

Summer 2019

# Preserving Ideals of Chemical Reaction Networks

Mark Curiel  
*San Jose State University*

Follow this and additional works at: [https://scholarworks.sjsu.edu/etd\\_theses](https://scholarworks.sjsu.edu/etd_theses)

---

## Recommended Citation

Curiel, Mark, "Preserving Ideals of Chemical Reaction Networks" (2019). *Master's Theses*. 5029.  
[https://scholarworks.sjsu.edu/etd\\_theses/5029](https://scholarworks.sjsu.edu/etd_theses/5029)

This Thesis is brought to you for free and open access by the Master's Theses and Graduate Research at SJSU ScholarWorks. It has been accepted for inclusion in Master's Theses by an authorized administrator of SJSU ScholarWorks. For more information, please contact [scholarworks@sjsu.edu](mailto:scholarworks@sjsu.edu).

PRESERVING IDEALS OF CHEMICAL REACTION NETWORKS

A Thesis

Presented to

The Faculty of the Department of Mathematics & Statistics

San José State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

by

Mark Curiel

Summer 2019

© 2019

Mark Curiel

ALL RIGHTS RESERVED

The Designated Thesis Committee Approves the Thesis Titled

PRESERVING IDEALS OF CHEMICAL REACTION NETWORKS

by

Mark Curiel

APPROVED FOR THE DEPARTMENT OF MATHEMATICS & STATISTICS

SAN JOSÉ STATE UNIVERSITY

Summer 2019

Elizabeth Gross, Ph.D.

Department of Mathematics & Statistics

Jordan Schettler, Ph.D.

Department of Mathematics & Statistics

Matthew D. Johnston, Ph.D.

Department of Mathematics & Statistics

## ABSTRACT

### PRESERVING IDEALS OF CHEMICAL REACTION NETWORKS

by Mark Curiel

Under the assumption of mass-action kinetics, every chemical reaction network has an associated polynomial dynamical system. Rather than study the dynamics of this system, we shall study the ideal generated by the polynomials in the system called the steady state ideal. In this thesis, we will show that there is a combinatorial way to determine the existence of monomials in the steady state ideal using the underlying structure of the network. This allows us to prove that there is a combinatorial condition that is enough to guarantee the steady state ideal is monomial. We introduce three operations on chemical reaction networks that preserve the steady state ideals. We are interested in classifying the chemical reaction networks with monomial ideals. In this thesis, we shall characterize a class of networks whose steady state ideal is monomial using the combinatorics on the network. This work can be viewed as the first step in the systematic study of steady state ideals. While we were able to define ideal preserving operations, the existence of a complete characterization of networks with monomial ideals still remains an open question.

## DEDICATION

To my cat, Peanut, and colored pens.

## TABLE OF CONTENTS

|   |      |
|---|------|
| List of Figures .....                                       | viii |
| 1 Introduction .....  | 1    |
| 1.1 Algebraic Structure of Chemical Reaction Networks ..... | 4    |
| 1.2 Chemical Reaction Networks .....                        | 7    |
| 2 From Chemical Reaction Networks to Hypergraphs .....      | 10   |
| 2.1 Hypergraphs .....                                       | 10   |
| 2.2 Multisets .....   | 11   |
| 2.3 Edge Coloring .....                                     | 11   |
| 2.4 Network Hypergraph .....                                | 13   |
| 3 Ideal Preserving Operations .....                         | 25   |
| 4 Monomial Steady State Ideals .....                        | 35   |
| 5 Conclusion .....  | 42   |
| Literature Cited .....                                      | 44   |
| Appendix A: Polynomial Rings, Ideals & Varieties .....      | 46   |

## LIST OF FIGURES

|          |   |    |
|----------|---|----|
| Fig. 1.  | Exchange of $CO_2$ in a closed carbonated beverage. ....  | 2  |
| Fig. 2.  | A chemical reaction network with its associated system of polynomials. ....   | 6  |
| Fig. 3.  | A 0,1-network with its associated system of polynomials. ....   | 8  |
| Fig. 4.  | The family $E = \{\{v_2, v_3, v_4\}, \{v_3\}, \{v_4, v_5\}\}$ is a set of hyperedges on $V = \{v_1, v_2, v_3, v_4, v_5\}$ . ....                              | 10 |
| Fig. 5.  | A bi-coloring of $\mathcal{E} = \{E_2^{(3)}, E_3^{(2)}\}$ where $\mathcal{E}_r = \{E_2^{(3)}, E_3\}$ and $\mathcal{E}_b = \{E_3\}$ . ....                     | 12 |
| Fig. 6.  | The vertex $v_1$ is almost balanced with respect to the minimal bi-colored edgeset $\mathcal{E} = \{E_1, E_2, E_3\}$ . ....                                   | 13 |
| Fig. 7.  | A chemical reaction network with its associated hypergraph. ....  | 14 |
| Fig. 8.  | A chemical reaction network, its network hypergraph, and a bi-coloring of its hyperedges. ....  | 15 |
| Fig. 9.  | Obtaining a bi-coloring of $\mathcal{E}'$ from a bi-coloring of $\mathcal{E}$ by swapping colors and adding red reactions edges. ....                         | 20 |
| Fig. 10. | A subhypergraph of a network hypergraph corresponding to a network containing a reversible reaction. ....   | 22 |
| Fig. 11. | Two chemical reaction networks with steady state ideal $\langle x_A, x_B \rangle$ . ....  | 25 |
| Fig. 12. | A sequence of network hypergraphs after performing four network operations. ....  | 26 |
| Fig. 13. | Examples of the addition of species $s$ to the complex $y$ . ....   | 28 |
| Fig. 14. | A network $N$ , whose steady state ideal is $\langle x_A, x_B \rangle$ , and its network hypergraph. ....   | 30 |
| Fig. 15. | A network $N_1$ , obtained from $N$ in Figure 14 by adding $B$ to the product complex of the reaction $A \rightarrow C$ , and its network hypergraph. ....    | 30 |
| Fig. 16. | A network $N_2$ , obtained from $N_1$ in Figure 15 by adding $B$ to the reactant complex of the reaction $A \rightarrow B$ , and its network hypergraph. .... | 32 |



|          |   |    |
|----------|---|----|
| Fig. 17. | A network $N_3$ , obtained from $N_2$ in Figure 16 by adding the reaction $A \rightarrow \emptyset$ , and its network hypergraph. ....                                    | 34 |
| Fig. 18. | A network, whose steady state ideal is symbolically monomial, and its network hypergraph. ....  | 40 |
| Fig. 19. | The vertex $u_4$ is almost balanced with respect to the bi-colorings $\mathcal{E}'_r \sqcup \mathcal{E}'_b$ (left) and $\mathcal{E}_r \sqcup \mathcal{E}_b$ (right). .... | 40 |

## 1 INTRODUCTION

Chemical reaction network theory attempts to model and predict chemical processes that can occur in the natural world and are involved in experiments such as, but not limited to, the oxidation of metals and the combustion of liquid ethanol. Many chemical processes, however, exhibit wild behavior and can range from destructive explosions or the inhalation of toxic fumes to harmless reactions such as the self-ionization of water. This calls for ways to place more control over a reaction to increase safety measures and prevent fatal events. Understanding the equilibria, or steady states, of a system of chemical reactions, called a chemical reaction network, is one way that a chemist can exert more control over manufactured reactions.

There are two ideas to bear in mind when thinking about steady states of chemical reactions: the concentration, or amount, of chemicals at a given time and the rate that the concentration changes with respect to the reactions taking place. A steady state is reached when the rate of change is zero. To demonstrate the idea of a steady state of chemical reactions, consider the mass production of a carbonated beverage. Before the beverage is bottled, it is injected with  $CO_2$ , carbon dioxide. This induces an interaction between the water and carbon dioxide molecules within the bottle. When the molecules collide, the carbon dioxide dissolves in the water to form  $H_2CO_3$ , carbonic acid. Hence, the concentrations are changing: we are losing a molecule of  $CO_2$  and a molecule of water but gaining a molecule of  $H_2CO_3$ . After this collision, the beverage is now carbonated and ready to be shipped to the consumer. As long as the beverage is kept bottled, notice that there will be no visual change in the beverage even though there are reactions still taking place. This phenomenon is unique since the reactions often cause materials to degrade completely and produce other chemically different objects. So why are the water and carbon dioxide molecules not being degraded completely? In this scenario, the carbon dioxide and water molecules are being consumed to produce carbonic acid while the

carbonic acid molecules break down into carbon dioxide and water molecules. This feedback loop causes the beverage to be in a steady state and is the reason that there is no visual change in the beverage. A summary of the exchange of carbon dioxide in this process is depicted in Figure 1.

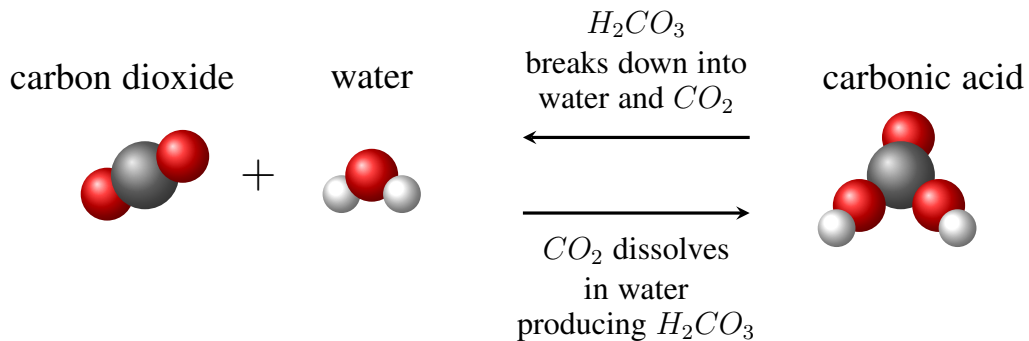


Fig. 1: Exchange of  $CO_2$  in a closed carbonated beverage.

By steady state, we mean that the rate of production equals the rate of consumption for each of the participants: water, carbon dioxide, and carbonic acid, called the *species* of the chemical reactions. The main interest of chemical reaction network theory is in the evolution of the concentrations of the species as the reactions take place. Researchers in the field are interested in the existence of steady states of a chemical reaction network where all species within the system have not degraded completely. A chemical reaction network is said to be *persistent* if all species that were initially present will not become extinct. In 1987, Feinberg conjectured a necessary condition for a network to be persistent called the Persistence Conjecture. A partial answer to this conjecture was proven by Horn, Jackson, and Feinberg and can be found in [1], [2], [3], [4], and [5].

In this thesis, we are interested in the rich algebra and combinatorics that arise from chemical reaction networks. The use of methods from algebraic geometry are discussed in [6] and other algebraic methods are summarized in Shiu's dissertation; see [7].

Algebraic methods are also used in [8] to study the Wnt Pathway model that arises in

biology from signaling patterns between proteins and cell receptors. In this paper, we shall study chemical reaction networks by studying the generators of an ideal that arises from the network called the steady state ideal. In particular, we are interested in answering three questions: 1) when is the steady state ideal monomial? 2) can we use the combinatorics of the chemical reaction network to search for monomial generators of the steady state ideal? and 3) are there operations on a chemical reaction network that preserve the steady state ideal?

Algebraically, monomials are interesting mathematical objects. In chemical reaction network theory, monomials appearing in the steady state ideal imply that, given a positive initial condition, the trajectory at steady state with respect to the dynamical system associated to the network will hit the boundary of the positive orthant. In other words, a monomial appearing in the steady state ideal implies that the network is not persistent. We shall study the monomial algebra nonetheless. Thus, by preserving steady state ideals containing a monomial, we maintain this property of the dynamics and the non-persistence of the network. Non-persistent networks have been studied before. In fact, Shiu and Sturmfels use the combinatorial structure of the network to study what is called a siphon, a set of a species that have a chance of being absent at steady state; see [9].

The focus of this thesis is to introduce operations on chemical reaction networks, such as adding species or reactions, that preserve the steady state ideal. Performing operations on the network is not a foreign concept. Gross, Harrington, Meshkat, and Shiu studied chemical reaction networks by gluing and breaking them down in [10]. In [11], they also introduce operations on chemical reaction networks that preserve identifiability, that is, the ability to determine the rate constants of the network using the associated dynamical system.

In Chapter 2, we will describe some of the combinatorics of chemical reaction networks and prove an essential lemma that relates the underlying structure of the

network to monomial generators of the steady state ideal. This result will serve as a support for defining the ideal preserving operations. Further, this chapter contains a collection of results that describe the theory of almost balanced vertices and a result that gives a necessary condition for determining when the steady state ideal is monomial.

In Chapter 3, we introduce and provide examples of the ideal preserving operations on chemical reaction networks. In Chapter 4, we characterize a class of chemical reaction networks with monomial steady state ideals. As a consequence, we can discuss the class of networks having the same steady state ideal. If the steady state ideal of the minimal network of such a class is monomial, then its steady state ideal is a Stanley-Riesner ideal. Thus, the minimal network is in correspondence with a simplicial complex and, indeed, it can be built from a simplicial complex.

Should the reader wish to recall the algebra of polynomial rings, ideals and varieties, see appendix A.

## 1.1 Algebraic Structure of Chemical Reaction Networks

To begin to understand the relationship between the underlying structure of a chemical reaction network and the generators of an algebraic ideal, a demonstration of how a chemical reaction network induces a system of polynomial equations (and thus, an algebraic ideal) is necessary. Consider two reactions that take place among the species labeled  $A$ ,  $B$ , and  $C$ . Suppose that the reactions are well-modeled by the following diagram:  $2A + 2C \rightleftharpoons B + C$ . We refer to  $2A + 2C$  and  $B + C$  as the *complexes* of the network. Specifically,  $2A + 2C$  is the *reactant complex* and  $B + C$  is the *product complex* of the forward reaction  $2A + 2C \rightarrow B + C$  and, in this scenario,  $B + C$  is the reactant complex and  $2A + 2C$  is the product complex of the backward reaction  $B + C \rightarrow 2A + 2C$ . We interpret the diagram to mean that two chemical units, or *moles*, of  $A$  and two moles of  $C$  are consumed to produce a mole of  $B$  and a mole of  $C$  and, likewise, a mole of  $B$  and a mole of  $C$  are consumed to produce two moles of  $A$  and two

moles of  $C$ . In terms of net change, the forward reaction implies that we are losing two moles of  $A$  and losing a mole of  $C$  but gaining a mole of  $B$ . The backward reaction implies we are losing a mole of  $B$  but gaining two moles of  $A$  and a mole of  $C$ .

