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Multilevel Coded Modulation for Unequal Error Protection and Multistage Decoding—Part I: Symmetric Constellations

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Abstract—In this paper, theoretical upper bounds and computer simulation results on the error performance of multilevel block coded modulations for unequal error protection (UEP) and multistage decoding are presented. The paper is divided into two parts. In part I, symmetric constellations are considered, while in the sequel, asymmetric constellations are analyzed. It is shown that nonstandard signal set partitionings and multistage decoding provide excellent UEP capabilities beyond those achievable with conventional coded modulation. The coding scheme is designed in such a way that the most important information bits have a lower error rate than other information bits. The large effective error coefficients, normally associated with standard mapping by set partitioning, are reduced by considering nonstandard partitionings of the underlying signal set. The bits-to-signal mappings induced by these partitionings allow the use of soft-decision decodings of binary block codes. Moreover, parallel operation of some of the staged decoders is possible, to achieve high data rate transmission, so that there is no error propagation between these decoders. Hybrid partitionings are also considered that trade off increased intraset distances in the last partition levels with larger effective error coefficients in the middle partition levels. The error performance of specific examples of multilevel codes over 8-PSK and 64-QAM signal sets are simulated and compared with theoretical upper bounds on the error performance.

Index Terms—Coded modulation, multistage decoding, unequal error protection.

I. INTRODUCTION

There are many practical applications, such as satellite broadcasting of digital high definition TV (HDTV) or digital speech transmission, where high bandwidth-efficient digital transmission systems must be designed to provide a gradual degradation of the received signal. It is proposed to combine coding and modulation in such a way that the required graceful degradation is achieved by error control coding. In this paper, we restrict ourselves to transmission over an additive white Gaussian noise (AWGN) channel.

Subsets of signal sequences, of increasing minimum squared Euclidean distances (MSED's), are associated with information bits of increasing importance level (e.g., decreasing image definition). Code sequences in correspondence to the least important part (e.g., the HDTV component) are clustered into clouds [1]. Each coded signal sequence in correspondence to a most important message part (e.g., the basic definition TV component) is associated with a cloud. The mapping of information bits to coded signal sequences is made in such a way that the minimum distance between coded signal sequences in different clouds is larger than the minimum distance between coded signal sequences within a cloud. This is an unequal error protection (UEP) coding scheme [2].

Nonstandard partitionings of signal sets for constructing coded modulations with UEP were first proposed in [3] and [4]. Also, nonstandard partitionings were considered to design good multilevel codes, based on rate and capacity arguments, in [5]. Coded modulation approaches for the terrestrial broadcasting of HDTV signals have been reported in [3], [4], and [6]. All of them, however, deal with asymmetric rectangular (M-QAM type) signal sets. A trellis coded modulation (TCM) scheme for UEP using asymmetric 8-PSK signal sets and nonstandard partitionings for satellite broadcasting is reported in [7]. The nonstandard and hybrid partitionings introduced in subsequent sections of this part of the paper have the advantage that conventional symmetric signal sets are used. This may result in a simpler implementation of the modulators and demodulators and easier synchronization. Asymmetric constellations are relegated to part II.

In part I, we consider multilevel coded modulation [8] over symmetric constellations with bits to signal mapping by set partitioning using a rule such that at each partition level, all the signal points within a subset are contained in disjoint half planes. This results in a small number of nearest neighbor (NN)
sequences as well as allowing the use of soft decision decoding procedures designed for binary linear block codes with binary transmission over an AWGN channel. This approach will be referred to as block partitioning.

A partitioning approach is introduced that constitutes a generalization of the block and Ungerboeck [9] partitioning rules. This kind of partitioning is suitable for coded modulation with a better average error performance with less levels of error protection. We call this a hybrid partitioning approach, because the higher partition levels are nonstandard, while at lower levels, partitioning is performed using Ungerboeck’s rules [9], i.e., to maximize the squared Euclidean distance (SED) between signal points within a subset. We will show that a good tradeoff is obtained between increasing the error coefficients in the middle partition levels and improving the error performance of the subsequent decoding stages.

The rest of the paper is organized as follows: In Section II, we present multilevel coded modulation and a design principle to achieve UEP. Theoretical analysis and computer simulations of block partitionings of 8-PSK and 64-QAM modulations for UEP are presented in Section III. A hybrid partitioning approach is introduced, and theoretical and simulation results for 8-PSK and 64-QAM modulations are presented in Section IV. Finally, Section V presents conclusions of this work.

II. MULTILEVEL CODED MODULATION

A. Definitions

Imai and Hirakawa [8] proposed a technique for constructing coded modulation schemes using binary block codes. For an \( M \)-level coded modulation, the codewords of \( M \) binary block codes are used to index code sequences of signal points in a \( 2^M \)-ary modulation signal constellation. The resulting signal sequences form a block modulation code (BCM) over the Euclidean space. A fundamental issue in the design of a multilevel coded modulation is the labeling of the signal set over which the component codes operate. Such labeling determines the MSED of the modulation code and, more generally, the distance structure of the set of coded sequences, as discussed below.

