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Computation in a Localization of the Free Group Algebra

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ABSTRACT

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The coefficients of a Taylor series expansion of any rational function in one variable satisfy a linear recurrence relation. Our main result is a generalization of this statement for rational functions of multiple non-commutative variables. We show that if such a function is represented in the form of a non-commutative formal power series via Magnus embedding, then the coefficients of this formal power series are determined by a finite set of linear homogeneous recurrence relations. This finite representation of an infinite series allows for efficient computation of operations (multiplication, addition, and in many cases inversion) on non-commutative rational functions.

DEDICATION

To my parents for their endless love and patience.

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INTRODUCTION

A real number is rational if and only if it has a recurring decimal representation. A parallel statement can be made about ordinary rational functions: a function in one variable is rational if and only if the coefficients of a Taylor series expansion of this function satisfy a linear recurrence relation.

In this thesis, we prove a generalization of this statement for rational functions of multiple non-commutative variables, as defined by Farber and Vogel [FV91]. Following their definition, we understand a non-commutative rational function to be a non-commutative formal power series whose FV-partial derivatives generate a finite dimensional vector space. Our main result proves that the coefficients of such formal power series are defined by a finite set of linear homogeneous recurrence relations.

This result allows us to represent a non-commutative rational function given by an infinite series with a finite amount of data, namely, an initial coefficient set and a set of linear recurrences. This representation in turn allows for efficient computation of operations (multiplication, addition, and in many cases inversion) on non-commutative rational functions.

The following is a brief summary of the rest of this thesis. First, in chapter 2, we offer an overview of basic topics from abstract algebra used in subsequent chapters, including modules, annihilators, and ideals. In chapter 3, we continue the theory overview and discuss the free monoid X^*, the free group F_m, and the group ring $R[F_m]$, where R is a commutative ring with unity. We also define formal power series (FPS) $\sum_{\alpha \in X^*} a_\alpha \alpha$ in commutative and non-commutative variables, and the

rings $R[[x_1, x_2, \ldots, x_m]]$ and $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ of commutative and non-commutative formal power series respectively, and discuss which elements of these rings have multiplicative inverses.

To define a rational function in non-commutative variables we also need a notion of a derivative in non-commutative structures such as $R[F_m]$ and $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$. In chapter 4 we introduce the Fox derivative [Fox53] D_{t_i} defined on $R[F_m]$, and the FV-derivative [FV91] ∂_{x_i} defined on $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$. We establish the rules that these derivatives follow and prove an analog of Leibniz's Rule for both of them. We also establish that $R[F_m]$ is embedded in $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ via Magnus embedding (denoted by μ), and show that the following diagram commutes.

$$\begin{array}{ccc} R[F_m] & \xrightarrow{\quad D_i \quad} & R[F_m] \\ \mu \downarrow & & \downarrow \mu \\ R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle & \xrightarrow[\quad \partial_i \quad]{} & R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle \end{array} .$$

In chapter 5 we explore periodicity of the formal power series representation $A(x)$ of a rational function in one variable. For this purpose, we introduce a measure of periodicity. To define it, we prove that the following three quantities are equal: the degrees of freedom of the linear recurrence sequence $\{a_i\}$, the dimension of the vector space generated by the FV-derivatives of A, and the degree of the monic polynomial that generates the annihilator of A. We call this periodicity measure the depth of A. Also, given the depth of two FPS A and B, we establish upper bounds for the depth of $A + B$, AB, and A^{-1} and give an algorithm to find their actual depth. Furthermore, given a recurrence relation representation of A and B, we find the analogous representations for $A + B$, AB, and A^{-1}.

In chapter 6, we explore periodicity of the formal power series representation $A(x, y)$ of a rational function in two non-commuting variables. We prove our main result that the coefficients of a FPS representation of a rational function in non-commuting variables satisfy a finite set of linear homogeneous recurrence relations. We introduce a notion analogous to depth in the one-variable case, and call it a minimal determining core of A, and investigate cores of of $A + B$, AB, and A^{-1} when cores of A and B are given. Also, as in the one variable case, we describe a finite set that must contain a minimal determining core of $A + B$, AB, and A^{-1}, and give an algorithm to reduce it to an actual minimal determining core. Furthermore, given a recurrence relation representation of A and B, we provide an algorithm to find the analogous representations for $A + B$, AB, and A^{-1}. We conclude the chapter with a detailed example.

CHAPTER 2

MODULES AND ANNIHILATORS

In this chapter, we offer a brief overview of modules, ideals, and annihilators. We assume familiarity with basic concepts from abstract algebra and linear algebra.

Definitions and results cited in this chapter are widely known and can be found in various books on algebra such as Jacobson [Jac76] and [Jac84]. In particular, we use Hungerford [Hun96] as a general reference text for this chapter.

Unless otherwise is stated, we assume that the ring R is a commutative ring with unity.

2.1 Modules

A module over a ring is a generalization of a vector space over a field. In a module the corresponding scalars are the elements of a ring R (rather then a field).

Definition 2.1.1. Let R be a ring, and 1_R be the multiplicative identity of R. A *left R-module M* is an additive abelian group $(M, +)$ and a left action

$$R \times M \to M$$

that satisfies the following axioms for all r, s in R and all x, y in M:

(1) $r(x + y) = rx + ry$,

(2) $(r + s)x = rx + sx$,

(3) $(rs)x = r(sx)$,

(4) $1_R x = x$.

A right R-module M is defined similarly, except that the ring acts on the right.

Definition 2.1.2. Let R be a ring, and 1_R be the multiplicative identity of R. A *right R-module M* is an additive abelian group $(M, +)$ and a right action

$$M \times R \to M$$

that satisfies the following axioms for all r, s in R and all x, y in M:

(1) $(x + y)r = xr + yr$,

(2) $x(r + s) = xr + xs$,

(3) $x(rs) = (xr)s$,

(4) $x1_R = x$.

Note that all future references to "module" will mean "left R-module", unless otherwise noted.

Theorem 2.1.3. *If M is a module with additive identity element 0_M over a ring R with additive identity 0_R, then the following identities hold for all x in M, r in R, and n in $\mathbb{Z}$.*

(1) $r0_M = 0_M$,

(2) $0_R x = 0_M$,

(3) $(-r)x = -(rx) = r(-x)$,

(4) $n(rx) = r(nx)$.

Proof. We prove (2) as an example. We see that

$$0_R x = (1_R - 1_R)x = 1_R x - 1_R x = x - x = 0_M. \tag{2.1}$$

□

Definition 2.1.4. Let M be a module over a ring R. A subset $S = \{x_1, x_2, \ldots, x_n\}$ of M is *linearly independent* if the equation

$$r_1x_1 + r_2x_2 + \cdots + r_nx_n = 0_M \tag{2.2}$$

can only be satisfied by $r_1 = r_2 = \cdots = r_n = 0_R$. This implies that no element in S can be represented as a linear combination of the remaining elements in S.

Definition 2.1.5. Let M be a module over a ring R. A subset $S = \{x_1, x_2, \ldots, x_n\}$ of M is a *generating set* of M if any element of M can be written as a linear combination of the elements of S, that is, for every y in M, there exist $r_1, r_2, \ldots, r_n$ in R such that

$$y = r_1x_1 + r_2x_2 + \cdots + r_nx_n. \tag{2.3}$$

Definition 2.1.6. A linearly independent generating set is called a *basis* of M.

Definition 2.1.7. Given a set X, a *free module* on X is a module with basis X.

Definition 2.1.8. A left module M over a commutative ring R equipped with an R-bilinear product is called an *algebra* over commutative ring R, or *R-algebra*. By R-bilinear product we understand a map $f : M \times M \to M$ that is linear in each argument separately, meaning that if we fix the first argument and let the second vary, or vary the first and fix the second, the resulting map is linear. In other words, the following properties hold for all x, x', y, y' in M, and r in R:

(1) $f(x, ry) = f(rx, y) = rf(x, y)$,

(2) $f(x, y) + f(x, y') = f\big(x, (y + y')\big)$,

(3) $f(x, y) + f(x', y) = f\big((x + x'), y\big)$.

2.2 Annihilators and Ideals

Definition 2.2.1. Let R be a ring, and let M be a left R-module. Suppose S is a nonempty subset of M. The *annihilator* of S, denoted $\mathrm{Ann}(S)$, is the set of all elements r in R such that for each s in S, $rs = 0$. It can be written in set notation as

$$\mathrm{Ann}(S) = \left\{ r \in R \;\middle|\; rs = 0,\ \forall s \in S \right\}. \tag{2.4}$$

Definition 2.2.2. A subring I of a ring R is a *right ideal* of R if for every r in R and every x in I, xr is in I. Equivalently, a right ideal of R is a right R-submodule of R. Similarly a subring I of a ring R is a *left ideal* of R if for every r in R and every x in I, rx is in I. Equivalently, a left ideal of R is a left R-submodule of R.

Definition 2.2.3. A subring I of a ring R is a *two-sided ideal* of R if it is both a left and a right ideal of R. If R is commutative, any ideal is two-sided, so we call it simply an *ideal*.

Example 2.2.4. It is easy to see that the set $aR = \{ar | \, r \in R\}$ is an ideal in a commutative ring R. It is said to be generated by the element a and is denoted by $\langle a \rangle$. Ideals generated by a single element of R are called *principal ideals*.

Definition 2.2.5. A nonzero commutative ring in which the product of any two nonzero elements is nonzero is called an *integral domain*, and an integral domain in which every ideal is principal is called a *principal ideal domain*, or PID.

Theorem 2.2.6. *The annihilator of a subset S of a left R-module M is a left ideal of R.*

Proof. $\mathrm{Ann}(S)$ contains 0, and is therefore nonempty. Let a and b be in $\mathrm{Ann}(S)$,

and let s be an element of S and r be an element of R. We see that

$$(a-b)s = as - bs = 0 - 0 = 0, \text{ so } a - b \in \text{Ann}(S); \text{ also,} \tag{2.5}$$

$$(ra)s = r(as) = r0 = 0, \text{ so } ra \in \text{Ann}(S). \tag{2.6}$$

Therefore, $\text{Ann}(S)$ is a left ideal of R. □

CHAPTER 3

GROUP RINGS AND FORMAL POWER SERIES

In this chapter we continue our overview of fundamental concepts and talk about free monoids and groups, group rings, and formal power series. All disclaimers from the previous chapter still hold. Unless otherwise stated, we use Hungerford [Hun96] as our standard reference.

Furthermore, R is assumed to be a commutative ring with unity, and $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers.

3.1 Free Monoids and Groups

Definition 3.1.1. A *monoid* is a nonempty set M together with a binary operation that satisfies the following two axioms for all a, b, and c in M:

(1) $a(bc) = (ab)c$;

(2) There exists an element called 1_M in M such that $a1_M = 1_M a = a$.

Definition 3.1.2. If for every element a in a monoid M there exists an inverse element a^{-1} in M such that

$$a^{-1}a = aa^{-1} = 1_M, \tag{3.1}$$

then M is called a *group*.

Definition 3.1.3. If the operation in a monoid (group) M is commutative, i.e., if for all a, b in M, $ab = ba$, we call M *abelian*.

Definition 3.1.4. Let X be a set that contains no inverses of its elements; i.e. if x is in X, then x^{-1} is not in X. The *free monoid* on a set X, usually denoted X^*, is the monoid whose elements are all the finite sequences (words or strings) of elements from X. The identity element, denoted 1_{X^*}, or simply 1, is the unique sequence of zero elements (often called the empty string), and the monoid operation is string concatenation.

Definition 3.1.5. Let $X = \{t_1, t_2, \ldots, t_m\}$ be a set that contains no inverses of its elements, and let $X^{-1} = \{t_1^{-1}, \ldots, t_m^{-1}\}$ be the set of the inverses of elements of X. The *free group* on the set X (of size m), denoted F_m, is the free monoid on the set $X \cup X^{-1}$ modulo the relation $t_i t_i^{-1} = t_i^{-1} t_i = 1$ for $i = 1, 2, \ldots, m$. The members of X are called *generators* of F_m.

Theorem 3.1.6. *Concatenation in Definitions 3.1.4 and 3.1.5 is associative, and the free monoid and free group, therefore, are well defined.*

Proof. See Hungerford [Hun96, p. 65, Thm. 9.1]. □

Remark 3.1.7. Theorem 3.1.6 is straightforward for the free monoid. However, in case of the free group where internal cancellation is possible, the statement becomes nontrivial to show.

Example 3.1.8. The additive group of integers, $\mathbb{Z}$, is the free group with a single generator (F_1), and the fundamental group of the figure eight graph is the free group with two generators (F_2).

Definition 3.1.9. We use the definition of a *reduced word* given in [Fox53]. Let F_m be the free group generated by $\{t_1, t_2, \ldots, t_m\}$. An element of F_m is an equivalence class α of words and is represented by a unique *reduced word* $\alpha = t_{i_1}^{k_1} \ldots t_{i_n}^{k_n}$, where $k = \pm 1$, and $k_j + k_{j+1} \neq 0$ if $i_j = i_{j+1}$.

Theorem 3.1.10. *Every element of F_m has a unique representation as a reduced word.*

Proof. See Hungerford [Hun96, p. 64]. □

Definition 3.1.11. If $\alpha = t_{i_1}^{k_1} \dots t_{i_n}^{k_n}$ is a reduced word in F_m, then its *length* is defined by $\text{length}(\alpha) = \sum_{j=1}^{n} |k_j|$, and the length of the identity element is 0. By the *length* of any element of F_m we mean the *length* of the representative reduced word.

3.2 Group Rings

A group ring is a free module and at the same time a ring, constructed in a natural way from any given ring R and any given group G. If the given ring is commutative, a group ring is also referred to as a group algebra.

Definition 3.2.1. Let R be a ring with unity (not necessarily commutative), and G be a group written multiplicatively. The *group ring* of G over R, denoted $R[G]$, is the free R-module with basis G. Elements of $R[G]$ are finite formal sums of the form $\sum_{g \in G} r_g g$, where finitely many r_g's are nonzero. Addition in the group ring $R[G]$ is given by:

$$\sum_{g \in G} r_g g + \sum_{g \in G} s_g g = \sum_{g \in G} (r_g + s_g) g. \tag{3.2}$$

(By inserting zero coefficients if necessary, we can ensure that the sums involve exactly the same indices).

Multiplication in $R[G]$ is given by

$$\sum_{g \in G} r_g g \sum_{h \in G} s_h h = \sum_{g \in G} \sum_{h \in G} (r_g s_h)(gh). \tag{3.3}$$

This makes sense since there is a product defined in both R and G, and both sums on the left have only finitely many nonzero coefficients. Therefore, the expression on the right is a finite formal sum as desired.

With these operations $R[G]$ is a ring, called the *group ring* of G over R. The ring $R[G]$ is commutative if and only if both R and G are commutative. The identity element of $R[G]$ is $1_R 1_G$, which we usually write as 1. More generally, for $r \in R$, we identify r with $r1_C$.