The amount, or *molar concentration*, of  $A$ ,  $B$ , and  $C$  present in the system at any given time  $t$  is described by  $x_A(t)$ ,  $x_B(t)$ , and  $x_C(t)$ , respectively. For simplicity we shall write  $x_A$ ,  $x_B$ , and  $x_C$  rather than  $x_A(t)$ ,  $x_B(t)$ , and  $x_C(t)$ . Further, we shall use  $\mathbf{x} = (x_A, x_B, x_C)$  to denote the vector of molar concentrations. To track the rate of production and consumption of each of the species we assume that  $\dot{x}_A$ ,  $\dot{x}_B$ , and  $\dot{x}_C$  exist and are in some way dependent on the reactions taking place and on the molar concentrations  $x_A$ ,  $x_B$ , and  $x_C$  at any given time. For example, since there is a loss of two moles of  $A$  in the reaction  $2A + 2C \rightarrow B + C$  and a gain of two moles of  $A$  in the reaction  $B + C \rightarrow 2A + 2C$ , then we expect we can write

$$\dot{x}_A = -2\mathcal{K}_{2A+2C \rightarrow B+C}(\mathbf{x}) + 2\mathcal{K}_{B+C \rightarrow 2A+2C}(\mathbf{x})$$

where  $\mathcal{K}_{2A+2C \rightarrow B+C}(\mathbf{x})$  and  $\mathcal{K}_{B+C \rightarrow 2A+2C}(\mathbf{x})$  are maps that measure the amount of  $A$  that is produced or consumed by the reactions  $2A + 2C \rightarrow B + C$  and  $B + C \rightarrow 2A + 2C$ , respectively. Similarly, the rate at which the molar concentrations change for the other two species are

$$\dot{x}_B = \mathcal{K}_{2A+2C \rightarrow B+C}(\mathbf{x}) + \mathcal{K}_{B+C \rightarrow 2A+2C}(\mathbf{x})$$

$$\dot{x}_C = -\mathcal{K}_{2A+2C \rightarrow B+C}(\mathbf{x}) + \mathcal{K}_{B+C \rightarrow 2A+2C}(\mathbf{x}).$$

The problem now is to determine an effective way to define the maps  $\mathcal{K}_{2A+2C \rightarrow B+C}(\mathbf{x})$  and  $\mathcal{K}_{B+C \rightarrow 2A+2C}(\mathbf{x})$  so as to accurately describe the rate of production of each species. A *kinetics* of a chemical reaction network is an assignment of a map  $\mathcal{K}_*(\mathbf{x})$  to each reaction. The kinetics in this paper will solely be of *mass-action type*. The law of mass-action kinetics states that the rate produced by a reaction is proportional to the

product of molar concentrations of reacting species. Mass-action kinetics captures the probability of a collision of the molecules. Since there are two moles of  $A$  and two moles of  $C$  reacting in the reaction  $2A + 2C \rightarrow B + C$ , then  $\mathcal{K}_{2A+2C \rightarrow B+C}(\mathbf{x}) = \beta x_A^2 x_C^2$  where  $\beta \in \mathbb{R}_{>0}$  is the proportionality constant for the reaction  $2A + 2C \rightarrow B + C$ ; the constant  $\beta$  will be called the *rate constant* for the reaction  $2A + 2C \rightarrow B + C$ . Similarly, since there is one mole of  $B$  and a mole of  $C$  reacting in  $B + C \rightarrow 2A + 2C$ , then  $\mathcal{K}_{B+C \rightarrow 2A+2C}(\mathbf{x}) = \kappa x_B x_C$  where  $\kappa \in \mathbb{R}_{>0}$  is the rate constant for the reaction  $B + C \rightarrow 2A + 2C$ . When a chemical reaction network is endowed with mass-action kinetics, it is customary to label the arrows of the diagram with the rate constants. Altogether, we have the following differential equations

$$\dot{x}_A = -2\beta x_A^2 x_C^2 + 2\kappa x_B x_C$$

$$\dot{x}_B = \beta x_A^2 x_C^2 - \kappa x_B x_C$$

$$\dot{x}_C = -\beta x_A^2 x_C^2 + \kappa x_B x_C$$

for the chemical reaction network  $2A + 2C \xrightleftharpoons[\kappa]{\beta} B + C$ . We regard the differential equations  $\dot{x}_A$ ,  $\dot{x}_B$ , and  $\dot{x}_C$  as elements of the polynomial ring  $\mathbb{Q}(\beta, \kappa)[\mathbf{x}]$ , that is, the ring of polynomials in  $x_A$ ,  $x_B$ , and  $x_C$  with coefficients in  $\mathbb{Q}$  and allow the division of the rate constants  $\beta$  and  $\kappa$ . For a larger example, Figure 2 contains a different chemical reaction network along with its associated system of polynomials.

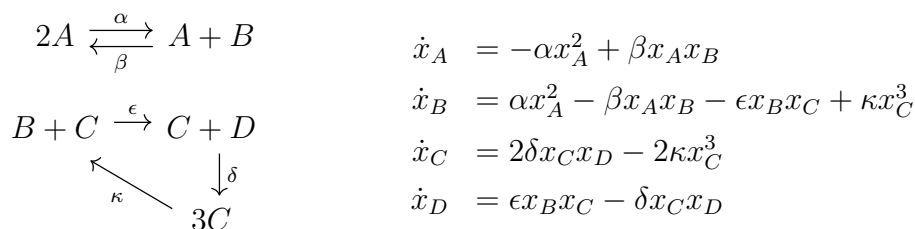


Fig. 2: A chemical reaction network with its associated system of polynomials.

## 1.2 Chemical Reaction Networks

Now what follows is a formal description of a chemical reaction network and notation that will be used for the remainder of the paper. We shall use similar definitions and notations as stated in [12]. For any monoid  $M$  and finite set  $I$ , we shall denote  $M^I$  the set containing all elements of the form  $\sum_{i \in I} m_i i$  where  $m_i \in M$ . The *support* of an element  $m \in M^I$  such that  $m = \sum_{i \in I} m_i i$  is the set  $\text{supp}(m) = \{i \in I : m_i \neq 0\}$ . Equipped with these notations, we establish a foundation for the language used in this thesis. First, a formal definition of a chemical reaction network is appropriate.

**Definition 1.1.** A *chemical reaction network*  $N$  is a triple  $(\mathcal{S}, \mathcal{C}, \mathcal{R})$  where  $\mathcal{S}$  is the finite set of chemical species, the finite subset  $\mathcal{C}$  of  $\mathbb{Z}_{\geq 0}^{\mathcal{S}}$  is the set of chemical complexes, and the set  $\mathcal{R}$  of chemical reactions is a relation on  $\mathcal{C}$  where reactions are denoted by  $y_i \rightarrow y_j$ . Moreover,  $N$  must satisfy the following:  $y \rightarrow y \notin \mathcal{R}$  for all  $y \in \mathcal{C}$ ; if  $y \in \mathcal{C}$ , then there is a  $y' \in \mathcal{C}$  such that either  $y \rightarrow y' \in \mathcal{R}$  or  $y' \rightarrow y \in \mathcal{R}$ ; and  $\mathcal{S} = \bigcup_{y \in \mathcal{C}} \text{supp}(y)$ .

The three conditions placed on the reaction network are set so that the polynomial equations that arise from the network remain meaningful. Condition 1 guarantees that each reaction will always change the reacting species to produce a different complex, thus, a meaningful change in concentration. Condition 2 guarantees that every complex must belong to some reaction, otherwise, there is an isolated complex that does not contribute to the reactions and can be ignored. Condition 3 guarantees that we consider the change of concentration of the species that take part in the reactions in some way.

For the remainder of the paper,  $n_{\mathcal{S}}$  will denote the number of chemical species,  $n_{\mathcal{C}}$  will denote the number of chemical complexes, and  $n_{\mathcal{R}}$  will denote the number of chemical reactions. Further, the set  $\mathcal{S}$  will almost always be taken to be the set  $\{s_1, \dots, s_{n_{\mathcal{S}}}\}$ , unless otherwise stated. Likewise,  $\mathcal{R} = \{y_1 \rightarrow y'_1, \dots, y_{n_{\mathcal{R}}} \rightarrow y'_{n_{\mathcal{R}}}\}$  and the complexes will be indexed according to the reaction they take place in.



**Remark 1.2.** We follow the convention that all reactant complexes have nonempty support.

It will be convenient to regard the set  $\mathcal{S}$  of species as a subset of vectors of  $\mathbb{Z}_{\geq 0}^{n_s}$ . In particular, for  $1 \leq i \leq n_s$ , we identify the species  $s_i \in \mathcal{S}$  as the  $i$ th standard unit vector in  $\mathbb{Z}_{\geq 0}^{n_s}$ . Consequently, each complex  $y = \sum_{s \in \mathcal{S}} y_s s \in \mathcal{C}$  will be regarded as the vector of coefficients  $[y_s]_{s \in \mathcal{S}}$ , that is, an element of  $\mathbb{Z}_{\geq 0}^{n_s}$ . In this way, it makes sense to write  $y + y'$  or  $y + s$  for any  $y, y' \in \mathcal{C}$  and  $s \in \mathcal{S}$  where the addition is taken coordinate-wise.

A chemical reaction network is called a *0,1-network* if the coefficient of the species  $s$  in the complex  $y$  is either 0 or 1 for any  $s \in \mathcal{S}$  and  $y \in \mathcal{C}$ . The network in Figure 2 is not a 0,1-network since the complexes  $2A$  and  $3C$  have coefficients that are not 0 or 1, however, the network in Figure 3 is a 0,1-network. This thesis focuses on chemical reaction networks that are 0,1-networks. Consequently, the polynomials associated to 0,1-networks will have square-free terms, that is, terms having powers of 0 or 1. To see this, we now discuss how to obtain the differential equations of an arbitrary chemical reaction network.

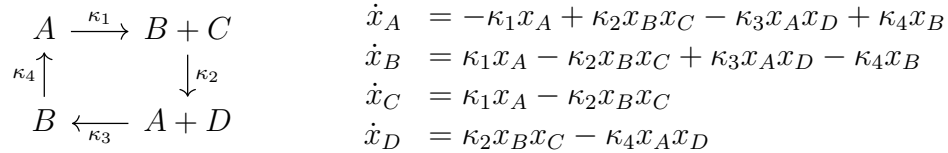


Fig. 3: A 0,1-network with its associated system of polynomials.

To each complex  $y_i = \sum_{s \in \mathcal{S}} y_{is} s \in \mathcal{C}$ , we associate the monomial  $x^{y_i} := \prod_{s \in \mathcal{S}} x_s^{y_{is}}$  where  $x_s$  is the molar concentration of the species  $s$ . Thus, under the assumption of mass action kinetics, we obtain the following differential equations:

$$\dot{x}_s = \sum_{y_i \rightarrow y'_i \in \mathcal{R}} \kappa_i x^{y_i} (y'_{is} - y_{is}), \quad (1)$$

where  $\kappa_i \in \mathbb{R}_{>0}$  is the rate constant for the reaction  $y_i \rightarrow y'_i$ . For each species  $s \in \mathcal{S}$ , we will refer to  $\dot{x}_s$  as the *steady state polynomial* for  $s$ . The *steady state ideal* for the reaction

network  $N$  is the ideal  $\mathbb{I}(N) = \langle \dot{x}_s : s \in \mathcal{S} \rangle \subseteq \mathbb{Q}(\boldsymbol{\kappa})[\mathbf{x}]$  where  $\mathbf{x} = (x_1, \dots, x_{n_{\mathcal{S}}})$  and  $\boldsymbol{\kappa}$  is the  $n_{\mathcal{R}}$ -vector of rate constants. A vector  $c$  of molar concentrations is a *steady state* of the reaction network  $N$  if  $c$  belongs to the affine variety  $\mathbb{V}(\mathbb{I}(N))$  of the ideal  $\mathbb{I}(N)$ ; see Appendix A.

The quantity  $y'_{is} - y_{is}$  from Equation 1 represents the net change of species  $s$  in the reaction  $y_i \rightarrow y'_i$ , it is referred to as the *stoichiometric coefficient* of  $s$  in the reaction  $y_i \rightarrow y'_i$  and will be denoted by  $\gamma_{si}$ . Proposition 1.3 follows from Equation 1 and the definition of a chemical reaction network.

**Proposition 1.3.** Reactants generate the support of all the polynomials in the steady-state ideal, i.e.  $\mathbb{I}(N) \subseteq \langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle$ . Moreover, if  $\mathbb{I}(N)$  is monomial, then  $\mathbb{I}(N) = \langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle$ .

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a chemical reaction network. The statement that  $\mathbb{I}(N)$  is a subideal of the ideal  $\langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle$  follows immediately from Equation 1. It remains to show that if  $\mathbb{I}(N)$  is monomial, then  $\mathbb{I}(N) = \langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle$ .

Assume  $\mathbb{I}(N)$  is monomial. We will show that  $x^{y_i} \in \mathbb{I}(N)$  for each  $y_i \in \mathcal{C}$  such that  $y_i \rightarrow y'_i \in \mathcal{R}$ . Consider the following equivalent notion of a monomial ideal, stated here in terms of the steady state ideal: for any polynomial  $f$  belonging to  $\mathbb{I}(N)$ , each nonzero term of  $f$  must also be an element of  $\mathbb{I}(N)$ . In particular, the nonzero terms of the generators  $\dot{x}_s$  of  $\mathbb{I}(N)$  are elements of  $\mathbb{I}(N)$ . Since the terms that make up  $\dot{x}_s$  are real multiples of  $x^{y_i}$  where  $y_i \in \mathcal{C}$  such that  $y_i \rightarrow y'_i \in \mathcal{R}$ , then we need only show that the real multiples are nonzero. Recall, the coefficient of  $x^{y_i}$  in  $\dot{x}_s$  is  $\kappa_i \gamma_{si}$ . Since  $\kappa_i$  is nonzero, we will show  $\gamma_{si}$  is also nonzero.

By the definition of a chemical reaction network, the reaction  $y_i \rightarrow y_i$  is not in  $\mathcal{R}$  for any  $y_i \in \mathcal{C}$ . Hence, for each reaction  $y_i \rightarrow y'_i \in \mathcal{R}$ , there is a species  $s \in \text{supp}(y'_i - y_i)$ . Necessarily,  $\gamma_{si}$  is nonzero and so the coefficient of  $x^{y_i}$  in  $\dot{x}_s$  is nonzero. Thus, since  $\mathbb{I}(N)$  is monomial, then  $x^{y_i} \in \mathbb{I}(N)$ .  $\square$

## 2 FROM CHEMICAL REACTION NETWORKS TO HYPERGRAPHS

In this chapter, we will give a brief overview of hypergraphs and discuss the combinatorics of chemical reaction networks by associating a hypergraph to each network. We will also introduce an application of edge coloring that will serve to support the ideal preserving operations in Chapter 3.

### 2.1 Hypergraphs

Given a finite set  $V = \{v_1, \dots, v_n\}$  and a nonempty family  $E = \{E_1, \dots, E_m\}$  of subsets of  $V$ , the pair  $\mathcal{H} = (V, E)$  is called a *hypergraph*. The elements of  $V$  are called *vertices* and the elements of  $E$  are called *hyperedges*, or in some cases, for simplicity, *edges*. If  $\mathcal{H}$  is a hypergraph, then  $V(\mathcal{H})$  and  $E(\mathcal{H})$  refer to the vertex set and hyperedge set of  $\mathcal{H}$ , respectively. Differing from edges in an ordinary graph, a hyperedge is drawn as a curve encircling all vertices in the hyperedge. Figure 4 provides an example of a hypergraph on  $V = \{v_1, v_2, v_3, v_4, v_5\}$ .

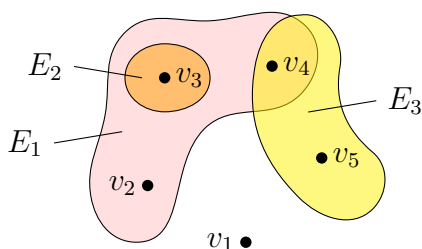


Fig. 4: The family  $E = \{\{v_2, v_3, v_4\}, \{v_3\}, \{v_4, v_5\}\}$  is a set of hyperedges on  $V = \{v_1, v_2, v_3, v_4, v_5\}$ .

In line with set theory vernacular, if a hyperedge  $E_i$  is a one element set then we shall say  $E_i$  is a *singleton*. If  $v \in E_i$ , then  $v$  is *covered* by  $E_i$ . If there is a vertex  $v$  that is not covered by any hyperedge, then  $v$  is said to be *isolated*. In Figure 4, the hyperedge  $E_2$  is a singleton and  $v_1$  is isolated in the hypergraph. Two vertices are *adjacent* if they are covered by the same edge. On the other hand, two hyperedges are adjacent if their

intersection is nonempty. A hyperedge  $E_i$  is an *incident* hyperedge of  $v$  if  $v$  is covered by  $E_i$ . The hypergraph in Figure 4 shows that  $v_2$  and  $v_3$  are adjacent vertices but  $v_2$  and  $v_5$  are not adjacent,  $E_1$  and  $E_3$  are adjacent hyperedges but  $E_2$  and  $E_3$  are not, and both  $E_1$  and  $E_3$  are incident to  $v_4$  but  $E_2$  is not. These are just a few basic concepts in the theory of hypergraphs. In this paper, we are mainly concerned with multisets whose elements are the hyperedges of a hypergraph.

## 2.2 Multisets

A multiset is a generalization of the concept of a set that allows for repetition of its elements. Formally, a multiset is defined as a pair  $\mathcal{E} = (A, \mu)$  where  $A$  is the underlying set for  $\mathcal{E}$  and  $\mu$  is a function from  $A$  to  $\mathbb{Z}_{\geq 0}$  that outputs the multiplicity  $\mu(a)$  for each element  $a \in A$  as an element of  $\mathcal{E}$ . Given a hypergraph  $\mathcal{H} = (V, E)$ , a multiset  $\mathcal{E}$  whose underlying set is  $E$  is said to be a *multiset over the edges in  $E$* . When an edge  $E_i \in E$  has nonnegative multiplicity  $\mu(E_i) = \mu_i$  in  $\mathcal{E}$ , we write  $\mathcal{E} = \{E_1^{(\mu_1)}, \dots, E_m^{(\mu_m)}\}$ . If  $\mathcal{A} = \{E_1^{(\mu_1)}, \dots, E_m^{(\mu_m)}\}$  and  $\mathcal{B} = \{E_1^{(\omega_1)}, \dots, E_m^{(\omega_m)}\}$  are multisets over the edges in  $E$ , the multiset union of  $\mathcal{A}$  and  $\mathcal{B}$  is  $\mathcal{A} \sqcup \mathcal{B} := \{E_1^{(\mu_1 + \omega_1)}, \dots, E_m^{(\mu_m + \omega_m)}\}$ .

