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On the Colorability of the Sphere Complex

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ON THE COLORABILITY OF THE SPHERE COMPLEX

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ABSTRACT

ON THE COLORABILITY OF THE SPHERE COMPLEX

by Bennett Haffner

One of the most prominently studied groups in geometric group theory is the outer automorphism group of the free group $\mathrm{Out}(F)$. The sphere complex provides a topological model for $\mathrm{Out}(F)$. We demonstrate the chromatic number of the sphere complex is finite.

DEDICATION

To my grandparents, who both encouraged and inspired this effort.

ACKNOWLEDGMENTS

I could not have done this without the generous support and guidance of my advisor, Dr. Bering. Special thanks to Dr. Campisi and Dr. Zhang for their feedback and insightful comments.

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1. Introduction

Geometric group theory studies geometries on the Caley graphs of infinite groups. One of the most prominent of these groups is the outer automorphisms of the free group $\mathrm{Out}(F)$. The sphere complex $\mathcal{S}(M_r)$ (Definition 1.3) (where M_r is the connect sum of S^3 with r copies of $S^1 \times S^2$) provides a topological model for $\mathrm{Out}(F_n)$ because $\mathrm{Out}(F_n)$ is isomorphic to the automorphisms of $\mathcal{S}(M_r)$ [AS11]. An important property of any complex is its chromatic number.

Definition 1.1. For a graph G and set of colors S, we say the function $\phi : G \to S$ is a *coloring* on G if whenever a pair of vertices $a, b \in G$ share an edge, $\phi(a) \neq \phi(b)$. The *chromatic number* of G, denoted $\chi(G)$, is the smallest $|S|$ over all sets S for which a coloring exists on G.

We show that the chromatic number of the sphere complex $\mathcal{S}(M_r)$ is finite.

Theorem A. *For every* $r \in \mathbb{N}$, $\chi(\mathcal{S}(M_r)) \leq 2^{(9r-4)2^g}$.

For $r = 1$, the sphere complex is a single vertex and thus trivially colorable, whereas for $r = 2$, the complex is the Farey graph, which is planar and thus 4-colorable. It is when $r = 3$ that the graph becomes not only locally infinite, but of infinite diameter as well. This is where we will focus our efforts.

We approach Theorem A by analogy with Bestvina, Bromberg, and Fujiwara's [BBF15] construction of a finite coloring on the curve graph $\mathcal{C}$ of a surface.

Definition 1.2. For a surface M, the *curve complex* $\mathcal{C}(M)$ is the graph with vertices given by isotopy classes of simple, closed curves in M. For any two curves $a, b \in \mathcal{C}(M)$, there is an edge between them only when there exists some disjoint representatives of $[a]$ and $[b]$.

The sphere graph is similarly defined on 3-manifolds.

Definition 1.3. For a 3-manifold M, the *sphere complex* $\mathcal{S}(M)$ is the graph with vertices given by isotopy classes of essential embedded spheres in M. For any two spheres $a, b \in \mathcal{S}(M)$, there is an edge between them only when there exists some disjoint representatives of $[a]$ and $[b]$.

To begin, we will introduce a set of colors defined by homology classes in double covers, then show this set is finite for any Σ_g. We then present a careful exposition of Bestvina, Bromberg, and Fujiwara's coloring of the curve complex, followed by a modified version of their argument designed to naturally translate to the sphere complex. Finally, we introduce the analogous set of colors and use it to prove Theorem A.

2. The Finite Colorability of $\mathcal{C}$

We begin with a detailed exposition of Bestvina, Bromberg, and Fujiwara's original proof. To do so, we first introduce a construction and a lemma useful in all dimensions.

Definition 2.1. Let A, B be manifolds with dimension n and P be a closed, smooth, orientable submanifold of dimension $n-1$ embedded in both A and B. The boundary of the cut manifold $\partial(A \setminus\!\!\setminus P)$ has two components a_1 and a_2 and similarly for B, $\partial(B \setminus\!\!\setminus P) = b_1 \sqcup b_2$ with a_1, a_2, b_1, b_2 all homeomorphic to P. We call the manifold constructed by identifying a_1 with b_2 and b_1 with a_2 the *crosswise cut-and-paste construction* (see Figure 1).

