^{\chi}(t^*)}{f_{N^i}^{\chi}(t^*)} - 1\right)$$

where, $f_i^{\chi}(t^*)$ is the height of the maximum of f_i^{χ}(t). E here stands for the mathematical expectation computed over many different network topologies, and $f_{N^i}^{\chi}(t^*)$ is the noise background in the proximity of t^*.

The term, $f_{N^i}^{\chi}(t^*)$ can be formally defined over a representative interval as.

$f_{N^i}^{\chi}(t^*) = \frac{\sum_{k=t^*-49}^{t^*-20} f_i^{\chi}(k) + \sum_{k=t^*+21}^{t^*+50} f_i^{\chi}(k)}{60}$. Following the extensive range of background studies, (see McDonnell, Stocks, et al. [13]), the SNR can then be plotted as a function of χ, the noise level in the time domain. The instance of stochastic resonance is identified with the existence of a clear maximum of the SNR at some strictly positive χ. This study follows previous investigators in also reporting a corresponding maximum for a noise level in the frequency domain.

4.4. Procedure in analysis

Agents selected for analysis ($n = 1000$) are chosen randomly from a uniform distribution of all agents in the networks that have been defined. Market data was integrated to operationalize the models and enact a direct search algorithm that iterates over defined ranges of parameters was enacted to instantiate a universe of estimates of expectations. The occurrence of SR as a function of χ, where χ is the noise level in the time and frequency domains in these estimates was subsequently examined.

Elaborating on standard definitions of SR, its incidence was defined in the time and frequency domains as a spike of at least twice the magnitude of the maximum of any observation of the SNR sequence generated. Given demonstrations of SR in the defined models of the SNR, estimates, and graphics of predictive models of SR in the SNR over defined model parameters will be examined with AUROC methodology that has had models extensive applications in economics and life sciences (e.g., [73–75,79]).

5. Implementing models of stochastic resonance in multicomponent expectations

Results for the incidence of SR with parameters that were identified in the deterministic and stochastic instantiations of multi-component models will next be reported. A direct search procedure (e.g., Nash [80]) sampled from uniform distributions of the parameter intervals in Tables 1 and 2 and estimated corresponding SNRs as formally defined in Section 4.3. The estimates of the SNR from defined variables and selected parameter levels and combinations in this part of the exercise were from a C++ program written to represent the SNR in the proposed models.

5.1. Estimates of stochastic resonance in the deterministic model of agent expectations

The incidence of SR was first examined with the deterministic model of Eqs. (2)–(5) in which market-based measures of price, earnings, and PE ratios were implemented. As noted, background studies establish that demonstrations of SR can be in stationary and/or non-stationary predictors (e.g., Fakir [71]). The deterministic model includes a forcing term for a sinusoidal signal and parameters of A as the magnitude of the signal and Ω as the angle of the sinusoidal signal in radians.

Figs. 3 and 4 exemplify instances of SR in the time and frequency domains, respectively, that were identified with C++ code for the deterministic model. In the time domain, the definition of the presence of SR required a spike in the SNR at a noise intensity that exceeded 2 times the magnitude of other spikes in the domain. Variation in the frequency domain was generally of a smaller magnitude than that observed in the time domain. Cases of SR in the frequency domain evidenced a spike of at least 3 times the magnitude of other spikes in this domain. Instances of SR were detected in 60 of 800 simulations (7.5 %). Parameters for the exemplary instantiations are reported in the figures.

As indicated in a previous discussion, the results in Figs. 3 and 4 correspond to the so-called "signature" of SR: a distinct spike in the SNR relative to the surrounding variation of the SNR as a function of noise intensities in the time domain and a corresponding maximum in the power function for the frequency domain (e.g., [13,54,81]).

5.2. Estimates of stochastic resonance in the stochastic model of agent expectations

For this exercise with a stochastic measure of market price as introduced in Eqs. (6)–(8), 340 instances of SR in the signal-to-noise ratios (SNR) were generated in the time domain with a matching number of Fourier transforms of the signal in the frequency domain. The parameters under study in this model are shown in Table 2. As indicated, this model follows historical traditions in implementing a price that in short intervals can be represented as Brownian motion with drift. A total of 60 (0.176) of the 340 networks we investigated evidenced SR. Results with the models indicated that the model with a stochastic term for price and fundamental value generated acceptable results in the SNR without a forcing term for signal.

