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Knight's Tours and Zeta Functions

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ABSTRACT

KNIGHT'S TOURS AND ZETA FUNCTIONS

by Alfred Brown

Given an $m \times n$ chessboard, we get an associated graph by letting each square represent a vertex and by joining two vertices if there is a valid move by a knight between the corresponding squares. A knight's tour is a sequence of moves in which the knight lands on every square exactly once, i.e., a Hamiltonian path on the associated graph. Knight's tours have an interesting history. One interesting mistake regarding Knight's Tours was made by the famous mathematician Euler. His mistake led to the further study of knight's tours on $3 \times n$ chessboards. We will explore and explain a method found by Donald Knuth for enumerating the number $k(n)$ of all closed knight's tours on a $3 \times (2n + 8)$ chessboard for an integer $n \geq 1$. Interestingly, there is a 21-term recurrence relation for $k(n)$ discovered independently by Knuth and Elkies. We conclude by noting that this relation comes from studying generating functions which can be interpreted in the context of the Ihara zeta function of a certain graph.

DEDICATION

To Rebecca, the love of my life, whose unfaltering support helped me get through my graduate studies with my sanity intact.

ACKNOWLEDGEMENTS

I want to thank both my mother and my father for believing in me and constantly pushing me further, for without them I would not have made it where I am today. To my brother who helped convince me to go to graduate school. A special thanks to the remainder of my family who have always been there for me. To Rebecca, her continued and undying patience, love and support was the foundation of this journey. Lastly, I would like to send a very special thanks to Jordan Schettler, for without his support none of this would have been possible.

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INTRODUCTION

A **graph** is a mathematical structure containing a vertex set, an edge set, and a relation that associates every edge to two vertices. The study of graphs began with the famous Swiss mathematician Euler, who needed an efficient method to resolve the Königsberg bridges problem. The problem involves a city in Prussia. The city sits on both sides of the Pregal River and included two small islands that were connected to each other. The problem was to find a way to walk through the city that would

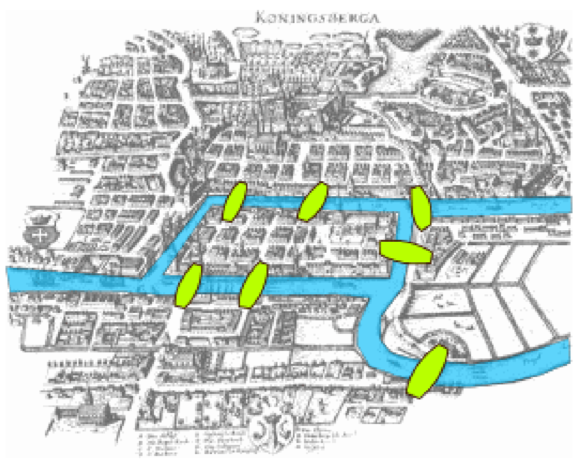

Figure 1.1: Königsberg

cross each of the bridges once and only once. Euler had proven that this problem had no solution. The issue Euler faced was that of proving this assertion mathematically, thus, the creation of graph theory. Euler regarded each one of the land masses as a vertex and the bridges between landmasses represented the edges between the associated vertices. He observed that whenever one enters a landmass by a bridge, one must leave that landmass using another previously untraveled bridge. Euler figured that if every bridge has been traveled once, it follows that for each landmass, the number of bridges connecting to that landmass must be even. In other words, each vertex/landmass must have an even degree in which the **degree** of a vertex v is the number of edges, or in this case bridges, that are incident with v. A **path** is defined as a finite sequence of edges connecting a sequence of vertices such that all edges and vertices are distinct from each other. A trail in a finite graph which visits every edge exactly once is called an **Eulerian paths**. The existence of a connected Eulerian path on a given graph is equivalent to the following condition: every vertex in a connected graph has even degree. The necessity of this condition was proven by Euler and sufficiency was established by Carl Hierholzer. One can also consider paths which visit every vertex exactly once. Such paths are known as **Hamiltonian paths**. Here we will look at Hamiltonian paths on graphs associated with chessboards.

Given any $m \times n$ chessboard, we can represent this chessboard as a graph with each square of our chessboard corresponding with a vertex in our graph. Our edges are drawn between a vertex that our knight starts at and the vertex our knight lands on after a legal knight move. The knight, more than other chess pieces, is an interesting study, not only because of the unique 'L' shaped move pattern that it must make (two squares vertically and one square horizontally, or two squares horizontally and one square vertically), but also in that it is the only piece in chess that can jump over other pieces to make its movement. This makes it essential for closed positions in which the pawns are locked on a chess board.

A **knight's tour** is a Hamiltonian path on the knight graph associated with a chessboard. The earliest known reference of the knight's tour dates back to 9th century A.D. in the form of a poem written so that it could be read using knight's moves. Another mention of the knight's tour is the use of the Turk machine. (See [Sta02]) The Turk was alleged to be an elaborate chess playing machine that turned out to be a hoax. The creators claimed that their machine would be able to perform a strong game of chess and be able to complete a knight's tour after placing a knight anywhere on the board. It was later found out that it was in fact not a machine and consisted of a man in the "machine" moving the pieces of the chessboard using magnets. He also had a diagram of a knight's tour in the compartment that he sat in, thus being able to always complete a knight's tour and impress the guests. The knight's tour was not extensively studied until Euler, and much work has gone into the generation of these knight's tours on not only an 8×8, but arbitrary $m \times n$ boards as well. Being able to count the number of cycles was also of great interest.

A **cycle** is a closed walk that does not allow for repetitions of vertices or edges, and a **closed walk** can be regarded as a sequence of adjacent vertices and corresponding edges starting and ending at the same vertex. A **closed knight's tour** is a Hamiltonian cycle on the knight graph associated with a chessboard. Many of the discoveries regarding knight's tours have been made within the last 30 years due in part to the age of computers. In fact, it was not until 1996 that the number of closed knight's tours on an 8×8 chessboard was finally calculated by Löbbing and Wegener in [LW96] who purported that this number equaled $33,439,123,484,294$. However, even this number was found shortly thereafter to be incorrect, although the methods of calculation were correct. Brendan McKay has proven in [McK97], using related methods, that the number of closed Knight's tours was actually $13,267,364,410,532$. This includes open and closed Knight's tours. If we look at $1 \times n$ and $2 \times n$ chessboards, we can see that no tours can ever exist. In $1 \times n$ chessboard, a knight cannot move from its original position due to the "L" shaped nature of the knight. With a $2 \times n$ chessboard a knight can move throughout the chessboard; however the knight can only backtrack on these chessboards. The first chessboard that allows for full movement of a knight resulting in a Knight's tour is a $3 \times n$ chessboard. This was the motivation for Euler when considering Knight's on Hamiltonian paths and cycles on $3 \times n$ chessboards. Euler found all of the Hamiltonian paths of the 3×4 chessboard. An example is shown below in Figure 1.2.

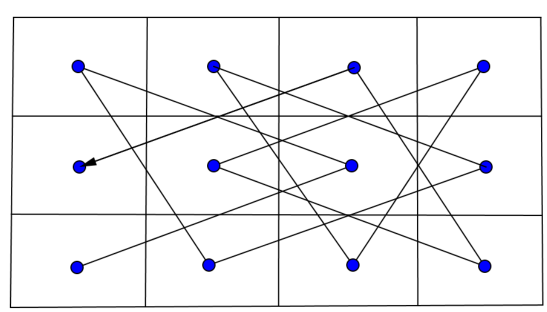

Figure 1.2: 3×4 Knight Tour

Euler also stated correctly that no such Hamiltonian paths or cycles exist on a 3×5 and a 3×6 chessboard. Euler, however, made false claims in 1759 about the non-existence of Hamiltonian cycles for arbitrary $3 \times n$ chessboards. It was not until 1917 that we realized Euler had made a mistake, when a man by the name of the Ernest Bergholt exhibited in [Ber18] that there existed Hamiltonian Knight's paths on the 3×10 chessboard. For an example of a closed Knight's tour, see Figure 1.3 below. One can compare the closed 3×10 closed Knight's tour with the open 3×10 Knight's tour in Figure 1.4

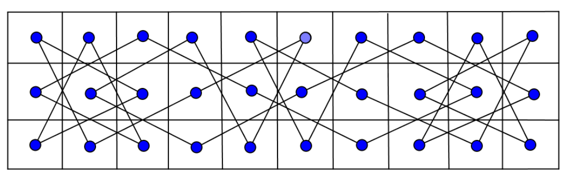

Figure 1.3: A 3×10 Closed Knight's Tour

The **order** of a graph is the number of vertices in a graph. A **bipartite graph** is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent. In this paper, we only look at $3 \times n$ chessboards for n even. The reason for this is that if n is odd, then our order would be $3n$ which would also be odd. However, every Hamiltonian cycle in our chessboard graph must be even since our chessboard graph is bipartite (since every Knight's move would alternate the color of the square the knight is in). Hence, our Hamiltonian cycle could not exist if n was odd.