In what follows, Ungerboeck’s well-known standard mapping-by-set partitioning [9] is briefly overviewed. A \( 2^M \)-ary modulation signal set \( S \) is partitioned into \( M \) levels. For \( 1 \leq i \leq M \), at the \( i \)-th partition level, the signal set is divided into two subsets \( S_i(0) \) and \( S_i(1) \), such that the intraset SED, \( \delta^2_i \), is maximized. A label bit \( b_i \in \{0, 1\} \) is associated with the subset choice \( S_i(b_i) \) at the \( i \)-th partition level. This partitioning process results in a labeling of the signal points. Each signal point in the set has a unique \( i \)-bit label and is denoted by \( \bar{c}_i \). With this standard partitioning of \( 2^M \)-ary modulation signal constellation, the intraset SED’s are in nondecreasing order \( \delta^2_1 \leq \delta^2_2 \leq \cdots \leq \delta^2_M \).

For \( 1 \leq i \leq M \), let \( C_i \) denote an \( (n_i, k_i, d_i) \) binary linear block code of length \( n_i \), dimension \( k_i \), and minimum Hamming distance \( d_i \). Also, let \( A_{M_i}^{(k_i)} \) denote the number of codewords in \( C_i \) of weight \( k_i \). Let

\[
\bar{c}_1 = (c_{11}, c_{12}, \ldots, c_{1n}) \\
\bar{c}_2 = (c_{21}, c_{22}, \ldots, c_{2n}) \\
\vdots \\
\bar{c}_M = (c_{M1}, c_{M2}, \ldots, c_{Mn})
\]

be \( M \) codewords in \( C_1, C_2, \ldots, C_M \), respectively. Form the following sequence:

\[
\bar{c}_1 \star \bar{c}_2 \star \cdots \star \bar{c}_M = (c_{11}c_{21} \cdots c_{M1}, c_{12}c_{22} \cdots c_{M2}, \ldots, c_{1n}c_{2n} \cdots c_{Mn}).
\]

Each component in \( \bar{c}_1 \star \bar{c}_2 \star \cdots \star \bar{c}_M \) is regarded as the label of a signal in the \( 2^M \)-ary modulation signal set \( S \). Then

\[
s(\bar{c}_1 \star \bar{c}_2 \star \cdots \star \bar{c}_M) = (s(c_{11}c_{21} \cdots c_{M1}), s(c_{12}c_{22} \cdots c_{M2}), \ldots, s(c_{1n}c_{2n} \cdots c_{Mn}))
\]

is a sequence of signal points in \( S \). The following collection of signal sequences over \( S \):

\[
\Lambda = \{s(\bar{c}_1 \star \bar{c}_2 \star \cdots \star \bar{c}_M); \bar{c}_i \in C_i, 1 \leq i \leq M\}
\]

forms an \( M \)-level modulation code over the signal set \( S \) or an \( M \)-level coded \( 2^M \)-ary modulation.

The rate, or spectral efficiency, of this coded modulation system in bits/symbol is \( \frac{R}{n} \). It is well known that the MSED of this system, denoted by \( d^2_\Lambda(\Delta) \), is given by [8]

\[
d^2_\Lambda(\Delta) \geq \min_{1 \leq i \leq M} \{d_i^2\}
\]

B. Asymptotic UEP Design Principle

In order to achieve UEP, the following design guideline for \( M \)-level coded \( 2^M \)-ary modulation is proposed [10].

For \( 1 \leq i \leq M \), the binary codes \( C_i \) are selected in such a way that the following inequalities are satisfied:

\[
d_i \delta^2_i \geq d_0 \delta^2_0 \geq \cdots \geq d_M \delta^2_M.
\]

For \( 1 \leq i \leq M \), let \( \bar{c}_i(\bar{m}_i) \) be the codeword of \( C_i \) in correspondence to a \( k_i \)-bit message vector \( \bar{m}_i \), and let \( \bar{s} = (\bar{s}(\bar{m})) \) and \( \bar{s}' = (\bar{s}'(\bar{m}')) \) denote coded \( 2^M \)-ary modulation signal sequences in \( \Lambda \) corresponding to message vectors \( \bar{m} = (\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_M) \) and \( \bar{m}' = (\bar{m}'_1, \bar{m}'_2, \ldots, \bar{m}'_M) \), respectively. The Euclidean separations [11] and [12] between coded sequences at the \( i \)-th partition level, for \( i = 1, \ldots, M \), are defined as

\[
s_i = \min \{d(\bar{s}, \bar{s}'); \bar{m}_j \neq \bar{m}'_j, \bar{m}_i = \bar{m}'_i, j \neq i\}.
\]

It follows from (1) that \( s_1 \geq d_1 \delta^2_1 \), \( s_2 \geq d_2 \delta^2_2 \), \ldots, \( s_M \geq d_M \delta^2_M \). For transmission over an AWGN channel, the set of inequalities (1) results in message vectors with decreasing error protection levels.

