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A Heuristic for Marketing-Production Decisions in Industrial Channels of Distribution

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10. The probability of observing payoff outcome on a reported high and indeed high value property is:
   \[ P(H, ND, H) = P(H \mid ND) P(H) = (1 - p_h) \eta. \]

11. The probability of observing payoff outcome on a reported low but in fact high value property is:
   \[ P(L, ND, H) = P(L \mid ND) P(H) = 0. \]

12. Lastly, the probability of observing payoff outcome on a reported low and indeed low value property is:
   \[ P(L, ND, L) = P(L \mid ND) P(L) = (1 - p_l) \eta (1 - \eta). \]
have proposed heuristics to arrive at optimal pricing and inventory policies in a distribution channel environment.

In a recent paper, Eliashberg and Steinberg [6]—henceforth referred to as "ES87"—employ an optimal control theoretical approach to derive optimal pricing, processing, and inventory policies for both manufacturer and distributor in an industrial channel of distribution using a Stackelberg game theoretical model. In that article, they propose a novel approach to provide explicit policies for the manufacturer and distributor operating in a vertical distribution channel environment. This was one of the first papers to use an "indirect adjoining" approach in their optimal control solution for this type of problem structure. The issue addressed in this paper deals with the specific assumption on the nature of the time horizon, from 0 to T, and its impact on the optimal policies. In their model, ES87 specify a quadratic formulation for the market potential term, \( a_0(t) \). They say:

In order to capture the seasonality effect, we have chosen to model the market potential term, \( a_0(t) \), through a quadratic formulation which provides interesting interpretations. That is

\[
a_0(t) = -\alpha_1 \alpha_2 t^2 + \alpha_3, \quad 0 \leq t \leq T,
\]

where \( T = \alpha_1/\alpha_2 \) and \( \alpha_1, \alpha_2, \alpha_3 > 0 \).

Here, \( \alpha_1 \) represents the "nominal" size of the market potential before the season begins. The parameters \( \alpha_1 \) and \( \alpha_2 \) determine the timing \( (\alpha_1/2\alpha_2) \) and the magnitude \( ((\alpha_1/4\alpha_4) + \frac{1}{2}) \) of the peak sales. It is straightforward to show that for larger values of \( \alpha_2 \), the peak sooner and will lower its magnitude, whereas larger values of \( \alpha_1 \) have opposite effects. Finally, \( T \) is set equal to \( \alpha_1/\alpha_2 \) in order to encompass the season in its entirety [p. 988].

ES87 assume that the season starts at 0 and terminates at \( T \). The length of the season, \( T \), equals to \( \alpha_1/\alpha_2 \), which is the time the market potential drops back to its "nominal" size \( (\alpha_1) \). Once the values of \( \alpha_1 \), \( \alpha_2 \), and \( \alpha_3 \) are specified, the start and terminal times for the season are fixed and are not sensitive to any changes in the remaining specific and firm-specific parameters such as \( b_0, K_0, h_0, etc. \).

Furthermore, in order to simplify their analysis, ES87 assume an interior solution while deriving the optimal policies of the distributor. They write:

In order to simplify the analysis below, we assume an interior solution. That is, \( a_2b_2 > b_0 > b_{20} \) and \( 0 < b_0' < b_0 \) for \( 0 \leq t \leq T \) [p. 997].

The assumption of an interior solution ensures the length of the season to be \( [0,T] \). This is a very restrictive assumption which can be violated very easily. In fact, the optimal policies in the numerical example provided in ES87 violate this assumption.

The violation of this assumption can occur when at least one of the constraints in the optimal control problem examined in ES87 is not being met. As an example, the demand for the product can drop to zero before the end of the season, presumably at \( T \), even if there is positive demand potential. To resolve this issue, we introduce two new variables \( t_5 \) and \( t_4 \), which represent the start and terminal values of the season. These values reflect the points in time at which the season "effectively starts and terminates." Note that \( t_4 \geq 0 \) and \( t_5 \leq T \). During this interval \([t_4,t_5]\), all constraints pertaining to both channel members are satisfied at the pre-specified parameter values (such as \( b_0, K_0, h_0, etc \)).