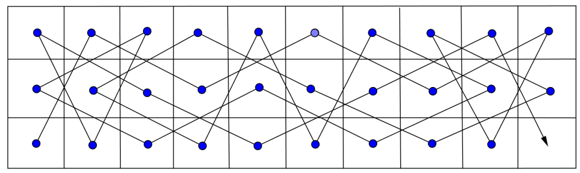

Figure 1.4: A 3×10 Knight's Tour

CHAPTER 2

KNUTH'S METHOD OF ENUMERATION

We are given a knight graph for a $3 \times n$ chessboard (in which n is even and greater than 8) and we dissect this graph into states as in [Knu11], which are basically chunks of our graph at any given time. These states can be separated into four classes, of which only one we have interest in. Using this one particular class, we can organize the associated states into a bipartite graph. This bipartite graph can then be used to create a generating function (a rational function later interpreted in the context of a zeta function) for counting the number of closed knight's tours on a $3 \times n$ chessboard.

2.1 States and Mates

In this section, the vertices of a certain directed graph K represent **states** that can arise when we take a $3 \times n$ chessboard and cut it into two separate pieces. We separate the chessboard into two pieces consisting of k columns on the left of the chessboard and $n - k$ columns on the right of the chessboard, for some k.

Let S_n be the knight graph of a $3 \times n$ chessboard and $\hat{S_n}$ be the extension obtained by adding a new vertex ∞ that is adjacent to all other vertices. Given a Hamiltonian cycle T on S_n or $\hat{S_n}$, let T_k be the edges of T that lie entirely in the first k columns of the chessboard possibly extended by ∞. So in the case of $\hat{S_n}$, this T_k would include edges connecting vertices in the first k columns with ∞. In general, every vertex of $\hat{S_k}$ will either be **covered** by exactly two edges in T_k or will be **uncovered**. If the vertex is uncovered, there are two possibilities we wish to distinguish. Namely, an uncovered vertex could be absent from all of T_k's edges (we call such a vertex **untouched**) or it could attached to some other vertex by a single edge of T_k. In the latter case, the vertex is said to be **mated** to a similarly defined attached vertex by a maximum subpath of T_k whose endpoints are the two mates mentioned. Note that a vertex that is uncovered must appear in the two rightmost columns or be ∞ itself.

<table>
<tr><td>4</td><td>1</td></tr>
<tr><td>5</td><td>2</td></tr>
<tr><td>6</td><td>3</td></tr>
<tr><td colspan="2">∞</td></tr>
</table>

Figure 2.1: Two Rightmost Columns with Numbered Vertices

Each of these vertices with number x is assigned a **mate number** $mate[x]$ by the following rules:

$$mate[x] = \begin{cases} 0 & \text{if the vertex is covered by } T_k \\ x & \text{if the vertex is untouched by } T_k \\ y & \text{if the vertex is uncovered and mated} \\ & \text{to a vertex with number } y \text{ by } T_k \end{cases} \tag{2.1}$$

The **state** corresponding with T_k will be denoted by a seven digit code

$$mate[1]mate[2]mate[3]mate[4]mate[5]mate[6]mate[\infty]$$

For example, if we take the graph in Figure 2.2, we can see it starts with the state code "6243∞15".

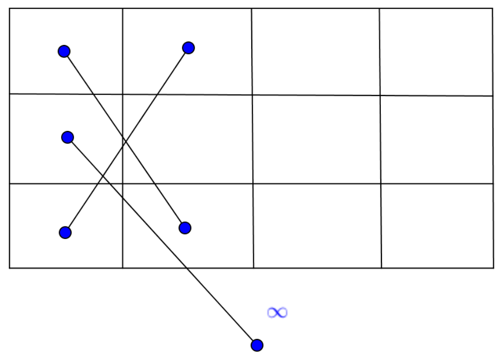

Figure 2.2: Two Column Graph

Now, to go from T_k to T_{k+1} we begin by increasing all of the vertex numbers by 3. Next, we append new untouched vertices with numbers 1,2,3, obtaining up to ten potentially relevant vertices as shown below in Figure 2.3.

7	4	1
8	5	2
9	6	3
∞		

Figure 2.3: Three Rightmost Columns with Numbered Vertices

Notice that we will only ever look at three columns at a time, because since we are dealing with Knight's, the knight can only move vertically or horizontally two spaces and thus needs enough vertices to make a valid knight move. Now, when we incorporate the next column, we connect new edges to the vertices that were just introduced in the third column by using valid knight moves as show below in Figure 2.4. As a result, the state code changes. We can use the mate rules to change the state code. If we look at vertex one in Figure 2.4, we can see it is uncovered since it does not contain two edges of T_k (thus not covered), and it is not untouched since it does contain at least one edge of T_k (not untouched). Thus, vertex one gets assigned ∞ since vertex one is attached to ∞. Just like the first vertex, we will go through every vertex in Figure 2.4 and go through the same process. We can see that vertex two has exactly two edges attached with it and thus vertex two is covered and assigned 0. Notice that vertex three is uncovered and attached to vertex six and thus assigned 6. Vertex four is covered and assigned a 0, while vertex five is untouched (has no edges attached to it) and assigned a 5. The last two vertices 6 and ∞ are both uncovered and attached to vertices 3 and 1, respectively, and thus assigned 3 and 1. One can now see that by adding a third column the state code changes from "6243∞15" to "∞060531."

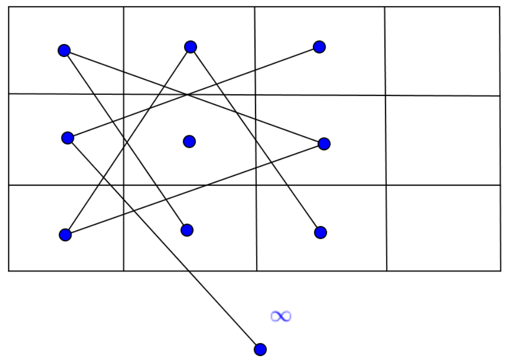

Figure 2.4: Graph with Third Column implemented

The next thing to do would be to take into account the edges in T that belong to T_{k+1} but not to T_k. We can take all the edges with the endpoints in column $k+1$ and another endpoint in the column k or $k-1$ or ∞. This new set of edges is encoded with three italic digits. The list is as follows:

100 for edge 8 to 1	*010* for edge 7 to 2	*001* for edge 4 to 3
200 for edge 6 to 1	*020* for edge 9 to 2	*002* for edge 8 to 3
400 for edge ∞ to 1	*040* for edge ∞ to 2	*004* for edge ∞ to 3

The sum of the italicized digits above represents the edges that were added through each transition. For example, the state transformation "6243∞15" $\to$ "∞060531" is achieved by the transition code *131*. This is the sum of the transitions (edges added) of $\mathit{100} + \mathit{010} + \mathit{020} + \mathit{001} = \mathit{131}$. We only need to keep track of the sum since the sum uniquely decomposes. This is exactly why Knuth chooses 1,2, and 4.

Instead of computing the mate values of each transition by hand, there is an easier way to compute them. As a new edge is added from x to y, we just need to do the following steps in order.

$$\text{Let } x' = mate[x] \text{ and } y' = mate[y]$$

$$\text{Now we let } mate[x] = 0 \text{ and } mate[y] = 0$$

$$\text{Then we let } mate[x'] = y' \text{ and } mate[y'] = x'$$

We will use a directed graph K to help us encapsulate all the information about a $3 \times n$ Knight's tour for an arbitrary even $n \geq 4$. This graph K is defined by letting the set of states be the vertices and joining two states s_1 and s_2 by a directed edge from s_1 to s_2 if we can obtain s_2 from s_1 by a transition from some T_k to T_{k+1}.