Figs. 5 and 6 exemplify instances of SR in the time and frequency domains, respectively, in the results with the C++ code for the model with a stochastic measure of price and their parameter levels.

Fig. 5 demonstrates a clear spike in the signal-to-noise ratio as a function of the noise intensity. The consensus of the literature cited on SR is that the presence of a unique and clear spike in the time domain of SNR is a signature of stochastic resonance in the system. Fig. 6 demonstrates a clear spike of power as a function of frequency. The literature consensus is that the presence of a unique and clear spike in the frequency domain is a further confirmation of the presence of stochastic resonance in the system. The exact parameter combination used to generate the figures is specified below the figures.

6. AUROC predictive models of the incidence of SR

Finally, estimates of the predictive power of these models were examined with Area Under the Receiver Operating Curve (AUROC) methodology that has now been applied across a range of applications in economics and life sciences and economics (e.g., [73–75,79,82]). AUROC metrics provide widely used numerical and graphic performance measures of the goodness of fit of discrete classifications in implemented models (e.g., [83,84]).

6.1. AUROC predictive estimate of the deterministic model

Table 3 reports the results of estimating an AUROC maximization model of predictive efficiency in detecting cases of SR with 10^3 cross-validations over the pool of variables defined in Table 1 for the deterministic model, AUROC models were estimated by an algorithm in the CARRoT package (Bazarova and Raseta, [85]) that implements "best subset" regressions (e.g., [86–88]) of binary model states on model parameters with cross-validation (e.g., [89,90]). The coefficients in these models were maximum likelihood estimates. Each of 10^3 cross-validations for a parameter set corresponded to the partition of an entire data set into training (90 %) and test sets (10 %). The classification model for the deterministic model had the largest AUROC of 0.66 in the (0.5,1) interval and was comprised of linear, quadratic, and interaction terms in the parameters of ρ, M_{ij}, A, Ω and Ω^2. Coefficients and their respective confidence intervals of the mixed AUROC model are reported in Table 3. The AUROC figure for the model itself is presented in Fig. 7. The best fit predictive model includes non-linear terms in the interaction between the small-world parameter (ρ) and the weight of the influence of other agents on an agent's expectations (M_{ij}) and the square the angle (Ω) in the sinusoidal signal in radians.

Fig. 7 reports the Area Under the Receiver Operating Curve (AUROC) for the best-performing model under deterministic price assumption. The AUROC was obtained by fitting the model with the best performance on the entire data set which is a logistic regression whose variables and coefficients are fully specified in Table 3. The convention of defining the x-axis as (1-Specificity) was followed.

6.2. AUROC predictive estimates of the stochastic model

This model was estimated over parameters in Table 2. The parameter ranges with the largest AUROC of 0.903 in the (0.5,1) interval were comprised of $\rho^2$and M_{ij}^2. Model coefficients and their respective confidence intervals for this model are shown in Table 4. The AUROC is presented in Fig. 8.

Fig. 8 reports the Area Under the Receiver Operating Curve (AUROC) for the best-performing model under the assumption that price follows a Brownian motion with drift. The AUROC was obtained by fitting the model of the binary states of SR to parameters with the best performance over the entire data set in logistic regression. The variables and coefficients are fully specified in Table 4.