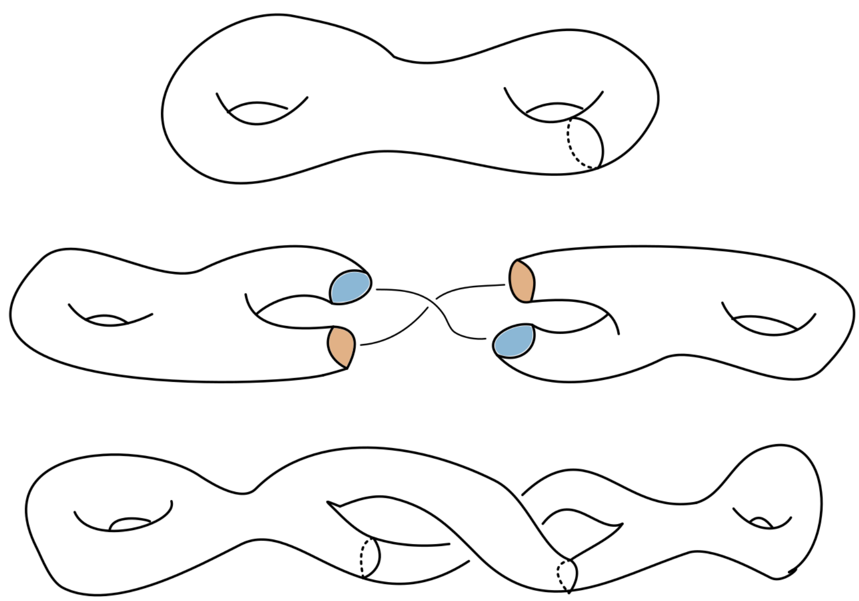

Figure 1. A connected cover created by Definition 2.1 using a non-separating curve

Lemma 2.2. *Let M be an n-dimensional closed, orientable, smooth manifold and P an orientable, closed $n-1$ dimensional smooth submanifold in M. If P is* *non-separating, then the crosswise cut-and-paste construction applied to two copies of M along P is a connected double cover $\tilde{M} \to M$.*

Proof. Let $\tilde{M}$ be the cover constructed as above. For any $x \in M$ not in any component of P, there exists some neighborhood N of x disjoint from P. The two copies of x in $\tilde{M}$, x_1 and x_2, will have their own neighborhoods N_1 and N_2 which are both homeomorphic to N and are disjoint by construction.

Let σ be a component of P and let $x \in \sigma$. Since M is a n-manifold, there is some neighborhood N of x homeomorphic to a ball in $\mathbb{R}^n$. Because the embedding of σ into M is smooth, we can decompose N into two neighborhoods A, B with $A \cap B = \sigma \cap N$. Their lifts A_1, B_1 and A_2, B_2 will be glued crosswise to create neighborhoods $N_1 = A_1 \cup B_2$ and $N_2 = A_2 \cup B_2$, which are both n-balls homeomorphic to N. ☐

There are other double covers that will not serve our purposes, namely disconnected covers. Notice that if P is a separating set, it will produce a disconnected cover (see Figure 2).

2.1. Coloring with double covers. We will use the following notation to define a set of colors $F(\Sigma_g)$ on the curve complex.

Notation 2.3. We denote the closed, orientable surface of genus g with d discs removed as $\Sigma_{g,d}$. If $d = 0$, we will simply use Σ_g. We introduce $T(\Sigma_g)$ as the set of all connected double covers of Σ_g. For any curve s embedded in Σ_g that lifts to some $\tilde{\Sigma}_g$, let the components of its inverse image under the covering map $\rho : \tilde{\Sigma}_g \to \Sigma_g$ be $\tilde{s}$ and $\tilde{s}'$. Let $X(\Sigma_g)$ be the set of $\{0\}$ and all subsets A of $H_1(\tilde{\Sigma}_g, \mathbb{Z}_2)$ with $|A| \leq 2$ across all $\tilde{\Sigma}_g \in T(\Sigma_g)$. Finally, we define our set of colors $F(\Sigma_g)$ to be the set of all functions $f : T(\Sigma_g) \to X(\Sigma_g)$.

First, we will show that the sets $T(\Sigma_g)$ and $F(\Sigma_g)$ are both finite.

Lemma 2.4. *For every $g \in \mathbb{N}$, $|T(\Sigma_g)| = 4^g$.*

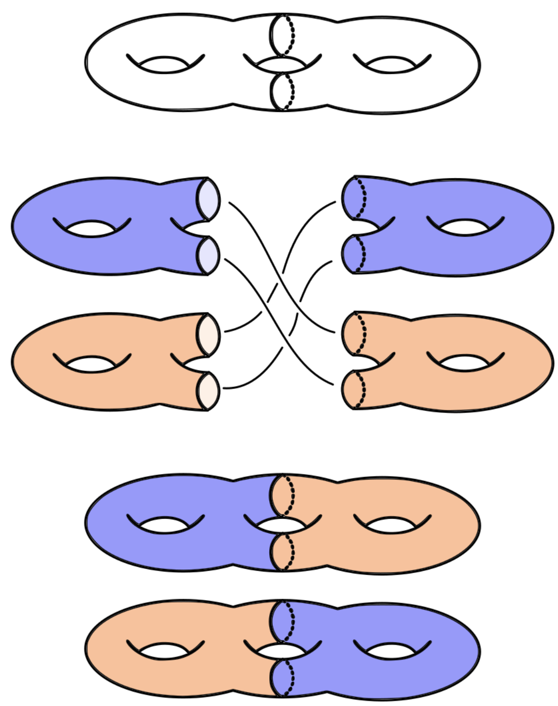

Figure 2. A disconnected cover produced by the construction in Lemma 2.2

Proof. Connected double covers correspond to the index-2 subgroups of the first fundamental group $\pi_1(\Sigma)$ [Hat02, Theorem 1.38]. There is a bijection between the set of index-2 subgroups of $\pi_1(\Sigma)$ and the set of homomorphisms $\phi : \pi_1(\Sigma) \to \mathbb{Z}_2$ because index-2 subgroups are normal. Each such homomorphism ϕ is determined by its values on a generating set of $\pi_1(\Sigma_g)$. Since $\pi_1(\Sigma_g)$ is generated by $2g$ curves, there are 2^{2g} such homomorphisms and therefore 4^g connected double covers. $\square$