The above principle is useful in specifying the asymptotic error performance of a coded modulation with UEP. However, as also shown in [5], design criteria based on intraset MSED’s are inappropriate for multistage decoding of multilevel coded modulations, at low to medium signal-to-noise ratios (SNR’s),
because of the large number of NN sequences in the first decoding stages. As an example, Fig. 1 shows simulation results on the performance of a three-level coded 8-PSK modulation with the \((64,18,22), (64,57,4),\) and \((64,63,2)\) extended BCH codes (ex-BCH codes) as component codes \(C_i, i = 1, 2, 3,\) respectively. In this figure as in the rest of the paper, the signal constellation considered is normalized to unit energy. The Euclidean separations are \(s_1 = 120, s_2 = s_3 = 8,\) for 18 and 120 information bits, respectively (asymptotic coding gains of 8.1 and 6 dB, respectively). The adverse effects of the number of NN (or error coefficient) in the first decoding stage are such that the coding gains are greatly reduced. With multistage decoding, the number of NN associated with the first stage is such that the coding gains are greatly reduced. With multistage decoding, the number of NN associated with the first stage is greatly reduced. With multistage decoding, the number of NN associated with the first stage is greatly reduced.

III. BLOCK PARTITIONING

In this section, a partitioning strategy is presented that reduces the number of NN at every partition level. At the \(q\)th partition level, the signal points within each subset \(S_q(b_i)\) are contained in disjoint planes of the two-dimensional Euclidean space. As a result, only a small number of signal points, those located near the decision boundary, will have neighbors at minimum distance. On the average, assuming equiprobable signaling, the number of NN associated with the \(q\)th level will be much less than with Ungerboeck partitioning. On the other hand, the price to pay is a constant minimum intraset distance at each level of the partition.

![Fig. 1. Simulation results of a coded 8-PSK modulation with Ungerboeck mapping.](image)

![Fig. 2. An 8-PSK constellation with block partitioning: (a) labeling, (b) \(X\)-coordinate projections, and (c) decoder structure.](image)

### TABLE I

<table>
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<th>(r_{eij})</th>
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</table>

A. Three-Level Coded 8-PSK Modulation with UEP Using Block Partitioning

The block partitioning shown in Fig. 2(a) is used to construct three-level coded 8-PSK modulation schemes with UEP. Note that this partitioning also corresponds to Gray mapping. In the figure, the color black is used to represent signal points whose label is of the form \(0_2 0_3\), with \(b_2, b_3 \in \{0, 1\}\). Similarly, the color white is used for points with labels \(1_2 0_3\). A circle indicates that the label is of the form \(0_2 0_3, b_1 = 0\), while a square is used to represent signal points with labels \(1_2 1_3\).

It can be seen from Fig. 2(a) that in order to determine the value of the first label bit \(b_1\), only the \(X\)-coordinate is sufficient. If a signal point is on the left half plane \((X < 0)\), then it corresponds to \(b_1 = 0\), otherwise, it corresponds to \(b_1 = 1\). In the same way, the \(Y\)-coordinate suffices to determine the value of the second label bit \(b_2\). If a signal point lies in the upper half plane \((Y > 0)\), then \(b_2 = 0\), otherwise \(b_2 = 1\). This is an important property of this block partitioning. It means that in a three-level coded 8-PSK modulation scheme using this partitioning, the first and second levels are independent. This in turn implies that the first and second level decoders can be implemented in parallel.

In the first and second decoding stages, the decision variable is just the projection of the received signal sequence onto the
or axis, respectively. Fig. 2(c) shows a block diagram of a multistage decoder for a three-level coded 8-PSK modulation with block partitioning. The decoders for the first and second stages operate independently on the in-phase and quadrature component of the received signal sequences $r_x$ and $r_y$, respectively. Once decisions are made as to the corresponding codewords $\mathbf{c}_1$ and $\mathbf{c}_2$, they are passed on to the third decoding stage. Let $\mathbf{c}_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \in C_i$ be the decoded codeword at the $i$th stage, $i = 1, 2$. Before the third-stage decoding, each two-dimensional coordinate $(r_{xj}, r_{yj})$ of the received signal $\mathbf{r} = (r_{x1}, r_{y1})$ is projected onto a one-dimensional coordinate $r_{xj}'$, $1 \leq j \leq n$. The values $r_{xj}'$ are the decision variables used by the decoder of $C_3$. The projection depends on the decoded quadrant, which is indexed by the pair $(\mathbf{c}_1, \mathbf{c}_2)$, $1 \leq j \leq n$, as shown in Table I. It corresponds to a scaled rotation of $\pi/4$. The rotated sequence $\mathbf{r}' = (r_{x1}', r_{x2}', \ldots, r_{xn}')$ is then decoded using a soft-decision procedure for component code $C_3$. The independence between the first and second levels also results in no error propagation from the first decoding stage to the second. For Ungerboeck partitionings, the opposite is always true.

1) Error Performance: In analyzing the error performance of the $i$th decoding stage it will be assumed, without loss of generality, that the all-zero sequence is transmitted in the $i$th level. Note that this assumption is different from assuming that the all-zero codeword is transmitted at all levels, which is not correct with the partitionings considered in this paper. Also, for multistage decoding, all sequences are possible and equally likely in the subsequent stages.

