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Blue Red Hackenbush Spiders

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BLUE RED HACKENBUSH SPIDERS

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Master of Arts

by

Ravi Cho

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BLUE RED HACKENBUSH SPIDERS

by

Ravi Cho

APPROVED FOR THE DEPARTMENT OF MATHEMATICS AND STATISTICS

SAN JOSÉ STATE UNIVERSITY

December 2021

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ABSTRACT

BLUE RED HACKENBUSH SPIDERS

by Ravi Cho

One of the goals of Combinatorial Game Theory is to find provable winning strategies for certain games. In this paper, we give winning strategies for certain spider positions played using the rules of Blue Red Hackenbush and a variant. Blue Red Hackenbush and its variants are played on a graph of a bLue and Red edges that are connected to a vertex called the ground. We will represent the ground as a horizontal black line. In this paper, we study spider graphs played under two different variants: Blue Red Hackenbush and Reverse Blue Red Hackenbush. Both variants are played by two players: Left and Right. On Left's turn, they must choose a blue edge to delete. Any edges no longer connected to the ground are also deleted. Right's turn is similar, except they must choose a red edge to delete. The first player unable to move loses. Every Blue Red Hackenbush position can be identified as a dyadic rational. This value completely determines who wins the game playing first and who wins the game playing second. It's been shown that determining this value is NP-hard for certain kinds of BR Hackenbush games, making the study of even certain classes of positions interesting. Reverse BR Hackenbush is played exactly the same as the usual BR Hackenbush, except after a player deletes an edge (and any subsequently unconnected to the ground edges) they reverse the color of any edges that were in the component they played in.

In this paper, we begin by introducing the essentials necessary to analyze spiders. In section 2, we analyze spiders in BR Hackenbush. In particular, we give a solution for a certain class of spiders called balanced spiders. In section 3, we turn our attention to Reverse BR Hackenbush. In particular, we give a solution for two legged spiders. Even though Reverse BR Hackenbush *seems* like it has similar rules to BR Hackenbush, the results of section 2 and section 3 appear remarkably different.

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1 QUICK START GUIDE TO BLUE RED HACKENBUSH

Our goal for this section is to introduce the reader to Combinatorial Game Theory with an emphasis on analyzing positions in Blue Red Hackenbush. In this section and the next, the reader should assume all game play is using the rules of the usual BR Hackenbush (and not Reverse BR Hackenbush). We will use the words game and position interchangeably. Almost all the material in this section can be found in [1] and [2].

1.1 Blue Red Hackenbush

The game of Blue Red Hackenbush (BR Hackenbush) is a game played on a graph of bLue and Red edges. There is a unique vertex that we label the body. It is convenient to draw the body vertex as a horizontal black line. The game is played by two players, Left and Right. On a player's turn, they must delete an edge of their color. Left can delete blue edges, and Right can delete red edges. After deleting an edge, any other edges no longer connected to the ground via a path are also deleted. The first player unable to move, loses.

An example of such a game is in Figure 1.1.

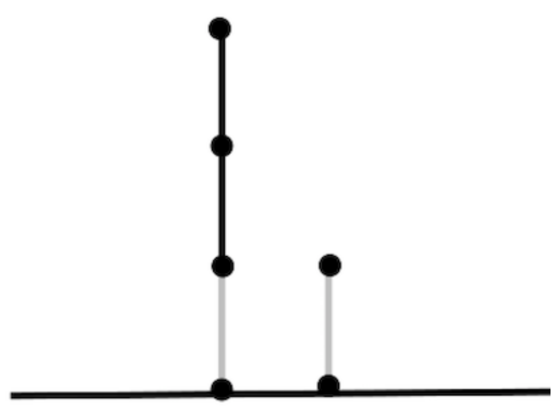

Fig. 1.1: Example of a Game of Blue Red Hackenbush

In Figure 1.1 and in the remainder of the paper, we will represent blue edges with black edges and red edges with grey edges. In some proofs, we may use black lines to represent blue edges and/or arbitrarily colored paths. In these instances, we will state clearly what the black lines represent. Returning to the above example, we can do a brute force analysis using pencil and paper to determine Right is the winner of this game. In particular, Right can win this game whether they go first or second.

Another example is given in Figure 1.2.

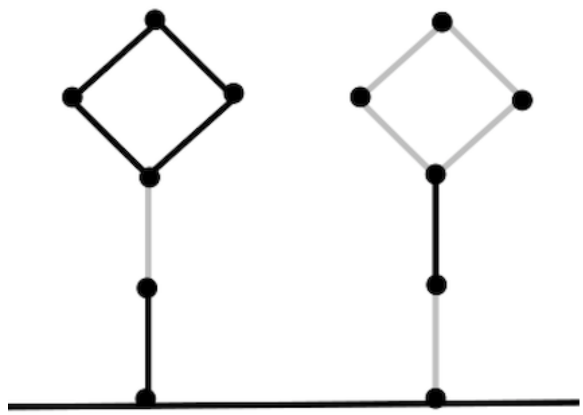

Fig. 1.2: Example of a Tweedle Dee Tweedle Dum Game

The game in Figure 1.2 is special because it has a natural strategy for the second player. The second player can mimic the first player's moves. In this way, the second player will never run out of moves before the first player. The strategy of one player mimicking the other player's moves is called the Tweedle Dee Tweedle Dum strategy (TDTD). The above game is a win for the second player to move.

1.2 Basic Definitions

Let G be a game. We write $G = \{\mathscr{G}^L | \mathscr{G}^R\}$ where
$\mathscr{G}^L = \{$ the set of sub positions of G when Left moves on G $\}$ and
$\mathscr{G}^R = \{$ the set of sub positions of G when Right moves on G $\}$. We define the elements of $\mathscr{G}^L$ as the Left options of G. We denote a Left option as G^L. We define analogous terms for Right. In this paper, we will write $G = \{G^L | G^R\}$, where G^L ranges over all possible Left options and G^R ranges over all possible Right options.

A game G is a short game if it meets the following two properties:

- G has finitely many sub positions
- There exists no infinite sequences of moves on G, i.e., the the game eventually ends.

All games in this paper are short games. We play under the convention that the first player unable to move loses. We also assume that every BR Hackenbush game we play has a finite number of edges.

Theorem 1.1. *Every Blue Red Hackenbush game G is a short game.*

Proof. Let G be a BR Hackenbush game. We'll view G as a graph of blue and red edges. By assumption, G has n edges where n some non negative integer. Every move on G reduces the number of edges by at least one. So in at most n moves, G will be empty. This implies there are no infinite sequences of moves on G. Moreover, G has n edges and so has a finite number of subgraphs. Every sub position G is a subgraph of G. We can conclude G is short.

□

1.3 Short Games

In this section we will construct all short games. We will define a certain relation between short games, and see that the set of equivalence classes along with a certain addition form a partially ordered abelian group.

Define $0 = \{|\}$ and $\tilde{\mathbb{G}}_0 = \{0\}$. We recursively define the short games born on day n by $\tilde{\mathbb{G}}_n = \{\{G^L|G^R\} : G^L, G^R \subset \tilde{\mathbb{G}}_{n-1}\}$. We define $\tilde{\mathbb{G}} = \bigcup_{n\in\mathbb{N}} \tilde{\mathbb{G}}_n$ as the set of short games. We define disjunctive sum on $\tilde{\mathbb{G}}$ as $G+H = \{G^L+H, G+H^L|G^R+H, G+H^R\}$.

Theorem 1.2. $(\tilde{\mathbb{G}}, +)$ *is a abelian semigroup.*

Proof. We first prove that addition is commutative by induction. Observe,

$$
\begin{aligned}
G+H &= \{G^L+H, G+H^L | G^R+H, G+H^R\} \\
&= \{H+G^L, H^L+G | G^R+H, H^R+G\} \quad \text{(By induction)} \qquad (1.1) \\
&= H+G
\end{aligned}
$$

The proof for associativity is similar. □

We define the outcome class of a game, denoted $o(G)$, in the following way:

$$
\begin{aligned}
o(G) = \mathscr{L} &\iff \text{Left can win going first or second} \\
o(G) = \mathscr{N} &\iff \text{The first player to move wins} \\
o(G) = \mathscr{P} &\iff \text{The second player to move wins} \\
o(G) = \mathscr{R} &\iff \text{Right can win going first or second}
\end{aligned}
$$

Theorem 1.3. *(Fundamental Theorem of Combinatorial Game Theory) For any short game G either Left can win playing first or Right can win playing second but not both.*

Proof. Let G be a short game and G^L be an arbitrary Left option of G. Then by induction and symmetry, Right can win G^L playing first or Left can win G^L playing second but not both. If, for all left options G^L, Right can win playing G^L first, then Right can win G playing second. On the other hand, if there exists a Left option G^L such that Left can win playing second, then Left can win G by moving to such a G^L. It is clear exactly one of these two cases must hold. This proves the claim. □

Corollary 1.4. *Every short game G belongs to exactly one outcome class.*

Proof. By Theorem 1.3, every short game G belongs to $\mathscr{L}, \mathscr{R}, \mathscr{N}$, or $\mathscr{P}$. If a game G belonged to more than one outcome class, then we would have a counter example to Theorem 1.3. □

Lemma 1.5. *If X belongs to $\mathscr{P}$ then $o(G+X) = o(G)$.*

Proof. There's two cases to consider:

Case 1: Suppose Left can win G playing second. Suppose Right moves first on $G+X$. Then Left is guaranteed a response on whichever component Right moved on. Left can continue this strategy until Right eventually runs out of moves.

Case 2: Suppose Left can win G playing first. Then Left has a winning move on G which we denote as G^L. Observe that Left can win G^L playing second. Hence, Left's move to $G^L + X$ is a winning move by the strategy in case 1. □

Lemma 1.6. *If G and H belong to $\mathscr{L}$ or $\mathscr{P}$, then $G+H$ also belongs to $\mathscr{L}$ or $\mathscr{P}$.*

Proof. It is enough to show Right cannot win going first. If Right does go first, then by our hypotheses, Left must have a response in the same component. So Left can keep responding in whichever component Right moves in. Therefore, Left will not run out of moves before Right, and since these games are short, Right must eventually run out of moves. □

Definition 1.7. We define the negative of a short game as $-G = \{-G^R | -G^L\}$.

Definition 1.8. (Preorder on $\tilde{\mathbb{G}}$) We define a relation $\geq$ on $\tilde{\mathbb{G}}$ by saying that $G \geq H$ if and only if $o(G + (-H)) = \mathscr{L}$ or $\mathscr{P}$.

Theorem 1.9. *$\geq$ is a preorder on $\tilde{\mathbb{G}}$.*

Proof. (Reflexive): Consider the game $G + (-G)$. Then any move by the first player can be mimicked by the second player in the other component. This shows the second player will never run out of moves before the first player. Since both games are short, the first player must eventually run out of moves. Therefore $G+(-G)$ belongs to $\mathscr{P}$ and $G \geq G$.

(Transitive): Suppose $G \geq H$ and $H \geq J$. Then $G+(-H)$ and $H+(-J)$ belong to $\mathscr{L}$ or $\mathscr{P}$. By Lemma 1.6, $G+(-H)+H+(-J)$ belongs to $\mathscr{L}$ or $\mathscr{P}$. By Lemma 1.5 and the fact that $H+(-H)$ belongs to $\mathscr{P}$, we have $G+(-J)$ belongs to $\mathscr{L}$ or $\mathscr{P}$. □

Lemma 1.10. *Let G,H be short games. Then $-(G+H) = (-G)+(-H)$.*

Proof. We have

$$\begin{aligned} -(G+H) &= -\{G^L+H, G+H^L | G^R+H, G+H^R\} \\ &= \{-(G^R+H), -(G+H^R) | -(G^L+H), -(G+H^L)\} \quad \text{(By Definition 1.7)} \\ &= \{-G^R-H, -G-H^R | -G^L-H, -G-H^L\} \quad \text{(By induction)} \\ &= (-G)+(-H) \end{aligned} \tag{1.2}$$

□

Theorem 1.11. *For all short games G,H,X, $G \geq H$ implies $G+X \geq H+X$.*

Proof. Using Theorem 1.2 and Lemma 1.10,
$(G+X)+(-(H+X)) = G+(X+(-X))-H$. Since $X+(-X)$ belongs to $\mathscr{P}$, Lemma 1.5. gives us $o(G-H) = o(G+X+(-(H+X))$. Since $G \geq H$, we have $G+X \geq H+X$. □

We have shown $(\tilde{\mathbb{G}},+)$ is a preordered abelian semigroup. Next, define a relation on $\tilde{\mathbb{G}}$ as $G = H$ if and only if $G \geq H$ and $H \geq G$.

Theorem 1.12. *The relation defined above is an equivalence relation on $\tilde{\mathbb{G}}$.*

Proof. Let $G,H,J \in \tilde{\mathbb{G}}$.

(Reflexive): We've shown $G+(-G)$ belongs to $\mathscr{P}$. Equivalently, $G \geq G$ which shows $G = G$.

(Symmetry): Suppose $G = H$. Then $G \geq H$ and $H \geq G$. Equivalently, $H \geq G$ and $G \geq H$. So, $H = G$.

(Transitivity): Suppose $G = H$ and $H = J$. Then $G \geq H$ and $H \geq J$. Since $\geq$ is transitive, we have $G \geq J$. Similarly, $J \geq G$. So, $G = J$. □

We define $\mathbb{G}$ to be the set of equivalence classes formed by $\tilde{\mathbb{G}}$ and the relation above. We refer to the elements of $\mathbb{G}$ as game values. We define addition in $\mathbb{G}$ as $[G]+[H] = [G+H]$. We also define a relation $\geq$ on $\mathbb{G}$ by $[G] \geq [H]$ if and only if $G' \geq H'$ for some $G' \in [G]$ and some $H' \in [H]$. We say $[G] \mathrel{|\!\triangleright} [H]$ if and only if $[G] \not\geq [H]$. We will show that $(\mathbb{G}, +)$ is a partially ordered abelian group.

Theorem 1.13. *The relation $[G] \geq [H]$ if and only if $G \geq H$ for some $G \in [G]$ and some $H \in [H]$ is well defined.*

Proof. Let $[G],[H]$ be game values and $G, G' \in [G]$ and $H, H' \in [H']$. Suppose $G \geq H$. We want to show $G' \geq H'$. Note that $H = H'$ implies $-H = -H'$. Hence, $-H' \geq -H$. We also have $G' \geq G$. Observe,

$$\begin{aligned} G'+(-H') &\geq G+(-H') && \text{(By Theorem 1.11)} \\ G+(-H') &\geq G+(-H) && \text{(By Theorem 1.11)} \\ G'+(-H') &\geq G+(-H) && \text{(Since } \geq \text{ in } \tilde{\mathbb{G}} \text{ is transitive)} \end{aligned} \tag{1.3}$$

Using transitivity again, we get $G'+(-H') \geq 0$. Hence, $G'+(-H')$ belongs to $\mathscr{L}$ or $\mathscr{P}$ which implies $G' \geq H'$. □

Theorem 1.14. *$\geq$ is a partial order on $\mathbb{G}$.*

Proof. Let $[G],[H],[J]$ be game values.

(Reflexive): We see that $G \geq G$ since $G+(-G)$ belongs to $\mathscr{P}$. Therefore, $[G] \geq [G]$.

(Transitive): Suppose $[G] \geq [H]$ and $[H] \geq [J]$. Then $(G+(-H))+(H+(-J))$ is the sum of two games that belong to $\mathscr{L}$ or $\mathscr{P}$. Hence, the sum also belongs to $\mathscr{L}$ or $\mathscr{P}$. We can rewrite the sum as $G+(-H+H)+(-J)$. Since $-H+H$ belongs to $\mathscr{P}$, we may conclude the outcome of $G+(-J)$ is the same as the outcome of $G+(-H+H)+(-J)$. Hence, $G \geq J$. This implies $[G] \geq [J]$.