Theorem 3.2.2. *$R[G]$ is a ring.*

Proof. This holds since ring axioms hold component-wise, and hence can be extended to finite sums. As an example we verify that multiplication distributes over addition; i.e. for any elements u, v, and w of the ring $R[G]$, $u(v+w) = uv + uw$. Let $u = \sum_{g \in G} r_g g$, $v = \sum_{h \in G} s_h h$, and $w = \sum_{h \in G} t_h h$. (Because we can insert terms with zero coefficients, we can assume that v and w have the same terms.) Then

$$\begin{aligned} u(v+w) &= \sum_{g \in G} r_g g \left(\sum_{h \in G} s_h h + \sum_{h \in G} t_h h \right) \\ &= \sum_{g \in G} r_g g \sum_{h \in G} (s_h + t_h) h \\ &= \sum_{g,h \in G} \big(r_g (s_h + t_h)\big) gh \\ &= \sum_{g,h \in G} \big(r_g s_h + r_g t_h\big) gh \\ &= \sum_{g,h \in G} (r_g s_h) gh + \sum_{g,h \in G} (r_g t_h) gh \\ &= \left(\sum_{g \in G} r_g g \sum_{h \in G} s_h h \right) + \left(\sum_{g \in G} r_g g \sum_{h \in G} t_h h \right) \\ &= uv + uw. \end{aligned} \tag{3.4}$$

The theorem follows. □

Example 3.2.3. Of particular interest to us is the case where the group G is the free group F_m, and correspondingly the group ring is $R[F_m]$, which we will sometimes denote by Λ.

Let F_m be a free group on the set $X = \{t_1, t_2, \ldots, t_m\}$. The *group ring* of F_m over R, denoted $R[F_m]$, is an R-module with basis F_m. Elements of $R[F_m]$ are sums of the form $\sum_{\alpha \in F_m} a_\alpha \alpha$, where $\alpha = t_{i_1}^{k_1} \ldots t_{i_n}^{k_n}$ is a word in F_m, and a_α is in R. Furthermore, only finitely many a_α are nonzero.

Addition in the group ring $R[F_m]$ is given by:

$$\sum_{\alpha \in F_m} a_\alpha \alpha + \sum_{\alpha \in F_m} a'_\alpha \alpha = \sum_{\alpha \in F_m} (a_\alpha + a'_\alpha)\alpha, \tag{3.5}$$

and multiplication in $R[F_m]$ is defined as follows:

$$\sum_{\alpha \in F_m} a_\alpha \alpha \sum_{\beta \in F_m} b_\beta \beta = \sum_{\alpha, \beta \in F_m} (a_\alpha b_\beta)(\alpha\beta) = \sum_{\gamma \in F_m} c_\gamma \gamma, \tag{3.6}$$

where $c_\gamma = \sum_{\alpha\beta = \gamma} a_\alpha b_\beta = \sum_{\alpha \in F_m} a_\alpha b_{\alpha^{-1}\gamma}$.

3.3 Formal Power Series (FPS)

Definition 3.3.1. Let R be a commutative ring with unity and X_m a free abelian monoid (written multiplicatively) on the set $X = \{x_1, x_2, \ldots, x_m\}$. A *commutative formal power series* is an infinite linear combination of elements of the monoid X_m with coefficients from the ring R. The set of all such sums is denoted $R[[x_1, x_2, \ldots, x_m]]$ and is called the *(commutative) ring of formal power series* over the ring R. The power series in $R[[x_1, x_2, \ldots, x_m]]$ is denoted by the formal sum $\sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} r_\varepsilon x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_m^{\varepsilon_m}$, where ε is an m-tuple of non-negative integers.

Addition and multiplication in $R[[x_1, x_2, \ldots, x_m]]$ are defined by:

$$\sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} r_\varepsilon x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m} + \sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} s_\varepsilon x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m} = \sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} (r_\varepsilon + s_\varepsilon) x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m}, \tag{3.7}$$

and

$$\sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} r_\varepsilon x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m} \sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} s_\varepsilon x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m} = \sum_{\varepsilon \in \mathbb{Z}^m_{\geq 0}} t_\varepsilon x_1^{\varepsilon_1} \ldots x_m^{\varepsilon_m}, \tag{3.8}$$

where $t_\varepsilon = \sum_{\varepsilon'+\varepsilon''=\varepsilon} r_{\varepsilon'} s_{\varepsilon''}$.

We define non-commutative formal power series similarly, except we do not assume the monoid to be abelian.

Definition 3.3.2. Let R be a commutative ring with unity and X^* a free monoid on the set $X = \{x_1, x_2, \ldots, x_m\}$. The *non-commutative formal power series*, denoted $\sum_{\alpha \in X^*} a_\alpha \alpha$, is an "infinite linear combination" of elements of the monoid X^* with coefficients from the ring R.

The set of all such sums is denoted $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ and is called the *non-commutative ring of formal power series* over the ring R. Addition and multiplication in $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ are defined by:

$$\sum_{\alpha \in X^*} a_\alpha \alpha + \sum_{\alpha \in X^*} a'_\alpha \alpha = \sum_{\alpha \in X^*} (a_\alpha + a'_\alpha)\alpha, \tag{3.9}$$

$$\sum_{\alpha \in X^*} a_\alpha \alpha \sum_{\beta \in X^*} b_\beta \beta = \sum_{\alpha,\beta \in X^*} (a_\alpha b_\beta)(\alpha\beta) = \sum_{\alpha,\beta \in X^*} c_\gamma(\alpha\beta), \tag{3.10}$$

where $c_\gamma = \sum_{\alpha\beta=\gamma} a_\alpha b_\beta$.

Theorem 3.3.3. *The sum $c_\gamma = \sum_{\alpha\beta=\gamma} a_\alpha b_\beta$ is finite. Therefore, multiplication in $\Gamma = R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ is well defined.*

Proof. Let α, β, and γ be elements of X^* such that $\gamma = \alpha\beta$. Suppose that length$(\gamma) = n$, $n \in \mathbb{Z}$. In the free monoid X^*, cancellation is not possible, which implies that there are at most $n+1$ ways to construct γ from 2 words by concatenation. Therefore, the sum $c_\gamma = \sum_{\alpha\beta=\gamma} a_\alpha b_\beta$ is finite; in fact, it has at most $n+1$ terms. □

Theorem 3.3.4. *Multiplication in $R[[x_1, x_2, \ldots, x_m]]$ is also well defined since the sum $\sum_{\varepsilon'+\varepsilon''=\varepsilon} r_{\varepsilon'} s_{\varepsilon''}$ is finite.*

Proof. An argument similar to the one given in the proof of Theorem 3.3.3 shows that the sum $\sum_{\varepsilon'+\varepsilon''=\varepsilon} r_{\varepsilon'} s_{\varepsilon''}$ has at most $\prod_{i=1}^{m}(\varepsilon_i+1)$ terms. □

Corollary 3.3.5. *$\Gamma = R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ and $R[[x_1, x_2, \ldots, x_m]]$ are rings.*

Proof. After Theorems 3.3.3 and 3.3.4, the remaining details are analogous to proof of Theorem 3.2.2 and will be omitted. □

Theorem 3.3.6. *Let $A = \sum_{\alpha\in X^*} a_\alpha \alpha$ be a formal power series (possibly non-commutative). A has an inverse if and only if its constant term is a unit in R..*

Proof. Suppose A is invertible, that is there exists a formal power series B such that $AB = 1$. Then, if $AB = \sum_{\alpha,\beta\in X^*} c_\gamma(\alpha\beta)$, where $c_\gamma = \sum_{\alpha\beta=\gamma} a_\alpha b_\beta$,

$$c_\gamma = \sum_{\alpha\beta=\gamma} a_\alpha b_\beta = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

In particular, $a_1 b_1 = 1$, and, hence, a_1 is a unit in R.

We will prove the converse by induction on the length of γ. Suppose a_1 is a unit in R, and $b_1 = \frac{1}{a_1}$, and assume that for any β such that $1 \leq \text{length}(\beta) \leq k$, $k \in \mathbb{Z}$, b_β's are defined so that $c_\beta = 0$. Let γ_0 be a word of length $k+1$. We know that

$$c_{\gamma_0} = a_1 b_{\gamma_0} + \sum_{\alpha\beta=\gamma_0,\, |\beta|\leq k} a_\alpha b_\beta. \tag{3.12}$$

Therefore, if we define b_{γ_0} to be

$$b_{\gamma_0} = -\frac{1}{a_1} \sum_{\alpha\beta=\gamma_0,\, \alpha\neq 1} a_\alpha b_\beta, \tag{3.13}$$

then $c_{\gamma_0} = 0$. □

Remark 3.3.7. The elements of $R[F_m]$ are not invertible in $R[F_m]$ unless the constant term is a unit in R and all other terms are equal to 0. The elements of

$R[[x_1, x_2, \ldots, x_m]]$ or $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$ are invertible if the constant term is a unit, which is a much weaker condition. For example $1 + t$ is not invertible in $R[F_m]$, yet $1 + x$ is invertible in $R[[x_1, x_2, \ldots, x_m]]$ or $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$;
$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots$.

CHAPTER 4

DERIVATIONS

To define a rational function in non-commutative variables we need a notion of a derivative in non-commutative structures such as $R[F_m]$ and $R\langle\langle x_1, x_2, \ldots, x_m\rangle\rangle$. In this chapter, we introduce two such derivatives, one due to Ralph Fox [Fox53], and the other due to M. Farber and P. Vogel [FV91].

Unless otherwise stated, we assume that the ring R is a commutative ring with unity and F_m is a free group on m generators $t_1, t_2, \ldots, t_m$.

4.1 The Fox Derivative

In the 1950's, Ralph Fox introduced a derivative over free groups that bears many similarities to the conventional derivative of calculus. The Fox derivative and related concepts are often referred to as *free differential calculus over free groups.* The Fox derivative was developed in a series of papers by Fox [Fox53], [Fox54], [Fox56], and [CFL58].

Definition 4.1.1. Let G be a group and $R[G]$ be a group ring. The *trivializer* (or augmentation homomorphism) is the map $t : R[G] \to R$ that maps any element of $R[G]$ to its coefficient sum. It is defined by

$$t\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g. \tag{4.1}$$

The sum on the right is finite because only a finite number of a_g's are nonzero by Definition 3.2.1.

Definition 4.1.2. A *derivation* in a group ring $R[G]$ is any mapping $D : R[G] \to R[G]$ which satisfies

(1) $D(u+v) = Du + Dv$ and

(2) $D(uv) = t(v)Du + uDv$,

where t is the trivializer and u and v are in $R[G]$. For elements of G, equation (2) takes the simpler form

(3) $D(gh) = Dg + gDh$,

where g and h are in G.

Definition 4.1.3. If F_m is a free group with identity element 1 and generators $\{t_1, \ldots, t_m\}$, then the *Fox derivative with respect to* t_i, denoted D_i or $\frac{\partial}{\partial t_i}$, is a derivation map from the group ring $R[F_m]$ into itself, $D_i : R[F_m] \to R[F_m]$, that satisfies the following axioms:

(1) $\dfrac{\partial}{\partial t_i}(1) = 0$,

(2) $\dfrac{\partial}{\partial t_i}(t_j) = \delta_{ij}$,

where δ_{ij} is the Kronecker delta: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, and

(3) $\dfrac{\partial}{\partial t_i}(\alpha\beta) = \dfrac{\partial}{\partial t_i}(\alpha) + \alpha\dfrac{\partial}{\partial t_i}(\beta)$ for any α, β in F_m.

Theorem 4.1.4. *To each generator t_j of F_m there corresponds a unique Fox derivative with respect to t_j.*

Proof. See [Fox53, p. 550]. □

Theorem 4.1.5. *The following hold for all generators of F_m and for all $n \in \mathbb{Z}_+$:*

(1) $\dfrac{\partial}{\partial t_i}(t_j^{-1}) = -\delta_{ij}t_j^{-1}$,

(2) $\dfrac{\partial}{\partial t_i}(t_j^n) = \delta_{ij}\left(1 + t_j + t_j^2 + \cdots + t_j^{n-1}\right) = \delta_{ij}\left(\sum_{k=0}^{n-1} t_j^k\right)$,

(3) $\dfrac{\partial}{\partial t_i}(t_j^{-n}) = -\delta_{ij}\left(t_j^{-1} + t_j^{-2} + \cdots + t_j^{-n}\right) = -\delta_{ij}\left(\sum_{k=1}^{n} t_j^{-k}\right)$.

Proof. (1) It follows from the axioms given in Definition 4.1.3 that

$$0 = \frac{\partial}{\partial t_i}(t_j t_j^{-1}) = \underbrace{\frac{\partial}{\partial t_i}(t_j)}_{\delta_{ij}} + t_j \frac{\partial}{\partial t_i}(t_j^{-1}). \tag{4.2}$$

Therefore,

$$-\delta_{ij} = t_j \frac{\partial}{\partial t_i}(t_j^{-1}), so - \delta_{ij} t_j^{-1} = \frac{\partial}{\partial t_i}(t_j^{-1}). \tag{4.3}$$

(2) The second equation is true for $n = 1$ by axiom (2) in Definition 4.1.3. Proceeding by induction, assume $\dfrac{\partial}{\partial t_i}(t_j^n) = \delta_{ij}\left(1 + t_j + t_j^2 + \cdots + t_j^{n-1}\right)$ for some $n \in \mathbb{Z}_+$. Then

$$\begin{aligned}
\frac{\partial}{\partial t_i}(t_j^{n+1}) &= \frac{\partial}{\partial t_i}(t_j^n t_j) \\
&= \frac{\partial}{\partial t_i}(t_j^n) + t_j^n \frac{\partial}{\partial t_i} t_j \text{ (by axiom (3) in 4.1.3)} \\
&= \delta_{ij}\left(\sum_{k=0}^{n-1} t_j^k\right) + \delta_{ij} t_j^n \\
&= \delta_{ij}\left(\sum_{k=0}^{n} t_j^k\right).
\end{aligned} \tag{4.4}$$

The result follows by induction on n.

The proof of (3) is similar. □

Corollary 4.1.6. *Suppose v and w are words in F_m that contain no t_i or t_i^{-1}, and let $\alpha_1 = v t_i w$ and $\alpha_2 = v t_i^{-1} w$. Then*

$$\frac{\partial}{\partial t_i}(\alpha_1) = \frac{\partial}{\partial t_i}(v t_i) = v, \tag{4.5}$$

$$\frac{\partial}{\partial t_i}(\alpha_2) = \frac{\partial}{\partial t_i}(v t_i^{-1}) = -v t_i^{-1}. \tag{4.6}$$

Remark 4.1.7. More generally, if α is a reduced word in F_m, its Fox derivative with respect to t_i, $\dfrac{\partial}{\partial t_i}(\alpha) = D_i(\alpha)$, contains exactly one term for each occurrence of t_i (or t_i^{-1}) in α. Suppose $w_1, w_2, \ldots, w_n$ are words in F_m that contain no t_i or t_i^{-1}, and let $\alpha = w_1 t_i^{k_1} w_2 t_i^{k_2} \ldots w_n t_i^{k_n}$ and $k_j \neq 0$ for any $1 \leq j \leq n$. (We do not assume that the k_j's are positive.) Then

$$\frac{\partial}{\partial t_i}(\alpha) = [w_1]\frac{\partial}{\partial t_i}(t_i^{k_1}) + [w_1 t_i^{k_1} w_2]\frac{\partial}{\partial t_i}(t_i^{k_2}) \cdots + [w_1 t_i^{k_1} w_2 t_i^{k_2} \ldots w_n]\frac{\partial}{\partial t_i}(t_i^{k_n}) \tag{4.7}$$

Note that $\dfrac{\partial}{\partial t_i}(\alpha)$ above has $\sum_{j=1}^{n} |k_j|$ terms.