With reference to Fig. 2(a), we observe that the projections of the four possible signals in the left half plane can take one of two values: $X = -\Delta_1 = -\sin(\pi/8)$ or $X = -\Delta_2 = -\cos(\pi/8)$, as shown in Fig. 2(b). Since messages are equally likely, the probability of a signal point having coordinate $(w, i)$ or $(w, i')$ is equal to $1/2$. A block error event will occur at the first stage whenever a codeword of nonzero weight is decoded. Let $X_j = \Delta_1 b_j + \Delta_2 b_j$, $1 \leq j \leq w$ denote the $X$ components of a coded sequence such that $b_1 = 1$, and let $w$ denote the Hamming weight of an incorrectly decoded codeword $\mathbf{c}_w$ in the first level code $C_1$. The two corresponding decision region occupy a $w$-dimensional space separated by the decision hyperplane $X_1 + X_2 + \cdots + X_w = 0$. With respect to $\mathbf{c}_w$, an error event occurs when a signal vector with $X$-coordinates

$$P = (-\Delta_1, -\Delta_2, \ldots, -\Delta_w)$$

is corrupted by AWGN noise and moves to the decision region specified by $X_1 + X_2 + \cdots + X_w > 0$. The probability of an erroneous decoding into $\mathbf{c}_w$ at the first stage is given by

$$P_r = \Pr \{X_1 + X_2 + \cdots + X_w > 0\}.$$  

For the $w$ nonzero positions of $\mathbf{c}_w$, the all-zero codeword can be mapped into $i$ components with $X$-coordinate $X = -\Delta_1$ and $(w-i)$ components with $X$-coordinate $X = -\Delta_2$, for $i = 0, 1, \ldots, w$. For multistage decoding, all $\binom{w}{i}$ possible points $P$ corresponding to a given value of $i$ are valid sequences. It is shown in Appendix A (for $j = 2$) that the smallest SED from the corresponding point $P$ to the hyperplane $X_1 + X_2 + \cdots + X_w = 0$ is

$$E_P(i) = \frac{1}{w} (i\Delta_1 + (w-i)\Delta_2)^2.$$  

Hence, with respect to the codeword $\mathbf{c}_w$, and for a given values of $i$ in $\{0, 1, \ldots, w\}$, it follows that

$$\Pr \{X_1 + X_2 + \cdots + X_w > 0, i\} = \binom{w}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{w-i} Q \left(\sqrt{\frac{2RE_b}{N_0}} E_P(i)\right)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-n^2/2} dn$$

and $E_b/N_0$ is the energy-per-bit-to-noise ratio. For all the code sequences associated with the codeword $\mathbf{c}_w$ of weight $w$ in the first level code, and due to the symmetry of the decision hyperplane, the union bound yields the following expression for the probability of a block error:

$$Pr \{X_1 + X_2 + \cdots + X_w > 0\} \leq \sum_{i=0}^{w} \binom{w}{i} 2^{-wQ \left(\sqrt{\frac{2RE_b}{N_0}} E_P(i)\right)}.$$  

Finally, when assuming a systematic encoding, the union bound on the bit-error probability of the first decoding stage can be written as [14]

$$H_b^{(NS)} \leq \sum_{w=1}^{n} \frac{w}{n} A^{(1)}_{w} \Pr \{X_1 + X_2 + \cdots + X_w > 0\} \leq \sum_{w=1}^{n} \frac{w}{n} A^{(1)}_{w} \sum_{i=0}^{w} \binom{w}{i} 2^{-wQ \left(\sqrt{\frac{2RE_b}{N_0}} E_P(i)\right)}.$$  

The bound (7) can be compared with a similar one for the Ungerboeck’s partitioning (UG) strategy

$$H_b^{(UG)} \leq \sum_{w=1}^{n} \frac{w}{n} A^{(1)}_{w} 2^{wQ \left(\sqrt{\frac{2RE_b}{N_0}} w\Delta_1^2\right)}.$$  

From (7) and (8), we observe that while Ungerboeck’s partitioning increases exponentially the effect of NN sequences, by a factor of $2^w$, the block partitioning has for $E_b/N_0$ an error coefficient term $2^{-wQ}$, which decreases exponentially with the distances of the first-level component code. As a result, for practical values of $E_b/N_0$, the block partitioning may yield, at the first stage, a real coding gain even greater than the asymptotic coding gain. This is a very desirable feature of a coded modulation with UEP.

Due to the independence and symmetry between the first and second stages, the probability of a bit error in the second decoding stage is also upper bounded by (7). On the other hand, the error performance of the third stage of a three-level coded 8-PSK modulation depends on that of the previous two stages. However, for the block partitioning for UEP, the first level codes
are the most powerful, and the effect of errors from the first decoding stages can be neglected. Under this assumption, a good approximation is obtained by assuming that decoding decisions in the first and the second decoding stages are correct. From a conventional union bound argument for binary linear block codes [14], it follows that:

$$I(NS)_{b_3} \leq \sum_{w=0}^{n} \frac{w}{n} A^{(3)}(w) Q \left( \sqrt{\frac{2RE_b}{N_0} w \Delta_i^2} \right). \quad (9)$$

2) Simulation Results: A three-level 8-PSK modulation for UEP was selected as an example with (64, 18, 22), (64, 45, 8), and (64, 63, 2) ex-BCH codes as the first-, second-, and third-level codes, respectively. This coding scheme, denoted $\Delta_1$, has rate equal to 1.97 bits/symbol and can be compared with uncoded QPSK modulation, which has approximately the same rate (a difference of only 0.06 dB).