Lemma 2.5. *For every $g \in \mathbb{N}$, $|F(\Sigma_g)|$ is finite and bounded above by $2^{(10g-4)4^g}$.*

Proof. Since the domain of every function is $T(\Sigma_g)$, which is finite, all that remains is to show that the codomain $X(\Sigma_g)$ is also finite, since $|F(\Sigma_g)| = |X(\Sigma_g)|^{|T(\Sigma_g)|}$.

Each nonzero element of $X(\Sigma_g)$ is determined by a choice of a double cover $\tilde{\Sigma}_g$ and some 1 or 2 element subset of $H_1(\tilde{\Sigma}_g, \mathbb{Z}_2)$. The genus of any $\tilde{\Sigma}_g$ will be $2g-1$, and thus its homology group will be generated by $4g-2$ curves. Therefore,

$|H_1(\tilde{\Sigma}_g, \mathbb{Z}_2)| = 2^{4g-2}$ and for every choice of $\tilde{\Sigma}_g$ we have

$$\binom{2^{4g-2}}{2} = 2^{8g-5} - 2^{4g-3}$$

choices of 2-element subsets and 2^{4g-2} choices of 1-element subsets. Using the number of possible choices of $\tilde{\Sigma}_g$ from Lemma 2.4 and including 0, we arrive at

$$|X(\Sigma_g)| = 1 + 4^g \left((2^{8g-5} - 2^{4g-3}) + 2^{4g-2}\right) \leq 2^{10g-4}.$$

Combined with our result from above, we conclude:

$$|F(\tilde{\Sigma}_g)| = |X(\tilde{\Sigma}_g)|^{|T(\Sigma_g)|} \leq 2^{(10g-4)4^g}.$$ □

2.2. Bestvina, Bromberg, and Fujiwara's proof. We now define the specific functions in $F(\Sigma_g)$ that we will use as colors on the sphere complex. For every simple, closed, essential curve a in Σ_g, define $f_a : T(\Sigma_g) \to X(\Sigma_g)$ by

$$f_a(\tilde{\Sigma}_g) = \begin{cases} \{[\tilde{a}], [\tilde{a}']\} & a \text{ lifts to } \tilde{\Sigma}_g \\ 0 & a \text{ does not lift to } \tilde{\Sigma}_g \end{cases}$$

where $[\tilde{a}]$ and $[\tilde{a}']$ are the homology classes of the lifts of a in $\tilde{\Sigma}_g$.

In the proof below, we make use of the equivalence of cobordism and homology classes on surfaces. This equivalence does not hold for 3-manifolds, so the technique will be reconsidered in the next proof.

Theorem 2.6. *For $g \geq 2$, the map $\phi : \mathcal{C}_g \to F(\Sigma_g)$ determined by $\phi(a) = f_a$ produces a coloring on $\mathcal{C}(\Sigma_g)$ such that if a and b share an edge in $\mathcal{C}_g$, $\phi(a) \neq \phi(b)$, that is, $\chi(\mathcal{C}(\Sigma_g)) \leq |F(\Sigma_g)|$.*

Proof. We will now show that for all adjacent curves a and b in $\mathcal{C}(\Sigma_g)$, there exists some $\tilde{\Sigma}_g \in T(\Sigma_g)$ such that $f_a(\tilde{\Sigma}_g) \neq f_b(\tilde{\Sigma}_g)$, and thus $f_a \neq f_b$.

For any choice of a, we can always construct a double cover where a lifts by choosing a single non-separating curve c disjoint from a and applying Lemma 2.2. We continue by cases:

(1) There is a cover $\tilde{\Sigma}_g$ where a lifts and b does not

(2) For every cover where a lifts, b also lifts and a and b are not homologous.

(3) For every cover where a lifts, b also lifts and a and b are homologous.

The first two cases are straighforward. In Item 1, for the $\tilde{\Sigma}_g$ where a lifts and b does not, $f_a(\tilde{\Sigma}_g) \neq 0 = f_b(\tilde{\Sigma}_g)$. In Item 2, their homology classes on any $\tilde{\Sigma}_g$ where a lifts are distinct and $f_a(\tilde{\Sigma}_g) \neq f_b(\tilde{\Sigma}_g)$.