Example 4.1.8. Let $yx^{-2}y^3x^3y^{-4}xy^5$ be an element of $F_2 = \{x, y\}$. Then,

$$\begin{aligned} D_x(yx^{-2}y^3x^3y^{-4}xy^5) =& y\underbrace{(-x^{-1} - x^{-2})}_{D_x(x^{-2})} + yx^{-2}y^3\underbrace{(1 + x + x^2)}_{D_x(x^3)} + yx^{-2}y^3x^3y^{-4} \\ =& - yx^{-1} - yx^{-2} + yx^{-2}y^3 + yx^{-2}y^3x + yx^{-2}y^3x^2 \\ &+ yx^{-2}y^3x^3y^{-4} \end{aligned} \tag{4.8}$$

Theorem 4.1.9 (Extended Leibniz's law)**.** *The following equation holds for any λ and η in $R[F_m]$:*

$$\frac{\partial}{\partial t_i}(\lambda\eta) = t(\eta)\frac{\partial}{\partial t_i}(\lambda) + \lambda\frac{\partial}{\partial t_i}(\eta) \tag{4.9}$$

Proof. Let $\lambda = \sum_{\alpha \in F_m} a_\alpha \alpha$ and $\eta = \sum_{\beta \in F_m} b_\beta \beta$ be elements of $R[F_m]$. Then

$$\begin{aligned}
\frac{\partial}{\partial t_i}(\lambda\eta) =& \frac{\partial}{\partial t_i}\left(\sum_{\alpha\in F_m} a_\alpha\alpha \sum_{\beta\in F_m} b_\beta\beta\right) \\
=& \frac{\partial}{\partial t_i}\left(\sum_{\alpha\in F_m}\sum_{\beta\in F_m} (a_\alpha b_\beta)\alpha\beta\right) \\
=& \sum_{\alpha\in F_m}\sum_{\beta\in F_m} (a_\alpha b_\beta)\frac{\partial}{\partial t_i}(\alpha\beta) \\
=& \sum_{\alpha\in F_m}\sum_{\beta\in F_m} (a_\alpha b_\beta)\left(\frac{\partial}{\partial t_i}(\alpha) + \alpha\frac{\partial}{\partial t_i}(\beta)\right) \\
=& \sum_{\alpha\in F_m}\sum_{\beta\in F_m} (a_\alpha b_\beta)\frac{\partial}{\partial t_i}(\alpha) + \sum_{\alpha\in F_m}\sum_{\beta\in F_m} (a_\alpha b_\beta)\alpha\frac{\partial}{\partial t_i}(\beta) \\
=& \underbrace{\sum_{\beta\in F_m} b_\beta}_{t(\eta)} \underbrace{\sum_{\alpha\in F_m} a_\alpha\frac{\partial}{\partial t_i}(\alpha)}_{\frac{\partial}{\partial t_i}(\lambda)} + \underbrace{\sum_{\alpha\in F_m} a_\alpha\alpha}_{\lambda} \underbrace{\sum_{\beta\in F_m} b_\beta\frac{\partial}{\partial t_i}(\beta)}_{\frac{\partial}{\partial t_i}(\eta)}. \qquad (4.10)
\end{aligned}$$

The theorem follows. □

4.2 The Farber-Vogel (FV) Derivative

This derivative was defined in Farber and Vogel [FV91], though they define it as a left cancellation, and we define it as a right cancellation, for compatibility with the Fox derivative.

Definition 4.2.1. The *partial FV-derivative* in the ring of non-commutative power series, Γ, with respect to x_i, is the (infinitely) R-linear map $\partial_i : \Gamma \to \Gamma$, that acts as a cancellation of x_i from the right on monomials containing x_i on the right-most position, and sends all other monomials to zero. That is, for any monomial αx_j^k in Γ,

$$\partial_i(\alpha x_j^k) = \delta_{ij}\alpha x_j^{k-1}. \qquad (4.11)$$

Theorem 4.2.2. *Let β be any element of X^*, and $A = \sum_{\alpha \in X^*} a_\alpha \alpha$. Then*

$$\partial_\beta A = \sum_{\alpha \in X^*} a_{\alpha\beta} \alpha. \tag{4.12}$$

Proof. We can write β as a product of individual generators $\beta = x_{i_1} x_{i_2} \dots x_{i_n}$. Then

$$\partial_\beta A = \partial_{x_{i_1} x_{i_2} \dots x_{i_n}} A = \partial_{x_{i_1}} \partial_{x_{i_2}} \dots \partial_{x_{i_n}} A \tag{4.13}$$

We can separate all monomials $a_\gamma \gamma$ of A into two disjoint groups: the ones that do not end on β, and the ones that do, and therefore, can be written as $a_\gamma \gamma = a_{\alpha\beta}(\alpha\beta)$.

All monomials that do not end on β are mapped to zero by ∂_β for the following reason: if a monomial γ does not end on β, it can be written as

$$\gamma = \gamma_0 \underbrace{x_{i_1} x_{i_2} \dots \overbrace{x^*_{i_k}}^{\neq x_{i_k} \text{ on this position of } \beta} \dots x_{i_n}}_{\text{positions corresponding to } \beta}, \tag{4.14}$$

and it is mapped to zero by $\partial_{x_{i_k}}$ that corresponds to a letter of β different from that of γ on the matching position.

On the other hand, if a monomial $a_\gamma \gamma$ can be written as $a_{\alpha\beta}(\alpha\beta)$,

$$\begin{aligned} \partial_\beta(a_\gamma \gamma) &= \partial_\beta(a_{\alpha\beta} \alpha\beta) \\ &= a_{\alpha\beta} \partial_\beta(\alpha\beta) \\ &= a_{\alpha\beta} \partial_{(x_{i_1} x_{i_2} \dots x_{i_n})}(\alpha(x_{i_1} x_{i_2} \dots x_{i_n})) \\ &= a_{\alpha\beta} \alpha. \end{aligned} \tag{4.15}$$

The theorem follows because ∂_i is an R-linear map for any i. □

Definition 4.2.3. Let ϵ be the augmentation homomorphism $\epsilon : \Gamma \to \Gamma$ that maps a power series to its constant term given by

$$\epsilon \left(\sum_{\alpha \in X^*} a_\alpha \alpha \right) = a_1. \tag{4.16}$$

Theorem 4.2.4. *The product (Leibniz) rule for the partial FV-derivative ∂_i is given by the formula*

$$\begin{aligned}\partial_i(AB) =&\epsilon(B)\partial_i(A) + A\partial_i(B)\\ =&b_1\partial_i(A) + A\partial_i(B),\end{aligned} \tag{4.17}$$

where A and B are in Γ.

Proof. Let A, B be elements of Γ given by

$$A = \sum_{\alpha} a_\alpha \alpha, \tag{4.18}$$

$$B = \sum_{\beta} b_\beta \beta = \epsilon(B) + \sum_{\beta \neq 1_\Lambda} b_\beta \beta. \tag{4.19}$$

Then

$$\begin{aligned}\partial_i(AB) &= \partial_i\left(\left(\sum_{\alpha} a_\alpha \alpha\right)\left(\epsilon(B) + \sum_{\beta\neq 1_\Lambda} b_\beta\beta\right)\right)\\ &= \partial_i\left(\epsilon(B)\sum_{\alpha} a_\alpha\alpha + \sum_{\alpha}\sum_{\beta\neq 1_\Lambda} a_\alpha b_\beta(\alpha\beta)\right)\\ &= \epsilon(B)\partial_i\left(\sum_{\alpha} a_\alpha\alpha\right) + \partial_i\left(\sum_{\alpha}\sum_{\beta\neq 1_\Lambda} a_\alpha b_\beta(\alpha\beta)\right)\\ &= \epsilon(B)\partial_i(A) + A\partial_i(B),\end{aligned} \tag{4.20}$$

where $\partial_i(\sum_{\alpha}\sum_{\beta\neq 1_\Lambda} a_\alpha b_\beta(\alpha\beta)) = A\partial_i(B)$ because the FV-derivative ∂_i of a monomial $ab(\alpha\beta)$, where β is not the identity element, is determined only by the letter (variable) on the right-most position of β. □

Corollary 4.2.5. *If A is in Γ, then*

$$\partial_i A^{-1} = -\frac{1}{\epsilon(A)} A^{-1}\partial_i A. \tag{4.21}$$

Proof. The proof is similar to the one given in Theorem 4.1.5(1). □

4.3 The Magnus embedding

Definition 4.3.1. Let the multiplicative homomorphism $\mu_{F_m} : F_m \to \Gamma$ be defined by:

$$t_i \mapsto 1 + x_i, \tag{4.22}$$

$$t_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \dots. \tag{4.23}$$

Also, for any finite product of $t_i^{\pm 1}$'s, we define

$$\mu_{F_m}\left(\prod t_i^{\pm 1}\right) = \prod \mu_{F_m}(t_i^{\pm 1}). \tag{4.24}$$

Theorem 4.3.2. *The map $\mu_{F_m} : F_m \to \Gamma$ is a well-defined monoid homomorphism.*

Proof. We can see that

$$\begin{aligned}
\mu_{F_m}(1_{F_m}) =& \mu_{F_m}(t_i t_i^{-1}) \\
=& \mu_{F_m}(t_i)\mu_{F_m}(t_i^{-1}) \\
=& (1 + x_i)(1 - x_i + x_i^2 - x_i^3 + \dots) \\
=& 1 - x_i + x_i^2 - x_i^3 + \dots \\
& + x_i - x_i^2 + x_i^3 - \dots \\
=& 1_\Gamma.
\end{aligned} \tag{4.25}$$

This together with (4.24) shows that μ_{F_m} is a monoid homomorphism. □

The monoid homomorphism μ_{F_m} can be extended in an R-linear manner to the group ring $R[F_m]$, giving a ring homomorphism $\mu : R[F_m] \to \Gamma$ called the *Magnus embedding.*

Theorem 4.3.3. *The relationship between the Fox derivative, D_i, defined on $R[F_m] = \Lambda$, and the partial FV-derivative ∂_i, defined on Γ is given by the equation:*

$$\mu(D_i\lambda) = \partial_i(\mu(\lambda)), \tag{4.26}$$

where μ is the Magnus embedding, and λ is in $R[F_m]$.

Equivalently, the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow{D_i} & \Lambda \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \Gamma & \xrightarrow[\partial_i]{} & \Gamma \end{array} .$$

Proof. Let $\lambda = \sum_{\alpha \in F_m} a_\alpha \alpha$ be an element of $R[F_m]$. Since μ D_i, and ∂_i are R-linear, and λ is a finite linear combination,

$$\mu(D_i\lambda) = \mu\left(D_i\ \left(\sum_{\alpha\in F_m} a_\alpha\alpha\right)\right) = \sum_{\alpha\in F_m} a_\alpha\left(\mu(D_i\alpha)\right), \tag{4.27}$$

$$\partial_i(\mu(\lambda)) = \partial_i\left(\mu\left(\sum_{\alpha\in F_m} a_\alpha\alpha\right)\right) = \sum_{\alpha\in F_m} a_\alpha\partial_i(\mu(\alpha)). \tag{4.28}$$

Therefore, it is sufficient to show that

$$\mu(D_i\alpha) = \partial_i[\mu(\alpha)] \tag{4.29}$$

for any $\alpha \in F_m$. We proceed by induction on the length of α.

Suppose α has length 0; i.e. $\alpha = a$ is a constant. Since

$$\mu(D_ia) = \mu(0) = 0 = \partial_i a = \partial_i[\mu(a)], \tag{4.30}$$

(4.29) holds for words of length 0.

Suppose (4.29) is true for any word α of length at most k, $k \geq 0 \in \mathbb{Z}$, and

assume that the word $\alpha t_j^{\pm 1}$ is reduced and has length $k+1$. We see that

$$\begin{aligned}
\mu(D_i(\alpha t_j)) =& \mu(D_i\alpha + \alpha D_i t_j) \\
=& \mu(D_i\alpha + \delta_{ij}\,\alpha) \\
=& \mu(D_i\alpha) + \delta_{ij}\,\mu(\alpha) \\
=& \partial_i(\mu(\alpha)) + \delta_{ij}\,\mu(\alpha), \qquad (4.31) \\
\partial_i(\mu(\alpha t_j)) =& \partial_i(\mu(\alpha)\mu(t_j)) \\
=& \partial_i(\mu(\alpha)(1+x_j)) \\
=& \partial_i(\mu(\alpha) + \mu(\alpha)x_j) \\
=& \partial_i(\mu(\alpha)) + \delta_{ij}\,\mu(\alpha),
\end{aligned}$$

and

$$\begin{aligned}
\mu(D_i(\alpha t_j^{-1})) =& \mu(D_i\alpha + \alpha D_i(t_j^{-1})) \\
=& \mu(D_i\alpha - \delta_{ij}\,\alpha t_j^{-1}) \\
=& \mu(D_i\alpha) - \delta_{ij}\,\mu(\alpha t_j^{-1}) \\
=& \partial_i(\mu(\alpha)) - \delta_{ij}\,\mu(\alpha)\mu(t_i^{-1}) \\
=& \partial_i(\mu(\alpha)) - \delta_{ij}\,\mu(\alpha)(1 - x_j + x_j^2 - x_j^3 + \cdots), \qquad (4.32) \\
\partial_i(\mu(\alpha t_j^{-1})) =& \partial_i(\mu(\alpha)\mu(t_j^{-1})) \\
=& \partial_i(\mu(\alpha)(1 - x_j + x_j^2 - x_j^3 + \cdots)) \\
=& \mu(\alpha)\underbrace{\partial_i(1 - x_j + x_j^2 - \cdots)}_{-\delta_{ij}\,(1 - x_j + x_j^2 - x_j^3 + \cdots)} + \underbrace{\epsilon(1 - x_j + x_j^2 - \cdots)}_{=1}\partial_i(\mu(\alpha)) \\
=& -\delta_{ij}\,\mu(\alpha)(1 - x_j + x_j^2 - x_j^3 + \cdots) + \partial_i(\mu(\alpha)). \qquad (4.33)
\end{aligned}$$

Therefore,

$$\mu(D_i(\alpha t_i^{\pm 1})) = \partial_i(\mu(\alpha t_i^{\pm 1})), \qquad (4.34)$$

and theorem follows. □

Theorem 4.3.4. *The map μ is injective, and, therefore, the ring $R[F_m]$ is embedded in the ring Γ via the Magnus embedding.*

Proof. Beyond the scope of this thesis, see [Fox53]. □

CHAPTER 5

ONE VARIABLE

Our ultimate goal is to investigate the periodicity of non-commutative formal power series. We, however, will start with a simpler problem and examine FPS of a single variable with complex coefficients. We denote them by capital letters, for instance, $A(x)$ (or just A). Naturally, any formal power series in a single variable is commutative (defined in 3.3.1). Furthermore, by ∂_x we denote the partial FV-derivative with respect to x (defined in 4.2.1), and, as before, we assume that the ring R is a commutative ring with unity and F_m is a free group on m generators.

5.1 Depth

Definition 5.1.1. Let $A(x)$ be a formal power series. By V_A we denote the subspace of $\mathbb{C}[[x]]$ generated by all the FV-derivatives of $A(x)$. In set notation,

$$V_A = \{p(\partial_x)A \mid p(\partial_x) \in \mathbb{C}[\partial_x]\}, \tag{5.1}$$

where $p(\partial_x) = p_0 + p_1\partial_x + \cdots + p_{d-1}\partial_x^{d-1} + p_d\partial_x^d$ is a polynomial in ∂_x with complex coefficients.

Definition 5.1.2 (Due to [FV91])**.** A (non-commutative) formal power series A in Γ represents a *rational function* if its partial FV-derivatives generate a finite dimensional vector space V_A.

Definition 5.1.3. Let $A(x) = a_0 + a_1x + a_2x^2 + \dots$ be a formal power series, and k be a nonnegative integer. We say that the first k coefficients *determine* $A(x)$ if there exist $r_0, \dots, r_{k-1} \in \mathbb{C}$ such that for any $n \geq k$,

$$a_n = \sum_{i=0}^{k-1} r_i a_{n-k+i}. \tag{5.2}$$

In other words, the a_i's satisfy an order k linear homogeneous recurrence relation with constant coefficients r_i, $i = 0, 1, \ldots, k-1$.