## 2.3 Edge Coloring

Let  $\mathcal{H} = (V, E)$  be a hypergraph. Consider the colors  $c_1, \dots, c_k$  and a multiset  $\mathcal{E}$  over the edges in  $E$ . A *k-coloring* of  $\mathcal{E}$  is an assignment of the edges in  $\mathcal{E}$  into  $k$  possibly empty submultisets. We write  $\mathcal{E} = \mathcal{E}_{c_1} \sqcup \dots \sqcup \mathcal{E}_{c_k}$  and think of the edges in  $\mathcal{E}_{c_i}$  as being colored  $c_i$ . When considering a coloring of a multiset  $\mathcal{E}$ , we adopt the convention that hyperedges with multiplicity zero in  $\mathcal{E}$  will not be drawn. Figure 5 depicts a 2-coloring of  $\mathcal{E} = \{E_2^{(3)}, E_3^{(2)}\}$  where  $E_2$  and  $E_3$  are the hyperedges in Figure 4. The *degree* of a vertex  $v \in V$  with respect to  $c_i$  is the number of edges in  $\mathcal{E}_{c_i}$  that cover  $v$  and will be denoted  $\deg_{\mathcal{E}_{c_i}}(v)$ . Every multiset  $\mathcal{E}$  has many  $k$ -colorings; however, we are interested in 2-colorings, or *bi-colorings*, that satisfy a particular property. When  $\mathcal{E}$  has a bi-coloring, we take  $c_1 = r$  to denote the color red and  $c_2 = b$  to denote the color blue.

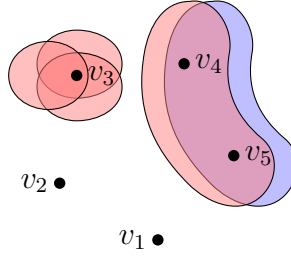


Fig. 5: A bi-coloring of  $\mathcal{E} = \{E_2^{(3)}, E_3^{(2)}\}$  where  $\mathcal{E}_r = \{E_2^{(3)}, E_3\}$  and  $\mathcal{E}_b = \{E_3\}$ .

**Definition 2.1.** Given a multiset  $\mathcal{E}$  over the edges in  $E$ , a vertex  $v \in V$  is said be *almost balanced with respect to a bi-coloring*  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  if  $\deg_{\mathcal{E}_r}(v) = \deg_{\mathcal{E}_b}(v) + k$  for some positive integer  $k$  and  $\deg_{\mathcal{E}_r}(u) = \deg_{\mathcal{E}_b}(u)$  for all  $u \in V \setminus \{v\}$ .

**Remark 2.2.** We shall say  $v \in V$  is *almost balanced* when we know there is a bi-colored edgeset  $\mathcal{E}$  such that  $v$  is almost balanced with respect to  $\mathcal{E}$ .

When  $\mathcal{E}$  has a bi-coloring  $\mathcal{E}_r \sqcup \mathcal{E}_b$ , we use  $\mathcal{E}' \subseteq \mathcal{E}$  to mean  $\mathcal{E}'$  is a submultiset of  $\mathcal{E}$  with respect to the bi-coloring of the edges of  $\mathcal{E}$ , that is,  $\mathcal{E}'$  has a natural bi-coloring  $\mathcal{E}'_r \sqcup \mathcal{E}'_b$  induced by the coloring of the edges in  $\mathcal{E}$  such that  $\mathcal{E}'_r$  is a submultiset of  $\mathcal{E}_r$  and  $\mathcal{E}'_b$  is a submultiset of  $\mathcal{E}_b$ . We say that  $v \in V$  is almost balanced with respect to a *minimal* bi-colored edgeset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  if  $v$  is almost balance with respect to  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  and for all  $\mathcal{E}' \subseteq \mathcal{E}$  such that  $v$  is almost balanced with respect to  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  implies  $\mathcal{E}' = \mathcal{E}$ .

The vertex  $v_3$  in Figure 5 is almost balanced since  $\deg_{\mathcal{E}_r}(v_3) = \deg_{\mathcal{E}_b}(v_3) + 3 = 3$ ,  $\deg_{\mathcal{E}_r}(v_1) = \deg_{\mathcal{E}_b}(v_1) = 0$ ,  $\deg_{\mathcal{E}_r}(v_2) = \deg_{\mathcal{E}_b}(v_2) = 0$ ,  $\deg_{\mathcal{E}_r}(v_4) = \deg_{\mathcal{E}_b}(v_4) = 1$ , and  $\deg_{\mathcal{E}_r}(v_5) = \deg_{\mathcal{E}_b}(v_5) = 1$ . The bi-colored edgeset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$ , however, is not minimal since  $\mathcal{E}' = \{E_2\} \subset \mathcal{E}$  and  $v_3$  remains almost balanced with respect to the bi-coloring  $\mathcal{E}'_r \sqcup \mathcal{E}'_b = \{E_2\} \sqcup \emptyset$ . The vertex  $v_1$  in Figure 6 is almost balanced and  $\mathcal{E} = \{E_1, E_2, E_3\}$  is minimal.

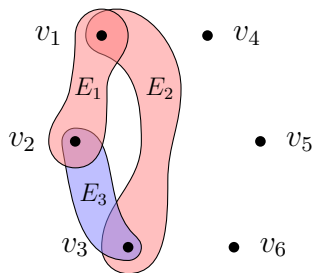


Fig. 6: The vertex  $v_1$  is almost balanced with respect to the minimal bi-colored edgeset  $\mathcal{E} = \{E_1, E_2, E_3\}$ .

Bi-colorings that satisfy the almost balanced condition are related to monomial walks on a  $d$ -uniform hypergraph; see [13]. Monomial walks imply the existence of binomial generators in the edge subring associated to the hypergraph.

## 2.4 Network Hypergraph

To capture combinatorial information from a chemical reaction network, we induce a hypergraph structure for each network and introduce an application of edge colorings in the hypergraph. The induced hypergraph on the network will include hyperedges corresponding to reactions in  $\mathcal{R}$  and hyperedges that associate complexes with a common chemical species. We therefore, due in part to convenience, choose to label the complexes corresponding to the reactions they take place in. That is, for each reaction  $y_i \rightarrow y'_i \in \mathcal{R}$ , let  $u_i$  be the distinct label for the reactant complex  $y_i$  and let  $v_i$  be the distinct label for the product complex  $y'_i$ . In this set-up, if two reactions  $y_i \rightarrow y'_i$  and  $y_j \rightarrow y'_j$  have the same reactant complex then we treat  $u_i$  and  $u_j$  as being different vertices even though they correspond to the same complex, i.e.  $u_i \neq u_j$  even though  $y_i = y_j$ . Equivalently, we do the same for the product complexes as well.

**Definition 2.3.** Given a chemical reaction network  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ , the *network hypergraph*  $\mathcal{H}_N$ , or simply  $\mathcal{H}$ , has vertex set  $V = \bigcup_{y_i \rightarrow y'_i} \{u_i, v_i\}$  and hyperedge set  $E$  that contains the hyperedges  $E_s := \bigcup_{y_i \rightarrow y'_i} \{u_i : s \in \text{supp}(y_i)\} \cup \{v_i : s \in \text{supp}(y'_i)\}$  for each

$s \in \mathcal{S}$  and the hyperedges  $E_i := \{u_i, v_i\}$  for each  $y_i \rightarrow y'_i \in \mathcal{R}$  if  $y'_i \neq \emptyset$ , else  $E_i = \emptyset$  if  $y'_i = \emptyset$ .

Consequently, the number of vertices in  $\mathcal{H}$  is  $2|\mathcal{R}|$  since every complex  $y$  gets a uniquely labeled vertex for each reaction containing  $y$ . A vertex  $v_i$  will be isolated in a network hypergraph if its corresponding reaction is  $y_i \rightarrow \emptyset$ . In other words, if a reaction has an empty product complex, then the corresponding hyperedge is empty, leaving  $v_i$  not contained in any hyperedge. For future reference, the vertex  $w_i$  will be used to refer to either  $u_i$  or  $v_i$ . An example of a chemical hypergraph is provided in Figure 7.

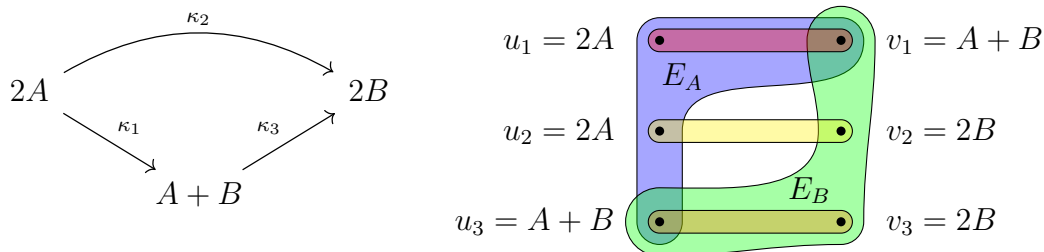


Fig. 7: A chemical reaction network with its associated hypergraph.

It will be convenient to refer to an edge  $E_s \in E(\mathcal{H}_N)$  as the *species edge* for  $s \in \mathcal{S}$ . Similarly, we refer to  $E_i$  as the *reaction edge* corresponding to the  $i$ th reaction  $y_i \rightarrow y'_i$ . Unless otherwise stated, given a bi-colored multiset  $\mathcal{E}$  over the edges in  $E(\mathcal{H}_N)$ ,  $m_s$  will always denote the multiplicity of the species edge  $E_s$  in  $\mathcal{E}$  and  $m_i$  will denote the multiplicity of the reaction edge  $E_i$  in  $\mathcal{E}$ .

The idea of the network hypergraph is to retain information about the individual reactions while also combinatorially encoding steady state polynomials. For an arbitrary chemical reaction network  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  and any species  $s \in \mathcal{S}$ , the steady state polynomial  $\dot{x}_s$  is determined by the hyperedge  $E_s$  and the stoichiometric coefficients  $\gamma_{si}$  for all  $i$  such that  $y_i \rightarrow y'_i \in \mathcal{R}$ . Since the hyperedge  $E_s$  contains the vertices, thought of as complexes, that  $s$  appears with a nonzero coefficient, then  $E_s$  encodes the reactions

that  $s$  appears. Indeed,  $\dot{x}_s = \sum_{u_i \text{ or } v_i \in E_s} \gamma_{si} \kappa_i x^{y_i}$ . If  $N$  is a 0,1-network, then  $\gamma_{si}$  is  $-1$ ,  $1$ , or  $0$ , and we rewrite the previous sum by summing over the reactant vertices in  $E_s$  and the product vertices in  $E_s$  separately. If  $s$  appears in the reaction  $y_i \rightarrow y'_i$ , then the rate that  $s$  is produced by the reaction  $y_i \rightarrow y'_i$  is  $\gamma_{si} \kappa_i x^{y_i} = \chi_{y'_i}(s) \kappa_i x^{y_i} - \chi_{y_i}(s) \kappa_i x^{y_i}$  where  $\chi$  is the indicator function defined as follows: if  $y \in \mathcal{C}$  and  $s \in \mathcal{S}$ , then  $\chi_y(s)$  is 1 if  $s \in \text{supp}(y)$  and 0 if  $s \notin \text{supp}(y)$ . If  $s \in \text{supp}(y_i)$ , then  $u_i \in E_s$ . Similarly, if  $s \in \text{supp}(y'_i)$ , then  $v_i \in E_s$ . Thus,

$$\dot{x}_s = \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i}. \quad (2)$$

The above equation suggests that the hyperedges in  $\mathcal{H}_N$  play a role in the algebraic structure of  $\mathbb{I}(N)$ . Indeed, under certain conditions, there is such a relationship. Recall, if a monomial  $x^y$  appears in  $\mathbb{I}(N)$  then  $x^y$  is generated by  $\dot{x}_{s_1}, \dots, \dot{x}_{s_n}$ . Thus, in general, in order for  $x^y$  to appear in  $\mathbb{I}(N)$ , many cancellations must occur. Equation 2 suggests that the hyperedges can help to determine when such cancellations occur. Almost balanced vertices give us the desired cancellation. To see this, consider the hypergraph in Figure 8, which is the hypergraph from Figure 6 but realized as the network hypergraph of the chemical reaction network in Figure 8.

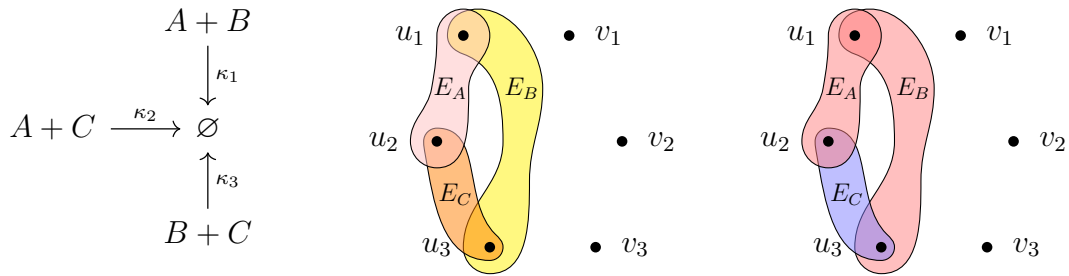


Fig. 8: A chemical reaction network, its network hypergraph, and a bi-coloring of its hyperedges.



From Figure 6, we have seen that  $v_1$ , realized as  $u_1$  in Figure 8, is almost balanced with respect to the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  where  $\mathcal{E}_r = \{E_A, E_B\}$  and  $\mathcal{E}_b = \{E_C\}$ . Notice that the difference of the sum of steady states polynomials  $\dot{x}_s$  such that  $E_s \in \mathcal{E}_b$  with the sum of steady state polynomials  $\dot{x}_s$  such that  $E_s \in \mathcal{E}_r$  is monomial:

$$\begin{aligned} \dot{x}_C - \dot{x}_A - \dot{x}_B &= (-\kappa_2 x_A x_C - \kappa_3 x_B x_C) \\ &\quad - (-\kappa_1 x_A x_B - \kappa_2 x_A x_C) \\ &\quad - (-\kappa_1 x_A x_B - \kappa_3 x_B x_C) \\ &= 2\kappa_1 x_A x_B. \end{aligned}$$

This computation generalizes to the following result.

**Lemma 2.4.** *Let  $N$  be a 0,1-network. If  $u_i \in V(\mathcal{H})$  is almost balanced with respect to a bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H})$ , then  $k\kappa_i x^{y_i} = \sum_{E_s \in \mathcal{E}_b} \dot{x}_s - \sum_{E_s \in \mathcal{E}_r} \dot{x}_s$  for some positive integer  $k$ .*

*Proof.* Let  $N$  be a 0,1-network and suppose  $u_{i_0} \in V(\mathcal{H})$  is almost balanced with respect to a bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$ . By the definition of almost balanced,  $\deg_{\mathcal{E}_r}(u_{i_0}) = \deg_{\mathcal{E}_b}(u_{i_0}) + k$  for some positive integer  $k$  and  $\deg_{\mathcal{E}_r}(w) = \deg_{\mathcal{E}_b}(w)$  for all  $w \in V \setminus \{u_{i_0}\}$ . Without loss of generality, suppose  $\mathcal{E}$  is minimal. Then for any edge  $e \in \mathcal{E}$ , either  $e \in \mathcal{E}_r$  or  $e \in \mathcal{E}_b$  but not both, otherwise, there is an edge  $e' \in \mathcal{E}_r \cap \mathcal{E}_b$  and, consequently,  $u_{i_0}$  is almost balanced with respect to  $(\mathcal{E}_r \setminus \{e'\}) \sqcup (\mathcal{E}_b \setminus \{e'\})$ , contradicting the minimality of  $\mathcal{E}$ . In particular, each species edge  $E_s \in \mathcal{E}$  is colored either red or blue but not both. Thus, the sets  $\mathcal{S}_r := \{s \in \mathcal{S} : E_s \in \mathcal{E}_r\}$  and  $\mathcal{S}_b := \{s \in \mathcal{S} : E_s \in \mathcal{E}_b\}$  are disjoint. We shall prove the following two claims:

- 1)  $\sum_{s \in \mathcal{S}_b} m_s \gamma_{s i_0} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{s i_0} = k$
- 2)  $\sum_{s \in \mathcal{S}_b} m_s \gamma_{s i} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{s i} = 0$  for all  $i \neq i_0$  such that  $y_i \rightarrow y'_i \in \mathcal{R}$ ,

where  $\gamma_{si}$  is the stoichiometric coefficient of  $s$  in the  $i$ th reaction and  $m_s$  denotes the multiplicity of the edge  $E_s$  in  $\mathcal{E}$ .