(Anti-symmetry): Suppose $[G] \geq [H]$ and $[H] \geq [G]$. Then $G \geq H$ and $H \geq G$. By our definition of $=$, we have $G = H$ which implies $[G] = [H]$. □

Theorem 1.15. *The addition defined above is well defined.*

Proof. Let $[G], [G'], [H], [H'] \in \mathbb{G}$ and suppose $[G] = [G']$ and $[H] = [H']$. We want to show $[G+H] = [G'+H']$. We see $G \geq G'$ and $H \geq H'$. So, $G - G'$ and $H - H'$ are in $\mathscr{L}$ or $\mathscr{P}$. Then the game $(G+H)-(G'+H') = (G-G')+(H-H')$ belongs to $\mathscr{L}$ or $\mathscr{P}$. Equivalently, $G+H \geq G'+H'$. Using a symmetric argument, $G'+H' \geq G+H$. This shows $[G+H] = [G'+H']$. □

Theorem 1.16. $(\mathbb{G}, +)$ *is a commutative semigroup.*

Proof. This is true because $(\tilde{\mathbb{G}}, +)$ is a commutative semigroup. □

Theorem 1.17. $[0]$ *is the identity in* $\mathbb{G}$.

Proof. Observe, $[G]+[0] = [G+0] = [G]$ for all $[G] \in \mathbb{G}$. □

Lemma 1.18. *If* $[G] \geq [H]$ *then* $[G]+[J] \geq [H]+[J]$.

Proof. First we'll show that if $G \geq H$ then $G+J \geq H+J$. Observe that $(G+J)+-(H+J) = (G+(-H)+(J+(-J))$ is a sum of games belonging to $\mathscr{L}$ or $\mathscr{P}$. Therefore, $G+J+-(H+J)$ also belongs to $\mathscr{L}$ or $\mathscr{P}$. This shows $G+J \geq H+J$. In particular, if $[G] \geq [H]$ then $G \geq H$. We've shown this implies $G+J \geq H+J$. And this implies $[G]+[H] \geq [H]+[J]$. □

Theorem 1.19. *If [G] = [H] then* $[G]+[J] = [H]+[J]$.

Proof. Now suppose $G = H$. Then $G + J \geq H + J$ and $G + J \leq H + J$. By definition, we have equality which proves the theorem. $\square$

Lemma 1.20. *$o(G) = \mathscr{P}$ if and only if $[G] = [0]$*

Proof. $(\Longrightarrow)$: Suppose $o(G) = \mathscr{P}$. Then G and $-G$ belong to $\mathscr{P}$. Therefore, $G \geq 0$ and $0 \geq G$. Equivalently, $[G] = [0]$.

$(\Longleftarrow)$: Suppose $[G] = 0$. Then G and $-G$ belong to $\mathscr{L}$ or $\mathscr{P}$. If one of them, say G, belongs to $\mathscr{L}$, then we must have $-G$ belongs to $\mathscr{R}$, a contradiction. Therefore, G and $-G$ belong to $\mathscr{P}$. In particular, G belongs to $\mathscr{P}$. $\square$

Theorem 1.21. *For any $[G] \in \mathbb{G}$, we have $[G] + [-G] = [0]$.*

Proof. We see that $G + (-G)$ is a second player win by a TDTD argument. By the previous lemma, $[G] + [-G] = [G + (-G)] = [0]$. $\square$

We may conclude $(\mathbb{G}, +)$ is a partially ordered abelian group. At this point, we will drop the bracket notation.

Corollary 1.22. *Let G, H be game values. Then $G + (-H) = 0$ if and only if $G = H$.*

Proof. We have

$$G + (-H) = 0 \iff G + (-H) + H = 0 + H \iff G + 0 = H \iff G = H.$$ $\square$

In future sections, we will use the above corollary to prove certain formulas.

1.4 Canonical Forms

Intuitively, some moves are more optimal than others. In these cases, it seems intuitive to ignore moves that are not optimal. What we mean by *optimal* moves will be the topic of this section.

Definition 1.23. Let G be a game and G^{L_1} and G^{L_2} be two Left options of G. We say G^{L_2} is dominated by G^{L_1} if $G^{L_1} \geq G^{L_2}$. (I.e., $G^{L_1} - G^{L_2}$ is a loss for Right going first).

Theorem 1.24. *Let G be a game. Suppose $G^{L'}$ is a dominated Left option. Write $G = \{G^{L'}, G^L | G^R\}$ and $H = \{G^L | G^R\}$, where G^L ranges over all Left options except for $G^{L'}$. Then $G = H$.*

Proof. We want to show $G - H = 0$ which is equivalent to showing $G - H$ is a second player win. Note that $G - H = \{G^{L'}, G^L | G^R\} + \{-G^R | -G^L\}$. Suppose Left moves first to $G^{L'} - H$. By assumption, there exists G^L such that $G^L \geq G^{L'}$. Hence, Right may move to $G^{L'} - G^L \leq 0$. Right has forced a win. Every other move by one player has a clear corresponding move by the other player to 0. In particular, if Left moves to $G^L - H$, then Right may respond to $G^L - G^L = 0$. And if Left moves to $G - G^R$, then Right may respond to $G^R - G^R = 0$. A similar argument applies to Right's opening moves. We may conclude that $G - H$ is a second player win, i.e., $G = H$. □

Example 1.25. We define $B = \{1|0\}$ and $A = \{1|0, -B\}$. We'll show $\{B, A|1\} = \{B|1\}$. By the previous theorem, it suffices to show $B - A \geq 0$, i.e., that Right cannot win playing first. We have

$$B - A = \{1|0\} + \{B, 0| -1\}$$

Right has two options: $0 + \{B, 0| -1\}$ or $\{1|0\} + (-1)$. In the former, Left can respond to $0 + 0$ and in the latter, Left can respond to $1 + (-1)$. Either way, Left has a winning move. This shows B dominates A and the equality follows.

Lemma 1.26. *For short games G, H, J if $G \mathrel{|\!\triangleright} H$ and $H \geq J$ then $G \mathrel{|\!\triangleright} J$. (i.e., Left has a winning move on $G - J$.*

Proof. Observe that $G - J = (G - H) + (H - J)$. We're given when playing $G - H$, Left has a winning move, say $(G - H)^L$. Fix this $(G - H)^L$. Then $(G - H)^L \geq 0$. The sum of games that are zero or positive is a game that is zero or positive. Since $(G - H)^L \geq 0$ and

$H-J \geq 0$, we have $(G-H)^L + (H-J) \geq 0$. This shows that $(G-H)^L + (H-J)$ is a winning move for Left, which proves the lemma. $\square$

Lemma 1.27. *If* $J \lhd\!| \, H$ *then* $G+J \lhd\!| \, G+H$.

Proof. Consider the difference game $(G+H)-(G+J)$. We can rewrite this as $(G-G)+(H-J)$. We would like to show $0 \ngeq (G+G)+(H-J)$ i.e., Left has a winning move as the first player. Since $J \lhd\!| \, H$, we have Left has a winning move on $H-J$. Fix such an option $(H-J)^L$. Consider Left's option $(G+G)+(H-J)^L$. This is a sum of games that are ≥ 0. Hence, $(G+G)+(H-J)^L \geq 0$. This shows Left has a winning move on $(G+H)-(G+J)$. $\square$

Definition 1.28. Let G be a game and G^L be a Left option. Suppose there exists Right option G^{LR} such that $G^{LR} \leq G$. We say G^L is reversible through G^{LR}.

Theorem 1.29. *Let* $G = \{G^{L'}, G^L | G^R\}$ *be a game. Suppose* $G^{L'}$ *is reversible through* $G^{L'R}$. *Let* $H = \{G^{L'RL}, G^L | G^R\}$ *where* $G^{L'RL}$ *ranges over all Left options of* $G^{L'R}$. *Then* $G = H$.

Proof. Let G and H be defined as above. We want to show $G-H=0$ i.e. $G-H$ is a second player win. Note, $G-H = \{G^{L'}, G^L | G^R\} + \{-G^R | -G^{L'RL}, G^L\}$. Consider Left's opening move to $G^{L'} - H$. Right can move to $G^{L'R} - H$. Left has two choices. If Left moves to $G^{L'RL} - H$, then Right has a winning move on H to 0. Otherwise, Left moves to $G^{L'R} - G^R$. We have $G^{L'R} \leq G \lhd\!| \, G^R$. By Lemma 1.26., Right has a win on $G^{L'R} - G^R$.

On the other hand, suppose Right moves first to $G - G^{L'RL}$. Observe $G^{L'RL} \lhd\!| \, G^{L'R}$. Hence, $G - G^{L'RL} \, |\!\rhd \, G - G^{L'R} \geq 0$. By Lemma 1.26., $G - G^{L'RL} \, |\!\rhd \, 0$ which implies Left has a winning response.

All other openings moves have a Tweedle Dee Tweedle Dum response to 0. We may conclude $G = H$. $\square$

Example 1.30. Let $G = \{0, A, \{1-A|0\}|-1+A\}$. Write $G^L = \{1-A|0\}$. We see $G^{LL} = 1-A$ and $G^{LLR} = 1-1 = 0$. So, Left cannot win playing first on G^L which implies $G^L \leq 0$. Since we can ignore dominated options, we have $G = \{0, A|-1+A\}$. Write $G^{L'} = A$. We show $G^{L'}$ is reversible through some $G^{L'R}$. Consider $G^{L'R} = 0$. To see that $G^{L'R} \leq G$, observe that if Right plays first on G, then they must play to $-1+A$. Left can respond to $-1+1 = 0$. This shows Right cannot win playing first on G. This proves the equality.

Definition 1.31. A short game G is in canonical form if for any sub position H of G (including G itself), H has no dominated options and no reversible options.

Theorem 1.32. *Let G be a short game. There exists a short game K in canonical form such that $G = K$.*

Proof. Write $G = \{G^L|G^R\}$. By induction, we can assume all proper sub positions of G are in canonical form. To put G in canonical form, we have the following method:

- 1) Replace all reversible options G^L with the appropriate G^{LRL}. Do the same for reversible options G^R.
- 2) Remove all dominated options.
- 3) If the resulting game has no reversible options, STOP. Otherwise, return to step 1).

The above process must end in a finite number of steps since G is short. Define the resulting game as G'. Then $G' = G$ and G' is in canonical form. □

Definition 1.33. Let G, H be games. We say $G \cong H$ if and only if their game trees are identical (i.e., they are equal in $\tilde{\mathbb{G}}$).

Theorem 1.34. *If $H = K$ and H, K are in canonical form, then $H \cong K$.*

Proof. Let H^L be a Left option of H. Then $H^L - K \lhd\!| \; 0$. If Right has a winning move on H^L, then $H^{LR} - K \leq 0$. But that implies $H^{LR} \leq K = H$, i.e., H^L is reversible through H^{LR}.

This contradicts our assumption that H is canonical. Hence, Right's winning move must be of the form $H^L - K^L \leq 0$. This implies $H^L \leq K^L$. Fix this K^L.

Now suppose Right were to move first to $H - K^L$. Then Left has a winning move. If it were to $H - K^{LR}$ then $H - K^{LR} \geq 0$ i.e., $K \geq K^{LR}$. This implies K^L is reversible through K^L which contradicts our assumption that K is canonical. Hence, Left's winning move must be of the form $H^{L'} - K^L$. This implies $H^L \leq K^L \leq H^{L'}$. But H has no dominated moves. It follows that $H^L = K^L = H^{L'}$.

We may conclude for all H^L, there exists K^L such that $H^L = K^L$ and vice versa. We may conclude the same for Right options. Next, we'll show $H \cong K$. By definition of canonical, every sub position of H and K is canonical. In particular, for each H^L and K^L such that $H^L = K^L$, we have $H^L \cong K^L$ by induction. It follows that $H \cong K$. □

1.5 Simplicity

Our main tools for studying Hackenbush games are the Simplicity Theorem and the Simplicity Rule. We will show that every BR Hackenbush position is a number (Definition 1.35). By the Simplicity Theorem, every BR Hackenbush position can be identified as a certain dyadic rational. The Simplicity Rule gives us a way to identify this dyadic rational. Note that the correspondence between numbers and dyadic rationals respects partial ordering. In particular, adding numbers (Definition 1.35) respects ordinary arithmetic. For example, if a BR Hackenbush position can be identified as $\frac{1}{2}$, then its outcome class is $\mathscr{L}$.

Definition 1.35. We say G is a number if $H^L < H^R$ holds for all sub positions H of G (including G itself).

Lemma 1.36. *If G is a number, then $G^L < G < G^R$ for all Left and Right options of G.*

Proof. Let G^L be a Left option of G. We'll show the difference $G - G^L$ is a win for Left. Left can win going first by moving to $G^L - G^L$. Suppose Right moves first. If Right moves on G, we have $G^R - G^L > 0$ by definition of number. Hence, Left has a winning response. Otherwise, Right moves on $-G^L$. The resulting game is $G+(-G^{LL})$. Left responds to $G^L+(-G^{LL})$. This game is a positive number and Left has a winning move by induction. We may conclude $G^L < G$ for all Left options of G. By a similar argument, $G < G^R$ for all Right options of G. □

Theorem 1.37. *If G is a number, then $-G$ is a number.*

Proof. Suppose G is a number. Then $G^L < G^R$ for all Left and Right options of G. We can rewrite the inequality as $-G^R < -G^L$. But the Left options of $-G$ look like $-G^R$ and the Right options of $-G$ look like $-G^L$. Hence, $(-G)^L < (-G)^R$ for all Left and Right options of $-G$. We may conclude $-G$ is a number. □

Theorem 1.38. *If G and H are numbers, then $G+H$ is a number.*

Proof. By definition of G and H being numbers, $G^L < G^R$ and $H^L < H^R$. This implies $G^L+H < G^R+H$ and $G+H^L < G+H^R$. Hence, it is enough to show $G^L+H < G+H^R$ and $G+H^L < G^R+H$. The first inequality can be rewritten as $0 < (G-G^L)+(H^R-H)$. This inequality is true by lemma 1.36. The second inequality can be rewritten as $0 < (G^R-G)+(H-H^L)$. This inequality is also true by Lemma 1.36.

□

The set of numbers is a non empty subset of $\mathbb{G}$ closed under addition and negatives. We may conclude that the set of numbers forms a subgroup of $\mathbb{G}$.

Definition 1.39. We define $\mathbb{D} = \{\frac{a}{2^b} : a, b \in \mathbb{Z}\}$ and we call this set the dyadic rationals. Note that $(\mathbb{D}, +)$ is a subgroup of $(\mathbb{Q}, +)$.

Our next goal is to show there is a group isomorphism from the subgroup of numbers to the group of dyadic rationals. The Simplicity Theorem and Simplicity Rule will follow shortly after, both of which are crucial to the study of Hackenbush.

We begin by defining the following short games: Recall $0 \cong \{|\}$. Next, we define $\left[1\right] = \{0|\}$. We can think of $\left[1\right]$ as being a game where Left has an advantage worth 1 move. By playing the difference game, we see that $\left[1\right] = \{0|\left[1\right]\} + \{0|\left[1\right]\}$. Hence, we define $\left[\frac{1}{2}\right] = \{0|\left[1\right]\}$. Similarly, we see that $\left[\frac{1}{2}\right] = \{0|\left[\frac{1}{2}\right]\} + \{0|\left[\frac{1}{2}\right]\}$. Hence, we define $\left[\frac{1}{4}\right] = \{0|\left[\frac{1}{2}\right]\}$. This suggests the more general definition:

Definition 1.40. For any nonnegative integer n, we define $\left[\frac{1}{2^n}\right] = \{0|\left[\frac{1}{2^{n-1}}\right]\}$.