2 Generalized Optimal Policies

We re-solve the optimal control problem of ES87 under the assumption that an interior solution exists for the interval \([t_4,t_5]\), which is a subset of \([0,T]\). The generalized optimal policies for distributor and manufacturer are provided below:

**Distributor’s Policies:**

Propositions 1, 2, and 3 of ES87 [p. 986], which allude to the nature of pricing, processing, and inventory policies of the distributor, also hold in the case of variable start and terminal time of the season. However, the condition in Proposition 4 of ES87 [p. 989], under which the distributor can smooth out his operations in contrast to when he should follow a stockless production policy, need to be revised as follows:

**Proposition 4 (revised).**

In general, if the distributor’s inventory holding cost per unit is sufficiently low, price sensitivity is low, processing efficiency is low, and the seasonal demand is volatile, he can smooth out his operations. In particular, if:

(i) \( h_0 < (\alpha_2 - 2\alpha_5)/3((b_0 + K_0)) \), the distributor can smooth out his operations.

(ii) \( h_0 \geq (\alpha_2 - 2\alpha_5)/3((b_0 + K_0)) \), the distributor should not smooth out his operations and act according to stockless production policy throughout the season.

**Corollary 5 (revised).**

The optimal pricing, processing, and inventory policies for the distributor are:

\[
Q_0(t) = \begin{cases} 
(\alpha_2/2(\alpha_5)(\alpha_5 - 1)(\alpha_5 - 2)b_0 + K_0)(\alpha_5 - 1) & 0 \leq t \leq t_4 \,
\end{cases}
\]

\[
P_0(t) = \begin{cases} 
(\alpha_2/2(\alpha_5)(\alpha_5 - 1)(\alpha_5 - 2)b_0 + K_0)(\alpha_5 - 1) & 0 \leq t \leq t_4 \,
\end{cases}
\]

where}

\[
I_0(t) = \begin{cases} 
(\alpha_2/2(\alpha_5)(\alpha_5 - 1)(\alpha_5 - 2)b_0 + K_0)(\alpha_5 - 1) & 0 \leq t \leq t_4 \,
\end{cases}
\]
HEURISTIC FOR MARKETING-PRODUCTION DECISIONS

\[ t^* = \left(3/(4a_1)\right)\left(\alpha_1 - (b_2 + K_0)h_0^* - (2a_1t_3)/2\right). \]  \hspace{1cm} (4)

Proof See Appendix A

Manufacturer’s Policies:
Propositions 6, 7, and 8 of ES87 [p. 190-1], which allude to the nature of pricing, processing, and inventory policies of the manufacturer, also hold in the case of variable start and terminal time of the season.

Corollary 9 (revised).
The optimal pricing, processing, and inventory policies for the manufacturer are:

\[ Q_M(t) = \left\{ \begin{array}{ll}
(K_0/(2(b_0+K_0))(\alpha_1 - (b_2 + K_0))h_0^* - (2a_1t_3)/3), & t_3 \leq t \leq t^*, \\
(K_0/(2(b_0+K_0))(\alpha_1 - (b_2 + K_0))h_0^* - (2a_1t_3)/3), & t_1^* \leq t \leq t_1,
\end{array} \right. \]  \hspace{1cm} (5)

where,

\[ t_3^* = \left(3/(4a_1)\right)\left(\alpha_1 - (b_2 + K_0)(b_2 + K_0)h_0^* - (2a_1t_3)/3\right), \]  \hspace{1cm} (6)

\[ t_1^* = \left(3/(4a_1)\right)\left(\alpha_1 - (b_2 + K_0)(b_2 + K_0)h_0^* - (2a_1t_3)/3\right), \]  \hspace{1cm} (7)

where,

\[ w_1 = (1/(2b_n/K_0))(2+2b_n/K_0), \quad w_2 = 1/(2+2b_n/K_0), \]  \hspace{1cm} (8)

and

\[ b_n = b_2K_0/(2(b_2+K_0)). \]  \hspace{1cm} (9)

Proof See Appendix B

Comparison with ES87
One distinct result is that both t_3 and t_1 computed in the generalized policies case are lower than those given in ES87 by the value \( (t_3-2a_1t_3)/3 \).

3 Derivation of Season’s Start and Terminal Times
In the above section, we have provided the generalized optimal policies for both the distributor and the manufacturer operating in an industrial channel. However, the correct values of the start and terminal times (\( t_s \) and \( t_f \)) of the season are yet to be determined. It is only logical to initially assume the values of \( t_s \) and \( t_f \) to be 0 and T respectively, which corresponds to the full length of the season. In this case, the generalized policies degenerate to those provided by ES87. Nevertheless, these values of \( t_s \) and \( t_f \) may violate one (or more) of the constraints of the optimal control problems listed in Appendices A and B. In such a case, the values of \( t_s \) and \( t_f \) need to be updated so as to satisfy all the constraints, and establish the applicability of the optimal policies for members in the industrial distribution channel. This is achieved through an iterative process as specified in the heuristic which is provided below.