For Item 3, we will construct a double cover $\tilde{\Sigma}_g$ where $f_a(\tilde{\Sigma}_g) \neq f_b(\tilde{\Sigma}_g)$. Note because a and b share an edge in $\mathcal{C}(\Sigma_g)$, they are disjoint. Therefore the complement $\Sigma_g \setminus (a \cup b)$ has at most three connected components S_1, S_2, S_3.

We claim that each S_i contains some non-separating curve σ_i. The only orientable surfaces without non-separating curves are the disc $\Sigma_{0,1}$, the annulus $\Sigma_{0,2}$, and the 3-holed sphere $\Sigma_{0,3}$. In the first case, either a or b would be non-essential. In the second case, a and b would be isotopic and belong to the same vertex in $\mathcal{C}(\Sigma_g)$. Finally, recall that curves are homologous if and only if they cobound a subsurface. In the third case, one of the two curves is separating and the other is non-separating, so they are not homologous.

Use the union of all σ_i to create a connected double cover $\tilde{\Sigma}_g$ by Lemma 2.2. Suppose, seeking contradiction, that $f_a(\tilde{\Sigma}_g) = f_b(\tilde{\Sigma}_g)$. Then there are lifts of a and b, $\tilde{a}$ and $\tilde{b}$, that are homologous and therefore cobound a subsurface $\tilde{S}$ of $\tilde{\Sigma}_g$ such that $\tilde{S}$ does not contain any other lifts of a or b. Under the covering map $\rho : \tilde{\Sigma}_g \to \Sigma_g$, the image $\rho(\tilde{S}) = S$ is a connected subsurface of Σ_g bounded by a and b. Thus S is a component of $\Sigma_g \setminus (a \cup b)$, that is, $S = S_i$ for some i. However, then the image of $\tilde{S}$ under the covering map includes σ_i, which means that the other lifts of a and b are part of the boundary of S_i (see Figure 3). This is a contradiction and thus $f_a(\tilde{\Sigma}_g) \neq f_b(\tilde{\Sigma}_g)$. $\square$

2.3. An alternative proof by counting intersections. We construct an alternative proof of Theorem 2.6 by modifying two key arguments with an eye towards translating the proof to the sphere complex.

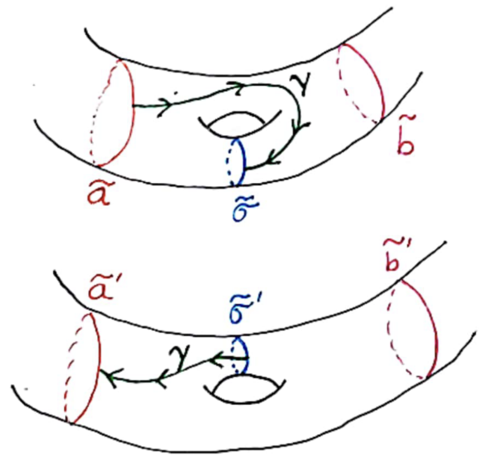

Figure 3. The existence of σ in S guarantees that $\rho^{-1}(S)$ is connected, shown by curve γ

First, we formalize the portion of the argument that relies on the classification of the spaces by examining the Euler characteristic $\chi(S_i)$ and leveraging χ's additivity over disjoint unions. When we shift to 3-manifolds, χ will be replaced by the 1^{st} Betti number β_1.

Second, unlike in surfaces, the equivalence of homology classes and cobordism classes does not hold for spheres in 3-manifolds. To distinguish between homology classes in a general setting, we use a theorem about counting intersections, for which we define the intersection number.

Definition 2.7. For any subspaces A, B in some manifold with homotopy classes $\mathcal{A}, \mathcal{B}$, we define the *intersection number* $\iota(A, b)$ as the following:

$$\iota(A, B) = \min\{|\pi_0(A' \cap B')| : A' \in \mathcal{A}, B' \in \mathcal{B}\}.$$

Now we can make use of the following consequence of Poincaré duality [Hat02]:

Theorem 2.8. *Let M be an n-manifold with some closed curve γ. If a is an embedded manifold of dimension $n-1$, the function $\iota(\gamma, a) \mod 2$ determines a homomorphism from $H_{n-1}(M, \mathbb{Z}_2) \to \mathbb{Z}_2$.*

Proof of Theorem 2.6. We proceed as in the previous proof up to the cases. Item 1 and Item 2 follow in exactly the same way as before, so it suffices to address Item 3. Thus we suppose that b lifts to every cover of Σ_g that a lifts to and $[a] = [b]$.

Notice that there are at most three components of $\Sigma_g \setminus (a \cup b)$ and call them S_1, S_2, S_3. Additionally, each S_i must be of the form $\Sigma_{h,d}$ for some $1 \leq d \leq 4$ and $0 \leq h \leq g$.