We note that $\left[\frac{1}{2^n}\right]$ is strictly greater than 0, by induction.

Lemma 1.41. $\left[\frac{1}{2^n}\right] + \left[\frac{1}{2^n}\right] = \left[\frac{1}{2^{n-1}}\right]$ *for any integer* $n \geq 1$.

Proof. We will show that the game $\left[\frac{1}{2^{n-1}}\right] - \left[\frac{1}{2^n}\right] - \left[\frac{1}{2^n}\right]$ is a second player win. Suppose Left moves first. If Left moves on $\left[\frac{1}{2^{n-1}}\right]$, then Right necessarily moves on one of the $-\left[\frac{1}{2^n}\right]$, which is negative. On the other hand, Left can open by moving to $\left[\frac{1}{2^{n-1}}\right] - \left[\frac{1}{2^{n-1}}\right] - \left[\frac{1}{2^n}\right] = -\left[\frac{1}{2^n}\right]$, which is negative. We may conclude Left cannot win going first.

Next, we show Right cannot win by going first. If Right moves on $\left[\frac{1}{2^{n-1}}\right]$, then the resulting game is $\left[\frac{1}{2^{n-2}}\right] - \left[\frac{1}{2^n}\right] - \left[\frac{1}{2^n}\right]$. By induction,

$$\begin{aligned}
[\frac{1}{2^{n-2}}] - [\frac{1}{2^n}] - [\frac{1}{2^n}] &= [\frac{1}{2^{n-1}}] + [\frac{1}{2^{n-1}}] - [\frac{1}{2^n}] - [\frac{1}{2^n}] \\
&= \left([\frac{1}{2^{n-1}}] - [\frac{1}{2^n}]\right) + \left([\frac{1}{2^{n-1}}] - [\frac{1}{2^n}]\right) \\
&> 0.
\end{aligned}$$

On the other hand, Right can open by moving on $-\left[\frac{1}{2^n}\right]$. The resulting game is $\left[\frac{1}{2^{n-1}}\right] + 0 - \left[\frac{1}{2^n}\right]$. Left wins by moving to $\left[\frac{1}{2^{n-1}}\right] + 0 - \left[\frac{1}{2^{n-1}}\right] = 0$. This shows Right cannot win by moving first. We may conclude the desired equality. $\square$

Lemma 1.42. *Let A, B be posets. If $f : A \to B$ is a mapping that respects partial ordering, then f is injective.*

Proof. Suppose $f(x) = f(y)$. Then $f(x) \geq f(y)$ and $f(x) \leq f(y)$. Since f respects partial ordering, we have $x \geq y$ and $x \leq y$. Equivalently, $x = y$. This shows f is injective. $\square$

Theorem 1.43. *There exists an injective group homomorphism from $\mathbb{D}$ to $\mathbb{G}$.*

Proof. For odd a, define a mapping by $\frac{a}{2^b} \mapsto a\left[\frac{1}{2^b}\right]$ where

$$a\left[\frac{1}{2^b}\right] = \underbrace{\left[\frac{1}{2^b}\right] + \cdots + \left[\frac{1}{2^b}\right]}_{a \text{ times}} \tag{1.4}$$

Let $\frac{a}{2^b}, \frac{m}{2^n} \in \mathbb{D}$. Then $\frac{a}{2^b} + \frac{m}{2^n}$ maps to $(a2^{n-b} + m)\left[\frac{1}{2^n}\right] = a2^{n-b}\left[\frac{1}{2^n}\right] + m\left[\frac{1}{2^n}\right] = a\left[\frac{1}{2^b}\right] + m\left[\frac{1}{2^n}\right]$. The last equality is true since repeated applications of Lemma 1.41 shows $2^{n-b}\left[\frac{1}{2^n}\right] = \left[\frac{1}{2^b}\right]$. We may conclude that this mapping is a group homomorphism. Moreover, this mapping respects the partial ordering of $\mathbb{G}$, and is therefore injective. $\square$

We can now identify the game $a[\frac{1}{2^b}]$ with the dyadic rational $\frac{a}{2^b}$. Because of this, we will now drop the bracket notation.

Definition 1.44. Let $I \subset \mathbb{D}$. Then I is an interval if for any $x, y \in I$ with $x > y$, we have $z \in I$ for all z such that $x > z > y$.

Definition 1.45. Let G be a short game. Define the birthday of G, denoted $b(G)$, as the smallest integer n such that $G \in \mathbb{G}_n$. For completeness, we also define the formal birthday of G, denoted $\tilde{b}(G)$, as the smallest integer n such that $G \in \tilde{\mathbb{G}}_n$.

Note that $b(G)$ only cares about the game value G, while $\tilde{b}(G)$ cares about the structure of G, i.e., its options.

Lemma 1.46. *Let $\frac{a}{2^b} \in \mathbb{D}$ where a is odd. Then $\frac{a}{2^b} = \{\frac{a-1}{2^b} | \frac{a+1}{2^b}\}$ is the canonical form of $\frac{a}{2^b}$.*

Proof. Write

$$\frac{a}{2^b} = \underbrace{\frac{1}{2^b} + \frac{1}{2^b} + \cdots + \frac{1}{2^b}}_{a \text{ times}} \tag{1.5}$$

Recall that $\frac{1}{2^b} = \{0|\frac{1}{2^{b-1}}\}$. Hence there is exactly one Left option, and it is $0 + \frac{1}{2^b} + \cdots + \frac{1}{2^b} = \frac{a-1}{2^b}$. Similarly, there is exactly one Right option, and it is $\frac{1}{2^{b-1}} + \frac{1}{2^b} + \cdots + \frac{1}{2^b} = \frac{2}{2^b} + \frac{1}{2^b} + \cdots + \frac{1}{2^b} = \frac{a+1}{2^b}$. We may conclude $\frac{a}{2^b} = \{\frac{a-1}{2^b}|\frac{a+1}{2^b}\}$.

Note that $\frac{a}{2^b}$ has no dominated options since that would require at least two Left options or at least two Right options. Next, we will show $\frac{a}{2^b}$ has no reversible options. For clarity of notation, we define $G = \frac{a}{2^b}$. Note that $\frac{a-1}{2^b} = \frac{a'}{2^{b'}}$ in reduced form where a' is odd and $b' < b$. We know $\frac{a'}{2^{b'}} = \{\frac{a-1}{2^b} - \frac{1}{2^{b'}}|\frac{a-1}{2^b} + \frac{1}{2^{b'}}\}$. Hence, $G^{LR} = \frac{a-1}{2^b} + \frac{1}{2^{b'}}$ where $b' < b$. Therefore, $G^{LR} \geq \frac{a-1}{2^b} + \frac{1}{2^{b-1}} = \frac{a+1}{2^b} > G$. It follows that G^L is not reversible through G^{LR}. We may conclude G has no reversible Left options. By a similar argument, we may conclude G has no reversible Right options. This proves the lemma. □

Lemma 1.47. *If G is in canonical form, then $\tilde{b}(G) = b(G)$.*

Proof. Let $\tilde{b}(G) = n$. Certainly the game value of G occurs at or before the formal birthing of G. Hence, $\tilde{b}(G) \geq b(G)$. Assume by contradiction that $\tilde{b}(G) > b(G)$. Then there is a short game H such that $b(G) = \tilde{b}(H)$. But the canonical form of H, call it K, is certainly born by the time H is born. So, $b(G) = \tilde{b}(H) \geq \tilde{b}(K)$. But $G \cong K$ since canonical forms are unique, and thus we have $b(G) \geq \tilde{b}(G)$. By our assumption, this gives $\tilde{b}(G) > b(G) \geq \tilde{b}(G)$, a contradiction. We may conclude $\tilde{b}(G) = b(G)$. □

Lemma 1.48. *If G is in canonical form, then $b(G^L) < b(G)$ and $b(G^R) < b(G)$ for all G^L, G^R.*

Proof. Let G^L and G^R be options of G. We have that G is in canonical form. By definition of canonical, we have that all of the options of G are canonical. Hence,

$$b(G) = \tilde{b}(G) > \tilde{b}(G^L) = b(G^L) \tag{1.6}$$

where the middle inequality follows by our recursive construction of the short games. A similar argument shows $b(G^R) < b(G)$. □

Lemma 1.49. *Let $I \subset \mathbb{D}$ be a non empty interval. Then there exists a unique $x \in I$ of minimal birthday.*

Proof. Suppose $x, y \in I$ have the same birthday, say n, and are in canonical form. We may assume $x > y$. Then $x - y > 0$. So, Left has a winning move of the form $x^L - y \geq 0$ or $x - y^R \geq 0$. In the former case, we have $x > x^L \geq y$ which implies $x^L \in I$. In the latter case, we have $x \geq y^R > y$ which implies $y^R \in I$. Observe that x^L and y^R have birthdays strictly less than n. So, we have found an element of strictly less birthday. Repeating this process at most n times will yield a unique element of minimal birthday. □

Theorem 1.50. *(Simplicity Theorem): Let G be a short game and define $I(G) = \{x \in \mathbb{D} : \forall G^L, G^R, G^L \lhd| \, x \lhd| \, G^R\}$. If $I(G) \neq \emptyset$, then $G = x$ where x is the unique element in I of minimal birthday.*

Proof. Suppose $I(G) \neq \emptyset$ and let x be the unique element in $I(G)$ of minimal birthday. Let us show $G - x = 0$. It suffices to show $G - x$ is a second player win. Suppose Left goes first. Left has two options: $G^L - x$ and $G - x^R$. But $G^L - x \lhd| \, 0$ by our definition of $I(G)$, so this is a losing move for Left. On the other hand, $x^R \notin I(G)$ since it has strictly less birthday than x. Hence, there is a Left option of G such that $G^L \geq x^R$ or there is a Right option of G such that $G^R \leq x^R$. If the latter is true, then Right has a winning response on $G - x^R$. If the former is true, then $G^L \geq x^R > x$ which contradicts the fact that

$x \in I(G)$. We may conclude Left cannot win going first. Using a similar argument, we have that Right cannot win going first. This shows $G = x$. □

Theorem 1.51. *Simplicity Rule: Let G be a short game and define* $I(G) = \{x \in \mathbb{D} : \forall G^L, G^R, G^L \lhd| \, x \lhd| \, G^R\}$. *Suppose* $I(G)$ *is non empty. Then the unique element in* $I(G)$ *of minimal birthday can be found as follows:*

1) If $I(G)$ *contains an integer, then x is the integer of least magnitude.*

2) Otherwise, x is the unique element of minimal denominator.

Proof. First, suppose that $I(G)$ contains a positive number and a negative number. Then $I(G)$ contains 0 and the unique element of minimal birthday is 0. Therefore, we may assume $I(G)$ contains only positive numbers. Suppose $I(G)$ contains an integer. Write the integers of $I(G)$ in ascending order as $n_1 < n_2 < n_3 < \dots$. We have $n_1 = \{n_1 - 1|\}$. We will show $G - n_1 = 0$. Suppose Left goes first. They have one option: $G^L - n_1$. Since $n_1 \in I(G)$, we have $G^L - n_1 \lhd| \, 0$. Hence, Right can force a win. Suppose Right goes first. They have two options: $G^R - n_1$ and $G - (n_1 - 1)$. In the former case, we have $G^R - n_1 \, |\rhd \, 0$ and so Left can force a win. In the latter case, we have $n_1 - 1 \notin I(G)$. Hence, we can find a Left option such that $G^L \geq n_1 - 1$ or we can find a Right option such that $G^R \leq n_1 - 1$. The latter case can't occur, since it would imply $G^R \leq n - 1 < n$. So, we must have $G^L - (n_1 - 1) \geq 0$. This shows Left has a winning response on $G - (n_1 - 1)$. We may conclude $G = n_1$.

Now suppose $I(G)$ contains no integers. Let $x = \frac{a}{2^b}$ be the unique element of minimal denominator. First, we prove that such an element exists. To do so, we will show that if there are two distinct elements with the same denominator in $I(G)$, then we can find an element in $I(G)$ of strictly less denominator. Suppose $\frac{m}{2^n}$ and $\frac{m+k}{2^n}$ are elements in $I(G)$, in reduced form. If $k \geq 2$, then $(m+1)/2^n \in I(G)$. On the other hand, if $k = 1$ then either $m/2^n$ or $(m+k)/2^n$ is not in reduced form. This proves the claim. Now we will show $G - x = 0$.

Suppose Left goes first. They have two options: $G^L - x$ and $G - x^R$. Since $G^L \lhd\!| \ x$, we have that Right has a winning response on $G^L - x$. As for $G - x^R$, since x^R is a number of strictly less denominator than x, we have $x^R \notin I$. Then we can find an option of G such that $G^L \geq x^R$ or $G^R \leq x^R$. But the former implies $G^L \geq x^R > x$, a contradiction. So, we must have $G^R - x^R \leq 0$, which shows Right has a winning response on $G - x^R$. A similar argument shows Right cannot win moving first. We may conclude $G - x$ is a second player win, i.e., $G = x$. □

Theorem 1.52. *If G is a number, then $I(G)$ is non empty. In particular, if G is a number, then G is a dyadic rational.*

Proof. Suppose G is a number. Then every option of G is a number. By induction, we may assume $I(G^L)$ and $I(G^R)$ are non empty for every option of G. By the Simplicity Theorem, every G^L and G^R is a dyadic rational. Since G is short, it must have finitely many options. Therefore, we can find a maximum G^L and a minimum G^R. Denote these as $G^L = \frac{a}{2^b}$ and $G^R = \frac{c}{2^d}$ where a, c are odd. Then $x = \frac{G^L + G^R}{2}$ is a dyadic rational in between. In particular, $G^L < x < G^R$ for all options of G. We may conclude $I(G)$ is non empty. □

Let G be the game in Figure 1.3.

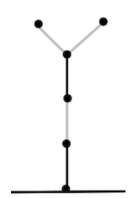

Fig. 1.3: Example of Simplicity Rule

We will determine which number G is equal to. We can rewrite G as in Figure 1.4. Evaluating the strings, we have the following. The empty string is worth 0. The string that is a single blue edge is equal to $\{0|\} = 1$. The string with one blue edge on the ground and one red edge on top is equal to $\{0|1\}$. By the Simplicity Rule, $\{0|1\} = \frac{1}{2}$. Evaluating

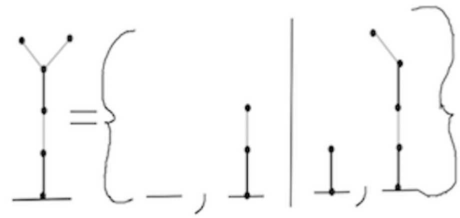

Fig. 1.4: Example of Simplicity Rule Continued

the longest string gives $\{0, \frac{1}{2}|1, \frac{3}{4}\} = \frac{5}{8}$. Removing dominated options, we have $G = \{\frac{1}{2}|\frac{5}{8}\}$. Using the Simplicity Rule, we have $G = \frac{9}{16}$.

We have shown that if G is a short game and a number, then G is a dyadic rational. But in general, $I(G)$ may be empty. In this case, G is not a number. Some examples of non numbers are $\{50|-50\}, \{0|0\}$ and $\{0|\{0|0\}\}$. The latter two games are usually referred to as $*$ and $\uparrow$.

2 BALANCED BLUE RED HACKENBUSH SPIDERS

2.1 Introduction

In this section we begin our study of Blue Red Hackenbush spiders. Various classes of spiders have been studied such as 1 legged spiders, 2 legged spiders, and redwood spiders. Building on this, we prove a solution to a certain kind of n-legged spiders.

2.2 Definitions

Definition 2.1. Let G be a Blue Red Hackenbush game. Let e be an edge in G. We define the height of e as the length of the shortest path from e to the ground.

In Figure 2.1, e_1 has height 1, e_2 has height 4 and e_3 has height 3.