We call the state "000000∞" the **open source vertex** of K, denoted by σ_o. This is always the state from which starts a Hamiltonian cycle T or $\hat{T}$. We call the state "0000000" the **sink vertex**, and we denote that as τ. This is the state that represents a complete graph, that is a Hamiltonian cycle T or $\hat{T}$ (thanks to the vertex ∞ which, by definition, forces there to be a Hamiltonian cycle from a Hamiltonian path). We call the state "1230000" the **closed source vertex**, denoted by σ_c. This state is adjacent to the sink vertex, but one step beyond a Hamiltonian cycle, as the new column can't connect with any of the vertices before it. We will show examples later on in this paper that better explain these states.

We need to be very careful when counting paths. In fact, Euler's $3 \times n$ chessboard has eight different shortest paths from σ_o to τ, not just three as exhibited by Euler. Euler was counting all of the geometrically distinct tours. If we take the three that were proposed by Euler, we can reflect the graphs both horizontally, vertically, or rotate the graph. It is in this way that we get 8 distinct paths instead of 3.

In general, every state α corresponds to a **flipped state** α' corresponding with a reflection across the horizontal midline, creating horizontal symmetry if dealing with a closed Knight's tour. Likewise, every transition state t formed by transitional triplets (the italicized triplets that add new edges) corresponds to a **flipped transition** t'. For example if α = abcdefg and t = ABC, then α' = a′b′c′d′e′f′g′ = cbafedg and t' = A′B′C′ = CBA . The vertices

$$0' = 0, 1' = 3, 2' = 2, 3' = 1, 4' = 6, 5' = 5, 6' = 4, \infty' = \infty$$

and the edge transition digits become,

$$\mathit{0}' = \mathit{0}, \mathit{1}' = \mathit{2}, \mathit{2}' = \mathit{1}, \mathit{3}' = \mathit{3}, \mathit{4}' = \mathit{4}, \mathit{5}' = \mathit{6}, \mathit{6}' = \mathit{5}$$

We will now show an example on a 3×4 chessboard. We start with the open source which is "000000∞". We then look at the first column T_1.

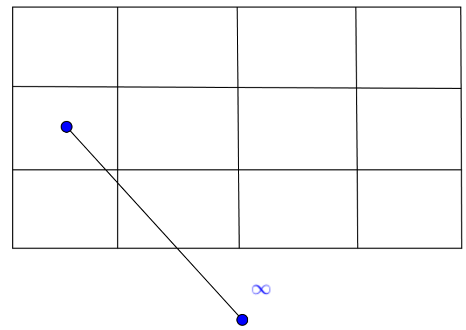

Figure 2.5: T_1

As we can see, there is one edge from the vertex $(1, 1)$ (first row, first column) to ∞. The state code has now changed to "1∞30002" using the mate definitions in Equation 2.1. We can now add the next column T_2.

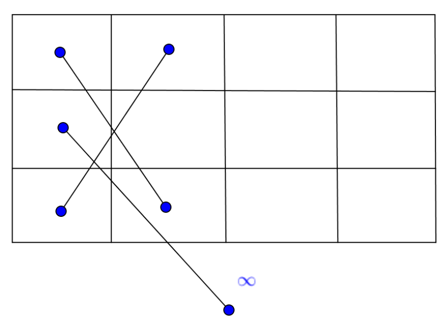

Figure 2.6: T_2

Now with the second column added we add all the edges that connect from the second column and up to two columns before it. Thus we only have to worry about the first column. In this case $(1, 2)$ and $(3, 2)$ have edges from their squares to the $(3, 1)$ and $(1, 1)$ squares respectively as shown in Figure 2.6. These edges are added using the code *201*, effectively changing the state code to "6243∞15". We now add the third column T_3.

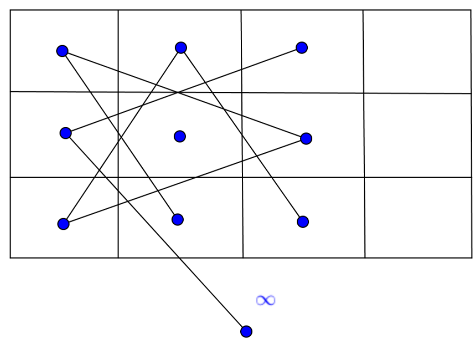

Figure 2.7: T_3

Each square in the third column has edges from it to the two columns before it. Using the edge code *131* we achieve the graph as in Figure 2.7 above. The state code is now changed to "∞060531". Finally we add the final column T_4, look at the two columns that lie before T_4 and use the edge generating code *363* to add edges to each vertex in the column T_4. The end result is that we get the complete Knight's graph of the 3×4 as seen Figure 2.8 below. The state code is thus changed to the final complete state $\tau =$ "0000000".

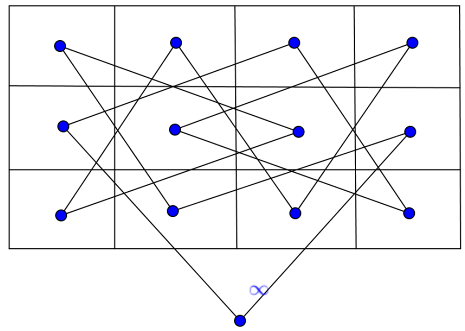

Figure 2.8: T_4

2.2 Classification of States

An **involution** is a function or transformation that is equal to its inverse, that is a function or transformation that produces the identity when applied to itself. Every state (vertex in K) corresponds to an involution in the symmetric group S_k with $k \leq 7$, in which two mated vertices correspond to a transposition and covered vertices correspond to fixed points of the permutation. Thus the entirety of the state codes of K are all obtainable by:

(a) starting with a sequence of seven digits 123456∞

(b) setting some subset of them to 0

(c) applying an involution permutation to the remaining non-zeros

We can use a generating function to count the number t_k of involution permutations in S_k. We have a recurrence relation $t_k = t_{k-1} + (k-1)t_{k-2}$ with initial conditions $t_0 = t_1 = 1$ which means the generating function

$$F(x) = \sum_{k\geq 0} t_k \frac{x^k}{k!}$$

satisfies the differential equation $F'(x) = (x+1)F(x)$ with initial condition $F(0) = 1$. The solution is $F(x) = e^{x+x^2/2}$. These involution numbers t_k also have another interesting connection to chess. In particular, t_n counts the numbers of ways to place n rooks on an $n \times n$ chessboard in such a way that no two rooks attack each other (the so-called 'n rooks puzzle') and such that the configuration of the rooks is symmetric under a diagonal reflection of the board. This follows since the non-attacking condition will force exactly one rook in each column, so each such arrangement will correspond with one of $n!$ permutation matrices and the diagonally symmetric ones will correspond with involutions. In [Hol74], D. F. Holt counted the number $A(n)$ of arrangements of mutually non-attacking rooks up to any symmetry of the chessboard (i.e., up to the action of the dihedral group) with the formula

$$A(n) = \frac{n! + f(n) + 2g(n) + 2t_n}{8}$$

where

$$f(n) = \begin{cases} (n/2)!2^{n/2} & \text{if } n \text{ is even} \\ ((n-1)/2)!2^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases} \tag{2.2}$$

and

$$g(n) = \begin{cases} (n-2)(n-6)\cdots 6\cdot 2 & \text{if 4 divides } n \\ (n-3)(n-7)\cdots 6\cdot 2 & \text{if 4 divides } n-1 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

This formula was proved rigorously by using a technique commonly used in group theory called the Pólya enumeration theorem. On a normal chessboard we have $n = 8$, $f(8) = 384$, $g(8) = 12$, and $t_8 = 764$. Thus the number of arrangements is equal to $A(8) = 5282$. In Figure 2.9, we give an example of a diagonally symmetric 5×5 illustration of one non-attacking arrangement.