Simulation results of this example are shown in Fig. 3. $S_1(n, k)$ and $UB(n, k)$ denote computer simulations and upper-bound evaluation, respectively, of the corresponding ex-BCH code. In the simulations, we used the ordered statistics soft-decision decoding procedures of [15]. The results agree with the theoretical bounds at the practical BER of $10^{-5}$. Three levels of error protection are achieved with the block partitioning. An impressive coding gain of 8.5 dB is achieved at the BER of $10^{-5}$ for 18 most important bits (14.3%) encoded in the first level. In the second and third stages, the corresponding values of coding gain are 2.5 and $-4.0$ dB, respectively. It is interesting to note that at this BER, the simulated coding gain at the first decoding stage is greater than the asymptotic coding gain (8.1 dB) due to the reduced error coefficients.

B. Six-Level Coded 64-QAM Modulation with UEP Using Block Partitioning

For a 64-QAM modulation, a six-level coded system can be constructed with the block partitioning. As in the case of 8-PSK, partitioning at each level is done such that signal points are contained in disjoint regions of the two-dimensional Euclidean space, as shown in Fig. 4. In this figure, the four less significant label bits $b_3b_2b_1b_0$ are shown in the quadrant corresponding to $b_1 = b_2 = 0$. In the other three quadrants, the same assignment of label bits $b_3b_1b_2b_0$ is used. The convention used to draw the points is the same as in Fig. 2, i.e., label bit $b_1$ determines the color and label bit $b_2$ the shape of the signal points. Once again, this partitioning approach results in a small number of NN for each stage of the multistage decoding.

1) Error Performance: The theoretical derivation of the probability of a bit error for six-level coded 64-QAM is similar to that for three-level coded 8-PSK. Based on the same method as in Section III-A-1 and in Appendix A with $j = 4$, it follows that at the first two decoding stages, the probability of a bit error, for $j = 1, 2$, is given by

$$P_{b_j} \leq \sum_{w=0}^{n} \frac{w}{n} A^{(j)}(w) \sum_{i_1=0}^{w-i_2} \sum_{i_3=0}^{w-i_1} \left( \begin{array}{c} w \\ i_1 \\ i_2 \\ i_3 \end{array} \right) Q \left( \sqrt{\frac{2RE_b}{N_0} d_{ij}(i_1, i_2, i_3)} \right) \quad (10)$$

where

$$d_{ij}(i_1, i_2, i_3) = \frac{1}{w} \left[ w - i_1 - i_2 - i_3 \right] \Delta_4 \left( w - i_1 - i_2 - i_3 \right) \Delta_4$$

and $\Delta_1 = 1/\sqrt{E_b}$, $\Delta_2 = 3\Delta_1$, $\Delta_3 = 5\Delta_1$, and $\Delta_4 = 7\Delta_1$.

Assuming correct decoding in the first and second stages, the bound on the bit-error probability in the third and fourth decoding stages becomes, for $j = 3, 4$

$$P_{b_j} \leq \sum_{w=0}^{n} \frac{w}{n} A^{(j)}(w) \sum_{i=0}^{w} \left( \begin{array}{c} w \\ i \end{array} \right) Q \left( \sqrt{\frac{2RE_b}{N_0} d_{ij}(i)} \right) \quad (11)$$

where

$$d_{ij}(i) = \frac{1}{w} \left[ i \Delta_1 + (w - i) \Delta_2 \right]^2.$$
Fig. 5. Error performance of a six-level coded 64-QAM modulation with block partitioning and four levels of error protection.

Finally, assuming that the previous decoding stages are correct, the probability of a bit error in the fifth and sixth stages is, for \( j = 5, 6 \)

\[
I_{b,j} \leq \sum_{w \in d_j} \frac{w}{n} A_w^j Q \left( \sqrt{\frac{2RE_b}{N_0}} w\Delta_t^2 \right). \tag{12}
\]

2) Simulation Results: Fig. 5 shows simulation results and theoretical upper bounds on the error performance of a six-level coded 64-QAM modulation with block partitioning. This scheme transmits 4.03125 bits/symbol. The component codes \( C_i \), \( 1 \leq i \leq 6 \), were selected as (64, 24, 16), (64, 24, 16), (64, 45, 8), (64, 51, 6), and (64, 57, 4) ex-BCH codes, respectively. As before, the simulation results agree with the upper bounds at the BER of \( 10^{-5} \) or less. A coding gain of 12 dB at the BER of \( 10^{-5} \), with respect to uncoded 16-QAM is obtained for 48 bits, or 18.6% of the information, encoded in the first two levels.