The Heuristic
Step 1: Read the parameter vector \( \Theta' = (b_0, K_0, b_n, K_n, h_n, \alpha_1, \alpha_2, \alpha_3) \). Set \( t_s^0 = 0, t_f^0 = T \) (n=1).

Step 2: Test if the condition \( h_0 < \left(\alpha_2 - 2a_1t_3\right)/(3(b_0+K_0)) \), holds [refer Proposition (revised); condition (l)]. If the condition is not satisfied, then go to Step 14.

Step 3: Use generalized policies of the manufacturer to obtain \( P_M^t \) [refer Corollary 9].

Step 4: Individually solve the constraints equations [refer distributor as manufacturer problems in Appendices A and B] for the inventory stocking period (denoted by subscript I). Obtain the boundary values of t from each equation.

Step 5: Select the maximum value of t from all roots of the constraints associate with the inventory stocking period. Call this \( t_3^0 \).

Step 6: If \( t_3^0 < 0 \) then \( t_3^0 = 0 \).

Step 7: Individually solve the constraints equations [refer distributor as manufacturer problems in Appendices A and B] for the stockless period (denoted by subscript 2). Obtain the boundary values of t from each equation.

Step 8: Select the minimum value of t from all roots of the constraints associate with the stockless period. Call this \( t_2^0 \).

Step 9: If \( t_2^0 \geq T \) then \( t_2^0 = T \).

Step 10: If \( t_3^0 = 0 \) and \( t_2^0 = T \). Go to Step 14.

Step 11: Compute the value of \( P_M^t \) from the following equation:

\[ P_M^t = w_1\left(\left(1/(t_f-t_3)\right)\int_{t_3}^{t_f} (a_0/(b_3+K_0))dt\right) + w_2c_n \]  \hspace{1cm} (refer equation 7). Here \( w_1, w_2, \) and \( c_n \) are computed from equations 8 and 9.

Step 12: Check if \( |P_M^t - P_M^{t-1}| \leq \epsilon \) (here \( \epsilon \) is a pre-specified infinitesimal value). If true, go to Step 14.

Step 13: Set n=n+1. Go to Step 2.
Step 14: Use the generalized optimal policies for both distributor and manufacturer (refer Corollaries 5 and 9), where \( t_s^* = t_s^0 \), \( t_r^* = t_r^0 \), and \( P_M = P_M^* \). Go to Step 16.

Step 15: Stop. The channel members should follow the stockless policy [refer Proposition 4 (revised): condition (ii)].

Step 16: End.

The intuition behind the heuristic is as follows: For some specific parameter values of the problem, we compute the solutions for the optimal control problem including \( P_M \) based upon the demand interval \([0,T]\). We then check if one or more constraint equations are violated and if the assumption of interior solution is invalid. In such a case, we get a boundary solution, taking into account the binding constraint. As a result, we obtain the values of \( t_s \) and \( t_r \) which will satisfy all the constraints. Note that the length of the time interval \([t_s, t_r]\) is a subset of the original interval \([0,T]\).

Since the manufacturer is the Stackelberg leader, she would revise (increase) the value of \( P_M \) based on the new information on \( t_s \) and \( t_r \). What follows is an iterative process of computing \( t_s \), \( t_r \) and \( P_M \) till a point of convergence is reached. This provides us with the final (equilibrium) values of \( t_s \) and \( t_r \) which when used in the generalized optimal policies would, in fact, result in true optimal profits for the channel members.

### 4 Numerical Example

To illustrate the heuristic, we use the example presented in ES87. Specifically,

\[
\begin{align*}
   h_0 &= 1/20, \quad K_M = 2, \quad a_0(t) = -t^4 + 6t + 12, \\
   h_M &= 1/30, \quad K_M = 2, \quad [a_0 = 1, \quad a_1 = 6, \quad \alpha_1 = 12], \\
   b_0 &= 1, \quad C_M = 3, \\
   w_1 &= 4/7, \quad w_2 = 3/7.
\end{align*}
\]

Here, \( T = a_1/\alpha_1 = 6 \), \( w_1 = 1/3 \), \( w_1 = 4/7 \), and \( w_2 = 3/7 \) [p. 992-3].