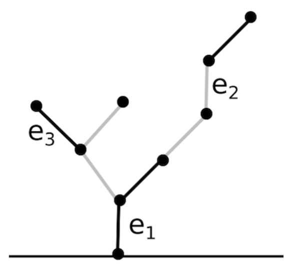

Fig. 2.1: Example of Height

Definition 2.2. Define $S = \varepsilon_0 : \varepsilon_1 : \varepsilon_2 : \cdots : \varepsilon_n$ to be a string of height $n+1$ where $\varepsilon_i \in \{-1, 1\}$. Each ε_i represents an edge at height $i+1$. If $\varepsilon_i = 1$, then ε_i is a blue edge. If $\varepsilon_i = -1$, then ε_i is a red edge. Note, ε_0 is the edge on the ground and ε_n is the highest edge. We define the ρ function on BR Hackenbush strings as

$$\rho(S) = \sum_{k=0}^{n} \varepsilon_k \frac{1}{2^k}$$

where $n+1$ is the length of the string.

Definition 2.3. We define a Blue Red Hackenbush spider as a collection of Blue Red Hackenbush strings attached at their highest edges. The vertex at which the strings attach is called the body. We allow for strings to be attached on top of the body; we refer to these strings as arms.

It is helpful to describe a spider in terms of its legs and arms. Let S be a spider with legs $X_1, \ldots, X_k$ and arms $A_1, \ldots, A_m$. We define the following notation $S = X_1/X_2/\ldots/X_k\backslash A_1\backslash\ldots\backslash A_m$. Figure 2.2 and Figure 2.3 are examples of spiders.

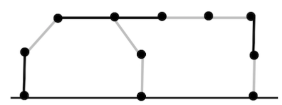

Fig. 2.2: Example of a Spider

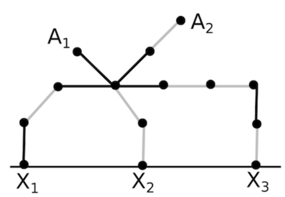

Fig. 2.3: A Spider of the Form $X_1/X_2/X_3\backslash A_1\backslash A_2$

2.3 Strings

In this section we give a solution to strings.

Theorem 2.4. *When playing in a string, higher moves dominate lower moves.*

Proof. We will show it is true for Left. Let S be an arbitrary string with at least two blue edges. We let S be the string in Figure 2.4, where m is some nonnegative integer.

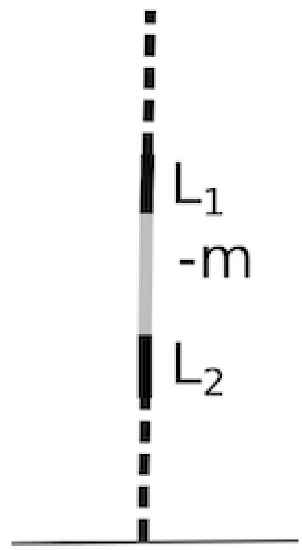

Fig. 2.4: A String With Two Blue Edges

Consider the difference game $S^{L_1} - S^{L_2}$ in Figure 2.5.

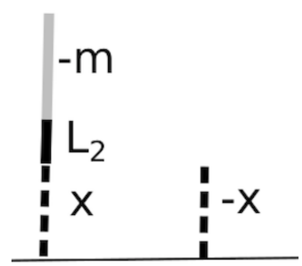

Fig. 2.5: Difference Game of S

For clarity, we label the bottom parts of the strings x and $-x$. Note that the negative of a Blue Red Hackenbush game is drawn by reversing the colors. We will show Right cannot win going first. If Right moves above the edge labelled L_2, then Left can respond by deleting L_2. The result is the 0 game with Right to move. Hence, Right loses. If Right moves at an edge of height strictly less than the height of the edge L_2, then Left plays at the same height in the other string. The result is the 0 game with Right to move. Hence, Right loses. We may conclude $S^{L_1} - S^{L_2} \geq 0$ i.e., S^{L_1} dominates S^{L_2}. $\square$

Lemma 2.5. *Let S be a string with a swap foot. WLOG, suppose the edge on the ground is blue. Then* $0 < S < 1$.

Proof. Let S be the game in Figure 2.6.

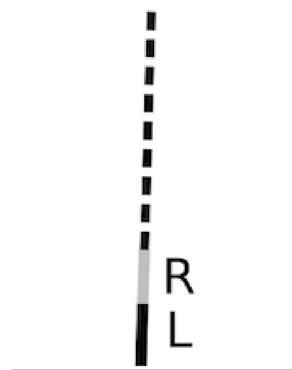

Fig. 2.6: A String with a Swap Foot

First we show $S > 0$. If Right goes first, then no matter where they move, Left can delete the edge labelled L and win. This shows $S \geq 0$. If Left goes first, they win by deleting the edge L. This shows $S \neq 0$. Hence, $S > 0$. Now consider the difference game $S - 1$. If Left goes first, there are two cases:

Case 1: Left moves above the swap foot. In this case, Right should delete the edge labelled R. The result is $1 - 1 = 0$.

Case 2: Left deletes the edge on the ground in S. The result is $0 + (-1) = -1$.

Either way, Left cannot win playing first. Hence, $S - 1 \leq 0$. But Right has a winning move as the first player by moving to $S^R - 1 = 1 - 1 = 0$. Hence, $S - 1 \neq 0$. We may conclude $S - 1 < 0$. This proves the lemma. $\square$

Theorem 2.6. *(vanRoode's string evaluation): For arbitrary $n \geq 1$, define a string $S = \varepsilon_0 : \varepsilon_1 : \varepsilon_2 : \cdots : \varepsilon_{n-1}$. Assume S has a swap foot. Then $S = \rho(S)$.*

Proof. We have seen this is true when $n-1 = 2$. Proceeding by strong induction, consider the string $S : 1$. By induction, we know $S = \frac{a}{2^{n-1}}$ for some odd integer a. We know players will prefer to move highest. Hence, Left will delete the highest edge which results in S. Right will delete their highest edge which results in $S + \frac{1}{2^{n-1}}$. Hence, $S : 1 = \{\frac{a}{2^{n-1}} | \frac{a}{2^{n-1}} + \frac{1}{2^{n-1}}\}$. We see that $I(S : 1)$ is non empty. By lemma 2.4., $S : 1$ is not equal to an integer. Hence, we seek the unique element of minimal denominator in $I(S : 1)$. In particular, we claim this number is $S + \frac{1}{2^n}$.

Any element in $I(S : 1)$ can be written as $\frac{a}{2^{n-1}} + \frac{b}{2^c}$ where $\frac{b}{2^c}$ is reduced. If $c \leq n-1$, then $\frac{a}{2^{n-1}} + \frac{b}{2^c}$ it not an element of $I(S : 1)$. If $c = n$, then we must have $b = 1$, for if $b \neq 1$ then $\frac{a}{2^{n-1}} + \frac{b}{2^c}$ is not an element of $I(S : 1)$. Hence, $I(S : 1)$ contains an element in reduced form whose denominator is 2^n. The only other cases are elements where $c > n$. By the Simplicity Rule, we have found the unique element of minimal denominator and so $S : 1 = S + \frac{1}{2^n}$.

If we had instead considered $S : (-1)$, then by a similar argument we have $S : (-1) = \{S - \frac{1}{2^{n-1}} | S\} = S - \frac{1}{2^n}$. This completes the induction. □

The following lemma is useful for finding upper and lower bounds.

Lemma 2.7. *An edge is worth at least as much on the ground as it is attached to a component.*

Proof. [3] □

Theorem 2.8. *We can always peel off edges on a string until the string has a swap foot in the following way:*

Note, G in Figure 2.7 is an arbitrary Blue Red Hackenbush position.

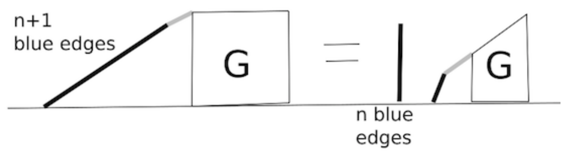

Fig. 2.7: Peeling Off Edges

Proof. [3] □

Theorem 2.6. and 2.8. combine into a solution for strings.

2.4 Arches

In this section we present a solution to two legged spiders. Proofs of these formulas can be in [4].

Definition 2.9. We refer to a spider with two legs as an arch.

Theorem 2.10. *Let A, B be strings with, both with a swap foot, and of the same length, say $n \geq 2$. Then $A/B = A + B$.*

Proof. [4]. □

Theorem 2.11. *Let A, B be balanced strings of length $n \geq 2$. Then* $(A:1)/B = A:1 + B:1$.

Proof. [4]. □

The above formulas say that solving arches reduces to solving strings, since every arch can be equated to a sum of strings. We can think of the extra $\frac{1}{2^{n-1}}$ in Theorem 2.11. as a 'bonus'. Why bonuses occur is still not fully understood, aside from "the math just works out".

To evaluate the arch in Figure 2.8, we need only evaluate the strings on the right hand side. The sum of strings evaluates to $\frac{5}{8}+\frac{1}{8}=\frac{3}{4}$. So, the arch is equal to $\frac{3}{4}$.

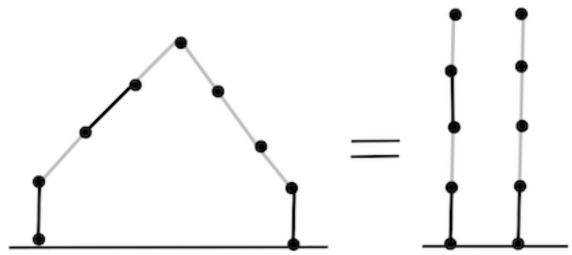

Fig. 2.8: An Arch with no Bonus

To evaluate the arch in Figure 2.9, we evaluate the strings on the right hand side. The sum of strings evaluates to $\frac{11}{16}+\frac{3}{16}=\frac{14}{16}=\frac{7}{8}$. So, the arch is equal to $\frac{7}{8}$. Note that the blue edge gets ”doubled”.

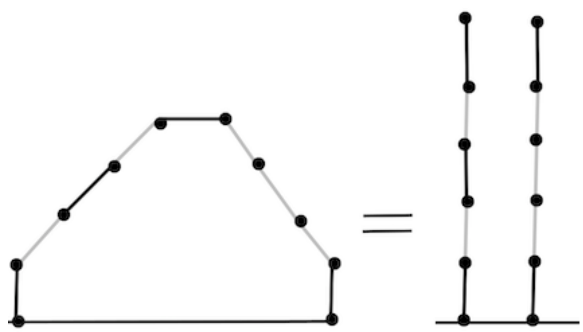

Fig. 2.9: An Arch with a Bonus

2.5 Balanced Spiders

In this section, we present a solution to balanced spiders. The results here are joint work with Tim Hsu and Ardak Kapbasov. We will present proofs of the author’s main contributions. For theorems that the author did not significantly contribute to, we will state them without proof. In particular, we will only state Theorem 2.16, which is due to Kapbasov.

Definition 2.12. A string has a swap foot if the two edges closest to the ground have different color. (i.e. one is red and the other blue).

Definition 2.13. A balanced spider is a spider such that every leg has a swap foot and all legs share the same number of edges. (i.e., all legs have the same height). We also allow the possibility for the spider to have arms.

Definition 2.14. We define the height of a balanced spider to be the height of one (and therefore all) of its legs.

Definition 2.15. Let S be a spider (not necessarily balanced). We define t_b to be the number of blue edges incident to the body and t_r to be the number of red edges incident the body.

Theorem 2.16. *(Optimal play is highest) Let S be a balanced spider. If* e_1 *and* e_2 *are two blue edges in S and* e_2 *has height strictly greater than* e_1, $S^{e_1} - S^{e_2} > 0$ *where* S^e *is the game that results by deleting edge e. If the height of* e_1 *equals the height of* e_2 *then* $S^{e_1} = S^{e_2}$.

Proof. The proof is due to Ardak Kapbasov. [3]. □

Consider the spider G in Figure 2.10.

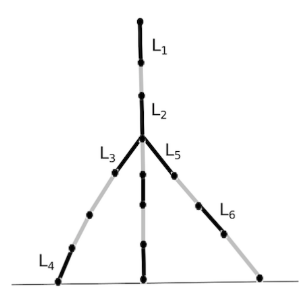

Fig. 2.10: Edges of Various Height

Then $G^{L_1} > G^{L_2} > G^{L_3} > G^{L_4}$. Also, $G^{L_3} = G^{L_5}$ and $G^{L_3} > G^{L_6} > G^{L_4}$. Knowing that the optimal strategy for balanced spiders is to delete the highest edge will be crucial for the remaining arguments in this section.

Lemma 2.17. *Let S_1 and S_2 be strings of height n_1 and n_2 respectively. If $n_1 \geq n_2$, then Left may as well play in S_1. In other words, $S_1^L + S_2 \geq S_1 + S_2^L$.*

Proof. We know that if a player plays in a string, then they will play highest in that string. Consider the sum $S_1 + S_2$ in Figure 2.11.

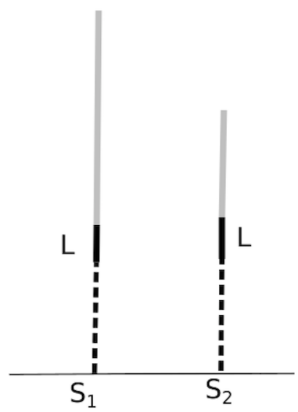

Fig. 2.11: The Game $S_1 + S_2$

It is enough to show $S_1^L + S_2 \geq S_1 + S_2^L$. We see that

$$S_1^L + S_2 = (S_1 + S_2) - \frac{1}{2^{n_1-1}} \geq S_1 + S_2 - \frac{1}{2^{n_2-1}} = S_1 + S_2^L$$

The middle inequality is true since $-\frac{1}{2^{n_1-1}} \geq -\frac{1}{2^{n_2-1}}$ when $n_1 \geq n_2$. This proves the lemma. We note that the height of the moves did not matter. □

We note that the above lemma shows that given a sum of strings with swap feet $G_1 + G_2 + \ldots$, a player may not necessarily play highest in $G_1 + G_2 + \ldots$, but rather play highest in whichever string they move in. And by the above lemma, a player will move in the string with the most edges.

Lemma 2.18. *Suppose $S = X_1/X_2\backslash(-1) : A$ is a balanced arch with an arm, i.e., X_1/X_2 forms a balanced spider and we then attach an arm $(-1) : A$ to that body. Then $S = X_1 + X_2 + \frac{1}{2^{n-1}}\rho((-1) : A)$ where n is the height of S.*

Proof. Note, we could replace $(-1) : A$ with A, but specifying the bottom edge of the arm is convenient in the following proof. We are also assuming, WLOG, that the bottom edge of the arm is red. We induct on the length of the arm and take the base case to be when the length of the arm is empty. This is the balanced arch formula which was proved by vanRoode [4].

Proceeding by induction, we want to show the game in Figure 2.12 is a second player win.

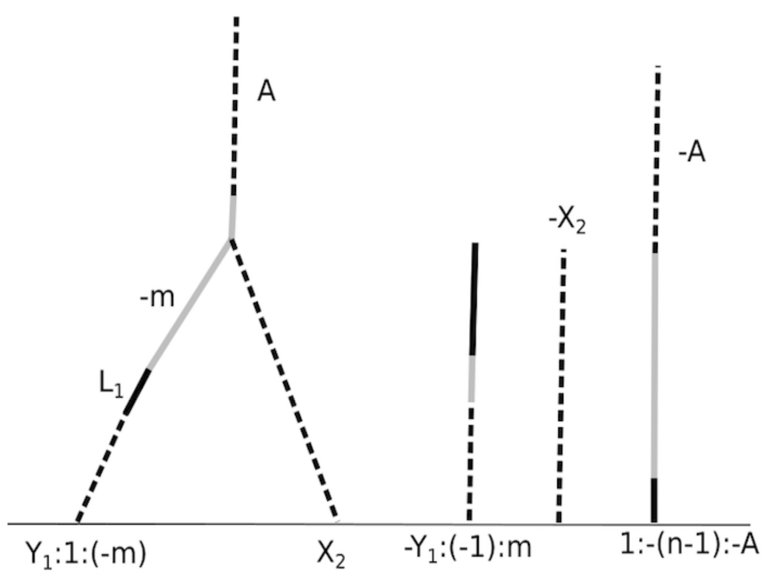

Fig. 2.12: Difference Game for Lemma 2.18.