Every orientable surface with $\chi \leq -2$ has some non-separating curve. We will now show that even if $\chi(S_i) > -2$, each S_i still contains some non-separating curve or contradicts our hypotheses. Consider each value of d in turn, supposing that $\chi(S_i) > -2$.

d=1 In this case, S_i is a subspace bounded by exactly one of a or b. Then $S_i = \Sigma_{h,1}$ so that $\chi(\Sigma_h) - 1 > -2$ and therefore $h \geq 0$. If $h = 0$, then $\Sigma_{h,1}$ is a disc and either a or b bounds the disc S_i and is non-essential, which negates our hypotheses (see Figure 4). If $h = 1$, then $S_i = \Sigma_{1,1}$ has a non-separating curve (see Figure 5).

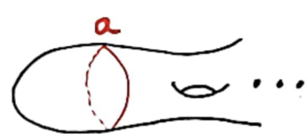

Figure 4. Example of non-essential a when $d = 1$

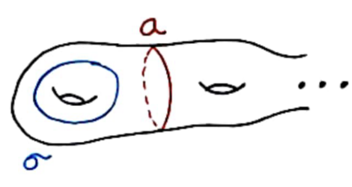

Figure 5. Example of nonseparating curve σ in S_i when $d = 1$

d=2 In this case, since Σ_g is connected, $\partial S_i = a \cup b$. Then we have for some $h \leq g$ that $\chi(S_i) = \chi(\Sigma_{h,d}) = \chi(\Sigma_h) - 2 > -2$ and thus $-2h > -2$, which means that $1 > h = 0$, Σ_h is a 2-sphere, and S_i is an annulus. However, then a and b cobound an annulus and are isotopic to each other, which negates our hypotheses (see Figure 6).

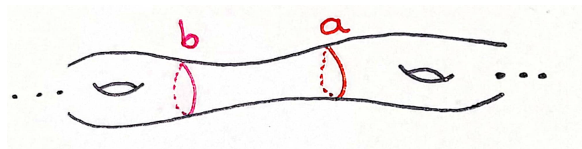

Figure 6. Example of isotopic a and b when $d = 2$

d=3 In this case, δS_i is three curves. Without loss of generality, let two copies belong to the boundary curve b and one belong to the curve a. We claim there is some curve γ in Σ_g that passes through b exactly once and has $a \cap \gamma = \emptyset$ Given a neighborhood N of b, select a pair of points p_1, p_2 in the two components of $N \setminus b$. Since N is path-connected and b separates N, there exists some path P_N between p_1 and p_2 that intersects b exactly once. Since S_i is path-connected and $N \setminus b \subset S_i$, there exists a path P_c connecting p_1, p_2 that does not intersect a or b. Take the union P to be γ. Then by Theorem 2.8, a and b are not homologous in Σ_g and this negates our hypotheses (see Figure 7).

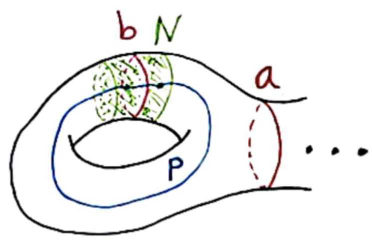

Figure 7. Construction of P to show a and b are non-homologous when $d = 3$

d=4 In this case, $\chi(\Sigma_{h,4}) = 2 - 2h - 4 \leq -2$ for every $h \geq 0$.

Thus, each S_i contains some non-separating curve σ_i disjoint from both a and b. Using this set of σ_i, construct $\tilde{\Sigma}_g$ with Lemma 2.2. Since a and b are homologous, they cobound some S_i, and a is part of ∂S_i for at least two components S_i, call them S_1 and S_2. Since each component S_i has some curve σ_i, the complement $\tilde{\Sigma}_g \setminus (\tilde{b} \cup \tilde{b}')$ does not separate $\tilde{a}$ and $\tilde{a}'$ and so there exists some closed path γ that passes through, in order, $\tilde{a}, \tilde{\sigma}_1, \tilde{a}', \tilde{\sigma}_2'$ and back to $\tilde{a}$ without intersecting either lift of b (Figure 8). By Theorem 2.8 $\iota(\gamma, \tilde{a}) \mod 2$ defines a homomorphism $\tau : H_1(\tilde{\Sigma}_g, \mathbb{Z}_2) \to \mathbb{Z}_2$. Then for the lifts $\tilde{a}, \tilde{b}, \tilde{b}'$, we have by construction $\tau(\tilde{a}) = 1 \neq 0 = \tau(\tilde{b}) = \tau(\tilde{b}')$ and $[\tilde{b}'] \neq [\tilde{a}] \neq [\tilde{b}]$ so that $f_a(\tilde{\Sigma}_g) \neq f_b(\tilde{\Sigma}_g)$. $\square$