The results obtained from ES87 policies and from the heuristic are tabulated in Table 1 for comparison purposes. In this example, the initial value of \( P_M \) is calculated to be 11.9571 for the interval \([0,6]\).

Following the heuristic, we find that the constraint, which requires the distributor's price to be less than what the market can bear at all times (refer equation (16); this constraint is similar to equation (2.10) in ES87, p. 988), is violated for the above parameter values. Hence, the correct start and terminal times of the "effective" season are not 0 and 6 as assumed by ES87. Therefore, the values of \( t_s \) and \( t_r \) need to be recalculated and \( P_M \) needs to be revised subsequently. The values of \( t_s^0 \) and \( t_r^0 \) (after the initial constraint validity check) are found to be 0.3738 and 6 respectively. Using these values in equation (7), \( P_M \) is calculated to be 12.1463. This \( P_M \) value, along with \( t_s^0 \) and \( t_r^0 \), is used to revise the optimal pricing, processing, and inventory constraint functions. Once again, a validity check on the revised constraint functions is conducted which provides \( t_s^* = 0.4495 \) and \( t_r^* = 5.9670 \). After subsequent iterations, the final values of \( t_s \) and \( t_r \) are computed to be 0.4495 and 5.9670. Also, the final value of \( P_M \) is 12.1970.

### 5 Variable Price Sensitivity Case

To further test the robustness of the heuristic, we preset the values of all parameters to those used in numerical example provided in section 4, except one—say the sensitivity of the distributor \( (b_0) \), which is treated as a variable.

| Table 2 Constraint validity check using heuristic for various values of \( b_0 \) (using \( t_s^0 \) and \( t_r^0 \)) |
|-----------------|-----------------|-----------------|-----------------|
| \( b_0 \)       | Eq. 13          | Eq. 14          | Eq. 15          |
| 0.25            | \( P_M \)       | \( t_s \)       | \( t_r \)       |
| 1.00            | \( P_M \)       | \( P_M \)       | \( P_M \)       |
| 2.00            | \( P_M \)       | \( P_M \)       | \( P_M \)       |
| 4.00            | \( P_M \)       | \( P_M \)       | \( P_M \)       |
| 6.00            | \( P_M \)       | \( P_M \)       | \( P_M \)       |

Inventory

<table>
<thead>
<tr>
<th>Period 1</th>
<th>( Q_{DI} )</th>
<th>( Q_{DJ} )</th>
<th>( Q_{DI} )</th>
<th>( Q_{DJ} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-76.1219f</td>
<td>-93.0638</td>
<td>-10.4891</td>
<td>-3.3063</td>
</tr>
<tr>
<td>11</td>
<td>-21.9375</td>
<td>-3.4156</td>
<td>-43.0638</td>
<td>-10.4891</td>
</tr>
<tr>
<td>12</td>
<td>-5.3063</td>
<td>-43.0638</td>
<td>-10.4891</td>
<td>-3.3063</td>
</tr>
</tbody>
</table>


The inventory constraints are used to arrive at \( t_0 \) and \( t_M \) values and, therefore, do not influence the constraint validity check process.

All the constraints have the right hand side as \( \geq 0 \) except the last constraint \( (P_0 - P_M) \) which has to be \( > 0 \).

The values provided in bold face characters are the relevant roots of \( t \) which represent the boundary point of the constraint.

The underlined values are the values of \( t \) that are most restrictive and belong to the most binding constraint.

Table 2 illustrates the procedure used by the heuristic to perform the constraint validity check for various values of \( b_0 \) for the initial run. For example, when \( b_0 =0.25 \), all the constraints listed in the first column of Table 2 are satisfied and, therefore, ESS policies are valid from \([0,6]\).

However, when \( b_0 =1 \), constraint equation (16) is violated in the inventory stocking period (Period 1) which results in \( t^* \geq 0.3788 \) (refer Table 2). After going through one iteration, we revise the optimal pricing, processing, and inventory constraint functions of both distributor and manufacturer and subsequently conduct a valid check on these constraints. This results in \( t^* =0.4370 \) and \( t_M =5.9755 \). The results from constraint validity check after one iteration are compiled in Table 3.

The heuristic computes the final values \( t_0, t_M \) through the iterative process. We find that the effective season is reduced on both ends to \([0.4495, 5.9670]\).