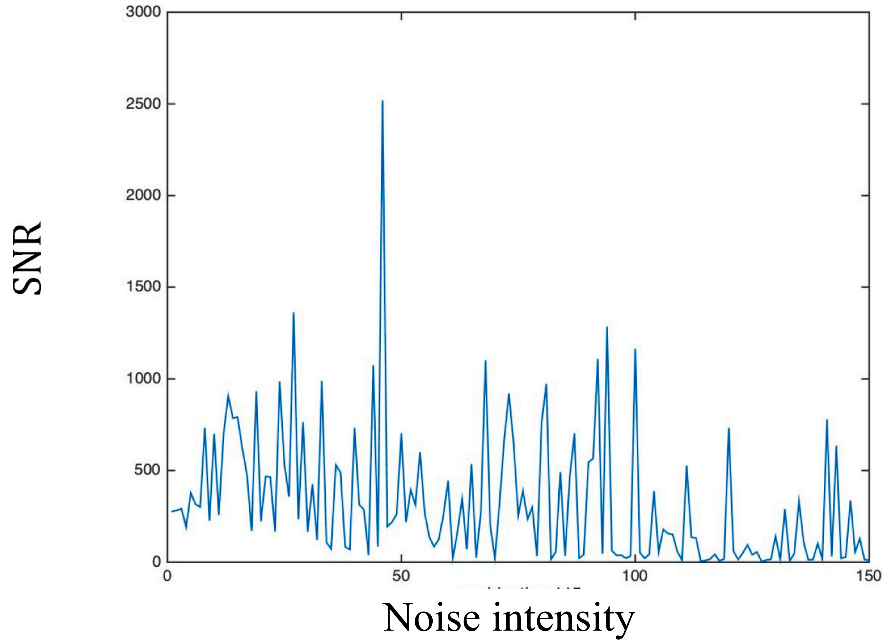

Fig. 3. SNR as a Function of Noise Intensity in the Time Domain of an SWN: Deterministic Model[1].
1 $\rho = 0.1681$; $M_{ij} = 0.0176$; $\alpha = 0.8134$; $\Omega = 0.2501$; $A = 791.637$.

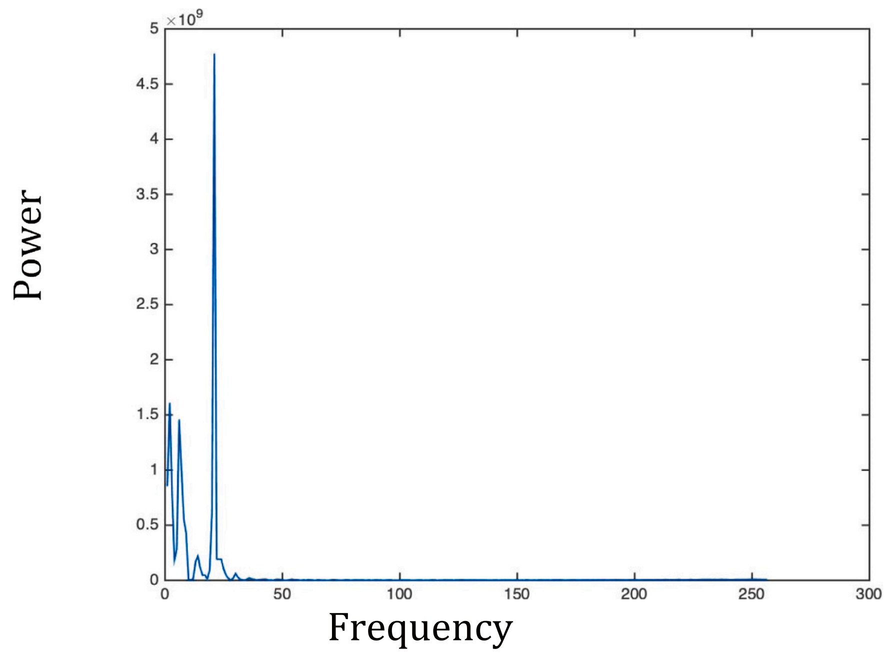

Fig. 4. SNR as a function of Noise Intensity in the Frequency Domain of an SWN: Deterministic Model[1]
1 $\rho = 0.1681$; $M_{ij} = 0.0176$; $\alpha = 0.8134$; $\Omega = 0.2501$; $A = 791.637$.

6.3. Summary of model estimates of SR

The results reported for models in both the deterministic and stochastic specifications are consistent with the presence of stochastic resonance in multicomponent models of agent expectations for financial markets. It has been shown that parameterized of the proposed deterministic and stochastic models demonstrate what is widely defined to be the signature of SR in the signal-to-noise ratio in both the time and frequency domains. These results support the capability of a noise term to increase the detection of signal in multicomponent expectations for price. Following these demonstrations, estimates of widely implemented AUROC models that support the predictive capabilities of the parameters in the models of SR in agent expectations have been reported.