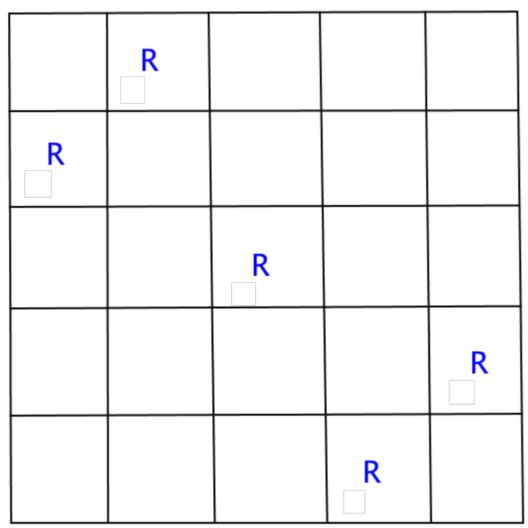

Figure 2.9: Rook Example Corresponding to the Involution $(1, 2)(4, 5)$

Using the generating function for the involution numbers allows us to calculate that the total number of potential states is

$$\sum_{k=0}^{7} \binom{7}{k} t_k = 1850$$

in which $\binom{7}{k}$ counts the number of subsets of size k (to be either covered or mated vertices) from the set of all vertices of size 7. However only 1406 of the states are actually reachable from the source states of σ_o and σ_c by legal transitions of Knight's moves. For example, using the legal edge transitions, that is the three digit number we used to add edges to our new vertices, we can never reach the chessboard state "1000000". This is not the only state which does not lead to a potential state. For example, the forms "***4***", "*****6*", and "***6*4*" have no successive moves (where '*' is considered any digit) as pointed out in [Knu11]. These states do not belong in K because they do not lead to τ. We can also knock out states of the form "1*3*5**" because in this state, any move made to a new state from "1*3*5**" will not have a move that leads to τ. In other words, if we are looking at the 9 vertices of any given state, the 1st, 3rd, and 5th vertices all have no edges attached. No matter what vertex you come from you will never be able to attach to those vertices to get to τ and thus they will remain vertices with no edges. Similar logic and reasoning can be applied to the other "unusable states." Finally, there are states which are reachable like "1035400" which fail to satisfy a certain condition as outlined in [Knu11]. All in all, this means that there are only 712 vertices in K. A computer program that Knuth created tells us that all of these vertices (except σ_c) occur on some path of length 16 or less from σ_o to τ. In fact, only two states have paths length of more than of 14.

We call vertex (i, j) of the chessboard *even* when $i + j$ is even, and *odd* otherwise. Note that if a knight is on an odd vertex, it can only move to an even vertex and vice versa. A $3 \times n$ chessboard has the ceiling of $3n/2$ even vertices and the floor of $3n/2$ odd vertices. Thus when n is odd, the number of even vertices is one more than the number of odd vertices, so the endpoints of an open tour must be even. Likewise, when n is even, the number of even vertices equals the number of odd vertices, so the endpoints of an open tour must have opposite parity. Thus there are four different possible open tours:

(1) Both endpoints appear in the same column; one of the endpoints being odd and the other endpoint is even. Also, n is even.

(2) Both endpoints appear in the same column; both of them are even and n is odd.

(3) The endpoints appear in different columns and the left endpoint is even. The right endpoint is even if n is odd and odd if n is even.

(4) The endpoints appear in different columns, the left endpoint is odd, the right endpoint is even and so is n.

We can divide the vertices of K into four classes: O for open, F for free, B for bound, C for closed. Class O consists of all states whose code names end in symbol ∞; these states occurred before any endpoints of the tour have appeared. Class C consists of all the states that end in 0. The states in class C are those of closed tours (they also occur after both endpoints of an open tour have been seen). Between the classes O and C we have two classes of states in which one endpoint has appeared but not the other. Class F corresponds to type (3) above following an even left endpoint, and class B corresponds to open tours of type (4) above following an odd left endpoint.

The graph states of the classes O, B, and C are bipartite, so we can subdivide them into partite classes: O_0, O_1, B_0, B_1, C_0 and C_1. An O_0 or B_0 always occur an even number of moves or steps after σ_0. In addition, a B_0 or a C_0 state always occurs an even number of moves or steps before τ. The classes of O_1, B_1, and C_1 are all similar but with an odd number of steps. The class F has no parity restrictions, and thus we do not need to include them with the partite classes. Below is a list of all the allowable transitions.

$$O_0 \to O_1 \ , O_0 \to B_1 \ , O_0 \to F, O_0 \to C_0, O_0 \to C_1;$$

$$O_1 \to O_0, O_1 \to B_0, O_1 \to F, O_1 \to C_0;$$

$$F \to F, F \to C_0, F \to C_1;$$

$$B_0 \to B_1, B_0 \to C_1;$$

$$B_1 \to B_0, B_1 \to C_0;$$

$$C_0 \to C_1, C_1 \to C_0$$

It is worthy to note however that the transition $O_1 \to C_1$ is illegal due to the fact that O_1 occurs an odd number of steps after τ. Therefore if O_1 transitions to C_1 it would occur an even number of steps after τ. By definition however, C_1 must occur an odd number of steps after τ. It turns out, there is a total of $(84, 75, 204, 110, 72, 91, 76)$ vertices belong with the respective classes $(O_0, O_1, F, B_0, B_1, C_0, C_1)$ as noted in [Knu11].

2.3 Computing the Parity

There is a way to determine the class of any vertex directly from its state code. First we will go over how we enumerate this, followed by how the values change with knight moves. Given the values of $mate[x]$ for $1 \leq x \leq 7$ in which '7' in this specific case refers to ∞, we define the following quantities:

$$z_0 = \text{the number of non-zero } x \text{ with } mate[x] = 0 \text{ and } x \text{ is even;}$$

$$z_1 = \text{the number of non-zero } x \text{ with } mate[x] = 0 \text{ and } x \text{ is odd;}$$

$$f_0 = \text{the number of non-zero } x \text{ with } mate[x] = x \text{ and } x \text{ is even;}$$

$$f_1 = \text{the number of non-zero } x \text{ with } mate[x] = x \text{ and } x \text{ is odd;}$$

$$p_{00} = \text{the number of } x < y \text{ with } mate[x] = y,\ x \text{ and } y \text{ are both even;}$$

$$p_{01} = \text{the number of } x < y \text{ with } mate[x] = y,\ x + y \text{ odd;}$$

$$p_{11} = \text{the number of } x < y \text{ with } mate[x] = y,\ x \text{ and } y \text{ both odd;}$$

This information is then put into a table. These quantities can then be changed based on a move made by a knight. We will organize these table changes based on whether or not x and y are "touched" by the edges in T_k.

If x and y are previously untouched, then

$$f_0 \leftarrow f_0 - 1, f_1 \leftarrow f_1 - 1, p_{01} \leftarrow p_{01} - 1$$

If only one was untouched, one of four possible changes will occur

$$f_0 \leftarrow f_0 - 1, z_1 \leftarrow z_1 + 1, p_{00} \leftarrow p_{00} + 1, p_{01} \leftarrow p_{01} - 1$$

$$f_0 \leftarrow f_0 - 1, z_1 \leftarrow z_1 + 1, p_{01} \leftarrow p_{01} + 1, p_{11} \leftarrow p_{11} - 1$$

$$f_1 \leftarrow f_1 - 1, z_0 \leftarrow z_0 + 1, p_{00} \leftarrow p_{00} - 1, p_{01} \leftarrow p_{01} + 1$$

$$f_1 \leftarrow f_1 - 1, z_0 \leftarrow z_0 + 1, p_{00} \leftarrow p_{01} - 1, p_{11} \leftarrow p_{11} + 1$$

If both were previously mated with other elements, we have either

$$z_0 \leftarrow z_0 + 1, z_1 \leftarrow z_1 + 1, p_{01} \leftarrow p_{01} - 1;$$

or

$$z_0 \leftarrow z_0 + 1, z_1 \leftarrow z_1 + 1, p_{00} \leftarrow p_{00} - 1, p_{01} \leftarrow p_{01} + 1, p_{11} \leftarrow p_{11} - 1.$$

In each case, the state code iterations leave the quantity

$$I = z_0 - z_1 + p_{00} - p_{11}$$

invariant, which is very interesting. In these cases I represents the difference between the number of odd and even vertices covered in the first $k-2$ columns. When k increases by 1, we promote vertices $\{1, \ldots, 7\}$ to $\{4, \ldots 10\}$; this "promotion" causes the even and odd vertices to switch roles.

$$z_0 \leftrightarrow z_1 \qquad\qquad p_{00} \leftrightarrow p_{11} \qquad\qquad I \leftarrow -I$$

We then attach $\{1, 2, 3\}$. This causes $f_0 \leftarrow f_0 + 1$, and $f_1 \leftarrow f_1 + 2$. As noted above, I is invariant while making knight moves, and this holds true until we shorten the table by removing the 0 mates of $\{7, 8, 9\}$. If the mate of ∞ is not 0, this shortening causes $z_0 \leftarrow z_0 - 1$ and $z_1 \leftarrow z_1 - 2$, thus making $I \leftarrow I + 1$.