Theorem 5.1.4. *If $A(x) = a_0 + a_1x + a_2x^2 + \ldots$ is a formal power series such that the a_i's satisfy an order k linear homogeneous recurrence relation with constant coefficients r_i, $i = 0, 1, \ldots, k-1$, then $A(x)$ is a rational function.*

Proof. We will give a constructive proof of the statement using generating functions. Suppose the recurrence relation is given by

$$a_{n+k} = r_{k-1}a_{n+k-1} + \cdots + r_1a_{n+1} + r_0a_n, \tag{5.3}$$

for $n = 0, 1, 2, \ldots$.

We multiply both sides of (5.3) by x^n and sum them from $n = 0$ to infinity. We obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n+k}x^n &= \sum_{n=0}^{\infty}(r_{k-1}a_{n+k-1} + \cdots + r_1a_{n+1} + r_0a_n)x^n, \\
\sum_{n=0}^{\infty} a_{n+k}x^n &= r_{k-1}\sum_{n=0}^{\infty} a_{n+k-1}x^n + \cdots + r_1\sum_{n=0}^{\infty} a_{n+1}x^n + r_0\sum_{n=0}^{\infty} a_nx^n, \\
\frac{1}{x^k}\underbrace{\sum_{n=0}^{\infty} a_{n+k}x^{n+k}}_{A - a_0 - \cdots - a_{k-1}x^{k-1}} &= \frac{r_{k-1}}{x^{k-1}}\underbrace{\sum_{n=0}^{\infty} a_{n+k-1}x^{n+k-1}}_{A - a_0 - \cdots - a_{k-2}x^{k-2}} + \ldots \\
&\quad \cdots + \frac{r_1}{x}\underbrace{\sum_{n=0}^{\infty} a_{n+1}x^{n+1}}_{A - a_0} + r_0\underbrace{\sum_{n=0}^{\infty} a_nx^n}_{A(x)}.
\end{aligned} \tag{5.4}$$

Therefore,

$$\begin{aligned}
\frac{1}{x^k}(A - a_0 - \cdots - a_{k-1}x^{k-1}) &= \frac{r_{k-1}}{x^{k-1}}(A - a_0 - \cdots - a_{k-2}x^{k-2}) + \ldots \\
&\quad \cdots + \frac{r_1}{x}(A - a_0) + r_0 A, \\
A - a_0 - \cdots - a_{k-1}x^{k-1} &= r_{k-1}x(A - a_0 - \cdots - a_{k-2}x^{k-2}) + \ldots \\
&\quad \cdots + r_1 x^{k-1}(A - a_0) + r_0 x^k A, \\
A - r_{k-1}xA - \cdots - r_1 x^{k-1}A - r_0 x^k A &= a_0 + (a_1 - r_{k-1}a_0)x \\
&\quad + (a_2 - r_{k-1}a_1 - r_{k-2}a_0)x^2 + \ldots \\
&\quad \ldots + (a_{k-1} - r_{k-1}a_{k-2} - \cdots - r_1 a_0)x^{k-1}.
\end{aligned} \tag{5.5}$$

It follows that

$$A(x) = \frac{a_0 + (a_1 - r_{k-1}a_0)x + \cdots + (a_{k-1} - r_{k-1}a_{k-2} - \cdots - r_1 a_0)x^{k-1}}{1 - r_{k-1}x - r_{k-2}x^2 - \cdots - r_1 x^{k-1} - r_0 x^k}. \tag{5.6}$$

□

Corollary 5.1.5. *If $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is a formal power series such that the a_i's satisfy* (5.3)*, where $r_0 \neq 0$, then $A(x)$ is a rational function with degree of denominator exactly k and degree of numerator at most $k - 1$.* □

Remark 5.1.6. It follows from Theorem 2.2.6 that if $A(x)$ is a formal power series in $\mathbb{C}[[x]]$, then $\text{Ann}\big(A(x)\big)$ is an ideal in $\mathbb{C}[x]$. Furthermore, since $\mathbb{C}[x]$ is a PID, $\text{Ann}\big(A(x)\big)$ is generated by a single element of $\mathbb{C}[x]$, i.e. $\text{Ann}\big(A(x)\big) = \langle m(x)\rangle$, for some $m(x)$ in $\mathbb{C}[x]$.

Theorem 5.1.7. *For $A(x) \neq 0$, let a monic polynomial $p(x)$ be in $\text{Ann}\big(A(x)\big)$, and let $\deg\big(p(x)\big) = k$, for some positive integer k. Then the first k coefficients of $A(x)$ determine $A(x)$.*

Proof. Let

$$p(\partial_x) = \partial_x^k - r_{k-1}\partial_x^{k-1} - \cdots - r_1\partial_x - r_0 \tag{5.7}$$

be in Ann $\big(A(x)\big)$. Then

$$\partial_x^k A(x) = r_{k-1}\partial_x^{k-1}A(x) + \cdots + r_1\partial_x A(x) + r_0 A(x), \tag{5.8}$$

where $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$.

We write out $r_i\partial_x^i A(x)$ for $i = 0, 1, \ldots, k-1$ and $\partial_x^k A(x)$, and consider column sums.

$$\begin{array}{rrrrr}
r_0A(x) = & r_0a_0+ & r_0a_1x+ & r_0a_2x^2+\ldots \\
+ & & & & \\
r_1\partial_xA(x) = & r_1a_1+ & r_1a_2x+ & r_1a_3x^2+\ldots \\
+ & & & & \\
r_2\partial_x^2A(x) = & r_2a_2+ & r_2a_3x+ & r_2a_4x^2+\ldots \\
+ & & & & \\
\vdots & \vdots & \vdots & \vdots \\
+ & & & & \\
r_{k-1}\partial_x^{k-1}A(x) = & r_{k-1}a_{k-1}+ & r_{k-1}a_kx+ & r_{k-1}a_{k+1}x^2+\ldots \\
\hline
= \quad \partial_x^kA(x) = & a_k+ & a_{k+1}x+ & a_{k+2}x^2+\ldots
\end{array}$$

Therefore,

$$a_k = r_0a_0 + r_1a_1 + \cdots + r_{k-1}a_{k-1} = \sum_{i=0}^{k-1} r_ia_i, \tag{5.9}$$

and

$$a_{k+j} = r_0a_j + r_1a_{j+1} + \cdots + r_{k+j-1}a_{k+j-1} = \sum_{i=0}^{k+j-1} r_ia_i, \tag{5.10}$$

where j is any nonnegative integer. □

Corollary 5.1.8. *If V_A is finitely generated, i.e., $\dim_{\mathbb{C}} V_A = d$ for some $d \in \mathbb{Z}_{\geq 0}$, then there exists a monic polynomial*

$$p(\partial_x) = r_0 + p_1\partial_x + \cdots + r_{d-1}\partial_x^{d-1} - \partial_x^d, \tag{5.11}$$

such that

$$\partial_x^d A = r_{d-1}\partial_x^{d-1} A + \cdots + r_1\partial_x A + r_0 A. \tag{5.12}$$

Proof. If $\dim_{\mathbb{C}} V_A = d$ then any set of $d+1$ elements of V_A is linearly dependent, i.e. there are complex numbers $h_0, \ldots, h_d$, not all zeroes, such that

$$h_d\partial_x^d A = h_{d-1}\partial_x^{d-1} A + \cdots + h_1\partial_x A + h_0 A. \tag{5.13}$$

Furthermore, by Theorem 5.1.7, if $\partial_x^i A = h_{i-1}\partial_x^{i-1} A + \cdots + h_1\partial_x A + h_0 A$, then the set $\{\partial_x^{i-1} A, \ldots, \partial_x A, A\}$ spans V_A. Therefore, $h_d \neq 0$ because otherwise we can find a spanning set for V_A with fewer then d elements. We obtain the desired result by dividing both sides of (5.13) by h_d. □

Theorem 5.1.9. *Let $A(x)$ be a formal power series whose derivatives generate a finite-dimensional vector space V_A, and let $\operatorname{Ann}\big(A(x)\big) = \langle m(\partial_x)\rangle$, where m is (monic) minimal polynomial of $\operatorname{Ann}\big(A(x)\big)$. If d, n, and k are nonnegative integers defined by:*

$d = \dim_{\mathbb{C}} V_A$,

$n = \deg\big(m(\partial_x)\big)$,

$k =$ *the smallest positive integer such that the first k coefficients determine $A(x)$,*

then $d = n = k$.

Proof. First we show that $d = n$.

(1) Given that $\dim_{\mathbb{C}} V_A = d$, there exist (by Corollary 5.1.8) complex numbers $r_0, \ldots, r_{d-1}$, not all equal to 0, such that

$$\partial_x^d A - r_{d-1}\partial_x^{d-1} A - \cdots - r_1 \partial_x A - r_0 A = p(\partial_x)A = 0; \tag{5.14}$$

i.e. $p(\partial_x) \in \mathrm{Ann}\big(A(x)\big)$, and $\deg\big(p(\partial_x)\big) = d$.

Therefore, $n = \deg\big(m(\partial_x)\big) \leq d$.

(2) Now, $\deg\big(m(\partial_x)\big) = n$, so

$$m(\partial_x) = m_0 + m_1\partial_x + \cdots + m_{n-1}\partial_x^{n-1} + \partial_x^n. \tag{5.15}$$

Therefore,

$$0 = m(\partial_x)A = \partial_x^n A + m_{n-1}\partial_x^{n-1} A + \cdots + m_1 \partial_x A + m_0 A, \tag{5.16}$$

so

$$\partial_x^n A = -m_{n-1}\partial_x^{n-1} A - \cdots - m_1 \partial_x A - m_0 A. \tag{5.17}$$

Higher derivatives are also linear combinations of $\partial_x^{n-1}A, \ldots, \partial_x A, A$ by Theorem 5.1.7, and therefore, $d = \dim_{\mathbb{C}} V_A \leq n$.

From (1) and (2) we can conclude that $d = n$.

Now we show that $d = k$.

(3) We know that k is the smallest positive integer such that there exists a finite set $\{r_0, \ldots, r_{k-1}\} \subseteq \mathbb{C}$ such that for any $n \geq k$,

$$a_n = \sum_{i=0}^{k-1} r_i a_{n-k+i}. \tag{5.18}$$

We write out (5.18) for $a_k, a_{k+1}, \ldots$, and obtain

$$\begin{aligned} a_k &= r_0 a_0 + r_1 a_1 + \ldots + r_{k-1} a_{k-1}, \\ a_{k+1} &= r_0 a_1 + r_1 a_2 + \ldots + r_{k-1} a_k, \\ &\vdots \qquad\quad \vdots \qquad\qquad \vdots \end{aligned} \tag{5.19}$$

.

Now we multiply equation $(k+i)$ by x^i, where $i = 0, 1, 2, , \ldots$, and (formally) sum the resulting equations column-wise. We obtain the following result:

$$\begin{array}{rrrrr}
a_k = & r_0 a_0 + & r_1 a_1 + \ldots & + r_{k-1} a_{k-1} \\
+ & & & & \\
a_{k+1}x = & r_0 a_1 x + & r_1 a_2 x + \ldots & + r_{k-1} a_k x \\
+ & & & & \\
a_{k+2}x^2 = & r_0 a_2 x^2 + & r_1 a_3 x^2 + \ldots & + r_{k-1} a_{k+1} x^2 \\
+ & & & & \\
\vdots & \vdots & \vdots & \vdots & \\
\hline
= \partial_x^k A(x) = & r_0 A(x) + & r_1 \partial_x A(x) + \ldots & + r_{k-1} \partial_x^{(k-1)} A(x) \\
. & & & &
\end{array} \tag{5.20}$$

Therefore, $d = \dim_{\mathbb{C}} V_A \leq k$.

(4) Since $\dim_{\mathbb{C}} V_A = d$, we know that

$$\partial_x^d A = r_{d-1} \partial_x^{d-1} A + \cdots + r_1 \partial_x A + r_0 A. \tag{5.21}$$

Consider the x_i term on both sides. We see that

$$a_{d+i} = r_{d+i-1} a_{d+i-1} + \cdots + r_{i+1} a_{i+1} + r_i a_i. \tag{5.22}$$

Therefore, $k \leq d$.

Items (3) and (4) imply that $d = k$, and the theorem follows. □

Definition 5.1.10. We call $d = n = k$ from Theorem 5.1.9 *the depth of* A, and denote it by $\operatorname{depth}\big(A(x)\big)$, or $\operatorname{depth}(A)$.

Corollary 5.1.11. *If* $\operatorname{depth}(A)$ *is at most* k*, and the first* k *coefficients of* $A(x)$ *all are equal to zero, then* $A(x) = 0$.

Proof. Suppose $\operatorname{depth}(A) \leq k$. Therefore there exist $r_0, r_1, \ldots, r_{k-1} \in \mathbb{C}$ such that $a_{k+m} = r_0 a_m + r_1 a_{m+1} + \cdots + r_{k-1} a_{m+k-1}$, for $m = 0, 1, 2, \ldots$. It follows that,

$$\begin{aligned} a_k &= r_0 a_0 + r_1 a_1 + \cdots + r_{k-1} a_{k-1} = 0 \sum_{i=0}^{k-1} r_i = 0, \text{so} \\ a_{k+1} &= r_0 a_1 + r_1 a_2 + \cdots + r_{k-1} a_k = 0 \sum_{i=0}^{k-1} r_i = 0. \end{aligned} \tag{5.23}$$

The result follows by induction. □

Remark 5.1.12. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series of depth $k \in \mathbb{Z}_+$. Then there exist complex numbers $r_0, r_1, \ldots, r_{k-1}$ such that

$$\begin{aligned} a_k &= r_0 a_0 + r_1 a_1 + \cdots + r_{k-1} a_{k-1}, \\ a_{k+1} &= r_0 a_1 + r_1 a_2 + \cdots + r_{k-1} a_k, \\ a_{k+2} &= r_0 a_2 + r_1 a_3 + \cdots + r_{k-1} a_{k+1}, \\ &\vdots \end{aligned} \tag{5.24}$$

We can write the above in matrix form:

$$\begin{pmatrix} a_k \\ a_{k+1} \\ a_{k+2} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ a_1 & a_2 & \cdots & a_k \\ a_2 & a_3 & \cdots & a_{k+1} \\ \vdots & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{k-1} \end{pmatrix}. \tag{5.25}$$

Theorem 5.1.13. *The following is in depth version of [FV91, Prop. 1.6]. Let $A(x)$ and $B(x)$ be formal power series of depth k and ℓ respectively, and let c be a nonzero complex number. Then*

(1) $\operatorname{depth}(cA) = k$;

(2) $\operatorname{depth}(\partial_x A) \leq k$;

(3) $\operatorname{depth}(A + B) \leq k + \ell$;

(4) $\operatorname{depth}(AB) \leq k + \ell$*; and*

(5) If $a_0 \neq 0$, $\operatorname{depth}(A^{-1}) \leq k + 1$.

Proof. Let $A(x)$ and $B(x)$ be formal power series of depth k and ℓ respectively.

(1) The fact that $\operatorname{depth}(A) = k$ implies (5.25). Multiplying both sides of (5.25) by a nonzero constant shows that $\operatorname{depth}(cA) \leq \operatorname{depth}(A)$. By a similar argument, $\operatorname{depth}(A) = \operatorname{depth}\left(\frac{1}{c}(cA)\right) \leq \operatorname{depth}(cA)$.

Therefore, $\operatorname{depth}(cA) = \operatorname{depth}(A) = k$.

(2) It follows from (5.25) that

$$\begin{pmatrix} a_{k+1} \\ a_{k+2} \\ a_{k+3} \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_2 & a_3 & \cdots & a_{k+1} \\ a_3 & a_4 & \cdots & a_{k+2} \\ \vdots & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_{k-1} \end{pmatrix}. \tag{5.26}$$

Therefore, $\operatorname{depth}(\partial_x A) \leq k$.