For  $c \in \{r, b\}$ , the sum  $\sum_{s \in \mathcal{S}_c} m_s \gamma_{si}$  can be partitioned into four sums, one for each of the following properties: 1)  $u_i \in E_s$  and  $v_i \notin E_s$ , 2)  $u_i \notin E_s$  and  $v_i \in E_s$ , 3)  $u_i, v_i \in E_s$ , and 4)  $u_i, v_i \notin E_s$ . Then, for each  $i$  such that  $y_i \rightarrow y'_i \in \mathcal{R}$ ,

$$\begin{aligned} \sum_{s \in \mathcal{S}_b} m_s \gamma_{si} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{si} &= \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \notin E_s}} m_s \gamma_{si} + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \notin E_s \\ v_i \in E_s}} m_s \gamma_{si} + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \in E_s}} m_s \gamma_{si} + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \notin E_s \\ v_i \notin E_s}} m_s \gamma_{si} \\ &\quad - \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \notin E_s}} m_s \gamma_{si} - \sum_{\substack{s \in \mathcal{S}_r \\ u_i \notin E_s \\ v_i \in E_s}} m_s \gamma_{si} - \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \in E_s}} m_s \gamma_{si} - \sum_{\substack{s \in \mathcal{S}_r \\ u_i \notin E_s \\ v_i \notin E_s}} m_s \gamma_{si}. \end{aligned}$$

Since  $N$  is a 0,1-network, if  $s$  is a species such that  $u_i$  and  $v_i$  both belong to  $E_s$ , then  $\gamma_{si} = 0$ . On the other hand, if  $u_i$  and  $v_i$  are both not in  $E_s$ , then  $\gamma_{si} = 0$ . Following suit,  $u_i \in E_s$  and  $v_i \notin E_s$  implies  $\gamma_{si} = -1$  and  $u_i \notin E_s$  and  $v_i \in E_s$  implies  $\gamma_{si} = 1$ . Then the previous equation becomes

$$\sum_{s \in \mathcal{S}_b} m_s \gamma_{si} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{si} = - \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \notin E_s \\ v_i \in E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \notin E_s}} m_s - \sum_{\substack{s \in \mathcal{S}_r \\ u_i \notin E_s \\ v_i \in E_s}} m_s.$$

For each vertex  $w$  in the reaction edge  $E_i = \{u_i, v_i\}$  where  $w' \in E_i \setminus \{w\}$ , if  $m_i$  denotes the multiplicity of  $E_i$  in  $\mathcal{E}$ , then notice that

$$\deg_{\mathcal{E}_r}(w) = \sum_{\substack{s \in \mathcal{S}_r \\ w \in E_s \\ w' \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_r \\ w \in E_s \\ w' \in E_s}} m_s + \sum_{\substack{y_i \rightarrow y'_i \in \mathcal{R} \\ E_i \in \mathcal{E}_r}} m_i \quad (3)$$

and

$$\deg_{\mathcal{E}_b}(w) = \sum_{\substack{s \in \mathcal{S}_b \\ w \in E_s \\ w' \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_b \\ w \in E_s \\ w' \in E_s}} m_s + \sum_{\substack{y_i \rightarrow y'_i \in \mathcal{R} \\ E_i \in \mathcal{E}_b}} m_i. \quad (4)$$

If  $i \neq i_0$ , then the subtraction of the two equations  $\deg_{\mathcal{E}_r}(u_i) = \deg_{\mathcal{E}_b}(u_i)$  and  $\deg_{\mathcal{E}_r}(v_i) = \deg_{\mathcal{E}_b}(v_i)$  and equating to zero gives the following equation

$$0 = -\deg_{\mathcal{E}_b}(u_i) + \deg_{\mathcal{E}_b}(v_i) + \deg_{\mathcal{E}_r}(u_i) - \deg_{\mathcal{E}_r}(v_i). \quad (5)$$

By substituting the equations 3 and 4 into 5, we have

$$\begin{aligned} 0 &= - \left( \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \in E_s}} m_s + \sum_{\substack{y_i \rightarrow y'_i \in \mathcal{R} \\ E_i \in \mathcal{E}_b}} m_i \right) + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \notin E_s \\ v_i \in E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \in E_s}} m_s + \sum_{\substack{y_i \rightarrow y'_i \in \mathcal{R} \\ E_i \in \mathcal{E}_b}} m_i \\ &+ \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \in E_s}} m_s + \sum_{\substack{y_i \rightarrow y'_i \in \mathcal{R} \\ E_i \in \mathcal{E}_r}} m_i - \left( \sum_{\substack{s \in \mathcal{S}_r \\ u_i \notin E_s \\ v_i \in E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \in E_s}} m_s + \sum_{\substack{y_i \rightarrow y'_i \in \mathcal{R} \\ E_i \in \mathcal{E}_r}} m_i \right) \\ &= - \sum_{\substack{s \in \mathcal{S}_b \\ u_i \in E_s \\ v_i \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_b \\ u_i \notin E_s \\ v_i \in E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_r \\ u_i \in E_s \\ v_i \notin E_s}} m_s - \sum_{\substack{s \in \mathcal{S}_r \\ u_i \notin E_s \\ v_i \in E_s}} m_s. \end{aligned}$$

On the other hand, if  $i = i_0$ , then in a similar way we can show that the equations  $\deg_{\mathcal{E}_r}(u_{i_0}) = \deg_{\mathcal{E}_b}(u_{i_0}) + k$  and  $\deg_{\mathcal{E}_r}(v_{i_0}) = \deg_{\mathcal{E}_b}(v_{i_0})$  together with 3 and 4 imply

$$k = - \sum_{\substack{s \in \mathcal{S}_b \\ u_{i_0} \in E_s \\ v_{i_0} \notin E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_b \\ u_{i_0} \notin E_s \\ v_{i_0} \in E_s}} m_s + \sum_{\substack{s \in \mathcal{S}_r \\ u_{i_0} \in E_s \\ v_{i_0} \notin E_s}} m_s - \sum_{\substack{s \in \mathcal{S}_r \\ u_{i_0} \notin E_s \\ v_{i_0} \in E_s}} m_s.$$

This proves the above two claims that allow for the following equalities:

$$\begin{aligned}
k\kappa_{i_0}x^{y_{i_0}} &= \kappa_{i_0}x^{y_{i_0}} \left( \sum_{s \in \mathcal{S}_b} m_s \gamma_{si_0} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{si_0} \right) \\
&+ \sum_{\substack{y_i \rightarrow y'_i \\ i \neq i_0}} \kappa_i x^{y_i} \left( \sum_{s \in \mathcal{S}_b} m_s \gamma_{si} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{si} \right) \\
&= \sum_{y_i \rightarrow y'_i} \kappa_i x^{y_i} \left( \sum_{s \in \mathcal{S}_b} m_s \gamma_{si} - \sum_{s \in \mathcal{S}_r} m_s \gamma_{si} \right) \\
&= \sum_{y_i \rightarrow y'_i} \kappa_i x^{y_i} \sum_{s \in \mathcal{S}_b} m_s \gamma_{si} - \sum_{y_i \rightarrow y'_i} \kappa_i x^{y_i} \sum_{s \in \mathcal{S}_r} m_s \gamma_{si} \\
&= \sum_{s \in \mathcal{S}_b} m_s \sum_{y_i \rightarrow y'_i} \gamma_{si} \kappa_i x^{y_i} - \sum_{s \in \mathcal{S}_r} m_s \sum_{y_i \rightarrow y'_i} \gamma_{si} \kappa_i x^{y_i} \\
&= \sum_{s \in \mathcal{S}_b} m_s \dot{x}_s - \sum_{s \in \mathcal{S}_r} m_s \dot{x}_s \\
&= \sum_{E_s \in \mathcal{E}_b} \dot{x}_s - \sum_{E_s \in \mathcal{E}_r} \dot{x}_s.
\end{aligned}$$

□

The same result is achieved with a near identical proof where “ $u_i$  is almost balanced” is replaced with “ $v_i$  is almost balanced”. The statement of the lemma with this change is therefore given without proof. It should be noted, however, that the difference of sums in Lemma 2.5 is “red minus blue” and not “blue minus red” as it is in Lemma 2.4. This is due to our choice in the definition of almost balanced that a vertex has red degree that is strictly larger than its blue degree.

**Lemma 2.5.** *Let  $N$  be a 0,1-network. If  $v_i \in V(\mathcal{H})$  is almost balanced with respect to the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$ , then  $k\kappa_i x^{y_i} = \sum_{E_s \in \mathcal{E}_r} \dot{x}_s - \sum_{E_s \in \mathcal{E}_b} \dot{x}_s$  for some positive integer  $k$ .*

**Corollary 2.6.** *Let  $N$  be a 0,1-network. If either  $u_i \in V(\mathcal{H})$  or  $v_i \in V(\mathcal{H})$  is almost balanced, then  $x^{y_i} \in \mathbb{I}(N)$ .*

Now that we have established a proof of the main lemma that relates the combinatorics of a reaction network to its associated algebra, what follows are three statements that follow from the definition of almost balanced and Corollary 2.6. The first result relates the two vertices belonging to a reaction edge via the almost balanced condition, the second gives a necessary condition for vertices that cannot be almost balanced, and the third is a necessary condition for the steady state ideal to be monomial. Note that Propositions 2.7 and 2.8 do not require the network to be a 0,1-network. This is because the statements are solely about the combinatorial structure of the network and not the implication it has for the associated algebra.

**Proposition 2.7.** *Let  $N$  be a chemical reaction network containing the reaction  $y \rightarrow y'$  such that  $y' \neq \emptyset$ . Then the vertex  $u \in V(\mathcal{H})$  corresponding to  $y$  in the reaction  $y \rightarrow y'$  is almost balanced if and only if the vertex  $v \in V(\mathcal{H})$  corresponding to  $y'$  in the reaction  $y \rightarrow y'$  is almost balanced.*

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a chemical reaction network such that  $y \rightarrow y' \in \mathcal{R}$  and  $y' \neq \emptyset$ . Let  $u$  and  $v$  be distinct vertices in  $V(\mathcal{H})$  corresponding to the complexes  $y$  and  $y'$  of the reaction  $y \rightarrow y'$ , respectively. Since  $y' \neq \emptyset$ , then the hyperedge  $e \in E(\mathcal{H})$  corresponding to the reaction  $y \rightarrow y'$  is nonempty, that is,  $e = \{u, v\}$ ; see Figure 9. Assume  $w \in e$  is almost balanced. We show that  $w' \in e \setminus \{w\}$  is almost balanced.

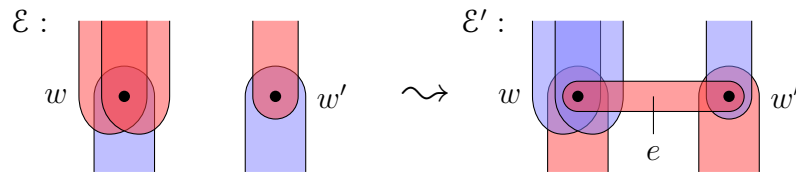


Fig. 9: Obtaining a bi-coloring of  $\mathcal{E}'$  from a bi-coloring of  $\mathcal{E}$  by swapping colors and adding red reactions edges.

Suppose  $w$  is almost balanced with respect to the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over the edges in  $E(\mathcal{H})$ . Hence,  $\deg_{\mathcal{E}_r}(w) = \deg_{\mathcal{E}_b}(w) + k$  for some positive integer  $k$  and  $\deg_{\mathcal{E}_r}(z) = \deg_{\mathcal{E}_b}(z)$  for all  $z \in V(\mathcal{H}) \setminus \{w\}$ . Set  $\mathcal{E}' = \mathcal{E} \sqcup \{e^{(k)}\}$ . Bi-color the edges in  $\mathcal{E}'$  so that  $\mathcal{E}'_r = \mathcal{E}_b \sqcup \{e^{(k)}\}$  and  $\mathcal{E}'_b = \mathcal{E}_r$ . Since  $e$  covers only  $w$  and  $w'$ , then  $\deg_{\mathcal{E}'_r}(w) = \deg_{\mathcal{E}_b}(w) + k = \deg_{\mathcal{E}_r}(w) = \deg_{\mathcal{E}'_b}(w)$ ,  $\deg_{\mathcal{E}'_r}(w') = \deg_{\mathcal{E}_b}(w') + k = \deg_{\mathcal{E}_r}(w') + k = \deg_{\mathcal{E}'_b}(w') + k$ , and  $\deg_{\mathcal{E}'_r}(z) = \deg_{\mathcal{E}_b}(z) = \deg_{\mathcal{E}_r}(z) = \deg_{\mathcal{E}'_b}(z)$  for all  $z \in V(\mathcal{H}) \setminus \{w, w'\}$ . Therefore,  $w'$  is almost balanced with respect to  $\mathcal{E}'$ .  $\square$

A reaction  $y_i \rightarrow y'_i \in \mathcal{R}$  is said to be *reversible* if  $y'_i \rightarrow y_i \in \mathcal{R}$ . In a reversible reaction we require both complexes belonging to a reversible reaction to be nonempty since we are not considering reactions of the form  $\emptyset \rightarrow y$ . Vertices in the network hypergraph that come from complexes belonging to a reversible reaction are never almost balanced.

**Proposition 2.8.** Let  $N$  be a chemical reaction network. If  $y \rightarrow y' \in \mathcal{R}$  is reversible, then the vertices in  $V(\mathcal{H}_N)$  corresponding to either  $y$  or  $y'$  in the reaction  $y \rightarrow y'$  or  $y' \rightarrow y$  is not almost balanced.

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a chemical reaction network such that  $\mathcal{R}$  contains the reactions  $y \rightarrow y'$  and  $y' \rightarrow y$ . Let  $u, v, u', v'$  be vertices in  $V(\mathcal{H}_N)$  such that  $u$  and  $v$  correspond to  $y$  and  $y'$  in the reaction  $y \rightarrow y'$ , respectively, and  $u'$  and  $v'$  correspond to  $y'$  and  $y$  in the reaction  $y' \rightarrow y$ , respectively. Note that  $u$  and  $v'$  correspond to the same complex  $y$  but in different reactions. Thus, for any  $s \in \text{supp}(y)$ ,  $E_s$  covers  $u$  if and only if  $E_s$  covers  $v'$ ; see the yellow edge in Figure 10. Similarly, both  $u'$  and  $v$  correspond to  $y'$  so, for any  $s \in \text{supp}(y')$ ,  $E_s$  covers  $u'$  if and only if  $E_s$  covers  $v$ ; see the orange edge in Figure 10.

By the symmetry of the reversible reactions, it is enough to show that  $u$  is not almost balanced. Seeking a contradiction, suppose  $u$  is almost balanced with respect to the

bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over the edges in  $E(\mathcal{H}_N)$ . Then  $\deg_{\mathcal{E}_r}(u) = \deg_{\mathcal{E}_b}(u) + k$  for some positive integer  $k$  and  $\deg_{\mathcal{E}_r}(w) = \deg_{\mathcal{E}_b}(w)$  for all  $w \in V(\mathcal{H}_N) \setminus \{u\}$ .

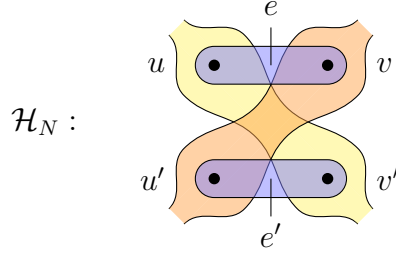


Fig. 10: A subhypergraph of a network hypergraph corresponding to a network containing a reversible reaction.