If ∞ was mated with ∞ (as in the O states) before and after this state transition then $I \leftarrow 1 - I$, thus we have O_0, which since this class occurs an even number of steps after σ_0 has $I = 1$. Then for O_0, we have $I = 0$. Similar logic can be used for the class O_1 and we find that $I = 1$.

If ∞ was mated with 0 before and after the state transition (as it is in the C states), the shortening operation causes $z_0 \leftarrow z_0 - 2$ and $z_1 \leftarrow z_1 - 1$ thus making $I \leftarrow -1 - I$. Using similar logic as above, we find that the subclass C_0 corresponds with $I = -1$ and the subclass C_1 corresponds with $I = 0$.

Finally if ∞ was mated with any of the other vertices in $\{1, \ldots, 6\}$ before and after the state transition (as it is in the F and B states), the process of demoting the 10 position to the 7 position has the overall effect of decreasing I by 1.

2.4 Enumeration

So far we have demonstrated a way to take the movement of knights on a chessboard and we associate them with a state code. Using a series of transitional triplets, we have a way of changing the state code for each new edge added to our graph. In doing so, we have also eliminated all possible states that can't be a part of a knight's tour. We then classified our states into 4 separate classes, most of which were then themselves separated into 2 sets of partite classes. Finally, we have figured out how to use the parity of our states to directly determine the class of our state codes. Now we can start the good stuff.

We let $N(\alpha, \beta, n)$ be the number of paths of length n in our directed graph K from state α to state β. Thus the number of knight's tours on a $3 \times n$ chessboard is then $N(\sigma_0, \tau, n)$ for open tours and $N(\sigma_c, \tau, n-1)$ for closed tours. The calculation of $N(\alpha, \beta, n)$ is not difficult to compute when n is not too large. First, we set up the adjacency matrix A of K, in which the matrix has one column and one row for each state of K.

The entry $A_{\alpha\beta}$ in row α and column β is 1 if $\alpha \to \beta$ (i.e., there is a directed edge from α to β) and 0 otherwise. Symbolically we write $A_{\alpha\beta} = [\alpha \to \beta]$. Then

$$N(\alpha, \beta, n) = (A^n)_{\alpha\beta}$$

is the entry in row α and column β of the nth power of the matrix A. We can get much more information by determining the generating function for $N(\alpha, \beta, n)$ instead of evaluating it for a particular n:

$$\begin{aligned} G(\alpha, \beta) &= N(\alpha, \beta, 0) + N(\alpha, \beta, 1)z + N(\alpha, \beta, 2)z^2 + \dots \\ &= \sum_{n=0}^{\infty} N(\alpha, \beta, n) z^n \end{aligned}$$

where $N(\alpha, \beta, 0) = 1$ by convention. This generating function $G(\alpha, \beta)$ represents all of $N(\alpha, \beta, n)$ at the same time. In addition, the formula $N(\alpha, \beta, n) = (A^n)_{\alpha\beta}$ tells us that the matrix of all these generating functions is

$$I + Az + A^2z^2 + \dots = \sum_{n=0}^{\infty} A^n z^n = (I - Az)^{-1},$$

where $I = A^0$ is the identity matrix of the same dimensions as A. An important consequence of this is that everything that we want to know about enumerating these paths comes from the inverse of the matrix $I - Az$. From the well-known formula for the inverse of a matrix, the individual entries of this inverse are

$$\pm g(\alpha, \beta, z) / \det(I - Az)$$

in which $g(\alpha, \beta, z)$ is the determinant of the sub-matrix obtained by deleting column α and row β. Since $\det(I - Az)$ and $g(\alpha, \beta, z)$ are polynomials in z, the generating functions $G(\alpha, \beta)$ are rational, and the numbers $N(\alpha, \beta, n)$ obey a linear recurrence relation with constant coefficients as explained in the next section. Since our matrix is so big, that is a 712×712 matrix with 5506 nonzero entries, it is rather comforting to know that we are really only interested in two of these generating functions, namely, $G(\sigma_0, \tau)$ and $G(\sigma_c, \tau)$.

We will create a toy example that will help illustrate the generating functions that we are talking about above. Let us assume for the sake of simplicity that you have a $3 \times n$ chessboard, and states that are defined by legal chess moves. We can then organize the states based on classes that the states live in. We are only interested in the C_0 and the C_1 states since those are the classes in which σ_o, σ_c, and τ, live. We now identify states up to dual, that is the states that are equivalent when reflected about the horizontal midline. This takes the 712 states to 376 states up to dual. If two states have the same set of successors, their generating functions are also identical. The successive grouping of states up to dual with the same successors is what we call **superstates**. These two reductions (duality and successors) give a reduced graph $\hat{K}$ with 220 inequivalent superstate vertices. It turns out (see [Knu11]) that there are 25 C_0 superstates and 27 C_1 superstates. Something worthy of note is that all C class superstates only have C class successors. This is a direct result from the allowable transitions between classes, namely that C_0 goes to C_1 and vice versa, which gives us a natural bipartite graph. We are interested in the generating functions

$$G(\alpha) = G(\alpha, \tau) = z \sum_{\alpha \to \beta} G(\beta)$$

in which the sum extends over all successors β of α.

We model this with an explicit small example. We can then organize them in a bipartite graph, labeling the states in C_0 as α_n and states in C_1 as β_n. To make things more simple, we created a mock example with only 6 α vertices (C_0 superstates) and 5 β vertices (C_1 superstates). The directed bipartite graph for this example is shown in Figure 2.10.

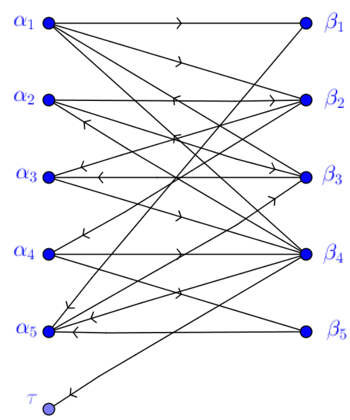

Figure 2.10: Bipartite Example

We can define $G(\alpha)$ formerly as

$$G(\alpha) = [\alpha = \tau] + z \sum_{\alpha \to \beta} G(\beta)$$

We can then roughly create the generating polynomials of the vertices based on the bipartite graph by expressing the generating function in terms of the generating functions of other vertices. You start with an α in this case, say α_2. The successors of α_2 are β_2 and β_3. We then take the successors of β_2 and β_3 which are α_3 , α_4 and α_1 , α_3 respectively.

$$G(\alpha_1) = z^2(G(\alpha_5) + G(\alpha_3) + G(\alpha_4))$$

$$G(\alpha_2) = z^2(2 \cdot G(\alpha_3) + G(\alpha_4) + G(\alpha_1))$$

$$G(\alpha_3) = z^2(G(\alpha_1) + G(\alpha_5) + G(\alpha_2) + G(\tau))$$

$$G(\alpha_4) = z^2(2 \cdot G(\alpha_5) + G(\alpha_1) + G(\alpha_2) + G(\tau))$$

$$G(\alpha_5) = z^2(G(\alpha_3) + G(\alpha_1))$$

$$G(\tau) = 1$$

We can calculate the generating polynomial precisely through some linear algebra. First we will need to calculate the generalized adjacency matrix of our directed α-graph obtained from our bipartite graph by joining α_i to α_j if α_j is a successor of a successor β_k of α_i.

Definition 2.4.1. Given a directed graph (possibly not simple), we define the **generalized adjacency matrix** as follows: if there are precisely n directed edges from a vertex i to vertex j then we put n in row i, column j.