IV. HYBRID PARTITIONING

In this section, we present a partitioning approach that takes advantage of both the reduction of error coefficients, achieved by the block partitioning, and the increasing minimum intraset distances associated with Ungerboeck partitioning.

A. Three-Level Coded 8-PSK Modulation with Two-Level Error Protection Using Hybrid Partitioning

Fig. 6 depicts an 8-PSK signal set with points labeled by a hybrid partitioning. The first partition level is identical to a block partitioning. In the remaining partition levels, Ungerboeck’s partitioning rules [9] are used. At the third level, the MSED between signal points is 2.0, as opposed to 0.586 for the block partitioning. The price to pay for the corresponding improvement in performance of the third level is: 1) an increased number of NN at the second level and 2) a slightly more complex decoder for the second level code, since at the second partition level the subsets are no longer contained in half planes.

1) Error Performance: For the hybrid partitioning, the bit-error probability of the first decoding stage is also upper bounded by (7). When the decoded sequence at the first stage is correct, the constellation associated with the second decoding stage becomes a half 8-PSK constellation. Consequently, each signal point has either one or two NN as observed from Fig. 6. Then an error event will occur at the second stage whenever the decoded codeword of the second level has nonzero weight. Let \( u \) denote the weight of the incorrectly decoded codeword \( \bar{c}_w \).

With probability 1/2, a signal point with one neighbor (or with two neighbors) is selected. If we assume that for the \( w \) nonzero positions of \( \bar{c}_w \), \( i \) signal points have two NN and \( w-i \) signal points have one NN in the corresponding code sequence, then the probability of a block error associated with \( \bar{c}_w \) is upper bounded as in (6) by

\[
\Pr \{ \bar{c}_w \} \leq \sum_{i=0}^{w} \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{w-i} \binom{w}{i}^2 Q \left( \sqrt{\frac{2RE_b}{N_0}} w\Delta_t^2 \right) = \left( \frac{3}{2} \right)^w Q \left( \sqrt{\frac{2RE_b}{N_0}} w\Delta_t^2 \right) \tag{13}
\]

where \( \Delta_t = \sin(\pi/8) \). The last equality follows from the fact that \( \sum_{i=0}^{w} \binom{w}{i}^2 = 3^w \). Note that in (13), the value 3/2 simply represents the average number of NN associated with a half 8-PSK constellation. As in the case of block partitioning, when assuming systematic encoding of the second-level code, the probability of a bit error in the second decoding stage is upper bounded by

\[
I_{b2}^{NS} \leq \sum_{w=0}^{n} \frac{w}{n} A_w^{(2)} Q \left( \sqrt{\frac{2RE_b}{N_0}} w\Delta_t^2 \right) \tag{14}
\]

From (14), we observe that the second level of the hybrid partitioning has a smaller error coefficient factor than that of Ungerboeck’s partitioning, \((1.5)^w \) compared to \(2^w \).

To estimate the bit-error probability in the third decoding stage, we note that the second stage is more likely to be in error than with block partitioning. However, on the average, given that a decoding error is made at the second stage, the probability of a decoding error at the third stage is 1/2. Therefore, the bit-error probability at the third stage can be expressed as the sum of the contributions of errors from the second stage plus a conventional union bound for the third-level code. For \( \Delta = \sqrt{2}/2 \), it follows the approximated upper bound on the probability of a bit error at the third decoding stage

\[
I_{b3}^{NS} \approx \sum_{w=0}^{n} \frac{w}{n} A_w^{(3)} Q \left( \sqrt{\frac{2RE_b}{N_0}} w\Delta_t^2 \right) + \frac{1}{2} I_{b2}^{NS}. \tag{15}
\]
2) Simulation Results: The same code selection as in the three-level 8-PSK modulation example presented in Section III-A-2 was first simulated with hybrid partitioning. Fig. 7 presents the corresponding simulation results, with the practically optimum decoding achieved by the algorithm of [15]. For the half 8-PSK constellation obtained in the second level decoding after removing the contribution from the first decoding stage, it is shown in Appendix B that the metrics to be used by this algorithm are simply the X- and Y-coordinates of the received signal points scaled and added. Once again, we observe that the bounds match the simulation results at BER <10^{-5}. In this case, by proper selection of the component codes, two levels of error protection are achieved. Note also that the average coding gain is greater than the one obtained with the block partitioning.

For the block partitioning, at a BER of 10^{-5}, the degradation in coding gain for the second level code, with respect to the block partitioning of Section III-A-2, is about 2.3 dB. However, the advantage in coding gain for the third level code is approximately 4.4 dB at a BER of 10^{-5}. A good tradeoff is thus obtained between error performance loss at the second level (a larger error coefficient) and increase in intraset distance at the third level.

It is also possible to use convolutional codes as component codes. Fig. 8 presents plots of simulations and bounds on the error performance of a coded 8-PSK modulation with hybrid partitioning. As for the component codes, C1 is a best rate-1/3 memory-6 convolutional code with generators (in octal) (554, 624, 764) and minimum free distance 15, C2 is a rate-2/3 memory-6 punctured convolutional code, obtained from a rate-1/2 convolutional code with generators (133, 171), with minimum free distance 6, and C3 is a (30, 29) single parity-check code.