For each  $w \in V(\mathcal{H})$  and  $c \in \{r, b\}$ , let  $\sigma_c(w)$  be the number of species edges in  $\mathcal{E}_c$  that cover  $w$  and let  $\rho_c(w)$  be the number of reaction edges in  $\mathcal{E}_c$  that cover  $w$ . By a previous observation, for any  $c \in \{r, b\}$ ,  $\sigma_c(u) = \sigma_c(v')$  and  $\sigma_c(u') = \sigma_c(v)$ . By definition of the reaction edge  $e := \{u, v\}$ ,  $e$  covers  $u$  if and only if  $e$  covers  $v$ . Similarly, the reaction edge  $e' := \{u', v'\}$  covers  $u'$  if and only if  $e'$  covers  $v'$ . Thus, for any  $c \in \{r, b\}$ ,  $\rho_c(u) = \rho_c(v)$  and  $\rho_c(u') = \rho_c(v')$ . Since  $\deg_{\mathcal{E}_r}(w) = \deg_{\mathcal{E}_b}(w)$ , for any  $w \in \{v, u', v'\}$ , then  $\sigma_r(w) + \rho_r(w) = \sigma_b(w) + \rho_b(w)$  or, equivalently,  $\sigma_r(w) - \sigma_b(w) = \rho_b(w) - \rho_r(w)$ . Therefore,

$$\begin{aligned}
k &= \deg_{\mathcal{E}_r}(u) - \deg_{\mathcal{E}_b}(u) \\
&= \sigma_r(u) + \rho_r(u) - \sigma_b(u) - \rho_b(u) \\
&= \sigma_r(v') + \rho_r(v) - \sigma_b(v') - \rho_b(v) \\
&= \rho_b(v') + \sigma_b(v) - \rho_r(v') - \sigma_r(v) \\
&= \rho_b(u') + \sigma_b(u') - \rho_r(u') - \sigma_r(u') \\
&= \deg_{\mathcal{E}_b}(u') - \deg_{\mathcal{E}_r}(u') \\
&= 0.
\end{aligned}$$

This is a contradiction since  $k$  is a strictly positive integer so the assumption that  $u$  is almost balanced must be false.  $\square$

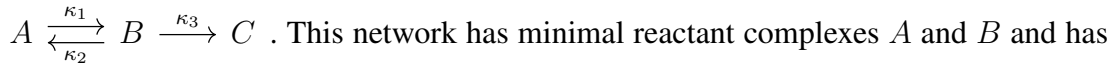
Consider two networks  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  and  $N' = (\mathcal{S}', \mathcal{C}', \mathcal{R}')$  such that  $\mathcal{S} \subseteq \mathcal{S}'$ . For complexes  $y \in \mathcal{C}$  and  $y' \in \mathcal{C}'$ , we write  $y \mid y'$  if the coefficient of  $s$  in  $y$  is at most the coefficient of  $s$  in  $y'$  for each species  $s \in \mathcal{S}$ . Divides as a relation between complexes matches our intuition of divides in the usual sense:  $y \mid y'$  if and only if  $x^y \mid x^{y'}$ . Similar notation is found in [9]. A complex  $y \in \mathcal{C}$  is said to be *minimal* in the network  $N$  if, for any  $y' \in \mathcal{C}$ , either 1)  $y \mid y'$  or 2)  $y \nmid y'$  and  $y' \nmid y$ .

**Theorem 2.9.** *Let  $N$  be a 0,1-network and suppose  $y_1, \dots, y_k$  are the minimal reactants of  $N$ . If, for all  $i = 1, \dots, k$ , either  $u_i$  or  $v_i$  in  $\mathcal{H}(N)$  is almost balanced, then*

$$\mathbb{I}(N) = \langle x^{y_1}, \dots, x^{y_k} \rangle.$$

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a 0,1-network and let  $y_1, \dots, y_k$  be the minimal reactants of  $N$ . By Proposition 1.3,  $\mathbb{I}(N) \subseteq \langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle$ . Without loss of generality, suppose the vertices  $u_1, \dots, u_k$  in  $\mathcal{H}(N)$  are almost balanced. By Corollary 2.6,  $x^{y_i} \in \mathbb{I}(N)$  for  $i = 1, \dots, k$ . Thus,  $\langle x^{y_i}, \dots, x^{y_k} \rangle \subseteq \mathbb{I}(N) \subseteq \langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle$ . We show  $\langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle \subseteq \langle x^{y_i}, \dots, x^{y_k} \rangle$ . Let  $y \rightarrow y'$  be any reaction in  $\mathcal{R}$ . Since  $y_1, \dots, y_k$  are the minimal reactants of  $N$ , then there is some  $\ell \in \{1, \dots, k\}$  such that  $y_\ell \mid y$ . Necessarily,  $x^{y_\ell} \mid x^y$  and  $y - y_\ell$  has nonnegative coefficients. Since  $x^{y_\ell} \in \mathbb{I}(N)$ , then  $x^y = x^{y_\ell} x^{y - y_\ell} \in \langle x^{y_1}, \dots, x^{y_k} \rangle$ . Thus,  $\langle x^{y_i} : y_i \rightarrow y'_i \in \mathcal{R} \rangle \subseteq \langle x^{y_i}, \dots, x^{y_k} \rangle$  and, consequently,  $\mathbb{I}(N) = \langle x^{y_1}, \dots, x^{y_k} \rangle$ .  $\square$

The converse of the previous theorem does not hold. Consider the 0,1-network:



This network has minimal reactant complexes  $A$  and  $B$  and has monomial steady state ideal  $\langle x_A, x_B \rangle$ . While the minimal reactant  $B$  corresponds to an almost balanced vertex in  $V(\mathcal{H})$ , the minimal reactant  $A$  does not. The minimal reactant



$A$  does not correspond to an almost balanced vertex by Proposition 2.8. In Chapter 4, we investigate a condition on the ideal  $\mathbb{I}(N)$  that guarantees the converse is true.

### 3 IDEAL PRESERVING OPERATIONS

Two different networks can have the same steady state ideal. Figure 11 depicts two chemical reaction networks with the same steady state ideal. In this chapter, we discuss three ways that a chemical reaction network can be modified so that the network ideal is preserved. The operations on the network we consider are: adding a single species to the reactant complex of a reaction, adding a single species to the product complex of a reaction, and adding a single reaction.

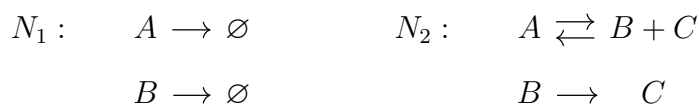


Fig. 11: Two chemical reaction networks with steady state ideal  $\langle x_A, x_B \rangle$ .

Since the goal of this chapter is to preserve the steady state ideal of a chemical reaction network, we seek to preserve the generators of the steady state ideal upon performing one of the three previously mentioned operations. In particular, if there are monomial generators in the ideal, then we wish to preserve these monomials. In Chapter 2, we saw that almost balanced vertices in the network hypergraph imply the existence of monomials in the steady state ideal. Thus, preserving monomials in the steady state ideal can be achieved by preserving almost balanced vertices in the network hypergraph.

First, let us illustrate how changes in the network affect its associated hypergraph. Observe that from Figure 11,  $N_2$  is obtained from  $N_1$  by the following sequence of operations: 1) add  $C$  to the product of the reaction  $A \rightarrow \emptyset$ , 2) add  $C$  to the product of the reaction  $B \rightarrow \emptyset$ , 3) add  $B$  to the product of the reaction  $A \rightarrow C$ , and 4) add the reaction  $B + C \rightarrow A$ . In general, we see that adding to a network will introduce vertices or edges in the hypergraph or cause edges to “expand”; that is, the edges present in the hypergraph will cover more vertices in the newly obtained hypergraph; see Figure 12.

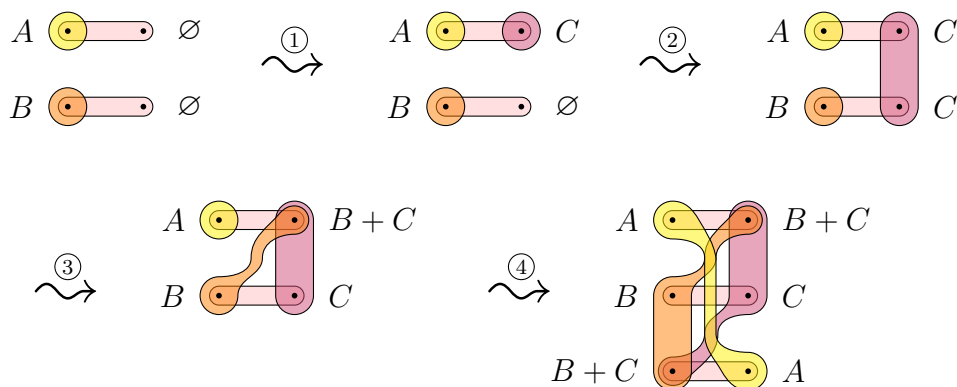


Fig. 12: A sequence of network hypergraphs after performing four network operations.

In each hypergraph from Figure 12, the vertex labeled  $B$  remains almost balanced: for the first three hypergraphs,  $E_B$  is a singleton so take  $\mathcal{E}_r = \{E_B\}$  and  $\mathcal{E}_b = \emptyset$ , for the fourth hypergraph, take  $\mathcal{E}_r = \{E_A, E_B\}$  and  $\mathcal{E}_b = \{E_1\}$  where  $E_1$  corresponds to the reaction  $A \rightarrow B + C$ , and for the fifth hypergraph, take  $\mathcal{E}_r = \{E_A, E_B\}$  and  $\mathcal{E}_b = \{E_1, E_3\}$  where  $E_3$  corresponds to the reaction  $B + C \rightarrow A$ . Therefore, the sequence of operations performed on  $N_1$  preserves the fact that  $B$  is almost balanced. Hence, the monomial  $x_B$  is in the steady state ideal of each corresponding network. The same, however, cannot be said about  $x_A$ . The vertex whose label is  $A$  only remains almost balanced after the first three operations since  $E_A$  is a singleton but is not almost balanced after the fourth operation. Thus,  $x_A$  is contained in the steady state ideal of  $N_2$  for other reasons that cannot be detected by the almost balanced condition; these reasons will be discussed in Chapter 4. Indeed, Proposition 2.8 confirms that the vertices covered by  $E_1$  corresponding to the complex  $A$  and  $B + C$  are almost balanced in the network hypergraph associated to  $N_2$ .

The operations we consider are always done reaction-wise. Thus, we need to discuss how to add and remove reactions. Removing the reaction  $y \rightarrow y'$  belonging to the network  $N$  will be denoted  $N \setminus \{y \rightarrow y'\}$ . Adding any reaction  $y \rightarrow y'$  to the network  $N$

will be denoted  $N \cup \{y \rightarrow y'\}$ . From the definition of a chemical reaction network, every complex must belong to some reaction. Then if deleting the reaction  $y \rightarrow y'$  from  $N$  results to either  $y$  or  $y'$  no longer belonging to some reaction, then they too must be deleted in the process. On the other hand, when adding an arbitrary reaction  $y \rightarrow y'$  to the network  $N$ , it is possible that neither  $y$  nor  $y'$  are complexes in the original network  $N$ . In such a case, both the reaction and complexes are added to the network. We can think of adding a species  $s$  to a reactant complex as deleting and then adding a reaction; for example, adding a species  $s$  to the reactant complex  $y$  belonging to the reaction  $y \rightarrow y'$  can now be denoted by  $(N \setminus \{y \rightarrow y'\}) \cup \{y + s \rightarrow y'\}$ . As seen in Figure 13, there are two cases to consider: 1)  $y$  belongs to one reaction or 2)  $y$  belongs to more than one reaction. The main distinction with the two cases is that if  $y$  belongs to only one reaction, then  $y$  must be deleted and replaced with  $y + s$ , meaning  $y$  by itself no longer appears as a complex in the network, whereas if  $y$  belongs to more than one reaction then  $y$  will remain as a complex in the network. Also, note that in either of the cases above,  $y + s$  may or may not have been a complex in the network prior to adding the species  $s$ . If  $y + s$  is already a complex, then by adding  $s$  to  $y$ , we do not require adding a new complex to the network, only a new reaction. If not, then  $y + s$  must be added to the set of complexes. The addition of a species to a product complex is done similarly.

**Remark 3.1.** When obtaining a network  $N' = (\mathcal{S}', \mathcal{C}', \mathcal{R}')$  from a network  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ , it is possible that  $N'$  contains a species that  $N$  does not, that is,  $\mathcal{S} \subsetneq \mathcal{S}'$ . Thus,  $\mathbb{I}(N)$  and  $\mathbb{I}(N')$  can reside in different rings. Namely,  $\mathbb{I}(N)$  is an ideal of the ring  $\mathbb{Q}(\kappa_1, \dots, \kappa_{n_{\mathcal{R}}})[x_1, \dots, x_{n_{\mathcal{S}}}]$  and, if  $N'$  is obtained from  $N$  by adding a species to a complex in the  $i$ th reaction, then  $\mathbb{I}(N')$  is an ideal of the ring  $\mathbb{Q}(\kappa_1, \dots, \kappa_{i-1}, \kappa_{i+1}, \dots, \kappa_{n_{\mathcal{R}}}, \kappa'_i)[x_1, \dots, x_{n_{\mathcal{S}'}}]$  where  $\kappa'_i$  is the rate constant for the newly added reaction. Otherwise, if  $N'$  is obtained from  $N$  by adding a reaction, then

$\mathbb{I}(N') \subseteq \mathbb{Q}(\kappa_1, \dots, \kappa_{n_{\mathcal{R}}}, \kappa'_i)[x_1, \dots, x_{n_{\mathcal{S}'}}]$ . In either case, when we write  $\mathbb{I}(N') = \mathbb{I}(N)$ , we mean they are equivalent in the larger ring  $\mathbb{Q}(\kappa_1, \dots, \kappa_{n_{\mathcal{R}}}, \kappa'_i)[x_1, \dots, x_{n_{\mathcal{S}'}}]$ .

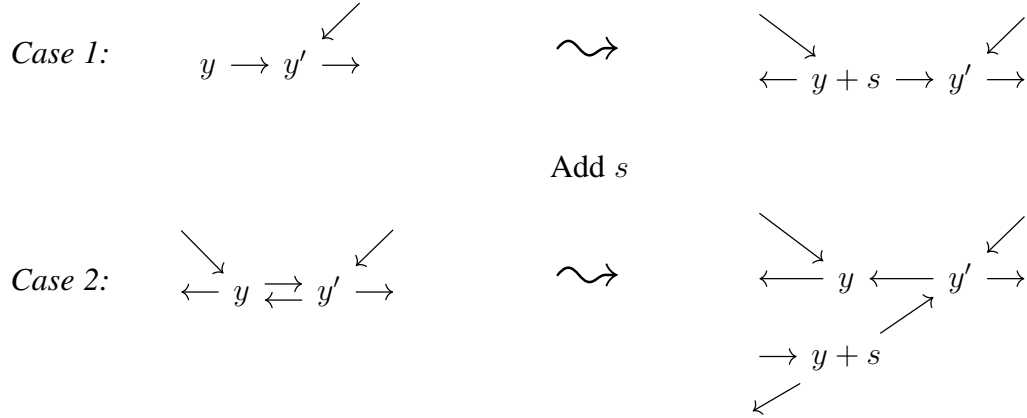


Fig. 13: Examples of the addition of species  $s$  to the complex  $y$ .

We now present the first ideal preserving operation of the thesis.

**Theorem 3.2.** (Adding species to a product complex) *Let  $N$  be a 0,1-network and suppose  $u_i \in V(\mathcal{H}_N)$  is almost balanced with respect to the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over the edges in  $E(\mathcal{H}_N)$ . Suppose  $s_k$  is a species such that  $s_k \notin \text{supp}(y'_i)$ , and let  $N' = (N \setminus \{y_i \rightarrow y'_i\}) \cup \{y_i \rightarrow y'_i + s_k\}$ . If  $E_{s_k} \notin \mathcal{E}$ , then  $\mathbb{I}(N') = \mathbb{I}(N)$ .*

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a 0,1-network and let  $u_i \in V(\mathcal{H}_N)$  be almost balanced with respect to the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H}_N)$ . Suppose  $s_k$  is a species such that  $s_k \notin \text{supp}(y'_i)$  and  $E_{s_k} \notin \mathcal{E}$ . Obtain a network  $N' = (\mathcal{S}', \mathcal{C}', \mathcal{R}')$  from  $N$  by removing the reaction  $y_i \xrightarrow{\kappa_i} y'_i$  and adding the reaction  $y_i \xrightarrow{\kappa'_i} y'_i + s_k$ . Since  $s_k \notin \text{supp}(y'_i)$ , then the coefficient of  $s_k$  in the complex  $y'_i + s_k$  is 1. Hence,  $N'$  is a 0,1-network.