(3) We know that $A + B$ is an element of $V_A + V_B$, a space closed under ∂_x. The dimension of the vector space $V_A + V_B$ is at most the sum of the individual dimensions of V_A and V_B. That is, $\dim(V_A + V_B) \leq k + l$, which implies $\operatorname{depth}(A + B) \leq k + l$.

(4) By Theorem 4.2.4,

$$\partial_x(AB) = \underbrace{\epsilon(B)\partial_x A}_{\in V_A} + \underbrace{A\partial_x B}_{\in AV_B}. \tag{5.27}$$

Furthermore, if $A_0 \in V_A$ and $B_0 \in V_B$, so that $A_0 + AB_0$ is an element of $V_A + AV_B$, then

$$\partial_x(A_0 + AB_0) = \underbrace{\partial_x A_0}_{\in V_A} + \underbrace{\epsilon(B_0)\partial_x A}_{\in V_A} + \underbrace{A\partial_x B_0}_{\in AV_B}, \tag{5.28}$$

and, therefore, AB and all its FV-derivatives are elements of $V_A + AV_B$. Therefore, $\operatorname{depth}(AB) \leq \dim(V_A + AV_B) = k + \ell$.

(5) By Corollary 4.2.5,

$$\partial_x A^{-1} = -\frac{1}{\epsilon(A)} \underbrace{A^{-1}\partial_x A}_{\in A^{-1}V_A}. \tag{5.29}$$

Similarly to (4), if $A_0 \in V_A$ and $c \in \mathbb{C}$, so that $cA^{-1} + A^{-1}A_0$ is an element of $A^{-1}V_A$, then

$$\begin{aligned} \partial_x(cA^{-1} + A^{-1}A_0) =& c\partial_x A^{-1} + \epsilon(A_0)\partial_x A^{-1} + A^{-1}\partial_x A_0 \\ =& -\frac{c + \epsilon(A_0)}{\epsilon(A)} \underbrace{A^{-1}\partial_x A}_{\in A^{-1}V_A} + \underbrace{A^{-1}\partial_x A_0}_{\in A^{-1}V_A}. \end{aligned} \tag{5.30}$$

Therefore, A^{-1} and all its FV-derivatives are elements of the vector space

$$W = \{cA^{-1} + A^{-1}A_0 \,|\, A_0 \in V_A,\, c \in \mathbb{C}\}. \tag{5.31}$$

Therefore, $\operatorname{depth}(A^{-1}) \leq \dim(W) = 1 + k$.

□

Corollary 5.1.14. *It follows from the proof of Theorem 5.1.13 (3) that*

$$\dim_{\mathbb{C}} V_{(A+B)} \leq \dim_{\mathbb{C}} V_A + \dim_{\mathbb{C}} V_B; \tag{5.32}$$

it follows from (5.27) *that*

$$\dim_{\mathbb{C}} V_{AB} \leq \dim_{\mathbb{C}} V_A + \dim_{\mathbb{C}} V_B; \tag{5.33}$$

and by (5.30)

$$\dim_{\mathbb{C}} V_{A^{-1}} \leq \dim_{\mathbb{C}} V_A + 1. \tag{5.34}$$

Remark 5.1.15. To establish Corollary 5.1.14 we do not need to assume that A and B are formal power series in a single variable or in commutative variables. Therefore, it holds for any A and B in $R\langle\langle x_1, \ldots, x_m\rangle\rangle$. We will use this fact in the next chapter.

5.2 Determining sets

Definition 5.2.1. As we saw in the previous section, if a formal power series $A(x)$ has depth k, then it is uniquely determined by two sets of complex numbers: namely, the sets $\boldsymbol{a} = \{a_0, a_1, \ldots, a_{k-1}\}$ and $\boldsymbol{r} = \{r_0, r_1, \ldots, r_{k-1}\}$, where $\boldsymbol{a}$ is the set of the first k coefficients of $A(x)$, and the set $\boldsymbol{r}$ defines a linear relation to recursively calculate all coefficients of $A(x)$. The recurrence relation is given by

$$a_n = \sum_{i=0}^{k-1} r_i a_{n-k+i}. \tag{5.35}$$

We call the set $\boldsymbol{a}$ an *initial coefficient set* and the set $\boldsymbol{r}$ a *determining set*.

Given such representations of formal power series A and B we are now interested in finding the analogous representations of $A + B$, AB, and A^{-1}.

Example 5.2.2 (Algorithm)**.** Let $C(x)$ be a formal power series such that depth(C) is at most k. We find a determining set of $C(x)$ by the following algorithm.

(1) Find the first $2k$ coefficients of C: $c_0, c_1, \ldots, c_{2k-1}$.

(2) Form a $k \times k$ matrix $\underset{k\times k}{M(C)}$ as follows:

$$\underset{k\times k}{M(C)} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} \end{pmatrix}. \tag{5.36}$$

(3) If $\underset{k\times k}{M(C)}$ is singular, we reduce it to a $(k-1) \times (k-1)$ matrix:

$$\underset{(k-1)\times(k-1)}{M(C)} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{k-2} \\ c_1 & c_2 & \cdots & c_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-2} & c_{k-1} & \cdots & c_{2k-4} \end{pmatrix}.$$

(4) Repeat the process until we obtain a non-singular matrix $\underset{m\times m}{M(C)}$.

(5) To find the recurrence set $\boldsymbol{t} = \{t_0, t_1, \ldots, t_{m-1}\}$, we solve the system

$$\underset{m\times m}{M(C)} \begin{pmatrix} t_0 \\ \vdots \\ t_{m-1} \end{pmatrix} = \begin{pmatrix} c_m \\ \vdots \\ c_{2m-1} \end{pmatrix}. \tag{5.37}$$

The matrix $\underset{m\times m}{M(C)}$ is non-singular, so it is invertible, and the solution to system (5.37) is given by

$$\begin{pmatrix} t_0 \\ \vdots \\ t_{m-1} \end{pmatrix} = \underset{m\times m}{M(C)^{-1}} \begin{pmatrix} c_m \\ \vdots \\ c_{2m-1} \end{pmatrix}. \tag{5.38}$$

Remark 5.2.3. Algorithm 5.2.2 is not computationally efficient. It can be easily improved by starting from $\underset{1\times 1}{M(C)}$ matrix and increasing its size by 1 at a time until we get the largest possible non-singular matrix $\underset{m)timesm}{M(C)}$. We prefer Algorithm 5.2.2 as stated for convenience of proof.

Theorem 5.2.4. *The algorithm given in Example 5.2.2 results in a minimal determining sets* $\boldsymbol{t}$ *and corresponding minimal initial coefficient set* $\boldsymbol{c}$.

Proof. Suppose we obtain determining sets $\boldsymbol{c}$ and $\boldsymbol{t}$ of size $m > 0$ using the algorithm. This implies that the matrix

$$\underset{m\times m}{M(\boldsymbol{c})} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{m-1} \\ c_1 & c_2 & \cdots & c_m \\ \vdots & \vdots & \ddots & \vdots \\ c_{m-1} & c_m & \cdots & c_{2m-2} \end{pmatrix} \tag{5.39}$$

is non-singular. However, if there is a pair of determining sets $\boldsymbol{c}^*$ and $\boldsymbol{u}$ of size at most $m-1$, then we can write

$$\begin{aligned} c_{m-1} &= u_0c_0 + u_1c_1 + \cdots + u_{m-2}c_{m-2}, \\ c_m &= u_0c_1 + u_1c_2 + \cdots + u_{m-2}c_{m-1}, \\ &\vdots \\ c_{2m-2} &= u_0c_{m-1} + u_1c_m + \cdots + u_{m-2}c_{2m-3}, \end{aligned} \tag{5.40}$$

and in matrix form:

$$\begin{pmatrix} c_{m-1} \\ c_m \\ \vdots \\ c_{2m-2} \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{m-2} \\ c_1 & c_2 & \cdots & c_{m-1} \\ \vdots & \vdots & & \vdots \\ c_{m-1} & c_m & \cdots & c_{2m-3} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-2} \end{pmatrix}. \tag{5.41}$$

Hence, the last column of $\underset{m\times m}{M(\boldsymbol{c})}$ is a linear combination of the other columns with coefficients u_i's, $i = 0, 1, \ldots, m-2$, which contradicts the fact that $\underset{m\times m}{M(\boldsymbol{c})}$ is non-singular. Therefore, the sets $\boldsymbol{c}$ and $\boldsymbol{t}$ obtained with the algorithm given in Example 5.2.2 are the minimal determining set and the corresponding minimal initial coefficient set. □

Corollary 5.2.5. *The minimal determining sets $\boldsymbol{c}$ and $\boldsymbol{t}$ obtained by Algorithm 5.2.2 are unique.* □

Example 5.2.6. Let $A(x)$ and $B(x)$ be formal power series of depth k and ℓ respectively. Suppose $A(x)$ is determined by sets $\boldsymbol{a} = \{a_0, a_1, \ldots, a_{k-1}\}$ and $\boldsymbol{r} = \{r_0, r_1, \ldots, r_{k-1}\}$, and $B(x)$ is determined by sets $\boldsymbol{b} = \{b_0, b_1, \ldots, b_{\ell-1}\}$ and $\boldsymbol{s} = \{s_0, s_1, \ldots, s_{\ell-1}\}$, and linear homogeneous recurrence relations are given by $a_n = \sum_{i=0}^{k-1} r_i a_{n-k+i}$ and $b_n = \sum_{i=0}^{\ell-1} s_i b_{n-\ell+i}$ respectively.

Now we want to find determining sets for $A + B$, AB, and, assuming $a_0 \neq 0$, A^{-1}. We find an initial coefficient set, and then use Theorem 5.1.13 and Algorithm 5.2.2 to estimate an upper bound for the depth and to find a determining set in each of these cases.

(1) Suppose $C(x) = A(x) + B(x)$, then depth(C) is at most $k + \ell$, and

$$c_n = a_n + b_n, \tag{5.42}$$

where $n = 0, 1, \ldots, 2k + 2\ell - 1$.

(2) Suppose $C(x) = A(x)B(x)$, then depth(C) is at most $k + \ell$, and

$$c_n = \sum_{i=0}^{n} a_{n-i} b_i, \tag{5.43}$$

where $n = 0, 1, \ldots, 2k + 2\ell - 1$.

(3) Suppose $C(x) = A^{-1}(x)$. We assume that $a_0 \neq 0$, and $A^{-1}(x)$, therefore, exists. In this case, depth(C) is at most $k + 1$, and

$$c_0 = \frac{1}{a_0}, \text{and } c_n = -\frac{1}{a_0} \sum_{i=0}^{n-1} a_{n-i} c_i, \tag{5.44}$$

where $n = 1, 2, \ldots, 2k + 1$.

Example 5.2.7. Let us work through a straightforward example. Suppose $A(x)$ is determined by sets $\boldsymbol{a} = \{1, 1\}$ and $\boldsymbol{r} = \{1, 1\}$, and $B(x)$ is determined by sets $\boldsymbol{b} = \{3\}$ and $\boldsymbol{s} = \{1\}$. We see that the coefficients of A are the Fibonacci numbers and the coefficients of B are all equal to 3. That is,
$A(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \ldots$, and
$B(x) = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + \ldots$.

In this example $k = \text{depth}(A) = 2$ and $\ell = \text{depth}(B) = 1$, so we need to calculate the first $2(k + \ell) = 6$ coefficients of $C = A + B$. They are $\boldsymbol{c} = \{4, 4, 5, 6, 8, 11\}$. Therefore,

$$\underset{3\times 3}{M(\boldsymbol{c})} = \begin{pmatrix} 4 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 8 \end{pmatrix}. \tag{5.45}$$

We see that $M(\boldsymbol{c})$ is invertible, and now we find the recurrence set $\boldsymbol{t} = \{t_0, t_1, t_3\}$ by solving

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 8 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 8 \\ 11 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & -2 & -1 \\ -2 & 7 & -4 \\ -1 & -4 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \\ 11 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}. \tag{5.46}$$

Therefore, $C(x) = A(x) + B(x)$ has depth 3, and is determined by sets $\boldsymbol{c} = \{4, 4, 5\}$ and $\boldsymbol{t} = \{-1, 0, 2\}$.

CHAPTER 6

TWO VARIABLES

In this chapter, we explore periodicity of the formal power series representation $A(x, y)$ of a rational function in two non-commuting variables. We show that the coefficients of a FPS representation of a rational function in non-commuting variables satisfy a finite set of linear homogeneous recurrence relations. We introduce a notion similar to that of depth in the one-variable case, and call it a core of A, and investigate cores of of $A + B$, AB, and A^{-1} when cores of A and B are given.

6.1 Structure and Definitions

Remark 6.1.1. A non-commutative formal power series $A(x, y) \in \mathbb{C}\langle\langle x, y\rangle\rangle$ is given by

$$\begin{aligned} A(x,y) &= a_1 + a_x x + a_y y + a_{xx}x^2 + a_{yx}yx + a_{xy}xy + a_{yy}y^2 + \dots \\ &= \sum_{\alpha \in X^*} a_\alpha \alpha, \end{aligned} \tag{6.1}$$

where α is an element of the non-commutative free monoid with 2 generators x and y.

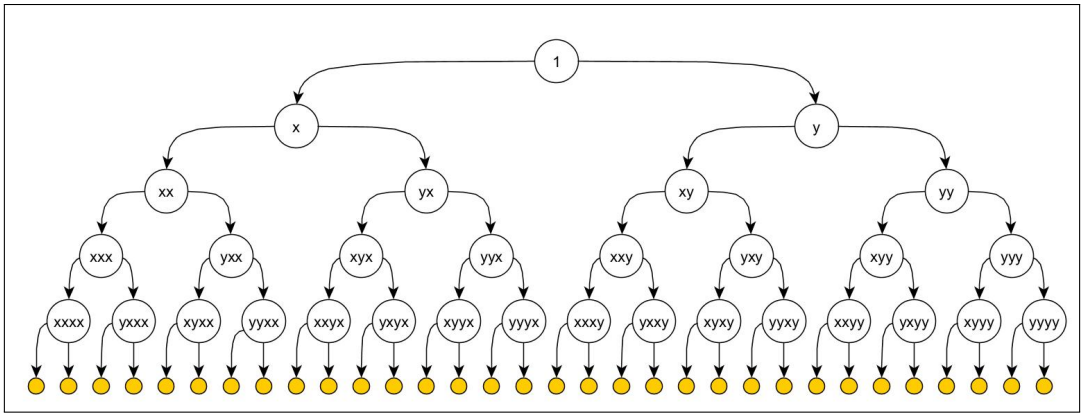

Figure 6.1: Diagram showing the hierarchical rooted binary tree structure ("1" is the root node) of the free monoid on two generators $X^* = \{x, y\}$. Each node represents an element of X^*, and each edge represents multiplication by a generator on the left.

Since there is a one to one correspondence between the elements of X^* and nodes on the tree shown on Figure 6.1, we will refer to elements of X^* as nodes when convenient.

Definition 6.1.2. A rooted tree naturally imposes a notion of levels (distance from the root); that is, for every node, *children* are defined as the nodes connected to it a level below, and *the parent* as the node connected to it a level above. Every node has exactly 2 children obtained by left-multiplying the parent node by a generator: the *left child* by x and the *right child* by y.

Definition 6.1.3. Every node α is connected to the root "1" by a unique path. We call all nodes on this path *ancestors* of α. Note that we can obtain a list of all ancestors of α by removing letters on the leftmost position of α one by one. For example, the ancestors of $xyxy$ are: yxy, xy, y, and 1. The first node obtained by this removal process is the parent of α.