Table 3 Constraint validity check using heuristic for various values of \( b_0 \) (using \( t_0^* \) and \( t_M^* \))

<table>
<thead>
<tr>
<th>Equation #</th>
<th>Constraints</th>
<th>( b_0 =0.25 )</th>
<th>( b_0 =1.00 )</th>
<th>( b_0 =2.00 )</th>
<th>( b_0 =3.00 )</th>
<th>( b_0 =4.00 )</th>
<th>( b_0 =5.00 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(13)</td>
<td>( t_0 )</td>
<td>0.44156</td>
<td>0.3788</td>
<td>0.8079</td>
<td>1.0796</td>
<td>1.2725</td>
<td></td>
</tr>
<tr>
<td>(13)</td>
<td>( t_1 )</td>
<td>4.1981</td>
<td>3.9461</td>
<td>3.7727</td>
<td>3.6388</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(14)</td>
<td>( t_2 )</td>
<td>6.3263</td>
<td>5.9755</td>
<td>5.5476</td>
<td>5.1280</td>
<td>4.6578</td>
<td></td>
</tr>
<tr>
<td>(15)</td>
<td>( t_3 )</td>
<td>6.3263</td>
<td>5.9755</td>
<td>5.5476</td>
<td>5.1280</td>
<td>4.6578</td>
<td></td>
</tr>
<tr>
<td>(16)</td>
<td>( t_4 )</td>
<td>6.3263</td>
<td>5.9755</td>
<td>5.5476</td>
<td>5.1280</td>
<td>4.6578</td>
<td></td>
</tr>
</tbody>
</table>

The initial and final values of \( t_0, t_M \) and \( P_M^* \), along with the resulting profits of channel members are listed in Table 4.

For \( b_0 =2 \), constraint equation (16) again proves to be most binding, now for inventory stocking and stockless periods (periods 1 and 2) which gives us \( t^*_0 =0.8C \) and \( t^*_1 =5.6608 \) (refer Table 2). After one iteration, we get \( t_0^* =1.0032 \) and \( t_1^* =5.54 \).
(refer Table 3). Finally, the effective length of the season is calculated to be [1.0570, 5.5144] (refer Table 4).

<table>
<thead>
<tr>
<th>Values</th>
<th>( b_0 = 0.25 )</th>
<th>( b_0 = 1.00 )</th>
<th>( b_0 = 2.00 )</th>
<th>( b_0 = 3.00 )</th>
<th>( b_0 = 4.00 )</th>
<th>( b_0 = 5.00 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>( t_0^0 )</td>
<td>( t_0^0 )</td>
<td>( t_0^0 )</td>
<td>( t_0^0 )</td>
<td>( t_0^0 )</td>
<td>( t_0^0 )</td>
</tr>
<tr>
<td>( t_0^0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( t_1^0 )</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( P_{M0}^0 )</td>
<td>39.7421</td>
<td>11.9571</td>
<td>6.9600</td>
<td>5.1923</td>
<td>4.2750</td>
<td>3.7105</td>
</tr>
<tr>
<td>After one iteration:</td>
<td>( t_0^1 )</td>
<td>0</td>
<td>0.3788</td>
<td>0.8079</td>
<td>1.0796</td>
<td>1.2725</td>
</tr>
<tr>
<td>( t_1^1 )</td>
<td>6</td>
<td>6</td>
<td>5.6608</td>
<td>5.3288</td>
<td>4.9748</td>
<td>4.5644</td>
</tr>
<tr>
<td>After two iterations:</td>
<td>( t_0^2 )</td>
<td>0</td>
<td>0.4370</td>
<td>1.0032</td>
<td>1.4113</td>
<td>1.7485</td>
</tr>
<tr>
<td>( t_1^2 )</td>
<td>6</td>
<td>5.9755</td>
<td>5.5476</td>
<td>5.1280</td>
<td>4.6578</td>
<td>4.1864</td>
</tr>
<tr>
<td>Results: ( b_0 )</td>
<td>0.1111</td>
<td>0.3333</td>
<td>0.5000</td>
<td>0.6000</td>
<td>0.6667</td>
<td>0.7143</td>
</tr>
<tr>
<td>( t_0 )</td>
<td>0</td>
<td>0.4495</td>
<td>1.0570</td>
<td>1.5166</td>
<td>1.9286</td>
<td>2.3077</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>6</td>
<td>5.9670</td>
<td>5.5144</td>
<td>5.0679</td>
<td>4.5560</td>
<td>4.0241</td>
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<tr>
<td>( P_{M0} )</td>
<td>39.7421</td>
<td>11.9571</td>
<td>6.9600</td>
<td>5.1923</td>
<td>4.2750</td>
<td>3.7105</td>
</tr>
<tr>
<td>( \Pi_{M} )</td>
<td>388.0200</td>
<td>41.6194</td>
<td>6.7678</td>
<td>1.3402</td>
<td>0.1998</td>
<td>0.0557</td>
</tr>
<tr>
<td>( \Pi_{M} + \Pi_{D} )</td>
<td>730.6790</td>
<td>82.0480</td>
<td>14.3582</td>
<td>2.9622</td>
<td>0.4539</td>
<td>0.1111</td>
</tr>
</tbody>
</table>