Note that m is a nonnegative integer. We also note that the right most string is equal to $\rho(1:(-(n-1)):-A)$. Furthermore, we have

$$\begin{aligned}\rho(1:(-(n-1)):-A) &= 1-\frac{1}{2}-\cdots-\frac{1}{2^{n-1}}+\frac{1}{2^n}\rho(-A)\\ &= \left(1-\frac{1}{2}-\cdots-\frac{1}{2^{n-1}}\right)+\frac{1}{2^n}\rho(-A)\\ &= \frac{1}{2^{n-1}}+\frac{1}{2^n}\rho(-A)\\ &= \frac{1}{2^{n-1}}\rho(1:-A)\end{aligned} \tag{2.1}$$

Suppose Left moves first. If they move in the spider, they have two kinds of moves. If they move in A, then Right can mimic this response in $-A$ which results in $G^{LR}=0$ by the induction hypothesis. If Left moves in the legs, they move highest. WLOG, we assume L_1 is a highest edge in the legs. It is enough to show G^{L_1} is negative. We have

$$\begin{aligned}G^{L_1} &= \frac{1}{2^n}\rho((-m-1):A)-\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\frac{1}{2^n}\rho(-A)\\ &\leq \frac{1}{2^n}\rho(-1:A)+\frac{1}{2^n}\rho(-A) && \text{(Assuming } m=0\text{)}\\ &\leq -\frac{1}{2^n}+\left(\frac{1}{2^n}-\frac{1}{2^{n+1}}\right)+\left(\frac{1}{2^{n+1}}-\frac{1}{2^{n+2}}\right)+\dots && \text{(If } -A \text{ all blue)}\\ &< 0\end{aligned} \tag{2.2}$$

By Lemma 2.17, if Left plays in the strings, then they necessarily play in the longest string. Whether they move in $-A$ or at the edge of lowest height, Right will have a winning corresponding move in $(-1):A$ in the spider. This shows Left cannot win moving first. Next, we'll show Right cannot win moving first.

We rewrite the game as shown in Figure 2.13.

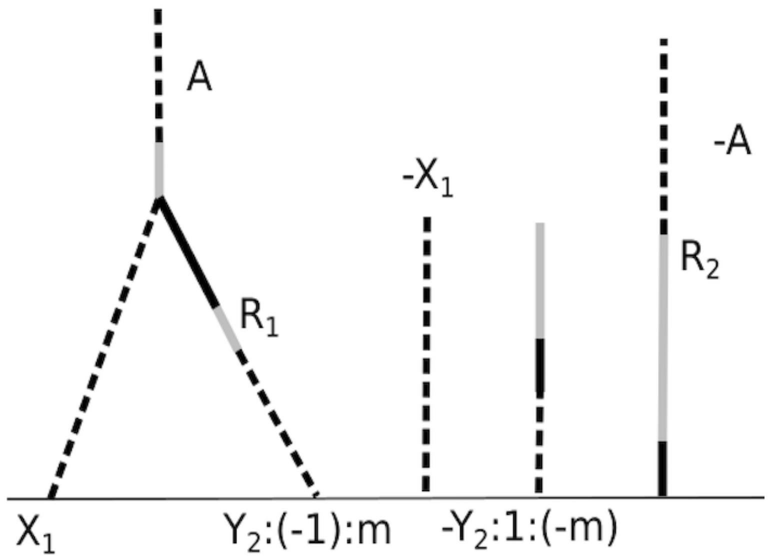

Fig. 2.13: An Equivalent Game

Note, m is a nonnegative integer. We have

$$\begin{aligned} G^{R_1} &\geq \frac{1}{2^n}\rho((-1):A)+\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}\rho(1:(-A)) && \text{(by letting } m=0) \\ &= \frac{1}{2^{n-1}}(\rho(1:(-1):(A))+\rho(1:(-A))) && \text{(Rearranging)} \\ &> 0 && \text{(sum of two positive numbers)} \end{aligned} \tag{2.3}$$

Lastly, if Right plays in $(-1):A$ or $-A$, then Left has a natural winning response. But, if $-A$ is all blue, then Right's best move in the strings is at R_2. Then Left can respond by playing highest in the spider's legs. Using our calculation from earlier, we have

$$\begin{aligned} G^{R_2L} &= \frac{1}{2^n}\rho((-m-1):A)-\frac{1}{2^{n-1}}+\frac{1}{2^{n-2}} \\ &= \frac{1}{2^{n-2}}-\frac{1}{2^{n-1}}+\frac{1}{2^n}\rho((-m-1):A) \\ &> 0 \end{aligned} \tag{2.4}$$

We may conclude Right cannot win going first. This completes the induction.

□

Lemma 2.19. *Let $S = X_1/X_2/X_3$ be a balanced spider with 3 legs. Then*

$$S = \begin{cases} X_1 + X_2 + X_3 & t_b = 1 \text{ or } 2 \\ X_1 + X_2 + X_3 + \frac{1}{2^{n-1}} & t_b = 3 \end{cases}$$

where the height of S is n. By symmetry (i.e., by considering negatives), the above formula solves all balanced spiders with three legs.

Proof. We consider two cases:

Case 1: $t_b = 1$ or 2. We want to show the game in Figure 2.14 is a second player win.

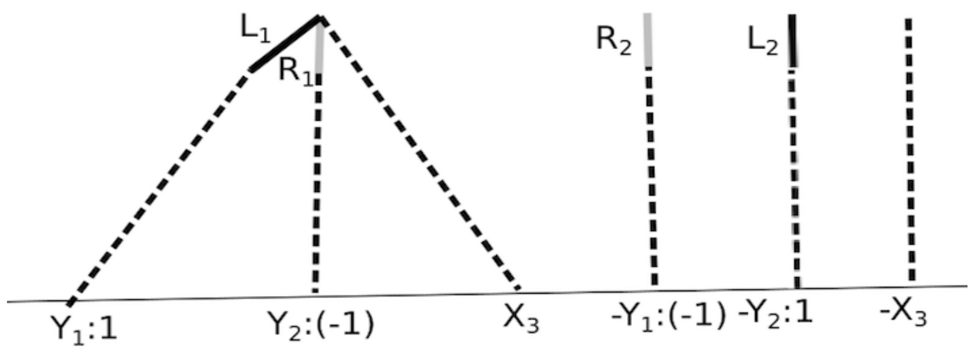

Fig. 2.14: Difference Game for Case 1 of Lemma 2.19.

Note $X_1 = Y_1 : 1$ and $X_2 = Y_2 : (-1)$. Since players will play highest in any given component, the only moves to consider are L_1, L_2, R_1, R_2. But $G^{L_1R_2} = G^{R_2L_1} = G^{R_1L_2} = G^{L_2R_1} = 0$, by vanRoode. This shows G is a second player win.

Case 2: $t_b = 3$.

We want to show the game G in Figure 2.15 is a second player win.

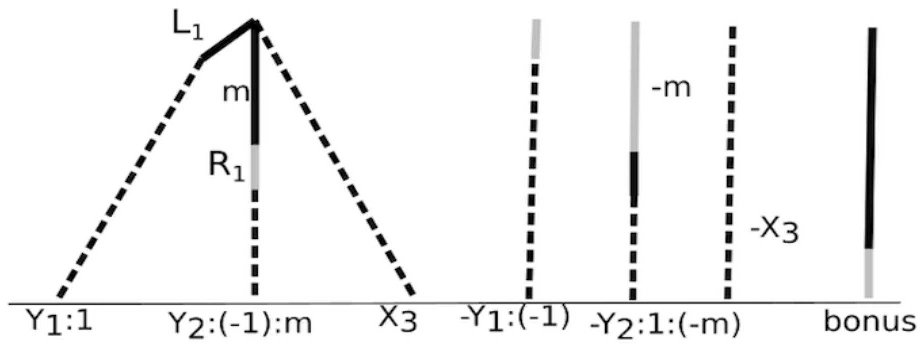

Fig. 2.15: Difference Game for Case 2 of Lemma 2.19.

We can rewrite Figure 2.15 as Figure 2.16 by adding the top edges of the strings $-X_1, -X_2, -X_3$ and the bonus (which is worth $-\frac{1}{2^{n-1}}$).

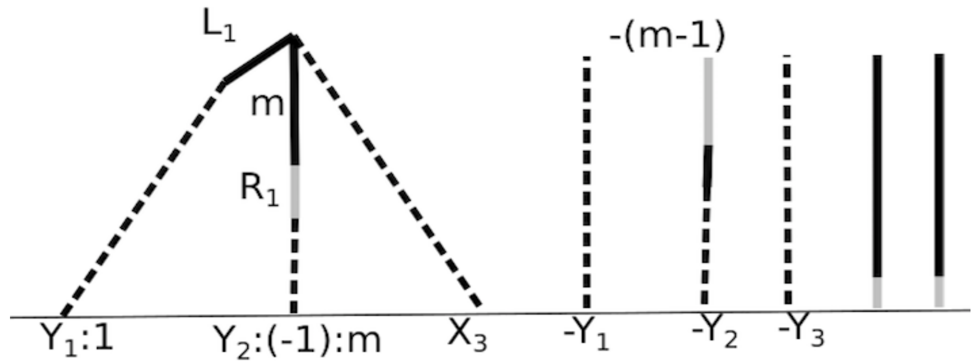

Fig. 2.16: A Game Equivalent to Figure 2.15

We note that $X_3 = Y_3 : 1$. In particular, the two rightmost strings have height $n-1$ and so are each worth $-\frac{1}{2^{n-2}}$. We will show that this game is a second player win. Suppose Left moves first. If Left moves in the spider then they move highest and we have $G^{L_1} = -\frac{1}{2^{n-2}}$. If Left moves in the strings, Right has a winning move on R_1. We have $G^{LR_1} = -\frac{1}{2^{n-2}} + \frac{1}{2^{n-1}}\rho(m) < 0$. This shows Left cannot win moving first.

Suppose Right moves first. If Right moves in the spider, we have $G^{R_1} = \frac{1}{2^{n-1}}\rho(m) > 0$. If Right moves in the strings, have Left move in the spider and we have $G^{RL_1} = 0$. This shows Right cannot win moving first which completes the proof. □

The above theorem and argument generalize to an arbitrary number of legs.

Theorem 2.20. *Let $S = X_1/X_2/\dots/X_k$ be a balanced spider. Define t_b to be the number of legs whose top edge (i.e. the edge containing the body) is blue. Define t_r to be the number of legs whose top edge (i.e. the edge containing the body) is red. WLOG assume $t_b \geq t_r$ (the case for $t_r \geq t_b$ will follow, since negatives of Blue Red Hackenbush positions are obtained by reversing colors). Define $t = t_b - t_r$. Then*

$$S = \begin{cases} X_1 + X_2 + \cdots + X_k & 0 \leq t \leq 2 \\ X_1 + X_2 + \cdots + X_k + \frac{t-2}{2^{n-1}} & 2 < t \end{cases}$$

where the height of S is n.

Proof. Let $A(k)$ say the above formula holds and let $B(k)$ say if $t_b = k$ and $t_r = 0$ and $M \geq 0$ is a string then $S\backslash M = S + \frac{1}{2^{n-1}}\rho(M)$. The statement $B(k)$ will be used when the edges incident to the body are all of the same color. We will show

$$A(k) \Longrightarrow B(k) \Longrightarrow A(k+1) \Longrightarrow B(k+1) \Longrightarrow \dots$$

for $k \geq 2$.

(base case): $A(2)$ is the arch formula, $B(2)$ is the arch with an arm formula, and $A(3)$ is the tripod formula.

(inductive step): Suppose $k \geq 3$. First we'll show $A(k) \Longrightarrow B(k)$. We induct on the length of the arm and take the base case to be when the arm is empty. Hence, the base case is true by $A(k)$. Proceeding by induction, we consider the game G in Figure 2.17.

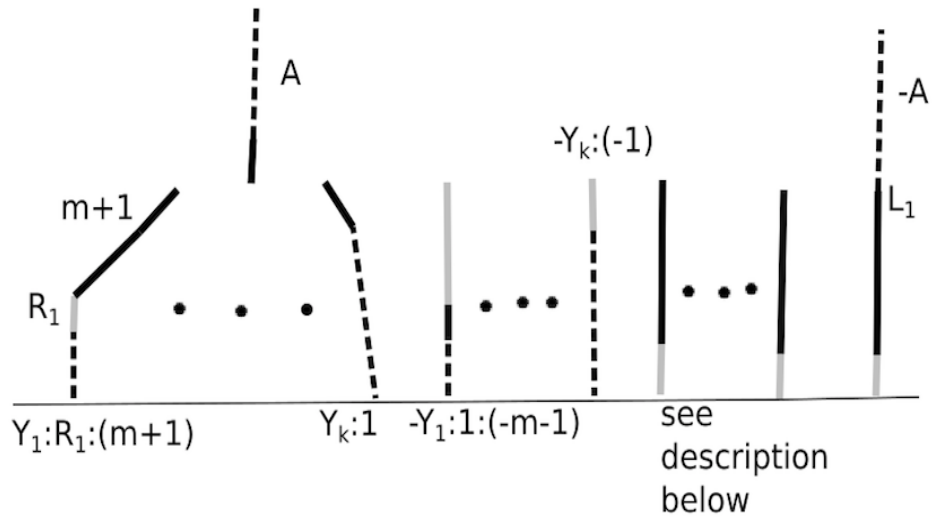

Fig. 2.17: Difference Game for $A(k) \Longrightarrow B(k)$

Note, the strings above 'see description below' represent $k-2$ strings of the form $(-1):(n-1)$ and the rightmost string is $(-1):(n-1):(-A)$ where A may be empty. Since we know the arm is nonnegative, it is convenient to write it as $1:A$ (as we have done above) instead of A. Furthermore, using vanRoode's string evaluation, we have $(-1):(n-1) = \rho(-1:(n-1)) = -\frac{1}{2^{n-1}}$ and
$(-1):(n-1):(-A) = \rho((-1):(n-1):(-A)) = -\frac{1}{2^{n-1}} + \frac{1}{2^n}\rho(-A) = \frac{1}{2^{n-1}}\rho(-1:(-A))$.

Suppose Left goes first. Suppose they move in the spider. Since Left is guaranteed a move in the arm and the fact that optimal moves are higher, we have that Left may as well play in the arm. We see that if Left plays in $1:A$ then Right has a response in the rightmost string to 0 by the induction hypothesis.

If Left plays in the strings, then they will play in the string of greatest height. Hence, Left may as well play in the rightmost string. If Left *can* move in $-A$, then they will. Then Right will respond in A and the resulting game is 0 i.e., a loss for Left. If Left cannot move in $-A$, then $-A$ must be all red. In this case, Left may as well move at L_1. We will show L_1 is a losing first move for Left in the next step.