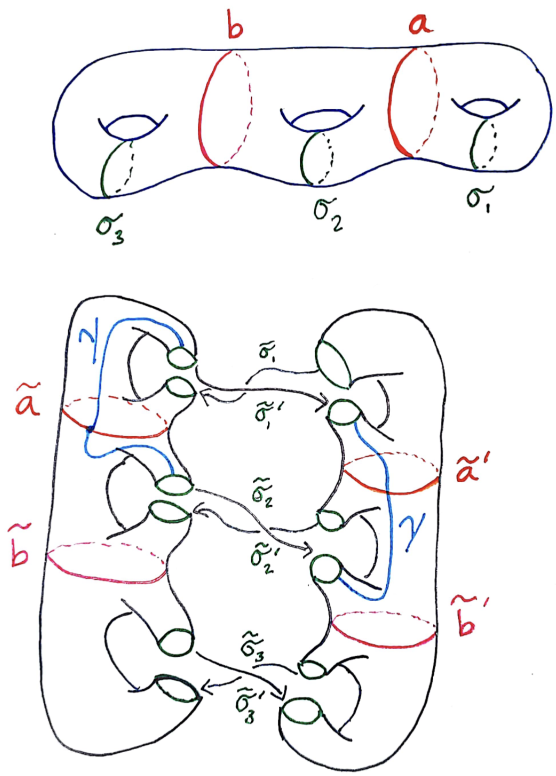

Figure 8. The construction of γ on a double cover of Σ_3

3. Coloring the Sphere Complex

We now show that the sphere complex is finitely colorable. Our proof follows the alternate proof of Theorem 2.6. As noted in the introduction, we only need to consider $r \geq 3$. There are a few properties of 3-manifolds that require us to modify the argument. First, an embedded sphere will always lift to a cover in 3-manifolds, which reduces the complexity of defining $X(M_r)$ (see Notation 3.1). This also removes a case when initially selecting spheres $a, b \in \mathcal{S}(M_r)$, since there are no covers where exactly one of a, b will lift. Additionally, homology classes and cobordism classes are not equivalent in 3-manifolds.

3.1. Coloring with double covers (again). Our previous notation in Section 2(Notation 2.3) translates straightforwardly with the exception of the codomain $X(M_r)$. Namely, there is no need to include 0 in the codomain as every sphere lifts to every cover in 3-manifolds.

Notation 3.1. Let $M_{r,b}$ be the connect sum of the 3-sphere with r copies of $S^1 \times S^2$ and b copies of B^3 removed. If $b = 0$, we simply use $M_{r,0} = M_r$. Define $T(M_r)$ as the set of all of connected double covers of M_r. For any sphere s embedded in M_r, let the components of it inverse image under the covering map $\rho : \tilde{M}_r \to M_r$ be $\tilde{s}$ and $\tilde{s}'$. Let $X(M_r)$ be the set of all subsets A of $H_1(\tilde{M}_r, \mathbb{Z}_2)$ with $|H| \leq 2$ across all $\tilde{M}_r \in T(M_r)$. Finally, let $F(M_r)$ be the set of all functions $f : T(M_r) \to X(M_r)$.

Here we translate Lemma 2.4 to the sphere complex, resulting in a slightly lower bound due to the smaller generating set of $\pi_1(M_r)$.

Lemma 3.2. *For every $r \in \mathbb{N}$, $|T(M_r)| = 2^r$.*

Proof. Double covers correspond to the index-2 subgroups of the first fundamental group of a manifold $\pi_1(M)$ [Hat02, Theorem 1.38]. There is a bijection between the set of index-2 subgroups of $\pi_1(M_r)$ and the set of homomorphisms $\phi : \pi_1(M_r) \to \mathbb{Z}_2$ because index-2 subgroups are normal. Each such homomorphism ϕ is determined by its values on a generating set of $\pi_1(M_r)$. Since $\pi_1(M_r) \cong F_r$ the free group of rank r, there are r generators of $\pi_1(M_r)$. Then there are 2^r homomorphisms ϕ and therefore 2^r connected double covers. $\square$

Continuing, we translate Lemma 2.5 to the sphere complex. The overall logic is the same as before, although we use an alternate way of determining the rank of $H_2(\tilde{M}_r, \mathbb{Z})$, and the lack of 0 in the codomain slightly modifies the calculations.

Lemma 3.3. *The set of colors $F(M_r)$ is finite for any given M_r and bounded above by $2^{(9r-4)2^r}$.*

Proof. Since the domain of every function is $T(M_r)$, which is finite, all that remains is to show that the codomain $X(M_r)$ is also finite, since $|F(M_r)| = |X(M_r)|^{|T(M_r)|}$.

Each element of $X(M_r)$ is determined by a choice of a double cover $\tilde{M}_r$ and some 1 or 2 element subset of $H_2(\tilde{M}_r, \mathbb{Z}_2)$. Recall that $\chi(M) = \sigma_{i=0}^{3}(-1)^i\beta_i(M) = 0$ for any closed 3-manifold. Then $\beta_2(M) = \beta_1(M)$ since both $\beta_0(M) = 0$ and $\beta_3(M) = 0$. So the free rank of $H_2(\tilde{M}_r, \mathbb{Z}) = \beta_2(\tilde{M}_r) = \beta_1(\tilde{M}_r) = 4r - 2$. Therefore, $|H_2(\tilde{M}_r, \mathbb{Z}_2)| = 2^{4r-2}$ and for every choice of $\tilde{M}_r$ we have