Definition 6.1.4. A finite nonempty set of nodes $I \subset X^*$, that has the property

that for any τ in I all ancestors of τ are also in I is called a *core*, and the elements of I are called *core-nodes* or *core-elements*.

Definition 6.1.5. Given a core I, the set nodes that are children of the nodes in I, but themselves are not in I is called a *cut*. We denote this set J or $\mathrm{cut}(I)$, and call its elements *cut-nodes* or *cut-elements*.

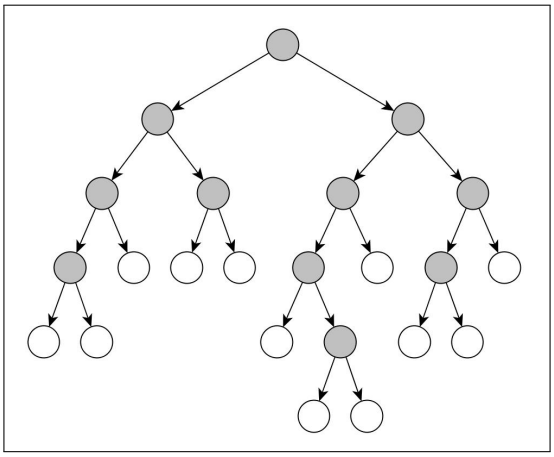

Figure 6.2: The gray nodes represent a *core*, and the white nodes represent a *cut*.

Remark 6.1.6. Any core determines a unique cut, and conversely, any proper cut determines a unique core. By a "proper cut" we understand a set of nodes, such that every path down from the root contains exactly one cut-node; i.e. every downward path is "cut" at some level. If a core contains n elements, then the matching cut contains $n + 1$ elements.

6.2 Core and Periodicity

We start with a straightforward example.

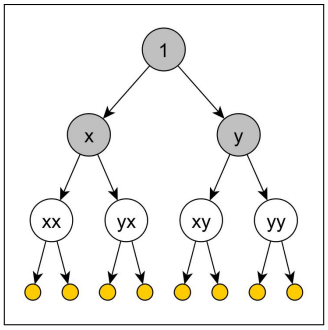

Figure 6.3: Simple cut. The gray nodes represent a *core*, and the white nodes represent a *cut*.

Example 6.2.1. Let the core $I = \{1, x, y\}$ and cut $J = \{xx, yx, xy, yy\}$ be as shown in Figure 6.3.

Suppose all the partial FV-derivatives with respect to the cut-elements are linear combinations of the FV-derivatives with respect to the core-elements; i.e. there exist complex numbers $r_\tau, s_\tau, t_\tau, u_\tau$, where $\tau \in I$, such that:

$$\partial_{xx} A = r_1 A + r_x \partial_x A + r_y \partial_y A, \tag{6.2}$$

$$\partial_{yx} A = s_1 A + s_x \partial_x A + s_y \partial_y A, \tag{6.3}$$

$$\partial_{xy} A = t_1 A + t_x \partial_x A + t_y \partial_y A, \tag{6.4}$$

$$\partial_{yy} A = u_1 A + u_x \partial_x A + u_y \partial_y A. \tag{6.5}$$

We can write the above equations in sum notation:

$$\partial_{xx} A = \sum_{\tau \in I} r_\tau \partial_\tau A,$$

$$\partial_{yx} A = \sum_{\tau \in I} s_\tau \partial_\tau A,$$

$$\partial_{xy} A = \sum_{\tau \in I} t_\tau \partial_\tau A,$$

$$\partial_{yy} A = \sum_{\tau \in I} u_\tau \partial_\tau A,$$

where $\partial_1 A = A$.

(6.2) implies that:

$$\begin{aligned}
&\underbrace{a_{xx} + a_{xxx}x + a_{yxx}y + a_{xxxx}xx + a_{yxxx}yx + \dots}_{\partial_{xx}A} \qquad (6.6)\\
=&\underbrace{r_1a_1 + r_1a_xx + r_1a_yy + r_1a_{xx}xx + r_1a_{yx}yx + \dots}_{r_1A}\\
+&\underbrace{r_xa_x + r_xa_{xx}x + r_xa_{yx}y + r_xa_{xxx}xx + r_xa_{yxx}yx + \dots}_{r_x\partial_xA}\\
+&\underbrace{r_ya_y + r_ya_{xy}x + r_ya_{yy}y + r_ya_{xxy}xx + r_ya_{yxy}yx + \dots}_{r_y\partial_yA}.
\end{aligned}$$

Therefore, by summing the above equation column-wise, we obtain:

$$a_{xx} = r_1a_1 + r_xa_x + r_ya_y, \qquad (6.7)$$

$$a_{xxx} = r_1a_x + r_xa_{xx} + r_ya_{xy}. \qquad (6.8)$$

$$\vdots$$

In general, if α is in X^*, then

$$\begin{aligned}
a_{\alpha xx} &= r_1a_\alpha + r_xa_{\alpha x} + r_ya_{\alpha y}\\
&= \sum_{\tau\in I} r_\tau a_{\alpha\tau}. \qquad (6.9)
\end{aligned}$$

If we perform similar calculations for $\partial_{yx}A$, $\partial_{xy}A$, and $\partial_{yy}A$, and for any $\alpha \in X^*$, we obtain the following equations:

$$a_{\alpha yx} = s_1a_\alpha + s_xa_{\alpha x} + s_ya_{\alpha y} = \sum_{\tau\in I} s_\tau a_{\alpha\tau}, \qquad (6.10)$$

$$a_{\alpha xy} = t_1a_\alpha + t_xa_{\alpha x} + t_ya_{\alpha y} = \sum_{\tau\in I} t_\tau a_{\alpha\tau}, \qquad (6.11)$$

$$a_{\alpha yy} = u_1a_\alpha + u_xa_{\alpha x} + u_ya_{\alpha y} = \sum_{\tau\in I} u_\tau a_{\alpha\tau}. \qquad (6.12)$$

Every non-core element of X^* can be written as $\alpha\pi$, where π is an element of $\{xx, yx, xy, yy\} = J$. Therefore, given sets $\{a_\tau\}$ and $\{r_\tau, s_\tau, t_\tau, u_\tau\}$, where $\tau \in I$, we can recursively find all the coefficients a_α, $\alpha \in X^*$ of A using equations (6.9), (6.10), (6.11), and (6.12).

Remark 6.2.2. Note that the set $\{a_\tau\}$ corresponds to an *initial coefficient set*, and the set $\{r_\tau, s_\tau, t_\tau, u_\tau\}$ corresponds to a *determining set* from Definition 5.2.1. We will return to this idea after we generalize Example 6.2.1.

Theorem 6.2.3 (Generalization of Example 6.2.1)**.** *Let I be a core set and $J = \mathrm{cut}(I)$, and suppose there exist complex numbers $r(\pi)_\tau$ for all $\tau \in I$ and $\pi \in J$ such that*

$$\partial_\pi A = \sum_{\tau \in I} r(\pi)_\tau \partial_\tau A, \tag{6.13}$$

where the $r(\pi)$'s correspond to different letters r, s, t, u from Example 6.2.1. Then all the non-core coefficients of $A(x, y)$ can be determined recursively for any α in X^* with the following equation

$$a_{\alpha\pi} = \sum_{\tau \in I} r(\pi)_\tau a_{\alpha\tau}. \tag{6.14}$$

Proof. Let $r(\pi)_\tau$ be complex numbers such that (6.13) holds for all $\tau \in I$ and $\pi \in J$. By Theorem 4.2.2,

$$\partial_\beta A = \sum_{\alpha \in X^*} a_{\alpha\beta}\alpha \tag{6.15}$$

for every β in X^*. If we apply (6.15) to $\partial_\pi A$ and $\partial_\tau A$, (6.13) becomes

$$\sum_{\alpha \in X^*} a_{\alpha\pi}\alpha = \sum_{\tau \in I} r(\pi)_\tau \sum_{\alpha \in X^*} a_{\alpha\tau}\alpha. \tag{6.16}$$

Consider the coefficient of some fixed α. On the left hand side it is $a_{\alpha\pi}$, and on the

right hand side it is $\sum_{\tau \in I} r(\pi)_\tau a_{\alpha\tau}$. Therefore,

$$a_{\alpha\pi} = \sum_{\tau \in I} r(\pi)_\tau a_{\alpha\tau}. \tag{6.17}$$

Since π is an element of $\mathrm{cut}(I)$, any element of X^* that is not in I can be written as $\alpha\pi$, and, hence, any non-core coefficient $a_{\alpha\pi}$ can be calculated by repeating (6.17). □

Corollary 6.2.4. *Given a core set I of $A(x, y)$ and $J = \mathrm{cut}(I)$, if there exist complex numbers $r(\pi)_\tau$ such that* (6.13) *holds for all $\tau \in I$ and $\pi \in J$. Then the set*

$$\boldsymbol{a} = \{a_\tau \,|\tau \in I\} \tag{6.18}$$

is an initial coefficient set, and the set

$$\boldsymbol{r} = \{r(\pi)_\tau \,|\tau \in I,\ \pi \in J\} \tag{6.19}$$

is a collection of determining sets. That is, $a_{\alpha\pi}$'s satisfy a set of linear homogeneous recurrence relations with constant coefficients $r(\pi)_\tau$'s. There is a linear recurrence relation for each π in $\mathrm{cut}(I)$. □

Definition 6.2.5. If for a core set I and $\mathrm{cut}(I)$ of $A(x, y)$, there exist complex numbers $r(\pi)_\tau$ such that (6.13) holds for all $\tau \in I$ and $\pi \in \mathrm{cut}(I)$, we call I a *determining core.*

Definition 6.2.6. If a core set I is a determining core such that no proper subset of I is a determining core, we call I a *minimal determining core.*

Corollary 6.2.7. *Given a determining core I of A, $\{\partial_\tau A,\ \tau \in I\}$ is a spanning set for V_A; in particular,* $\dim V_A \leq |I|$. □

Theorem 6.2.8. *Let I be a core set, such that $\partial_\tau A$, $\tau \in I$, form a basis for V_A. Then there is no proper subset K of I such that a_κ, $\kappa \in K$ determine A, i.e., I is a minimal determining core of A.*

Proof. Suppose there exists a core set K, a proper subset of I, such that a_κ, $\kappa \in K$ determine A. We can choose a node ω in $\mathrm{cut}(K)$ such that that ω is in I.

Since ω is in $\mathrm{cut}(K)$, there exist complex numbers r_κ, $\kappa \in K$ such that

$$a_{\alpha\omega} = \sum_{\kappa \in K} r_\kappa a_{\alpha\kappa}. \tag{6.20}$$

Now,

$$\partial_\omega A = \sum_{\alpha \in X^*} a_{\alpha w}\alpha, \tag{6.21}$$

so by substituting (6.20) into (6.22), we get:

$$\begin{aligned} \partial_\omega A &= \sum_{\alpha \in X^*} \sum_{\kappa \in K} r_\kappa a_{\alpha\kappa}\alpha \\ &= \sum_{\kappa \in K} \sum_{\alpha \in X^*} r_\kappa a_{\alpha\kappa}\alpha \\ &= \sum_{\kappa \in K} r_\kappa \sum_{\alpha \in X^*} a_{\alpha\kappa}\alpha \\ &= \sum_{\kappa \in K} r_\kappa \partial_\kappa A \end{aligned} \tag{6.22}$$

This contradicts the assumption that elements of I form a basis for V_A because ω is in I and K is a subset of I, and K together with ω form a linearly dependent set. □

Corollary 6.2.9. *If I is a core set of A, such that $\partial_\tau A$, $\tau \in I$, form a basis for V_A, then I is a minimal determining core of A.* □

Corollary 6.2.10. *Given any determining core set of A, we can reduce it to a minimal determining core of A.*

Proof. Let I be a determining core of A. By Corollary 6.2.7, $S = \{\partial_\tau A \,|\, \tau \in I\}$ is a spanning set for V_A. Therefore, we can find a basis for V_A that is a subset of S, and obtain the corresponding minimal determining core of A (Theorem 6.2.8).

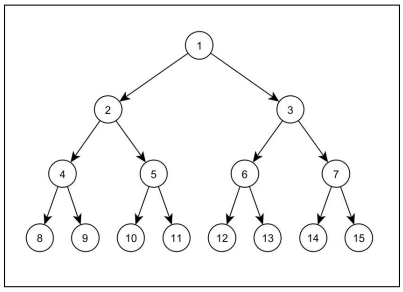

Figure 6.4: An ordering of elements of X^*: first by length, and then by some alphabet.

To ensure that the basis we find corresponds to a proper core, we execute the following steps. We start with a would-be basis set W containing only the root node i.e. $\partial_1 A$, and a"candidate for basis list" containing all the other elements of I, call this set Q. We pick the next (according to the order given in Figure 6.4) node α in Q, and check if $\partial_\alpha A$ can be written as a linear combination of the elements already in W. If it can, then, by Theorem 6.2.3, all its decedents also can be written as a linear combination of the elements in W, and we remove α and all its decedents from Q. If not, we adjoin α to W without damaging its linear independence. We terminate the process when there is no more nodes in Q. Since we started with a spanning set, the resulting set W will be a basis for V_A, and since we added only the nodes whose ancestors are already in W, this basis corresponds to a proper core set. □

Theorem 6.2.11. *Let α be a word in X^*; $\alpha = \alpha_1\alpha_2 \ldots \alpha_k$, where $\alpha_i \in \{x, y\}$. Then,*

$$\begin{aligned}\partial_\alpha AB =& A\partial_\alpha B + b_{\alpha_2\alpha_3\ldots\alpha_k}\partial_{\alpha_1}A + b_{\alpha_3\alpha_4\ldots\alpha_k}\partial_{\alpha_1\alpha_2}A + \ldots \\ &+ b_{\alpha_k}\partial_{\alpha_1\alpha_2\ldots\alpha_{k-1}}A + b_1\underbrace{\partial_{\alpha_1\alpha_2\ldots\alpha_k}A}_{\partial_\alpha A}.\end{aligned} \tag{6.23}$$

Proof. We proceed by induction on the length k of α. The $k = 1$ case is the product rule (4.2.4). Assume the statement is true for $\alpha = \alpha_1\alpha_2\ldots\alpha_{k-1}$. Let α_0 be a single letter. Then

$$\begin{aligned}\partial_{\alpha_0\alpha}AB =& \partial_{\alpha_0}(\partial_\alpha AB) \\ =& \partial_{\alpha_0}(A\partial_\alpha B + b_{\alpha_2\alpha_3\ldots\alpha_k}\partial_{\alpha_1}A + \cdots + b_1\partial_\alpha A) \\ =& \underbrace{A\partial_{\alpha_0\alpha}B + b_\alpha\partial_{\alpha_0}A}_{\text{product rule (4.2.4)}} + b_{\alpha_2\alpha_3\ldots\alpha_k}\partial_{\alpha_0\alpha_1}A + \cdots + b_1\partial_{\alpha_0\alpha}A.\end{aligned} \tag{6.24}$$

The theorem holds. □

Corollary 6.2.12. *The following inclusion holds*

$$V_{AB} \subseteq A(V_B) + V_A. \tag{6.25}$$

for any A and B in $\mathbb{C}\langle\langle x, y\rangle\rangle$.