Suppose Right moves first. Suppose Right moves in the spider. If they move in A, then Left has a winning response in $-A$ to 0. Right's only other reasonable move in the spider is at R_1. We have

$$\begin{aligned} G^{R_1} &= \frac{1}{2^{n-1}}(\rho((1+m):A)+\rho(1:(-A))) \\ &> 0 \text{ (observing that making } A \text{ all blue gives a lower bound on } G^{R_1}) \end{aligned} \tag{2.5}$$

This calculation also tells us $G^{L_1R_1} = \frac{1}{2^{n-1}}(\rho((1+m):A) - \frac{1}{2^{n-2}} < 0$. Hence, L_1 is a losing first move for Left. Lastly, if Right moves in the strings, they will prefer to move in the longest string. Whether they move in $-A$ or delete the whole string; either way Left has a winning response in $1:A$. We may conclude $G = 0$.

Next, we show $B(k) \implies A(k+1)$. We consider three cases:

Case 1: $t_r = 0$. Consider the game G in Figure 2.18, where the strings above 'see description below' represent $k-1$ strings of the form $(-1):(n-1)$.

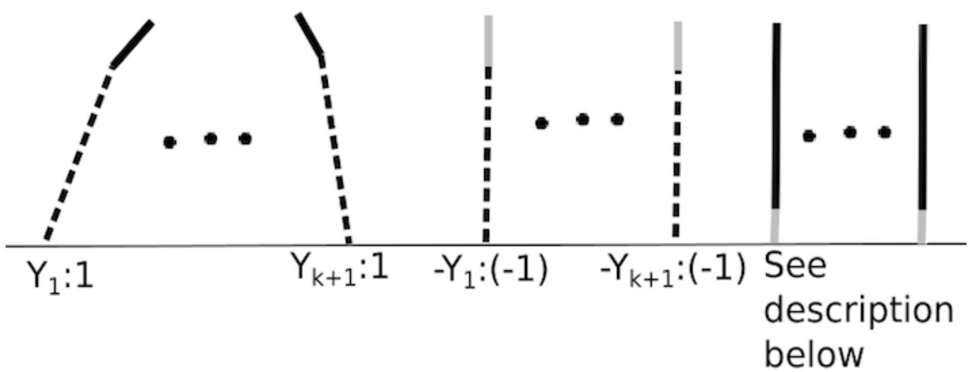

Fig. 2.18: Case 1 for $B(k) \implies A(k+1)$

We noted previously that $(-1):(n-1) = -\frac{1}{2^{n-1}}$. We note that
$Y_i:(-1) - \frac{1}{2^{n-1}} = Y_i - \frac{1}{2^{n-1}} - \frac{1}{2^{n-1}} = Y_i - \frac{1}{2^{n-2}}$. We also note
$Y_i:(-1)+Y_j:(-1) = Y_i+Y_j - \frac{1}{2^{n-2}}$. Lastly, we note $-\frac{1}{2^{n-2}} = (-1):(n-2)$. Therefore

we can rewrite G as the game in Figure 2.19, where the strings above 'see description below' are k strings of the form $(-1):(n-2)$, i.e., k strings each worth $-\frac{1}{2^{n-2}}$.

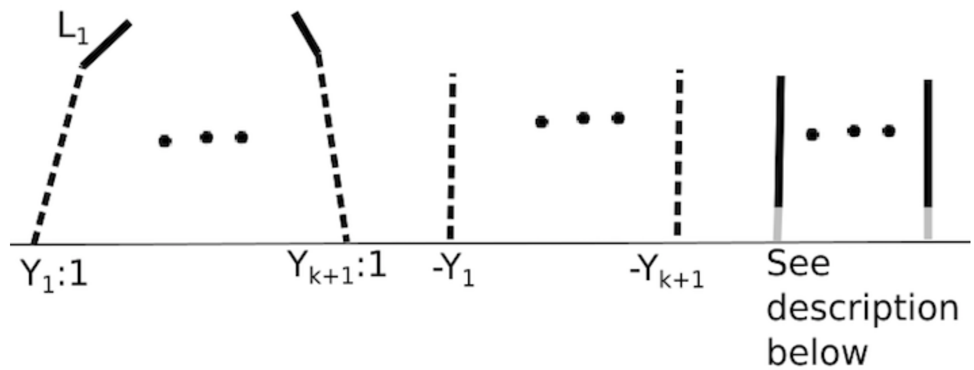

Fig. 2.19: A Game Equivalent to Figure 2.18

Suppose Left moves first. If Left moves in the spider, they may as well move at L_1. We have $G^{L_1} = -\frac{1}{2^{n-2}}$. In the next step, we will show that if Left instead moves in the strings, then this is also a losing move.

Suppose Right moves first. We can rewrite G as the game in Figure 2.20, where the strings above 'see description below' are k strings of the form $(-1):(n-2)$, i.e., k strings each worth $-\frac{1}{2^{n-2}}$.

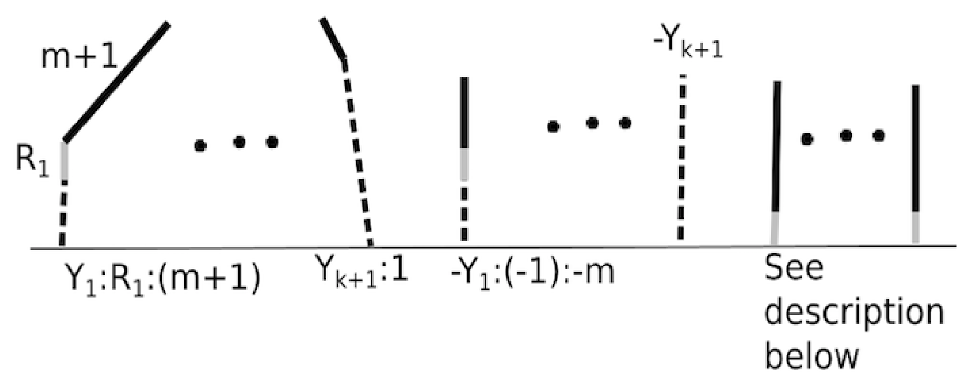

Fig. 2.20: A Game Equivalent to Figure 2.18

Using $B(k)$, we have $G^{R_1} = \frac{1}{2^{n-1}}\rho(m+1)$. Let G^{L_2} be an option of Left moving highest in the strings and G^{R_2} be a right option of Right moving highest in the strings. Then

$G^{L_2R_1} = -\frac{1}{2^{n-2}} + \frac{1}{2^{n-1}}\rho(m+1) < 0$ and $G^{R_2L_1} = \frac{1}{2^{n-2}} - \frac{1}{2^{n-2}} = 0$. This shows G is a second player win, i.e., $G = 0$.

case 2: $t_b, t_r \neq 0$ and $0 \leq t \leq 2$. Let G be the game in Figure 2.21.

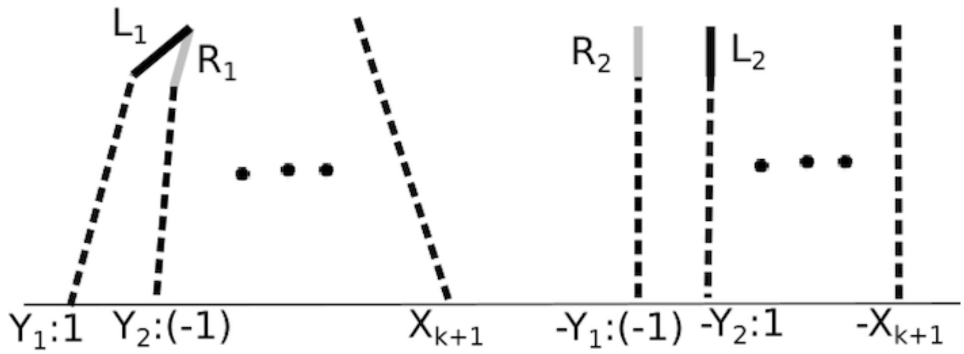

Fig. 2.21: Case 2 for $B(k) \Longrightarrow A(k+1)$

Then $G^{L_1R_1} = G^{R_1L_1} = 0$. Also, $G^{L_2R_1} = 0$. If $t = 0$ or $t = 1$, then $G^{R_2L_1} = 0$. Suppose $t = 2$. Then G^{R_2} results in a spider with $t = 3$. Observe, $G^{R_2L_1} = \frac{1}{2^{n-1}}$. This shows $G = 0$.

case 3: $t_b, t_r \neq 0$ and $t > 2$. First, let $S = X_1/X_2/\ldots/X_{k+1}$ be the spider Figure 2.22.

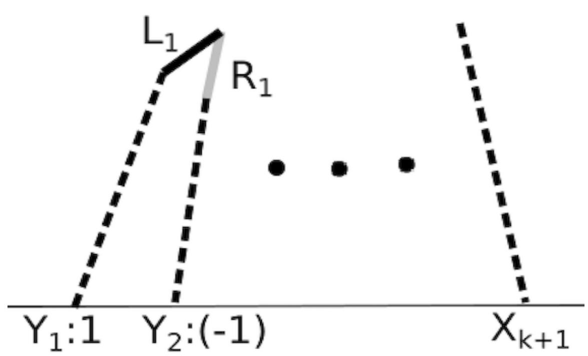

Fig. 2.22: Case 3 for $B(k) \Longrightarrow A(k+1)$

We define $X_i = Y_i : \varepsilon_i$ where $\varepsilon_i = 1$ or -1 depending on the highest edge of that leg. Then $S^{L_1} = Y_1 + \cdots + Y_{k+1} + \frac{t-2}{2^{n-2}}$. Similarly, $S^{R_1} = Y_1 + \cdots + Y_{k+1} + \frac{t}{2^{n-2}}$. For this case, we would like to show the game in Figure 2.23 is equal to 0, where we let L_2, R_2 be optimal moves for Left, resp. Right, in the strings.

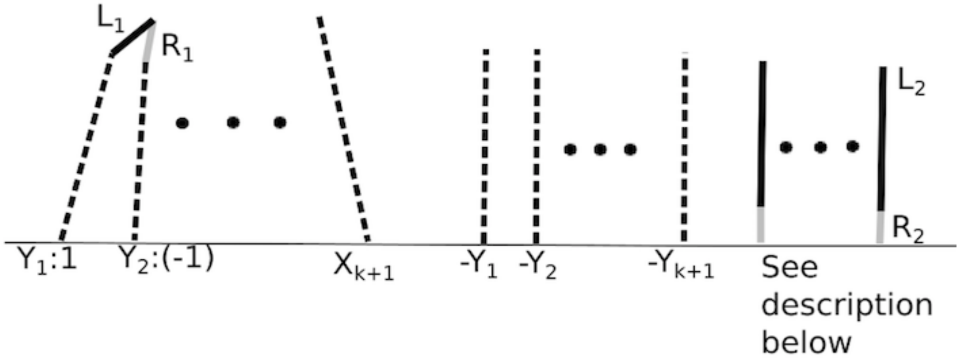

Fig. 2.23: The Game G in Case 3 for $B(k) \Longrightarrow A(k+1)$

The spider is S and the strings above 'see description below' represent $t-1$ strings of the form $(-1):(n-2)$. In particular, the strings are each worth $-\frac{1}{2^{n-2}}$. Hence, $G^{L_1R_2} = G^{R_2L_1} = 0$ and $G^{R_1L_2} = G^{L_2R_1} = 0$. This shows $G = 0$. □

2.6 Example

In this section, we apply the previous section to some examples. In particular, if we are playing a spider (or sum of spiders), it is optimal to move highest in whichever component we decide to move in. If we only care about the outcome of a spider (or sum of spiders) we can use the previous section, assuming that the spiders are strings, arches, or balanced spiders.

By Theorem 2.20, we have the equality in Figure 2.24. Adding up the strings, we find the spider is equal to $\frac{7}{8}+\frac{3}{8}-\frac{5}{8}-\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{6}{8}=\frac{3}{4}$. Therefore, the spider is a win for Left whether they play first or second.

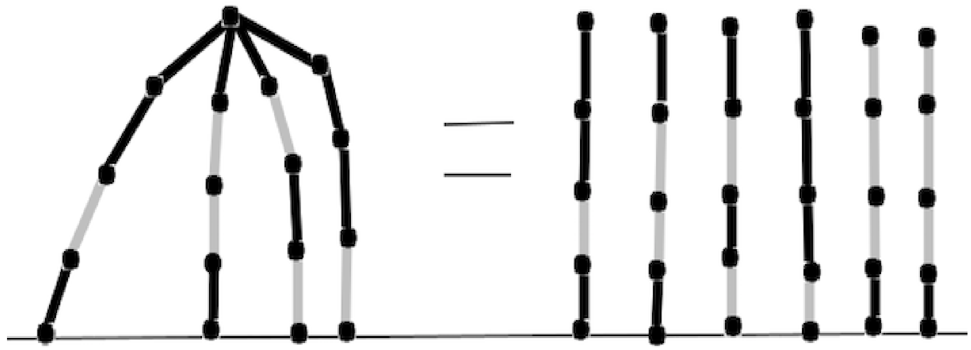

Fig. 2.24: Decomposing a Spider Into a Sum of Legs and Bonuses

2.7 Conclusion

Our approach to solving balanced spiders was to first determine the optimal strategy, and then prove the formula. It turned out that the optimal strategy was for players to delete the highest edge available to them. In general, this does not hold when the spider is unbalanced, i.e., when the legs have different height. Moreover, by our formula, it turned out that a balanced spider could be rewritten as a sum of strings. This was ideal, because strings are easy to evaluate. It is unclear unclear how to rewrite an unbalanced spider as a sum of easy to evaluate games. Another loose end is finding a formula for balanced spiders with an arm (by Colon Principle, the case of multiple arms reduces to the case of exactly one arm). In this thesis, we solved a couple special cases but a general formula for games of the form $S\backslash A$ remains unknown (Here, S is a balanced spider and A is an arm).

3 REVERSE HACKENBUSH

The ruleset Reverse Hackenbush was introduced by vanRoode [4], where she also gives a solution to strings. A solution to arches and trees were then posed as problems. In this section, we give a solution to arches. A complete analysis of trees remains open. A position in Reverse Hackenbush is a graph of blue and red edges which are all connected to a vertex called the ground. We typically draw the ground as a horizontal black line for aesthetic purposes. The game is played between two players: Left and Right. On a player's turn, they delete an edge of their respective color: Left is bLue and Right is Red. Additionally, if an edge was in the component a player moved in, then that edge's color is reversed. (Blue edges reverse to red edges and vice versa). We emphasize that the only edge's whose color changes are ones in the component (which may now be disconnected) that was played in. The first player unable to move loses.

Remark 3.1. In [4], playing a game with multiple components is not explicitly mentioned. Therefore there is some ambiguity. For example, in the game of two blue edges connected to the ground, does moving in one reverse the color of the other? If this were so, then the position would be a second player win, i.e., $1+1=0$. Therefore, we explicitly state that a player reverses the color of edges **only** in the component that was moved in.

As an example, we have the game G in Figure 3.1 and a sequence of moves on G described in Figures 3.2 - 3.4. Indeed, Left has won this game since it will be shown that $G^{L_1R_1L_2} = 1$, i.e., a lone blue edge is worth 1 in Reverse Hackenbush.

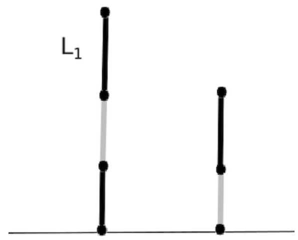

Fig. 3.1: A Sample Game G of Reverse Hackenbush

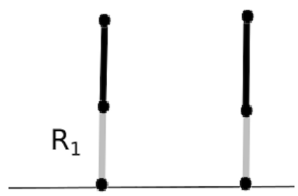

Fig. 3.2: An Option G^{L_1}

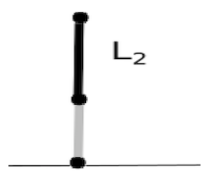

Fig. 3.3: An Option $G^{L_1 R_1}$

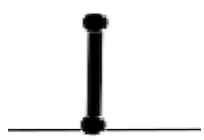

Fig. 3.4: An Option $G^{L_1 R_1 L_2}$

As another example, we consider the tree T in Figure 3.5 and a sequence of moves on T described in Figures 3.6 - 3.8. Since Left has no options in $T^{R_1 L_1 R_2}$, Right has won the game. We note that Left playing L_1 was a mistake on their part. If they deleted the lower edge, they would have won.