In the results for the AUROC deterministic model, non-linearity was detected in an interaction between the SWN parameter of remoteness in the SWN (ρ) and the parameter for the magnitude of the influence of other agents (M_{ij}) and a squared term in the angle of the signal in radians (Ω). In the results with the stochastic AUROC model, squared terms in the structural parameter of the SWN (ρ) and the magnitude of other agents' influence (M_{ij})were the strongest predictors of the occurrence of

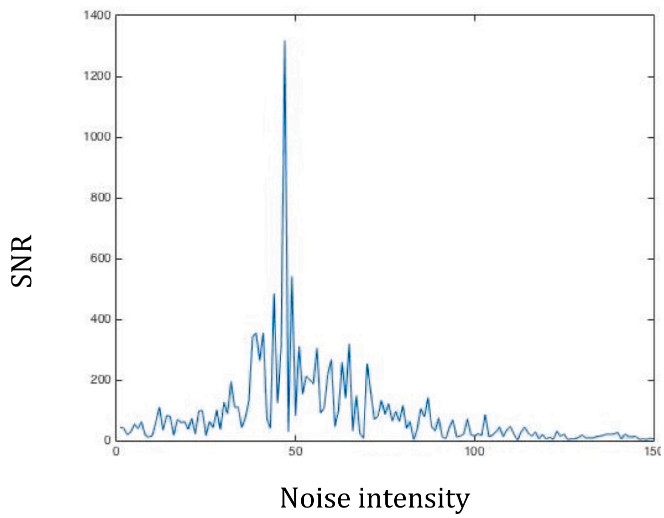

Fig. 5. The SNR as a Function of Noise Level in the Time Domain: Stochastic Model
[1] $\rho = 0.1890$; $M_{ij} = 0.0066$; $\alpha = 0.9556$; $\gamma = 0.8900$.

SR. The results of the AUROC models indicate the importance that the representation of the network organization of agents in a component for "herding" can have to the representation of agent expectations.[3]

Table 3
Coefficient Estimates and Confidence Intervals in an AUROC deterministic model.

Variable	Coefficient	95 % CI
ρM_{ij}	68.03689	(68.03688, 68.03690)
A	-2.165204×10^{-4}	$(-4.904985\times10^{-4}, 5.745767\times10^{-5})$
Ω	639.5538	(639.5537, 639.5539)
Ω^2	−1321.397	(−1321.398, −1321.396)

7. Summary and discussion

Following a review of expectations as precursors of price levels in financial markets, a multi-component model of price expectations was introduced. In the multi-component model, a component in bounded rational processing and a component that represents the influence of other agents on an agent's expectations and generates commonly cited "herding" in financial markets were represented. A transition function for component dominance that depends on the distance between market price and metrics of fundamental value was further specified. A behavioral basis for transitions in component dominance was elaborated in support of the form given in the specification of this function. Following a background review of multi-component models of expectations, we have defined an agent objective in the detection of signal on the direction and level of expectations for market price.

While in most models of expectations, exogenous randomness is assumed to interfere with detecting the equivalent of signal for market price levels, conditions in which randomness can facilitate the detection