$$\binom{2^{4r-2}}{2} = 2^{8r-5} - 2^{4r-3}$$

choices of 2-element subsets and 2^{4r-2} choices of 1-element subsets. Using the number of possible choices of $\tilde{M}_r$ from Lemma 3.2, we arrive at

$$|X(M_r)| = 2^r \left((2^{8r-5} - 2^{4r-3}) + 2^{4r-2}\right) \leq 2^{9r-4}.$$

Then we conclude that

$$|F(\tilde{M}_r)| = |X(\tilde{M}_r)|^{|T(M_r)|} \leq 2^{(9r-4)2^r}.$$ $\square$

3.2. Proof of Theorem A. Similar to Section 2.2, we define the function $f_a : T(M_r) \to X(M_r)$ by

$$f_a(\tilde{\Sigma}_g) = \{[\tilde{a}], [\tilde{a}']\}$$

where $[\tilde{a}]$ and $[\tilde{a}']$ are the homology classes of the lifts of a to $\tilde{\Sigma}_g$. We now prove that $\phi : \mathcal{S}(M_r) \to F(M_r)$ given by $\phi(a) \mapsto f_a$ is a coloring.

Theorem A. *For every $r \geq 3$, there is a finite coloring $\phi : \mathcal{S}(M_r) \to F(M_r)$ of the set of embedded essential spheres in M_r so that if a, b span an edge, $\phi(a) \neq \phi(b)$. Thus $\chi(\mathcal{S}(M_r)) \leq 2^{(9r-4)2^r}$ by Lemma 3.3.*

Proof. By Lemma 3.2 and Lemma 3.3, it remains to show that for any spheres a, b adjacent in $\mathcal{S}(M_r)$, $f_a \neq f_b$. To do so, we will show there exists some $\tilde{M}_r \in T(M_r)$ such that $f_a(\tilde{M}_r) \neq f_b(\tilde{M}_r)$.

Let a, b be adjacent spheres in $\mathcal{S}(M_r)$. We will construct a cover $\tilde{M}_r \in T(M_r)$ such that for any choice of lifts of a and b in $\tilde{M}_r$, call them $\tilde{a}, \tilde{b}$, will always have $[\tilde{a}] \neq [\tilde{b}]$ within $H_2(\tilde{M}_r, \mathbb{Z}_2)$.

Since a, b are disjoint, there are at most three components of $M_r \setminus (a \cup b)$. Label the components W_1, W_2, W_3 and notice that each one is of the form $M_{k,b}$ with $\beta_1(W_i) = k$ and $1 \leq b \leq 4$. We will next show that each W_i contains some non-separating sphere by showing that $\beta_1(W_i) \geq 1$. To do so, we show that for every value of b, if $\beta_1(W_i) = 0$, that is, $W_i = M_{0,b}$, then we contradict our hypotheses.

- b=1 In this case, W_i is a subspace bounded by exactly one of a or b. Then $W_i = M_{0,1}$, that is S^3 with a ball removed, i.e., W_i is a ball itself. Then either a or b bounds a ball and is non-essential, which negates our hypotheses.
- b=2 In this case, the two boundaries of W_i belong one each to a and b, since W_i is connected. Then $W_i = M_{0,2}$, that is, S^3 with two balls removed which is homeomorphic to $S^2 \times I$. However, this implies that a and b are isotopic to each other, which negates our hypotheses.

b=3 In this case, δS_i is three spheres. Without loss of generality, let two copies belong to the boundary sphere b and one belong to the sphere a. We claim there is some curve γ in Σ_g that passes through b exactly once and has $a \cap \gamma = \emptyset$. Given a neighborhood N of b, select a pair of points p_1, p_2 in the two components of $N \setminus b$. Since N is path-connected and b separates N, there exists some path P_N between p_1 and p_2 that intersects b exactly once. Since S_i is path-connected and $N \setminus b \subset S_i$, there exists a path P_c connecting p_1, p_2 that does not intersect a or b. Take the union $\gamma = P_N \cup P_c$. Then by Theorem 2.8, a and b are not homologous in Σ_g, contradicting our hypotheses.

b=4 In this case, there is only one component W_1 and $\delta S_1 = a \cup b$. If $W_1 = M_{0,4}$, then the original space $M_r = M_2$. However, we are only considering $r \geq 3$, so again we have negated our hypotheses.

Thus, each W_i contains some non-separating sphere σ_i. Using this set of σ_i, construct $\tilde{M}_r$ using Lemma 2.2. Notice that if $a \cup b$ does not separate M_r, then we can show that $[a] \neq [b]$ via similar argument to case $b = 3$ above. Therefore $a \cup b$ separates M_r.