Upon calculation using the above equations for $G(\alpha)$ our generalized adjacency matrix A appears below with columns and rows labeled from α_1 to α_5 with the exception that τ is in the last row and column of the matrix:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let

$$\vec{G} = \begin{bmatrix} G(\alpha_1) \\ G(\alpha_2) \\ G(\alpha_3) \\ G(\alpha_4) \\ G(\alpha_5) \\ G(\tau) \end{bmatrix} \quad \text{and} \quad \vec{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\vec{G} = z^2 A \vec{G} + \vec{C},$$

so

$$\vec{G} = (I - z^2 A)^{-1} \vec{C}$$

Thus we know that the generating functions can be found by

$$G(\alpha_i) = \text{ the } (i, \tau) \text{ entry of } (I - z^2 A)^{-1}$$

$$= \pm g(\alpha_i)/P(z^2)$$

in which $g(\alpha_i)$ is the determinant of the sub-matrix with the α_i row and the τ column removed and $P(z^2)$ is the determinant of $(I - z^2A)$. In this case I is just the identity matrix. By using Sage and the code created below we can compute $\pm g(\alpha_i)$ and $P(z^2)$:

```
sage: A = matrix([
[0,0,1,1,1,0],[1,0,2,1,0,0],
[1,1,0,0,1,1],[1,1,0,0,2,1]
,[1,0,1,0,0,0],[0,0,0,0,0,0]
])
sage: C = matrix([[0,0,0,0,0,1]]).transpose()
sage: z = var('z')
sage: z^2*A
[    0     0   z^2   z^2   z^2     0]
[  z^2     0 2*z^2   z^2     0     0]
[  z^2   z^2     0     0   z^2   z^2]
[  z^2   z^2     0     0 2*z^2   z^2]
[  z^2     0   z^2     0     0     0]
[    0     0     0     0     0     0]
sage: I=matrix.identity(6)
sage: B=I-z^2*A
```

```
sage: B

[     1      0   -z^2   -z^2   -z^2      0]

[  -z^2      1 -2*z^2   -z^2      0      0]

[  -z^2   -z^2      1      0   -z^2   -z^2]

[  -z^2   -z^2      0      1 -2*z^2   -z^2]

[  -z^2      0   -z^2      0      1      0]

[     0      0      0      0      0      1]

sage: B = B.transpose()

sage: g_1=B.matrix_from_rows_and_columns([1,2,3,4,5], [0,1,2,3,4])

sage: g_1

[     0      1   -z^2   -z^2      0]

[  -z^2 -2*z^2      1      0   -z^2]

[  -z^2   -z^2      0      1      0]

[  -z^2      0   -z^2 -2*z^2      1]

[     0      0   -z^2   -z^2      0]

sage: det(g_1)

-z^8 - z^6 - 2*z^4

sage: det(B)

-6*z^6 - 7*z^4 + 1

sage: f(z) = det(g_1)/det(B)

(z^8 + z^6 + 2*z^4)/(6*z^6 + 7*z^4 - 1)
```

```
sage: f.taylor(z,0,25)

z |--> -66387*z^24 - 20935*z^22 - 7575*z^20 - 2227*z^18 - 891*z^16 -

223*z^14 - 111*z^12 - 19*z^10 - 15*z^8 - z^6 - 2*z^4
```

Thus we can calculate that

$$\pm g(\alpha_1) = z^8 + z^6 + 2z^4$$

and,

$$P(z^2) = 6z^6 + 7z^4 - 1$$

So the generating function for α_1 is

$$\pm\frac{z^8 + z^6 + 2z^4}{6z^6 + 7z^4 - 1} = 2z^4 + z^6 + 15z^8 + 19z^{10} + 111z^{12} + 233z^{14} + 891z^{16} + \cdots$$

Now for any other generating function computation, we change the values for our g_1 matrix. For example, if we compute for g_2 we take the submatrix

```
g_2 = B.matrix\_from\_rows\_and\_columns([0,2,3,4,5],[0,1,2,3,4])
```

and complete the rest of the computations normally.

In the case of the actual graph $\hat{K}$, Knuth [Knu11] finds that $P(z) =$

$$1 - 6z - 64z^2 + 200z^3 + 1000z^4 - 3016z^5 - 3488z^6 + 24256z^7 - 23776z^8 - 104168z^9$$

$$+203408z^{10} + 184704z^{11} - 443392z^{12} - 14336z^{13} + 151296z^{14} - 145920z^{15}$$

$$+263424z^{16} - 317440z^{17} - 36864z^{18} + 966656z^{19} - 573440z^{20} - 131072z^{21}$$

is a polynomial of degree 21. As we will see in the following section, this implies that the number $k(n)$ of knight's tours on a $3 \times n$ chessboard satisfies a 21-term recurrence relation. To complete the enumeration of closed tours on $3\times n$ chessboards, Knuth computes the numerator $g(\alpha)$ corresponding with the generating function of $\alpha =$ "6243510″, the successor of the initial state σ_c. Knuth finds that $G(\alpha) =$

$$\frac{16z^{10} + 80z^{12} - 544z^{14} + 8080z^{16} - \cdots}{1 - 6z^2 - 64z^4 + 200z^6 + 1000z^8 - \cdots} = 16z^{10} + 176z^{12} + 1536z^{14} + 25360z^{16} \cdots$$

This implies that there are no closed knight's tours on a $3\times n$ chessboard with $n < 10$ since all the coefficients of z^n are zero for $n < 10$. Furthermore, we get 16 closed tours up to symmetry on a 3×10 chessboard, 176 closed tours up to symmetry on a 3×12 chessboard, and so on. Knuth uses the same methods to enumerate open tours as well.

CHAPTER 3

RECURRENCE RELATIONS AND ZETA FUNCTIONS

In this chapter, we will explore how the rational generating functions $G(\alpha)$ of the previous chapter give rise to recurrence relations and have a natural interpretation in the context of the 'zeta function' of a certain directed graph.

3.1 Maclaurin Series of Rational Functions

A **recurrence relation** is an equation $r_n = f(r_{n-1}, r_{n-2}, \ldots, r_{n-k})$ that recursively defines a sequence r_n as a function $f(\bullet)$ of the previous k terms with respect to some **initial conditions** $r_0 = c_0$, $r_1 = c_1$, ..., $r_{k-1} = c_{k-1}$. We can can consider the Fibonacci Sequence F_n as our primary example. The initial conditions are

$$F_0 = 0, F_1 = 1$$

and the recurrence relation is

$$F_n = F_{n-1} + F_{n-2}.$$

So $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$ and so on. If we take the Taylor series of the rational function

$$\frac{-x}{x^2 + x - 1}$$

and expand it, we get

$$0 + 1 \cdot x + 1 \cdot x^2 + 2 \cdot x^3 + 3 \cdot x^4 + 5 \cdot x^5 + 8 \cdot x^6 + 13 \cdot x^7 + \cdots$$

As you can see, the coefficients are the first 8 numbers of the Fibonacci sequence. Of course, if one continues to expand the sequence further than 8 terms, the coefficients will still follow the form of the Fibonacci numbers due to the recursive nature of the rational function above. We can now write the function in more abstract terms

$$\frac{-x}{x^2+x-1} = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \cdots .$$

We are trying to show that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ and that $a_0 = 0$ and $a_1 = 1$. We have

$$\begin{aligned} -x &= (x^2+x-1)(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) \\ &= -a_0 + (-a_1 + a_0)x + (-a_2 + a_1 + a_0)x^2 + \ldots + (-a_n + a_{n-1} + a_{n-2})x^n. \end{aligned}$$

By comparing coefficients, we can see that $0 = -a_0$ and that $-1 = -a_1 + a_0 = -a_1$, so our initial conditions $a_0 = 0$ and $a_1 = 1$ are satisfied. If we go even further we see that $0 = -a_n + a_{n-1} + a_{n-2}$ for $n \geq 2$, thus following the desired pattern $a_n = a_{n-1} + a_{n-2}$.

Interestingly enough, recurrence relations appear more than once in the study of chess. We have talked about the recurrence relation found when studying Knight's tours. The other is found in the placement of rooks, which is the number of arrangements of n non-attacking rooks symmetric to the diagonal which creates the recurrence relation $Q_n = Q_{n-1} + (n-1)Q_{n-2}$. If we place a rook on the board anywhere, we are left with $n-1$ rooks to be placed on an $(n-1) \times (n-1)$ board. We will call this quantity Q_{n-1}. For the second case, there is another rook that is symmetric to the first rook along the same chosen diagonal. Removing the row and column of this rook leads to a symmetric arrangement of $n-2$ rooks on a $(n-2)\times(n-2)$ board. The number of these arrangements is called Q_{n-2}, which can be placed on the $n-1$ square of the first column. We can immediately see the recurrence relation since there are $(n-1)Q_{n-2}$ terms.