The modification of  $N$  by a single species implies that  $\mathcal{S} \subseteq \mathcal{S}'$ . For  $\ell = 1, \dots, n_{\mathcal{S}}$ , let  $\dot{x}_{s_\ell}$  denote the steady state polynomial for species  $s_\ell$  in  $\mathcal{S}$  and, for  $\ell = 1, \dots, n_{\mathcal{S}'}$ , let  $\dot{x}'_{s_\ell}$  denote the steady state polynomial for species  $s_\ell$  in  $\mathcal{S}'$ . To prove  $\mathbb{I}(N') = \mathbb{I}(N)$  it is sufficient to show  $\dot{x}'_{s_\ell} \in \mathbb{I}(N)$  for all  $1 \leq \ell \leq n_{\mathcal{S}'}$  and  $\dot{x}_{s_\ell} \in \mathbb{I}(N')$  for all  $1 \leq \ell \leq n_{\mathcal{S}}$ .

Note that if  $y_j \rightarrow y'_j$  is a reaction in  $\mathcal{R}$  different from  $y_i \rightarrow y'_i$ , then  $y_j \rightarrow y'_j$  is also in  $\mathcal{R}'$ , and the stoichiometric coefficients of the species do not change for  $y_j \rightarrow y'_j$  in  $N'$ . On the other hand, the stoichiometric coefficient of the species  $s_k$  in the reaction  $y_i \rightarrow y'_i + s_k$  has increased by one. Therefore,  $\dot{x}'_s = \dot{x}_s$  for any  $s \in \mathcal{S} \setminus \{s_k\}$  and  $\dot{x}'_{s_k} = \dot{x}_{s_k} + \kappa'_i x^{y_i}$  where  $\dot{x}_{s_k} = 0$  if  $s_k \notin \mathcal{S}$ . Hence,  $\dot{x}'_s \in \mathbb{I}(N)$  and  $\dot{x}_s \in \mathbb{I}(N')$  for any  $s \in \mathcal{S} \setminus \{s_k\}$ .

It remains to show that  $\dot{x}'_{s_k} \in \mathbb{I}(N)$  and  $\dot{x}_{s_k} \in \mathbb{I}(N')$ . Since  $u_i$  is almost balanced, then  $x^{y_i} \in \mathbb{I}(N)$  by Corollary 2.6. Then  $\dot{x}'_{s_k} = \dot{x}_{s_k} + \kappa'_i x^{y_i} \in \mathbb{I}(N)$ .

To show  $\dot{x}_{s_k} \in \mathbb{I}(N')$ , notice that  $\mathcal{H}_N$  and  $\mathcal{H}_{N'}$  have the same vertex set, since  $N$  and  $N'$  have the same number of reactions, moreover, the only difference between  $E(\mathcal{H}_N)$  and  $E(\mathcal{H}_{N'})$  is that  $E_{s_k}$  is extended to include  $v_i$ . Since  $E_{s_k} \notin \mathcal{E}$ , then  $u_i \in V(\mathcal{H}_{N'})$ , the vertex corresponding to the reactant complex of  $y_i \rightarrow y'_i + s_k$ , is almost balanced in  $\mathcal{H}_{N'}$  with respect to  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  where  $\mathcal{E}'_r = \mathcal{E}_r$  but as a multiset over  $E(\mathcal{H}_{N'})$  and  $\mathcal{E}'_b = \mathcal{E}_b$  but as a multiset over  $E(\mathcal{H}_{N'})$ . Thus, by Corollary 2.6,  $x^{y_i} \in \mathbb{I}(N')$  and, hence,  $\dot{x}_{s_k} = \dot{x}'_{s_k} - \kappa'_i x^{y_i} \in \mathbb{I}(N')$ . Therefore,  $\dot{x}'_\ell \in \mathbb{I}(N)$  for all  $1 \leq \ell \leq n_{S'}$  and  $\dot{x}_\ell \in \mathbb{I}(N')$  for all  $1 \leq \ell \leq n_S$  or, equivalently,  $\mathbb{I}(N') = \mathbb{I}(N)$ .  $\square$

**Example 3.3.** Consider the running example in Figure 14 to demonstrate how adding a species to the product of a reaction under the conditions described in Theorem 3.2 preserves the steady state ideal but changes the network hypergraph. Reaction edges are drawn as line segments in the network hypergraph to emphasize their distinction from the species edges.

By Proposition 1.3,  $\mathbb{I}(N) \subseteq \langle x_A, x_B \rangle$ . Indeed,  $\mathbb{I}(N)$  equals  $\langle x_A, x_B \rangle$  but requires verification. The vertex labeled  $A$  that is incident to  $E_3$  is almost balanced with respect to the multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H}_N)$  where  $\mathcal{E}_r = \{E_3\}$  and  $\mathcal{E}_b = \{E_C\}$ . By Corollary 2.6,  $x_A \in \mathbb{I}(N)$ . Further, notice that  $x_B = \frac{\kappa_1}{\kappa_2 \kappa_3} \dot{x}_C - \frac{1}{\kappa_2} \dot{x}_B$  so  $x_B \in \mathbb{I}(N)$ . Hence,  $\mathbb{I}(N) = \langle x_A, x_B \rangle$ . Since  $E_B \notin \mathcal{E}$ , then  $\mathbb{I}(N_1) = \mathbb{I}(N)$  where  $N_1 = (N \setminus \{A \rightarrow C\}) \cup \{A \rightarrow B + C\}$ ; see Figure 15.



multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H}_N)$ . Suppose  $s_k$  is a species, not necessarily in  $\mathcal{S}$ , such that  $s_k \notin \text{supp}(y_i)$  and  $E_{s_k} \notin \mathcal{E}$ . Obtain a network  $N' = (\mathcal{S}', \mathcal{C}', \mathcal{R}')$  from  $N$  by removing the reaction  $y_i \rightarrow y'_i$  and adding the reaction  $y_i + s_k \rightarrow y'_i$ . Since  $s_k \notin \text{supp}(y_i)$ , then the coefficient of  $s_k$  in  $y_i + s_k$  is one so  $N'$  is a 0,1-network.

Let  $S'_1 = \text{supp}(y_i + s_k + y'_i)$  and  $S'_2 = \mathcal{S}' \setminus S'_1$ . If  $s \in S'_2$ , then  $s$  is not involved in either of the reactions  $y_i \rightarrow y'_i$  or  $y_i + s_k \rightarrow y'_i$ . Necessarily, the rate at which the concentration of  $s$  changes in  $N$  remains the same as in  $N'$ , that is,  $\dot{x}'_s = \dot{x}_s$ . Thus,  $\dot{x}'_s \in \mathbb{I}(N)$  and  $\dot{x}_s \in \mathbb{I}(N')$  for all  $s \in S'_2$ .

On the other hand, if  $s \in S'_1$ , then  $\dot{x}'_s$  and  $\dot{x}_s$  differ by the rate at which the concentration of  $s$  changes by the reaction  $y_i \rightarrow y'_i$  and the rate at which the concentration  $s$  changes by the reaction  $y_i + s_k \rightarrow y'_i$ . Specifically,  $\dot{x}'_s - \gamma'_{si} \kappa'_i x^{y_i + s_k} = \dot{x}_s - \gamma_{si} \kappa_i x^{y_i}$ . Since  $u_j \in V(\mathcal{H}_N)$  is almost balanced, then  $x^{y_j} \in \mathbb{I}(N)$ . Since  $y_j \mid y_i$ , then  $y_i - y_j$  is nonnegative and  $x^{y_i} = x^{y_j} x^{y_i - y_j} \in \mathbb{I}(N)$  and so  $x^{y_i + s_k} = x^{y_i} x_{s_k} \in \mathbb{I}(N)$ . Thus,  $\dot{x}'_s = \dot{x}_s - \gamma_{si} \kappa_i x^{y_i} + \gamma'_{si} \kappa'_i x^{y_i + s_k} \in \mathbb{I}(N)$ .

Now notice that  $\mathcal{H}_N$  and  $\mathcal{H}_{N'}$  have the same vertex set, moreover, the only difference between  $E(\mathcal{H}_N)$  and  $E(\mathcal{H}_{N'})$  is that  $E_{s_k}$  is extended to include  $u_i$ . Since  $E_{s_k} \notin \mathcal{E}$ , then  $u_j \in V(\mathcal{H}_{N'})$ , the vertex corresponding to the reactant complex of  $y_j \rightarrow y'_j$ , is almost balanced in  $\mathcal{H}_{N'}$  with respect to  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  where  $\mathcal{E}'_r = \mathcal{E}_r$  but as a multiset over  $E(\mathcal{H}_{N'})$  and  $\mathcal{E}'_b = \mathcal{E}_b$  but as a multiset over  $E(\mathcal{H}_{N'})$ . By Corollary 2.6,  $x^{y_j} \in \mathbb{I}(N')$ . Since  $x^{y_i} = x^{y_j} x^{y_i - y_j} \in \mathbb{I}(N')$ , then we have  $\dot{x}_s = \dot{x}'_s - \gamma'_{si} \kappa'_i x^{y_i + s_k} + \gamma_{si} \kappa_i x^{y_i} \in \mathbb{I}(N')$ . Thus,  $\dot{x}_s \in \mathbb{I}(N')$  and  $\dot{x}'_s \in \mathbb{I}(N)$  for all  $s \in S'_1$ . Therefore,  $\mathbb{I}(N') = \mathbb{I}(N)$ .  $\square$

**Example 3.5.** From Figure 15, the reactions  $A \rightarrow B$  and  $A \rightarrow B + C$  are distinct and certainly  $A \mid A$ . Since the vertex  $u_3$ , labeled  $A$  and incident to  $E_3$ , is almost balanced in  $\mathcal{H}_{N_1}$ , then obtaining a network  $N_2$  from  $N_1$  by adding  $B$  to the reactant of  $A \rightarrow B$  preserves the steady state ideal; see Figure 16. The vertex  $u_3 \in V(\mathcal{H}_{N_2})$ , labeled  $A$  and incident to  $E_3$ , remains almost balanced in  $\mathcal{H}_{N_2}$  with the same colored edges as for



$u_3 \in V(\mathcal{H}_{N_1})$ , so  $x_A \in \mathbb{I}(N_2)$ . Notice the vertex  $u_2 \in V(\mathcal{H}_{N_1})$ , labeled  $B$  and incident to  $E_2$ , is almost balanced in  $\mathcal{H}_{N_2}$ . To be specific,  $u_2$  is almost balanced in  $\mathcal{H}_{N_2}$  with respect to the bi-colored edgeset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H}_{N_2})$  where  $\mathcal{E}_r = \{E_B\}$  and  $\mathcal{E}_b = \{E_1, E_C\}$ . Hence,  $x_B \in \mathbb{I}(N_2)$ .

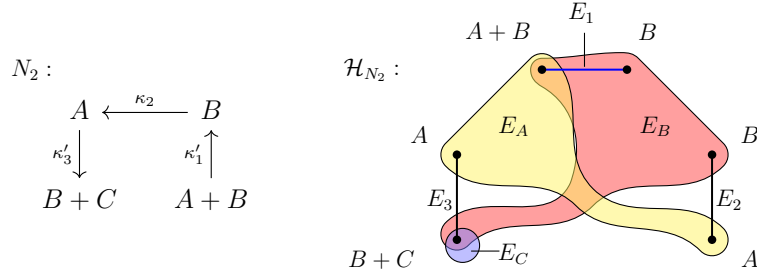


Fig. 16: A network  $N_2$ , obtained from  $N_1$  in Figure 15 by adding  $B$  to the reactant complex of the reaction  $A \rightarrow B$ , and its network hypergraph.

**Theorem 3.6.** (Adding reactions) Let  $N$  be a 0,1-network. Suppose there is some vertex  $v_i \in V(\mathcal{H}_N)$  such that  $v_i$  is almost balanced with respect to the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H}_N)$ . Let  $N' = N \cup \{y_i \rightarrow \emptyset\}$ . If  $E_i \notin \mathcal{E}$ , then  $\mathbb{I}(N') = \mathbb{I}(N)$ .

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a 0,1-network and suppose  $v_i \in V(\mathcal{H}_N)$  is almost balanced with respect to a bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  over  $E(\mathcal{H}_N)$  such that  $E_i \notin \mathcal{E}$ . Obtain a network  $N' = (\mathcal{S}', \mathcal{C}', \mathcal{R}')$  from  $N$  by adding the reaction  $y_i \xrightarrow{\kappa'_i} \emptyset$ . Since  $N'$  and  $N$  contain the same nonempty complexes and  $N$  is a 0,1-network, then  $N'$  must also be a 0,1-network.

Let  $S'_1 = \text{supp}(y_i)$  and  $S'_2 = \mathcal{S} \setminus S'_1$ . If  $s \in S'_2$ , then  $s$  is not involved in the reaction  $y_i \rightarrow \emptyset$ . Necessarily, the rate of change of the concentration of species  $s$  in  $N$  is the same as the rate of change of its concentration in  $N'$ . That is,  $\dot{x}'_s = \dot{x}_s$  for any  $s \in S'_2$ . Thus,  $\dot{x}'_s \in \mathbb{I}(N)$  and  $\dot{x}_s \in \mathbb{I}(N')$  for all  $s \in S'_2$ .

Now, if  $s \in S'_1$ , then  $\dot{x}'_s$  and  $\dot{x}_s$  differ by the rate at which the concentration of  $s$  changes by the reaction  $y_i \rightarrow \emptyset$ . More specifically,  $\dot{x}'_s = \dot{x}_s - \kappa'_i x^{y_i}$ . Since  $v_i$  is almost

balanced in  $\mathcal{H}_N$ , then  $x^{y_i} \in \mathbb{I}(N)$  by Corollary 2.6. Thus,  $\dot{x}'_s = \dot{x}_s - \kappa'_i x^{y_i} \in \mathbb{I}(N)$  for any  $s \in S'_1$ .

Notice that the vertex set of  $\mathcal{H}_{N'}$  is  $V(\mathcal{H}) \cup \{u, v\}$  where  $u$  and  $v$  correspond to the complexes  $y_i$  and  $\emptyset$  in the reaction  $y_i \rightarrow \emptyset$ , respectively. Moreover, there are two differences between  $E(\mathcal{H}_N)$  and  $E(\mathcal{H}_{N'})$ : first,  $E(\mathcal{H}_{N'})$  contains the empty reaction edge corresponding to  $y_i \rightarrow \emptyset$  and, second, if  $s \in \text{supp}(y_i)$ , then  $E_s$  is extended to include  $u$ . We will show that the bi-colored multiset  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  extends to a bi-colored multiset  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  over  $E(\mathcal{H}_{N'})$  such that  $v_i$  is almost balanced in  $\mathcal{H}_{N'}$  with respect to  $\mathcal{E}'$ . By the previous observations on the differences between  $E(\mathcal{H}_N)$  and  $E(\mathcal{H}_{N'})$ , we need only show that  $\mathcal{E}$  extends to  $\mathcal{E}'$  in such a way that  $\deg_{\mathcal{E}'_r}(u) = \deg_{\mathcal{E}'_b}(u)$  and  $\deg_{\mathcal{E}'_r}(v) = \deg_{\mathcal{E}'_b}(v)$ . Since  $v$  corresponds to the empty complex, then  $v$  is isolated and, hence,  $\deg_{\mathcal{E}'_r}(v) = \deg_{\mathcal{E}'_b}(v)$ , regardless of the choice of  $\mathcal{E}'$ .