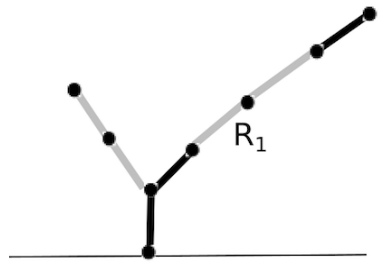

Fig. 3.5: A Sample Game T of Reverse Hackenbush

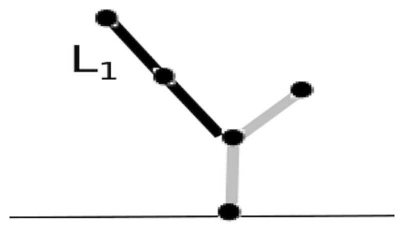

Fig. 3.6: An Option T^{R_1}

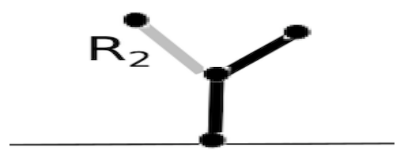

Fig. 3.7: An Option $T^{R_1L_1}$

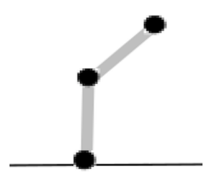

Fig. 3.8: An Option $T^{R_1L_1R_2}$

3.1 Strings

Definition 3.2. For any position in Reverse Hackenbush, we define a foot to be an edge containing the ground as a vertex.

Definition 3.3. Let G be a position in Reverse Hackenbush. We define a blue segment to be a nonempty, maximal path of blue edges and we define a red segment to be a nonempty, maximal path of red edges. We define s_G to be the number of red or blue segments that G contains. When clear by context, we'll refer to s_G by s.

Definition 3.4. Let G, H, K be games. We define $\{G||H|K\}$ as a game whose Left option is G and whose Right option is the game $\{H|K\}$.

Definition 3.5. For the remaining sections, the following games appear frequently: Define $A = \{1||0, \{0|-1\}\}$ and $B = \{1|0\}$. Note, A and B are in canonical form.

We note that the Left option of A is 1 and the Right options of A are $\{0|-1\}$ and 0. Examples of these game values are given in Figure 3.9.

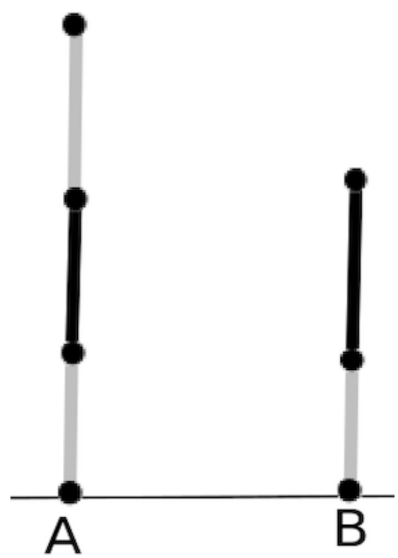

Fig. 3.9: Examples of A and B

The following theorem is vanRoode's solution to strings [4].

Theorem 3.6. *(vanRoode's Solution to Strings): Let S be a string with a blue foot. Then*

$$S = \begin{cases} 0 & s = 0 \\ 1 & s = 1 \\ -B & s = 2 \\ -A & s \geq 3 \end{cases}$$

We see that the value of a string is completely determined by the number of segments it has, along with the color of its foot. Since we can omit dominated options, the following theorem will be helpful in future calculations.

Theorem 3.7. *We have $B > A$.*

Proof. We play the difference game $B - A$. Note, $-A = \{0, B | -1\}$. It is sufficient to show Left can win playing first, and Right cannot win playing first. If Left goes first, then they can move on $-A$ to the position $B + B$. Right plays and Left responds to a position equal to 1. If Right goes first and moves on B, then move to the position $0 - A$. Left can respond to $0 + 0 = 0$. If Right moves on $-A$, then Left can respond on B to the position $1 - 1 = 0$. The proves the inequality. □

3.2 Arches

Similar to strings, the value of an arch is related to its segments. We will first solve arches for $s = 1, 2, 3, \ldots, 7$. Let e_G denote the number of edges of a graph G. Unless otherwise stated, WLOG, we assume all arches have at least one blue foot. We note that arches are referred to as *grounded loops* in other sources such as [4].

Definition 3.8. Recall, we define the game $* = \{0|0\}$. This game is pronounced 'star'.

Definition 3.9. For a short game G, we define $\pm G = \{G | -G\}$.

Remark 3.10. In some of the following proofs, our strategy will be to calculate options in certain segments. We note that moving in a segment results in a sum of two strings.

Since the value of a string is determined by the number of segments, and not on the number of edges contained in a given segment, we may assume the number of edges in each segment (except the one we moved in) is one.

In many of the proofs we will calculate certain options, then argue that any other possible options must be dominated. We will also break our solution to arches into cases based on the number of segments the arch contains. We saw that the solution to strings breaks into four cases depending on the number of segments contained in the string. Because of this, we will only have to consider finitely many cases in our solution to arches.

Lemma 3.11. *We may assume all segments have at most* 3 *edges. This is described in Figure 3.10 below.*

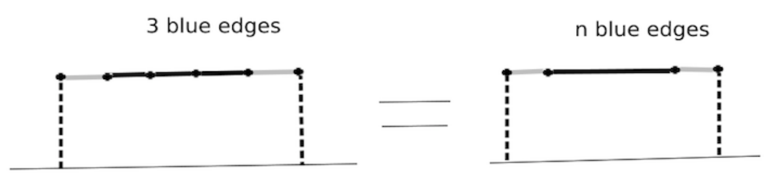

Fig. 3.10: Lemma 3.11.

where $n \geq 3$.

Proof. In Figure 3.10, the two arches are equal as graphs except in the blue segments of length 3 and n, respectively. We will show that the above games have essentially (i.e., considering the options as game values) the same set of Left options and same set of Right options. Any option in a lined segment in one arch is equal to the deleting the corresponding edge in the other arch, since the arches have the same number of segments. Similarly, the set of options by moving in the red edges for both games are equal.

Lastly, we denote the game to the left in Figure 3.10 as G and the game to the right as H. Suppose Left plays in the blue segment in G. Call this move G^L. If the edge deleted was adjacent to a red edge, then let H^L be an option of Left deleting a blue edge in the blue segment adjacent to a red edge. These options are described in Figure 3.11.

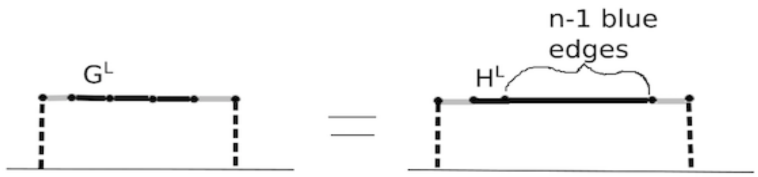

Fig. 3.11: G^L and H^L Are Adjacent to Red Edges

Then $G^L = H^L$. On the other hand, if they deleted the edge in the middle, then we have $G^L = H^L$ where H^L is any option of deleting a blue edge not adjacent to a red edge in H. This situation is represented in Figure 3.12. The preceding statement and a similar argument shows that for every Left option HL there exists a Left option GL such that HL = GL. This proves the lemma.

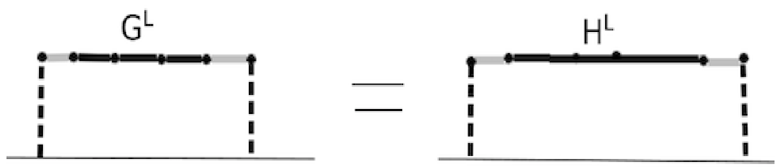

Fig. 3.12: G^L and H^L Are Not Adjacent to Red Edges

□

Theorem 3.12. *Let G be an arch with $s = 1$ (i.e., G contains only blue edges). Let e_G be the number of edges in G. Then*

$$G = \begin{cases} 1 & e_G = 1 \\ 0 & e_G \geq 2 \end{cases}$$

Proof. Observe,

Case 1: $e_G = 1$. We have $G = \{0|\} = 1$.

Case 2: $e_G = 2$. We have $G = \{-1|\} = 0$.

Case 3: $e_G = 3$. We have $G = \{-2, -1|\} = 0$. □

Theorem 3.13. *Let G be an arch with $s = 2$. Let x be the blue leg and y be the red leg. Then*

$$G = \begin{cases} \pm 1 & e_x = 1 = e_y \\ \{0, -B| -1\} & e_x \geq 2, e_y = 1 \\ * & e_x, e_y \geq 2 \end{cases}$$

Proof. Let e_x denote the number of edges in the blue segment and e_y denote the number of edges in the red segment. Observe,

Case 1: $e_x = 1$ and $e_y = 1$. Then $G = \pm 1$.

Case 2: $e_x \geq 2$ and $e_y = 1$. Then $G = \{0, \{0| -1\}| -1\} = \{0, -B| -1\}$.

Case 3: $e_x \geq 2$ and $e_y \geq 2$. Then $G = \{0, -B|0, B\} = *$. To see the last equality, we play the difference game $\{0, -B|0, B\} + *$. If the first player plays on $*$, the second can play 0 in G (and vice versa). On the other hand, if the first player, say Left, plays to $-B + *$, then Right can move to $-1 + *$ which is strictly less than 0. Lastly, if Left were to have played to $0 + *$, then Right can play on $*$ and win the game. It follows that Left cannot win playing first. By a symmetric argument, neither can Right.

□

Theorem 3.14. *Let G be an arch with $s = 3$ (that is, two blue segments and one red segment). Let e_x be the minimum length of the two blue segments and let e_y be the length red segment. Then*

$$G = \begin{cases} \{B|-2\} & e_x = 1, e_y = 1 \\ 0 & e_x = 1, e_y > 1 \\ \{-1+B|-2\} & e_x > 1, e_y = 1 \\ 0 & e_x > 1, e_y > 1 \end{cases}$$

Proof. We will first calculate the Left options. There are two cases:

Case 1: Both feet have exactly one edge. Let G be the game in Figure 3.13.

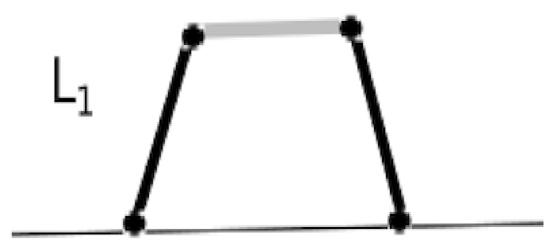

Fig. 3.13: Both Feet Have Exactly One Edge

Then $G^{L_1} = B$. In this case, $G = \{B|\dots\}$.

Case 2: At least one foot has more than one edge. Let G be the game in Figure 3.14.

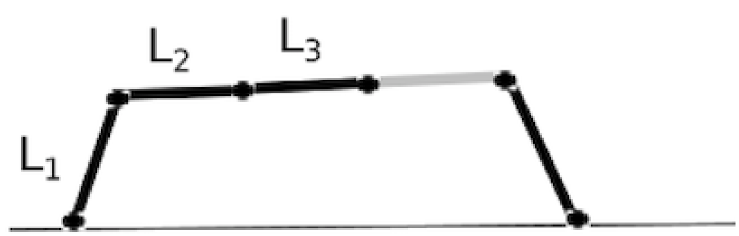

Fig. 3.14: At Least One Foot Has More Than One Edge

Note that G^{L_2} may or may not exist depending on the number of edges in the segments, but we will see if it does exist then it is dominated. We have $G^{L_1} = A, G^{L_2} = -1 + A$, and $G^{L_3} = -1 + B$. We see $G^{L_2} < G^{L_1} < G^{L_3}$. In this case, $G = \{-1 + B | \dots\}$.

We can conclude if one of the blue segments has length 1, then Left's only worthwhile option is equal to B. Otherwise, each blue segment has length ≥ 2 and Left's only worthwhile option is to $-1 + B$.

Next, we calculate the Right options. There are two cases:

Case 1: There is exactly one red edge. Then $G^{R_1} = -2$.

Case 2: There is more than one red edge. Let G be the game in Figure 3.15.

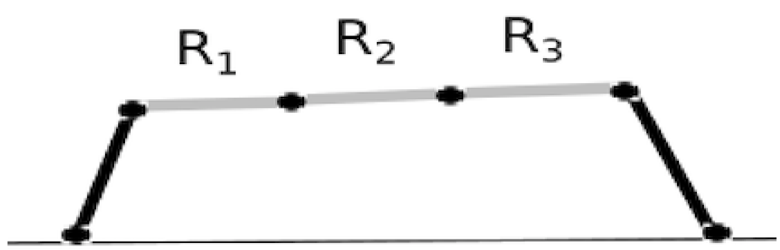

Fig. 3.15: More Than One Red Edge

Note, G^{R_2} may or may not exist depending on the number of edges in the segments, but we will see if it does exist then it is dominated. We have $G^{R_1} = -1 + B = G^{R_3}$ and $G^{R_2} = B + B = 1$. In this case, $G = \{\dots | -1 + B\}$. Going through cases and putting games into canonical forms gives the theorem. □

Lemma 3.15. *Suppose G is an arch with* $s \geq 4$. *Then we may assume the feet segments contain exactly one edge.*

Proof. Let x represent a foot. WLOG, suppose the foot is blue and x is the segment containing this foot. Let G^L be the option where Left deletes the lowest edge in x, and let $G^{L'}$ be any other option of Left deleting an edge in x. By Theorem 3.6, $G^{L'} = -1 + G^L < G^L$. We may ignore dominated options. It follows that $G = G'$ where G'

is the arch G where x is replaced with exactly one blue edge. A symmetric argument works if x had been a red foot. □

Theorem 3.16. *Let G be an arch with $s = 4$. Let x be the blue segment that does not contain a foot and y be the red segment that does not contain a foot. Then*

$$G = \begin{cases} \pm\{2|1\} & e_x = 1 = e_y \\ \{\{2|1\}|-A-1\} & e_x = 1, e_y > 1 \\ \{A+1|\{-1|-2\}\} & e_x > 1, e_y = 1 \\ \pm(A+1) & e_x, e_y > 1 \end{cases}$$

Proof. By the lemma, we may assume the feet have exactly one edge. As before, we will calculate the options, starting with Left's. Note that since we are calculating Left options, we may assume all red segments have only one red edge in them.

If Left deletes the foot, we have $G^{L_1} = -A$. We will see that G^{L_1} is dominated by some other option. Suppose Left plays on x. Consider two cases:

Case 1: $e_x = 1$. Let G be the game in Figure 3.16.

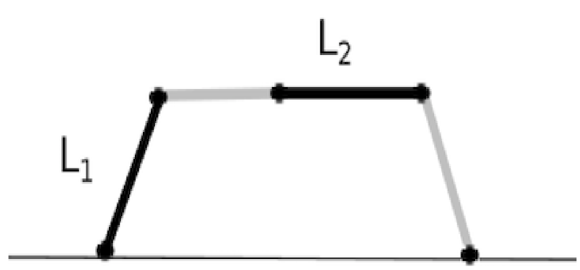

Fig. 3.16: The Segment x Has Length 1

If Left plays in this segment, we have $G^{L_2} = B + 1 = \{2|1\}$. Since $-A < \{2|1\}$, we have that Left's only worthwhile move is $\{2|1\}$.

Case 2: $e_x = 2$. Let G be the game in Figure 3.17.