3.2 The Ihara Zeta Function

We will now define and explain the Ihara zeta function of a graph, a meromorphic function which encodes information about certain so-called prime paths on that graph. The Ihara zeta function is an analog for graphs of the famous Riemann zeta function $\zeta(s)$, which for a complex variable s with real part greater than 1 is given by the product formula

$$\zeta(s) = \prod_{p \text{ prime}} (1-p^{-s})^{-1}$$

in which the product ranges over all prime's numbers p. Riemann showed that $\zeta(s)$ can be extended as a meromorphic function to the whole complex plane with only one pole at $s=1$. He used methods of complex analysis like Cauchy's residue theorem to study the prime's p. This section is largely motivated by Audrey Terras' book *Zeta Functions of Graphs: A Stroll through the Garden* [Ter10] and her work on the Ihara zeta function. The Ihara zeta function, put simply, also counts primes (primes of a graph).

To define the zeta function analog for a graph, we need some additional definition and theorems from graph theory. We will assume that the graphs that we will deal with are undirected and will not contain any vertices of degree zero. Additionally, we will allow our graphs to have loops or multiple edges between pairs of vertices and we call graphs that do not have loops or multiple edges **simple graphs**. We will call any graph that does not meet these conditions a "bad" graph. All the above are necessary conditions for many of the main theorems regarding the Ihara zeta function. We call a graph a **regular graph** when its vertices each have the same degree. A k**-regular graph** is a graph in which each vertex has degree k.

Definition 3.2.1. Let V denote the vertex set of a graph G with $n = |V|$. The **adjacency matrix** A of G is an $n \times n$ matrix with $(i, j)th$ entry,

$$a_{ij} = \begin{cases} \text{number of undirected edges connecting vertex } i \text{ to vertex } j, \text{ if } i \neq j \\ \\ 2 \times \text{number of loops at vertex } i, \text{ if } i = j \end{cases}$$

In order to formally define the Ihara zeta function, we need to define a prime in a graph G with an edge set of $m = |E|$ elements. We will orient the edges of our graph (either arbitrarily or use some natural orientation) and label the edges as follows:

$$e_1, \ldots, e_m, e_{m+1} = e_1^{-1}, \ldots, e_{2m} = e_m^{-1}$$

Here $e_j^{-1} = e_{j+m}$ denotes the edge e_j with the opposite orientation. Consider a path C viewed as a sequence $C = (a_1, \ldots, a_s)$, in which each a_j is an oriented edge of G. Such a C is said to have a **backtrack** if $a_{j+1} = a_j^{-1}$ for some $j = 1, \ldots, s-1$. Likewise, C is said to have a **tail** if $a_s = a_1^{-1}$. The **length** of $C = (a_1, \ldots, a_s)$ is s, the number of edges in C. We say that $v(C)$ is the length of the path C and similarly $v(P)$ denotes the length of the prime path P. A **closed path** is when the starting vertex is the same as our end vertex. If a closed path has no backtrack or tail, then we call the path a **prime path**. In other words, the path can only be traversed once. We can call two closed paths **equivalent** if we can get one path from the other by changing the starting vertex. We write $[C]$ for the equivalence class of a closed path C. We call the equivalence class of a prime path a **prime**. Given a prime path P, there are exactly $v(P)$ prime paths equivalent to P.

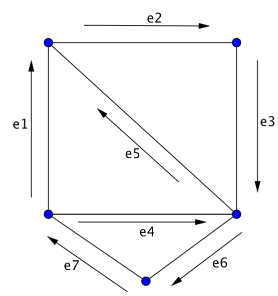

Figure 3.1: An Example of a Graph with a Choice of Oriented Edges

We will now give some example of primes using Figure 3.1. Our first primes are the paths $C = (e_2, e_3, e_5)$, $D = (e_1, e_2, e_3, e_4)$, $E = (e_4, e_6, e_7)$, $F = (e_4, e_5, e_2, e_3, e_6, e_7)$. These prime paths C, D, E, F have lengths 3, 4, 3, 7, respectively. The path $E^2 = (e_7, e_4, e_6, e_7, e_4, e_6)$ is not a prime path since it traverses a previously known prime path multiple times. However the path E^2 can be decomposed into two instances of the prime path E. Lets look at the path $J = (e_4, e_5, e_2, e_3, e_6, e_7)$. This path J is obviously prime since it is a closed path, has no backtrack and has no tail. In addition, our path J can be decomposed into two smaller prime paths, that is prime paths C and E. Let's look at the path $H = (e_2, e_3, e_6, e_7, e_4, e_5)$. The path H and our preexisting path F are called equivalent prime paths. We can say that both of these prime paths are of the same equivalence class. Based on the definition of a prime graph we know that on Figure 3.1 $e_{13} = e_6^{-1}$. That is, for every labeled orientation in our example, there exists another label with an opposite orientation. Thus, the path $K = (e_1, e_2, e_3, e_4^{-1})$ is not a prime path due to the backtrack and the tail.

Definition 3.2.2. The **Ihara zeta function** for a finite connected graph G (without degree-1 vertices) is defined to be the following function of the complex number u with $|u|$ sufficiently small:

$$\zeta_G(u) = \zeta(u, G) = \prod_{[P]} (1 - u^{v(P)})^{-1}$$

in which the product is over all primes $[P]$ in G [Ter10]. Here, we note that $v(P)$ denotes the length of the prime path P. Also take note that the prime $[P^{-1}]$ is just the path $[P]$ traversed in the opposite direction.

It is worthy of note that this product converges absolutely in a circular disc $|u| \leq R_G$. We call this R_G, the radius of convergence.

Theorem 3.2.3. *Let G be an undirected graph with a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and let the adjacency matrix $A = (a_{ij})$. Let Q be the diagonal matrix whose ith diagonal entry is one less than the degree of the vertex v_i. Then*

$$\zeta_G(u)^{-1} = (1 - u^2)^{m-n} \det(I - Au + Qu^2)$$

in which $m = |E|$, $n = |V|$ are the sizes of the edge and vertex set, respectively.

We will go over the directed version of this theorem shortly. Take note of what the Ihara zeta function actually counts. If we take a look at the generating function that the Ihara zeta function produces, the coefficient of the term of power n is the number of prime paths of length n up to equivalence. The enumeration of these Knight's tours comes directly from the coefficients of this generating polynomial. The recurrence relation appears in these generating functions and is determined by the denominator. Knuth's polynomial is actually 1 over the zeta function of the reduced adjacency matrix, that is, the determinant of the sub matrix with a row and the τ column removed. In the Knuth case, the denominator of our generating function for the Knuth example is $6z^6 + 7z^4 - 1$. As a result, the zeta function of the Knuth example is $\dfrac{1}{6z^6 + 7z^4 - 1}$. Thus the determinant of the reduced sub matrix (zeta function) is directly connected to the generating function of the bipartite graph.

We can demonstrate the Ihara Zeta function by looking at a couple of examples. For our first example we will look at a simple 3×3 chessboard and create a graph that expresses every possible knight move on said chessboard. The chessboard is shown in Figure 3.2.