We will construct a closed curve γ in M_r that intersects both $\tilde{a}$ and $\tilde{a}'$ exactly once while never intersecting either lift of b. If such a γ exists, then by Theorem 2.8 $\iota(\gamma, ta_1) \mod 2$ defines a homomorphism $\tau : H_1(\tilde{M}_r, \mathbb{Z}_2) \to \mathbb{Z}_2$. Then for this γ by construction we will have $\tau(\tilde{a}) = 1 \neq 0 = \tau(\tilde{b}) = \tau(\tilde{b}')$ and the homology classes of the lifts of a and b are distinct so that $f_a(\tilde{M}_r) \neq f_b(\tilde{M}_r)$.

Choose two components S_1 and S_2 such that $a \subset \partial S_1$ and $a \subset \partial S_2$. Notice that $\tilde{M}_r \setminus (\tilde{b} \cup \tilde{b}')$ does not separate $\tilde{a}$ and $\tilde{a}'$, so there exists some closed path γ that passes through, in order, $\tilde{a}, \tilde{\sigma}_1, \tilde{a}', \tilde{\sigma}_2'$ and returning to $\tilde{a}$ without intersecting either lift of b (see Figure 9). $\square$

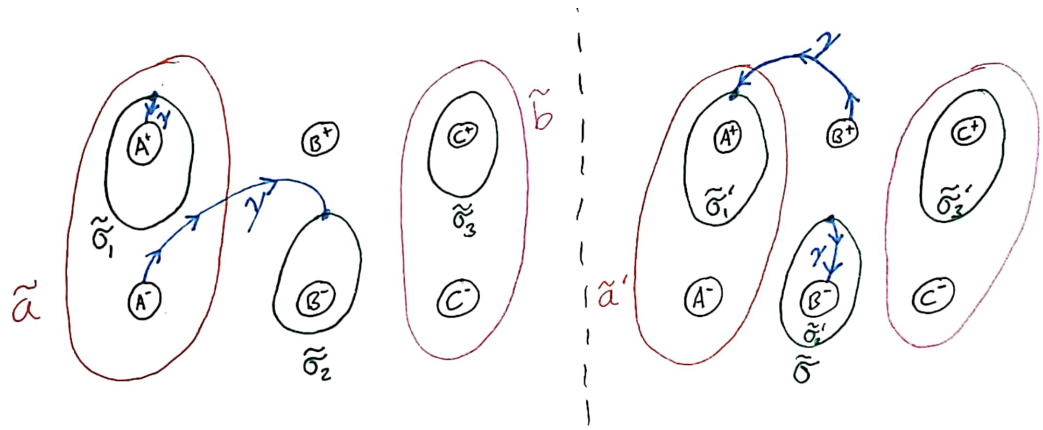

Figure 9. Construction of γ in a double cover of M_3

4. Conclusion

This work establishes the upper bound of the chromatic number of $\mathcal{S}(M_r)$. A companion lower bound was obtained by Bering, Ortiz, and Sanchez. Combined, we obtain:

$$r\log(r) \leq \chi(\mathcal{S}(M_r) \leq 2^{9r-4}.$$

Tightening this vast gap between these two estimates is a natural next task. The techniques to use may be gathered from work by Gaster, Greene, and Vlamis [GGV18, Theorem 1.5], who have found a stricter upper bound on $\chi(\mathcal{C}(\Sigma_g))$. Their methods use induced subgraphs generated from primitive elements of $H_1(\Sigma_g, \mathbb{Z}_2)$, allowing them to conclude that $\chi(\mathcal{C}(\Sigma_g)) \leq g4^g$.

The argument that gave us the upper bound for $\chi(M_r)$ should also naturally extend to prove an upper bound for $\chi(M_{r,d})$. This would then prove that the sphere complex of any compact and orientable 3-manifold is finitely colorable. This follows from Kneser's Prime Decomposition theorem by virtue of the fact that any sphere outside of the prime $M_{r,d}$ components will be one of finitely many prime-decomposition spheres and thus only increase the chromatic number of the sphere complex by a finite amount.

Unlike in surfaces, up to homeomorphism there is more than one connected orientable codimension 1 manifold that can be embedded in a 3-manifold. Instead of using spheres to build $\mathcal{S}(M_r)$, fix $g \geq 1$ and construct the graph with vertices determined isotopy classes of essential embedded surfaces Σ_g and edges between classes with disjoint representatives. The major difference in employing the approach we use here will be in ensuring that the non-separating spheres still exist to be able to construct an appropriate double cover. Additionally, while spheres will always lift in a double cover, some Σ_g may not always lift and so the techniques applied in the proof of Theorem 2.6 may be needed.

Another analog of the curve complex to investigate is the free factor complex of F_n. Its vertices are the proper free factors of F_n and its edges exist between two factors when one is a proper factor of the other. Like the sphere complex, $\mathrm{Out}(F_n)$ also acts on the free factor complex, but is directly related to F_n and as such, provides greater algebraic insight into $\mathrm{Out}(F_n)$ while the sphere complex gives more insight into $\mathrm{Out}(F_n)$'s geometric nature. Bestvina and Feighn [BF14] give a finite coloring of the free factor complex. Their proof relies on the $\mathbb{Z}_2$ homology and differences in double covers, the same two factors which allowed this proof to succeed.

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