Proof. By Lemma 6.2.11,

$$\partial_\alpha AB = \underbrace{A\partial_\alpha B}_{\in A(V_B)} + b_{\alpha_2\alpha_3\ldots\alpha_k}\underbrace{\partial_{\alpha_1}A}_{\in V_A} + b_{\alpha_3\alpha_4\ldots\alpha_k}\underbrace{\partial_{\alpha_1\alpha_2}A}_{\in V_A} + \cdots + b_1\underbrace{\partial_\alpha A}_{\in V_A}. \tag{6.26}$$

The Corollary follows. □

Theorem 6.2.13. *If we know determining cores (not necessarily minimal) of formal power series A and B, we can find minimal determining cores of $A + B$, AB, and A^{-1}.*

Proof. Suppose determining cores of A and B have k and ℓ elements respectively. Recall that by Corollary 5.1.14,

$$\dim_{\mathbb{C}} V_{(A+B)} \leq \dim_{\mathbb{C}} V_A + \dim_{\mathbb{C}} V_B, \tag{6.27}$$

$$\dim_{\mathbb{C}} V_{AB} \leq \dim_{\mathbb{C}} V_A + \dim_{\mathbb{C}} V_B, \qquad \text{and} \tag{6.28}$$

$$\dim_{\mathbb{C}} V_{A^{-1}} \leq \dim_{\mathbb{C}} V_A + 1. \tag{6.29}$$

Also, by Corollary 6.2.7, any determining core of A corresponds to a spanning set for V_A. Hence,

$$\dim_{\mathbb{C}} V_A \leq k, \qquad \text{and} \tag{6.30}$$

$$\dim_{\mathbb{C}} V_B \leq \ell. \tag{6.31}$$

Therefore,

$$\dim_{\mathbb{C}} V_{(A+B)} \leq k + \ell, \tag{6.32}$$

$$\dim_{\mathbb{C}} V_{AB} \leq k + \ell, \qquad \text{and} \tag{6.33}$$

$$\dim_{\mathbb{C}} V_{A^{-1}} \leq k + 1. \tag{6.34}$$

This implies that minimal determining cores of $A + B$ and AB can have at most $k + \ell$ elements, so they can contain elements of length at most $k + \ell - 1$. Furthermore, a minimal determining core of A^{-1} can have at most $k + 1$ elements, so it can contain elements of length at most k. Therefore,

$$I_{AB} = \{\tau \in X^* \mid \text{length}(\tau) \leq k + \ell - 1\} \qquad \text{and} \tag{6.35}$$

$$I_{A^{-1}} = \{\tau \in X^* \mid \text{length}(\tau) \leq k\} \tag{6.36}$$

are determining cores of $A + B$, AB, and A^{-1} respectively.

Now we can reduce them to minimal determining cores using the algorithm given in the proof of Corollary 6.2.10. □

6.3 Example

In this section we illustrate the results of this chapter by working through a numeric example. Suppose we have a rational function in 2 non-commutative variables a and b, given by

$$\gamma(a,b) = [2 - a(1+b^2)^{-1}a - b(1+a^2)^{-1}b]^{-1}. \tag{6.37}$$

This expression is a slightly modified example of a rational non-commutative function given by Farber and Vogel [FV91, p. 437, Exmp. 2.4]. We wish to find the formal power series representation of γ, call it $C(x,y)$, a minimal determining core of C, and the set of linear recurrence relations that together with this core coefficients determine C.

Remark 6.3.1. The expression given by Farber and Vogel [FV91, p. 437, Exmp. 2.4] (up to renaming the variables) is

$$\gamma_{FV}(a,b) = [1 - a(1-b^2)^{-1}a - b(1-a^2)^{-1}b]^{-1}. \tag{6.38}$$

We modified it for the following reasons:

(1) Under the Magnus embedding,

$$\begin{aligned} 1 - a^2 &\mapsto -2x - x^2, \\ 1 - b^2 &\mapsto -2y - y^2. \end{aligned} \tag{6.39}$$

We can see that $1 - a^2$ and $1 - b^2$ are not invertible in Γ because their constant terms are not invertible. To make them invertible, we changed "–" to "+" in both expressions.

(2) Now,

$$1 - a(1+b^2)^{-1}a - b(1+a^2)^{-1}b \mapsto -\frac{1}{2}x - \frac{1}{2}y - \frac{3}{4}x^2 + yx + \ldots, \tag{6.40}$$

which is again not invertible. Hence, we changed "1" to "2" in this expression.

Example 6.3.2 (Finding the formal power series representation of γ)**.** Recall that the Magnus embedding $\mu : R[F_2] \to R\langle\langle x, y\rangle\rangle$ is given by:

$$\begin{aligned} a &\mapsto 1 + x & a^{-1} &\mapsto 1 - x + x^2 - x^3 + \dots \\ b &\mapsto 1 + y & b^{-1} &\mapsto 1 - y + y^2 - y^3 + \dots \end{aligned}$$

Suppose,

$$\mu\big((1 + a^2)^{-1}\big) = t_0 + t_1 x + t_2 x^2 + \dots, \tag{6.41}$$

for some $t_0, t_1, t_2, \dots$. Then with some basic algebra we find that

$$\begin{aligned} t_0 = \frac{1}{2}, \;\; t_1 = -\frac{1}{2}, \text{ and} \\ t_k = -t_{k-1} - \frac{1}{2} t_{k-2}. \end{aligned} \tag{6.42}$$

Therefore,

$$\mu\big((1 + a^2)^{-1}\big) = \frac{1}{2} - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \frac{1}{32}x^8 - \frac{1}{32}x^9 + \dots \tag{6.43}$$

It follows that

$$\begin{aligned} A(x, y) =& \mu\big(b(1 + a^2)^{-1} b\big) \\ =& (1 + y)\left(\frac{1}{2} - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \dots\right)(1 + y) \\ =& \frac{1}{2} + y + \frac{1}{2}y^2 \\ & - \frac{1}{2}x - \frac{1}{2}yx - \frac{1}{2}xy - \frac{1}{2}yxy \\ & + \frac{1}{4}x^2 + \frac{1}{4}yx^2 + \frac{1}{4}x^2 y + \frac{1}{4}yx^2 y \\ & - \frac{1}{8}x^4 - \frac{1}{8}yx^4 - \frac{1}{8}x^4 y - \frac{1}{8}yx^4 y \dots, \end{aligned} \tag{6.44}$$

and if $B(x, y) = \mu\big(a(1 + b^2)^{-1} a\big)$, then

$$B(x, y) = A(y, x). \tag{6.45}$$

Therefore, the expression in the outermost parenthesis,

$2 - a(1 + b^2)^{-1}a - b(1 + a^2)^{-1}b = 2 - A - B$ has the following the formal power series representation:

$$\begin{aligned} 2 - A - B =& 1 - \frac{1}{2}x - \frac{1}{2}y \\ & - \frac{3}{4}x^2 + yx + xy - \frac{3}{4}y^2 \\ & - \frac{1}{4}yx^2 + \frac{1}{2}xyx - \frac{1}{4}y^2x - \frac{1}{4}x^2y + \frac{1}{2}yxy - \frac{1}{4}xy^2 \\ & + \frac{1}{8}x^4 - \frac{1}{4}xy^2x - \frac{1}{4}yx^2y + \frac{1}{8}y^4 \end{aligned} \tag{6.46}$$

$$\ldots,$$

which can in turn be represented by the following tree diagram.

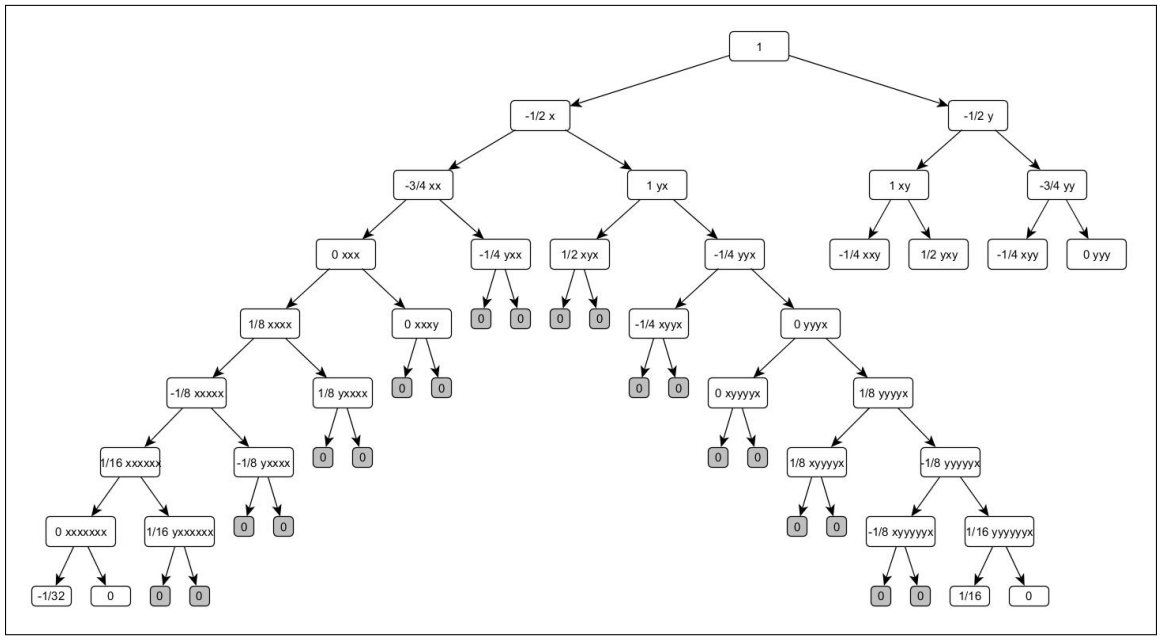

Figure 6.5: $[2 - a(1 + b^2)^{-1}a - b(1 + a^2)^{-1}b]$ The right half of the tree is a reflection of the left half in the vertical axis. All decedents of the gray 0 nodes are also 0's.

Let $D = 2 - A - B$, and $C = D^{-1}$, then (since $d_0 = 1$),

$$c_0 = 1, \text{and } c_\lambda = -\sum_{\lambda=\alpha\beta} d_\alpha c_\beta. \tag{6.47}$$

Now we calculate the coefficients of C with (6.47):

$$\begin{aligned} C =& 1+\frac{1}{2}x+\frac{1}{2}y+x^2-\frac{3}{4}yx-\frac{3}{4}xy+y^2+\frac{7}{8}x^3+\frac{1}{4}yx^2-\frac{11}{8}xyx \\ &+\frac{1}{4}y^2x+\frac{1}{4}x^2y-\frac{11}{8}yxy+\frac{1}{4}xy^2+\frac{7}{8}y^3+\frac{17}{16}x^4-\frac{7}{16}yx^3 \\ &-\frac{9}{8}xyx^2+y^2x^2-\frac{9}{8}x^2yx-\frac{3}{16}yxyx+\frac{5}{4}xy^2x-\frac{7}{16}y^3x-\frac{7}{16}x^3y \\ &+\frac{5}{4}yx^2y-\frac{3}{16}xyxy-\frac{9}{8}y^2xy+x^2y^2-\frac{9}{8}yxy^2-\frac{7}{16}xy^3+\frac{17}{16}y^4+\dots \end{aligned} \tag{6.48}$$

[Our special thanks to Boris Zamoruev, who helped writing a JavaScript program to calculate the coefficients of C. For the program see Appendix 6.3. For a list of the the first 127 coefficients of $C(x,y)$ (for words of length 0 - 6) see Appendix 6.3.]

Example 6.3.3 (Finding a minimal determining core of C)**.** To find a minimal determining core of C and the corresponding set of linear recurrences, we follow the procedure described in Corollary 6.2.10. We form square coefficient matrices, where each column represents ordered coefficients of the corresponding FV-derivative, and each row represents the coefficients of the specified monomial across derivatives. We start with a 2×2 matrix and increase the number of columns by 1 (keeping the matrix square):

$$\begin{array}{c} \begin{array}{cc} C & \partial_x C \end{array} \\ \begin{array}{c} 1 \\ x \end{array}\!\!\begin{pmatrix} c_1 & c_x \\ c_x & c_{xx} \end{pmatrix} \end{array}, \quad \begin{array}{c} \begin{array}{ccc} C & \partial_x C & \partial_y C \end{array} \\ \begin{array}{c} 1 \\ x \\ y \end{array}\!\!\begin{pmatrix} c_1 & c_x & c_y \\ c_x & c_{xx} & c_{xy} \\ c_y & c_{yx} & c_{yy} \end{pmatrix} \end{array}, \quad \begin{array}{c} \begin{array}{cccc} C & \partial_x C & \partial_y C & \partial_{xx} C \end{array} \\ \begin{array}{c} 1 \\ x \\ y \\ xx \end{array}\!\!\begin{pmatrix} c_1 & c_x & c_y & c_{xx} \\ c_x & c_{xx} & c_{xy} & c_{xxx} \\ c_y & c_{yx} & c_{yy} & c_{yxx} \\ c_{xx} & c_{xxx} & c_{xxy} & c_{xxxx} \end{pmatrix} \end{array}, \dots \end{array} \tag{6.49}$$

We repeat the process until we add a column that can be written as a linear

combination of the previous columns The first linearly dependent matrix we find is

$$\begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array}\begin{array}{c} \begin{array}{cccccc} C & \partial_x C & \partial_y C & \partial_{xx} C & \partial_{yx} C & \partial_{xy} C \end{array} \\ \begin{pmatrix} c_1 & c_x & c_y & c_{xx} & c_{yx} & c_{xy} \\ c_x & c_{xx} & c_{xy} & c_{xxx} & c_{xyx} & c_{xxy} \\ c_y & c_{yx} & c_{yy} & c_{yxx} & c_{yyx} & c_{yxy} \\ c_{xx} & c_{xxx} & c_{xxy} & c_{xxxx} & c_{xxyx} & c_{xxxy} \\ c_{yx} & c_{yxx} & c_{yxy} & c_{yxxx} & c_{yxyx} & c_{yxxy} \\ c_{xy} & c_{xyx} & c_{xyy} & c_{xyxx} & c_{xyyx} & c_{xyxy} \end{pmatrix} \end{array} = \tag{6.50}$$

$$\begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array}\begin{array}{c} \begin{array}{cccccc} C & \partial_x C & \partial_y C & \partial_{xx} C & \partial_{yx} C & \partial_{xy} C \end{array} \\ \begin{pmatrix} 1 & {}^1\!/_2 & {}^1\!/_2 & 1 & {}^{-3}\!/_4 & {}^{-3}\!/_4 \\ {}^1\!/_2 & 1 & {}^{-3}\!/_4 & {}^7\!/_8 & {}^{-11}\!/_8 & {}^1\!/_4 \\ {}^1\!/_2 & {}^{-3}\!/_4 & 1 & {}^1\!/_4 & {}^1\!/_4 & {}^{-11}\!/_8 \\ 1 & {}^7\!/_8 & {}^1\!/_4 & {}^{17}\!/_{16} & {}^{-9}\!/_8 & {}^{-7}\!/_{16} \\ {}^{-3}\!/_4 & {}^1\!/_4 & {}^{-11}\!/_8 & {}^{-7}\!/_{16} & {}^{-3}\!/_{16} & {}^5\!/_4 \\ {}^{-3}\!/_4 & {}^{-11}\!/_8 & {}^1\!/_4 & {}^{-9}\!/_8 & {}^5\!/_4 & {}^{-3}\!/_{16} \end{pmatrix} \end{array}.$$

We find its *rref* (omitted), and write the last column as a linear combination of the first five linearly independent columns as follows:

$$\partial_{xy} C = \frac{1}{2} C + \partial_x C - \frac{3}{4} \partial_y C - \partial_{xx} C + \frac{1}{2} \partial_{yx} C. \tag{6.51}$$