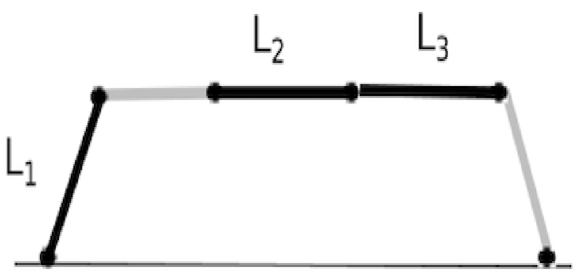

Fig. 3.17: The Segment x Has Length 2

If Left plays in the higher edge, we have $G^{L_2} = 0$. If Left plays in the lower edge, we have $G^{L_3} = 1 + A$. Now, G^{L_3} dominates G^{L_2} and G^{L_1} which shows Left's only worthwhile move is $1 + A$.

Case 3: $e_x = 3$. Let G be the game in Figure 3.18.

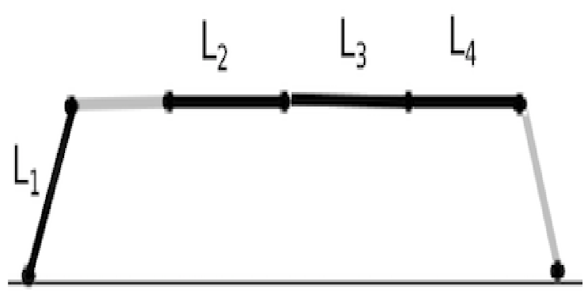

Fig. 3.18: The Segment x Has Length 3

We have $G^{L_2} = 0$. If Left plays in the middle edge, we have $G^{L_3} = -A - B$. If Left plays in the lowest edge, we have $G^{L_4} = 1 + A$. Now, G^{L_4} dominates G^{L_3}, G^{L_2}, and G^{L_1}. In this case, Left's only worthwhile move is $1 + A$.

A symmetric argument proves the claim about Right's options. The theorem follows. □

Definition 3.17. Let S be a segment in an arch G. We define the segment height of S as the minimal number of segments (inclusive) from S to the ground.

Theorem 3.18. *Let G be an arch with $s = 5$. WLOG, assume the feet are blue and let x be the blue segment of greatest height (the unique non-foot blue segment). Then*

$$G = \begin{cases} \{1|-1+A\} & e_x = 1 \\ \{A, B+A|-1+A\} & e_x \geq 2 \end{cases}$$

Proof. As before, we will calculate the options. Let G be the game in Figure 3.19.

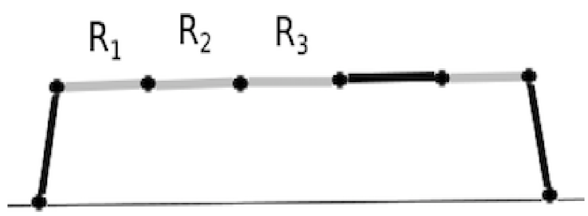

Fig. 3.19: A 5 Segment Arch Whose Right Options Are Emphasized

Calculating the Right options, we have $G^{R_1} = -1+A, G^{R_2} = B+A = G^{R_3}$ where G^{R_1} necessarily exists and G^{R_2}, G^{R_3} exist depending on the number of edges in the segments. Note, $G^{R_1} < G^{R_2} = G^{R_3}$. We may conclude $G = \{\ldots|-1+A\}$. Next, we calculate the Left options.

Case 1: $e_x = 1$. Let G be the game in Figure 3.20.

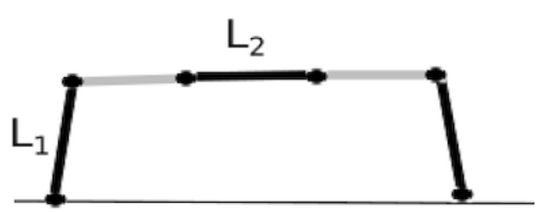

Fig. 3.20: The Segment x Has Length 1

Calculating the Left options, we have $G^{L_1} = A, G^{L_2} = 1$. Note, $A < 1$. We may conclude $G = \{1|-1+A\}$.

Case 2: $e_x = 2$. Let G be the game in Figure 3.21.

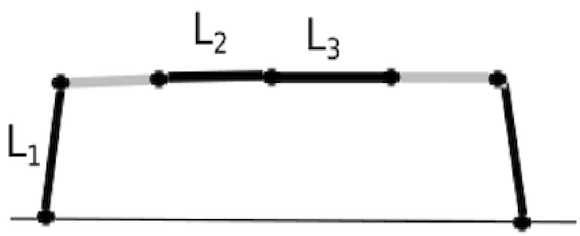

Fig. 3.21: The Segment x Has Length 2

We have $G^{L_1} = A, G^{L_2} = B + A, G^{L_3} = A + B$. It follows that $G = \{A, A+B | -1+A\}$.

Case 3: $e_x = 3$. Let G be the game in Figure 3.22.

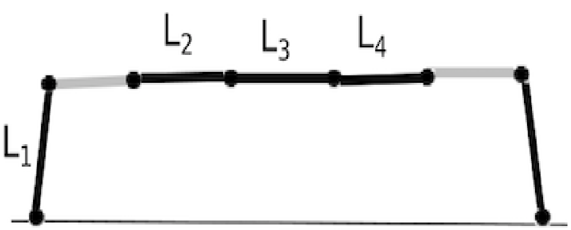

Fig. 3.22: The Segment x Has Length 3

We have $G^{L_1} = A, G^{L_2} = B + A = G^{L_4}, G^{L_3} = A + A$. Note, $A + B > A + A$. It follows that $G = \{A, A+B | -1+A\}$. □

Theorem 3.19. *Let G be an arch with $s = 6$. Then $G = \pm(1 + A)$.*

Proof. We will calculate the Left options and by symmetry this also calculates the Right options. Let G be the game in Figure 3.23.

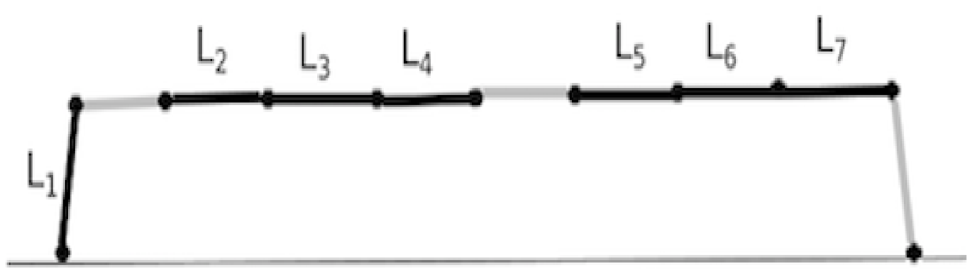

Fig. 3.23: A 6 Segment Arch Whose Left Options Are Emphasized

Note that $G^{L_3}, G^{L_4}, G^{L_5}, G^{L_6}$ may or may not exist depending on the number of edges in the segments, but we'll see they are dominated by G^{L_7} which necessarily exists. We have

$$G^{L_1} = -A$$
$$G^{L_2} = B - A$$
$$G^{L_3} = 0$$
$$G^{L_4} = 0$$
$$G^{L_5} = A - B$$
$$G^{L_6} = A - B$$
$$G^{L_7} = A + 1$$

Ignoring dominated options, we have $G = \{A+1|\dots\}$. Using a symmetric argument to determine G^R, we have $G = \pm(A+1)$. □

Theorem 3.20. *Let G be an arch with $s = 7$. Then $G = \{A, A+B| -1+A\}$.*

Proof. We first calculate the Left options. Let G be the game in Figure 3.24.

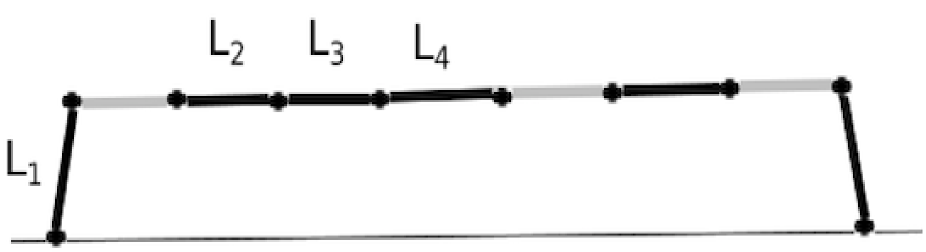

Fig. 3.24: A 7 Segment Arch Whose Left Options Are Emphasized

Note that G^{L_3} and G^{L_4} may or may not exist depending on the number of edges in the segments, but we will see they are dominated. We have

$$G^{L_1} = A$$

$$G^{L_2} = B + A$$

$$G^{L_3} = A + A$$

$$G^{L_4} = A + A$$

Since $A + A < B + A$, we have that G^{L_3} and G^{L_4} are dominated by G^{L_2}. We can conclude $G = \{A, A + B | \dots\}$. Next, we calculate the Right options. Let G be the game in Figure 3.25.

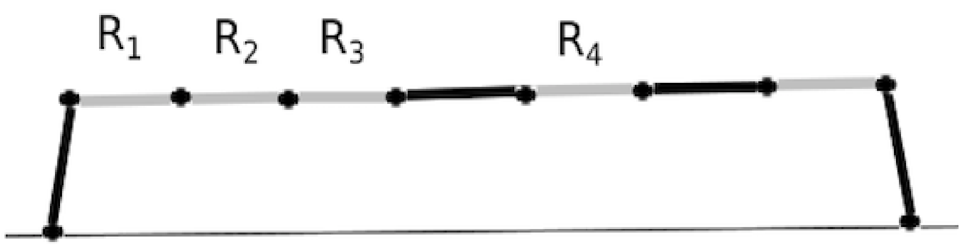

Fig. 3.25: A 7 Segment Arch Whose Right Options Are Emphasized

Note if the segment containing R_4 contained another edge R, we'd have $G^{R_4} = G^R$. So we may assume that the segment containing R_4 contains no other edges. We also note that G^{R_2} and G^{R_3} may or may not exist depending on the number of edges in the segments, but we'll show these options are dominated. We have

$$G^{R_1} = -1 + A$$

$$G^{R_2} = B + A$$

$$G^{R_3} = B + A$$

$$G^{R_4} = A + A$$

We see G^{R_1} dominates $G^{R_2}, G^{R_3}, G^{R_4}$. We may conclude $G = \{A, A+B \mid -1+A\}$. □

Lemma 3.21. *WLOG, assume G is an arch with at least one blue foot. If e is an edge in a segment with segment height* ≥ 4*, then*

$$G' = \begin{cases} A + A & G \text{ has two blue foot} \\ 0 & G \text{ has only one blue foot} \end{cases}$$

where G' is the result of a player deleting edge e.

Proof. By assumption, e is an edge in a segment with segment height ≥ 4. Hence, G' is the sum of two strings, each of segment height ≥ 3. Write this sum as $G' = T_1 + T_2$. Therefore, each string is equal to A or $-A$. By assumption, G has at least one blue foot. So T_1 or T_2 has a red foot. Hence, T_1 and T_2 are both equal to A, or one of them equals A and the other $-A$. This proves the lemma. □

Theorem 3.22. *Let G be an arch with* $s \geq 8$*. Then*

$$G = \begin{cases} \pm(1 + A) & s \text{ is even} \\ \{A, A+B \mid -1+A\} & s \text{ is odd} \end{cases}$$

Proof. We consider two cases.

Case 1: s is even. Then G has a blue foot and a red foot. We calculate the Left options first. We consider the options existing in segments of height $1,2,3$, and 4. The possible options in a segment of height 1 are equal to $-1-A$ or $-A$. The possible options in segment height 2 are equal to $1+A$ or $-B+A$. Note that $1+A$ is guaranteed to exist, and we will show it dominates all other options. The possible options in segment height 3 are equal to $B-A$ or 0. Lastly, the previous lemma shows an option in segment height 4 or higher is equal to 0. Hence, the possible options for Left are $-1-A, -A, 1+A, -B+A, B-A$ and 0. Since $1+A$ dominates all other options, and is guaranteed to exist (it is the option of deleting the edge of lowest height in the segment of height 2), we may ignore all other Left options.

By a symmetric argument to calculate the Right options, we have $G = \pm(1+A)$.

Case 2: s is odd. Then G has two blue feet. We calculate the Left options first. We consider options in segments of height $1,3$, and 5. The options in segments of height 1 are equal to $-1+A$ or A. The options in segments of height 3 are equal to $B+A$ or $A+A$. The options in segment 5 are all equal to $A+A$. The option equal to A exists and corresponds to deleting an edge containing the ground as a vertex. The option equal to $B+A$ exists and corresponds to deleting the lowest edge in a segment of height 3. We see that A and $B+A$ dominate $-1+A, A+A$. Moreover, A and $B+A$ are confused with each other. Hence, we may ignore all Left options besides A and $B+A$.

Next, we calculate the Right options. We consider options in segments of height 2 and 4. The options in segments of height 2 are equal to $-1+A$ or $B+A$. The options in segments of height 4 equal to $A+A$. Note that an option equal to $-1+A$ exists and corresponds to deleting the lowest edge in a segment of height 2. Moreover, $-1+A$ dominates $B+A$ and $A+A$. Hence, we may ignore all Right options besides $-1+A$. We can conclude $G = \{A, B+A \mid -1+A\}$. □

3.3 Colon Principle

In this section we show that the Colon Principle [1] does not hold in Reverse Blue Red Hackenbush.

Definition 3.23. We define the trunk of a tree T as the string of edges from the ground up to the first vertex of degree ≥ 3. If no vertex of degree ≥ 3 exists, then T is a string and we say T is its own trunk.

Let G be the game in Figure 3.26.

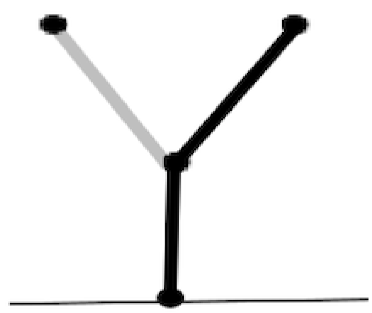

Fig. 3.26: Counter Example to the Colon Principle in Reverse Hackenbush

Then $G = \{0, B| - 1\}$. If the Colon Principle did hold, then we would have $G = 1$. But G is not even positive since Right has a winning move as the first player. This shows the usual Colon Principle in the usual Blue Red Hackenbush does not hold in the Reverse variant. This is because the branches play differently on the trunk than they do on the ground. More specifically, when playing in the branches on the trunk, any move in a branch will reverse the colors in the other branches. On the other hand, playing the branches on the ground, any move in one branch does not reverse the colors in the other branches. Branches are independent of each other on the ground, and are dependent on each other on a trunk.

Theorem 3.24. *(A Weaker Colon Principle): Let S be a string. For a game H, define the reverse ordinal sum* $S : H = \{S^L, (-S) : (H^L) | S^R, (-S) : (H^R)\}$. *Let G be a graph of blue and red edges. Then* $S : (G + G) = S$.

Proof. We play the difference game $S : (G+G) + (-S)$. If the first player moves in S or $-S$, then the second player has a clear TDTD response to 0. Suppose Left goes first and that they move in one of the copies of G. Then Right can TDTD in the other copy of G. By induction on the size of G, we have $(S : (G+G))^{LR} + (-S) = 0$. By a symmetric argument, if Right goes first and plays on one of the copies of G, then Left can force a win. This shows that $S : (G+G) + (-S)$ is a second player win, i.e., equal to 0. □

3.4 Conclusion

We have seen that arches and strings in Reverse Blue Red Hackenbush have a nice contrast to the same positions in the usual ruleset. More specifically, positions can take on values that are not numbers in Reverse Blue Red Hackenbush. For arches and strings, we have seen these positions can only take on finitely many values.

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