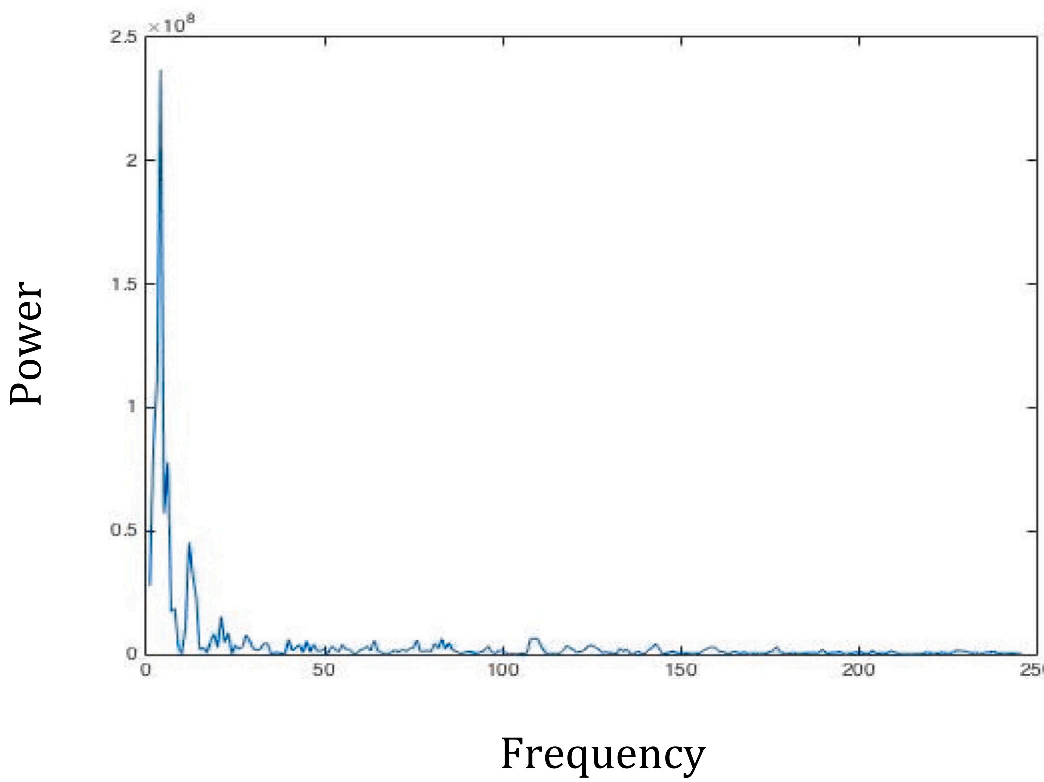

Fig. 6. Power Function in the Frequency Domain: Stochastic Model[1]
[1] *$\rho = 0.1890$; $M_{ij} = 0.0066$; $\alpha = 0.9556$; $\gamma = 0.8900$.*

[3] The M_{ij} parameter in the models index the influence of referent agents (i.e., *a* cohort) on an agent in a component of "herding" [29,30].

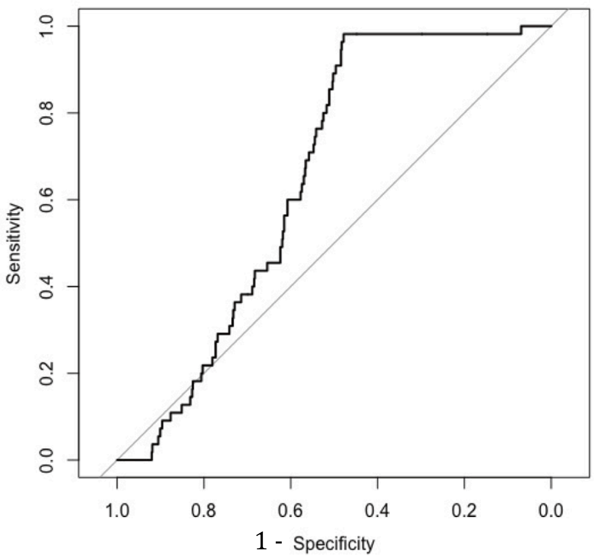

Fig. 7. AUROC for Binary Classifiers of SR: Deterministic Model[1]
[1] By convention, sensitivity is the proportion of the cases that were correctly identified by the model as evidencing SR. Specificity is the proportion of the cases identified by the model as not evidencing SR.

Table 4
Coefficient estimates and coefficient intervals: AUROC stochastic model.

Variable	Coefficient	95 % CI
ρ^2	−8.364157	(−11.20621, −5.522101)
$M_{ij}{}^2$	−120.4154	(−162.6336, −78.19708)

of signal have been reviewed. Stochastic resonance has been elaborated as a general process through which randomness can increase efficiency in the objective of detecting signal across a wide range of applications. In subsequent sections, SR in a signal-to-noise ratio for expectations was defined and directly investigated in alternative operationalizations of the multi-component models of agent price expectations.