Let  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  where  $\mathcal{E}'_r = \mathcal{E}_r$  but viewed as a multiset over  $E(\mathcal{H}_{N'})$  and  $\mathcal{E}'_b = \mathcal{E}_b$  but viewed as a multiset over  $E(\mathcal{H}_{N'})$ . Since  $v_i$  is almost balanced with respect to  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$ , then it is almost balanced with respect to  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$ , in particular,  $\deg_{\mathcal{E}'_r}(u_i) = \deg_{\mathcal{E}'_b}(u_i)$ . Since  $E_i \notin \mathcal{E}$ , it is the case  $E_i \notin \mathcal{E}'$ , so  $\deg_{\mathcal{E}'_r}(u_i) = \deg_{\mathcal{E}'_b}(u_i)$  solely counts the number of red and blue species edges that cover  $u_i$ . Note that both  $u_i$  and  $u$  correspond to the complex  $y_i$ . Then the number of red and blue species edges that cover  $u$  is the same as the number of red and blue species edges that cover  $u_i$ . That is,  $\deg_{\mathcal{E}'_r}(u) = \deg_{\mathcal{E}'_b}(u)$ . Thus,  $v_i$  is almost balanced in  $\mathcal{H}_{N'}$  with respect to  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$ . By Corollary 2.6,  $x^{y_i} \in \mathbb{I}(N')$  and so  $\dot{x}'_s = \dot{x}_s + \kappa'_i x^{y_i} \in \mathbb{I}(N')$  for any  $s \in S'_1$ . Therefore,  $\mathbb{I}(N) = \mathbb{I}(N')$ .  $\square$

**Example 3.7.** In the hypergraph  $\mathcal{H}_{N_2}$  in Figure 16, we see that  $v_3$ , the vertex labeled  $B + C$ , is almost balanced with respect to  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  where  $\mathcal{E}_r = \{E_C\}$  and  $\mathcal{E}_b = \emptyset$ . Since  $E_3 \notin \mathcal{E}$ , then the network  $N_3$  obtained from  $N_2$  by adding the reaction  $A \rightarrow \emptyset$  has the same steady state ideal as  $N_2$ ; see Figure 17. Further,  $v_3$  remains almost balanced in

$\mathcal{H}_{N_3}$  with respect to its previous bi-coloring of  $\mathcal{E}$  so  $x_A \in \mathbb{I}(N_3)$ . To see  $x_B \in \mathbb{I}(N_3)$ , notice that  $u_2 \in V(\mathcal{H}_{N_3})$  is almost balanced for the same reason it is almost balanced in Example 3.5.

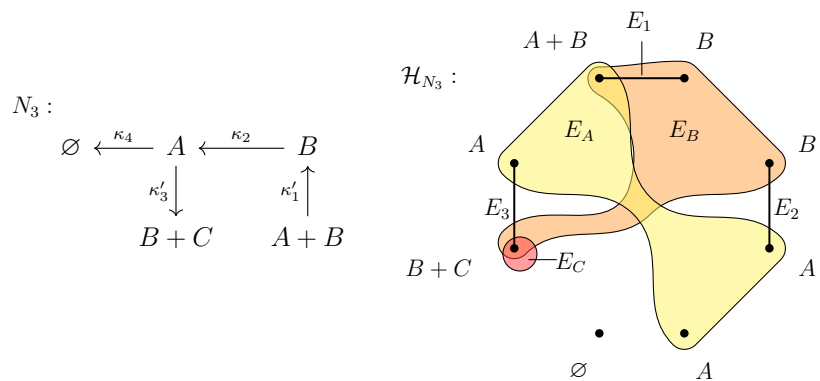


Fig. 17: A network  $N_3$ , obtained from  $N_2$  in Figure 16 by adding the reaction  $A \rightarrow \emptyset$ , and its network hypergraph.

## 4 MONOMIAL STEADY STATE IDEALS

The goal of this chapter is to characterize a class of 0,1-networks with monomial steady state ideals. In doing so, we emphasize the role that the rate constants play in searching for monomial generators of the steady state ideal. By Lemmas 2.4 and 2.5, integer multiples of  $\kappa_i x^{y_i}$  are integer combinations of the steady state polynomials. Thus, we obtain the monomial  $x^{y_i}$  in the steady state ideal upon division of an integer multiple of  $\kappa_i$ . In some cases, it is not possible to write a constant multiple of  $\kappa_i x^{y_i}$  as an integer combination of the steady state polynomials but rather as an honest  $\mathbb{Q}(\boldsymbol{\kappa})[\mathbf{x}]$ -combination. For instance, from Example 3.3, the monomial  $\kappa_2 x_B$  is written as the combination  $\frac{\kappa_1}{\kappa_3} \dot{x}_C - \dot{x}_B$ . Notice in this example that neither of the vertices corresponding to the reaction  $B \rightarrow A$  is almost balanced since  $B \rightarrow A$  is reversible, so we cannot guarantee the existence of  $x_B$  in the steady state ideal by almost balanced vertices. As it turns out, if an integer multiple of  $\kappa_i x^{y_i}$  can be written as an integer combination of the steady state polynomials, then  $y_i$  corresponds to an almost balanced vertex.

**Remark 4.1.** We will refer to the steady-state ideal as  $\mathbb{J}(N)$  when viewed as an ideal of the ring  $\mathbb{Q}[\boldsymbol{\kappa}, \mathbf{x}]$ .

**Definition 4.2.** Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a chemical reaction network. Given a complex  $y_i \in \mathcal{C}$ , the monomial  $x^{y_i}$  is *symbolic* if  $\kappa_i x^{y_i} \in \mathbb{J}(N) \subseteq \mathbb{Q}[\boldsymbol{\kappa}, \mathbf{x}]$ . The ideal  $\mathbb{I}(N)$  is *symbolically monomial* if  $\mathbb{I}(N)$  is monomial and  $x^{y_i}$  is symbolic for each minimal  $y_i \in \mathcal{C}$ .

**Remark 4.3.** The definition for a symbolic monomial can be written as “...  $x^{y_i}$  is *symbolic* if  $\kappa_j x^{y_i} \in \mathbb{J}(N)$  for some  $j$ .” This necessarily means that  $y_i = y_j$ ; that is,  $y_i$  is the reactant complex of the  $j$ th reaction  $y_j \rightarrow y'_j$  since we have chosen to index the complexes based on the reactions they appear in.

The motivation behind a symbolic monomial is that we would like to obtain the monomial  $x^{y_i}$  in  $\mathbb{I}(N)$  for some complex  $y_i$  solely by dividing an element of  $\mathbb{Q}[\boldsymbol{\kappa}, \mathbf{x}]$  by  $\kappa_i$ . This is exactly what happens when we obtain a monomial in the steady state ideal

from an almost balanced vertex since an almost balanced vertex implies an integer multiple of  $\kappa_i x^{y_i}$  is an integer combination of the steady state polynomials.

**Theorem 4.4.** *Let  $N$  be a 0,1-network. Then  $\mathbb{I}(N)$  is symbolically monomial if and only if for every minimal reactant complex  $y \in \mathcal{C}$  there is a vertex  $u$  in  $V(\mathcal{H}_N)$  such that  $u$  corresponds to  $y$  and  $u$  is almost balanced.*

*Proof.* Let  $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$  be a 0,1-network.

( $\Leftarrow$ ) Assume for every minimal reactant complex  $y \in \mathcal{C}$  there is a vertex  $u$  in  $V(\mathcal{H}_N)$  such that  $u$  corresponds to  $y$  and  $u$  is almost balanced. Then Theorem 2.9 gives us that  $\mathbb{I}(N)$  is monomial, and Lemma 2.4 guarantees that  $x^{y_i}$  is symbolic for every minimal complex  $y_i$ .

( $\Rightarrow$ ) Assume  $\mathbb{I}(N)$  is symbolically monomial. Fix a minimal reactant  $y \in \mathcal{C}$ . Since  $\mathbb{I}(N)$  is symbolically monomial, then  $\mathbb{I}(N)$  is monomial. In particular, the monomial  $x^y$  is an element of  $\mathbb{I}(N)$  by Proposition 1.3. Furthermore, since  $x^y \in \mathbb{I}(N)$ , then  $\kappa_j x^y \in \mathbb{J}(N)$  for some  $j$ . Necessarily,  $y = y_j$  so  $y$  belongs to the  $j$ th reaction  $y_j \rightarrow y'_j$ . Since  $\kappa_j x^{y_j} \in \mathbb{J}(N)$ , then we can write  $k \kappa_j x^{y_j} = \sum_{s \in \mathcal{S}} f_s \dot{x}_s$  where  $k \in \mathbb{Z}_{>0}$  and  $f_s \in \mathbb{Z}[\kappa, \mathbf{x}]$  for all  $s \in \mathcal{S}$ . We will show that  $u_j$  is almost balanced.

For each  $s \in \mathcal{S}$ , write  $f_s = c_s + g_s$  where  $c_s$  is the constant portion of  $f_s$  and  $g_s$  is the nonconstant portion of  $f_s$ . If  $\mathcal{S}_1 = \text{supp}(y_j + y'_j)$  and  $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$ , then

$$\begin{aligned}
k \kappa_j x^{y_j} &= \sum_{s \in \mathcal{S}} f_s \dot{x}_s \\
&= \sum_{s \in \mathcal{S}} c_s \dot{x}_s + \sum_{s \in \mathcal{S}} g_s \dot{x}_s \\
&= \sum_{s \in \mathcal{S}_1} c_s \dot{x}_s + \sum_{s \in \mathcal{S}_2} c_s \dot{x}_s + \sum_{s \in \mathcal{S}} g_s \dot{x}_s \\
&= \sum_{s \in \mathcal{S}_1} c_s \gamma_{sj} \kappa_j x^{y_j} + \sum_{s \in \mathcal{S}_1} c_s (\dot{x}_s - \gamma_{sj} \kappa_j x^{y_j}) \\
&\quad + \sum_{s \in \mathcal{S}_2} c_s \dot{x}_s + \sum_{s \in \mathcal{S}} g_s \dot{x}_s.
\end{aligned}$$

By equating like-terms in the above equation, we have  $k\kappa_j x^{y_j} = \sum_{s \in \mathcal{S}_1} c_s \gamma_{sj} \kappa_j x^{y_j}$  and  $0 = \sum_{s \in \mathcal{S}_1} c_s (\dot{x}_s - \gamma_{sj} \kappa_j x^{y_j}) + \sum_{s \in \mathcal{S}_2} c_s \dot{x}_s + \sum_{s \in \mathcal{S}} g_s \dot{x}_s$ . Since each  $c_s$  is constant and  $g_s$  is nonconstant, then the sums  $\sum_{s \in \mathcal{S}_1} c_s (\dot{x}_s - \gamma_{sj} \kappa_j x^{y_j}) + \sum_{s \in \mathcal{S}_2} c_s \dot{x}_s$  and  $\sum_{s \in \mathcal{S}} g_s \dot{x}_s$  do not share like-terms; to see this, note that every term of the first sum is a constant multiple of  $\kappa_i x^{y_i}$  and there is no way to obtain a constant multiple of  $\kappa_i x^{y_i}$  in the second sum. This implies, in particular, that  $\sum_{s \in \mathcal{S}} g_s \dot{x}_s = 0$ . Thus, without loss of generality, we can take each  $g_s$  to be the zero polynomial so that  $f_s = c_s$  for  $s \in \mathcal{S}$ .

For each species  $s \in \mathcal{S}$ , set  $m_s$  to be the value  $|f_s|$ . Define a bi-coloring of the multiset  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  over  $E(\mathcal{H}_N)$  by letting the red edges be defined by the multiset  $\mathcal{E}'_r = \{E_s^{(m_s)} : f_s < 0\}$  and letting the blue edges be defined by the multiset  $\mathcal{E}'_b = \{E_s^{(m_s)} : f_s > 0\}$ . Recall from Equation 2, for every species  $s \in \mathcal{S}$ , we have the equation  $\dot{x}_s = \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i}$ . Using this equation, we will re-write the monomial  $k\kappa_j x^{y_j}$  in the following way

$$\begin{aligned}
k\kappa_j x^{y_j} &= \sum_{s \in \mathcal{S}} f_s \dot{x}_s \\
&= \sum_{s \in \mathcal{S}} f_s \left( \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i} \right) \\
&= \sum_{\substack{s \in \mathcal{S} \\ E_s \in \mathcal{E}'_b}} m_s \left( \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i} \right) \\
&\quad - \sum_{\substack{s \in \mathcal{S} \\ E_s \in \mathcal{E}'_r}} m_s \left( \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i} \right).
\end{aligned}$$

Since  $E_s$  has multiplicity  $m_s$  in  $\mathcal{E}'$  for each  $s \in \mathcal{S}$ , then the previous equation becomes

$$k\kappa_j x^{y_j} = \sum_{E_s \in \mathcal{E}'_b} \left( \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i} \right) - \sum_{E_s \in \mathcal{E}'_r} \left( \sum_{v_i \in E_s} \kappa_i x^{y_i} - \sum_{u_i \in E_s} \kappa_i x^{y_i} \right).$$

From the previous equation, distribution gives the following

$$k\kappa_j x^{y_j} = \sum_{E_s \in \mathcal{E}'_b} \sum_{v_i \in E_s} \kappa_i x^{y_i} + \sum_{E_s \in \mathcal{E}'_r} \sum_{u_i \in E_s} \kappa_i x^{y_i} - \sum_{E_s \in \mathcal{E}'_b} \sum_{u_i \in E_s} \kappa_i x^{y_i} - \sum_{E_s \in \mathcal{E}'_r} \sum_{v_i \in E_s} \kappa_i x^{y_i}. \quad (6)$$

For each  $i$  such that  $y_i \rightarrow y'_i \in \mathcal{R}$ , comparing the coefficients of  $\kappa_i x^{y_i}$  on both sides of Equation 6 gives the following two equations

$$k = \deg_{\mathcal{E}'_b}(v_j) + \deg_{\mathcal{E}'_r}(u_j) - \deg_{\mathcal{E}'_b}(u_j) - \deg_{\mathcal{E}'_r}(v_j) \quad \text{for } i = j,$$

and

$$0 = \deg_{\mathcal{E}'_b}(v_i) + \deg_{\mathcal{E}'_r}(u_i) - \deg_{\mathcal{E}'_b}(u_i) - \deg_{\mathcal{E}'_r}(v_i) \quad \text{for } i \neq j.$$

For all  $i \neq j$ , let  $r_i = \deg_{\mathcal{E}'_b}(u_i) - \deg_{\mathcal{E}'_r}(u_i) = \deg_{\mathcal{E}'_b}(v_i) - \deg_{\mathcal{E}'_r}(v_i)$  and  $m_i = |r_i|$ . To construct a bi-coloring of  $\mathcal{E}'$  such that  $u_j$  almost balanced with respect to this coloring, then we must ensure that each  $r_i$  is zero so that  $\deg_{\mathcal{E}'_r}(u_i) = \deg_{\mathcal{E}'_b}(u_i)$  and  $\deg_{\mathcal{E}'_r}(v_i) = \deg_{\mathcal{E}'_b}(v_i)$ . This is not always the case, but it is possible to color the reaction edges to get the desired coloring. Unfortunately, reaction edges are not always available, namely, if the product complex of the reaction  $y_i \rightarrow y'_i$  is empty, then  $E_i = \emptyset$ . Fortunately, this implies that  $v_i$  is isolated in  $\mathcal{H}$  so  $r_i = \deg_{\mathcal{E}'_b}(v_i) - \deg_{\mathcal{E}'_r}(v_i) = 0 - 0 = 0$ . Thus, the availability of reaction edges is actually not a concern for achieving the correct bi-coloring of  $\mathcal{E}'$ . Now to deal with the case when  $r_i$  is nonzero. If  $r_i$  is positive for some  $i$ , then both  $u_i$  and  $v_i$  are covered by more blue species edges than red species edges belonging to  $\mathcal{E}'$ . Then the reaction edge  $E_i$  must be added  $m_i$  times to the edgeset  $\mathcal{E}'$  and

each  $E_i$  must be colored red. On the other hand, if  $r_i$  is negative, then both  $u_i$  and  $v_i$  are covered by more red species edges than blue species edges belonging to  $\mathcal{E}'$ . In this case, the reaction edge  $E_i$  must be added  $m_i$  times to  $\mathcal{E}'$  where each edge  $E_i$  is colored blue. Similarly, for  $i = j$ , set  $r_j = \deg_{\mathcal{E}'_b}(u_j) - \deg_{\mathcal{E}'_r}(u_j) + k = \deg_{\mathcal{E}'_b}(v_j) - \deg_{\mathcal{E}'_r}(v_j)$  and  $m_j = |r_j|$ . Define a bi-colored edgeset  $\mathcal{E} = \mathcal{E}' \sqcup \bigsqcup_{E_i \in E} \{E_i^{(m_i)}\}$  by  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  where

$$\mathcal{E}_r = \mathcal{E}'_r \sqcup \bigsqcup_{E_i \in E} \{E_i^{(m_i)} : r_i > 0\} \quad \text{and} \quad \mathcal{E}_b = \mathcal{E}'_b \sqcup \bigsqcup_{E_i \in E} \{E_i^{(m_i)} : r_i < 0\}.$$