We repeat the process for ∂_{yy}, ∂_{xxx}, ∂_{yxx}, ∂_{xyx}, and ∂_{yyx} with the new coefficient

matrix for ∂_{yy} given by

$$\begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array} \begin{array}{c} \begin{array}{cccccc} C & \partial_x C & \partial_y C & \partial_{xx} C & \partial_{yx} C & \partial_{yy} C \end{array} \\ \begin{pmatrix} c_1 & c_x & c_y & c_{xx} & c_{yx} & c_{yy} \\ c_x & c_{xx} & c_{xy} & c_{xxx} & c_{xyx} & c_{xyy} \\ c_y & c_{yx} & c_{yy} & c_{yxx} & c_{yyx} & c_{yyy} \\ c_{xx} & c_{xxx} & c_{xxy} & c_{xxxx} & c_{xxyx} & c_{xxyy} \\ c_{yx} & c_{yxx} & c_{yxy} & c_{yxxx} & c_{yxyx} & c_{yxyy} \\ c_{xy} & c_{xyx} & c_{xyy} & c_{xyxx} & c_{xyyx} & c_{xyyy} \end{pmatrix} \end{array}. \tag{6.52}$$

For the other FV-derivatives we keep the first five linearly independent columns fixed, and change the last column to one of these:

$$\begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array} \begin{array}{c} \partial_{xxx} C \\ \begin{pmatrix} c_{xxx} \\ c_{xxxx} \\ c_{yxxx} \\ c_{xxxxx} \\ c_{yxxxx} \\ c_{xyxxx} \end{pmatrix} \end{array}, \begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array} \begin{array}{c} \partial_{yxx} C \\ \begin{pmatrix} c_{yxx} \\ c_{xyxx} \\ c_{yyxx} \\ c_{xxyxx} \\ c_{yxyxx} \\ c_{xyyxx} \end{pmatrix} \end{array}, \begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array} \begin{array}{c} \partial_{xyx} C \\ \begin{pmatrix} c_{xyx} \\ c_{xxyx} \\ c_{yxyx} \\ c_{xxxyx} \\ c_{yxxyx} \\ c_{xyxyx} \end{pmatrix} \end{array}, \text{ or } \begin{array}{c} \\ 1 \\ x \\ y \\ xx \\ yx \\ xy \end{array} \begin{array}{c} \partial_{yyx} C \\ \begin{pmatrix} c_{yyx} \\ c_{xyyx} \\ c_{yyyx} \\ c_{xxyyx} \\ c_{yxyyx} \\ c_{xyyyx} \end{pmatrix} \end{array}. \tag{6.53}$$

We find the following equations:

$$\begin{aligned} \partial_{yy} C &= \frac{3}{4} C - \frac{1}{4} \partial_x C + \frac{5}{8} \partial_y C - \frac{1}{2} \partial_{xx} C - \frac{3}{4} \partial_{yx} C, \\ \partial_{xxx} C &= \partial_x C + \frac{3}{8} \partial_y C - \frac{1}{4} \partial_{yx} C, \\ \partial_{yxx} C &= \frac{1}{4} C + \frac{3}{4} \partial_y C + \frac{1}{2} \partial_{yx} C, \\ \partial_{xyx} C &= -\frac{1}{4} C - \frac{1}{2} \partial_x C - \frac{3}{8} \partial_y C - \frac{1}{2} \partial_{xx} C + \frac{1}{4} \partial_{yx} C, \qquad \text{and} \\ \partial_{yyx} C &= \frac{5}{8} C - \frac{3}{8} \partial_x C - \frac{9}{16} \partial_y C - \frac{3}{4} \partial_{xx} C - \frac{9}{8} \partial_{yx} C. \end{aligned} \tag{6.54}$$

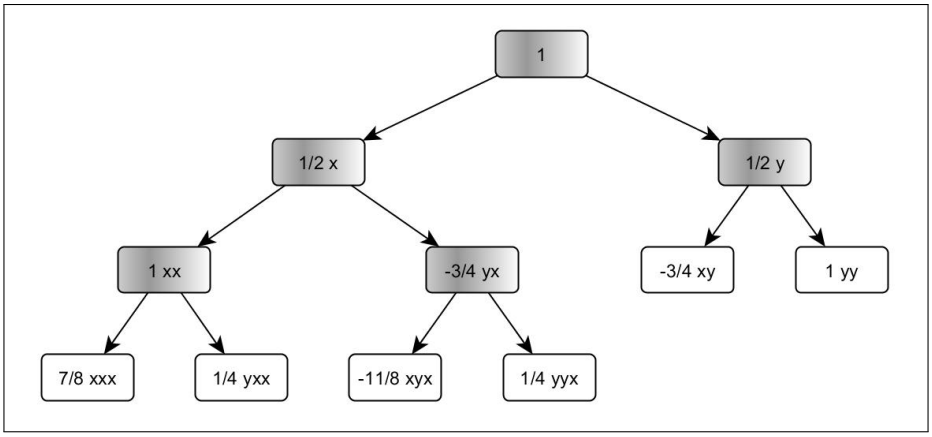

Figure 6.6: Tree diagram of the formal power series representation of $\gamma = [2 - a(1+b^2)^{-1}a - b(1+a^2)^{-1}b]^{-1}$. The gray nodes represent a minimal core, and the white ones represent the corresponding cut.

It follows from (6.51) and (6.54) that a determining coefficient set for this non-commutative formal power series is $\{c_1 = 1, c_x = \frac{1}{2}, c_y = \frac{1}{2}, c_{xx} = 1, c_{yx} = -\frac{3}{4}\}$, and the corresponding set of homogeneous linear recurrences is

$$c_{\alpha xy} = \frac{1}{2}c_\alpha + c_{\alpha x} - \frac{3}{4}c_{\alpha y} - c_{\alpha xx} + \frac{1}{2}c_{\alpha yx}, \tag{6.55}$$

$$c_{\alpha yy} = \frac{3}{4}c_\alpha - \frac{1}{4}c_{\alpha x} + \frac{5}{8}c_{\alpha y} - \frac{1}{2}c_{\alpha xx} - \frac{3}{4}c_{\alpha yx}, \tag{6.56}$$

$$c_{\alpha xxx} = c_{\alpha x} + \frac{3}{8}c_{\alpha y} - \frac{1}{4}c_{\alpha yx}, \tag{6.57}$$

$$c_{\alpha yxx} = \frac{1}{4}c_\alpha + \frac{3}{4}c_{\alpha y} + \frac{1}{2}c_{\alpha yx}, \tag{6.58}$$

$$c_{\alpha xyx} = -\frac{1}{4}c_\alpha - \frac{1}{2}c_{\alpha x} - \frac{3}{8}c_{\alpha y} - \frac{1}{2}c_{\alpha xx} + \frac{1}{4}c_{\alpha yx}, \qquad \text{and} \tag{6.59}$$

$$c_{\alpha yyx} = \frac{5}{8}c_\alpha - \frac{3}{8}c_{\alpha x} - \frac{9}{16}c_{\alpha y} - \frac{3}{4}c_{\alpha xx} - \frac{9}{8}c_{\alpha yx}. \tag{6.60}$$

This information is sufficient to recursively determine all the coefficients of C. For

instance, to calculate $c_{xyyxyx} = c_{(xyy)(xyx)}$ we use (6.59) for $\alpha = xyy$:

$$\begin{aligned} c_{xyyyyx} &= -\frac{1}{4}c_{xyy} - \frac{1}{2}c_{xyyx} - \frac{3}{8}c_{xyyy} - \frac{1}{2}c_{xyyxx} + \frac{1}{4}c_{xyyyx} \\ &= -\frac{1}{4}0.25 - \frac{1}{2}1.25 - \frac{3}{8}(-0.4375) - \frac{1}{2}0.625 + \frac{1}{4}(-0.65625) \\ &= -1. \end{aligned} \tag{6.61}$$

Our result agrees with the coefficient table in Appendix 6.3.

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APPENDIX

A.1 JavaScript code

This is the JavaScript code used to calculate the coefficients of the formal power series C in Section 6.3. Our special thanks to Boris Zamoruev, who helped writing this program.

```
// Input A
var a = {
    "1":  1,
    "x":  -1/2,
    "y":  -1/2,
    "xx": -3/4,
    "yx": 1,
    "xy": 1,
    "yy": -3/4,
    "yxx": -1/4,
    "xyx": 1/2,
    "yyx": -1/4,
    "xxy": -1/4,
    "xyy": -1/4,
    "yxy": 1/2,
    "xxxx": 1/8,
    "yyyy": 1/8,
    "xyyx": -1/4,
    "yxxy": -1/4,
```

```
    "xxxxx": -1/8,
    "yxxxx": 1/8,
    "yyyyx": 1/8,
    "xyyyy": 1/8,
    "xxxxy": 1/8,
    "yyyyy": -1/8,
    "xxxxxx": 1/16,
    "yxxxxx": -1/8,
    "xyyyyx": -1/8,
    "xyyyyy": -1/8,
    "yxxxxx": -1/8,
    "yxxxxy": -1/8,
    "xyyyyy": -1/8,
    "yyyyyy": 1/16,
};
var st = comp_subs(6);
/*************  GENERATE SUBSCRIPTS  *************/
var b = {};
b["1"] = 1/a["1"];
function comp_subs(max) {
    st = [];

    rec(1, "x");
    rec(1, "y");

    function rec(cur, xy) {
```

```
        var r;
        if (cur == 1) {
            r = xy;
        } else {
            r = xy + cur;
        }
        st.push(r);

        if (r.length >= max)
            return;

        rec(r, "x");
        rec(r, "y");
    }
    return st;
}
console.log(st);
/****************** FIND INVERSE ******************/
for (var i in st) {
    var sum = 0;
    console.log("Computing B" + st[i]);
    for (var j = 0; j < st[i].length; ++j) {
        Asub = st[i].substring(0, st[i].length - j);
        Bsub = st[i].substring(st[i].length - j, st[i].length);
        if (Bsub.length == 0)
            Bsub = "1";
```

```
            sum = sum + (b[Bsub]*(a[Asub] ? a[Asub] : 0));


            console.log("Adding: A" + Asub + "B" + Bsub);
        }
        b[st[i]] = -b["1"]*sum;
        console.log("------------------");
    }
    console.log(b);
    /****************** PRINT RESULT ******************/
    var res = "";
    for (key in b) {
        $("#result").append("B<sub>" + key + "</sub> = " + b[key] + "<br>");
    }
```

A.2 Program Output

Here we list the the first 127 coefficients of $C(x, y)$ (for words of length 0 to 6) as calculated by the program. We list them in column-wise alphabetical order.

$C_1 = 1$
$C_x = 0.5$
$C_{xx} = 1$
$C_{xxx} = 0.875$
$C_{xxxx} = 1.0625$
$C_{xxxxx} = 1.25$
$C_{xxxxxx} = 1.296875$
$C_{xxxxxy} = -0.28125$
$C_{xxxxy} = -0.21875$
$C_{xxxxyx} = -1.609375$
$C_{xxxxyy} = 0.90625$
$C_{xxxy} = -0.4375$
$C_{xxxyx} = -1.59375$
$C_{xxxyxx} = -0.90625$
$C_{xxxyxy} = -0.796875$
$C_{xxxyy} = 0.6875$
$C_{xxxyyx} = 1.25$
$C_{xxxyyy} = 0.015625$
$C_{xxy} = 0.25$
$C_{xxyx} = -1.125$
$C_{xxyxx} = -0.125$
$C_{xxyxxx} = -0.90625$
$C_{xyxyxx} = 0.34375$
$C_{xyxyxy} = 1.703125$
$C_{xyxyy} = -1.3125$
$C_{xyxyyx} = -1$
$C_{xyxyyy} = -0.796875$
$C_{xyy} = 0.25$
$C_{xyyx} = 1.25$
$C_{xyyxx} = 0.625$
$C_{xyyxxx} = 1.25$
$C_{xyyxxy} = -1.0625$
$C_{xyyxy} = 0.75$
$C_{xyyxyx} = -1$
$C_{xyyxyy} = 1.375$
$C_{xyyy} = -0.4375$
$C_{xyyyx} = -0.65625$
$C_{xyyyxx} = -0.59375$
$C_{xyyyxy} = -0.015625$
$C_{xyyyy} = -0.21875$
$C_{xyyyyx} = 0.453125$
$C_{xyyyyy} = -0.15625$
$C_y = 0.5$
$C_{yx} = -0.75$
$C_{yxyxyy} = 0.34375$
$C_{yxyy} = -1.125$
$C_{yxyyx} = 0.75$
$C_{yxyyxx} = -0.8125$
$C_{yxyyxy} = 2.1875$
$C_{yxyyy} = -1.59375$
$C_{yxyyyx} = -0.015625$
$C_{yxyyyy} = -1.609375$
$C_{yy} = 1$
$C_{yyx} = 0.25$
$C_{yyxx} = 1$
$C_{yyxxx} = 0.6875$
$C_{yyxxxx} = 0.90625$
$C_{yyxxxy} = -0.59375$
$C_{yyxxy} = 0.625$
$C_{yyxxyx} = -0.8125$
$C_{yyxxyy} = 1.125$
$C_{yyxy} = -1.125$
$C_{yyxyx} = -1.3125$
$C_{yyxyxx} = -1.4375$
$C_{yyxyxy} = 0.34375$
$C_{yyxyy} = -0.125$

$C_{xxyxxy} = 1.375$
$C_{xxyxy} = -1.3125$
$C_{xxyxyx} = 0.34375$
$C_{xxyxyy} = -1.4375$
$C_{xxyy} = 1$
$C_{xxyyx} = 0.625$
$C_{xxyyxx} = 1.125$
$C_{xxyyxy} = -0.8125$
$C_{xxyyyy} = 0.6875$
$C_{xxyyyyx} = -0.59375$
$C_{xxyyyyy} = 0.90625$
$C_{xy} = -0.75$
$C_{xyx} = -1.375$
$C_{xyxx} = -1.125$
$C_{xyxxx} = -1.59375$
$C_{xyxxxx} = -1.609375$
$C_{xyxxxy} = -0.015625$
$C_{xyxxy} = 0.75$
$C_{xyxxyx} = 2.1875$
$C_{xyxxyy} = -0.8125$
$C_{xyxy} = -0.1875$
$C_{xyxyx} = 1.65625$
$C_{yxx} = 0.25$
$C_{yxxx} = -0.4375$
$C_{yxxxx} = -0.21875$
$C_{yxxxxx} = -0.15625$
$C_{yxxxxy} = 0.453125$
$C_{yxxxy} = -0.65625$
$C_{yxxxyx} = -0.015625$
$C_{yxxxyy} = -0.59375$
$C_{yxxy} = 1.25$
$C_{yxxyx} = 0.75$
$C_{yxxyxx} = 1.375$
$C_{yxxyxy} = -1$
$C_{yxxyy} = 0.625$
$C_{yxxyyx} = -1.0625$
$C_{yxxyyy} = 1.25$
$C_{yxy} = -1.375$
$C_{yxyx} = -0.1875$
$C_{yxyxx} = -1.3125$
$C_{yxyxxx} = -0.796875$
$C_{yxyxxy} = -1$
$C_{yxyxy} = 1.65625$
$C_{yxyxyx} = 1.703125$
$C_{yyxyyx} = 1.375$
$C_{yyxyyy} = -0.90625$
$C_{yyy} = 0.875$
$C_{yyyx} = -0.4375$
$C_{yyyxx} = 0.6875$
$C_{yyyxxx} = 0.015625$
$C_{yyyxxy} = 1.25$
$C_{yyyxy} = -1.59375$
$C_{yyyxyx} = -0.796875$
$C_{yyyxyy} = -0.90625$
$C_{yyyy} = 1.0625$
$C_{yyyyx} = -0.21875$
$C_{yyyyxx} = 0.90625$
$C_{yyyyxy} = -1.609375$
$C_{yyyyy} = 1.25$
$C_{yyyyyx} = -0.28125$
$C_{yyyyyy} = 1.296875$