In a deterministic model of the SNR, data on the S&P index was used for market price, mean earnings, and historical PE ratios of the companies in the index to operationalize fundamental value in the model. In a stochastic model, a derived price following historical representations of price dynamics in short intervals as Brownian motion with drift was specified. As indicated in Eq. (1) and (3) of the general model of the deterministic model, and Eq. (8) of the stochastic model, the difference between market price and fundamental value defines component weights through a proposed transition function.

In exercises with both models, small-world networks (SWN) models were implemented to represent structure in agent interactions within a component that generates "herding" in agent price expectations. A direct search algorithm in C++ code was used to identify instances of SR in the parameter spaces of the signal-to-noise ratios of defined models. Results with both of the models we implement support SR as a process through which noise can contribute to the detection of signal on price in financial markets.

AUROC of predictive models of the discrimination of SR over the defined parameter ranges of model parameters for both the deterministic and stochastic models were subsequently reported. Across all estimations of both the models, parameters of network structure in the SWN (ρ) and the magnitude of inter-agent influence (M_{ij}) in the component that generates "herding" was indicated to be among the significant predictors of SR. Taken together, our results provide strong support for the capability of the proposed system to accommodate SR in the recovery of signal on expectations for price in financial markets.

These results, in turn, further indicate the importance that randomness can have to efficient markets. The recovery of signal on price is a common objective in a range of applications to financial, and commodity markets ([18,50]). In most market applications, as in most engineering applications, the objective has been in recovering signal in cases where noise can obscure the signal. Here, as in applications in a range of disciplines that have been cited, there is a basis to anticipate that under certain conditions, randomness in expectations for financial markets can increase rather than detract from efficiency in agent objectives of detecting signal. The results that have been reported add to evidence that cases in which noise can contribute to the recovery of the signal in financial markets are likely to be more general than had previously been recognized.

The contribution that randomness can make to detecting signal

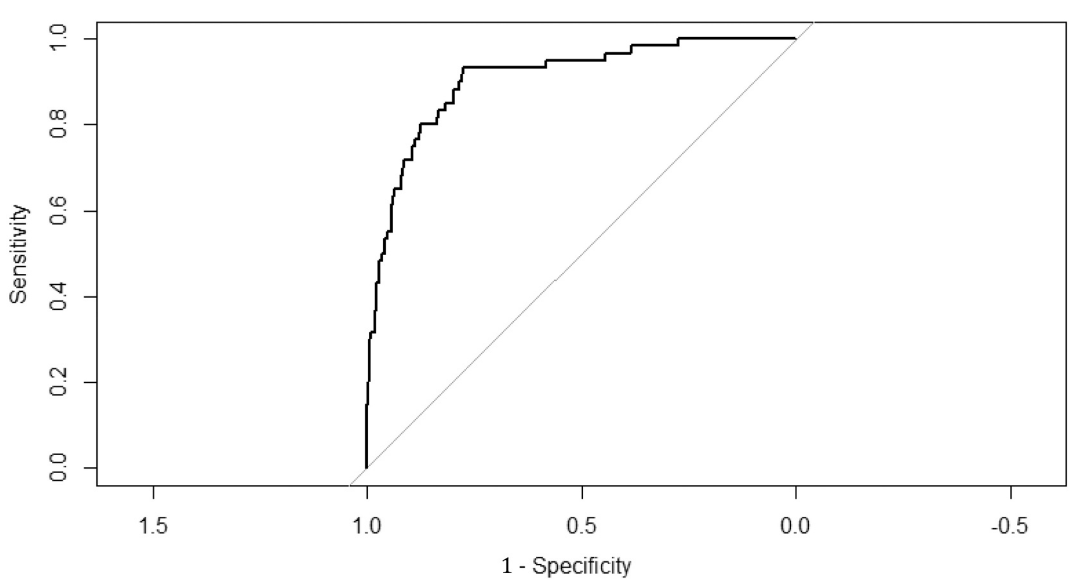

Fig. 8. AUROC for Binary Classifiers of SR: Stochastic Model[2]
[2]By convention, sensitivity is the proportion of the cases that were correctly identified by the model as evidencing SR. Specificity is the proportion of the cases identified by the model as not evidencing SR.

under certain conditions does not necessarily mean that one can directly "tune" a representation of randomness to increase efficiency in the detection of signal on market price [13]. In market applications as in other applications, randomness is likely to be embedded in structural determinations of price. It does further support the indication that the extensive treatment of randomness as decreasing efficiency in the detection of signal on price in financial markets can be modified for the case of stochastic resonance. Previous results with binary states of "bubbles" and "crashes" in financial markets (e.g., [43]) have been extended to a case of continuous processing in multi-component expectations in the results that we report.