Let  $i$  be so that  $i \neq j$ . Without loss of generality, suppose if  $E_i$  is colored, then it is red. Then  $m_i - r_i = 0$  and  $\deg_{\mathcal{E}'_b}(w) = \deg_{\mathcal{E}_b}(w)$  for  $w \in E_i$ . Thus, if  $w \in E_i$ , then

$$\deg_{\mathcal{E}_r}(w) = \deg_{\mathcal{E}'_r}(w) + m_i = (\deg_{\mathcal{E}'_b}(w) - r_i) + m_i = \deg_{\mathcal{E}'_b}(w) = \deg_{\mathcal{E}_b}(w).$$

Suppose  $i = j$ . Again, assume without loss of generality if  $E_j$  is colored, then it is colored red so that  $m_j - r_j = 0$  and  $\deg_{\mathcal{E}'_b}(w) = \deg_{\mathcal{E}_b}(w)$  for  $w \in E_j$ . As before, we have

$$\deg_{\mathcal{E}_r}(u_j) = \deg_{\mathcal{E}'_r}(u_j) + m_j = (\deg_{\mathcal{E}'_b}(u_j) + k - r_j) + m_j = \deg_{\mathcal{E}'_b}(u_j) + k = \deg_{\mathcal{E}_b}(u_j) + k$$

and

$$\deg_{\mathcal{E}_r}(v_j) = \deg_{\mathcal{E}'_r}(v_j) + m_j = (\deg_{\mathcal{E}'_b}(v_j) - r_j) + m_j = \deg_{\mathcal{E}'_b}(v_j) = \deg_{\mathcal{E}_b}(v_j).$$

Therefore,  $u_j$  is almost balanced with respect to the bi-colored edgeset  $\mathcal{E}$ .  $\square$

**Example 4.5.** To demonstrate the construction of a bi-coloring of a multiset of hyperedges from the proof of Theorem 4.4, consider the following network  $N$  given in Figure 18. The ideal  $\mathbb{I}(N)$  is symbolically monomial since  $\mathbb{I}(N) = \langle x_A, x_C \rangle$  and  $\mathbb{J}(N) = \langle \kappa_3 x_A, \kappa_4 x_C, \kappa_1 x_A x_B - \kappa_2 x_C x_D \rangle$ . This example also shows that if  $\mathbb{I}(N)$  is symbolically monomial, then  $\mathbb{J}(N)$  need not be monomial.



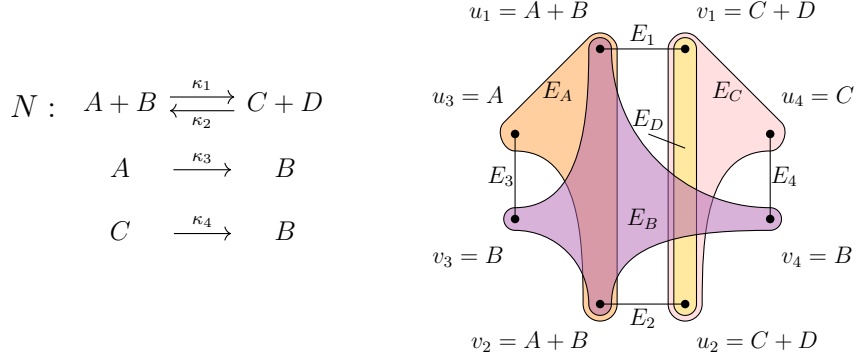


Fig. 18: A network, whose steady state ideal is symbolically monomial, and its network hypergraph.

Notice that  $\kappa_4 x_C$  is equivalent to two different integer combinations of the steady state polynomials:  $\kappa_4 x_C = \dot{x}_D - \dot{x}_C = \sum_{s \in \mathcal{S}} f_s \dot{x}_s$  and  $\kappa_4 x_C = \dot{x}_A + \dot{x}_B + 2\dot{x}_D = \sum_{s \in \mathcal{S}} f'_s \dot{x}_s$ . We will demonstrate how to obtain the correct bi-coloring for each integer combination of the steady state polynomials. For the former, our construction begins by bi-coloring the multiset  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  defined by  $\mathcal{E}'_r = \{E_s^{(m_s)} : f_s < 0\} = \{E_C\}$  and  $\mathcal{E}'_b = \{E_s^{(m_s)} : f_s > 0\} = \{E_D\}$  where  $m_s = |f_s|$ . Since  $\deg_{\mathcal{E}'_b}(v_1) - \deg_{\mathcal{E}'_r}(v_1) = 1 - 1 = 0$ ,  $\deg_{\mathcal{E}'_b}(u_2) - \deg_{\mathcal{E}'_r}(u_2) = 1 - 1 = 0$ , and  $\deg_{\mathcal{E}'_b}(u_4) - \deg_{\mathcal{E}'_r}(u_4) + k = 0 - 1 + 1 = 0$ , then there is no need to color the adjacent edges  $E_1, E_2$ , or  $E_4$ . Indeed, the vertex  $u_4$  is almost balanced with respect to the bi-coloring of  $\mathcal{E}'$ ; see a Figure 19.

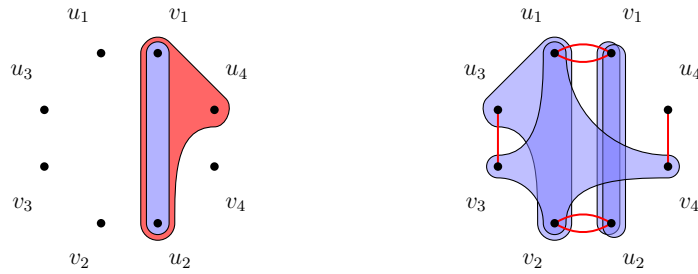


Fig. 19: The vertex  $u_4$  is almost balanced with respect to the bi-colorings  $\mathcal{E}'_r \sqcup \mathcal{E}'_b$  (left) and  $\mathcal{E}_r \sqcup \mathcal{E}_b$  (right).

As for  $\kappa_4 x_C = \dot{x}_A + \dot{x}_B + 2\dot{x}_D = \sum_{s \in \mathcal{S}} f'_s \dot{x}_s$ , we begin by bi-coloring the multiset  $\mathcal{E}' = \mathcal{E}'_r \sqcup \mathcal{E}'_b$  defined by  $\mathcal{E}'_r = \{E_s^{(m_s)} : f'_s < 0\} = \emptyset$  and  $\mathcal{E}'_b = \{E_s^{(m_s)} : f'_s > 0\} = \{E_A, E_B, E_D^{(2)}\}$  where  $m_s = |f'_s|$ . Since  $\deg_{\mathcal{E}'_b}(u_1) - \deg_{\mathcal{E}'_r}(u_1) = 2 - 0 = 2$ ,  $\deg_{\mathcal{E}'_b}(u_2) - \deg_{\mathcal{E}'_r}(u_2) = 2 - 0 = 2$ ,  $\deg_{\mathcal{E}'_b}(u_3) - \deg_{\mathcal{E}'_r}(u_3) = 1 - 0 = 1$ ,  $\deg_{\mathcal{E}'_b}(u_4) - \deg_{\mathcal{E}'_r}(u_4) + k = 0 - 0 + 1 = 1$ , then we must color the corresponding reaction edges  $E_1^{(2)}$ ,  $E_2^{(2)}$ ,  $E_3$ , and  $E_4$  red. Thus,  $u_4$  is almost balanced with respect to  $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$  where  $\mathcal{E}_r = \{E_1^{(2)}, E_2^{(2)}, E_3, E_4\}$  and  $\mathcal{E}_b = \{E_A, E_B, E_D^{(2)}\}$ .

## 5 CONCLUSION

We were able to answer the three questions stated in the introduction of the thesis. First, for determining when the steady state ideal is monomial, we showed that this property follows if the network is a 0,1-network and the minimal reactants of the network correspond to almost balanced vertices. Indeed, if the minimal reactants of a 0,1-network correspond to almost balanced vertices, then the steady state ideal is symbolically monomial.

Second, we were able to combinatorially capture information from the structure of the network to determine the existence of monomials in the steady state ideal. In other words, we can look at the reaction diagram and tell which monomials belong to the network ideal. Further, we can explicitly write these monomials as elements of the steady state ideal, specifically, as integer combinations of the steady state polynomials.

Lastly, we used the almost balanced condition on the vertices of the network hypergraph to define three ideal preserving operations: adding a species to a reactant, adding a species to a product, and adding a reaction of the form  $y \rightarrow \emptyset$ .

While we answered these three questions, there still remain two questions to consider. First, is there a characterization of networks whose ideal is monomial but not symbolically monomial, e.g.  $A \xrightleftharpoons[\kappa_2]{\kappa_1} B \xrightarrow{\kappa_3} C$ . Second, since the work in this thesis is done in the context of 0,1-networks, we can ask ourselves what a natural generalization of these definitions and results will look like for a general network. In particular, Lemma 2.4 supported all of the work in this thesis so we would like to generalize this result to arbitrary networks. That is, can we remove the 0,1-network condition on the network from Lemma 2.4 and still determine the existence of monomials in the steady state ideal? There are perhaps a few directions we can take here:

- Given an arbitrary network  $N$ , is there a way to reduce the network down to a 0,1-network  $N'$  so that the algebra of  $N'$  corresponds to the algebra of  $N$ ?

- Is there an alternative definition of the network hypergraph so that the species edge  $E_s$  will “see” the coefficients of  $s$  in the complexes that it appears in as opposed to asking if  $s$  is in the complex or if it is not in the complex? Perhaps this could occur by considering the edges as multisets?
- Is there an alternative definition of an almost balanced vertex that incorporates the coefficients of the complexes? Perhaps one could introduce weights on the vertices?

## Literature Cited

- [1] M. Feinberg, “Complex balancing in general kinetic systems,” *Archive for Rational Mechanics and Analysis*, vol. 49, no. 3, pp. 187–194, 1972.
- [2] M. Feinberg, “The existence and uniqueness of steady states for a class of chemical reaction networks,” *Archive for Rational Mechanics and Analysis*, vol. 132, no. 4, pp. 311–370, 1995.
- [3] M. Feinberg and F. J. Horn, “Dynamics of open chemical systems and the algebraic structure of the underlying reaction network,” *Chemical Engineering Science*, vol. 29, no. 3, pp. 775–787, 1974.
- [4] F. Horn, “Necessary and sufficient conditions for complex balancing in chemical kinetics,” *Archive for Rational Mechanics and Analysis*, vol. 49, no. 3, pp. 172–186, 1972.
- [5] F. Horn and R. Jackson, “General mass action kinetics,” *Archive for Rational Mechanics and Analysis*, vol. 47, no. 2, pp. 81–116, 1972.
- [6] A. Dickenstein, “Biochemical reaction networks: An invitation for algebraic geometers,” in *Mathematical Congress of the Americas*, vol. 656, pp. 65–83, American Mathematical Soc., 2016.
- [7] A. Shiu, “Algebraic methods for biochemical reaction network theory”. PhD dissertation, Dept. Math., Univ. California, Berkeley, 2010.
- [8] E. Gross, H. A. Harrington, Z. Rosen, and B. Sturmfels, “Algebraic systems biology: a case study for the wnt pathway,” *Bulletin of Mathematical Biology*, vol. 78, no. 1, pp. 21–51, 2016.
- [9] A. Shiu and B. Sturmfels, “Siphons in chemical reaction networks,” *Bulletin of Mathematical Biology*, vol. 72, no. 6, pp. 1448–1463, 2010.
- [10] E. Gross, H. A. Harrington, N. Meshkat, and A. Shiu, “Joining and decomposing reaction networks,” *arXiv preprint arXiv:1810.05575*, 2018.
- [11] E. Gross, H. A. Harrington, N. Meshkat, and A. Shiu, “Linear compartmental models: input-output equations and operations that preserve identifiability,” *arXiv preprint arXiv:1808.00335*, 2018.

- [12] G. Craciun and C. Pantea, “Identifiability of chemical reaction networks,” *Journal of Mathematical Chemistry*, vol. 44, no. 1, pp. 244–259, 2008.
- [13] S. Petrović and D. Stasi, “Toric algebra of hypergraphs,” *Journal of Algebraic Combinatorics*, vol. 39, no. 1, pp. 187–208, 2014.

## Appendix A

### POLYNOMIAL RINGS, IDEALS & VARIETIES

A *ring* is a set together with two binary operations,  $+$  and  $\cdot$ , such that  $+$  is associative and commutative,  $\cdot$  is associative, there is an additive identity and additive inverses, and multiplication is distributive over addition. The set  $\mathbb{Q}$  of rational numbers is a ring when taken with the usual addition and multiplication of fractions. Rings are abstract mathematical objects and, given an arbitrary ring, understanding its algebraic structure can be a difficult task. In hopes of understanding the algebraic structure of a ring, perhaps it is reasonable to understand a part of the structure. A *subring*  $S$  of a ring  $R$ , is a subset  $S$  of  $R$  such that  $S$  is a ring together with the binary operations on  $R$ . For example,  $\mathbb{Q}$  is a subring of the ring  $\mathbb{R}$  of real numbers taken with usual addition and multiplication. Subrings are interesting algebraic objects in and of themselves that can provide some insight into the larger ring. Another interesting algebraic object that this paper is mainly concerned with is called an ideal. An *ideal*  $I$  of a ring  $R$  is a subring of  $R$  such that  $xr \in I$  and  $rx \in I$  whenever  $x \in I$  and  $r \in R$ .

In this thesis, we fix our ring to be the ring of polynomials in a finite number of variables. Given a ring  $R$ , the *polynomial ring*  $R[x_1, \dots, x_n]$  is the ring of polynomials in the  $n$  unknowns  $x_1, \dots, x_n$  with coefficients in  $R$ . The Hilbert Basis Theorem says that every ideal  $I$  of  $R[x_1, \dots, x_n]$  is finitely generated, that is, there are polynomials  $f_1, \dots, f_m \in I$  such that any polynomial  $f \in I$  can be written as  $f = \sum_{i=1}^m f_i g_i$  where  $g_i \in R[x_1, \dots, x_n]$ . We write  $I = \langle f_1, \dots, f_m \rangle$  to express that the ideal  $I$  is generated by the polynomials  $f_1, \dots, f_m$ . A polynomial is a *monomial* in  $R[x_1, \dots, x_n]$  if it can be written as a product  $x_1^{a_1} \cdots x_n^{a_n}$  where  $(a_1, \dots, a_n)$  is an  $n$ -tuple in  $\mathbb{Z}_{\geq 0}^n$  of nonnegative integers. An ideal is a *monomial ideal* if it is generated by monomials. Notice that a polynomial can be regarded as a map from the ring  $R^n$ , of  $n$ -tuples with entries in  $R$ , to

the ring  $R$ . The *affine variety* of an ideal  $I = \langle f_1, \dots, f_m \rangle$  is the set

$$\mathbb{V}(I) = \{x \in R^n : f_i(x) = 0 \text{ for all } 1 \leq i \leq m\}.$$

The core of the work in this thesis is preserving ideals of chemical reaction networks. This suggests the idea of a *ring isomorphism*, that is, a bijective function  $f$  from a ring  $R$  to a ring  $S$  such that  $f(r + r') = f(r) + f(r')$  and  $f(rr') = f(r)f(r')$  for any  $r$  and  $r'$  in  $R$ . If there is an isomorphism from  $R$  to  $S$ , we say  $R$  is *isomorphic* to  $S$  and write  $R \cong S$ . The existence of an isomorphism from  $R$  to  $S$  ensures that the algebraic structure of  $R$  and  $S$  are the same regardless of the difference of  $R$  and  $S$  as sets. Indeed, preserving ideals requires a ring isomorphism. Thus, by preserving ideals, we mean that both the algebraic structure of the ideals is the same *and* the ideals are the same as sets, that is,  $R = S$ . If  $R = \langle r_1, \dots, r_n \rangle$  and  $S = \langle s_1, \dots, s_m \rangle$ , then  $R = S$  if and only if  $r_i \in S$  for  $1 \leq i \leq n$  and  $s_i \in R$  for all  $1 \leq i \leq m$ .