We arrive at similar results for \( b_0 \) of 3 and 4 as constraint equation (16) again is violated for both periods. In contrast, for \( b_0 = 5 \), similar analysis cannot be done since constraint equation (30) [this constraint is similar to equation (3.5) in ES87] is violated, which implies that the manufacturer has a negative profit margin. As a result, no solution exists for this (or a larger) value of distributor's price sensitivity, \( b_0 \).

6 Implications and Conclusions

One of the implications of only considering the season interval to be \([t_0, t_1]\) (instead of \([0, T]\) as in ES87) to determine the channel member policies is that the resulting optimal profits of the channel members are lower than what were originally claimed by ES87. Also, from Table 4, note that as \( b_0 \) increases, the 'effective' length of the season decreases, which in turn, causes \( P_M \) to increase from its initial value. Also observe that, with an increase in the value of \( b_0 \), the time for which the stockless policies are in effect gets diminished. This time interval is \([t_0, t_1]\) for the distributor and \([t_0, t_2]\) for the manufacturer.

Although, not exhibited in Table 4, for certain parameter values, it is quite possible that the length of season dictates the employment/non-employment of stock policy by the manufacturer (in mathematical notations, the possibility that \( t_M = t_1 \) even the distributor. In such a scenario, both distributor's and manufacturer's inventory will become zero at the same point at the end of season, i.e., \( t_0 = t_1 = t_T \) implies that, for certain parametric conditions, the channel members are potentially implement single part policies, rather than two-part policies as posited by ES87.

In summary, in this paper, we have provided the generalized optimal policies (ES87) for the channel members where the start and terminal times of a season considered variables. We have also proposed a heuristic which, when used in conjunction with the generalized optimal policies, will compute the actual horizon during which the optimal policies will provide correct results. We demonstrated the appropriateness of this heuristic through a numerical example similar to the one presented by ES87. The robustness of the heuristic is subsequently tested by varying the distributor's price sensitivity, then computing effective season \([t_0, t_T]\) and the corresponding profits of the channel members. This paper has similar implications to the research article by Eliashberg and Steinberg [7].

7 References

9. Khouja, M., Saydam, C. Vergara, F.E., and Rajagopalan, H.K., "Optimal inventory policy under continuous unit cost decrease and risk of sur


Appendix A

The distributor's revised continuous profit maximization problem is formulated as:

\[
\max_{P_D(t), Q_D(t)} \int_{t_0}^{t_f} \left( (P_D(t) - P_M)(Q_D(t) - a_0 + b_0 P_D(t)) - (1/K_0)(Q_D(t))^2 - b_0 I_D(t) \right) dt
\]

s.t.

\[
I_D(t) = Q_D(t) - a_0 + b_0 P_D(t),
\]

\[
I_D(t) \geq 0,
\]

\[
Q_D(t) \geq 0,
\]

\[
P_D(t) > P_M,
\]

\[
P_D(t) < a_0 + b_0 P_M,
\]

\[
I_D(t) = I_D(t_0) = 0.
\]

The solution procedure of the revised problem is the same as the one presented in Appendix A of ES87 [p. 996-8]. For this reason, the nature of the optimal pricing and processing policies remain the same as ES87. In summary these policies are:

**Unconstrained Segment**

(Inventory is positive, \(\lambda_D(t) = \lambda_D(t_0) + b_0 P_D(t)\)):

\[
Q_D^* = Q_D^*(t_0) + K_0 \lambda_D^* / 2,
\]

\[
P_D^* = P_D^*(1/(2) \lambda_D^* + a_0 + b_0 P_M).
\]