B. Six-Level 64-QAM with UEP Using Hybrid Partitioning

The principle of hybrid partitioning described in Section IV-A can be extended in a straightforward way to QAM signaling. In this section, we consider a 64-QAM squared constellation, while generalization to any QAM constellation follows the same lines.

1) Error Performance: For 0 ≤ i ≤ 5, the nonstandard labeling described in Fig. 4 is applied to the first i levels of the partitioning only, while Ungerboeck’s partitioning is applied to the remaining 6 − i levels. As a result, the intraset distance associated with this hybrid partitioning remains constant in the first i levels, and for 6-stage decoding, the error probabilities of these stages are upper-bounded by (10), (11), or (12). On the other hand, the intraset distance associated with the last 6 − i levels increases at the expense of the corresponding effective error coefficients. Table II summarizes the average number of NN β associated with level-(i+1).

Since for multistage decoding, each stage is decoded based on the assumption that any sequence is possible at the following stages, the bit-error probability for stage-(i+1) of this hybrid decoding is bounded by

\[ P_{b_{i+1}} \leq \sum_{w=\tilde{c}_{i+1}}^{n} \frac{w}{n} A_w^{(i+1)} 3^w Q \left( \sqrt{\frac{2R E_b}{N_0}} w \Delta_1^2 \right) \]  

(16)

with Δ1 = 1/\sqrt{42}. In (16), we assume that the probability of decoding errors propagating from the previous decoded stages is negligible. The bit-error probabilities of the remaining stages are evaluated based on the same principle, after proper choice of the corresponding values of β and Δ.

As an example, consider the case i = 2 for which the 64-QAM constellation is first partitioned into four 16-QAM constellations as in Fig. 4. Each 16-QAM constellation is then partitioned using Ungerboeck’s rules. Hence \( P_{b_3} \) and \( P_{b_2} \) are upper-bounded by (10), while from (16)

\[ P_{b_3} \leq \sum_{w=\tilde{c}_{i+1}}^{n} \frac{w}{n} A_w^{(3)} 3^w Q \left( \sqrt{\frac{2R E_b}{N_0}} w \Delta_1^2 \right) \]  

with \( \Delta_1 = 1/\sqrt{42} \). In (17), we assume that the probability of decoding errors propagating from the previous decoded stages is negligible. The bit-error probabilities of the remaining stages are evaluated based on the same principle, after proper choice of the corresponding values of β and Δ.

As an example, consider the case i = 2 for which the 64-QAM constellation is first partitioned into four 16-QAM constellations as in Fig. 4. Each 16-QAM constellation is then partitioned using Ungerboeck’s rules. Hence \( P_{b_3} \) and \( P_{b_2} \) are upper-bounded by (10), while from (16)

\[ P_{b_3} \leq \sum_{w=\tilde{c}_{i+1}}^{n} \frac{w}{n} A_w^{(3)} 3^w Q \left( \sqrt{\frac{2R E_b}{N_0}} w \Delta_1^2 \right) \]  

(17)
For level 4, we evaluate the corresponding average number of NN as 2.25, so that

$$P_{b4} \leq P_{b3} + \sum_{w=d_{4}}^{n} \frac{w}{n} A_{0}^{(4)}(2.25)^{w} Q \left( \sqrt{\frac{2RE_{b}}{N_{0}} w \Delta_{2}^{2}} \right)$$  \hspace{1cm} (18)$$

with $\Delta_{2} = 3 \Delta_{1}$. Note that the average number of NN associated with level 4 and $\tilde{i} = 2$ differs from the value of Table II for $\tilde{i} = 3$. Finally, for $\Delta_{3} = 5 \Delta_{1}$ and $\Delta_{4} = 7 \Delta_{1}$, we obtain

$$P_{b5} \leq P_{b4} + \sum_{w=d_{5}}^{n} \frac{w}{n} A_{0}^{(5)}(2)^{w} Q \left( \sqrt{\frac{2RE_{b}}{N_{0}} w \Delta_{3}^{2}} \right)$$  \hspace{1cm} (19)$$

$$P_{b6} \leq \frac{1}{2} P_{b5} + \sum_{w=d_{6}}^{n} \frac{w}{n} A_{0}^{(6)} Q \left( \sqrt{\frac{2RE_{b}}{N_{0}} w \Delta_{4}^{2}} \right).$$  \hspace{1cm} (20)$$

2) Simulation Results: Fig. 9 depicts the simulation results for hybrid partitioning with $\tilde{i} = 2$ of the BCM scheme with component codes: the $(64,45,8)$ ex-BCH code at levels 1, 2, and 3, the $(64,57,4)$ ex-BCH code at level 4, the $(64,63,2)$ ex-BCH code at level 5, and the $(64,64,1)$ ex-BCH code at level 6. This scheme of rate 4.984375 bits/symbol is compared with uncoded 32 cross-QAM signaling. We obtain a UEP scheme with 2 levels of protection. At the BER $10^{-5}$, coding gains of 7.4 and 1.6 dB over uncoded 32 cross-QAM signaling. We observe that at this BER, the upper bounds derived previously are very tight. The dominance of the effective error coefficients in the error performance is emphasized by this example, in which stages 1, 2, and 3 have the same asymptotic coding gain of 5.95 dB, a value totally irrelevant for describing the error performance of this scheme at practical BER’s.