7.1. Directions for subsequent studies

Finally, several directions for the discourse on models of SR in price expectations and their implementation in financial markets are noted. First, it is noted that contrary to strict perspectives in natural selection that others have applied to the evolution of economic institutions, including those that govern markets (e.g., [91,92]), it can be a sufficient condition for efficiency-increasing noise to have been coordinated with other processes in dynamic systems that increased efficiency in signal detection. Once in place, efficiency-increasing noise can become structurally integrated in dynamics and remain operative. The basis for such an observation can be found in a number of examples of "hitchhiking" in sensory and physical processes. (e.g., Barton [93]).

Second, there are alternatives in representations of how agents converge to levels of noise that maximize an SNR that merit being investigated. As now well documented, self-organization (e.g., [94,95]) can allow convergence to an equilibrium from semi-autonomous behavior (i.e., behavior that uses greatly simplified rules to coordinate with other agents) that results in organized behavior at a group level (e.g., [96–98]). In well-recognized examples in studies of "flocking" (e.g., [99,100]), entities in these studies are commonly used rules to organize that are simple for agents, (e.g., distance from others) but have emergent properties at the group level that do benefit individual entities (e.g., resistance to predators). As such, there may be a basis to examine corresponding applications of "swarm-based" coordination of the expectations of agents in financial markets that can contribute to the efficiency that noise can have in signal detection.

Third, while closed systems have understandingly predominated in the demonstration of SR [13], there is a basis to anticipate that further investigation of SR in open systems (Trushechkin et al. [101]) can advance our capabilities in integrating sources of randomness in SR in financial markets and other applications. In such cases, SR can occur through internal processing that accommodates characteristic randomness from environments. While such processing has been recognized in closed systems of experimental studies [13], the representation of open system processing is further by recent representations. See, [101] on dynamics in open quantum systems. Also, see Rodrigo's [102] integration of intrinsic and extrinsic noise in the efficient processing of genetic signals that include SR.

Fourth, and finally, somewhat counter-intuitively, it appears that increased randomness in agent characteristics that result in increased variance in an emergent mean signal can increase subsequent accuracy in the prediction of a signal. Elton, Guber, and Gultekin [103] demonstrate this directly in the convergence of analysts in a financial market. Harnett et al. [104] demonstrate this in their application to collective decision-making. More fine-grained representation of heterogeneity in interactive agents (e.g. Oliva, Panzieri, and Setola [105]) possibly through social media [106] may further our understanding of how even random differentiation of agents in a financial market can contribute to accuracy in the recovery of signal through SR.

8. Conclusions

The importance of agent expectations in understanding dynamics of financial markets continues to be demonstrated by a range of investigators. Multicomponent models remain conceptually and empirically relevant in the study of expectations for price in financial markets. The results that we have reported with multicomponent models contribute to the evidence that stochastic resonance, as initially demonstrated in biological and physical systems ([13,40,104]) is a much more general mechanism that can have an important function in detecting signals in financial markets. Whereas background studies have reported SR in market cycles of "bubbles and crashes", we have examined SR in continuous, multi-component models of price expectations. Results reported for both deterministic and stochastic models suggest that under certain conditions the randomness that is endemic to financial markets can contribute rather than detract from the recovery of signal in agent expectations. The information-theoretic framework referenced in defining the SNR can provide a useful starting point to investigate the mechanisms of SR and their generality in other competitive markets. From the above, processing in multicomponent models for signal from expectations and the study of conditioning variables that include randomness continue to be directions that contribute to our understanding of the dynamics of financial markets.

CRediT authorship contribution statement

Steven D Silver and Marko Raseta are responsible for conceptualization of the model and model estimation.

Alina Bazarova is a contributor to model estimation and AUROC estimation.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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