**On a Boundary Segment**

(Inventory is zero, \(\lambda_D(t_0) + b_0 P_D(t_0) = \psi_D(t_0) = (a_0 - b_0 P_M) / (b_0 + K_0)\)):

\[
Q_D^* = Q_D^*(t_0) + K_0 \psi_D^* / 2 = (a_0 - b_0 P_M) / (2(b_0 + K_0)),
\]

\[
P_D^* = P_D^*(1/(2) \psi_D^* + a_0 + b_0 P_M).
\]

The determination of \(t_D^*\), the time at which entry to the boundary occurs (the point at which the distributor moves from a stocking to a stockless inventory policy), is achieved through the simultaneous solution of the following two equations:

\[
Q_D^*(t_D^*) = Q_D^*(t_D^*),
\]

\[
\int_{t_0}^{t_D^*} I_D(t) dt = 0.
\]

The first equation ensures that at the boundary point, the production level \(P_D\) is the same, whereas the second equation ensures that inventory is carried over in the stockless period after \(t_D^*\). Note that equation differs from the respective one presented in ES87 since the lower limit of the integral is set to \(t_0\) to accommodate a solution in the new interval \([t_0, t_f]\).

The solution of equations (25) through (26) results in \(\lambda_D(t_0)\) and \(t_D^*\), which substituted in equations (18) through (24) yield the distributor's optimal pricing and processing policies as presented in corollary 5.

**Proof of Proposition 4**

For \(t_D^*\) to exist, it must lie to the right of the point at which \(\psi_D\) reaches its maximum. This point is \(\psi_D(2a_D)\). Therefore, \(t_D^* > \psi_D(2a_D)\). Substituting equation (4) gives the necessary parametric condition under which the distributor can follow the optimal policies of the revised problem.
Appendix B

Proof of Corollary 9

After rearranging equation (1) the optimal quantity produced by the distributor can be written as:

\[ Q'_d(t) = \begin{cases} 
Q_0^n(0) - a_{u}(0) - b_u P_u & 0 \leq t \leq t'_0 \\
Q_0^n(0) - a_{c}(0) - b_c P_c & t'_0 \leq t \leq T, 
\end{cases} \]

where \( b_u = \frac{b_c K_u}{(b_0 + K_u)} \), and

\[ a_u(t) = \begin{cases} 
-\frac{K_u}{b_c} (a_0(t) + K_u a_0(0)) & 0 \leq t \leq t'_0 \\
-\frac{K_u}{b_c} (a_0(t) + K_u a_0(0)) & t'_0 \leq t \leq T. 
\end{cases} \]

Therefore the manufacturer's problem can be written as:

\[
\begin{align*}
\max & \quad \mathcal{M}(P_u) \\
\text{s.t.} & \quad C_u < P_u < \mathcal{M}_u, \\
& \quad a_u(0) b_u, \\
& \quad Q_0^n(0) - a_{c}(0) - b_c P_c, \\
& \quad t_0 < t \leq t'_0, \\
& \quad t'_0 \leq t \leq T, \\
& \quad a_{u}(t) = a_{u}(0) b_u, \\
& \quad t'_0 \leq t \leq T, \\
& \quad Q_0^n(0) - a_{c}(0) - b_c P_c, \\
& \quad t'_0 \leq t \leq T. 
\end{align*}
\]

where

\[ \mathcal{M}(P_u) = \max \quad Q_0^n(t) \int_{t_0}^{t_0'} (P_u - C_u) (a_u(t) + b_u P_u - Q_0^n(t)) \, dt \quad \text{s.t.} \quad Q_0^n(t) = \min \quad a_u(0) b_u, \]

The obtained solution for \( t'_0 \), the time at which entry to the boundary occurs where the inventory is zero, and the value of \( \mathcal{M}(P_u) \), we need to solve the following two equations:

\[
\begin{align*}
Q_0^n(t'_0) & = Q_0^n(t_0) \\
\mathcal{M}(P_u) & = \mathcal{M}(P_u),
\end{align*}
\]

Following a similar solution procedure as in Appendix B of ES87 [p. 998-9], and assuming an interior solution, we get the following manufacturer's optimal processing policies:

Unconstrained Segment

Inventory is positive, \( a_{u}(t) > a_{u}(0) b_u \) for:

\[ Q'_u = Q_0^n - K_u \lambda_u/2, \quad (37) \]