V. CONCLUSIONS

Theoretical upper bounds and simulation results for multistage decoding of multilevel coded modulations for UEP have been presented. Bits-to-signal mappings by block and hybrid set partitionings of $2^{M}$-ary modulations were used to construct good coding schemes with UEP capabilities. In all cases, a very large coding gain is achieved for the most important bits.

The theoretical bounds derived in this paper are very tight and consequently constitute a powerful tool for designing good multilevel coded modulation schemes for UEP with multistage decoding. Based on the various hybrid partitionings of the signal constellation associated with an $M$-level coded modulation scheme, a large choice of UEP schemes with up to $M$ distinct levels of protection can be devised. The approach provides a generalization of the set partitioning method proposed in [9] for multistage decoding of multilevel coded modulation schemes with UEP properties. The conventional set partitioning of [9] simply corresponds to the special case where one level of protection, i.e., uniform error protection (or no UEP) is required.

APENDIX A

DETERMINATION OF THE MINIMUM DISTANCE BETWEEN A CODE-SEQUENCE AND ITS ASSOCIATED DECISION HYPERPLANE

In this appendix, we determine the minimum distance between the point

$$P = (\Delta_{1}, \Delta_{2}, \cdots, \Delta_{1}, \Delta_{2}, \Delta_{2}, \cdots, \Delta_{j}, \cdots, \Delta_{j})$$

and the hyperplane ($\pi$) of equation $\sum_{i=1}^{w} x_{i} = 0$ in the $w$-dimensional Euclidean space. The point $P$ is chosen such that the first $\alpha_{1}$ coordinates have value $\Delta_{1}$, the $\alpha_{2}$ following coordinates have value $\Delta_{2}$, ..., the last $\alpha_{j}$ coordinates have value $\Delta_{j}$ with $\sum_{i=1}^{j} \alpha_{i} = w$. Without loss of generality, any of the $\alpha_{1} \cdots \alpha_{j} \cdots \alpha_{j}$ points obtained by permutation of the coordinates of $P$ can also be considered due to the symmetry of the hyperplane ($\pi$) with respect to any axis $X_{i} = 0$. Let $X = (x_{1}, x_{2}, \cdots, x_{w})$ be the projection of the point $P$ onto the hyperplane ($\pi$). For $\beta_{l} = \sum_{i=1}^{l} \alpha_{m}$ with $l = 1, \cdots, j$, the point $X$ is determined by solving the optimization problem ($P$)

Minimize $f(X) = \sum_{l=1}^{j} \sum_{i=\beta_{l-1}+1}^{\beta_{l}} (x_{i} - \Delta_{l})^{2}$

subject to

$$\sum_{i=1}^{w} x_{i} = 0$$

with $\beta_{0} = 0$. By solving ($P$) with the Lagrange multiplier method, we obtain for $l = 1, \cdots, j$ and $i = \beta_{l-1} + 1, \cdots, \beta_{l}$

$$x_{i} = \Delta_{l} - \frac{1}{w} \cdot \sum_{m=1}^{j} \alpha_{m} \Delta_{m}$$

from which it follows that

$$f(X) = \left( \sum_{m=1}^{j} \alpha_{m} \Delta_{m} \right)^{2}$$.
APPENDIX B
DIFFERENTIAL COST EVALUATION FOR PSK SIGNALING
AND UNGERBOECK-TYPE PARTITIONING

Let \( \mathbf{r} = (r_{x}, r_{y}) \) represent the received symbol and let \( \mathbf{s}_1 = (A_{c1}, A_{s1}) \) and \( \mathbf{s}_2 = (A_{c2}, A_{s2}) \) be the two closest points to \( \mathbf{r} \) in the transmitted PSK signal constellation, where \( A \) represents the transmitted signal energy and \( c_i^2 + s_i^2 = 1 \), for \( i = 1, 2 \). Then if \( d_{E}(\mathbf{x}, \mathbf{y}) \) represents the SED between \( \mathbf{x} \) and \( \mathbf{y} \), we obtain

\[
  d_{E}^2(\mathbf{r}, \mathbf{s}_1) - d_{E}^2(\mathbf{r}, \mathbf{s}_2) = 2A(r_{c}(c_2 - c_1) + r_{s}(s_2 - s_1))
\]

which is proportional to \( \delta(\mathbf{r}) = r_{c}(c_2 - c_1) + r_{s}(s_2 - s_1) \).

The value \( \delta(\mathbf{r}) \), which is independent of \( A \), can be used as the differential cost in the algorithm of [15] applied to multistage decoding of a BCM scheme based on a PSK constellation with Ungerboeck partitioning. The values \( c_i - c_j \) and \( s_i - s_j \) are preprocessed so that \( r_{c} \) and \( r_{s} \) are simply scaled and added to evaluate the corresponding \( \delta(\mathbf{r}) \).

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