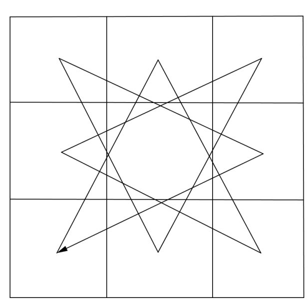

Figure 3.2: A 3x3 Complete Knight Graph

We can then create the matrix A and the matrix Q (the adjacency matrix and the diagonal matrix with the diagonals to be deg(v_i - 1)). Finally we let N_m be the number of closed paths of length m without backtracking or tails in the graph G. Then

$$\log \zeta_G(x) = \sum_{m \geq 1} \frac{N_m}{m} u^m$$

which effectively determines the number of closed paths of a given length m. Once both of the matrices are created we can then put them into Sage and compute the Ihara Zeta function.

```
sage: x = var('x')

sage: A =matrix([

[1*x^2+1,0,0,0,0,-x,0,-x,0],[0,1*x^2+1,0,0,0,0,-x,0,-x],

[0,0,1*x^2+1,-x,0,0,0,-x,0],[0,0,-x,1*x^2+1,0,0,0,0,-x],

[0,0,0,0,-1*x^2+1,0,0,0,0],[-x,0,0,0,0,1*x^2+1,-x,0,0],

[0,-x,0,0,0,-x,1*x^2+1,0,0],[-x,0,-x,0,0,0,0,1*x^2+1,0],

[0,-x,0,-x,0,0,0,0,1*x^2+1]

])

sage: f=det(A)

sage: factor(f)

-(x^4 + 1)^2*(x^2 + 1)^2*(x + 1)^3*(x - 1)^3

sage: g=(x^4 + 1)^2*(x^2 + 1)^2*(x + 1)^2*(x - 1)^2

sage: g.expand()
```

```
x^16 - 2*x^8 + 1

sage: f=ln(1/(x^16 - 2*x^8 + 1))

sage: (f).taylor(x,0,25)

2/3*x^24 + x^16 + 2*x^8
```

Since our graph has only 9 vertices, the biggest path it could have would be a path of length 8. If we look at the coefficient of x^8, it tells us that there are exactly 2 closed paths of length 8. We can confirm this result by arbitrarily orienting the edges and using the formula in [Hor07]. If we look at Definition 3.2.2, $|u|$ "sufficiently small" for a very large graph may be in fact incredibly small. In fact this definition can be extended even further to its analytic continuation, as it is done in [Hor07]. Since the reciprocal of this polynomial agrees with our original definition of the Ihara zeta function within a small circle about zero in the complex plane and is analytic everywhere but at the isolated zeros of the polynomial, we take this analytic continuation as our new definition of the Ihara zeta function of a graph.

One may wonder, is the only purpose of the zeta function to be a tool of combinatoric significance? The short answer is no. Here we will go into another situation in which the zeta function is used. First we will define what the prime number theorem is. Let $\pi(x)$ be the prime counting function that counts the number of primes less than or equal to x.

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log(x)}} = 1$$

What this says is that $\pi(x)$ is asymptotic to $\frac{x}{\log(x)}$ as x goes to ∞ and that the quotient $\frac{\pi(x)\log(x)}{x}$ approaches 1 as x goes to ∞. The proof of this theorem was considered a major achievement. It was proved by both Jacques Hadamard and Charles Jean de la Vallée-Poussin in 1896 using ideas introduced by Bernhard Riemann and the Riemann zeta function. The classic prime number theorem was the main motivation behind the prime number theorem of graphs, which coincidentally also has a connection with the zeta function. The prime number theorem of graphs is defined as

$$\pi(m) \sim \frac{\triangle_G}{mR_G^m} \text{ as } m \to \infty$$

in which $\pi(m) =$ the number of primes of length$P$. and $\triangle_G =$ g.c.d. of the prime path lengths. If we use the fact that the Ihara zeta function is meromorphic, one can prove the prime number theorem of graphs.

Definition 3.2.4. Arbitrarily orient the edges $e_1, e_2, \ldots, e_{|E|}$ of an undirected graph G and let $e_{|E|+i} = e_i^{-1}$ for all i, $1 \leq i \leq |E|$. The $2|E| \times 2|E|$ matrix M given by

$$(M)_{ij} = \begin{cases} 1 \text{ if } t(e_i) = s(e_j) \text{ and } s(e_i) \neq t(e_j), \\ \\ 0 \text{ otherwise} \end{cases}$$

is defined to be the directed edge matrix of G. A directed edge matrix of a graph G is related to the Ihara Zeta function of G by the following theorem written by [Hor07].

Theorem 3.2.5. *If M is a directed edge matrix of the graph G, then $\zeta_G(u)^{-1} = det(I - Mu)$*

We are going to compute the 3×3 example above using Theorem 3.2.5 to make sure that both polynomials match. This is straightforward once we find M. All we have to do is throw M into our formula and compute the polynomial using Sage.

```
sage: x = var('x')

sage: A = matrix([

[1,-x,0,0,0,0,0,0,0,0,0,0,0,0,0,0],[0,1,-x,0,0,0,0,0,0,0,0,0,0,0,0,0],

[0,0,1,-x,0,0,0,0,0,0,0,0,0,0,0,0],[0,0,0,1,-x,0,0,0,0,0,0,0,0,0,0,0],

[0,0,0,0,1,-x,0,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,1,-x,0,0,0,0,0,0,0,0,0],

[0,0,0,0,0,0,1,-x,0,0,0,0,0,0,0,0],[-x,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0],

[0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,-x],[0,0,0,0,0,0,0,0,-x,1,0,0,0,0,0,0],

[0,0,0,0,0,0,0,0,0,-x,1,0,0,0,0,0],[0,0,0,0,0,0,0,0,0,0,-x,1,0,0,0,0],

[0,0,0,0,0,0,0,0,0,0,0,-x,1,0,0,0],[0,0,0,0,0,0,0,0,0,0,0,0,-x,1,0,0],

[0,0,0,0,0,0,0,0,0,0,0,0,0,-x,1,0],[0,0,0,0,0,0,0,0,0,0,0,0,0,0,-x,1],

])

sage: det(A)

x^16 - 2*x^8 + 1

sage: f=ln(1/(x^16 - 2*x^8 + 1))

sage: (f).taylor(x,0,25)

2/3*x^24 + x^16 + 2*x^8
```

This is precisely what we achieved before. We will quickly demonstrate a 3×4 example just to see a more interesting example. We take the complete graph of a

3×4 graph as shown in Figure 3.4. Just like the 3×3 example, we need to find the

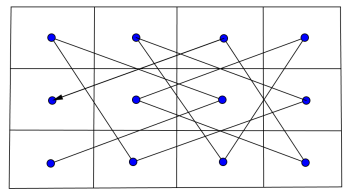

Figure 3.3: 3×4 Open Knight's Tours

adjacency matrix and the diagonal matrix with the diagonals to be $deg(v_i) - 1$. Once that is done we can throw them into Sage and compute the Ihara Zeta function using the following code:

```
sage: x=var('x')

sage: A = matrix([

[1*x^2+1,0,0,0,0,-x,0,-x,0,0,0,0],[0,1*x^2+1,0,0,0,0,-x,0,-x,0,0,0],

[0,0,1*x^2+1,-x,0,0,0,-x,0,0,0,0],[0,0,-x,2*x^2+1,0,0,0,0,-x,0,-x,0],

[0,0,0,0,1*x^2+1,0,0,0,0,-x,0,-x],[-x,0,0,0,0,2*x^2+1,-x,0,0,0,-x,0],

[0,-x,0,0,0,-x,2*x^2+1,0,0,0,0,-x],[-x,0,-x,0,0,0,0,1*x^2+1,0,0,0,0],

[0,-x,0,-x,0,0,0,0,2*x^2+1,-x,0,0],[0,0,0,0,-x,0,0,0,-x,1*x^2+1,0,0],

[0,0,0,-x,0,-x,0,0,0,0,1*x^2+1,0],[0,0,0,0,-x,0,-x,0,0,0,0,1*x^2+1]

])

sage: f

(16*x^24 + 24*x^22 + 25*x^20 + 8*x^18 - 4*x^16 - 18*x^14 -

22*x^12 - 22*x^10 - 11*x^8 - 2*x^6 + 3*x^4 + 2*x^2
```

```
+ 1)*(x^2 - 1)^2

sage: g=ln(1/f)

sage: g

log(1/((16*x^24 + 24*x^22 + 25*x^20 + 8*x^18 - 4*x^16 -

18*x^14 - 22*x^12 - 22*x^10 - 11*x^8 - 2*x^6 + 3*x^4

+ 2*x^2 + 1)*(x^2 - 1)^2))

sage: (g).taylor(x,0,25)

461/6*x^24 + 88*x^22 + 73*x^20 + 16*x^18 + 10*x^16

+ 20*x^14 + 7*x^12 + 2*x^10 + 4*x^8 + 6*x^6
```

We can approach the meaning of the polynomial much like we did with the 3×3 example. Some interesting consequences that we found during our research include that 1/denominator of the Knuth generating function is equal to the Ihara zeta function. This could be the subject of further studies. In addition, the recurrence relation in both functions can be looked at further.

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